On Ultrabarrelled Spaces, their Group Analogs and Baire Spaces

In Honour of Manuel López-Pellicer, Loyal Friend and Indefatigable Mathematician

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Abstract Let E and F be topological vector spaces and let G and Y be topological abelian groups. We say that E is *sequentially barrelled with respect to* F if every sequence $(u_n)_{n\in\mathbb{N}}$ of continuous linear maps from E to F which converges pointwise to zero is equicontinuous. We say that G is *barrelled with respect to* F if every set $\mathscr{H}$ of continuous homomorphisms from G to F, for which the set $\mathscr{H}(x)$ is bounded in F for every $x \in E$, is equicontinuous. Finally, we say that G is g*-barrelled with respect to* Y if every $\mathscr{H} \subseteq \mathrm{CHom}(G, Y)$ which is compact in the product topology of Y^G is equicontinuous. We prove that

- a barrelled normed space may not be sequentially barrelled with respect to a complete metrizable locally bounded topological vector space,
- a topological group which is a Baire space is barrelled with respect to any topological vector space,
- a topological group which is a Namioka space is g-barrelled with respect to any metrizable topological group,
- a protodiscrete topological abelian group which is a Baire space may not be g-barrelled (with respect to $\mathbb{R}/\mathbb{Z}$).

We also formulate some open questions.

Keywords Topological vector space · Locally convex space · Equicontinuity · Barrelledness · Ultrabarrelledness · Topological group · Baire space · Namioka space

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1 Main Results

A locally convex space E is said to be barrelled if every closed, absorbing and absolutely convex subset of E is a neighborhood of zero. Barrelled (real) locally convex spaces were introduced by N. Bourbaki in [3] and they have been extensively studied in many references as the monographs [10, 16].

Already in [3] the following characterization of barrelled spaces can be found:

Theorem 1.1 *Let E be a locally convex space. The following properties are equivalent:*

(i) E is barrelled.
(ii) If F is a nontrivial locally convex Hausdorff topological vector space and $\mathscr{H}$ *is a set of continuous linear mappings from E to F for which the set*

$$\mathscr{H}(x) = \bigcup_{u \in \mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$*, then* $\mathscr{H}$ *is equicontinuous.*

Local convexity of F is essential for the validity of implication $(i) \Rightarrow (ii)$ of Theorem 1.1. It seems that this fact was pointed out for the first time in [23].

W. Robertson obtained in [17, Theorem 4] the following characterization: a locally convex space E with topology η is barrelled if and only if the only locally convex vector space topologies with bases of η-closed neighborhoods of the origin are those coarser than η. This motivated the following definition, included in the same reference:

Definition 1.1 Let E be a topological vector space under the topology η. We say that E is *ultrabarrelled* if the only vector space topologies on E, compatible with the algebraic structure of E and in which there is a base of η-closed neighbourhoods of the origin, are those coarser than η.

For ultrabarrelled spaces we have the following nice analogue of Theorem 1.1:

Theorem 1.2 (W. Robertson, L. Waelbroeck) *For a topological vector space E the following properties are equivalent.*

(i) E is ultrabarrelled.
(ii) If F is a topological vector space and $\mathscr{H}$ *is a set of continuous linear mappings from E to F, for which the set*

$$\mathscr{H}(x) = \bigcup_{u \in \mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$*, then* $\mathscr{H}$ *is equicontinuous.*

The implication $(i) \Rightarrow (ii)$ was proved in [17], where the validity of $(ii) \Rightarrow (i)$ was posed as a question as well. It seems that a (rather delicate) proof of the implication $(ii) \Rightarrow (i)$ appeared for the first time in [22, Proposition I.5]. A proof of Theorem 1.2 is presented also in [1, §7.3] (where the term 'barrelled' is used instead of 'ultrabarrelled').

It is clear that any ultrabarrelled locally convex space is barrelled. In [17] an argument based on an idea from [23] was used to show that the converse implication may fail:

Theorem 1.3 ([17, p. 256]) *There is a normed space which is barrelled but not ultrabarrelled.*

To formulate our first theorem we need to introduce the following concept:

Definition 1.2 Let E be a topological vector space and F a Hausdorff topological vector space. We say that E is *sequentially barrelled with respect to F* if every sequence $(u_n)_{n\in\mathbb{N}}$ of continuous linear maps from E to F which converges pointwise to zero is equicontinuous.

Clearly every ultrabarrelled space is sequentially barrelled with respect to any topological vector space. Hence the following result is a refinement of Theorem 1.3:

Theorem 1.4 *A barrelled normed space need not be sequentially barrelled with respect to a complete metrizable, locally bounded topological vector space.*

Question 1.1 Let E be a normed space which is sequentially barrelled with respect to every complete metrizable (locally bounded) topological vector space. Is then E ultrabarrelled?

The following result establishes a natural connection between ultrabarrelledness and the property of being a Baire space:

Theorem 1.5 ([17, Proposition 12]) *Let E be a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If E as a topological space is a Baire space, then E is ultrabarrelled.*

In view of Theorem 1.5 the underlying topological space of a barrelled normed space which is not ultrabarrelled cannot be a Baire space. According to [7], the first example of a normed barrelled space which is not Baire appeared in [9]; see also [8, 20] for more examples of this sort.

Definition 1.3 Let G be a topological group and F be a topological vector space over a nontrivially valued division ring $\mathbb{K}$. We say that G is *barrelled with respect to F* if every set $\mathscr{H}$ of continuous homomorphisms from G to F, for which the set

$$\mathscr{H}(x) = \bigcup_{u\in\mathscr{H}} \{u(x)\}$$

is bounded in F for every $x \in E$, is equicontinuous.

We shall prove the following statement, which generalizes a similar one obtained in [13] for the case of a normed space F.

Theorem 1.6 *Let G be a topological group and F be a topological vector space over $\mathbb{R}$. If G as a topological space is a Baire space, then G is barrelled with respect to F.*

Question 1.2 Let G be a topological group and F be a topological vector space over a nontrivially valued division ring $\mathbb{K}$. If G as a topological space is a Baire space, is then G barrelled with respect to F?

Definition 1.4 Let X be a topological space.

- X is called *a Namioka space* ([6]), or is said to have *the Namioka property*, if for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$, there exists a dense G_δ-subset A of X such that f is continuous at every point of $A \times K$.
- X is called *a weak Namioka space*, or is said to have *the weak Namioka property*, if for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$, there exists $a \in X$ such that f is continuous at every point of $\{a\} \times K$.

Proposition 1.1 *(a) Let X be a topological space. Assume that for every compact Hausdorff space K, every metrizable space Z and every separately continuous $f : X \times K \to Z$ there exists a dense subset A of X such that f is continuous at every point of $A \times K$. Then X is a Namioka space.*

(b) (A. Bouziad, oral communication) Let X be a topological space. Assume that every element of X admits a neighborhood which is a Namioka space. Then X is a Namioka space.

Proof (a) This follows from the following known fact (see [15, p. 518]): for a separately continuous $f : X \times K \to Z$ the set

$$A(f) := \{a \in X : f \text{ is continuous at every point of } \{a\} \times K\}$$

is always a G_δ-subset of X.

(b) Let $f : X \times K \to Z$ be a separately continuous map, where Z is a metric space and K is a compact space. Taking into account (a), we only have to show that the set $A(f)$ is dense in X. Let U be a nonempty open subset of X. Choose x in U and let V be a neighborhood of x in X such that V is a Namioka space. Choose also an open subset W of X such that $x \in W$ and $W \subset V$. The mapping $g := f|_{V \times K} \to Z$ is separately continuous; since V is a Namioka space, the set

$$A(g) := \{a \in V : g \text{ is continuous at every point of } \{a\} \times K\}$$

is dense in V. From this, since $U \cap W$ is a nonempty open subset of V, we get that $A(g) \cap (U \cap W) \neq \emptyset$. Fix an element $a \in A(g) \cap (U \cap W)$. Clearly, f is jointly continuous at each point of $\{a\} \times K$.

The next theorem contains several known results about Namioka spaces.

Theorem 1.7 *The following statements hold:*

(a) [15, Theorem 1.2] If X is a strongly countably complete regular space, then X is a Namioka space. In particular, if X is a Čech complete Tychonoff space, then X is a Namioka space.
(b) [19, Théorème 3] If X is a completely regular Namioka space, then X is a Baire space.
(c) [19, Théorème 7] If X is a metrizable Baire space, then X is a Namioka space.
(d) [21, Théorème 2] There exists a completely regular Hausdorff Baire space, which has not the weak Namioka property.
(e) [21, Corollaire 6] If X is a Baire space which contains a dense σ-compact subset, then X is a Namioka space.
(f) [18] If X is a Baire space which contains a dense $\mathscr{K}$-countably determined subset, then X is a Namioka space.
(g) [2, p. 333] If X is a pseudocompact space, then X is a Namioka space.

Proposition 1.2 *If X is a locally pseudocompact space, then X is a Namioka space.*

Proof This follows from Theorem 1.7(g) and Proposition 1.1(b).

Definition 1.5 Let G be a topological group and Y a Hausdorff topological group. We say that

- G is *g-barrelled with respect to Y* if every $\mathscr{H} \subseteq \mathrm{CHom}(G, Y)$ which is compact in the product topology of Y^G is equicontinuous.
- G is *sequentially g-barrelled with respect to Y* if every sequence $\{u_n\}_{n\in\mathbb{N}}$ contained in $\mathrm{CHom}(G, Y)$ which converges pointwise to zero, is equicontinuous.

In the case where Y is the compact group $\mathbb{R}/\mathbb{Z}$ we will drop the reference to Y and use the shorter expression "(sequentially) g-barrelled group".

g-barrelled topological abelian groups were introduced in [5]. Corollary 1.6 in this reference provides some classes of g-barrelled groups. Also, several permanence properties of this class were established in [5], but only recently it was proved that the class of g-barrelled groups is closed with respect to Cartesian products [4].

For our purposes it is convenient to highlight the following results from [5]:

Theorem 1.8 *Let G and Y be topological groups.*

(a) If G as a topological space is a Baire space, then G is sequentially g-barrelled with respect to Y (cf. [5, Proposition 1.4]).
(b) If G and Y are metrizable and all closed separable subgroups of G are Baire spaces, then G is g-barrelled with respect to Y (cf. [5, Theorem 1.5]).

Here we shall prove the following statements:

Theorem 1.9 *Let G be a topological group and Y be a metrizable topological group.*

(a) If G as a topological space is a Namioka space, then G is g-barrelled with respect to Y.
(b) If G as a topological space is locally pseudocompact, then G is g-barrelled with respect to Y.

Remark 1.1 The following particular case of Theorem 1.9(b) was obtained earlier in [11, Proposition 4.4]: every pseudocompact topological abelian group is g-barrelled.

Question 1.3 Let G be a Hausdorff (locally quasi-convex abelian) topological group, which is g-barrelled with respect to every metrizable (abelian) topological group Y. Is then G as a topological space a weak Namioka space?

It was shown in [21] that the Namioka spaces form a proper subclass of the class of Baire spaces. By using a construction of [21], we will show that Theorem 1.9(a) is no longer true if we replace "Namioka space" with "Baire space", thus answering the question posed in [14, Remark 2.2]. We denote by $\mathbb{Z}(2)$ the 2-element abelian group $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.10 *There exists a protodiscrete (in particular, locally quasi-convex) Hausdorff topological abelian group G with the following properties:*

(a) G as a topological space is a Baire space.
(b) G is not g-barrelled with respect to the discrete group $\mathbb{Z}(2)$. *In particular, G is not g-barrelled.*

There also exists a submetrizable topological abelian group which as a topological space is a Baire space, but which is not g-barrelled (A. Bouziad, personal communication).

Remark 1.2 In [12] it was introduced a notion of a *g-ultrabarrelled* topological group. This class admits the following remarkable characterization: a Hausdorff topological abelian group G is g-ultrabarrelled iff every closed group homomorphism from G into any separable complete metrizable topological group is continuous [12, Theorem 3.1]. In [12] it is noticed also that any topological group which is a Baire space, is g-ultrabarrelled. From this and Theorem 1.10 it follows that a Hausdorff topological abelian protodiscrete (hence, locally quasi-convex) g-ultrabarrelled group may not be g-barrelled.

2 The Proofs

Proof of Theorem 1.4

Fix a number p with $0 < p \leq 1$ and consider the sequence space

$$l_p = \{\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

endowed with the p-norm

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}, \quad \mathbf{x} \in l_p .$$

Let us write

$$(l_p)_1 := (l_p, \|\cdot\|_1) .$$

Now we can formulate the following statement, which implies Theorem 1.4.

Theorem 2.1 *Let* $0 < p < 1$. *Then*

(a) $(l_p)_1$ *is a barrelled normed space.*
(b) $(l_p)_1$ *is not sequentially barrelled with respect to* l_p.

Proof (a) is proved in [17, 23].
(b) Fix $n \in \mathbb{N}$ and consider the linear mapping $u_n : (l_p)_1 \to l_p$ defined by the equality

$$u_n(\mathbf{x}) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad \mathbf{x} \in l_p .$$

We have:

$$\|u_n(\mathbf{x})\|_p \leq \|\mathbf{x}\|_p, \quad \mathbf{x} \in l_p , \tag{1}$$

and

$$\|u_n\| := \sup\{\|u_n(\mathbf{x})\|_p : \mathbf{x} \in l_p,\ \|\mathbf{x}\|_1 \leq 1\} = n^{\frac{1}{p}-1} . \tag{2}$$

Fix now a number r with $0 < r < \frac{1}{p} - 1$ and write

$$v_n = \frac{1}{n^r} u_n .$$

Then we have
(C1) $v_n : (l_p)_1 \to l_p$ is a continuous linear mapping.
(C2) $\lim_n \|v_n(\mathbf{x})\|_p = 0$ for every $\mathbf{x} \in l_p$. This follows from (1).
(C3) The sequence $(v_n)_{n\in\mathbb{N}}$ is not equicontinuous at $0 \in (l_p)_1$. In fact, from (2) we have

$$\|v_n\| = n^{\frac{1}{p}-(r+1)} . \tag{3}$$

The equicontinuity of $(v_n)_{n\in\mathbb{N}}$ at $0 \in (l_p)_1$ would imply that

$$\sup_n \|v_n\| < \infty$$

in contradiction with (3).

Proof of Theorem 1.6

Let $\mathscr{H}$ be a set of continuous homomorphisms from G to F for which $\mathscr{H}(x)$ is bounded in F for every $x \in G$. Fix a zero neighborhood $W \in \mathscr{N}(F)$. We are going to find $O \in \mathscr{N}(G)$ with $u(O) \subset W$ for every $u \in \mathscr{H}$, which means that $\mathscr{H}$ is equicontinuous at $0 \in G$.

Fix a symmetric closed $W_1 \in \mathscr{N}(F)$ with $W_1 + W_1 \subset W$. Write

$$X_n = \bigcap_{u \in \mathscr{H}} u^{-1}(nW_1), \quad n = 1, 2, \ldots$$

The boundedness in F of $\mathscr{H}(x)$ for every $x \in G$ implies

$$G = \bigcup_{n \in \mathbb{N}} X_n . \tag{4}$$

Since the sets $X_n,\ n = 1, 2, \ldots$ are closed and G is a Baire space, we can find and fix $n_0 \in \mathbb{N}$ such that

$$U := \mathrm{Int}(X_{n_0}) \neq \emptyset .$$

Pick $x_0 \in U$. Then

$$V := U - x_0 \in \mathscr{N}(G) .$$

It is easy to check that

$$u(x) = u(x + x_0) - u(x_0) \in n_0 W_1 + n_0 W_1 \subset n_0 W, \quad \forall x \in V,\ \forall u \in \mathscr{H} . \tag{5}$$

Find and fix now $O \in \mathscr{N}(G)$ such that $O + \overset{n_0}{\cdots} + O \subset V$. As

$$x \in O \Rightarrow n_0 x \in V ,$$

from (5) we get

$$n_0 u(x) = u(n_0 x) \in n_0 W \quad \forall x \in O,\ \forall u \in \mathscr{H} .$$

Hence $u(O) \subset W\ \forall u \in \mathscr{H}$, as required.

Proof of Theorem 1.9

We will prove the following stronger version of Theorem 1.9:

Theorem 2.2 *Let G be a topological group and Y be a metrizable topological group.*

(a) If G as a topological space is a weak Namioka space, then G is g-barrelled with respect to Y.

(b) If G as a topological space is locally pseudocompact, then G is g-barrelled with respect to Y.

Proof (a) Fix a set $\mathscr{H}$ of continuous homomorphisms from G to Y which is compact in the product topology of Y^G. Consider the mapping $f : G \times \mathscr{H} \to Y$ defined as follows:

$$f(x, u) = u(x), \quad x \in G,\ u \in \mathscr{H}.$$

Then f is separately continuous. Since G has the weak Namioka property, there exists an element $a \in G$ such that f is continuous at every point of $\{a\} \times Y$. From this, according to [15, Lemma 2.1] we can conclude that the set

$$\{f(\cdot, u) : u \in \mathscr{H}\} = \mathscr{H}$$

is equicontinuous at a. Since $\mathscr{H}$ consists of homomorphisms, we obtain that Y is equicontinuous.
(b) This follows from (a) and Proposition 1.2.

Proof of Theorem 1.10

Let I be a fixed uncountable set. For $f \in \mathbb{Z}(2)^I$ we denote by supp f the support of f, i. e. the set of all $i \in I$ such that $f(i) = 1$. Write

$$G = \{f \in \mathbb{Z}(2)^I : \text{card}(\text{supp } f) \le \aleph_0\}.$$

Consider on G the group topology which admits as a basis of neighborhoods of zero the sets of the form

$$U_J := \{f \in G : f(i) = 0\ \forall i \in J\}$$

where J runs through all subsets of I with card $(J) \le \aleph_0$. Since U_J is a subgroup of G for every J, this is a protodiscrete group topology.

(1) G is a Baire space [21].

Let $K = \beta I$ be the Stone-Čech compactification of the discrete space I and let $C(K, \mathbb{Z}(2))$ be the the set of all continuous mappings $h : K \to \mathbb{Z}(2)$. Let us identify G with a subset of $C(K, \mathbb{Z}(2))$ as follows: to each $f \in G$ corresponds its unique continuous extension $\tilde{f} : K \to \mathbb{Z}(2)$. Consider the mapping $\Phi : G \times K \to \mathbb{Z}(2)$ defined by the equality

$$\Phi(f, h) = \tilde{f}(h) \quad \forall (f, h) \in G \times K\,.$$

We have

(2) For a fixed $h \in K$ the mapping $\Phi(\cdot, h)$ is continuous on G [21].
(3) For each $f \in G$ there exists $h \in K$ such that Φ is not continuous at (f, h) [21]. Consequently G is not a weak Namioka space.

Note also that for each $h \in K$ the mapping $\Phi(\cdot, h)$ is a group homomorphism from G to $\mathbb{Z}(2)$ (indeed, this is so when $h = i \in I$ by the definition of the group operation of G; the general case follows from the density of I in K).

Clearly the set of continuous homomorphisms

$$\mathscr{H} = \{u \in \mathbb{Z}(2)^G : \exists h \in K,\ u(\cdot) = \Phi(\cdot, h)\}$$

is pointwise compact, but it is not equicontinuous (as the equicontinuity of $\mathscr{H}$ at 0 would imply that Φ is continuous at each point of $\{0\} \times K$, which is not the case by (3) above).

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