II) Geometry of the symmetries in dimension 4 = (1+[1)+"2"] , and general Time-Space-Spin vectors (matrices).

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We write the vectors in dimension three in terms of square matrices, which we diagonalize. We propose a parametrization for these vectors, with angle variables $(\mathbf{2}\varphi\,,\,\phi)$, different to the usual one. We append a time type dimension, we define a second parametrization for their 4-vectors, and we write the vectors (the spin type matrices) as functions of these new angle variables $(\varphi\,,\,\phi)$. We study the symmetries for some specially related values of one of the angle variables $(\,\varphi\,)$, and we also consider various specific values of this variable. We define an intermediated product and anticommutators. We include a first brief approximation to the spin, the vector-spin and the chirality with their possible implications at the end.

Keywords: time and space spin matrices, symmetries, anticommutators.

I - INTRODUCTION

We express the three and four dimensional vectors, geometrical objects, in an algebraic way in relation to square $2x2$ matrices, via the identity and the Pauli matrices [1] [2]. We do not use quaternions, but the geometrical treatment underneath is obvious. In a first stage this representation has the meaning of Cartesian coordinates. In a second stage we transform it by defining a new parametrization with the angle type variables 2φ and ϕ. The diagonatilzation of these vectors is straightforward, and it provides two matrices, $\mathbf{R}_o^{\varphi}(\phi)$ and $\mathbf{R}_e^{\varphi}(\phi)$, which are one of the keystones for this research. The ϕ angle has an essential role at the beginning but afterwards it is almost ignored. It is also immediate the extension of the Pauli spin matrices to spin matrices related to any direction in the three dimensional space. We handle these questions in sections II, III and IV.

We include a time type variable with implications in both, the geometrical representation of the angle variable φ, and the form of some matrices, denoted as *Time-Space-Spin vectors (matrices)*. The form of these matrices suggests to consider various symmetries after $\frac{\pi}{2}$ differences in the φ angle and with its opposite $-\varphi$. The inclusion of some $\frac{\pi}{4}$ differences in φ drives us to specific values of this angle variable (see table 2). These symmetries, which imply the stated values of the φ angle, suggest us the definitions and classification of the elementary fermions in *Study III* **1**. This is the content of sections V, VI and VII.

Up to this point, this work can be considered another form of doing mathematics for some geometrical objects. Our interest settles in the physic of the fermions. This motivates the definition of anticommutators with the introduction of an antisymmetrical product and of the spin, the vector-spin and the chirality. This is in sections VIII, IX and X.

There are two sets of statements which are succinctly presented: a) (43) with the rest of the paragraphs in that Section, and b) the ones in the Sections XI (the digression). We require the other parts of this set of studies in order to properly work out them. The program of studies is in the Appendix C.

II - PRELIMINARY: ROTATIONS IN $\Re^3$.

Let w_t be the time-like and w_x, w_y, w_z the space-like coordinates of a vector in $\Re x \Re^3$. We write these vectors in the form:

$$\mathbb{w} \;\equiv\; w_t\,\mathbb{1} \;+\; w_x\,\sigma^x + w_y\,\sigma^y + w_z\,\sigma^z \;=\; \begin{pmatrix} w_t + w_z & w_x - i w_y \\ w_x + i w_y & w_t - w_z \end{pmatrix} \,, \tag{1}$$

where the identity with the Pauli matrices constitute a basis of the linear space $\Re x \Re^3$, [1] [2] and they are:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \;. \tag{2}$$

These matrices are Hermitian matrices, therefore $\mathbb{w}$ is Hermitian. Also $\mathrm{Tr}\,\sigma^z = \mathrm{Tr}\,\sigma^x = \mathrm{Tr}\,\sigma^y = 0$. They satisfy:

$$\sigma^j\,\sigma^k = \delta^{jk}\;\mathbb{1} + \mathrm{i}\,\epsilon^R j\,k\,l\sigma^l, \qquad \text{so that:} \qquad \begin{cases} [\,\sigma^j, \sigma^k\,] \;=\; \mathrm{i}\,2\,\epsilon^{jkl}\;\sigma^l \\ \{\,\sigma^j, \sigma^k\,\} \;=\; 2\,\delta^{jk}\;\mathbb{1} \end{cases} ; \qquad \{j, k, l\} = \{x, y, z\}\,,$$

with $\{A, B\} \equiv A\,B{+}B\,A$ and $[A, B] \equiv A\,B{-}B\,A$. Using these matrices, we define the Jordan Wigner matrices σ^+, σ^- [3][4][5]:

$$\sigma^+ \;\equiv\; \tfrac{1}{2}(\sigma^x{+}\,\mathrm{i}\,\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- \;\equiv\; \tfrac{1}{2}(\sigma^x{-}\,\mathrm{i}\,\sigma^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3a}$$

$$\text{and} \qquad \overset{\wedge}{\sigma} \equiv \sigma^+\sigma^- {=}\; \tfrac{1}{2}(\mathbb{1}{+}\sigma^z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \overset{\vee}{\sigma} \equiv \sigma^-\sigma^+ {=}\; \tfrac{1}{2}(\mathbb{1}{-}\sigma^z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3b}$$

where $\overset{\wedge}{\sigma}$, $\overset{\vee}{\sigma}$ are the number matrices. Verifying: $\overset{\wedge}{\sigma}{}^2 = \overset{\wedge}{\sigma}$, $\overset{\vee}{\sigma}{}^2 = \overset{\vee}{\sigma}$, $\sigma^{+2} = \sigma^{-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{0}$, $\{\sigma^+, \sigma^-\} = \mathbb{1}$, $[\sigma^+, \sigma^-] = \sigma^z$.

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We define the matrices $\mathbf{R}_o^{\varphi}(\phi)$ and $\mathbf{R}_e^{\varphi}(\phi)$:

$$\begin{cases} \mathbf{R}_o^{\varphi}(\phi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi\, e^{-i\phi} \\ \sin\varphi\, e^{i\phi} & \cos\varphi \end{pmatrix} = \mathbf{R}_e^{\varphi}(\phi)\, \sigma^Z \\ \mathbf{R}_e^{\varphi}(\phi) \equiv \begin{pmatrix} \cos\varphi & \sin\varphi\, e^{-i\phi} \\ \sin\varphi\, e^{i\phi} & -\cos\varphi \end{pmatrix} = \mathbf{R}_o^{\varphi}(\phi)\, \sigma^Z \end{cases} , \tag{4}$$

with $\varphi \in [-\pi, \pi)$ and $\phi \in [0, \pi)$, in order to have also negative values for a new variable τ. We write useful formulas in the Appendix A.

We express a rotation over the axis defined with the unit vector $\mathbf{n} \equiv (n_x, n_y, n_z)$ and the angle 2α :

$$\mathbb{w}' \equiv \mathcal{R}[\mathbb{w}] \equiv \mathbb{R}\, \mathbb{w}\, \mathbb{R}^{\dagger} , \tag{5}$$

$\mathbb{R}^{\dagger}$ the rotation matrix:

$$\mathbb{R}^{\dagger} \equiv e^{i\alpha\mathbb{n}} = \cos\alpha\, \mathbb{1} + i \sin\alpha\, \mathbb{n} = \begin{pmatrix} \cos\alpha + i\sin\alpha\, n_z & i\sin\alpha\,(n_x - in_y) \\ i\sin\alpha\,(n_x + in_y) & \cos\alpha - i\sin\alpha\, n_z \end{pmatrix} , \tag{6}$$

and

$$\mathbb{n} = \mathbb{n}(n_x, n_y, n_z) \equiv [\mathbf{n}\,.\,\boldsymbol{\sigma}] \equiv n_z\sigma^Z + n_x\sigma^X + n_y\sigma^Y \equiv n_z\sigma^Z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}} = \begin{pmatrix} n_z & \tau\, e^{-i\phi} \\ \tau\, e^{i\phi} & -n_z \end{pmatrix} , \tag{7}$$

with $n_x = \tau\cos\phi$, $n_y = \tau\sin\phi$ and $|\tau| = \sqrt{1 - n_z^2}$, it is

$$\|\mathbf{n}\| = \sqrt{-\det(\mathbb{n})} \equiv \sqrt{n_x^2 + n_y^2 + n_z^2} \equiv \sqrt{\tau^2 + n_z^2} = 1, \qquad -1 \le n_j \le 1, \quad j = \{x, y, z\} . \tag{8}$$

$\mathbb{w}'$ is Hermitian: the realness of the quantities $w_j{}'$ is guaranteed by the form of the transformation (5), once we depart from a Hermitian $\mathbb{w}$ (realness of the w_j). It has to be remarked the introduction of the imaginary unit i, even working with the reals. Also, σ^+ and σ^- have a key role in our study. They are not Hermitian. Therefore, our framework space is $\mathfrak{C}^4$, not $\mathfrak{R}^4$. This point deserves more attention and we treat it in *Study I.2*.

III - DIAGONALIZING THE MATRICES $\mathbb{n}$ (UNIT VECTORS $\mathbf{n}$): $\mathbb{n} = \mathbf{P}_{\pm}^{\varphi}\, \sigma^Z\, \mathbf{P}_{\pm}^{\varphi^{-1}}$.

We refer in what follows to the Figure 1 in Appendix B. The labels n_x, n_y, n_z, τ denote both, the axes and specific values over them (we have to take care of the signs). The label 2φ indicates the value of an angle variable. $\mathbb{n}$ is an actual function of 2φ. We denote a reference value of 2φ as $2\varphi_R$, such that: $2\varphi_R \in [0, \frac{\pi}{2}]$.

By using this angle we will accomplish the formulation of various symmetries:

in the axis n_z with $-2\varphi_R$, in the *'axis'* τ (plane {X,Y}) with $-2(\varphi_R - \frac{\pi}{2})$, and in the origin with $2(\varphi_R - \frac{\pi}{2})$.

We define τ and n_z in terms of the new angle variable φ by means of:

$$n_z(\varphi) \equiv \cos(2\varphi) , \qquad \tau(\varphi) \equiv \sin(2\varphi) \;,, \qquad -\pi \le 2\varphi < \pi , \qquad -1 \le \tau \le 1. \tag{9}$$

This parametrization is different to the usual one, for which one it is imposed: $0 \le 2\varphi \le \pi$ and $\begin{cases} 0 \le \tau \le 1 \\ 0 \le \phi < 2\pi \end{cases}$.

The values in $(\tau, n_z) = (\pm 1, 0)$ are related to $2\varphi = \pm\frac{\pi}{2}$, and in $(\tau, n_z) = (0, \pm 1)$ with $2\varphi \in \{0, -\pi\}$.

In relation to the third dimension, we define the polar decompositions in the plane {X,Y} ($\tau \ne 0$):

$$v = n_x + i n_y = \tau\cos\phi + i\tau\sin\phi = \tau e^{i\phi} = e^{\log\tau + i\phi 1} = \text{sign}(\tau)\, e^{\log|\tau| + i\phi 1}$$

$$\text{with} \quad \tau^2 = n_x^2 + n_y^2 , \quad \tau \in [-1, 0) \cup (0, 1] \quad \text{and} \quad \text{sign}(\tau) \equiv \begin{cases} +1 & \text{if } \tau > 0 \quad (n_y > 0) \\ -1 & \text{if } \tau < 0 \quad (n_y < 0) \end{cases}$$

$$\begin{cases} \tau = +\sqrt{n_x^2 + n_y^2} \quad \text{with} \begin{cases} \phi = 0 & n_x > 0,\ n_y = 0 \\ \phi \in (0, \pi) & n_y > 0 \end{cases} & \phi \quad \text{counterclockwise} \quad (\text{from } \mathfrak{R}^+) \\ \tau = -\sqrt{n_x^2 + n_y^2} \quad \text{with} \begin{cases} \phi \in (0, \pi) & n_y < 0 \\ \phi = 0 & n_y = 0,\ n_x < 0 \end{cases} & \text{either } \phi \begin{cases} a) \text{ counterclockwise} & (0, \pi) \\ b) \text{ clockwise} & (\pi, 0) \end{cases} \end{cases} . \tag{10}$$

If $\Delta\phi = \phi_2 - \phi_1 > 0$ (counterclockwise) then $\phi_2 \pm \pi - (\phi_1 \pm \pi) = \Delta\phi > 0$ (counterclockwise).

Or also, in the plane {X,Y} $\left(\| \mathbf{r} \| = +\sqrt{n_z^2+\tau^2} = 1 \right)$:

$$\text{for} \quad 0 \le 2\varphi < \pi \quad \Longrightarrow \quad \tau \in [\,0,1]\,, \qquad \begin{matrix} 0 \le \phi < \pi \\ \tau \ne 0 \end{matrix} \qquad \begin{cases} \tau \ge (n_x = \tau\cos\phi) > -\tau, \\ n_y = \tau\sin\phi \in [0\,,\tau] \end{cases},$$

$$\text{for} \quad -\pi \le 2\varphi < 0 \quad \Longrightarrow \quad \tau \in [-1,0]\,, \qquad \begin{matrix} \pi \le \phi+\pi < 2\pi \\ \tau \ne 0 \end{matrix} \qquad \begin{cases} -|\tau| \le (n_x = |\tau|\cos(\phi+\pi)) < |\tau|, \\ n_y = |\tau|\sin(\phi+\pi) \in [-|\tau|\,,0] \end{cases}.$$

Finally, $(n_x,n_y) = (\pm1,0)$ with $(\phi = 0\,,\,\tau = \pm1)$, and $(n_x,n_y) = (0,\pm1)$ with $(\phi = \frac{\pi}{2}\,,\,\tau = \pm1)$.

Summarizing:

$$\begin{cases} n_z \equiv r\cos(2\varphi), \quad \tau \equiv r\sin(2\varphi), \quad r \equiv 1, \qquad -\pi \le 2\varphi < \pi, \quad \Longrightarrow \begin{cases} -1 \le n_z \le 1 \\ -1 \le \tau \le 1 \end{cases}, \\[2ex] \left.\begin{matrix} \tau > 0, & n_x \equiv \tau\cos\phi, & n_y \equiv \tau\sin\phi, & 0 \le \phi < \pi \\ \tau < 0, & n_x \equiv |\tau|\cos(\phi+\pi), & n_y \equiv |\tau|\sin(\phi+\pi), & \pi \le \phi+\pi < 2\pi \end{matrix}\right\} \begin{matrix} \mathrm{sign}(n_y) = \mathrm{sign}(\tau) \\ (\tau \ne 0) \end{matrix} \end{cases}. \tag{11}$$

The matrix $\mathbb{n}$ in (7), for the previous $(2\varphi,\phi)$, and with $s = \mathrm{sign}(n_z)\,\mathrm{sign}(n_y)$, is:

$$\mathbb{n} \equiv \mathbb{n}(2\varphi,\phi) = \mathrm{sign}(n_z)\begin{pmatrix} |n_z| & s\,|\tau|\,\mathrm{e}^{-\mathrm{i}\phi} \\ s\,|\tau|\,\mathrm{e}^{\mathrm{i}\phi} & -|n_z| \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi)\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin(2\varphi)\,\mathrm{e}^{\mathrm{i}\phi} & -\cos(2\varphi) \end{pmatrix}, \tag{12}$$

We keep the notation $\mathbb{n}$ for the values of $2\varphi \in [0,\frac{\pi}{2}] \cup [-\pi,-\frac{\pi}{2}]$ (red vectors in Figure 1); for them $s = +$. In the following two formulas we introduce the notation: "•" for the values with $s = -$ which correspond to $2\dot{\varphi} \in [0,-\frac{\pi}{2}] \cup (\pi,\frac{\pi}{2}]$ (green vectors in Figure 1). We use a doted vector with the vector in (7), denoted as $\dot{\mathbb{n}} \equiv \dot{\mathbb{n}}(2\dot{\varphi},\phi) = \mathbb{n}(-2\varphi,\phi) = \sigma^Z\,\mathbb{n}(2\varphi,\phi)\,\sigma^Z$, with opposite values in τ to the previous ones for a given n_z, or also in 2φ. We write these vectors in the form:

$$\dot{\mathbb{n}} \equiv \dot{\mathbb{n}}(2\dot{\varphi},\phi) \equiv \begin{pmatrix} \cos(2\dot{\varphi}) & \sin(2\dot{\varphi})\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin(2\dot{\varphi})\,\mathrm{e}^{\mathrm{i}\phi} & -\cos(2\dot{\varphi}) \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi)\,\mathrm{e}^{-\mathrm{i}\phi} \\ -\sin(2\varphi)\,\mathrm{e}^{\mathrm{i}\phi} & -\cos(2\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{sign}(n_z)\,|n_z| & -|\tau|\,\mathrm{e}^{-\mathrm{i}\phi} \\ -|\tau|\,\mathrm{e}^{\mathrm{i}\phi} & -\mathrm{sign}(n_z)\,|n_z| \end{pmatrix}, \tag{13}$$

$2\dot{\varphi} \equiv -2\varphi$. 2φ either $2\varphi_R$ or $2(\varphi_R-\frac{\pi}{2})$. They are diagonalized in the following forms:

$$\mathbb{n} = \mathbf{P}_{\pm}^{\varphi}\,\sigma^Z\,\mathbf{P}_{\pm}^{\varphi\;-1}\,, \qquad \dot{\mathbb{n}} = \mathbf{P}_{\pm}^{\dot{\varphi}}\,\sigma^Z\,\mathbf{P}_{\pm}^{\dot{\varphi}\;-1} = \mathbf{P}_{\pm}^{-\varphi}\,\sigma^Z\,\mathbf{P}_{\pm}^{-\varphi\;-1}\,, \tag{14}$$

where

$$\mathbf{P}_{\pm}^{\varphi} = \begin{pmatrix} \cos\varphi & \mp\sin\varphi\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\varphi\,\mathrm{e}^{\mathrm{i}\phi} & \pm\cos\varphi \end{pmatrix}. \tag{15}$$

$\mathbf{P}_{\pm}^{\varphi}$ is unitary. We also have: $\mathbf{P}_{+}^{\varphi}$, denoted as $\mathbf{R}_O^{\varphi}$, satisfies $\mathbf{P}_{+}^{\varphi}\mathbf{P}_{+}^{-\varphi} = \mathbb{1}$ and $\mathbf{P}_{-}^{\varphi}$, denoted $\mathbf{R}_e^{\varphi}$; satisfies $(\mathbf{P}_{-}^{\varphi})^2 = \mathbb{1}$. The formulas for these matrices are in (4).

Therefore, we diagonalize $\mathbb{n}(\tau,\phi,n_z) = \mathbb{n}(2\varphi,\phi)$ with $\tau^2+n_z^2 = 1$, by using two different types of matrices, $\mathbf{R}_O^{\varphi}$ and $\mathbf{R}_e^{\varphi}$. Their dependence in ϕ, in general, will not be explicitly showed. The relevant formulas for these matrices are in the Appendix A.

If we let $-\pi \le \varphi < \pi$, then we have for $n_z(\varphi)$ and $\tau(\varphi)$:

in φ and in $\varphi \pm \frac{\pi}{2}$, opposite values: $n_z(\varphi \pm \frac{\pi}{2}) = -n_z(\varphi)$ and $\tau(\varphi \pm \frac{\pi}{2}) = -\tau(\varphi)$,

in φ and in $\varphi \pm \pi$, equal values: $n_z(\varphi \pm \pi) = n_z(\varphi)$ and $\tau(\varphi \pm \pi) = \tau(\varphi)$,

and for $\mathbf{R}_O^{\varphi}$ and $\mathbf{R}_e^{\varphi}$ opposite values of: $\mathbf{R}_O^{\varphi\pm\pi} = -\mathbf{R}_O^{\varphi}$ and $\mathbf{R}_e^{\varphi\pm\pi} = -\mathbf{R}_e^{\varphi}$.

The matrices $\mathbf{R}_O^{\varphi}$ *and* $\mathbf{R}_e^{\varphi}$ are unitary. They are not orthogonal matrices, they have complex values not just real values, except for $\phi = 0$ or π. Although this, we design these matrices as **C-Rotation** and **C-Reflection** matrices, respectively, due to their similar form with the ones in $O(2)$. The determinant of $\mathbf{R}_O^{\varphi}$ is $+1$, meanwhile the determinant of $\mathbf{R}_e^{\varphi}$ is -1; in this way $\mathbf{R}_O^{\varphi}$ belongs to $\mathbf{SU}(2)$. In a bit different context, working with $\mathbf{O}(3)$ and $\mathbf{SO}(3)$, Altmann and Herzig proposed for the unitary matrices with determinant ±1 the notation $\mathbf{SU}'(2)$, "as the name $\mathbf{U}(2)$ is pre-empted". Pages 118-119 in [6]. Also [7].

IV - CONSTRUCTION OF A NEW BASIS FOR THE LINEAR SPACE OF 2x2 MATRICES RELATED TO AN ARBITRARY VECTOR $\mathbb{n}$: SPACE-SPIN VECTORS (MATRICES).

With the canonical basis of $\mathfrak{C}^2$, $\left\{ \mathrm{e}_1 = \begin{pmatrix}1\\0\end{pmatrix} \;,\; \mathrm{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} \right\}$, we consider a new set of orthonormal vectors, formed by the eigenvectors of $\mathbb{n}$ appearing in (15) ((14)):

$$\{ v_1^{\varphi},\, v_2^{\varphi} \} \equiv \{v_1 \,, v_2\}(\varphi, \phi) \equiv \left\{ \begin{pmatrix}\cos\varphi \\ \sin\varphi\, \mathrm{e}^{\mathrm{i}\phi}\end{pmatrix} , \begin{pmatrix}-\sin\varphi\, \mathrm{e}^{-\mathrm{i}\phi} \\ \cos\varphi\end{pmatrix} \right\}, \qquad v_i^{\varphi\dagger}\, v_j^{\varphi} = \delta_{ij}, \qquad \{i, j\} = \{1, 2\} \,. \tag{16}$$

Using these vectors, we will define a new basis for the vectors in $\Re \mathrm{x} \Re^3$, in terms of φ, although most of the final matrices are functions of 2φ. In a second step there will be another extension in Section V.

We write the vectors v_1^{φ} and v_2^{φ} in the form:

$$v_1^{\varphi} \;=\; \mathbf{R}_o^{\varphi}\, \mathrm{e}_1 = \; \mathbf{R}_e^{\varphi}\, \mathrm{e}_1 \,, \qquad\qquad v_2^{\varphi} \;=\; \mathbf{R}_o^{\varphi}\, \mathrm{e}_2 = -\,\mathbf{R}_e^{\varphi}\, \mathrm{e}_2 \,. \tag{17}$$

After them, we define:

$$s^{+\varphi} \equiv v_1^{\varphi}\, v_2^{\varphi\dagger} \,, \qquad s^{-\varphi} \equiv v_2^{\varphi}\, v_1^{\varphi\dagger} \,, \qquad \hat{s}{}^{\varphi} \equiv v_1^{\varphi}\, v_1^{\varphi\dagger} = s^{+\varphi}\, s^{-\varphi} \,, \qquad \overset{\vee}{s}{}^{\varphi} \equiv v_2^{\varphi}\, v_2^{\varphi\dagger} = s^{-\varphi}\, s^{+\varphi} \,, \tag{18}$$

or else:

$$\begin{aligned} \mathbb{1} \;&=\; \hat{s}{}^{\varphi} + \overset{\vee}{s}{}^{\varphi} = \left\{ s^{+\varphi}, s^{-\varphi} \right\}, & s^{x\,\varphi} \;&\equiv\; s^{+\varphi} + s^{-\varphi} \\ s^{z\,\varphi} &\equiv \hat{s}{}^{\varphi} - \overset{\vee}{s}{}^{\varphi} = \left[s^{+\varphi}, s^{-\varphi} \right] = 2\hat{s}{}^{\varphi} - \mathbb{1}\,, & s^{y\,\varphi} \;&\equiv -\,\mathrm{i}\,(s^{+\varphi} - s^{-\varphi}) \end{aligned} \;. \tag{19}$$

We can define the Pauli matrices as well and in the same way. For them, we impose $\varphi = 0$ ($n_z = 1$, $\tau = 0$).

$s^{+\varphi}$, $s^{-\varphi}$ are shift matrices and with $\hat{s}{}^{\varphi}$, $\overset{\vee}{s}{}^{\varphi}$. They satisfy:

$$\begin{aligned} & s^{+\varphi}\, v_1^{\varphi} = \vec{0} \,, && s^{+\varphi}\, v_2^{\varphi} = v_1^{\varphi} \,, && s^{-\varphi}\, v_1^{\varphi} = v_2^{\varphi} \,, && s^{-\varphi}\, v_2^{\varphi} = \vec{0} \\ & \hat{s}{}^{\varphi}\, v_1^{\varphi} = 1\, v_1^{\varphi} \,, && \hat{s}{}^{\varphi}\, v_2^{\varphi} = \vec{0} \,, && \overset{\vee}{s}{}^{\varphi}\, v_1^{\varphi} = \vec{0} \,, && \overset{\vee}{s}{}^{\varphi}\, v_2^{\varphi} = 1\, v_2^{\varphi} \,, \\ & \left(s^{+\varphi}\right)^2 = \mathbb{0} \,, && && \left(s^{-\varphi}\right)^2 = \mathbb{0} && \end{aligned} \tag{20}$$

and

$$\begin{aligned} \hat{s}{}^{\varphi}\, s^{+\varphi} \;&=\; s^{+\varphi} \overset{\vee}{s}{}^{\varphi} \;=\; s^{+\varphi} \,, & \overset{\vee}{s}{}^{\varphi}\, s^{+\varphi} \;&=\; s^{+\varphi} \hat{s}{}^{\varphi} \;=\; \mathbb{0} \\ \hat{s}{}^{\varphi}\, s^{-\varphi} \;&=\; s^{-\varphi} \overset{\vee}{s}{}^{\varphi} \;=\; \mathbb{0} \,, & \overset{\vee}{s}{}^{\varphi}\, s^{-\varphi} \;&=\; s^{-\varphi} \hat{s}{}^{\varphi} \;=\; s^{-\varphi} \\ s^{z\,\varphi}\, s^{+\varphi} &= (+1)\; s^{+\varphi} \,, & s^{+\varphi}\, s^{z\,\varphi} &= s^{+\varphi}\; (-1) \\ s^{z\,\varphi}\, s^{-\varphi} &= (-1)\; s^{-\varphi} \,, & s^{-\varphi}\, s^{z\,\varphi} &= s^{-\varphi}\; (+1) \end{aligned} \;. \tag{21}$$

The explicit forms for these matrices are:

$$\begin{aligned} s^{+\varphi} &= v_1^{\varphi} v_2^{\varphi\dagger} = \mathbf{R}_o^{\varphi}\, e_1 e_2^{\dagger}\, \mathbf{R}_o^{-\varphi} = \mathbf{R}_o^{\varphi}\, \sigma^{+} \mathbf{R}_o^{-\varphi} = -\,\mathbf{R}_e^{\varphi}\, \sigma^{+}\, \mathbf{R}_e^{\varphi} = \frac{\mathrm{e}^{\,\mathrm{i}\,\phi}}{2} \left\{ -\sin(2\varphi)\, \sigma^{z} - \mathbf{R}_o^{\frac{\pi}{2}} + \cos(2\varphi)\, \mathbf{R}_e^{\frac{\pi}{2}} \right\} \\ s^{-\varphi} &= v_2^{\varphi} v_1^{\varphi\dagger} = \mathbf{R}_o^{\varphi}\, e_2 e_1^{\dagger}\, \mathbf{R}_o^{-\varphi} = \mathbf{R}_o^{\varphi}\, \sigma^{-}\, \mathbf{R}_o^{-\varphi} = -\,\mathbf{R}_e^{\varphi}\, \sigma^{-}\, \mathbf{R}_e^{\varphi} = \frac{\mathrm{e}^{-\mathrm{i}\,\phi}}{2} \left\{ -\sin(2\varphi)\, \sigma^{z} + \mathbf{R}_o^{\frac{\pi}{2}} + \cos(2\varphi)\, \mathbf{R}_e^{\frac{\pi}{2}} \right\} \end{aligned} \,, \tag{22}$$

$$\begin{aligned} \hat{s}{}^{\varphi} &= s^{+\varphi} s^{-\varphi} = v_1^{\varphi} v_1^{\varphi\dagger} = \mathbf{R}_o^{\varphi}\, \overset{\wedge}{\sigma}\, \mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{\varphi}\, \overset{\wedge}{\sigma}\, \mathbf{R}_e^{\varphi} = \tfrac{1}{2}\,(\mathbb{1} + \mathbf{R}_e^{2\varphi}) = \tfrac{1}{2}(\mathbb{1} + s^{z\,\varphi}) = \tfrac{1}{2}(\mathbb{1} + \mathbb{n}) \,, \qquad \left(\equiv\; \tfrac{1}{2}\, \widetilde{\mathbb{n}}_{\mu}\right) \\ \overset{\vee}{s}{}^{\varphi} &= s^{-\varphi} s^{+\varphi} = v_2^{\varphi} v_2^{\varphi\dagger} = \mathbf{R}_o^{\varphi}\, \overset{\vee}{\sigma}\, \mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{\varphi}\, \overset{\vee}{\sigma}\, \mathbf{R}_e^{\varphi} = \tfrac{1}{2}(\mathbb{1} - \mathbf{R}_e^{2\varphi}) = \tfrac{1}{2}(\mathbb{1} - s^{z\,\varphi}) = \tfrac{1}{2}\,(\mathbb{1} - \mathbb{n}) \,, \qquad \left(\equiv -\tfrac{1}{2}\, \widetilde{\mathbb{n}}^{\mu}\right) \end{aligned} \,, \tag{23}$$

and:

$$\begin{aligned} & \mathbb{1} = \hat{s}{}^{\varphi} + \overset{\vee}{s}{}^{\varphi} \,, && \left(\equiv \tfrac{1}{2}\, (\widetilde{\mathbb{n}}_{\mu} - \widetilde{\mathbb{n}}^{\mu})\right) \\ & s^{z\,\varphi} = s^{z\varphi}(\phi) = \hat{s}{}^{\varphi} - \overset{\vee}{s}{}^{\varphi} = \mathbf{R}_o^{\varphi}\, \sigma^{z}\, \mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{\varphi}\, \sigma^{z}\, \mathbf{R}_e^{\varphi} = \mathbf{R}_e^{2\varphi} = \mathbb{n} \,, && \left(\equiv \tfrac{1}{2}\, (\widetilde{\mathbb{n}}_{\mu} + \widetilde{\mathbb{n}}^{\mu})\right) \end{aligned} \;. \tag{24}$$

This final equality expresses the relation of a unit vector $\mathbb{n}$ with a component of the set of spin matrices, $s^{z\,\varphi}$. It also expresses that σ^{z} diagonalizes $\mathbb{n}$. It has to be remarked that the determinants of σ^{z}, $s^{z\,\varphi}$, $\mathbf{R}_e^{2\varphi}$ and $\mathbb{n}$ are equal to -1 . For these spin matrices a π difference in φ is not relevant.

Also, the second equation in (24) lets us diagonalize the exponential forms appearing in the rotation operators, matrices (6):

$$\mathrm{e}^{\mathrm{i}\alpha\mathbb{n}} = \cos\alpha\; \mathbb{1} + \mathrm{i}\sin\alpha\; \mathbb{n} \;=\; \mathbf{R}_e^{\varphi}\; \mathrm{e}^{\mathrm{i}\,\alpha\sigma^{z}}\; \mathbf{R}_e^{\varphi} \;=\; \mathbf{R}_o^{\varphi}\; \mathrm{e}^{\mathrm{i}\,\alpha\sigma^{z}}\; \mathbf{R}_o^{-\varphi} \;=\; \mathbf{R}_o^{\varphi} \begin{pmatrix} \mathrm{e}^{\,\mathrm{i}\,\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\alpha} \end{pmatrix} \mathbf{R}_o^{-\varphi}. \tag{25}$$

We consider the set: $\left\{ \hat{s}^{\varphi},\ s^{+\varphi},\ \ s^{-\varphi},\ \ \overset{\vee}{s}^{\varphi} \right\}(\phi)$. We define: $s^{x\varphi} \equiv s^{+\varphi} + s^{-\varphi}$, $s^{y\varphi} \equiv \mathrm{i}\,(s^{+\varphi} - s^{-\varphi})$ and the basis:

$$\left\{ \ \mathbb{1}\ ,\ \ s^{z\,\varphi},\ \ s^{x\varphi},\ \ s^{y\varphi}\ \right\}(\phi)\,. \tag{26}$$

For a later use, based in arguments of symmetry, it is interesting to specifically show the forms of these vectors and matrices for the -2φ value ($\mathbb{n}(-2\varphi,\phi) \equiv \dot{\mathbb{n}}(2\dot{\varphi},\phi) \equiv \sigma^z\,\mathbb{n}(2\varphi,\phi)\,\sigma^z$):

$$v_1^{-\varphi} = \mathbf{R}_O^{-\varphi} e_1 = \mathbf{R}_e^{-\varphi} e_1 = \sigma^z\, v_1^{\varphi}\,, \qquad v_2^{-\varphi} = \mathbf{R}_O^{-\varphi} e_2 = -\,\mathbf{R}_e^{-\varphi} e_2 = -\,\sigma^z\, v_2^{\varphi}\,, \tag{27}$$

with:

$$\begin{cases} \mathbf{R}_O^{-\varphi} = \begin{pmatrix} \cos\varphi & \sin\varphi\,\mathrm{e}^{-\mathrm{i}\phi} \\ -\sin\varphi\,\mathrm{e}^{\mathrm{i}\phi} & \cos\varphi \end{pmatrix} = \sigma^z \mathbf{R}_O^{\varphi} \sigma^z \\[2ex] \mathbf{R}_e^{-\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi\,\mathrm{e}^{-\mathrm{i}\phi} \\ -\sin\varphi\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\varphi \end{pmatrix} = \sigma^z \mathbf{R}_e^{\varphi} \sigma^z \end{cases}\,,$$

we write (28)

$$\begin{aligned} s^{+-\varphi} &= \mathbf{R}_O^{-\varphi}\sigma^+\,\mathbf{R}_O^{\varphi} = -\,\mathbf{R}_e^{-\varphi}\sigma^+\,\mathbf{R}_e^{-\varphi} = \frac{\mathrm{e}^{\mathrm{i}\,\phi}}{2}\left\{ \sin(2\varphi)\,\sigma^z - \mathbf{R}_O^{\frac{\pi}{2}} + \cos(2\varphi)\,\mathbf{R}_e^{\frac{\pi}{2}} \right\} = -\sigma^z s^{+\varphi}\sigma^z \\ s^{-\,-\varphi} &= \mathbf{R}_O^{-\varphi}\sigma^-\,\mathbf{R}_O^{\varphi} = -\,\mathbf{R}_e^{-\varphi}\sigma^-\,\mathbf{R}_e^{-\varphi} = \frac{\mathrm{e}^{-\mathrm{i}\,\phi}}{2}\left\{ \sin(2\varphi)\,\sigma^z + \mathbf{R}_O^{\frac{\pi}{2}} + \cos(2\varphi)\,\mathbf{R}_e^{\frac{\pi}{2}} \right\} = -\sigma^z s^{-\varphi}\sigma^z \end{aligned}\,, \tag{29}$$

$$\begin{aligned} \hat{s}^{-\varphi} &= \mathbf{R}_O^{-\varphi}\,\overset{\wedge}{\sigma}\,\mathbf{R}_O^{\varphi} = \mathbf{R}_e^{-\varphi}\,\overset{\wedge}{\sigma}\,\mathbf{R}_e^{-\varphi} = \tfrac{1}{2}(\mathbb{1} + \mathbf{R}_e^{-2\varphi}) = \tfrac{1}{2}(\mathbb{1} + \sigma^z s^{z\,\varphi}\sigma^z) = \tfrac{1}{2}(\mathbb{1} + \sigma^z \mathbb{n}\,\sigma^z)\,, \qquad (\equiv\ \tfrac{1}{2}\,\dot{\widetilde{\mathbb{n}}}_\mu\ ) \\ \overset{\vee}{s}^{-\varphi} &= \mathbf{R}_O^{-\varphi}\,\overset{\vee}{\sigma}\,\mathbf{R}_O^{\varphi} = \mathbf{R}_e^{-\varphi}\,\overset{\vee}{\sigma}\,\mathbf{R}_e^{-\varphi} = \tfrac{1}{2}(\mathbb{1} - \mathbf{R}_e^{-2\varphi}) = \tfrac{1}{2}(\mathbb{1} - \sigma^z s^{z\,\varphi}\sigma^z) = \tfrac{1}{2}(\mathbb{1} - \sigma^z \mathbb{n}\,\sigma^z)\,, \qquad (\equiv -\tfrac{1}{2}\,\dot{\widetilde{\mathbb{n}}}^{\mu}\ ) \end{aligned}\,, \tag{30}$$

and:

$$\begin{aligned} \mathbb{1} &= \hat{s}^{-\varphi} + \overset{\vee}{s}^{-\varphi}\,, \qquad\qquad (\equiv \tfrac{1}{2}\,(\widetilde{\mathbb{n}}_\mu - \widetilde{\mathbb{n}}^\mu) = \tfrac{1}{2}\,(\dot{\widetilde{\mathbb{n}}}_\mu - \dot{\widetilde{\mathbb{n}}}^{\mu}\,)\,) \\ \dot{s}^{z\,\varphi} &\equiv s^{z\,-\varphi} = \sigma^z\, s^{z\,\varphi}\sigma^z = \hat{s}^{-\varphi} - \overset{\vee}{s}^{-\varphi}\,, \qquad\qquad (\equiv \tfrac{1}{2}\,(\dot{\widetilde{\mathbb{n}}}_\mu + \dot{\widetilde{\mathbb{n}}}^{\mu}\,)\,) \end{aligned}\,. \tag{31}$$

Looking for symmetries, we consider the following displacements in the φ variable:

$$\left.\begin{aligned} s^{+\varphi+\epsilon\frac{\pi}{2}} &= \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, s^{+\varphi}\,\mathbf{R}_O^{-\epsilon\frac{\pi}{2}} = \mathbf{m}^{+2}\, s^{-\varphi} \\ s^{-\varphi+\epsilon\frac{\pi}{2}} &= \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, s^{-\varphi}\,\mathbf{R}_O^{-\epsilon\frac{\pi}{2}} = \mathbf{m}^{-2}\, s^{+\varphi} \end{aligned}\right\}, \qquad \left.\begin{aligned} s^{+\varphi\pm\pi} &= s^{+\varphi} \\ s^{-\varphi\pm\pi} &= s^{-\varphi} \end{aligned}\right\}, \tag{32}$$

$$\left.\begin{aligned} \hat{s}^{\varphi+\epsilon\frac{\pi}{2}} &= \overset{\vee}{s}^{\varphi} \\ \overset{\vee}{s}^{\varphi+\epsilon\frac{\pi}{2}} &= \hat{s}^{\varphi} \end{aligned}\right\}\ s^{z\,\varphi+\epsilon\frac{\pi}{2}} = -\,s^{z\varphi}\,, \qquad \left.\begin{aligned} \hat{s}^{\varphi\pm\pi} &= \hat{s}^{\varphi} \\ \overset{\vee}{s}^{\varphi\pm\pi} &= \overset{\vee}{s}^{\varphi} \end{aligned}\right\}\ s^{z\,\varphi\pm\pi} = s^{z\varphi}\,, \tag{33}$$

with $\epsilon \in \{+1,-1\}$. We have denoted:

$$\mathbf{m}^+ \ \equiv\ \mathrm{e}^{\,\mathrm{i}\,(\phi - \frac{\pi}{2})}\,, \qquad\qquad \mathbf{m}^- \ \equiv\ \mathrm{e}^{-\mathrm{i}\,(\phi - \frac{\pi}{2})}\,. \tag{34}$$

and therefore: $\mathbf{m}^{+2} = -\mathrm{e}^{\,\mathrm{i}\,2\,\phi}$ and $\mathbf{m}^{-2} = -\mathrm{e}^{\,-\,\mathrm{i}\,2\,\phi}$.

Also:

$$\begin{cases} s^{+\varphi} = \left(\mathbf{m}^+\ \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\right) s^{-\varphi}\left(\mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\ \mathbf{m}^+\right) \\ s^{-\varphi} = \left(\mathbf{m}^-\ \mathbf{R}_\mathbf{o}^{\epsilon\frac{\pi}{2}}\right) \mathbf{s}^{+\varphi}\left(\mathbf{R}_\mathbf{o}^{-\epsilon\frac{\pi}{2}}\ \mathbf{m}^-\right) \\ s^{z\varphi} = \left(\mathrm{e}^{\mathrm{i}\,\widetilde{\epsilon}\frac{\pi}{2}}\ \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\right) s^{z\varphi}\left(\mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\ \mathrm{e}^{\mathrm{i}\,\widetilde{\epsilon}\frac{\pi}{2}}\right) \end{cases}\,, \tag{35}$$

with $\widetilde{\epsilon} \in \{+1,-1\}$. $s^{z\,\varphi}$ already in (24). Or, in terms of σ^+,σ^- :

$$\begin{cases} s^{+\varphi} = \mathbf{R}_O^{\varphi}\,\sigma^+\,\mathbf{R}_O^{-\varphi} = \left(\mathbf{m}^+\ \mathbf{R}_O^{\varphi+\,\epsilon\frac{\pi}{2}}\right)\sigma^-\left(\mathbf{R}_O^{-(\varphi+\,\epsilon\frac{\pi}{2})}\ \mathbf{m}^+\right) \\ s^{-\varphi} = \mathbf{R}_O^{\varphi}\,\sigma^-\,\mathbf{R}_O^{-\varphi} = \left(\mathbf{m}^-\ \mathbf{R}_\mathbf{o}^{\varphi+\,\epsilon\frac{\pi}{2}}\right)\sigma^+\left(\mathbf{R}_\mathbf{o}^{-(\varphi+\,\epsilon\frac{\pi}{2})}\ \mathbf{m}^-\right) \end{cases}\,. \tag{36}$$

Pay attention to the fact that all these spin matrices depend exclusively in n_z, $\tau(n_z)$, ϕ or in the 2φ and ϕ variables, (action in "$2d$"), i.e. in a spherical surface in the "3d" space (equation (8)), or in a 2 dimensional "hipersurface" in a "(1+1+ "2")d" space ($T \otimes Z \otimes \{X,Y\}$), but they do not depend in a time type n_t and in a $\|\mathbf{r}\|$ variables. The dependence in a time type variable n_t will be added in Section VII.

V - DEFINING NULL VECTORS IN DIMENSION 4.

We generalize the unit vector $\mathbf{n} \equiv (n_x, n_y, n_z) \in \Re^3$ in (7), by adding a discrete time-like component:

$\widetilde{\mathbf{n}} \equiv (n_t, n_x, n_y, n_z) \in \Re^4$ ($\|\mathbf{n}\| = \sqrt{-\det(\mathbb{n})} = 1$). These vectors, with $n_t = \{+1, -1\}$, are light-type:

$$\|\widetilde{\mathbf{n}}\|_\sim \equiv \sqrt{-\det(\widetilde{\mathbb{n}})} = \sqrt{-n_t^2 + n_x^2 + n_y^2 + n_z^2} = \sqrt{-n_t^2 + \tau^2 + n_z^2} = 0 \qquad \text{(a Minkwoskian-metric).}$$

Also, this is suggested in (24) and (31). We denote these vectors as **characteristic vectors** in **characteristic axes**.

$\widetilde{\mathbf{n}}$ in square matrix form is: $\widetilde{\mathbb{n}} = n_t \mathbb{1} + \mathbb{n} = n_t\, \mathbb{1} + n_z\, \sigma^Z + \tau\, \mathbf{R}_O^{\frac{\pi}{2}}$:

$$\widetilde{\mathbb{n}} = \begin{pmatrix} n_t + n_z & \tau\, e^{-i\phi} \\ \tau\, e^{i\phi} & n_t - n_z \end{pmatrix}, \qquad \left[\ \dot{\widetilde{\mathbb{n}}} = \sigma^Z \widetilde{\mathbb{n}}\, \sigma^Z = \begin{pmatrix} n_t + n_z & -\tau\, e^{-i\phi} \\ -\tau\, e^{i\phi} & n_t - n_z \end{pmatrix}\ \right]. \tag{37}$$

With $s_k = \text{sign}(n_k)$, and $k \in \{t, z, x, y\}$:

$$\widetilde{\mathbb{n}} = \begin{pmatrix} s_t 1 + s_z |n_z| & s_y |\tau|\, e^{-i\phi} \\ s_y |\tau|\, e^{i\phi} & s_t 1 - s_z |n_z| \end{pmatrix} = \begin{pmatrix} s_t 1 + \cos(2\varphi) & \sin(2\varphi)\, e^{-i\phi} \\ \sin(2\varphi)\, e^{i\phi} & s_t 1 - \cos(2\varphi) \end{pmatrix}, \tag{38}$$

Thanks to s_t we can define $\widetilde{\mathbb{n}}$ as a function of φ, with the assignation defined in (42):

$$\widetilde{\mathbb{n}} = \widetilde{\mathbb{n}}(\varphi) = \mathbf{R}_O^{\varphi}\, (n_t \mathbb{1} + \sigma^Z)\, \mathbf{R}_O^{-\varphi}, \qquad \varphi \in [-\pi, \pi),$$

in contrast to $\mathbb{n}$, which actually is a function of 2φ ($\mathbb{n} = \mathbf{R}_e^{2\varphi} = \mathbf{R}_O^{\varphi}\, \sigma^Z\, \mathbf{R}_O^{-\varphi}$, $2\varphi \in [-\pi, \pi)$).

We introduce the notations:

$$\left\{ \begin{array}{lll}
1) & \widetilde{\mathbb{n}}^\mu \equiv \quad -\mathbb{1} + \mathbb{n} = & -\mathbb{1} + n_z\, \sigma^Z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
3) & -\widetilde{\mathbb{n}}_\mu \equiv -(\ \mathbb{1} + \mathbb{n}) = & -\mathbb{1} - n_z\, \sigma^Z - \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
⑤ & \widetilde{\mathbb{n}}_\mu \equiv \quad \mathbb{1} + \mathbb{n} = & \mathbb{1} + n_z\, \sigma^Z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
7) & -\widetilde{\mathbb{n}}^\mu \equiv -(-\mathbb{1} + \mathbb{n}) = & \mathbb{1} - n_z\, \sigma^Z - \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
-1) & \dot{\widetilde{\mathbb{n}}}^\mu \equiv \sigma^Z \widetilde{\mathbb{n}}^\mu \sigma^Z = -\mathbb{1} + \dot{\mathbb{n}} = & -\mathbb{1} + n_z\, \sigma^Z - \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
-3) & -\dot{\widetilde{\mathbb{n}}}_\mu \equiv -\sigma^Z \widetilde{\mathbb{n}}_\mu \sigma^Z = -\mathbb{1} - \dot{\mathbb{n}} = & -\mathbb{1} - n_z\, \sigma^Z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
-5) & \dot{\widetilde{\mathbb{n}}}_\mu \equiv \sigma^Z \widetilde{\mathbb{n}}_\mu \sigma^Z = \mathbb{1} + \dot{\mathbb{n}} = & \mathbb{1} + n_z\, \sigma^Z - \tau\, \mathbf{R}_e^{\frac{\pi}{2}} \\
-7) & -\dot{\widetilde{\mathbb{n}}}^\mu \equiv -\sigma^Z \widetilde{\mathbb{n}}^\mu \sigma^Z = \mathbb{1} - \dot{\mathbb{n}} = & \mathbb{1} - n_z\, \sigma^Z + \tau\, \mathbf{R}_e^{\frac{\pi}{2}}
\end{array} \right. . \tag{39}$$

We denote this **8** vectors as *building null vectors*. We have defined them considering the coordinates of an arbitrary vector, n_t, n_z and τ with + and − signs and having in mind the perspective of using some arguments based in the symmetries: 1) time opposition ($\mathcal{T}$), 2) n_z-opposition, ('**mirroring**' in the plane XY) ($\mathcal{R}^Z$) and 3) τ-opposition, (π-rotation or '**mirroring**' in the line Z) ($\mathcal{M}^\tau$). Space opposition is the product of the last two elements ($\mathcal{P} = \mathcal{R}^Z \mathcal{M}^\tau = \mathcal{M}^\tau \mathcal{R}^Z$).

Therefore, after a value $\varphi_R \in [0, \frac{\pi}{4}]$, we can write:

$$\mathbb{n}(2\varphi_R) \equiv \mathbb{n}, \qquad \mathbb{n}(2\varphi_R - \pi) = -\mathbb{n}, \qquad \mathbb{n}(-2\varphi_R) = \dot{\mathbb{n}}, \qquad \mathbb{n}(-2\varphi_R + \pi) = -\dot{\mathbb{n}}, \tag{40}$$

and with (37) , (38) and (39) we make the following assignments (see Figures 2, 3 and 4):

$$\begin{array}{llll}
\widetilde{\mathbb{n}}(\varphi_R) \equiv \widetilde{\mathbb{n}}_\mu, & \widetilde{\mathbb{n}}(\varphi_R - \frac{\pi}{2}) \equiv -\widetilde{\mathbb{n}}_\mu, & \widetilde{\mathbb{n}}(-\varphi_R) = \dot{\widetilde{\mathbb{n}}}_\mu, & \widetilde{\mathbb{n}}(-\varphi_R + \frac{\pi}{2}) = -\dot{\widetilde{\mathbb{n}}}_\mu \\
\widetilde{\mathbb{n}}(\varphi_R - \pi) \equiv \widetilde{\mathbb{n}}^\mu, & \widetilde{\mathbb{n}}(\varphi_R + \frac{\pi}{2}) \equiv -\widetilde{\mathbb{n}}^\mu, & \widetilde{\mathbb{n}}(-\varphi_R + \pi) = \dot{\widetilde{\mathbb{n}}}^\mu, & \widetilde{\mathbb{n}}(-\varphi_R - \frac{\pi}{2}) = -\dot{\widetilde{\mathbb{n}}}^\mu
\end{array} . \tag{41}$$

For every value of 2φ (a vector $\mathbb{n}$) we get two vectors $\widetilde{\mathbb{n}}$, one with $n_t = +1$ and another one with $n_t = -1$. In order to have a one to one correspondence, we proceed in the following form:

a) we modify the domain of φ in (9): $-\pi \le \varphi < \pi$ (or $\le \pi$), and
we maintain the domain of $0 \le \phi < \pi$, although essentially it will be ignored,

b) we define two subsets in this interval. with the following assignments and notations (see Figures 2, 3 and 4):

$$\begin{array}{llll}
I_+ = (-\frac{3\pi}{4}, -\frac{\pi}{2}) \cup (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{3\pi}{4}), & \text{for } \varphi_+ \equiv \varphi \in I_+ & \text{it is} & n_t(\varphi_+) = +1 \\
I_- = [-\pi, -\frac{3\pi}{4}) \cup (-\frac{\pi}{2}, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{3\pi}{4}, \pi), & \text{for } \varphi_- \equiv \varphi \in I_- & \text{it is} & n_t(\varphi_-) = -1
\end{array} . \tag{42}$$

We consider the values of φ given by $k\frac{\pi}{4}$ $(k=\pm1,\pm2,\pm3)$ as limiting values. In *Study III* we show that although there is not a one to one correspondence with the Cartesian coordinates at these values, it makes sense to define the stated correspondence.

With these assignments for φ we get the same values of n_z and τ in $\varphi_\pm$ and in $\varphi_\pm+\pi$ or in $\varphi_\pm-\pi$. In general we write φ instead of $\varphi_\pm$. These π-differences in the φ variable are meaningless for the spin matrices (they depend in 2φ): $s^{*^{(\varphi_\pm+\epsilon\pi)}} = s^{*^{\varphi_\pm}} = s^{*^{\varphi}}$, $\epsilon=\pm1$.

In the Figures 1 and 2, the arrows in the circles indicate the increasing values of φ_R from 0 to $\frac{\pi}{4}$.

Considering the vectors $\widetilde{\mathfrak{n}}_\mu$, ⑤ $(\varphi\in[0,\frac{\pi}{4}])$, as reference vectors, we define the following correspondences:

					Value of φ referred to ⑤	Symmetry	Vectors
ν-11)	$\tau=0$	$n_z=1$	$n_t=-1$	$\varphi=-\pi$	$0-\pi$		
1)	$\tau>0$	$n_z>0$	$n_t=-1$	$-\pi<\varphi<-\frac{3\pi}{4}$	$\varphi_R-\pi$	$\mathcal{T}$	$\widetilde{\mathfrak{n}}^\mu$
l 1-3)	$\tau=1$	$n_z=0$	$n_t=-1$	$\varphi=-\frac{3\pi}{4}$	$\frac{\pi}{4}-\pi$		
l 5-7)	$\tau=1$	$n_z=0$	$n_t=1$	$\varphi=-\frac{3\pi}{4}$	$-(\frac{\pi}{4}+\frac{\pi}{2})$	$\mathcal{R}^{\mathcal{Z}}$	$-\dot{\widetilde{\mathfrak{n}}}^\mu$
-7)	$\tau>0$	$n_z<0$	$n_t=1$	$-\frac{3\pi}{4}<\varphi<-\frac{\pi}{2}$	$-(\varphi_R+\frac{\pi}{2})$		
ν-77)	$\tau=0$	$n_z=-1$	$n_t=1$	$\varphi=-\frac{\pi}{2}$	$-(0+\frac{\pi}{2})$		
ν-33)	$\tau=0$	$n_z=-1$	$n_t=-1$	$\varphi=-\frac{\pi}{2}$	$0-\frac{\pi}{2}$		
3)	$\tau<0$	$n_z<0$	$n_t=-1$	$-\frac{\pi}{2}<\varphi<-\frac{\pi}{4}$	$\varphi_R-\frac{\pi}{2}$	$\mathcal{T}P$	$-\widetilde{\mathfrak{n}}_\mu$
l 3-1)	$\tau=-1$	$n_z=0$	$n_t=-1$	$\varphi=-\frac{\pi}{4}$	$\frac{\pi}{4}-\frac{\pi}{2}$		
l 7-5)	$\tau=-1$	$n_z=0$	$n_t=1$	$\varphi=-\frac{\pi}{4}$	$-(\frac{\pi}{4})$	$\mathcal{M}^{\mathcal{T}}$	$\dot{\widetilde{\mathfrak{n}}}_\mu$
-5)	$\tau<0$	$n_z>0$	$n_t=1$	$-\frac{\pi}{4}<\varphi<0$	$-\varphi_R$		
ν-55)	$\tau=0$	$n_z=1$	$n_t=1$	$\varphi=0$	0		
⑤	$\tau>0$	$n_z>0$	$n_t=1$	$0<\varphi<\frac{\pi}{4}$	$\varphi=\varphi_R$	REF.	$\widetilde{\mathfrak{n}}_\mu$
l 5-7)	$\tau=1$	$n_z=0$	$n_t=1$	$\varphi=\frac{\pi}{4}$	$\frac{\pi}{4}$		
l 1-3)	$\tau=1$	$n_z=0$	$n_t=-1$	$\varphi=\frac{\pi}{4}$	$-(\frac{\pi}{4}-\frac{\pi}{2})$	$\mathcal{T}\mathcal{R}^{\mathcal{Z}}$	$-\dot{\widetilde{\mathfrak{n}}}_\mu$
-3)	$\tau>0$	$n_z<0$	$n_t=-1$	$\frac{\pi}{4}<\varphi<\frac{\pi}{2}$	$-(\varphi_R-\frac{\pi}{2})$		
ν-33)	$\tau=0$	$n_z=-1$	$n_t=-1$	$\varphi=\frac{\pi}{2}$	$-(0-\frac{\pi}{2})$		
ν-77)	$\tau=0$	$n_z=-1$	$n_t=1$	$\varphi=\frac{\pi}{2}$	$0+\frac{\pi}{2}$		
7)	$\tau<0$	$n_z<0$	$n_t=1$	$\frac{\pi}{2}<\varphi<\frac{3\pi}{4}$	$\varphi_R+\frac{\pi}{2}$	$\mathcal{P}$	$-\widetilde{\mathfrak{n}}^\mu$
l 7-5)	$\tau=-1$	$n_z=0$	$n_t=1$	$\varphi=\frac{3\pi}{4}$	$\frac{\pi}{4}+\frac{\pi}{2}$		
l 3-1)	$\tau=-1$	$n_z=0$	$n_t=-1$	$\varphi=\frac{3\pi}{4}$	$-(\frac{\pi}{4}-\pi)$	$\mathcal{T}\mathcal{M}^{\mathcal{T}}$	$\dot{\widetilde{\mathfrak{n}}}^\mu$
-1)	$\tau<0$	$n_z>0$	$n_t=-1$	$\frac{3\pi}{4}<\varphi<\pi$	$-(\varphi_R-\pi)$		

Table 1. (See Figures 4 and 5).

Cases with an "=" value for φ are specially interesting. We represent these cases with blue-ν and red-l characters, and we obtain them with φ_R tending to 0 and to $\frac{\pi}{4}$ respectively. $\varphi = -\pi$ and $\varphi = \pi$ represent the same point. We write νa-a to indicate where the point a joints the point -a, and with l a-b to indicate where the point a joints the point -b, for the specified values of φ. We handle the symmetries in Section VI.

We impose for our four variables, related to the four coordinates n_j, the following:

$$\begin{cases} n_t = \pm 1 & \text{already fixed and discretized,} \\ \varphi \in [-\pi, \pi) \quad i.e. \quad n_z, \tau, (n_t) & \text{we fix and discretize it (them) bellow and elsewhere, \quad and} \\ \phi & \text{will remain hidden} \end{cases}$$
$$2\alpha \quad (\, \|\mathbb{n}\| = 1 \,) \qquad \text{fixed and discretized,} \tag{43}$$

2α has the meaning of a rotation angle and it is fixed and discretized in *Studies I.2* and *III,2*.

The vectors $\widetilde{\mathbb{n}}$ satisfies two constraints: $\|\widetilde{\mathbb{n}}\|_{\sim} = 0$ with $\{\, \|\mathbb{n}\| = 1$ and $n_t = \pm 1 \,\}$.

Adding conditions on the value of φ (*i.e.* n_t, n_z, τ), we get the above mentioned discretization for the φ angle variable. We relate these discrete values with the values of the *electric charge, the spin, the vector-spin and the chirallity* of the elementary fermions (leptons and quarks). We advance them here:

	τ	n_z	φ	2φ	$\mathbf{R}_e^{2\varphi}$	
A)	0	1	$\{0, \pi\}$	0	$\{\, \sigma^Z \,\}$	**Pauli**
A')	0	− 1	$\{\pm\frac{\pi}{2}\}$	π	$\{-\sigma^Z\}$	**Jordan Wigner**
B)	$\{\pm 1\}$	0	$\{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$	$\{\pm\frac{\pi}{2}\}$	$\{\pm \mathbf{R}_e^{\frac{\pi}{2}}\}$	
C1)	$\{\pm\frac{1}{2}\}$	$\{\pm\frac{\sqrt{3}}{2}\}$	$\{\pm\frac{\pi}{12}, \pm\frac{5\pi}{12}, \pm\frac{7\pi}{12}, \pm\frac{11\pi}{12}\}$	$\{\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}\}$	$\{\pm[\frac{\sqrt{3}}{2}\sigma^Z \pm \frac{1}{2}\mathbf{R}_e^{\frac{\pi}{2}}]\}$	
C2)	$\{\pm\frac{\sqrt{3}}{2}\}$	$\{\pm\frac{1}{2}\}$	$\{\pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}\}$	$\{\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\}$	$\{\pm[\frac{1}{2}\sigma^Z \pm \frac{\sqrt{3}}{2}\mathbf{R}_e^{\frac{\pi}{2}}]\}$	
D)	$\{\pm\frac{\sqrt{2}}{2}\}$	$\{\pm\frac{\sqrt{2}}{2}\}$	$\{\pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \pm\frac{7\pi}{8}\}$	$\{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}$	$\{\pm\frac{\sqrt{2}}{2}[\sigma^Z \pm \mathbf{R}_e^{\frac{\pi}{2}}]\}$	

Table 2: Discrete values of φ ($n_t \in \{+1, -1\}$). See Figure 5.

Other values of φ do not provide interesting relations, as the previous ones do. Relationships involving the 2φ values are also important.

Note that, adding or subtracting a $\frac{\pi}{4}$ value to the values of φ in A), A') we get the values of φ in B), and vice versa.

The same happens with the values in C1), with odd numbers in Figure 5 ($\{\pm1, \pm3, \pm5, \pm7, \}$), and C2), these ones with even numbers ($\{\pm2, \pm4, \pm6, \pm8, \}$); half of them with $+\frac{\pi}{4}$, and the other half with $-\frac{\pi}{4}$. There are also interesting relationships related to the values $\frac{\pi}{6}$ and $\frac{\pi}{3}$ (with 2φ).

But the values of D) keep in D), although in two separate subsets, in particular, the ones with the + signs (2 , 4 , 6 , 8) and the ones with the − signs ($-2, -4, -6, -8$).

Here, departing from the value $\psi = 0$, and adding successively $\Delta\psi = \frac{\pi}{12}$, we represent all the elementary fermions of the first family that we know at present time. Including the absolute value of their electric charges, they are:

with $\varphi_R = 0\,\psi = 0$ (A , A'), $\quad |q_e(\nu)| = 0\,\psi\,\frac{4}{\pi} = 0$, $\quad$ neutrinos
with $\varphi_R = 1\,\psi = \frac{\pi}{12}$ (C1), $\quad |q_e(d)| = 1\,\psi\,\frac{4}{\pi} = \frac{1}{3}$, $\quad$ d-quarks
with $\varphi_R = 2\,\psi = \frac{\pi}{6}$ (C2), $\quad |q_e(u)| = 2\,\psi\,\frac{4}{\pi} = \frac{2}{3}$, $\quad$ u-quarks $\quad$ and
with $\varphi_R = 3\,\psi = \frac{\pi}{4}$ (B), $\quad |q_e(e)| = 3\,\psi\,\frac{4}{\pi} = 1$, $\quad$ electrons and positrons.

And, even more, possible ones related to $\varphi_R = \frac{\pi}{8}$ (D) (unknown).

With this scheme we can consider the electro-magneto-weak interaction), but neither the strong interaction (color) and neither the gravity. The gravity will be studied in *Study V Addenda*.

We develop these questions in the *Study III*, the physics of the model, and we apply them in the *Study IV*. We do so by constructing linear chains with m elements, Jordan Wigner type chains whose elements are the ones defined in this *Study*, our Time-Space-Spin vectors.

VI - SYMMETRY RELATIONS WITH AXES $((n_t,\ [n_z),\ \tau\,])$, IN TERMS OF φ, AS RELATED TO A PRIMMER ONE φ_R.

We do not refer these operators to the parity and time inversion operators in the **PCT** theorem.

We obtain various symmetry relationships between different characteristic axes in terms of relations for different values of φ ($\left\{\pm\varphi,\ \pm(\varphi\pm\frac{\pi}{2}),\ \pm\varphi\mp\pi\right\}$). We write ϵ and $\widetilde{\epsilon}$ for $+$ or $-$. We consider the following ones [7]:

⑤ REFERENCE φ_R .

$$\widetilde{\mathbb{m}}_5=\widetilde{\mathbb{m}}(\varphi_R)=\widetilde{\mathbb{m}}_\mu=\mathbb{1}+\mathbb{m}=\begin{pmatrix}1+n_z & \tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ \tau\,\mathrm{e}^{\mathrm{i}\,\phi} & 1-n_z\end{pmatrix}=\begin{pmatrix}1+\cos(2\varphi_R) & \sin(2\varphi_R)\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ \sin(2\varphi_R)\,\mathrm{e}^{\mathrm{i}\,\phi} & 1-\cos(2\varphi_R)\end{pmatrix}. \tag{44}$$

3) $\mathcal{TP}=\mathcal{PT}=-\mathbb{1}$, time and space oppositions: point mirroring in the origin in ("1+3d").
Or opposition in "time": $n_t=\pm1$ to $n_t=\mp1$, and for the "3d" part of a characteristic axis: $\mathbb{m}$ to $-\mathbb{m}$:

$(n_t,n_z,n_x,n_y)\overset{\mathcal{TP}}{\longrightarrow}(-n_t,-n_z,-n_x,-n_y)$ accomplished with $\varphi_R\overset{\mathcal{TP}}{\longrightarrow}\varphi=\varphi_R-\frac{\pi}{2}$.

$$\widetilde{\mathbb{m}}_3=\widetilde{\mathbb{m}}(\varphi_R-\tfrac{\pi}{2})=-\,\widetilde{\mathbb{m}}_\mu=\begin{pmatrix}-1-n_z & -\tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ -\tau\,\mathrm{e}^{\mathrm{i}\,\phi} & -1+n_z\end{pmatrix}=(\,\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\,)\,\widetilde{\mathbb{m}}_\mu\,(\,\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\,)\equiv\mathcal{TP}\,\widetilde{\mathbb{m}}_\mu=-\,\widetilde{\mathbb{m}}_5\,. \tag{45}$$

An argument to see the importance of the values $n_t=\pm1$ derives from the following simple equations: $[\mathrm{e}^{\mathrm{i}\alpha_t}]^2=\mathrm{e}^{\mathrm{i}4\alpha_t}[\mathrm{e}^{-\mathrm{i}\alpha_t}]^2$ and $[\mathrm{e}^{\mathrm{i}\,(\alpha_t\pm\frac{\pi}{2})}]^2=-[\mathrm{e}^{\mathrm{i}\,\alpha_t}]^2$, with $\alpha_t=n_t\frac{\pi}{4}$, then $[\mathrm{e}^{\pm\mathrm{i}\,\frac{\pi}{4}}]^2=-[\mathrm{e}^{\mp\mathrm{i}\,\frac{\pi}{4}}]^2$ (also, $=[\mathrm{e}^{\mp\mathrm{i}\frac{3\pi}{4}}]^2$). We have: $\mathcal{TP}=-\mathbb{1}$.

7) $\mathcal{P}$, $\mathcal{P}^2=\mathbb{1}$, opposition in space, point reflection in the origin in ("3d") :

$(n_t,n_z,n_x,n_y)\overset{\mathcal{P}}{\longrightarrow}(n_t,-n_z,-n_x,-n_y)$ accomplished with $\varphi_R\overset{\mathcal{P}}{\longrightarrow}\varphi=\varphi_R+\frac{\pi}{2}$.

$$\widetilde{\mathbb{m}}_7=\widetilde{\mathbb{m}}(\varphi_R+\tfrac{\pi}{2})=-\,\widetilde{\mathbb{m}}^\mu=\begin{pmatrix}1-n_z & -\tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ -\tau\,\mathrm{e}^{\mathrm{i}\,\phi} & 1+n_z\end{pmatrix}=\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\widetilde{\mathbb{m}}_\mu\,\mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}}=\mathcal{P}\,\widetilde{\mathbb{m}}_\mu=-\mathcal{T}\,\widetilde{\mathbb{m}}_\mu=-\widetilde{\mathbb{m}}_1\,. \tag{46}$$

1) $\mathcal{T}$, $\mathcal{T}^2=\mathbb{1}$, time opposition or time reflection :

$(n_t,n_z,n_x,n_y)\overset{\mathcal{T}}{\longrightarrow}(-n_t,n_z,n_x,n_y)$ accomplished with $\varphi_R\overset{\mathcal{T}}{\longrightarrow}\varphi=\varphi_R-\pi$.

$$\widetilde{\mathbb{m}}_1=\widetilde{\mathbb{m}}(\varphi_R-\pi)=\widetilde{\mathbb{m}}^\mu=\begin{pmatrix}-1+n_z & \tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ \tau\,\mathrm{e}^{\mathrm{i}\,\phi} & -1-n_z\end{pmatrix}=(\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}})\,\widetilde{\mathbb{m}}_\mu(\,\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}})=\mathcal{T}\widetilde{\mathbb{m}}_\mu=-\mathcal{P}\widetilde{\mathbb{m}}_\mu=-\,\widetilde{\mathbb{m}}_7\,. \tag{47}$$

τ OPPOSITION. φ OPPOSITION: $\varphi\longrightarrow-\varphi$:

−5) $\mathcal{M}^\tau$, mirroring in the "plane" $(\mathbf{n}_t,\mathbf{n}_z)$ (π rotation in the $\mathbf{n}_z$ axis, and no time opposition):
$(n_t,n_z,n_x,n_y)\longrightarrow(n_t,n_z,-n_x,-n_y)$ $(\tau\longrightarrow-\tau)$ accomplished with $\varphi_R\longrightarrow\varphi=-\,\varphi_R$.

$$\widetilde{\mathbb{m}}_{-5}=\widetilde{\mathbb{m}}(-\varphi_R)=\dot{\widetilde{\mathbb{m}}}_\mu=\begin{pmatrix}n_t+n_z & -\tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ -\tau\,\mathrm{e}^{\mathrm{i}\,\phi} & n_t-n_z\end{pmatrix}=\sigma^Z\,\widetilde{\mathbb{m}}_\mu\,\sigma^Z\equiv\mathcal{M}^\tau\,\widetilde{\mathbb{m}}_\mu=-\widetilde{\mathbb{m}}_{-3}\,. \tag{48}$$

−3) $\mathcal{TR}^Z$, mirroring in the plane $(\mathbf{n}_x,\mathbf{n}_y)$, or time opposition and mirroring in the "hiperplane" $(\mathbf{n}_t,\mathbf{n}_x,\mathbf{n}_y)$:
$(n_t,n_z,n_x,n_y)\longrightarrow(-n_t,-n_z,n_x,n_y)$ accomplished with $\varphi_R\longrightarrow\varphi=-(\varphi_R-\frac{\pi}{2})$.

$$\widetilde{\mathbb{m}}_{-3}=\widetilde{\mathbb{m}}(-(\varphi_R-\tfrac{\pi}{2}))=-\dot{\widetilde{\mathbb{m}}}_\mu=\begin{pmatrix}-n_t-n_z & \tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ \tau\,\mathrm{e}^{\mathrm{i}\,\phi} & -n_t+n_z\end{pmatrix}=(\,\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\sigma^Z)\widetilde{\mathbb{m}}_\mu(\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\sigma^Z)\equiv\mathcal{TR}^Z\,\widetilde{\mathbb{m}}_\mu=-\,\mathcal{M}^\tau\,\widetilde{\mathbb{m}}_\mu=-\,\widetilde{\mathbb{m}}_{-5}\,. \tag{49}$$

−7) $\mathcal{R}^Z$, mirroring in the "hiperplane" $(\mathbf{n}_t,\mathbf{n}_x,\mathbf{n}_y)$ (there is no possibility for a rotation) :
$(n_t,n_z,n_x,n_y)\longrightarrow(n_t,-n_z,n_x,n_y)$ accomplished with $\varphi_R\longrightarrow\varphi=-(\varphi_R+\frac{\pi}{2})$.

$$\widetilde{\mathbb{m}}_{-7}=\widetilde{\mathbb{m}}(-(\varphi_R+\tfrac{\pi}{2}))=-\dot{\widetilde{\mathbb{m}}}^\mu=\begin{pmatrix}n_t-n_z & \tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ \tau\,\mathrm{e}^{\mathrm{i}\,\phi} & n_t+n_z\end{pmatrix}=\mathbf{R}_e^{\widetilde{\epsilon}\frac{\pi}{2}}\,\widetilde{\mathbb{m}}_\mu\,\mathbf{R}_e^{\widetilde{\epsilon}\frac{\pi}{2}}\equiv\mathcal{R}^Z\,\widetilde{\mathbb{m}}_\mu=-\,\widetilde{\mathbb{m}}_{-1}\,. \tag{50}$$

−1) $\mathcal{TM}^\tau$, mirroring in the axis $\mathbf{n}_z$, or time opposition and mirroring in the "plane" $(\mathbf{n}_t,\mathbf{n}_z)$:
$(n_t,n_z,n_x,n_y)\longrightarrow(-n_t,n_z,-n_x,-n_y)$ accomplished with $\varphi_R\longrightarrow\varphi=-(\varphi_R-\pi)$.

$$\widetilde{\mathbb{m}}_{-1}=\widetilde{\mathbb{m}}(-(\varphi_R-\pi))=\dot{\widetilde{\mathbb{m}}}_\mu=\begin{pmatrix}-n_t+n_z & -\tau\,\mathrm{e}^{-\mathrm{i}\,\phi}\\ -\tau\,\mathrm{e}^{\mathrm{i}\,\phi} & -n_t-n_z\end{pmatrix}=(\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\mathbf{R}_e^{\widetilde{\epsilon}\frac{\pi}{2}})\,\widetilde{\mathbb{m}}_\mu\,(\mathrm{e}^{\epsilon\mathrm{i}\frac{\pi}{2}}\,\mathbf{R}_e^{\widetilde{\epsilon}\frac{\pi}{2}})\equiv\mathcal{TM}^\tau\widetilde{\mathbb{m}}_\mu=-\mathcal{R}^Z\,\widetilde{\mathbb{m}}_\mu=-\,\widetilde{\mathbb{m}}_{-7}\,. \tag{51}$$

VII - TIME-SPACE-SPIN VECTORS (MATRICES).

Using the notation in (42) and the spin matrices in (22) and (23), we define:

$$\begin{aligned}
&\widetilde{s}^{+\varphi_\pm} \equiv e^{\,i\frac{\pi}{2}n_t(\varphi_\pm)}\, e^{-i\phi}\, s^{+\varphi} = n_t(\varphi_\pm)\, i\, e^{-i\phi}\, s^{+\varphi} = n_t(\varphi_\pm)\, \mathbf{m}^-\, \mathbf{s}^{+\varphi} \;;\\
&\widetilde{s}^{-\varphi_\pm} \equiv e^{-i\frac{\pi}{2}n_t(\varphi_\pm)}\, e^{\,i\phi}\, s^{-\varphi} = -n_t(\varphi_\pm)\, i\, e^{\,i\phi}\, s^{-\varphi} = n_t(\varphi_\pm)\, \mathbf{m}^+\, s^{-\varphi} \;;\\
&\qquad \hat{s}{}^{\varphi_\pm} \equiv \widetilde{s}^{+\varphi_\pm}\, \widetilde{s}^{-\varphi_\pm} = \hat{s}{}^{\varphi} \;; \qquad\qquad \overset{\vee}{s}{}^{\varphi_\pm} \equiv \widetilde{s}^{-\varphi_\pm}\, \widetilde{s}^{+\varphi_\pm} = \overset{\vee}{s}{}^{\varphi} \;; \qquad (52)\\
&\widetilde{\mathbb{1}}^{\varphi_\pm} \equiv \left\{\widetilde{s}^{+\varphi_\pm}\,,\ \widetilde{s}^{-\varphi_\pm}\right\} = \left\{s^{+\varphi},\, s^{-\varphi}\right\} = \mathbb{1} \;;\\
&\widetilde{s}^{\,z\,\varphi_\pm} \equiv \left[\widetilde{s}^{+\varphi_\pm}\,,\ \widetilde{s}^{-\varphi_\pm}\right] = \left[s^{+\varphi},\, s^{-\varphi}\right] = s^{z\,\varphi} = \mathbf{R}_e^{2\varphi} = \mathbb{n}(2\varphi)\,.
\end{aligned}$$

We establish the Time-Space-Spin vectors (matrices) $\widetilde{s}^{*\varphi_\pm}$, in relation to a value $\varphi_R \in [0, \frac{\pi}{4}]$ and with $\lambda \in \{+,-\}$:

$$\begin{aligned}
&\lambda 1 \quad \varphi = \lambda(\varphi_R - \pi):\\
&\qquad \widetilde{s}^{\,-}_{\lambda 1} \equiv \widetilde{s}^{+\lambda(\varphi_R-\pi)} = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^+ \sigma^-\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\qquad \widetilde{s}^{\,+}_{\lambda 1} \equiv \widetilde{s}^{-\lambda(\varphi_R-\pi)} = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^- \sigma^+\, \mathbf{R}_{\mathbf{O}}^{-\epsilon\frac{\pi}{2}}\, \mathbf{R}_{\mathbf{O}}^{-\lambda\varphi_{\mathbf{R}}}\\
&\qquad \widetilde{s}^{\,z}_{\lambda 1} \equiv \widetilde{s}^{\,z\,\lambda(\varphi_R-\pi)} = \mathbf{R}_e^{2\lambda\varphi_R} = \mathbb{n}(\lambda\varphi_R) = \mathbf{R}_O^{\lambda\varphi_R}\, \sigma^z\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\lambda 3 \quad \varphi = \lambda(\varphi_R - \tfrac{\pi}{2}):\\
&\qquad \widetilde{s}^{\,-}_{\lambda 3} \equiv \widetilde{s}^{+\lambda(\varphi_R-\frac{\pi}{2})} = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^+ \sigma^-\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\qquad \widetilde{s}^{\,+}_{\lambda 3} \equiv \widetilde{s}^{-\lambda(\varphi_R-\frac{\pi}{2})} = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^- \sigma^+\, \mathbf{R}_{\mathbf{O}}^{-\lambda\varphi_{\mathbf{R}}}\\
&\qquad \widetilde{s}^{\,z}_{\lambda 3} \equiv \widetilde{s}^{\,z\,\lambda(\varphi_R-\frac{\pi}{2})} = -\,\mathbf{R}_e^{2\lambda\varphi_R} = -\,\mathbb{n}(\lambda\varphi_R) = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \sigma^z\, \mathbf{R}_O^{-\lambda\varphi_R} \qquad (53)\\
&\textcircled{\lambda 5} \quad \varphi = \lambda\varphi_R:\\
&\qquad \widetilde{s}^{\,+}_{\lambda 5} \equiv \widetilde{s}^{+\lambda\varphi_R} = \mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^- \sigma^+\, \mathbf{R}_{\mathbf{O}}^{-\lambda\varphi_{\mathbf{R}}}\\
&\qquad \widetilde{s}^{\,-}_{\lambda 5} \equiv \widetilde{s}^{-\lambda\varphi_R} = \mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^+ \sigma^-\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\qquad \widetilde{s}^{\,z}_{\lambda 5} \equiv \widetilde{s}^{\,z\,\lambda\varphi_R} = \mathbf{R}_e^{2\lambda\varphi_R} = \mathbb{n}(\lambda\varphi_R) = \mathbf{R}_O^{\lambda\varphi_R}\, \sigma^z\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\lambda 7 \quad \varphi = \lambda(\varphi_R + \tfrac{\pi}{2}):\\
&\qquad \widetilde{s}^{\,+}_{\lambda 7} \equiv \widetilde{s}^{+\lambda(\varphi_R+\frac{\pi}{2})} = \mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^- \sigma^+\, \mathbf{R}_{\mathbf{O}}^{-\epsilon\frac{\pi}{2}}\, \mathbf{R}_{\mathbf{O}}^{-\lambda\varphi_{\mathbf{R}}}\\
&\qquad \widetilde{s}^{\,-}_{\lambda 7} \equiv \widetilde{s}^{-\lambda(\varphi_R+\frac{\pi}{2})} = \mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^+ \sigma^-\, \mathbf{R}_O^{-\epsilon\frac{\pi}{2}}\, \mathbf{R}_O^{-\lambda\varphi_R}\\
&\qquad \widetilde{s}^{\,z}_{\lambda 7} \equiv \widetilde{s}^{\,z\,\lambda(\varphi_R+\frac{\pi}{2})} = -\,\mathbf{R}_e^{2\lambda\varphi_R} = -\,\mathbb{n}(\lambda\varphi_R) = -\,\mathbf{R}_O^{\lambda\varphi_R}\, \sigma^z\, \mathbf{R}_O^{-\lambda\varphi_R}
\end{aligned}$$

In a first wording, we did not include the factors $e^{\pm i\phi}$ in the definitions of $\widetilde{s}^{\pm\varphi_\pm}$. But after defining the creation and annihilation operators (in *Study III*) we realized a gain in symmetry and physical significance in their formulas by using these factors. Also, the formulas in (22) and (23) suggest these forms.

For the equations (32) and (33), with $\epsilon \in \{+1,-1\}$, and with (34), $\mathbf{m}^{+2} = -e^{\,i\,2\phi}$, $\mathbf{m}^{-2} = -e^{-i\,2\phi}$, we show:

$$\begin{cases}
s^{+\varphi} = s^{+\varphi-\epsilon\pi}\,, & s^{-\varphi} = s^{-\varphi-\epsilon\pi}\,, \qquad s^{z\varphi} = s^{z\,\varphi-\epsilon\pi}\\
s^{+\varphi} = \mathbf{m}^{+2} s^{-\varphi-\epsilon\frac{\pi}{2}}\,, & s^{-\varphi} = \mathbf{m}^{-2} s^{+\varphi-\epsilon\frac{\pi}{2}}\,, \qquad s^{z\varphi} = -\,s^{z\varphi-\epsilon\frac{\pi}{2}}\,.
\end{cases} \qquad (54)$$

For every $\hat{\varphi}_+$ there is a $\hat{\varphi}_-$, such that $\hat{\varphi}_+ = \hat{\varphi}_- + \pi$, or $\hat{\varphi}_+ = \hat{\varphi}_- - \pi$. They verify:

$$\widetilde{s}^{+\hat{\varphi}_+} = -\widetilde{s}^{+\hat{\varphi}_-}\,, \qquad \widetilde{s}^{-\hat{\varphi}_+} = -\widetilde{s}^{-\hat{\varphi}_-}\,, \qquad \widetilde{s}^{\,z\hat{\varphi}_+} = \widetilde{s}^{\,z\hat{\varphi}_-}\,. \qquad (55)$$

VIII - THE • -PRODUCT.

With $\{r, s\} = \{1, 3, 5, 7\}$ or $\{2, 4, 6, 8\}$, $\lambda \in \{+, -\}$, and $*$ indicating either $+$ or $-$. In a similar way for $**$, we define the following product:

$$\begin{cases} \widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} \equiv \widetilde{s}^{\,*}_{\lambda r}\, \mathbf{I}_{\lambda r \lambda s}\, \widetilde{s}^{\,**}_{\lambda s}\,, & r \neq s \\ \widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} \equiv \widetilde{s}^{\,*}_{\lambda r}\, \widetilde{s}^{\,**}_{\lambda r}\,, & r = s \end{cases}, \tag{56}$$

where:

$$\mathbf{I}_{\lambda r \lambda s} \equiv \mathbf{I}_{\mathbf{rs}} \equiv \mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}} = \mathrm{sign}\,(s-r)\;\, \mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}} = -\mathrm{sign}\,(s-r)\;\, \mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}} = -\mathbf{I}_{\lambda s \lambda r}\,, \tag{57}$$

and $\begin{cases} \mathrm{sign}\,(r-s) = +1 & \text{for } r > s \\ \mathrm{sign}\,(r-s) = -1 & \text{for } r < s \end{cases}$; also, we have used: $-\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}} = \mathbf{R}_O^{-\widetilde{\epsilon}\frac{\pi}{2}}$, and $\widetilde{\epsilon} = \{+1, -1\}$.

With $r = s$ it is the usual product. For $r \neq s$ the action is:

A) for the $\widetilde{s}^{\,*}_{\lambda r}$ and $\widetilde{s}^{\,**}_{\lambda s}$ which are expressed in terms, both of $s^{\pm^{\lambda\varphi_R}}$ or both with $s^{\pm^{\lambda(\varphi_R+\frac{\pi}{2})}}$:

1) if: $\pm$ for $*$, and $\mp$ for $**$:

$$\widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} = K' s^{\pm^{\lambda\varphi_R}}\, \mathbf{I}_{\lambda r\lambda s}\, s^{\mp^{\lambda\varphi_R}} = K'\mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\mathbf{R}_O^{-\lambda\varphi_R}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\mathbf{R}_O^{\lambda\varphi_R}\sigma^{\mp}\mathbf{R}_O^{-\lambda\varphi_R} =$$

$$= -\,K'\mathrm{sign}\,(s-r)\, \mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\sigma^{\mp}\,\mathbf{R}_O^{-\lambda\varphi_R} = \mathbb{0} = -\,\widetilde{s}^{\,**}_{\lambda s} \bullet \widetilde{s}^{\,*}_{\lambda r} \tag{58}$$

as it is: $\sigma^{\pm}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\sigma^{\mp} = \mathbb{0}$. Similarly if we use $\lambda\varphi_R' = \lambda(\varphi_R+\frac{\pi}{2})$ instead of $\lambda\varphi_R$.

2) if: $\pm$ for $*$, and $\pm$ for $**$:

$$\widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} = K'' s^{\pm^{\lambda\varphi_R}}\, \mathbf{I}_{\lambda r\lambda s}\, s^{\pm^{\lambda\varphi_R}} = -\,K'' s^{\pm^{\lambda\varphi_R}}\, \mathbf{I}_{\lambda s\lambda r}\, s^{\pm^{\lambda\varphi_R}} = -\,\widetilde{s}^{\,**}_{\lambda s} \bullet \widetilde{s}^{\,*}_{\lambda r} \tag{59}$$

Similarly if we use $\lambda\varphi_R' = \lambda(\varphi_R+\frac{\pi}{2})$ instead of $\lambda\varphi_R$.

B) for the $\widetilde{s}^{\,*}_{\lambda r}$ and $\widetilde{s}^{\,**}_{\lambda s}$ which are expressed in terms, one of $s^{\pm^{\lambda\varphi_R}}$ and the other one of $s^{\pm^{\lambda(\varphi_R+\frac{\pi}{2})}}$,

1) if: $\pm$ for $*$, and $\pm$ for $**$:

$$\widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} = K''' s^{\pm^{\lambda\varphi_R}}\, \mathbf{I}_{\lambda r\lambda s}\, s^{\pm^{\lambda(\varphi_R+\frac{\pi}{2})}} = K'''\mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\mathbf{R}_O^{-\lambda\varphi_R}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\mathbf{R}_O^{\lambda\frac{\pi}{2}}\mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\mathbf{R}_O^{-\lambda\varphi_R}\mathbf{R}_O^{-\lambda\frac{\pi}{2}} =$$

$$= K'''\mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\,\mathbf{R}_O^{(\widetilde{\epsilon}+\lambda)\frac{\pi}{2}}\,\sigma^{\pm}\,\mathbf{R}_O^{-\lambda\frac{\pi}{2}}\mathbf{R}_O^{-\lambda\varphi_R} = \mathbb{0} = -\,\widetilde{s}^{\,**}_{\lambda s} \bullet \widetilde{s}^{\,*}_{\lambda r} \tag{60}$$

as it is: $\mathbf{R}_O^{(\widetilde{\epsilon}+\lambda)\frac{\pi}{2}} = \begin{cases} \mathbb{1}\,, & \text{if: } \widetilde{\epsilon}+\lambda = 0 \\ -\,\mathbb{1}\,, & \text{if: } \widetilde{\epsilon}+\lambda = \{-2, 2\} \end{cases}$, and $\sigma^{\pm}\,\mathbb{1}\,\sigma^{\pm} = \mathbb{0}$.

2) if: $\pm$ for $*$, and $\mp$ $**$:

$$\begin{aligned} \widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} &= K'''' s^{\pm^{\lambda\varphi_R}}\, \mathbf{I}_{\lambda r\lambda s}\, s^{\mp^{\lambda(\varphi_R+\frac{\pi}{2})}} = \\ &= K''''\mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\lambda\varphi_R}\sigma^{\pm}\mathbf{R}_O^{-\lambda\varphi_R}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\lambda\frac{\pi}{2}}\sigma^{\mp}\mathbf{R}_O^{-\lambda\frac{\pi}{2}}\mathbf{R}_O^{-\lambda\varphi_R} = \\ &= K''''\mathrm{sign}\,(r-s)\, \mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\lambda\frac{\pi}{2}}\,\mathbf{m}^{\pm^2}\sigma^{\mp}\,\mathbf{R}_O^{-\lambda\frac{\pi}{2}}\mathbf{R}_O^{-\lambda\varphi_R}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\mathbf{R}_O^{\lambda\varphi_R}\,\mathbf{m}^{\mp^2}\sigma^{\pm}\,\mathbf{R}_O^{-\lambda\varphi_R} = \\ &= -\,K''''\mathrm{sign}\,(s-r)\, \mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\lambda\frac{\pi}{2}}\,\sigma^{\mp}\,\mathbf{R}_O^{-\lambda\frac{\pi}{2}}\mathbf{R}_O^{-\lambda\varphi_R}\,\mathbf{R}_O^{\widetilde{\epsilon}\frac{\pi}{2}}\,\mathbf{R}_O^{\lambda\varphi_R}\,\sigma^{\pm}\,\mathbf{R}_O^{-\lambda\varphi_R} = -\,\widetilde{s}^{\,**}_{\lambda s} \bullet \widetilde{s}^{\,*}_{\lambda r} \end{aligned} \tag{61}$$

as, from (36), we have: $\mathbf{R}_O^{\varphi}\,\sigma^{\pm}\,\mathbf{R}_O^{-\varphi} = \left(\mathbf{m}^{\pm}\,\mathbf{R}_O^{\varphi+\epsilon\frac{\pi}{2}}\right)\sigma^{\mp}\left(\mathbf{R}_O^{-(\varphi+\epsilon\frac{\pi}{2})}\,\mathbf{m}^{\pm}\right)$, and from (34): $\mathbf{m}^{\pm}\,\mathbf{m}^{\mp} = 1$.

Similarly with $s^{\pm^{\lambda(\varphi_R-\frac{\pi}{2})}}$ instead of $s^{\pm^{\lambda(\varphi_R+\frac{\pi}{2})}}$; and also with $s^{\pm^{\lambda(\varphi_R-\pi)}}$ (using (55)). Finally, we obtain the same results starting with $\varphi_R+\frac{\pi}{2}$ or $\varphi_R-\frac{\pi}{2}$ or $\varphi_R-\pi$ instead of φ_R .

We postpone the study of the geometrical and physical arguments underneath the definition (56) with (57), and henceforth in the symmetry , antisymmetry and in the commutators , anticommutators to *Study III* **2**.

IX - ANTICOMMUTATORS.

We extend the product defined in the previous section for the Time-Space-Spin vectors (matrices):

for r and s equal or different:

$$1) \qquad \widetilde{s}^{\,z}_{\lambda r} \bullet \widetilde{s}^{\,z}_{\lambda s} \equiv \widetilde{s}^{\,z}_{\lambda r}\ \widetilde{s}^{\,z}_{\lambda s} \in \left\{ +\mathbb{1}\ ,\ -\mathbb{1} \right\}, \tag{62}$$

taking account of $\sigma^{z2} = \mathbb{1}$;

$$2) \qquad \begin{cases} \widetilde{s}^{\,\pm}_{\lambda r} \bullet \widetilde{s}^{\,z}_{\lambda s} \equiv \widetilde{s}^{\,\pm}_{\lambda r}\ \widetilde{s}^{\,z}_{\lambda s} = \mp K\, \widetilde{s}^{\,\pm}_{\lambda r} \\ \widetilde{s}^{\,z}_{\lambda s} \bullet \widetilde{s}^{\,\pm}_{\lambda r} \equiv \widetilde{s}^{\,z}_{\lambda s}\ \widetilde{s}^{\,\pm}_{\lambda r} = \pm K\, \widetilde{s}^{\,\pm}_{\lambda r} \end{cases}, \tag{63}$$

with a common non null factor K, their anticommutators are :

$$\left\{ \widetilde{s}^{\,\pm}_{\lambda r}\ ;\ \widetilde{s}^{\,z}_{\lambda s} \right\} = \left\{ \widetilde{s}^{\,\pm}_{\lambda r}\ ,\ \widetilde{s}^{\,z}_{\lambda s} \right\} = \mathbb{0}\ , \tag{64}$$

taking account of $\sigma^{+}\sigma^{z} = -\sigma^{z}\sigma^{+} = -\sigma^{+}$ or of $\sigma^{-}\sigma^{z} = -\sigma^{z}\sigma^{-} = \sigma^{-}$,

for r and s equal :

$$3) \qquad r = s \qquad \widetilde{s}^{\,+}_{\lambda r} \bullet \widetilde{s}^{\,+}_{\lambda r} \equiv \widetilde{s}^{\,+}_{\lambda r}\ \widetilde{s}^{\,+}_{\lambda r} = \widetilde{s}^{\,-}_{\lambda r} \bullet \widetilde{s}^{\,-}_{\lambda r} \equiv \widetilde{s}^{\,-}_{\lambda r}\ \widetilde{s}^{\,-}_{\lambda r} = \mathbb{0}\ , \tag{65}$$

their anticommutators are:

$$\left\{ \widetilde{s}^{\,\pm}_{\lambda r}\ ;\ \widetilde{s}^{\,\pm}_{\lambda r} \right\} \equiv \widetilde{s}^{\,\pm}_{\lambda r} \bullet \widetilde{s}^{\,\pm}_{\lambda r} + \widetilde{s}^{\,\pm}_{\lambda r} \bullet \widetilde{s}^{\,\pm}_{\lambda r} \equiv \widetilde{s}^{\,\pm}_{\lambda r}\ \widetilde{s}^{\,\pm}_{\lambda r} + \widetilde{s}^{\,\pm}_{\lambda r}\ \widetilde{s}^{\,\pm}_{\lambda r} = \mathbb{0}\ , \tag{66}$$

we also have :

$$\left\{ \widetilde{s}^{\,+}_{\lambda r}\ ;\ \widetilde{s}^{\,-}_{\lambda r} \right\} \equiv \widetilde{s}^{\,+}_{\lambda r} \bullet \widetilde{s}^{\,-}_{\lambda r} + \widetilde{s}^{\,-}_{\lambda r} \bullet \widetilde{s}^{\,+}_{\lambda r} \equiv \widetilde{s}^{\,+}_{\lambda r}\ \widetilde{s}^{\,-}_{\lambda r} + \widetilde{s}^{\,-}_{\lambda r}\ \widetilde{s}^{\,+}_{\lambda r} = \mathbb{1}\ , \tag{67}$$

as it is $\sigma^{+2} = \sigma^{-2} = \mathbb{0}$, and $\{\sigma^{+}\ ,\ \sigma^{-}\} = \mathbb{1}$.

These three points express the essence of the Jordan Wigner method (particularly 2).

For r and s different, using the definitions and the equations of the previous section we have:

$$4) \qquad r \neq s \qquad \left\{ \widetilde{s}^{\,*}_{\lambda r}\ ;\ \widetilde{s}^{\,**}_{\lambda s} \right\} = \left\{ \widetilde{s}^{\,**}_{\lambda s}\ ;\ \widetilde{s}^{\,*}_{\lambda r} \right\} = \widetilde{s}^{\,*}_{\lambda r} \bullet \widetilde{s}^{\,**}_{\lambda s} + \widetilde{s}^{\,**}_{\lambda s} \bullet \widetilde{s}^{\,*}_{\lambda r} = \mathbb{0}\ , \tag{68}$$

X - SPIN, VECTOR-SPIN AND CHIRALITY. *TOWARDS PHYSICS.*

Tentatively, we present a first approach to these concepts.

Spin. The sense of the action of generalized rotations, with a privileged semi-sphere with $\mathbf{n_z}$ $(\, n_z \geq 0\ ,\ n_z \leq 0\,)$ *and in a time type variable* $\mathbf{n_t}$ $(\, n_t = +1\ ,\ n_t = -1\,)$.

In order to obtain the eigenvalues of a spin operator for the different $\widetilde{s}^{\,+}_{\lambda r}$ Time-Space-Spin vectors (matrices) we define the following spin operator (a matrix):

$$\mathbb{S}^{z}_{\lambda} \equiv \mathbf{R}_e^{\lambda 2\varphi_R} = \mathbf{R}_O^{\varphi_R}\, \sigma^{z}\, \mathbf{R}_O^{-\varphi_R}\ , \tag{69}$$

which for $\varphi_R = 0$ is just σ^{z}. It essentially acts over two types of $+$Time-Space-Spin vectors (matrices $\widetilde{s}^{\,+}$):

$$\begin{cases} \mathbb{S}^{z}_{\lambda}\ \widetilde{s}^{\,+}_{\lambda 5} = \mathbf{R}_e^{\lambda 2\varphi_R}\left(\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}}\right) = \mathbf{R}_\mathbf{O}^{\lambda\varphi_\mathbf{R}}\, \sigma^{\mathbf{z}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = (+\mathbf{1})\ \widetilde{\mathbf{s}}^{\,+}_{\lambda 5} \\ \mathbb{S}^{z}_{\lambda}\ \widetilde{s}^{\,+}_{\lambda 3} = \mathbf{R}_e^{\lambda 2\varphi_R}\left(-\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}}\right) = -\mathbf{R}_\mathbf{O}^{\lambda\varphi_\mathbf{R}}\, \sigma^{\mathbf{z}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = (+\mathbf{1})\ \widetilde{\mathbf{s}}^{\,+}_{\lambda 3} \end{cases}$$

$$\begin{cases} \mathbb{S}^{z}_{\lambda}\ \widetilde{s}^{\,+}_{\lambda 7} = \mathbf{R}_e^{\lambda 2\varphi_R}\left(\mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}}\right) = \mathbf{R}_\mathbf{o}^{\lambda\varphi_\mathbf{R}}\sigma^{\mathbf{z}}\, \mathbf{R}_\mathbf{o}^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = \\ \qquad = \mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, (-\sigma^{z})\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = (-\mathbf{1})\ \widetilde{\mathbf{s}}^{\,+}_{\lambda 7} \\ \mathbb{S}^{z}_{\lambda}\ \widetilde{s}^{\,+}_{\lambda 1} = \mathbf{R}_e^{\lambda 2\varphi_R}\left(-\mathbf{R}_O^{\lambda\varphi_R}\, \mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\, \mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}}\right) = -\mathbf{R}_\mathbf{o}^{\lambda\varphi_\mathbf{R}}\sigma^{\mathbf{z}}\, \mathbf{R}_\mathbf{o}^{\epsilon\frac{\pi}{2}}\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = \\ \qquad = -\,\mathbf{R}_O^{\lambda\varphi_R}\mathbf{R}_O^{\epsilon\frac{\pi}{2}}\, (-\sigma^{z})\, \mathbf{m}^{-}\sigma^{+}\, \mathbf{R}_\mathbf{O}^{-\epsilon\frac{\pi}{2}}\mathbf{R}_\mathbf{O}^{-\lambda\varphi_\mathbf{R}} = (-\mathbf{1})\ \widetilde{\mathbf{s}}^{\,+}_{\lambda 1} \end{cases} . \tag{70}$$

Accordingly to this we can state the following: in the 3d sphere ($\| \mathbb{n} \| = 1$), we establish:

spin up if the 3d part of the vector $\widetilde{\mathbf{n}}$ is in the upper half semi-sphere $(n_z > 0)$ and there is a positive time-type component $(n_t > 0)$; or, in the lower half semi-sphere $(n_z < 0)$ and there is a negative time-type component $(n_t < 0)$,

spin down with the upper half semi-sphere $(n_z > 0)$ but with a negative time component $(n_t < 0)$, or finally in the lower half semi-sphere $(n_z < 0)$ with a positive time component $(n_t > 0)$.

In brief:

if $\text{sign}(n_t)\,\text{sign}(n_z) = +$ them $\uparrow$ (spin up) , and

if $\text{sign}(n_t)\,\text{sign}(n_z) = -$ them $\downarrow$ (spin down).

Values in the equatorial plane of the sphere, $n_z = 0$, are obtained after the values of φ (or n_z) at each side, taking account of the two values of n_t. We get the actual values by applying (69)-(70) directly.

The key points are: $\text{sign}(n_t)$ and $\text{sign}(n_z)$. This drives to the two values: discretization.

Is is worth noting that although this spin operator does not depend in n_t , its action produces different results which depend on the n_t value of the generalized vector-spins (matrices) $\widetilde{s}\,^{+}_{\lambda r}$.

vector-spins $(s^{z\,\varphi})$. *Timeless, opposition in space (parity).* (Figure 4).

In the Section VII we have defined the new notions of Time-Space-Spin vectors (matrices) $\widetilde{s}\,^{\pm\,\varphi}{}_{\pm}$, $\widetilde{s}\,^{z\,\varphi}{}_{\pm}$. We have studied the first ones (with $+$) in the previous paragraphs, presenting them in relation to the spin operator. The third ones (with z) suggest the named *vector-spin* (see in (52)):

$$\widetilde{s}\,^{z\,\varphi}{}_{\pm} \equiv \left[\widetilde{s}\,^{+\,\varphi}{}_{\pm} \, , \, \widetilde{s}\,^{-\,\varphi}{}_{\pm} \right] = s^{z\,\varphi} = \mathbf{R}_e^{2\varphi} = \mathbb{n}(2\varphi) \, .$$

These unity vectors $\pm\mathbb{n}(\lambda 2\varphi_R)$ have the form:

$$\pm\, s^{z\,\lambda\varphi_R} = \pm\mathbb{n}(\lambda 2\varphi_R) \equiv \pm\mathbf{R}_e^{\lambda 2\varphi_R} \equiv \mathbf{R}_O^{\lambda\varphi_R} \, (\pm\,\sigma^z)\, \mathbf{R}_O^{-\lambda\varphi_R} \tag{71}$$

which for a given φ_R provides 4 different values, except for $2\varphi_R = 0$ and for $2\varphi_R = \frac{\pi}{2}$. The $\pm$ previous to a three dimensional semi-direction $\mathbb{n}(2\varphi_R)$ gives us a whole 3 dimensional part of a characteristic axis. And the value $\lambda = -1$ drives to one more characteristic axis, symmetric in the $\mathbf{n}_z$-axis to the one with $\lambda = +1$, as well it is π-rotated over such $\mathbf{n}_z$-axis, also providing another two values. They verify:

$$\text{sign}(n_z)\,\text{sign}(\tau) = \text{sign}(\lambda) = +\,, \quad \begin{cases} 5 \text{ and } 1 & \mathbb{n}(\,2\varphi_R) = \mathbf{R}_e^{2\varphi_R} \\ 7 \text{ and } 3 & -\,\mathbb{n}(\,2\varphi_R) = -\mathbf{R}_e^{2\varphi_R} \end{cases} \tag{72a}$$

$$\text{sign}(n_z)\,\text{sign}(\tau) = \text{sign}(\lambda) = -\,, \quad \begin{cases} -5 \text{ and } -1 & \dot{\mathbb{n}}(-2\varphi_R) = \mathbf{R}_e^{-2\varphi_R} \\ -7 \text{ and } -3 & -\,\dot{\mathbb{n}}(-2\varphi_R) = -\mathbf{R}_e^{-2\varphi_R} \end{cases} \tag{72b}$$

This discretization is similar to the one of the electric charge. We depict these 3-d vectors in Figures 1, 2, 4 and 5. For $\tau = 0$ and for $n_z = 0$ we are forced to apply directly (71), obtaining only two values, for each one of them.

Chirality.

We assign a right or left chirality in terms of the angle φ with the following rule:

$$\begin{cases} \text{left chirality} & \text{for} \quad \varphi \in [\,-\pi \, , \, 0\,) \\ \text{right chirality} & \text{for} \quad \varphi \in [\;\; 0 \, , \, \pi\,) \end{cases} , \tag{73}$$

In $\varphi = \pm\frac{\pi}{2}$ $(n_z = -1)$ we assign with $n_t = +1$ a *right chirality* and with $n_t = -1$ a *left chirality* . Alternatively:

$$\text{sign}(\varphi) = \text{sign}(n_t)\,\text{sign}(n_z)\,\text{sign}(\tau) = \begin{cases} - \quad \longleftrightarrow \quad \text{left chirality} \quad \text{for} \;\; \{\,1\,,\,-7\,,\,3\,,\,-5\,\} \\ + \quad \longleftrightarrow \quad \text{right chirality} \quad \text{for} \;\; \{\,-1\,,\,7\,,\,-3\,,\,5\,\} \end{cases} . \tag{74}$$

A special treatment for the values: $\tau = 0$ or $n_z = 0$ $(\,\varphi \in \{\,0\,,\,\pm\frac{\pi}{4}\,,\,\pm\frac{\pi}{2}\,,\,\pm\frac{3\pi}{4}\,,\,-\pi\,\}\,)$.

We have advanced some considerations for the *electrical charges* in Section V.

XI - A DIGRESSION. DEFINITION OF PRIVILEGED DIRECTIONS: [$(\mathbf{n_t})$, $\mathbf{n_x}, \mathbf{n_y}, \mathbf{n_z}$ $\longleftrightarrow$ $(\mathbf{n_t}, \{\mathbf{n_z}\}, (\mathbf{n_x}, \mathbf{n_y})\}$] . CHIRALITY? (FIGURE 6). IS IT ALL GEOMETRY?. FINAL REMARKS.

We consider three elements:

I and II) Algebra - geometry: the Pauli basis distinguishes two algebraic parts: the diagonal and the anti-diagonal. Are there correlations of these two parts distinction with the geometry?

The diagonal one. Underneath the different character of a time and a space in the metric (Minkowski). See *Study I.2*. Associated to it a time ($\mathbb{1}$) and a space (σ^z) type dimensions. Algebraically, it is a linear subspace. Under the matrix product it keeps in the diagonal. A boost (in **z**) mixes the two components (the time and the space). Fixed the time type as positive ($+\mathbb{1}$), we have two distinct space semi-directions to which we assign the coordinate letters **+z** and **−z**, or $\mathbf{n_z}$ and $-\mathbf{n_z}$.

The anti-diagonal part: its elements are two space type dimensions (σ^x , σ^y). Again a linear subspace. With the metric they are with the previous space type dimension (σ^z). But now, under the matrix product between their elements they produce matrix elements of the diagonal type, either a multiple of the identity (a time?, a scalar?, the scalar product?) or with complex coefficients for the space type vector (with the included vector product), which is not the same type of vector space: we would be mixing the polar vectors and the axial vectors. We already have seen the mixing of the real and the imaginary in the quaternion and biquaternion formulation of Hamilton [8]. This is clarified in *Study I,2*.

We can now post two questions (and a subsequent third):

If the σ^z coordinate-direction appears, in this formulation, in a different way to the σ^x, σ^y coordinates-directions,

1) are there any privileged direction(s) of a local space? Could we privilege a σ^z-{$\mathbf{n_z}$ space coordinate-direction}?

In a parallel way, as we differentiate the $\mathbb{1}$-{time coordinate-direction} and the σ^z-{$\mathbf{n_z}$ space coordinate-direction}, what does it happen with a σ^x and a σ^y coordinates-directions (this last one containing complex numbers)?,

2) is it possible to have different privileged $\mathbf{n_x}$ and $\mathbf{n_y}$ coordinates-directions of this two dimensional space? Is it possible to have some property distinguishing between them? Is this shown by the weak interaction?

3) does all of this mean a kind of local Newtonian absolute space?

III) Physics. Most of our observations show a privileged unique-direction with an “arrow of time”, but they do not show any privileged directions of space: there are three equally independent space directions.

The object of our *Studies* concerns “the very small”. We state our formulation for the elementary particles. We consider them as isolated objects, after their production. This creation provides the particle with a direction of movement, **its** $+n_z$ direction. In *Study III* we show another way for fixing this direction in an intrinsic way. The $\mathbb{n}$, space parts of the characteristic vectors, defined with respect to a $+\mathbf{n_z}$ axis of the particle, and adding also a type of time ($\mathbf{n_t}$). With these elements we have obtained the spin, an intrinsic physical quantity. Does this help answering the first question? **For 1): the helicity?**

For the second question, let us present an analogy: fixing the longitude over the surface of the earth. To define the latitude over the equatorial circle is simple. The longitude is fixed once we *arbitrarily “draw a”* Greenwich meridian. The *arbitrariness* is due to the kinematic circular symmetry (a South-North poles rotation axis). This symmetry helps for the fixing of the Equator.

The parametrization in (10), associated to the Pauli basis, suggest that we have two handed tetrahedrons: a right-handed one $\{n_x, n_y, n_z\}$, and a left-handed one $\{n_x, -n_y, n_z\}$. We already have a fixed $+\mathbf{n_z}$. And **here** we also have fixed $+\mathbf{n_x}$.

Now, let us examine the Figure 6. We have assumed $n_t = +1$. We can do something similar with $n_t = -1$. We represent two vectors (characteristic), both right handed: $\mathbf{n}$ and $\dot{\mathbf{n}}$ in the upper half semi-sphere ($n_z \geq 0$) which indicates **spin up** for both. **+z** represents, also, a direction of movement. The yellow and blue circles with their respective arrows indicate their equal senses of rotation (right-handed). One is a rotated axis of the other one under a π-rotation over $\mathbf{n_z}$, additionally they are symmetric with respect to the line defined by $\mathbf{n_z}$. We construct a “fictitious” plane with the $\mathbf{n_z}$ and an “undetermined” $\mathbf{n_x}$, therefore $\mathbf{n}$ *and* $\dot{\mathbf{n}}$, each at a side of this plane. The neutrinos represent an exceptional case (A and A’ in table 2); due to the geometrical situation just stated. We pay attention to the arrows over the circles in these two rotations, considered at the closest positions to this plane. Viewed from a distant point in **+z** they have opposite senses. This is easily generalized with $-n_z$ (and with $-n_t$).

Could these arguments drive us to consider **for 2): the chirality?**, Take it as a relationship that privileges a certain coordinate semi-axis $\mathbf{n_x}$ in the local universe of the isolated elementary particle, although we do not know where. To complete these arguments we have to add the electrical charges (bring here the meaning of the γ_y for them). See *Studies I,2* and *III*.

We say that a physical quantity is a property that can be measured in a laboratory and / or concluded after the body of a physical theory (the astronomical data and the chirality concept included) which can be counter-checked with nature. We have envisaged a correspondence between our physical quantities and the geometrical structure of a local universe for the isolated quarks and leptons. The electrical charge, the spin, the vector-spin and the chirality are the physical quantities. Let us remember the weak interaction. Interactions would be added “a posteriori” as modifications of the geometry of the time-space. In our treatment we have worked with the rotations but we have not deal with the displacements, we have only assumed them.

Therefore, **for the question 3)**, this author suggests: *our large scale Universe is relational (Leibnitzian)*, perhaps if larger than a fraction of the nucleus of an atom, and *the local universe of the quarks and leptons is locally absolute.*

Locally-Newtonian: [charge, spin, vector-spin, chirality] $\leftrightarrow$ *[* $(\mathbf{n_t}, \{\mathbf{n_z}\}, (\mathbf{n_x}, \mathbf{n_y})\}$ *] (a local 'timespace' for every particle.*

More on this in the other *Studies*. We should compare the pathway suggested here to the one shown by Hestenes [9] and by Thorne and Blanford [10]. These authors claim for *coordinate free* systems, meaning without accounting for any basis vectors; basis vectors can be arbitrarily identified or defined afterwards.

The reader might wonder: why $\{t, z, x, y\}$ and $\{n_t, n_z, n_x, n_y\}$? Polar and axial or real and imaginary vectors. We need a $\mathfrak{C}^4$ space, an algebraically eight dimensional space over the reals. See *Studies I,1* and *I,2*.

APPENDIX A: C-ROTATION AND C-REFLECTION MATRICES.

$$\mathbf{R}_o^{\varphi}(\phi) = \begin{pmatrix} \cos\varphi & -\sin\varphi\, e^{-i\phi} \\ \sin\varphi\, e^{i\phi} & \cos\varphi \end{pmatrix}, \qquad \mathbf{R}_e^{\varphi}(\phi) = \begin{pmatrix} \cos\varphi & \sin\varphi\, e^{-i\phi} \\ \sin\varphi\, e^{i\phi} & -\cos\varphi \end{pmatrix}.$$

$$\left\{ \mathbb{1} = \mathbf{R}_o^{0}(\phi), \quad \sigma^z = \mathbf{R}_e^{0}(\phi), \quad \sigma^x = \mathbf{R}_e^{\frac{\pi}{2}}(0), \quad \sigma^y = \mathbf{R}_e^{\frac{\pi}{2}}(\tfrac{\pi}{2}) \right\} \quad \text{(Pauli basis).}$$

$$\begin{cases} \mathbf{R}_o^{\varphi\dagger} = \mathbf{R}_o^{\varphi\,-1} = \mathbf{R}_o^{-\varphi}, & \mathbf{R}_e^{\varphi\dagger} = \mathbf{R}_e^{\varphi\,-1} = \mathbf{R}_e^{\varphi} \\ \mathbf{R}_e^{\varphi} = \mathbf{R}_o^{\varphi}\sigma^z = \sigma^z\mathbf{R}_o^{-\varphi}, & \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{\varphi} = \mathbf{R}_o^{-\varphi}\,\mathbf{R}_e^{\varphi} = \sigma^z \\ \sigma^z\,\mathbf{R}_o^{\varphi}\,\sigma^z = \mathbf{R}_o^{-\varphi}, & \sigma^z\,\mathbf{R}_e^{\varphi}\,\sigma^z = \mathbf{R}_e^{-\varphi} \\ \det[\,\mathbf{R}_o^{\varphi}\,] = 1, & \det[\,\mathbf{R}_e^{\varphi}\,] = -1 \end{cases}.$$

$$\begin{cases} \mathbf{R}_o^{\varphi_1}\,\mathbf{R}_o^{\varphi_2} = \mathbf{R}_o^{\varphi_1+\varphi_2} \\ \mathbf{R}_e^{\varphi_1}\,\mathbf{R}_e^{\varphi_2} = \mathbf{R}_o^{\varphi_1-\varphi_2} \\ \mathbf{R}_o^{\varphi_1}\,\mathbf{R}_e^{\varphi_2} = \mathbf{R}_e^{\varphi_1+\varphi_2} \\ \mathbf{R}_e^{\varphi_1}\,\mathbf{R}_o^{\varphi_2} = \mathbf{R}_e^{\varphi_1-\varphi_2} \end{cases}, \qquad \begin{cases} \mathbf{R}_o^{\varphi\,2} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_e^{-\varphi} = \mathbf{R}_o^{2\varphi} \\ \mathbf{R}_e^{\varphi\,2} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_o^{0} = \mathbb{1} \\ \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{\varphi} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{2\varphi} \\ \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{\varphi} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{-\varphi} = \mathbf{R}_e^{0} = \sigma^z \end{cases}.$$

It is clear that:

$$\mathbf{R}_o^{\frac{\pi}{2}} \equiv \mathbf{R}_o^{\varphi=\frac{\pi}{2}} = \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} = -\mathbf{R}_o^{-\frac{\pi}{2}}, \qquad \mathbf{R}_e^{\frac{\pi}{2}} \equiv \mathbf{R}_e^{\varphi=\frac{\pi}{2}} = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} = -\mathbf{R}_e^{-\frac{\pi}{2}}, \quad \text{and}$$

$$\mathbf{R}_o^{0} = -\mathbf{R}_o^{\pm\pi} = -\{\mathbf{R}_o^{\pm\frac{\pi}{2}}\}^2 = \mathbb{1}, \quad \{\mathbf{R}_e^{\pm\frac{\pi}{2}}\}^2 = \mathbb{1}, \quad \mathbf{R}_e^{0} = -\mathbf{R}_e^{\pm\pi} = \sigma^z, \quad \mathbf{R}_e^{\frac{\pi}{2}}\,\mathbf{R}_o^{\frac{\pi}{2}} = -\mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{R}_e^{\frac{\pi}{2}} = \sigma^z,$$

so that:

$$\begin{aligned} \mathbf{R}_o^{\varphi}(\phi) &\equiv \cos\varphi\,\mathbf{R}_o^{0} + \sin\varphi\,\mathbf{R}_o^{\frac{\pi}{2}} = \cos\varphi\,\mathbb{1} + \sin\varphi\,\mathbf{R}_o^{\frac{\pi}{2}} = e^{\varphi\mathbf{R}_o^{\frac{\pi}{2}}} \\ \mathbf{R}_e^{\varphi}(\phi) &\equiv \cos\varphi\,\mathbf{R}_e^{0} + \sin\varphi\,\mathbf{R}_e^{\frac{\pi}{2}} = \cos\varphi\,\sigma^z + \sin\varphi\,\mathbf{R}_e^{\frac{\pi}{2}} = \mathbf{R}_o^{\varphi}(\phi)\,\sigma^z \end{aligned}.$$

Acting over diagonal and anti-diagonal matrices:

$$\begin{cases} \mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{D}\,\mathbf{R}_o^{-\frac{\pi}{2}} = \mathbf{R}_o^{-\frac{\pi}{2}}\,\mathbf{D}\,\mathbf{R}_o^{\frac{\pi}{2}} = \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}; \\ \mathbf{R}_o^{\frac{\pi}{2}}\,\mathbf{A}\,\mathbf{R}_o^{-\frac{\pi}{2}} = -\mathbf{R}_e^{\frac{\pi}{2}}\,\mathbf{A}\,\mathbf{R}_e^{\frac{\pi}{2}} = \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{m}^-\mathbf{c} \\ \mathbf{m}^+ d & 0 \end{pmatrix}\begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{m}^-\mathbf{d} \\ \mathbf{m}^+ c & 0 \end{pmatrix} \end{cases},$$

in particular

$$\begin{cases} \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\sigma^z\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = \mathbf{R}_e^{\pm\frac{\pi}{2}}\,\sigma^z\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = \mathbf{R}_e^{\pm\pi} = -\sigma^z \\ \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\mathbf{m}^-\sigma^+\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = -\mathbf{R}_e^{\pm\frac{\pi}{2}}\mathbf{m}^-\sigma^+\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = -e^{i2\phi}\,\mathbf{m}^-\sigma^- = \mathbf{m}^+\sigma^- \\ \mathbf{R}_o^{\pm\frac{\pi}{2}}\,\mathbf{m}^+\sigma^-\,\mathbf{R}_o^{\mp\frac{\pi}{2}} = -\mathbf{R}_e^{\pm\frac{\pi}{2}}\mathbf{m}^+\sigma^-\,\mathbf{R}_e^{\pm\frac{\pi}{2}} = -e^{-i2\phi}\,\mathbf{m}^+\sigma^+ = \mathbf{m}^-\sigma^+ \end{cases}.$$

Also

$$\begin{cases} \mathbf{R}_o^{\pm\pi}\,\mathbf{D}\,\mathbf{R}_o^{\mp\pi} = (-\mathbb{1})\,\mathbf{D}\,(-\mathbb{1}) = \mathbf{R}_e^{\pm\pi}\,\mathbf{D}\,\mathbf{R}_e^{\pm\pi} = (-\sigma^z)\,\mathbf{D}\,(-\sigma^z) = \mathbf{D}; \\ \mathbf{R}_o^{\pm\pi}\,\mathbf{A}\,\mathbf{R}_o^{\mp\pi} = (-\mathbb{1})\,\mathbf{A}\,(-\mathbb{1}) = -\mathbf{R}_e^{\pm\pi}\,\mathbf{A}\,\mathbf{R}_e^{\pm\pi} = -(-\sigma^z)\,\mathbf{A}\,(-\sigma^z) = \mathbf{A} \end{cases}.$$

σ^z diagonalizes $\mathbf{R}_e(2\varphi) = \mathbb{n} = n_z\sigma^z + n_x\sigma^x + n_y\sigma^y = n_z\,\sigma^z + \tau\,\mathbf{R}_e^{\frac{\pi}{2}}$:

$$\mathbf{R}_o^{\varphi}\,\sigma^z\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{\varphi}\,\sigma^z\,\mathbf{R}_e^{\varphi} = \mathbf{R}_o^{\varphi}\,\mathbf{R}_e^{\varphi} = \mathbf{R}_e^{\varphi}\,\mathbf{R}_o^{-\varphi} = \mathbf{R}_e^{2\varphi} = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi\, e^{-i\phi} \\ \sin 2\varphi\, e^{i\phi} & -\cos 2\varphi \end{pmatrix} = \mathbb{n}.$$

APPENDIX B: FIGURES.

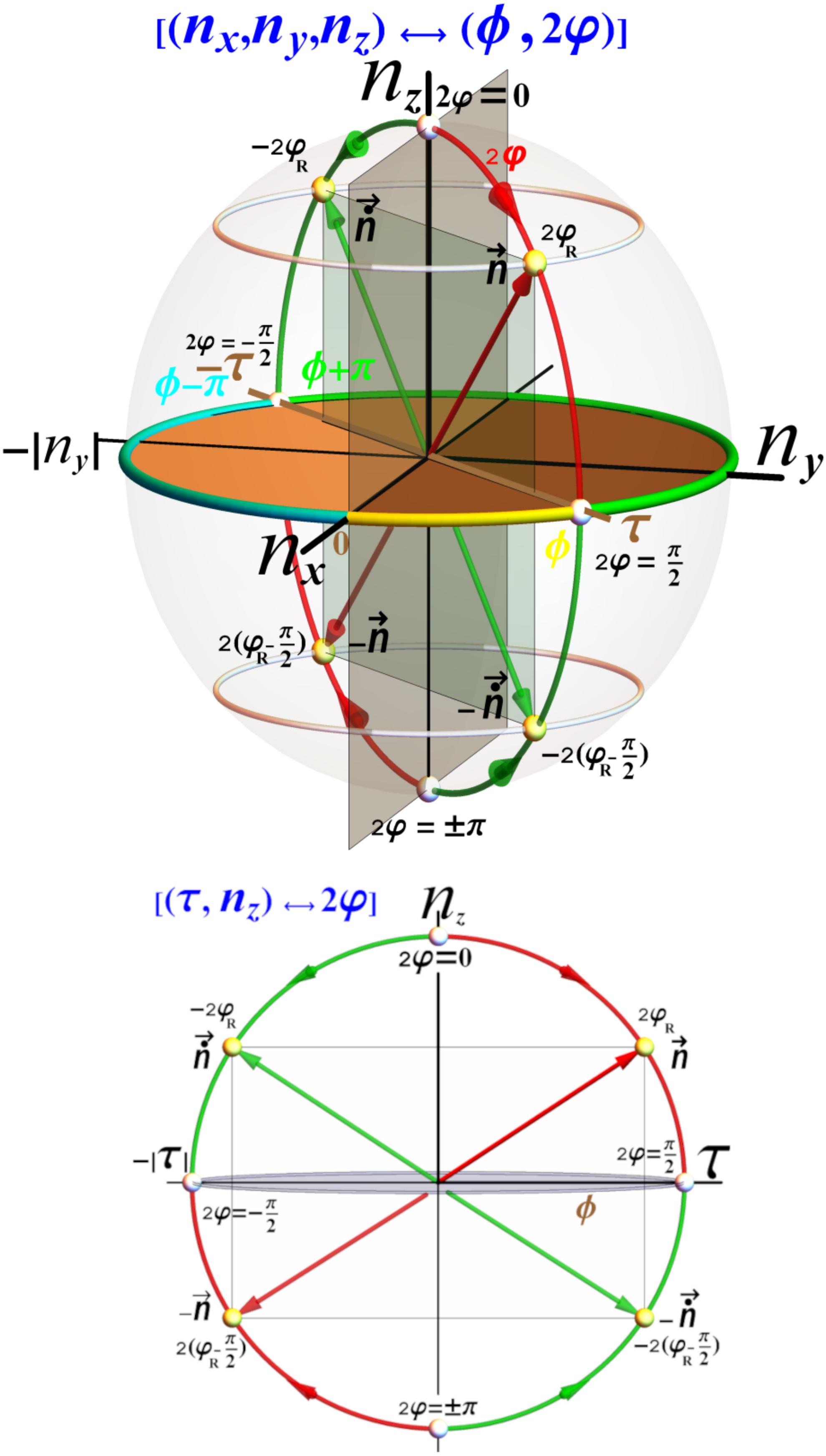

FIG 1: Representation of unity space vectors.

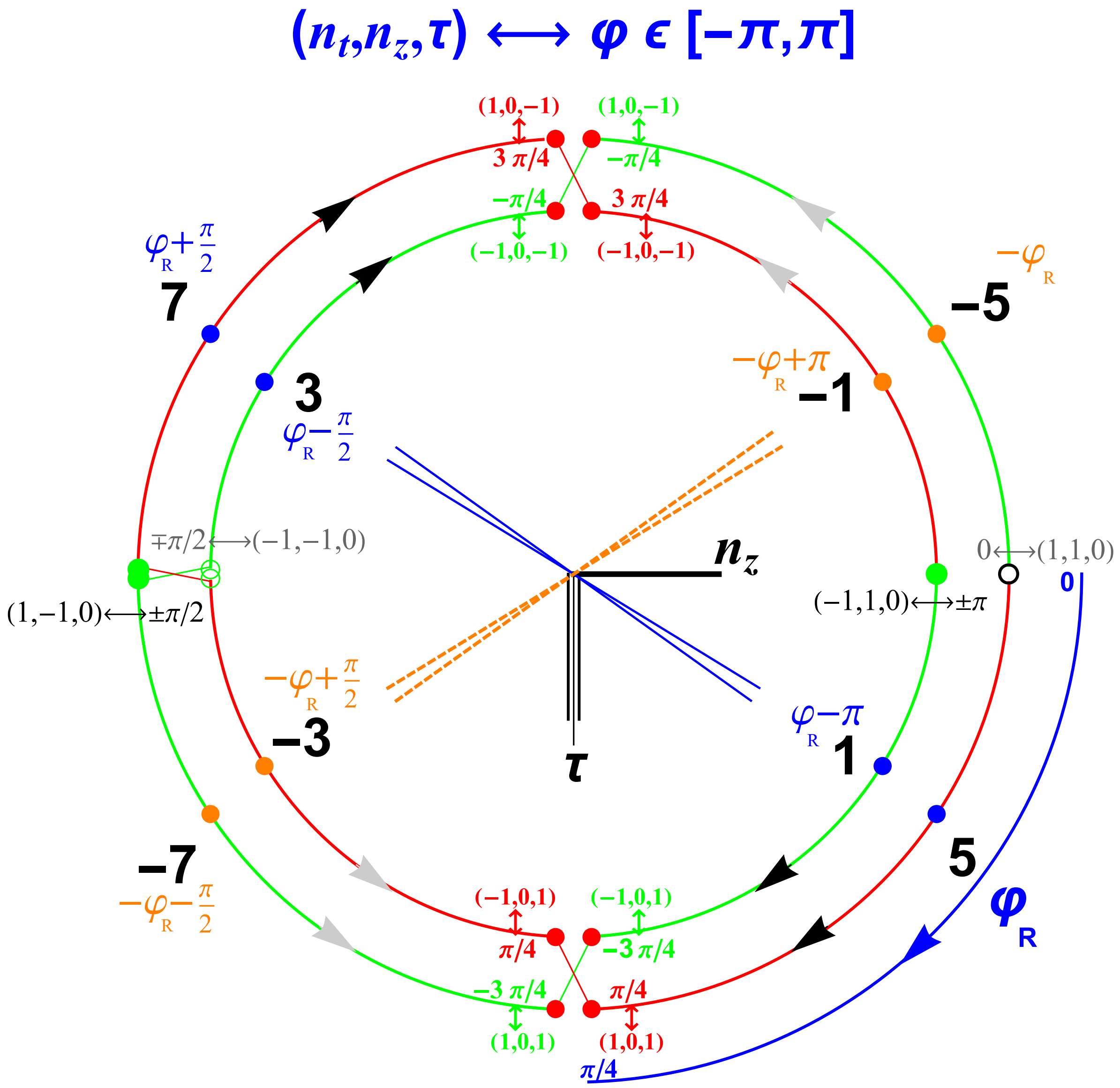

FIG 2: $(n_t, n_z, \tau) \longleftrightarrow \varphi$. Representation of null time-space vectors.
The exterior circle with $n_t = +1$ and the interior circle with $n_t = -1$.

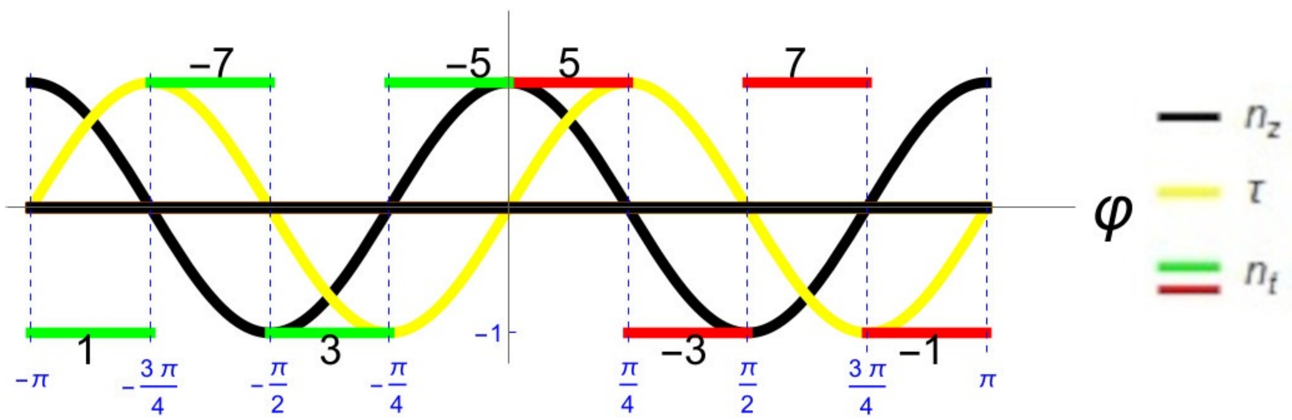

FIG 3: The graphs of $n_z(2\varphi)$, $\tau(2\varphi)$ and n_t as functions of φ .

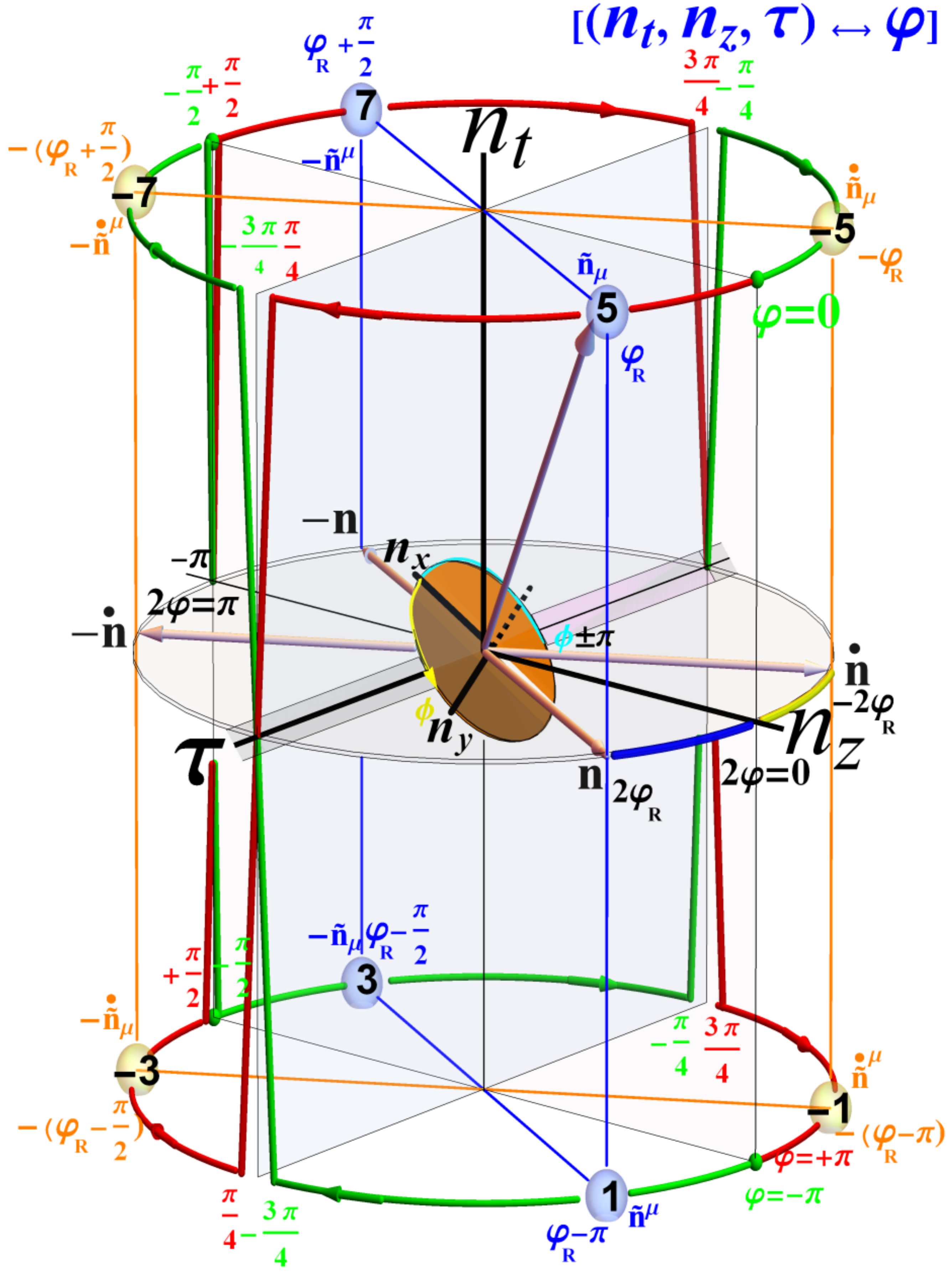

FIG 4: Representation of the characteristic axes (odd numbers) in a (1+1+‘2’)d time-space.

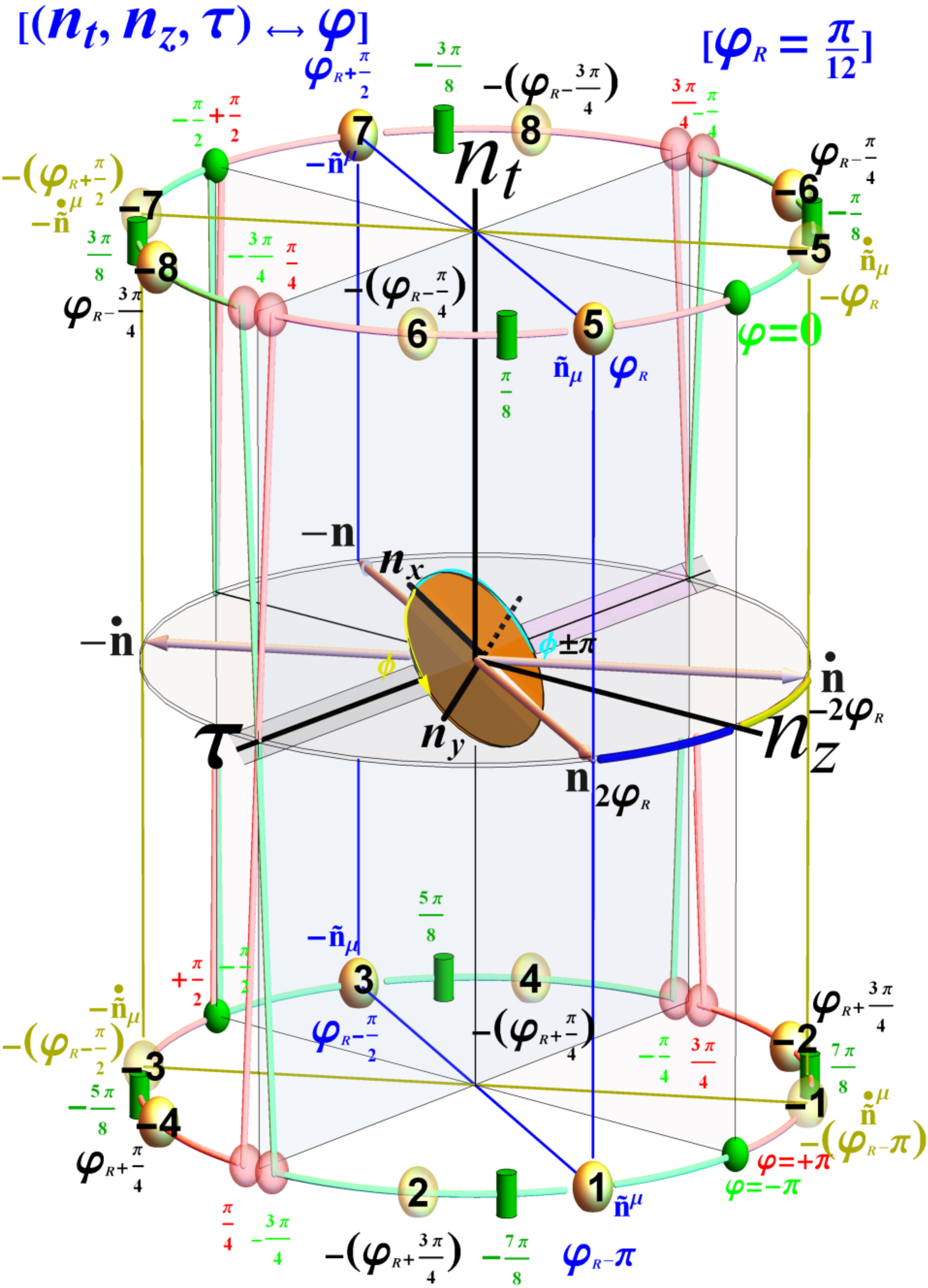

FIG 5: Representation of the quarks and leptons in a (1+1+'2')d time-space.

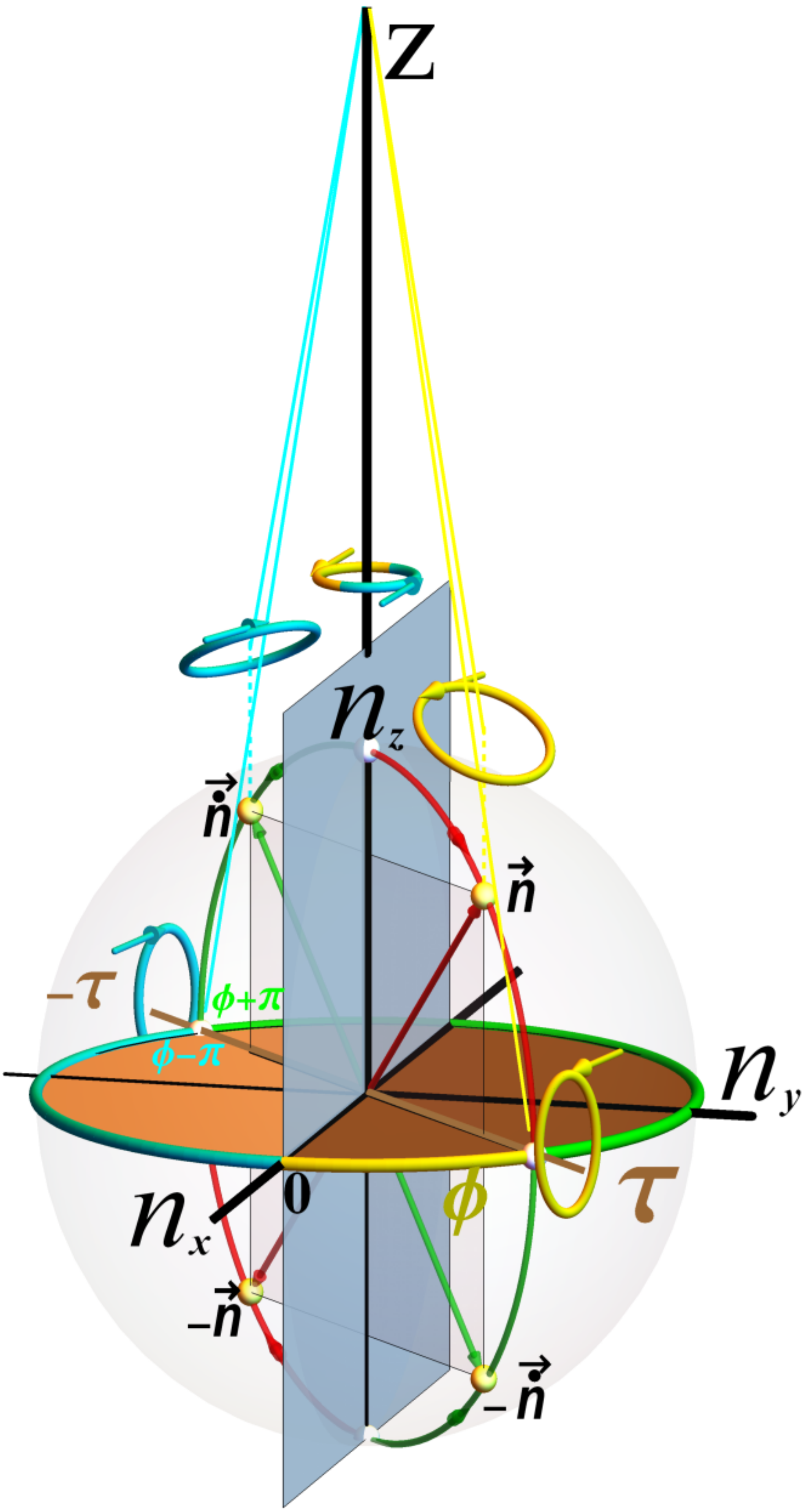

FIG 6: Chirallity ?

The horizontal circle at the top, yellow and blue, for the neutrinos. The vertical circles at the equator, yellow or blue with opposite senses of rotation relative to the "mirror" plane $[n_z, n_x]$, for the charged leptons. Similarly for the inclined circles in the intermediate position for quarks.

APPENDIX C: PROGRAM OF THE *STUDIES* CONTAINING THIS RESEARCH.

On the fermionization of the XYZ spin Heisenberg chain (algebra).
(2022) https://eprints.ucm.es/id/eprint/72882/ Study -2)

The JordanWigner transformations and the fermionization of the XYZ spin Heisenberg chain.
Algebra, geometry and physics? (2022) https://eprints.ucm.es/id/eprint/74550/ *Study -1)*

A tentative model of creation and annihilation operators for neutrinos.
(2021) https://eprints.ucm.es/id/eprint/65151/ *Study 0)*

Expression of the 3- and 4-dimensional vectors in total polar exponential form.
(2021) https://eprints.ucm.es/id/eprint/65825/ *Study I,1)*

Vectors. Dimensions 4 and 8. (2023) https://eprints.ucm.es/id/eprint/76327/ *Study I,2)*

Geometry of the time and the space. *Study I)*

Geometry of the symmetries in dimension 4=(1+[1)+"2"], and general Time-Space-Spin vectors.
(This study). (2023) https://eprints.ucm.es/id/eprint/76328/ *Study II)*

Geometry and Physics of the Elementary Fermions. (On pride of Jordan Wigner Pauli Weyl Dirac). **1**.
(2021) https://eprints.ucm.es/id/eprint/69295/ *Study III)*

Geometry and Physics of the Elementary Fermions. **2**. *Study III)*

Axial vector magnetic charge and magnetic moment. Maxwell's equations and Lorentz force law.
(2021) https://eprints.ucm.es/id/eprint/69294/ *Study IV)*

Addenda. *Study V)*

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(First paragraph in page 23 and pages 118-119).

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(Chapter 1, first paragraph in the seventh page, pages 36-37 and 227-228 (note34)).

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[10] K. S. Thorne and R. D. Blandford. *Modern Classical Physics.* Princeton University Press (2017). (Page 5).