

Difference of cross-spectral densities

M. Santarsiero,^{1,*} G. Piquero,² J. C. G. de Sande,³ and F. Gori¹

¹Dipartimento di Ingegneria, Università Roma Tre, and CNISM, via V. Volterra, 62, 00146 Rome, Italy

²Departamento de Óptica, Universidad Complutense de Madrid, 28040 Madrid, Spain

³Departamento de Circuitos y Sistemas, Universidad Politécnica de Madrid, Campos Sur, 28031 Madrid, Spain

*Corresponding author: msantarsiero@uniroma3.it

Received December 9, 2013; accepted December 29, 2013;

posted February 12, 2014 (Doc. ID 202784); published March 17, 2014

Generally speaking, the difference between two cross-spectral densities (CSDs) does not represent a correlation function. We will furnish a sufficient condition so that such difference be a valid CSD. Using such a condition, we will show through some examples how new classes of CSDs can be generated. © 2014 Optical Society of America
OCIS codes: (030.0030) Coherence and statistical optics; (030.1640) Coherence; (030.4070) Modes.
<http://dx.doi.org/10.1364/OL.39.001713>

In problems of coherence theory [1] we occasionally have to deal with the difference of correlation functions [2–6]. In particular, while the sum of two cross-spectral densities (CSDs) simply accounts for the superposition of two uncorrelated fields, their difference, generally speaking, does not have a clear physical meaning and does not represent another CSD. In fact, its use has caused discussion [7,8]. The problem of finding conditions under which the difference of two CSDs is a CSD itself could be tackled by using the theory of reproducing kernel Hilbert spaces [9]. For the sake of simplicity, we shall make use of an alternative approach. On exploiting recent results of coherence theory [10,11], we will derive a sufficient condition through which entire classes of cases can be constructed where the difference of two CSDs generates a new, genuine CSD. Beyond clarifying an old issue, these results permit new structures of CSDs to be studied. It is to be noted that an important first step in solving the problem of our interest has recently been given by Sahin and Korotkova [5] who considered the combination with alternate sign of Gaussian Schell-model CSDs with equal intensity profiles and different coherence widths.

The CSD [1] at two points ρ_1 and ρ_2 of a given plane will be denoted by $W(\rho_1, \rho_2)$, omitting the explicit dependence from the temporal frequency. When the two points coincide, the CSD reduces to the optical intensity $I(\rho) = W(\rho, \rho)$. Given two CSDs $W_1(\rho_1, \rho_2)$ and $W_2(\rho_1, \rho_2)$, we shall denote by $F(\rho_1, \rho_2)$ their difference:

$$F(\rho_1, \rho_2) = W_1(\rho_1, \rho_2) - W_2(\rho_1, \rho_2). \quad (1)$$

The obvious arithmetic meaning of F is that it represents the function to be added to W_2 to obtain W_1 . We wonder whether F itself can represent a CSD.

To see immediately a class of cases in which F cannot be a CSD, imagine W_1 and W_2 have the factorized expressions

$$W_i(\rho_1, \rho_2) = U_i^*(\rho_1)U_i(\rho_2); \quad (i = 1, 2), \quad (2)$$

where U_1 and U_2 are arbitrary nonproportional functions. Both functions in Eq. (2) are surely valid CSDs because they correspond to perfectly coherent fields. Remember [1] that a genuine CSD has to be a nonnegative

definite kernel. To check the nonnegativeness of $W_1 - W_2$, let us take any function $g(\rho)$ orthogonal to U_1 while not orthogonal to U_2 . Then, on multiplying both sides of Eq. (1) by $g(\rho_1)g^*(\rho_2)$ and integrating with respect to ρ_1 and ρ_2 we obtain

$$\iint F(\rho_1, \rho_2)g(\rho_1)g^*(\rho_2)d^2\rho_1d^2\rho_2 = - \left| \int U_2(\rho)g^*(\rho)d^2\rho \right|^2. \quad (3)$$

Now, if F is a CSD the left-hand side is nonnegative. On the other hand, the right-hand side is negative. Therefore, F cannot be a CSD.

Without making any hypothesis about the functional form of W_1 and W_2 , it could be argued that the function F specified by Eq. (1) gives a genuine CSD if the optical intensity it predicts is nowhere negative throughout space. Unfortunately, this is not a sufficient condition. As a matter of fact, even if this condition is satisfied, other elements could reveal F not to be a CSD. For example, the spectral degree of coherence [1] could exceed one at certain pairs of points [10].

Now we will establish a sufficient condition in order to guarantee that the difference of two CSDs is itself a valid CSD. Let us recall some recent results of coherence theory. The first is the superposition rule [10], according to which a function $W(\rho_1, \rho_2)$ is a genuine CSD if it can be expressed in the form

$$W(\rho_1, \rho_2) = \int p(v)H^*(\rho_1, v)H(\rho_2, v)d^2v, \quad (4)$$

where p is a nonnegative function that will be referred to as the weight function. Further, H is an arbitrary kernel. Another relevant result is that any valid CSD can be expressed, in an infinite number of ways, through a representation of the form of Eq. (4) [11]. The superposition rule played a significant role in recent studies of partially coherent fields [12–22].

Once H is chosen, a whole class of CSDs is obtained by varying the weight function. All of these CSDs will be said to belong to that class with respect to the kernel H .

It is useful to note the effect of applying a linear integral transformation to $W(\rho_1, \rho_2)$. Let $K(\rho, r)$ be the kernel of the transformation and compute the transformed function, say $W_K(r_1, r_2)$. We obtain

$$W_K(r_1, r_2) = \iint W(\rho_1, \rho_2) K^*(\rho_1, r_1) K(\rho_2, r_2) d^2 \rho_1 d^2 \rho_2. \quad (5)$$

On expressing W through Eq. (4) we have

$$W_K(r_1, r_2) = \int p(v) H_K^*(r_1, v) H_K(r_2, v) d^2 v, \quad (6)$$

where H_K denotes the kernel obtained from H under the action of K , namely,

$$H_K(r, v) = \int H(\rho, v) K(\rho, r) d^2 \rho. \quad (7)$$

Equation (6) shows that if W is a genuine CSD (and hence a suitable representation (4) exists) the same occurs for W_K . This generalizes a known result holding when K is a propagation kernel [13].

A special case of Eq. (4) is obtained when the weight function is a sequence of delta functions centered at a given set of points v_n , ($n = 0, 1, \dots$):

$$p(v) = \sum_{n=0}^{\infty} p_n \delta(v - v_n); \quad p_n \geq 0, \quad \forall n. \quad (8)$$

Equation (4) then becomes

$$W(\rho_1, \rho_2) = \sum_{n=0}^{\infty} p_n H^*(\rho_1, v_n) H(\rho_2, v_n). \quad (9)$$

In particular, the functions $H(\rho, v_n)$ can be the eigenfunctions, and the weights p_n the corresponding (nonnegative) eigenvalues of the homogeneous Fredholm integral equation with kernel given by $W(\rho_1, \rho_2)$ [1]. In such a case, Eq. (9) is nothing else than the Mercer's expansion of the CSD [1].

On using these results, we can derive the properties of our interest. The first is that while the process of subtracting one CSD from another generally leads to a function that is not a CSD, the converse process is always possible. In other words, the following proposition holds true:

Proposition 1. *Any CSD can be expressed, in an infinite number of ways, as the difference of two CSDs belonging to the same class.*

With reference to Eq. (4), the function $p(v)$ can be written, in an infinite number of ways, as the difference between two nonnegative functions $p_1(v)$ and $p_2(v)$ such that

$$p(v) = p_1(v) - p_2(v). \quad (10)$$

Consequently, the CSD takes on the form

$$W(\rho_1, \rho_2) = W_1(\rho_1, \rho_2) - W_2(\rho_1, \rho_2), \quad (11)$$

where

$$W_j(\rho_1, \rho_2) = \int p_j(v) H^*(\rho_1, v) H(\rho_2, v) d^2 v; \quad (j = 1, 2). \quad (12)$$

This proves the proposition.

Finally, we can immediately establish a condition ensuring that the difference of two CSDs is itself a CSD. We present it as the following proposition:

Proposition 2. *If there exists a kernel with respect to which two CSDs W_1 and W_2 belong to the same class and have weight functions $p_1(v)$ and $p_2(v)$ satisfying the inequality $p_1 \geq p_2$ for any v , then $W_1 - W_2$ is a genuine CSD.*

This is the most relevant result of the present Letter. It allows us to introduce entire new classes of CSDs. In the following, some classes of CSDs obtained through this type of difference will be examined. Obvious extensions of the previous propositions are obtained on replacing p_1 and p_2 by sums or series of nonnegative functions.

For the sake of simplicity, in the forthcoming examples we shall refer to cases in which one of the Cartesian coordinates (say y) of the considered points is irrelevant. Then, the CSD will be a function, $W(x_1, x_2)$, of two scalar variables.

Let us first refer to the so-called Schell-model (SM) sources. We recall that the CSD is said to be of the SM type [1] if it has the form

$$W(x_1, x_2) = q(x_1) q(x_2) \mu(x_1 - x_2), \quad (13)$$

where $q(x)$ is the square root of the optical intensity. The spectral degree of coherence in Eq. (13), μ , is a function of $x_1 - x_2$ only, i.e., it is shift-invariant. For any SM source a representation of the form of Eq. (4) is obtained through a Fourier representation. In fact, on letting

$$H(x, v) = q(x) \exp(2\pi i x v); \quad p(v) = \tilde{\mu}(v), \quad (14)$$

where the tilde denotes Fourier transform, Eq. (4) leads at once to Eq. (13). The positiveness condition on $p(v)$ is transferred to $\tilde{\mu}(v)$. Making an appeal to Bochner's theorem it is proved that, for SM sources, the nonnegativeness of $p(v)$ is a necessary condition too [23]. When we are interested in the difference of two CSDs, the function $p(v)$ is replaced by $p_1(v) - p_2(v)$, which has, of course, to be nonnegative for any v . Note instead that $q(x)$ is the same for both terms of the difference.

In Ref. [5], the case in which an infinite number of GSM CSDs are combined with alternate signs is discussed. In that case, the intensity profile $q^2(x)$ is the same for all contributions.

Let us pass to different classes of sources. For a very simple example, we take

$$H(x, v) = q(x) \exp(2\pi i x^2 v) \quad (15)$$

with v having proper dimension. Note that, even in this case, the kernel is of the Fourier type, but now the exponential exhibits a different dependence on x [13]. The exponential function in Eq. (15) can essentially be

seen as the expression of a cylindrical wave whose radius of curvature depends on v . Furthermore, suppose p to be of the form $p(v) = A \text{rect}(v/c)$, with $A, c > 0$, where $\text{rect}(t)$ equals 1 for $|t| \leq 1/2$ and 0 otherwise. On using Eq. (15), Eq. (4) leads to

$$W(x_1, x_2) = A c q(x_1) q(x_2) \text{sinc}[c(x_1^2 - x_2^2)], \quad (16)$$

where $\text{sinc}(t) = \sin(\pi t)/(\pi t)$. This is not a SM CSD because of the quadratic dependence on x_1 and x_2 of the sinc function, which represents the spectral degree of coherence.

If we now want to devise a CSD as the difference of two CSDs of the above form, we have, with evident meaning of the symbols,

$$W(x_1, x_2) = q(x_1) q(x_2) \{A_1 c_1 \text{sinc}[c_1(x_1^2 - x_2^2)] - A_2 c_2 \text{sinc}[c_2(x_1^2 - x_2^2)]\}, \quad (17)$$

and the nonnegativeness of the weight function requires $A_1 \geq A_2$ and $c_1 \geq c_2$.

In Fig. 1 the spectral degree of coherence, μ , associated to Eq. (17) is shown as a function of x_1 for $x_2 = 0$ (dashed line) and $x_2 = 1.5$ (solid line). Two features are worth noting. First, the single central lobe of μ splits into two symmetrical lobes when x_2 differs from zero. Second, the lobe width reduces when the midpoint abscissa grows up, which is due to the dependence of the sinc functions in Eq. (17) on the quantity $(x_1^2 - x_2^2) = (x_1 + x_2)(x_1 - x_2)$.

The previous examples rely on the Fourier transform. To see a different type of transformation, let us consider a kernel of the Laplace form, namely,

$$H(x, v) = \text{step}(x) \exp(-vx), \quad (18)$$

where $\text{step}(x)$ equals 1 for $x \geq 0$ and 0 otherwise, which could be realized through the impulse response of a suitable optical system.

Let us further choose a weight function as follows:

$$p(v) = A \text{step}(v) \exp(-vL), \quad (19)$$

where A and L are positive constants. Evaluating the corresponding CSD through Eq. (4) we easily obtain

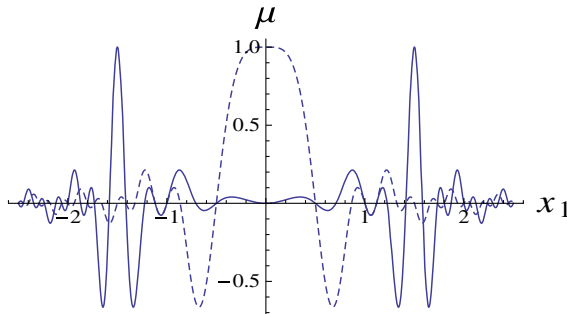


Fig. 1. Spectral degree of coherence, as a function of x_1 , corresponding to the CSD in Eq. (17), with $A_1 = A_2 = 1$, $c_1 = 3$, $c_2 = 1$, and $x_2 = 0$ (dashed), $x_2 = 1.5$ (solid line).

$$W(x_1, x_2) = \frac{A \text{step}(x_1) \text{step}(x_2)}{L + x_1 + x_2}. \quad (20)$$

We now consider the difference of two CSDs of the above form, with suitable parameters, i.e.,

$$W(x_1, x_2) = \text{step}(x_1) \text{step}(x_2) \times \left(\frac{A_1}{L_1 + x_1 + x_2} - \frac{A_2}{L_2 + x_1 + x_2} \right). \quad (21)$$

Such a CSD is obtained by superposing the kernels in Eq. (18) with a weight given by

$$p(v) = \text{step}(v) [A_1 \exp(-vL_1) - A_2 \exp(-vL_2)], \quad (22)$$

so that the nonnegativeness constraint is satisfied, provided that $A_1 \geq A_2$ and $L_1 \leq L_2$. In Fig. 2 the spectral degree of coherence is drawn as a function of x_1 , for several values of x_2 .

Let us finally recall that a significant case of the superposition rule is obtained when $H(x, v)$ is simply a shifted copy of a basic coherent field [12,15,24,25]. As an example, we shall briefly consider the case of GSM sources. It is known that the associated CSD can be represented through the superposition of shifted copies of a basic Gaussian field. Such copies are mutually uncorrelated and are weighted by a Gaussian function [24]. The kernel H to be used is itself a Gaussian function, i.e.,

$$H(x, v) = \exp[-\alpha(x - v)^2], \quad (23)$$

where α is a positive constant, and the weight function is

$$p(v) = A \exp(-\gamma v^2) \quad (24)$$

with A and γ positive parameters. On using the superposition rule in Eq. (4), after some calculations the CSD turns out to be [24]

$$W(x_1, x_2) = B \exp \left[-\frac{x_1^2 + x_2^2}{4\sigma^2} - \frac{(x_1 - x_2)^2}{2\delta^2} \right], \quad (25)$$

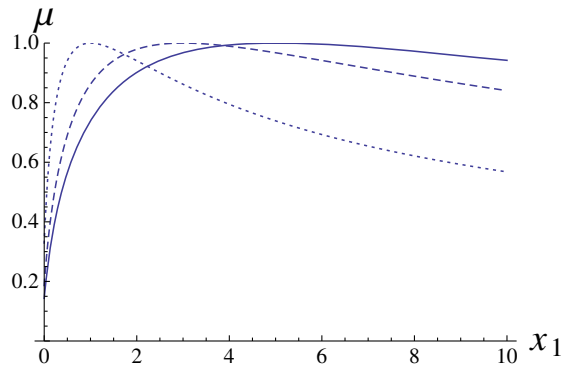


Fig. 2. Spectral degree of coherence, as a function of x_1 , corresponding to the CSD in Eq. (21), with $A_1 = 1.5$, $A_2 = 1$, $L_1 = 0.1$, $L_2 = 0.2$, and $x_2 = 1$ (dotted), 3 (dashed), 5 (solid line).

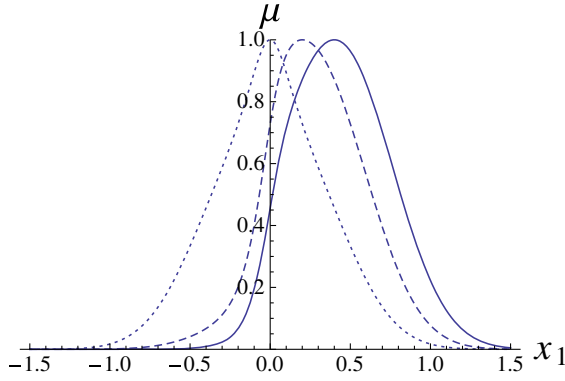


Fig. 3. Spectral degree of coherence as a function of x_1 , for the difference of two CSDs of the GSM type, with $\alpha = 10$, $\gamma_1 = 1$, $\gamma_2 = 4$, $A_1 = A_2 = 1$, and $x_2 = 0$ (dotted), 0.2 (dashed), 0.4 (solid line).

where the following quantities have been introduced:

$$B = A \sqrt{\frac{\pi}{\gamma + 2\alpha}}; \quad \frac{1}{4\sigma^2} = \frac{\alpha\gamma}{\gamma + 2\alpha}; \quad \frac{1}{2\delta^2} = \frac{\alpha^2}{\gamma + 2\alpha}. \quad (26)$$

We are now going to subtract two such CSDs. The non-negativeness constraint of the resulting CSD is fulfilled if the corresponding weight function, that is,

$$p(v) = A_1 \exp(-\gamma_1 v^2) - A_2 \exp(-\gamma_2 v^2). \quad (27)$$

is nonnegative for any v , and this implies $A_1 \geq A_2$ and $\gamma_1 \leq \gamma_2$.

A significant result of the present approach has to be stressed: while W_1 and W_2 are of the SM type, the CSD obtained from their difference is no longer of such type, because the two intensity profiles are different. This means that the standard techniques used for determining its nonnegativeness [23] cannot be used. As an example, the spectral degree of coherence is drawn in Fig. 3 as a function of x_1 , for a typical choice of the parameters, letting $x_2 = 0, 0.2, 0.4$. It is evident that none of these curves is Gaussian shaped. Moreover, except for $x_2 = 0$, the curves of μ are not symmetrical with respect to $x_1 = x_2$.

In conclusion, we examined the problem of determining in which cases the difference of two CSDs furnishes a valid CSD. The superposition rule allowed us to establish a sufficient condition for this to occur. This result

establishes a significant tool to study new classes of CSDs. A few simple examples for fields depending on only one Cartesian coordinate were given. For brevity, we limited ourselves to examining the difference of scalar CSDs in a given plane, but interesting results can be expected when propagation phenomena are considered, or when vectorial aspects are taken into account.

References

1. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, 1995).
2. D. Ge, Y. Cai, and Q. Lin, *Appl. Opt.* **43**, 4732 (2004).
3. Z.-H. Gu, E. R. Méndez, M. Ciftan, T. A. Leskova, and A. A. Maradudin, *Opt. Lett.* **30**, 1605 (2005).
4. E. E. Garcia-Guerrero, E. R. Méndez, Z.-H. Gu, T. A. Leskova, and A. A. Maradudin, *Opt. Express* **18**, 4816 (2010).
5. S. Sahin and O. Korotkova, *Opt. Lett.* **37**, 2970 (2012).
6. J. Cang, X. Fang, and X. Liu, *Opt. Laser Technol.* **50**, 65 (2013).
7. G. Wu, H. Guo, and D. Deng, *Appl. Opt.* **45**, 366 (2006).
8. D. Xu, Y. Cai, D. Ge, and Q. Lin, *Appl. Opt.* **45**, 369 (2006).
9. A. Berlinet and C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics* (Kluwer Academic, 2004).
10. F. Gori and M. Santarsiero, *Opt. Lett.* **32**, 3531 (2007).
11. R. Martínez-Herrero, P. M. Mejías, and F. Gori, *Opt. Lett.* **34**, 1399 (2009).
12. J. Turunen and P. Vahimaa, *Opt. Express* **16**, 6433 (2008).
13. F. Gori, V. Ramírez-Sánchez, M. Santarsiero, and T. Shirai, *J. Opt. A* **11**, 085706 (2009).
14. V. Torres-Company, A. Valencia, and J. P. Torres, *Opt. Lett.* **34**, 1177 (2009).
15. J. Tervo, J. Turunen, P. Vahimaa, and F. Wyrowski, *J. Opt. Soc. Am. A* **27**, 2004 (2010).
16. J. Turunen and A. T. Friberg, in *Progress in Optics*, E. Wolf, ed. (Elsevier, 2009), Vol. **54**, pp. 1–88.
17. R. Zhang, X. Wang, X. Cheng, and Z. Qiu, *J. Opt. Soc. Am. A* **27**, 2496 (2010).
18. H. Lajunen and T. Saastamoinen, *Opt. Lett.* **36**, 4104 (2011).
19. Z. Tong and O. Korotkova, *Opt. Lett.* **37**, 3240 (2012).
20. O. Korotkova, S. Sahin, and E. Shchepakina, *J. Opt. Soc. Am. A* **29**, 2159 (2012).
21. Z. Mei and O. Korotkova, *Opt. Lett.* **38**, 91 (2013).
22. H. Lajunen and T. Saastamoinen, *Opt. Express* **21**, 190 (2013).
23. P. De Santis, F. Gori, G. Guattari, and C. Palma, *J. Opt. Soc. Am. A* **3**, 1258 (1986).
24. F. Gori and C. Palma, *Opt. Commun.* **27**, 185 (1978).
25. P. Vahimaa and J. Turunen, *Opt. Express* **14**, 1376 (2006).