INTEGRAL MAPPINGS BETWEEN BANACH SPACES

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Abstract. We consider the classes of "Grothendieck-integral" (G-integral) and "Pietsch-integral" (P-integral) linear and multilinear operators (see definitions below), and we prove that a multilinear operator between Banach spaces is G-integral (resp. P-integral) if and only if its linearization is G-integral (resp. P-integral) on the injective tensor product of the spaces, together with some related results concerning certain canonically associated linear operators. As an application we give a new proof of a result on the Radon-Nikodym property of the dual of the injective tensor product of Banach spaces. Moreover, we give a simple proof of a characterization of the G-integral operators on $C(K, X)$ spaces and we also give a partial characterization of P-integral operators on $C(K, X)$ spaces.

1. Introduction

In [9], Grothendieck introduced the *integral* operators, which we call *G-integral*, between Banach spaces (in the more general context of locally convex spaces). Later on, Pietsch presented another (more restrictive) definition of integral operators, which we call P-integral, closely related to the previous one. Both notions have been deeply studied and applied by many authors in the theory of Banach spaces. More recently, Alencar [1] extended the definition of P-integral operators to multilinear operators and polynomials, and that notion has been studied by several authors since then. In Section 2, we introduce a generalization of G-integral operators modelled on Alencar's, and we show that a multilinear operator on a product of Banach spaces is P-integral (resp. G-integral) if and only if its *linearization* is a P-integral (resp. G-integral) operator on the injective tensor product of the spaces, together with some related results concerning certain canonically associated linear operators. As an application we obtain, with a completely new approach, a result on the Radon-Nikodym property of the dual of the injective tensor product of Banach spaces which had already been obtained in [13].

In Section 3 we use the previous results to obtain a simple proof of a result of [14], characterizing the G-integral operators on spaces of vector valued continuous functions in terms of their representing measures, and we

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present a similar result partially characterizing the P-integral operators on these same spaces.

The notations and terminology used along the paper will be the standard in Banach space theory, as for instance in [6] or [7]. However, before going any further, we shall clear out some terminology: $\mathcal{L}^k(X_1 \dots, X_k; Y)$ will be the Banach space of all the continuous k-linear mappings from $X_1 \times \cdots \times X_k$ into Y. When $Y = \mathbb{K}$ or $k = 1$, we will omit them. If $T \in \mathcal{L}^k(X_1 \dots, X_k; Y)$ we shall denote by $\hat{T} : X_1 \otimes \cdots \otimes X_k \to Y$ its linearization. As usual, $X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k$ stands for the injective tensor product of the Banach spaces $X_1, \dots, X_k$ and $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_k$ stands for their projective tensor product. The sign $\approx$ between two Banach spaces indicates that they are isomorphic. If X is a Banach space and Σ is a σ-algebra, $bvrca(\Sigma; X)$ denotes the Banach space of the regular measures with bounded variation $\mu : \Sigma \longrightarrow X$ endowed with the variation norm. For any Banach space X, B_{X^*} is a compact set when we endow it with the weak* topology; we write Σ_{X^*} for the Borel σ-algebra of B_{X^*}. For any Banach space X, $k_X : X \hookrightarrow C(B_{X^*})$ and $i_X : X \hookrightarrow X^{**}$ will denote the canonical isometric inclusions. We will often use that, if K_1, K_2 are compact Hausdorff spaces, then $C(K_1 \times K_2)$, $C(K_1, C(K_2))$ and $C(K_1)\hat{\otimes}_\epsilon C(K_2)$ are isometrically isomorphic. Throughout the paper the expression $j : C(K) \longrightarrow L_1(\mu)$ will denote that μ is a scalar regular Borel measure on a compact set K and j is the canonical mapping. This mapping is known to be G-integral, equivalently P-integral, in the sense of Definitions 2.2 and 2.3 below.

2. Integral mappings

There are several definitions of "integral" applications which have already been used in the literature. We state presently those which we will need.

Definition 2.1. *A multilinear form $T \in \mathcal{L}^k(X_1, \dots, X_k)$ is integral if $\hat{T}$ (i.e., its linearization) is continuous for the injective (ϵ) topology on $X_1 \otimes \cdots \otimes X_k$. Its norm (as an element of $(X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k)^*$) is the integral norm of T, $\|T\|_{int} := \|\hat{T}\|_\epsilon$.*

Definition 2.2. *An operator $T \in \mathcal{L}(X; Y)$ is G-integral (G for Grothendieck) if the associated bilinear form*

$$\begin{array}{rccc} B_T : & X \times Y^* & \longrightarrow & \mathbb{K} \\ & (x, y) & \mapsto & y(T(x)) \end{array}$$

is integral. In that case the G-integral norm of T, $\|T\|_{Gint} := \|B_T\|_{int}$. $\mathcal{I}(X; Y)$ denotes the Banach space of the integral operators from X into Y, endowed with the integral norm.

It is known ([6, Theorem 5.6]) that $T : X \longrightarrow Y$ is G-integral if and only if for any weak* compact norming subset $K \subset B_{X^*}$, there exists a scalar regular measure μ on K such that T admits a factorization

$$\begin{array}{ccc} X & \overset{T}{\longrightarrow} Y \overset{i_Y}{\hookrightarrow} & Y^{**} \\ \downarrow k & & \uparrow b \\ C(K) & \overset{j}{\longrightarrow} & L_1(\mu) \end{array}$$

where k is the natural isometric isomorphism defined by $k(x)(x^*) = x^*(x)$. This is equivalent to the existence of a regular Borel measure of bounded variation G defined on K and with values in Y^{**} such that, for every $x \in X$.

$$T(x) = \int_K x(x^*) dG(x^*).$$

In that case, $\|T\|_{Gint} = \inf\{v(G);$ where G represents T as above$\}$. It follows from the proof of [6, Theorem 5.6] that this factorization result remains true if K is *any* compact set (not necessarily contained in B_{X^*}) such that X is isometrically contained in $C(K)$. If X is isomorphically (but not isometrically) contained in $C(K)$, then the result remains true except for the statement about the norm.

Since j is always G-integral, it follows trivially from the ideal property of G-integral operators that the result is also true if there exists one such K for which the previous factorization holds.

We will also use later the known fact, that, for any $T \in \mathcal{I}(Y; Z^*)$, the bilinear form $B_T : Y \times Z \longrightarrow \mathbb{K}$ given by $B_T(y, z) = T(y)(z)$ is integral.

Definition 2.3. *An operator $T \in \mathcal{L}(X; Y)$ is said to be P-integral (P for Pietsch) if there exists a regular Y-valued Borel measure G of bounded variation on B_{X^*} such that, for every $x \in X$,*

$$T(x) = \int_{B_{X^*}} x^*(x) dG(x^*).$$

In that case the P-integral norm of T, $\|T\|_{Pint} := \inf\{v(G)$, where G represents T as above$\}$. $\mathcal{PI}(X; Y)$ denotes the Banach space of the P-integral operators from X into Y, endowed with the P-integral norm.

It is known ([6, p. 99]) that T is P-integral if and only if for any weak* compact norming subset $K \subset B_{X^*}$ there exists a scalar regular measure μ on K such that T admits a factorization

$$\begin{array}{ccc} X & \overset{T}{\longrightarrow} & Y \\ \downarrow k & & \uparrow b \\ C(K) & \overset{j}{\longrightarrow} & L_1(\mu) \end{array}$$

This is equivalent to the existence of a regular Borel measure of bounded variation G defined on K and with values in Y such that, for every $x \in X$.

$$T(x) = \int_K x(x^*) dG(x^*).$$

As in the case of G-integral operators, the result remains true if K is any compact set such that X is contained in $C(K)$. Again, it is clear that T is P-integral if and only if there exists one K and μ as above.

It follows immediately from these comments the existence of a norm one surjective operator $q : bvrca(\Sigma_{X^*}; Y) \longrightarrow \mathcal{PI}(X; Y)$.

It is obvious from the definitions that $C(K)$ spaces play a prominent role in the study of integral operators. It is known (and we will often use it) that, on these spaces, P-integral, G-integral and absolutely summing operators coincide (see [7, Chapter VI], [6]) and that, given a compact Hausdorff space K and a Banach space X, an operator $T \in \mathcal{L}(C(K); X)$ is P-integral (equivalently G-integral) if and only if its representing measure μ has bounded variation, and, in that case, $v(\mu) = \|T\|_{Pint} = \|T\|_{Gint}$.

P-integral operators are obviously G-integral. If the image space is complemented in its bidual (for example if it is a dual space), then the converse is easily seen to be true, but in the general there are G-integral, not P-integral operators, although there seem to be no easy examples of this. In [8] (see also [5, Appendix D]), the authors show the existence of a G-integral operator failing to be P-integral. Their example relies on the existence of a Banach space with the Approximation Property but without the Bounded Approximation Property.

In [1], Alencar introduced the following extension of the previous definition

Definition 2.4. *A multilinear operator $T \in \mathcal{L}^k(X_1, \dots, X_k; Y)$ is said to be P-integral if there exists a regular Y-valued Borel measure G of bounded variation on the product $B_{X_1^*} \times \cdots \times B_{X_k^*}$ such that*

$$T(x_1, \dots, x_k) = \int_{B_{X_1^*} \times \cdots \times B_{X_k^*}} x_1^*(x_1) \cdots x_k^*(x_k) dG(x_1^*, \dots, x_k^*)$$

for all $(x_1, \dots, x_k) \in X_1 \times \cdots X_k$. The space of P-integral multilinear operators $\mathcal{L}^k_{PI}(X_1, \dots, X_k; Y)$ is a Banach space with the norm $\|T\|_{Pint} = \inf\{v(G)$, where G represents T as above$\}$.

Looking at Definition 2.4 and the comments following Definition 2.2, the following extension of Definition 2.2 seems to be natural.

Definition 2.5. *A multilinear operator $T \in \mathcal{L}^k(X_1, \dots, X_k; Y)$ is said to be G-integral if there exists a regular Y^{**}-valued Borel measure G of bounded variation on the product $B_{X_1^*} \times \cdots \times B_{X_k^*}$ such that*

$$T(x_1, \dots, x_k) = \int_{B_{X_1^*} \times \cdots \times B_{X_k^*}} x_1^*(x_1) \cdots x_k^*(x_k) dG(x_1^*, \dots, x_k^*)$$

for all $(x_1, \dots, x_k) \in X_1 \times \cdots X_k$. The space of G-integral multilinear operators $\mathcal{L}^k_{GI}(X_1, \dots, X_k; Y)$ is a Banach space with the norm $\|T\|_{Gint} = \inf\{v(G)$, where G represents T as above$\}$.

Clearly, as in the linear case, every P-integral multilinear operator T is G-integral, and $\|T\|_{Gint} \leq \|T\|_{Pint}$; moreover, if Y is complemented in its bidual, then, $\mathcal{L}^k_{GI}(X_1, \ldots, X_k; Y)$ and $\mathcal{L}^k_{PI}(X_1, \ldots, X_k; Y)$ are identical spaces with identical norms.

We state now a first result.

Proposition 2.6. *Let $X_1, \ldots, X_k, Y$ be Banach spaces and consider a multilinear operator $T \in \mathcal{L}^k(X_1, \ldots, X_k; Y)$. Then T is G-integral if and only if its linearization $\hat{T}$ its continuous for the injective topology and $\hat{T} \in \mathcal{I}(X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k; Y)$. In that case $\|T\|_{Gint} = \|\hat{T}\|_{Gint}$, so $\mathcal{L}^k_{GI}(X_1, \ldots, X_k; Y)$ and $\mathcal{I}(X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k; Y)$ are isometrically isomorphic. The result remains true word by word if we replace "G-integral" for "P-integral" throughout.*

Proof. If T is G-integral, then T factorizes as

$$\begin{array}{ccc} X_1 \times \cdots \times X_k & \xrightarrow{T} Y \xrightarrow{i_Y} & Y^{**} \\ k_{X_1} \times \cdots \times k_{X_k} \downarrow & & \uparrow b \\ C(B_{X_1^*}) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(B_{X_k^*}) & \xrightarrow{j} & L_1(\mu) \end{array}$$

where $k_{X_1} \times \cdots \times k_{X_k}$ is the multilinear operator given by $k_{X_1} \times \cdots \times k_{X_k}(x_1, \ldots, x_k) = k_{X_1} x_1 \otimes \cdots \otimes k_{X_k} x_k$. Therefore, $\hat{T}$ factorizes as

$$\begin{array}{ccc} X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k & \xrightarrow{\hat{T}} Y \xrightarrow{i_Y} & Y^{**} \\ k_{X_1} \otimes \cdots \otimes k_{X_k} \downarrow & & \uparrow b \\ C(B_{X_1^*}) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(B_{X_k^*}) & \xrightarrow{j} & L_1(\mu) \end{array}$$

Since $k_{X_1} \otimes \cdots \otimes k_{X_k} : X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k \longrightarrow C(B_{X_1^*}) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(B_{X_k^*})$ is an isometry, we get that $\hat{T}$ is G-integral and $\|\hat{T}\|_{Gint} = \|T\|_{Gint}$ (see the comments after Definition 2.2). For the converse implication we just need to follow backwards this same reasoning. The case of P-integral operators is entirely analogous. $\square$

This proposition is crucial for the rest of the paper. It also provides a plentiful of examples of integral multilinear operators. For example, let $X_1, \ldots, X_{n+1}$ be Banach spaces, and consider any $S \in (X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_{n+1})^*$. Define now

$$T : X_1 \times \cdots \times X_n \longrightarrow X_{n+1}^*$$

by

$$T(x_1, \ldots, x_n)(x_{n+1}) = S(x_1 \otimes \cdots \otimes x_{n+1}).$$

Since $\hat{T} : X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_n \longrightarrow X_{n+1}^*$ is G-integral (Definition 2.2), using Proposition 2.6 we get that T is a G-integral multilinear operator. In fact, since it takes values in a dual space, it is also P-integral.

We can also prove the following

Corollary 2.7. *Let $X_1, \dots, X_k, Y$ be Banach spaces, $T \in \mathcal{L}^k(X_1, \dots, X_k; Y)$ and let $K_1, \dots, K_k$ be compact Hausdorff spaces such that X_i is isomorphically contained in $C(K_i)$ $(1 \leq i \leq k)$. Call $k_i : X_i \longrightarrow C(K_i)$ to the embeddings. Then T is G-integral (resp P-integral) if and only if there exists a regular Borel measure G of bounded variation defined on $K_1 \times \cdots \times K_k$ and with values in Y^{**} (resp. with values in Y) such that*

$$T(x_1, \dots, x_k) = \int_{K_1 \times \cdots \times K_k} j_1(x_1)(t_1) \cdots j_k(x_k)(t_k) dG(t_1, \dots, t_k).$$

Moreover, if every k_i is an isometry, then $\|T\|_{Gint} = \inf\{v(G),$ where G represents T as above $\}$, and the same holds for $\|T\|_{Pint}$.

Proof. If T is G-integral, then $\hat{T}$ is G-integral. Since

$$k_1 \otimes \cdots \otimes k_k : X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k \longrightarrow C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k)$$

is an isomorphic embedding (and an isometry if every k_i is an isometry), we get that $\hat{T}$ factorizes as

$$\begin{array}{ccc} X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_k & \stackrel{\hat{T}}{\longrightarrow} Y \stackrel{i_Y}{\longrightarrow} & Y^{**} \\ k_1 \otimes \cdots \otimes k_k \downarrow & & \uparrow b \\ C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k) & \stackrel{j}{\longrightarrow} & L_1(\mu) \end{array}$$

so T factorizes as

$$\begin{array}{ccc} X_1 \times \cdots \times X_k & \stackrel{T}{\longrightarrow} Y \stackrel{i_Y}{\longrightarrow} & Y^{**} \\ k_1 \times \cdots \times k_k \downarrow & & \uparrow b \\ C(K_1) \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon C(K_k) & \stackrel{j}{\longrightarrow} & L_1(\mu) \end{array}$$

which proves what we wanted. For the converse implication we just have to follow the reasoning backwards. The result about the norms in the isometric case is easy. The case of P-integral multilinear operators is entirely analogous. □

Recall that in [1], Alencar defines a k-homogeneous polynomial P between X and Y to be *P-integral* if there exists a measure Y-valued regular Borel measure of bounded variation defined on B_{X^*} such that, for every $x \in X$,

$$P(x) = \int_{B_{X^*}} x^*(x)^k dG(x^*),$$

and he proves in [2, Proposition 2] that P is P-integral if and only if its associated symmetric multilinear operator $T \in \mathcal{L}^k(X; Y)$ is P-integral. Using this and Proposition 2.6 we get

Corollary 2.8. *A k-homogeneous polynomial P between X and Y is P-integral if and only if its associated linear operator $\hat{T} \in \mathcal{L}(\hat{\otimes}^k_{\epsilon,s} X; Y)$ is P-integral.*

Similarly we could define G-integral polynomials and obtain a similar result for them.

We recall that an operator $T : X \longrightarrow Y$ is called *nuclear* if there exists sequences $(x_n^*) \subset X^*$, and $(y_n) \subset Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$ and such that

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$$

for all $x \in X$. The known relations between nuclear and integral operators which we use in this paper are the following (X and Y Banach spaces)

X^* has the Radon-Nikodym property if and only if, for every Y, each $T \in PI(X;Y)$ is nuclear [1, Theorem 1.3]

Let X be such that X^* has the approximation property. Then X^* has the Radon-Nikodym property if and only if, for every Y, every $T \in I(X;Y)$ is nuclear [7, Theorem VIII.4.6].

Y has the Radon-Nikodym property if and only if for every X, every $T \in PI(X;Y)$ is nuclear [7, Theorem VI.4.8] or [5, D7].

Using results of [1] we obtain the following corollary to Proposition 2.6, a result which had been already obtained, with a totally different approach, in [13, Theorem 1.9]

Corollary 2.9. *Given X and Y Banach spaces, X^* and Y^* have the Radon-Nikodym property if and only if $(X\hat{\otimes}_\epsilon Y)^*$ also has the Radon-Nikodym property.*

Proof. Suppose X^* and Y^* have the Radon-Nikodym property. According to [1, Theorem 1.3], it suffices to see that, for every Banach space Z, every P-integral operator $\hat{T} : X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is nuclear. Let then $\hat{T}$ be one such operator. According to Proposition 2.6, $T : X \times Y \longrightarrow Z$ is P-integral. Then, [1, Theorem 2.3] states that T is nuclear (for the definition of nuclear bilinear operator see [1]). It follows immediately from the definitions that, in that case, $\hat{T}$ is nuclear, which finishes one half of the proof. The other implication is clear since the Radon-Nikodym property is stable under closed subspaces. □

G-integral operators form an operator ideal. This ideal is not injective, but, if Y, Z are Banach spaces and $i_Z : Z \longrightarrow Z^{**}$ is the canonical injection, then an operator $T : Y \longrightarrow Z$ is integral if and only if $i_Z \circ T : Y \longrightarrow Z^{**}$ is integral, and, in that case, the integral norms of T and $i_Z \circ T$ coincide ([7, Theorem VIII.2.8]). Hence, we can define an isometric isomorphism into

$$h : \mathcal{I}(Y;Z) \longrightarrow \mathcal{I}(Y;Z^{**})$$

by

$$h(T) = i_Z \circ T.$$

If $T : X \times Y \longrightarrow Z$ is a bilinear operator, then we can consider a linear operator $T_1 : X \longrightarrow \mathcal{L}(Y;Z)$ given by $T_1(x)(y) = T(x,y)$. With this notation, we can state the following result.

Proposition 2.10. *Let X, Y, Z be Banach spaces and let $T \in \mathcal{L}^2(X,Y;Z)$. Consider the following statements:*

(a) T_1 is $\mathcal{I}(Y;Z)$-valued and G-integral when considering with values in this space

(b) $\hat{T}$ is continuous for the ϵ topology and $\hat{T} : X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is G-integral

(c) $T : X \times Y \longrightarrow Z$ is G-integral

(d) T_1 is $\mathcal{I}(Y;Z)$-valued and $h \circ T_1 : X \longrightarrow \mathcal{I}(Y;Z^{**})$ is G-integral.

Then, (b), (c) and (d) are equivalent and (a) implies all of them.

If (b) holds, then $\|\hat{T}\|_{Gint} = \|T\|_{Gint} = \|h \circ T_1\|_{Gint}$, and, if (a) holds, then $\|T\|_{Gint} \leq \|T_1\|_{Gint}$.

Moreover, consider the following conditions

(1) X is an $\mathcal{L}_\infty$ space

(2) Y^* has the approximation property and the Radon Nikodym property

(3) Z is complemented in its bidual.

Then, any of them suffices to guarantee that (d) implies (a).

Proof. Clearly (a) implies (d)

Suppose now that (d) holds. Then, the form $T_2 : X\hat{\otimes}_\epsilon(Y\hat{\otimes}_\epsilon Z^*) \longrightarrow \mathbb{K}$ associated to $h \circ T_1$ is continuous for the ϵ-topologies. So, the operator $T_3 : X\hat{\otimes}_\epsilon Y \longrightarrow Z^{**}$ is G-integral, and clearly $T_3 = i_Z \circ \hat{T}$. So, according to the comments preceding this proposition, $\hat{T}$ is G-integral and that is (b).

Now, if (b) holds, we can define the associated (continuous) operator

$$T_2 : X\hat{\otimes}_\epsilon Y\hat{\otimes}_\epsilon Z^* \longrightarrow \mathbb{K}.$$

Now we can consider the G-integral operator

$$T_3 : X \longrightarrow (Y\hat{\otimes}_\epsilon Z^*)^* \approx \mathcal{I}(Y;Z^{**})$$

canonically associated to T_2. It is easy to check that, for every $x \in X$ and $y \in Y$, $T_3(x)(y) = i_Z(T_1(x)(y))$. Hence, T_1 is $\mathcal{I}(Y;Z)$-valued, and $h \circ T_1 = T_3$ is G-integral, so (d) holds, and

$$\|\hat{T}\|_{Gint} = \|T_2\|_{Gint} = \|T_3\|_{Gint} = \|h \circ T_1\|_{Gint}.$$

The equivalence between (c) and (b) together with the equality $\|\hat{T}\|_{Gint} = \|T\|_{Gint}$ follows from Proposition 2.6.

For the rest of the proof, if (1) holds, then [12, Theorem III.3] states that the G-integral operators on X are exactly the absolutely summing operators on X. Since absolutely summing operators are an *injective* operator ideal, if (d) holds, then $j \circ T_1$ is absolutely summing, so T_1 is absolutely summing, hence G-integral.

Suppose that (2) holds and let us call $\mathcal{N}(Y;Z)$ to the space of nuclear operators between Y and Z. Then

$$\mathcal{I}(Y;Z) = \mathcal{N}(Y;Z) \approx Y^*\hat{\otimes}_\pi Z$$

and

$$\mathcal{I}(Y;Z^{**}) = \mathcal{N}(Y;Z^{**}) \approx Y^*\hat{\otimes}_\pi Z^{**}$$

(see [7, Theorem VIII.4.6] and [5, Corollary 1, p. 65]).

Let us recall that if $B : E \times F \longrightarrow \mathbb{K}$ is a bilinear form, we can define canonically an extension $\overline{B} : E \times F^{**} \longrightarrow \mathbb{K}$, so that $\overline{B}$ is weak* continuous in the second variable and $\|\overline{B}\| = \|B\|$ (see, f.i., [7, VIII.2]). Hence, if we consider $T : E\hat{\otimes}_\pi F \longrightarrow \mathbb{K}$, we can canonically extend T to $\overline{T} : E\hat{\otimes}_\pi F^{**} \longrightarrow \mathbb{K}$, with $\|\overline{T}\| = \|T\|$

So, we can define the operator

$$e : Y^*\hat{\otimes}_\pi Z^{**} \longrightarrow (Y^*\hat{\otimes}_\pi Z)^{**}$$

by

$$e(g)(T) = \overline{T}(g), \text{ for every } g \in Y^*\hat{\otimes}_\pi Z^{**} \text{ and } T \in (Y^*\hat{\otimes}_\pi Z)^*.$$

It is easy to see that e is continuous and $\|e\| \leq 1$. Moreover, it is clear that the restriction of e to $Y^*\hat{\otimes}_\pi Z$ is the canonical inclusion of $Y^*\hat{\otimes}_\pi Z$ into its bidual.

So, if we consider e as an operator from $\mathcal{I}(Y;Z^{**})$ into $\mathcal{I}(Y;Z)^{**}$, $e \circ h : \mathcal{I}(Y;Z) \longrightarrow \mathcal{I}(Y;Z)^{**}$ (h defined as above) is the canonical injection of a space into its bidual.

Therefore, if $h \circ T_1 : X \longrightarrow \mathcal{I}(Y;Z^{**})$ is G-integral, then $e \circ h \circ T_1 : X \longrightarrow \mathcal{I}(Y;Z)^{**}$ is G-integral, and, as follows from the comments preceding this proposition, $T_1 : X \longrightarrow \mathcal{I}(Y;Z)$ is integral.

If (3) holds, and $\pi : Z^{**} \longrightarrow Z$ is a projection, then the operator $p : \mathcal{I}(Y;Z^{**}) \longrightarrow \mathcal{I}(Y;Z)$ defined by $p(T) = \pi \circ T$ is also a projection. The result follows now easily. □

In the next section we will apply this proposition using the fact that $C(K)$ spaces are $\mathcal{L}_\infty$ spaces.

After writing a preliminary version of this paper we have learnt that parts of Proposition 2.10 can be seen in [11].

For P-integral operators we can give a similar result (but not identical; note that the implication which we can not always prove now is reversed).

Proposition 2.11. *Let X, Y, Z be Banach spaces and let $T \in \mathcal{L}^2(X,Y;Z)$. Consider the following statements:*

(a) T_1 is $\mathcal{PI}(Y;Z)$-valued and P-integral when considered with values in this space

(b) $\hat{T}$ is continuous for the injective topology and $\hat{T} : X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is P-integral

(c) $T : X \times Y \longrightarrow Z$ is P-integral

Then (b) and (c) are equivalent and they both imply (a). Moreover, if Z is complemented in its bidual, then (a) is equivalent to (b) and (c).

Proof. The equivalence between (b) and (c) is Proposition 2.6.

Let us now suppose that (c) holds. Then T factorizes as

$$\begin{array}{ccc} X\times Y & \stackrel{T}{\longrightarrow} & Z \\ k_X\times k_Y\downarrow & & \uparrow b \\ C(B_{X^*})\hat{\otimes}_\epsilon C(B_{Y^*}) & \stackrel{j}{\longrightarrow} & L_1(\mu) \end{array}$$

The operator $b\circ j : C(B_{X^*})\hat{\otimes}_\epsilon C(B_{Y^*}) \longrightarrow Z$ is P-integral, equivalently G-integral, so, by Proposition 2.10, the operator $S : C(B_{X^*}) \longrightarrow \mathcal{I}(C(B_{Y^*}); Z)$ given by $S(f)(g) = b\circ j(f\otimes g)$ is G-integral, hence P-integral. So, $T_1 = a\circ q\circ S\circ k_X$ (look at the diagram)

$$X \stackrel{k_X}{\to} C(B_{X^*}) \stackrel{S}{\to} \mathcal{I}(C(B_{Y^*}); Z) \approx bvrca(\Sigma_{Y^*}; Z) \stackrel{q}{\to} \mathcal{PI}(Y; Z) \stackrel{a}{\to} \mathcal{L}(Y; Z)$$

where a is the natural mapping and q is the already mentioned quotient. This proves (a).

Finally, if (a) holds we always have that $T_1 : X \longrightarrow \mathcal{I}(Y; Z)$ is integral, so $\hat{T}$ is integral by Proposition 2.10. Hence, if Z is complemented in its bidual, then $\hat{T}$ is P-integral, and

$$\|\hat{T}\|_{Pint} = \|\hat{T}\|_{Gint} \leq \|T_1\|_{Gint} \leq \|T_1\|_{Pint},$$

which finishes the proof. □

REMARK 2.12. The difficulty when trying to prove that (a) implies (b) in Proposition 2.11 above is that, given $A \in \mathcal{PI}(Y; Z)$, and a compact set K with Borel σ-algebra Σ such that Y is contained in $C(K)$, we do not know how to select *linearly* a measure $G \in bvrca(\Sigma; Z)$ which represents A. In the next proposition we show some cases in which we can surpass this difficulty.

Proposition 2.13. *With the notation of Proposition 2.11, if any one of the following conditions holds*

1. *Y is isomorphic to a $C(K)$ space*
2. *X is isomorphic to a closed subspace of a $C(K)$ space with K scattered*
3. *$\mathcal{PI}(Y, Z)$ has the Radon-Nikodym property*
4. *X^* has the Radon-Nikodym property*

then (a) implies (b) and (c).

Proof. Let us suppose that (a) holds and $Y \approx C(K)$. T_1 factorizes as

$$\begin{array}{ccc} X & \stackrel{T_1}{\longrightarrow} & \mathcal{PI}(Y; Z) \approx \mathcal{I}(C(K); Z) \\ k_X\downarrow & & \uparrow b \\ C(B_{X^*}) & \stackrel{j}{\longrightarrow} & L_1(\mu) \end{array}$$

Since $b\circ j : C(B_{X^*}) \longrightarrow I(C(K); Z)$ is G-integral, we get that the operator $S : C(B_{X^*})\hat{\otimes}_\epsilon C(K) \longrightarrow Z$ is G-integral, hence P-integral. So, T factorizes as

$$\begin{array}{ccc} X \times Y & \xrightarrow{T} & Z \\ k_X \times i \downarrow & & \uparrow b \\ C(B_{X^*}) \hat{\otimes}_\epsilon C(K) & \xrightarrow{j} & L_1(\mu) \end{array}$$

where $i : Y \longrightarrow C(K)$ is the isomorphism. So T is P-integral.

Suppose now that X is isomorphically contained in $C(K)$ with K scattered. In that case any regular countably additive Borel measure μ defined on K is purely atomic [10, §8 Theorem 10], hence $L_1(\mu)$ is isomorphic to ℓ_1. Suppose then that (a) holds. By the definitions and the previous comments, we get that there exist an operator b and a G-integral operator h such that $T_1 = b \circ h$ (see the diagram)

$$\begin{array}{ccc} & & bvrca(\Sigma_{Y^*}; Z) \\ & & q \downarrow \\ X & \xrightarrow{T_1} & \mathcal{PI}(Y, Z) \\ k_X \downarrow & & \uparrow b \\ C(K) & \xrightarrow{h} & \ell_1 \end{array}$$

Applying the lifting property of ℓ_1, we get that there exists $b' : \ell_1 \to bvrca(\Sigma_{Y^*}; Z)$ such that $q \circ b' = b$, where q is the canonical quotient mapping. So, $b' \circ h : C(K) \longrightarrow bvrca(\Sigma_{Y^*}; Z) \approx \mathcal{I}(C(B_{Y^*}); Z)$ is G-integral; hence, the associated operator $S : C(K) \hat{\otimes}_\epsilon C(B_{Y^*}) \longrightarrow Z$ is G-integral, equivalently P-integral. So there exist μ and $\tilde{b}$ such that T factorizes as

$$\begin{array}{ccc} X \times Y & \xrightarrow{T} & Z \\ i \times k_Y \downarrow & & \uparrow \tilde{b} \\ C(K) \hat{\otimes}_\epsilon C(B_{Y^*}) & \xrightarrow{j} & L_1(\mu) \end{array}$$

where $i : X \longrightarrow C(K)$ is the isomorphic embedding. So T is P-integral.

Suppose now that $\mathcal{PI}(Y; Z)$ has the Radon-Nikodym property and that (a) holds. In that case, there exist i, j, b and μ such that $T_1 = b \circ j \circ i$ (see the diagram)

$$\begin{array}{ccc} & & bvrca(\Sigma_{Y^*}; Z) \\ & & q \downarrow \\ X & \xrightarrow{T_1} & \mathcal{PI}(Y, Z) \\ \downarrow i & & \uparrow b \\ C(B_{X^*}) & \xrightarrow{j} & L_1(\mu) \end{array}$$

Since $\mathcal{PI}(Y; Z)$ has the Radon-Nikodym property, we know that b is representable. Representable operators factor through ℓ_1, so again the lifting property of ℓ_1 allows us to assure the existence of an operator $b' : L_1(\mu) \longrightarrow bvrca(\Sigma_{Y^*}; Z)$ such that $q \circ b' = b$. The proof now proceeds similarly to the previous cases.

Finally, suppose that X^* has the Radon-Nikodym property. In that case, [1, Theorem 1.3] states that, for every Banach space E, every $T \in \mathcal{PI}(X; E)$ is a nuclear operator. Suppose then that (a) holds. Then, T_1 is nuclear. So, there exist bounded sequences $(x_n^*) \in X^*$ and $(S_n) \in \mathcal{PI}(Y; Z)$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \|S_n\|_{Pint} < \infty$ and so that T_1 can be written as

$$T_1(x) = \sum_{n=1}^{\infty} x_n^*(x) S_n.$$

For every $n \in \mathbb{N}$, choose $\mu_n \in bvrca(\Sigma_{Y^*}; Z)$ such that μ_n represents S_n and such that $v(\mu_n) < \|S_n\|_{Pint} + 2^{-n}$. Then we can define the (clearly nuclear) operator

$$\tilde{S} : C(B_{X^*}) \longrightarrow bvrca(\Sigma_{Y^*}; Z) = \mathcal{I}(C(B_{Y^*}; Z)$$

by

$$\tilde{S}(f) = \sum_{n=1}^{\infty} \delta_{\frac{x_n^*}{\|x_n^*\|}}(f) \mu_n,$$

where $\delta_{x^*} : \Sigma_{X^*} \longrightarrow \mathbb{K}$ is the measure given by

$$\delta_{x^*}(A) = \begin{cases} 1 & \text{if } x^* \in A \\ 0 & \text{if } x^* \notin A \end{cases}$$

Then $T_1 = q \circ \tilde{S} \circ k_X$. Since $\tilde{S}$ is nuclear, it is G-integral, so the associated operator

$$S : C(B_{X^*}) \hat{\otimes}_\epsilon C(B_{Y^*}) \longrightarrow Z$$

is G-integral, equivalently P-integral. So, there exists b such that T factorizes as

$$\begin{array}{rcl} X \times Y & \xrightarrow{T} & Z \\ k_X \otimes k_Y \downarrow & & \uparrow b \\ C(B_{X^*}) \hat{\otimes}_\epsilon C(B_{Y^*}) & \xrightarrow{j} & L_1(\mu) \end{array}$$

and (c) holds. It is easily seen that the measure G associated to S is the only Borel measure of bounded variation which verifies that, for every $A \in \Sigma_X$ and $B \in \Sigma_Y$,

$$G(A \times B) = \sum_{n=1}^{\infty} \|x_n^*\| \delta_{\frac{x_n^*}{\|x_n^*\|}}(A) \mu_n(B).$$

(See [4] for the uniqueness of this measure). □

We leave certain questions without answer

QUESTION 2.14. *In Proposition 2.10, does (d) always imply (a), i.e., is T_1 G-integral when considered with values in $\mathcal{I}(Y;Z)$ whenever T_1 is $\mathcal{I}(Y;Z)$-valued and $j \circ T_1 : X \longrightarrow \mathcal{I}(Y;Z^{**})$ is G-integral?*

Question 2.14 would have a positive answer if there was a linear operator $S : \mathcal{I}(Y;Z^{**}) \longrightarrow \mathcal{I}(Y;Z)^{**}$ such that its restriction to $\mathcal{I}(Y;Z)$ is the canonical inclusion into the bidual.

The other open question refers to P-integral operators

QUESTION 2.15. *In Proposition 2.11, does (a) always imply (b) and (c), i.e., is $T : X \times Y \longrightarrow Z$ P-integral whenever T_1 is $\mathcal{PI}(Y;Z)$-valued and P-integral when considered with values in this space?*

It is easy to see that all that needs to be considered to answer this question in full generality is the case when X is isomorphic to a $C(K)$ space.

Note that the proof of case (4) in Proposition 2.13 above proves that if $T_1 : X \longrightarrow \mathcal{P}I(Y;Z)$ is nuclear, then T is P-integral, so a counterexample providing a negative answer to Question 2.15 should start out by being a P-integral, not nuclear operator $T_1 : X \longrightarrow \mathcal{P}I(Y;Z)$.

3. INTEGRAL MAPPINGS ON $C(K,X)$ SPACES

In this section, K will always be a compact Hausdorff space and Σ will be its Borel σ-algebra. If X is a Banach space, $C(K,X)$ is the Banach space of the X-valued continuous functions, endowed with the supremum norm. $S(\Sigma,X)$ is the space of the X-valued Σ-simple functions defined on K and $B(\Sigma,X)$ is the completion of $S(\Sigma,X)$ under the supremum norm. It is well known that $C(K,X)^* = bvrca(\Sigma;X^*)$.

If Σ is a σ-algebra, X a Banach space and $Y \subset X^*$, we say that a finitely additive vector measure $m : \Sigma \longrightarrow X$ is $\sigma(X,Y)$-regular if, for every $y \in Y$, the measure $y \circ m : \Sigma \longrightarrow \mathbb{K}$ is regular. We will later need the following well known lemma, which can be found, for instance, in [3].

Lemma 3.1. *Let Σ be a σ-algebra, X a Banach space and $Y \subset X^*$ a subspace norming X. If $m : \Sigma \longrightarrow X$ is a strongly additive and $\sigma(X,Y)$-regular measure, then m is regular.*

It is well known that $C(K,X) \approx C(K)\hat{\otimes}_\epsilon X$ (see, f. i., [7, Example VIII.1.6]). It is also well known that any operator $T \in \mathcal{L}(C(K,X);Y)$ can be canonically represented through a measure $m : \Sigma \longrightarrow \mathcal{L}(X;Y^{**})$ [7, p. 182].

The following corollary to Proposition 2.10 is the main result of [14]. The proof given in [14] is much longer and, in our opinion, more complicated, relying on measure theoretic methods rather than tensor product techniques.

Corollary 3.2. *Let $T \in \mathcal{L}(C(K,X);Y)$ and let m be its representing measure. Then T is G-integral if and only if m is $\mathcal{I}(X;Y)$-valued and it has bounded variation when considered with values in this space.*

Proof. If T is G-integral then, according to Proposition 2.10,

$$\tilde{T} : C(K) \longrightarrow \mathcal{I}(X;Y)$$

is G-integral (and therefore weakly compact). So, if $\mu : \Sigma \longrightarrow \mathcal{I}(X;Y)$ is the measure associated to $\tilde{T}$, then μ has bounded variation and $v(\mu) = \|\tilde{T}\|_{Gint} = \|T\|_{Gint}$. From regularity it follows that $\mu = m$, which finishes this part of the proof.

Conversely, let $T \in \mathcal{L}(C(K,X);Y)$ be an operator such that its associated measure m is as in the hypothesis. Let us see that m is regular when considered with values in $\mathcal{I}(X;Y)$: according to Lemma 3.1, we just have to check that m is $\sigma(\mathcal{I}(X;Y), D)$-regular, with $D \subset \mathcal{I}(X,Y)^*$ a subspace norming $\mathcal{I}(X;Y)$. It is clear that $D' = X \otimes Y^* \subset \mathcal{I}(X;Y^{**})^* \approx (X \hat{\otimes}_\epsilon Y^*)^{**}$ is a subspace norming $\mathcal{I}(X;Y^{**})$. If we call h to the canonical isometric injection of $\mathcal{I}(X;Y)$ into $\mathcal{I}(X;Y^{**})$, it follows from the properties of m that

$$h \circ m : \Sigma \longrightarrow \mathcal{I}(X;Y^{**})$$

is $\sigma(\mathcal{I}(X;Y^{**}), D')$-regular. If we call $D = h^*(D')$, then m is $\sigma(\mathcal{I}(X;Y), D)$-regular, and $D \subset \mathcal{I}(X;Y)^*$ is clearly a subspace norming $\mathcal{I}(X;Y)$. So, m is regular and with bounded variation, and now we can consider the G-integral operator

$$T_1 : C(K) \longrightarrow \mathcal{I}(X;Y)$$

associated to it; then we consider the operator

$$T_2 : C(K) \hat{\otimes}_\epsilon X \longrightarrow Y$$

associated to T_1. By Proposition 2.10, T_2 is G-integral, and clearly $T_2 = T$, which finishes the proof. □

A similar result can be given now (although not in full generality) for P-integral operators on $C(K,X)$ spaces.

Corollary 3.3. *Let $T \in \mathcal{L}(C(K,X);Y)$ and let m be its representing measure. If T is P-integral then m is $\mathcal{PI}(X;Y)$-valued and it has bounded variation when considered with values in this space. If K is a scattered compact space or if $\mathcal{PI}(X;Y)$ has the Radon-Nikodym property then the converse also holds.*

Proof. If T is P-integral then, Proposition 2.11 states that

$$\tilde{T} : C(K) \longrightarrow \mathcal{PI}(X;Y)$$

is P-integral. The proof proceeds now as in Corollary 3.2

Conversely, let $T \in \mathcal{L}(C(K,X);Y)$ be an operator such that its associated measure m is as in the hypothesis and let $T_1 : C(K) \longrightarrow \mathcal{L}(X;Y)$ be the associated operator. The measure $m : \Sigma \longrightarrow PI(X;Y)$ defines an operator $T_m : B(\Sigma) \longrightarrow PI(X;Y)$ (see [7, Section VI.1]). Since $v(m) < \infty$, [7, Corollary VI.1.4] states that T_m is absolutely summing. Now, using the fact that $C(K)$ is isometrically contained in $B(\Sigma) := B(\Sigma, \mathbb{K})$, we define the operator $T'_m = T_{m|_{C(K)}} : C(K) \longrightarrow PI(X;Y)$, which is also absolutely summing, hence P-integral. Since, for every $A \in \Sigma$, $T_m(\chi_A) = m(A)$, it follows that $T'_m = T_1$. Now we just need to apply Proposition 2.13 to finish the proof. □

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