ON A DOUBLY NONLINEAR PARABOLIC OBSTACLE PROBLEM MODELLING ICE SHEET DYNAMICS *

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Abstract. This paper deals with the weak formulation of a free (moving) boundary problem arising in theoretical glaciology. Considering shallow ice sheet flow we present the mathematical analysis and the numerical resolution of the second order, nonlinear, degenerate parabolic equation modelling, in the isothermal case, the ice sheet non-newtonian dynamics. An obstacle problem is then deduced and analysed. The existence of a free boundary generated by the support of the solution is proved and its location and evolution are qualitatively described by using a comparison principle and an energy method. Then the solutions are numerically computed with a method of characteristics and a duality algorithm to cope with the resulting variational inequalities. The weak framework we introduce and its analysis (both qualitative and numerical) are not restricted to the simple physics of the ice sheet model we consider nor to the model dimension. They can be applied succesfully to more realistic and sophisticated models related to other geophysical settings.

Key words. Ice sheet models, nonlinear degenerate equations, free boundaries, weak solutions, finite elements, duality methods.

AMS subject classifications. 35K65, 35K85, 65C20, 65N30

1. Introduction.

Modelling ice sheet dynamics has been a challenging problem since the beginning of the century, but nowadays the scientific community is showing a renewed, growing interest toward this problem. In fact, our understanding of climate system dynamics depends on the comprehension and predictability of the ice sheet dynamics because large ice sheets influence, and are influenced by, climate. The Antarctic and Greenland ice sheets are the two mayor present day examples of ice sheets but during the last age (terminating about 10.000 years ago) ice sheets existed in North America (the Laurentide) and northen Europe (the Fennoscandian), the ice extending into Southern England and Northen Europe. These ice sheets interact with climate, and their oscillations may be responsible for sudden shifts in climate in the recent geological past (see Fowler [24], chapter 18 and references therein for a more detailed introduction to the problem). Moreover is well known that large ice sheets reflect a (great) part of the solar energy received by the Earth so governing its energy balance. Their extension or retreat are considered as good indicators of the climate system future behaviour and modifying the values of the albedo function they contribute to determine finally the distribution of the Earth temperature.

The study of the Ice Sheet Models (ISM) is then fundamental to the construction and comprehension of the global Energy Balance Models (EBM) and of the General Circulation Models (GCM) (see, for example Tarasov and Peltier [30] for a coupled

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ISM/EBM model). Solving ice sheets problems has also a multi disciplinary approach and in fact glaciology spreads on continuum mechanics, fluid dynamics, geology, geophysic, material science, hydrology and applied mathematics. As a consequence of this effort, various physically based theories have appeared during the last decades in order to explain the flow of these large ice masses. Nowadays more and more sophisticated models can be tackled thanks to the new technologies (interferometric radar observations, satellite or aerial photographs, geodetic global positioning system (GPS) methods, chemical analysis of firn cores, etc.) which provide a great amount of field data as well as a basis for more realistic modelling. Processing such quantities of data is also possible due to the increasing computing power and the parallel computing techniques now availables. More sophisticated numerical methods can be applied. In a parallel way new mathematical techniques have been developed in the last years to deal with complicated nonlinear dynamics and this introduce and justify the basic aim of this paper which is to provide an example of how such powerful mathematical methods (multivalued equations, variational inequalities, renormalization solutions, energy methods, finite elements, duality methods) can be applied to the resolution of the *shallow* ice sheet flow problem. Our proposal is also a constructive one, whereas a numerical method is proposed and succesfully implemented.

This paper is organized as follows: after a brief description of the model equation and its strong formulation (section 2), we introduce in section 3 some weak formulations that we shall use later. The well-posedness of the model is also considered. Section 4 is devoted to the (qualitative) study of the free (moving) boundary defined by the model. The quantitative analysis is done in section 5 where we solve, numerically, the problem.

Some comparison tests are then performed. Finally, in section 6, we discuss our results and their scope.

2. Model equation and strong formulation. The model equation is that of Fowler [23], describing the evolution of the ice thickness $h(t,x)$ for a two-dimensional plane ice sheets (the coordinates are (x,z)) where $z = h(t,x)$ is the top surface of the ice sheet). For three-dimensional geometries the equations are analogous (just replacing ∂_x with ∇ or $\nabla\cdot$ as appropiate). For more general introductions to glaciology see also Hutter [26], Paterson [28], Lliboutry [27] and Fowler [24] among others. Ice is taken to be incompressible and the flow is very slow. It flows as a viscous medium under its own weight, owing to gravity. Then, the model equation can be deduced starting with the usual equations of conservation of mass, momentum and energy written in terms of a slow, gravity driven, viscous flow. The (unknown) domain where the equations hold has to be determined as part of the solution and it is characterized by the kinematic boundary condition (or surface kinematic condition). The resulting continuity equation, equilibrium equation and energy equation, complemented with the constitutive Glen's flow law, the Arrhenius' rate law (which is usually written in terms of the Frank-Kamenetskii approximation) and the kinematic boundary condition generate the basic system to solve. A Stokes problem arise coupled with the energy equation (via the Arrhenius term) in a *a priori* unknown domain. Assuming the usual hypothesis of shallow flow (when l and d are typical longitudinal and depth scales of the problem (of the Antarctic, for example), the lateral extent is $l \sim 3.000$ kilometers and its thickness is typically $d \sim 3$ kilometers), we can use the lubrication approximation to simplify the equations.

The variables are then scaled to non-dimensionalise and the reduced model (which is obtained from the equations by neglecting $\mathrm{O}(\epsilon^2)$ where $\epsilon = d/l \sim 10^{-3} \ll 1$) is considered.

Using the (additional) hypothesis of isothermal flow corresponding to temperature independence of the ice viscosity, the flow problem uncouples from the energy equation and an integrated mass conservation equation is deduced. Two integrations of the Glen's law provide an expression for the ice flux which, together with the kinematic boundary condition, yield a nonlinear diffusion equation for $h(t,x)$ (the unknown local thickness of the ice sheet)

$$h_t = \left(\frac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - u_b h\right)_x + a, \quad \text{on } I(t), \tag{2.1}$$

where $a = a(t,x)$ is a scaled fixed, given accumulation rate function ($a < 0$ signifies ablation) and u_b is a (given) function representing the basal velocity. For each fixed t the domain $I(t)$ represents the (unknown) bounded real interval where $h(t,x) > 0$, (i.e., $I(t) := \{x \,/\, h(x,t) > 0\}$). Notice that the physically relevant rate functions $a(t,x)$ are changing sign functions which are positive in the central (accumulation) region of the ice sheet and negative near the margins (the boundaries of $I(t)$, i.e., in the ablation region), see Fowler [24], pp. 95. The exponent n which appears in (2.1) represents Glen's exponent and it is usually assumed $n \approx 3$. Infact, we shall assume that $n = 3$ but the qualitative analysis remains unchanged for any $n > 1$ (non-newtonian case). For simplicity we also assume a flat, rigid and impermeable base (the bedrock). As regards the appropriate (mechanical) boundary condition it depends on the thermal regime that we consider at the base. There are two possible geophysical situations corresponding to slip or no slip conditions.

When basal ice reaches the melting point, there is a net heat flux arriving at the bed of the ice sheet, and consequently basal melt water is produced: the ice begins to slide. Sliding is expected only where the basal ice is at the melting point. When u_b (the sliding velocity) is a prescribed function of (t,x) (i.e., $u_b = u_b(t,x)$), this equation is a nonlinear diffusion-convection equation for h. It corresponds to slip conditions along an assumed temperate bed (warm-based ice sheet). Once the base reaches the melting point we assume that the ice above remains cold. Here we are not concerned with the functional dependence of the sliding velocity on the shear stress or other physical variables, i.e. water flux or effective pressure, which, in turn, depend on the kind of bed considered, hard or soft. Our aim is to show how it is possible to solve this model for a general, prescribed, velocity field.

In the diffusive case the ice sheet is supposed to be cold-based. For slow, shallow flow over a flat base, with no sliding (i.e., $u_b \equiv 0$), the isothermal ice sheet equation (2.1) is just

$$h_t = \left(\frac{h^{n+2}}{n+2}|h_x|^{n-1}h_x\right)_x + a, \quad \text{on } I(t). \tag{2.2}$$

Some remarks about the equation (2.2) are in order. As discussed in Fowler [23], where the no slip condition is prescribed, singularities appear at the margins (due to infinite slope) during the advance of fronts of a land based ice sheet (such that which covered North America in the last ice age). Classical (finite-differences) numerical methods can fail. A further complication is due to the fact that the domain where the equation holds is unknown. In fact it has to be determined as part of the solution.

The original *strong* formulation can be stated in the following terms: let $T > 0$, $L > 0$ be positive fixed real numbers and let $\Omega = (-L, L)$ be an open bounded interval of $I\!R$ (a sufficently large, fixed spatial domain). Given an accumulation/ablation rate function $a = a(t,x)$ and a function $u_b = u(t,x)$ (a sliding velocity eventually zero) defined on $Q = (0,T) \times (-L, L)$ (a large, fixed, parabolic domain) and an initial thickness $h_0 = h_0(x) \geq 0$ (bounded and compactly supported) on Ω, find two curves $S_+, S_- \in C^0([0,T])$, with $S_-(t) \leq S_+(t)$, $I(t) := (S_-(t), S_+(t)) \subset \Omega$ for any $t \in [0,T]$, and a sufficently smooth function $h(t,x)$ defined on the set $Q_T := \bigcup_{t\in(0,T)} I(t)$ such that

$$
(2.3) \qquad (SF) := \begin{cases} h_t = \left[\dfrac{h^{n+2}}{n+2} |h_x|^{n-1} h_x - u_b h \right]_x + a & \text{in} \quad Q_T, \\ h = 0 \quad \text{on} \quad \{S_-(t)\} \cup \{S_+(t)\}, & t \in (0,T), \\ h = h_0, & \text{on} \quad I(0), \end{cases}
$$

and $h(t,x) > 0$ on Q_T. Notice that, for each fixed $t \in (0,T)$, $I(t) = (S_-(t), S_+(t)) = \{x \in \Omega : h(t,x) > 0\}$ denotes the ice covered region. The curves $S_\pm(t)$ are called the interface curves or free boundaries associated to the problem and are defined by:

$$
S_-(t) = \mathrm{Inf}\{x \in \Omega \, : \, h(t,x) > 0\}, \qquad S_+(t) = \mathrm{Sup}\{x \in \Omega \, : \, h(t,x) > 0\}
$$

These curves defines the interface separating the regions in which $h(t,x) > 0$ (i.e, ice regions) from those where $h(t,x) = 0$ (i.e. ice-free regions). In the physical context they represent the propagation fronts of the ice sheet.

Many other examples of the class of degenerate equations are typical (see e.g. Díaz [13]) of slow phenomena and satisfy the finite speed of propagation property (although this property must be shown for each special formulation as (SF)). Assuming, for instance, $a \equiv 0$ if $h(0,x)$ has compact support then $h(t,x)$ has also a compact support in $I\!R$, for any $t \in [0,T]$. So, if $a \equiv 0$, the domain Q_T can be found through the support of the solution $h(t,x)$ of the doubly nonlinear parabolic equation over the whole space $(0,T) \times I\!R$ and satisfying the initial condition $h(0,x) = h_0(x)$, $x \in I\!R$. Unfortunately, the physically relevant case, $a \not\equiv 0$, is much more complicated. Indeed, the finite speed of propagation still holds if $a(t,.)$ has compact support in $I\!R$ (for fixed $t \in (0,T)$). Moreover, in that case, it can be shown that support $h(t,.) \subset$ support $a(t,.)$ and so $a(t,.)$ vanishes on the free boundary. Nevertheless, in glaceological models it is well-known (see Fowler [24], pp 95) that $a(t,.) < 0$ near the free boundaries (i.e. the margins of the ice-sheet) and so there must exist another reason (other than the degenerate character of the equation) justifying the occurrence of the free boundaries $S_-(t), S_+(t)$. It is a mathematical modelling problem. We must insure that the mathematical solutions are no negative, compactly supported (i.e., physically admissible) solutions. In short, fixed a sufficently large spatial domain, the physically admissible solutions are compactly supported non-negative bounded functions such that $a < 0$ where $h = 0$ (in particular in the free boundary) and this is not predicted by the solutions of the diffusion equation (despite of its degenerate character). Mathematically is then possible (for special choices of the accumulation/ablation rate function) to have negative (no physically admissible solutions) solutions corresponding to negative thickness! A practical way to overcome this inconsistence is proposed in the following section.

3. Weak formulations. In this section we show that a proper mathematical modelling of the physical problem must be considered if the assumed physics have to be respected (i.e. if just physically admissible solutions have to be computed). This introduces the need of a weak formulation for the ISM. Notice that we shall consider an isothermal model but the technique can be applied to more general cases provided a free boundary problem is considered.

Let T, L and Ω as before and set $Q := (0,T) \times \Omega \subset I\!R^2$. The new model we present is based upon the fact that we can extend the function $h(t,x)$ outside of Q_T (the ice covered regions) by zero on $Q \setminus Q_T = [(0,T) \times \Omega] \setminus Q_T$ (the complementary ice-free region) and that this extension still satisfies a nonlinear equation (this time of multivalued type) having the great advantage of being defined on an *a priori* known domain $Q = (0,T) \times \Omega$ (whose parabolic boundary is $\Sigma = \partial Q = (0,T) \times \partial\Omega$). This type of problem is known in the literature as an *obstacle problem* (in our case the obstacle function is $\psi \equiv 0$, the null function) and it arises in many contexts related to friction, elasticity, thermodynamics and so on (see e.g. Duvaut and Lions [21], for further details). The multivalued formulation we propose appeared first in Díaz and Schiavi [17] (where the no slip condition was considered) to describe the slow, isothermal, one-dimensional flow of *cold* (i.e., all the ice is below melting point and the melting point is reached only at the bed) ice along a *hard* (i.e., rigid, impermeable) bed. Our results can be generalized to deal with the two-dimensional case which describes the evolution of a 3-D ice sheet. Here we extend that model to consider sliding (prescribed) along a temperate base. This introduces a nonlinear convective term in the multivalued equation which describes the movement of the ice masses. Introducing the maximal monotone graph of $I\!R^2$, β, defined by

$$\text{(3.1)}\quad \beta(r) = \emptyset \text{ (the empty set) if } \ r<0,\quad \beta(0) = (-\infty, 0],\quad \beta(r) = 0 \text{ if } \ r>0,$$

the *obstacle formulation* (written in terms of a multivalued equation) is: given a bounded, sufficently large interval $\Omega = (-L, L) \subset I\!R$, a rate function $a \in L^\infty(Q)$, a sliding velocity $u_b \in L^\infty(Q)$ and a compactly supported initial data $h_0 \in L^\infty(\Omega)$, find a sufficently smooth function $h(t,x)$ solution of

$$\text{(3.2)}\,(MF) := \begin{cases} h_t - \left(\dfrac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - u_b h\right)_x + \beta(h) & \ni a(t,x) \quad \text{in} \quad Q, \\ h(t,x) & = 0 \quad \text{on} \quad \Sigma, \\ h(0,x) & = h_0(x) \quad \text{on} \quad \Omega. \end{cases}$$

Notice that β is multivalued just where h is zero, i.e., at the free boundaries. Moreover, by definition (3.1), we have that: $0 \in \beta(0)$. Now, let h be a solution (in a weak sense to be precised later) of (3.2), a.e. $x \in \Omega$ and $\forall t \in (0,T)$. It is clear that in the null set $Q \setminus Q_T$ we must have $\beta(0) \ni a(t,x)$. This condition shows that, if β is multivalued at the origin, then it is possible to have solutions with a non empty null set (i.e., $Q \setminus Q_T \neq \emptyset$) corresponding to equations in which $a \neq 0$ on $Q \setminus Q_T$ and thus new results are possible with respect to the single valued case ($\beta \equiv 0$).

Details on this kind of (multivalued) formulations, (MF), and on maximal monotone graphs can be found in Brezis [8]. It is well known that the multivalued equation (3.2) can be written in terms of the so called *complementary formulation* for obstacle problems which states: given Ω, a, u_b and h_0 as before, find a sufficently smooth function h such that:

$$
(3.3) \qquad (CF) := \begin{cases} h_t - \left(\dfrac{h^{n+2}}{n+2} |h_x|^{n-1} h_x - u_b h \right)_x - a \geq 0 & \text{in} \quad Q, \\ \left[h_t - \left(\dfrac{h^{n+2}}{n+2} |h_x|^{n-1} h_x - u_b h \right)_x - a \right] h = 0 & \text{in} \quad Q, \\ h \geq 0 & \text{in} \quad Q, \\ h = 0 & \text{on} \quad \Sigma, \\ h = h_0(x) & \text{on} \quad \Omega. \end{cases}
$$

It is obvious that if a regular function h verifies the strong formulation then its extension by zero over $Q \setminus Q_T$ (which we will denote again by h) satisfies trivially the complementary formulation, assuming that $a(t,x)$ satisfies the condition

$$
a(t,x) \leq 0 \qquad \text{on} \quad Q \setminus Q_T
$$

A more general framework is obtained if we define $\phi(r) = |r|^{n-1} r$, $r \in I\!R$, $n > 1$ and $\psi(s) = s^{1/m}$ with $s \geq 0$ and $m = 2(n+1)/n > 1$ (infact the existence and the uniqueness of solutions and some qualitative properties remain true if we replace ϕ by any real continuous strictly increasing convex function such that $\phi(0) = 0$, and β as before, see (3.1)). Introducing the new variable $u = u(t,x)$ and the real function $b(s)$ in form

$$
(3.4) \qquad u := h^m = \psi(h), \qquad \Longrightarrow \qquad u^{1/m} = h = \psi^{-1}(u) := b(u)
$$

we have $\phi(\psi(h)_x) = \phi(u_x) = |u_x|^{p-2} u_x$, where $p = n+1$. The previous *multivalued formulation* is the following: given Ω, a, u_b and $u_0 = \psi(h_0)$ as before and a constant $\mu = n^n / [2^n (n+1)^n (n+2)]$, determine a function $u(t,x) = \psi(h(t,x))$ solution of

$$
(3.5) \quad (GF) := \begin{cases} b(u)_t - [\mu \phi(u_x) - u_b b(u)]_x + \beta(u) & \ni a(t,x) \quad \text{in} \quad Q, \\ u(t,x) & = 0 \quad \text{on} \quad \Sigma, \\ b(u(0,x)) & = b(u_0(x)) \quad \text{on} \quad \Omega. \end{cases}
$$

This *general formulation*, (GF), is that we shall use to deal with the well-posedness of the model problem (3.2). Notice that we can write $\beta(u)$ instead of $\beta(b(u))$ because $\beta(u) \equiv \beta(h) := \beta(b(u))$ in Q. The equivalence is readly understood observing that h (i.e., the original variable) and u have exactly the same support. The same remark applies to the boundary condition on Σ.

3.1. On the existence and uniqueness of weak solutions. Problem (3.5) admits various notions of solutions according to the required spatial and time regularity. In any case we must start by assuming some regularity on the data $a(t,x)$, $u_b(t,x)$ and $u_0(x)$. In our case it will be enough to assume that

$$
(3.6) \qquad a \in L^\infty(Q), \qquad u_b \in L^\infty(0,T : W^{1,\infty}(\Omega)) \qquad \text{and} \quad u_0 \in L^\infty(\Omega).
$$

Then a natural notion of weak solution is the following:

DEFINITION 3.1. *A function $u \in L^1(Q)$ is a weak solution of (3.5) if $u \in L^p(0,T : W_0^{1,p}(\Omega))$, $b(u) \in L^1(Q)$, $u(t,x) \geq 0$ a.e. $(t,x) \in Q$ and there exists a function $j \in L^1(Q)$ such that $j(x,t) \in \beta(u(t,x))$ a.e. $(t,x) \in Q$ and*

$$\int_Q (\zeta_t b(u) - \zeta j + \zeta a) dx dt + \int_\Omega (\zeta(0,\cdot) b(u_0) dx = \int_Q \zeta_x [\mu\phi(u_x) - u_b b(u)] dx dt$$

for any $\zeta \in L^p(0,T : W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\zeta_t \in L^\infty(Q)$ and $\zeta(T,\cdot) = 0$.

The existence of a weak solution (assumed (3.6) and b, ϕ being the power functions indicated above) can be obtained by different methods. For instance, it is an easy modification of Theorem 2.3 and Proposition 3.2 of Benilan and Wittbold [3]. In fact, in order to check assumption (H1) of this paper it is useful to replace function b by its truncation

$$b_M(r) := \begin{cases} b(r) & \text{if } r \in [0, M], \\ b(M) & \text{if } r \in [M, +\infty), \end{cases}$$

with $M > 0$ a upper bound of any weak solution (we shall come back to this point later: see Proposition 1). The uniqueness of (and the comparison principle for) weak solutions is a more delicate task due to the presence of the nonlinear term $b(u)$. This type of results are well known (see, for instance Díaz and de Thelin [14]) when we know, additionally, that the weak solution is time differentiable in the sense that $b(u)_t \in L^1(Q)$ (we recall that from the definition of weak solutions we merely know that $b(u)_t \in L^{p'}(0,T : W^{-1,p'}(\Omega)) + L^1(Q)$ with $p' := p/(p-1)$). In order to get such results a weaker notion was introduced in previous works by different authors (see, Boccardo, Giachetti, Díaz and Murat [7] for the case of $b(u) = u$ and Carrillo and Wittbold [12] for a general nondecreasing function $b(u)$). It is the notion of *renormalized solution* coming originally from a different context (Di Perna and Lions [20]). In fact both notions coincide in the class of bounded functions $u \in L^\infty(Q)$ which is our case as we shall prove in this Section.

In order to apply the methods of proofs by Carrillo and Wittbold [12] we need to assume a technical assumption on the data

$$u_b(t,\cdot) \quad \text{is spatially constant.} \tag{3.7}$$

(we point out that the assumption (3.7) is not needed in the linear case $b(u) = u$ and that we suspect that this technical assumption can be avoid by means of some refinement which we do not try to develop in this paper). So, by some trivial modifications of the results of Carrillo and Wittbold [12] (see Section 4 of that paper and use the monotonicity of $\beta(u)$) we arrive to the following result:

THEOREM 3.2. *Assume a_i, u_b and $u_{0,i}$ satisfying (3.6) and (3.7) for $i = 1, 2$. Let u_i be weak solutions of problem (3.5) associated to the data a_i and $u_{0,i}$. Then for any $t \in [0,T]$*

$$\int_\Omega [b(u_1(t,\cdot)) - b(u_2(t,\cdot))]_+ dx \leq \int_\Omega [b(u_{0,1}) - b(u_{0,2})]_+ dx + \int_0^t \int_\Omega [a_1(s,x) - a_2(s,x)]_+ dx ds,$$

where $[f]_+ = max(f,0)$. *In particular,* $b(u_{0,1}) \leq b(u_{0,2})$ *and* $a_1(t,x) \leq a_2(t,x)$, *on their respective domains of definition, implies that* $b(u_1(t,x)) \leq b(u_2(t,x))$ *for any* $t \in [0,T]$ *and a.e.* $x \in \Omega$. *In consequence, there is at most one weak solution of problem (3.5).*

We point out that the above comparison remains true even if functions u_i are not homogeneous at the boundary but they satisfy $u_1(t,x) \leq u_2(t,x)$ for any $t \in [0,T]$ and a.e. $x \in \partial\Omega$ (this is again a trivial modification of the result by Carrillo and Wittbold [12]).

The boundedness of the associate weak solution can be deduced from the above comparison principle:

PROPOSITION 3.3. *Let* u *be any weak solution of problem (3.5) then*

$$\|u\|_{L^\infty(Q)} \leq M_0, \tag{3.8}$$

with

$$M_0 := b^{-1}\left(\max\{\|b(u_0)\|_{L^\infty(\Omega)},1\}\exp T\left\{\left[\frac{\|a\|_{L^\infty(Q)}}{\max\{\|b(u_0)\|_{L^\infty(\Omega)},1\}} + \|(u_b)_x\|_{L^\infty(\Omega)}\right]\right\}\right).$$

Proof. We take as candidate to *supersolution* a spatially constant function of the form $u_2(t,x) := b^{-1}(Ce^{\lambda t})$ for some $C > 0$ and $\lambda \in I\!R$ to be determined. Then $u_1(t,x) \leq u_2(t,x)$ for any $t \in [0,T]$ and a.e. $x \in \partial\Omega$ and $b(u_{0,1}) \leq b(u_{0,2})$ holds if

$$C = \max\{\|b(u_0)\|_{L^\infty(\Omega)},1\}.$$

Finally, by substituting at the equation of (3.5) is easy to check that

$$a_2(t,x) := \lambda Ce^{\lambda t} + (u_b)_x(x)Ce^{\lambda t}$$

and so condition $a_1(t,x) \leq a_2(t,x)$ is satisfied if, for instance,

$$\lambda = C^{-1}\|a\|_{L^\infty(Q)} + \|(u_b)_x\|_{L^\infty(\Omega)}$$

which implies the result.□

We point out that although the application of the present version of the comparison principle given at the above theorem requires condition (3.7), the *a priori* estimate (3.8) can be obtained without passing by a comparison principle (see, for instance formula (13) of Benilan and Wittbold [3]) and so the boundedness of u remains true also when $(u_b)_x \neq 0$.

4. On the free boundary. In this section we shall consider both thermal regimes at the base. In the first case the bed is assumed to be cold (below melting point). No sliding is prescribed (i.e. $u_b \equiv 0$) and the pure diffusive case is analysed. Then we assume the ice sheet to be warm-based; the bed is then temperate and sliding is prescribed (i.e. $u_b = u_b(t,x)$). Here we are not concerned with the switching mechanism between cold-temperate dynamics (results in that direction can be found in Fowler and Schiavi [25] and Díaz and Schiavi [18]). Our aim is to describe qualitatively the behaviour of the free boundaries by means of *a priori* estimates on the support of the solution.

4.1. The no slip condition (pure diffusive case): Existence of the free boundary and the waiting time property. In this section we shall prove the existence of a non empty null set

$$N(h(t,\cdot)) := \{(t,x) \in \{t\} \times \Omega /\, h(t,x) = 0\}$$

for the (unique) solution $h(t,x)$ of problem

$$\left\{\begin{array}{rclll} h_t - \mu\phi(\psi(h)_x)_x + \beta(h) & \ni & a(t,x) & \text{in} & Q, \\ h(t,x) & = & 0 & \text{on} & \Sigma, \\ h(0,x) & = & h_0(x) & \text{on} & \Omega, \end{array}\right. \tag{4.1}$$

which can be deduced from the general formulation (GF) written in terms of the original variable h and of the functions ϕ and ψ introduced before. Assuming extra regularity of the solution (i.e, $h \in C(\bar{Q})$), we are able to analyse a great number of geophysical fenomenous related to location and evolution of the free boundary and associated with the behaviour of the function $a(t,x)$.

We shall now deal with the existence and location of the free boundary defined by problem (4.1). To show the existence of a free boundary as well as to estimate locally its location, we will use a technique based on the comparison result of section (3.1) and consisting of the construction of appropriated local super-sub solutions having compact support. We define, $\forall \epsilon > 0$ the set

$$N_\epsilon(a(t,\cdot)) := \{(t,x) \in \{t\} \times \Omega /\, a(t,x) \leq -\epsilon\} \tag{4.2}$$

and also $S_\epsilon(a(t,.)) = Q \setminus N_\epsilon(a(t,\cdot)$. Then we have:

THEOREM 4.1. *Let $h \in C(\bar{Q})$, $h \geq 0$ be a solution of (4.1) and let $\epsilon > 0$ (a small real positive number) such that the set $N_\epsilon(a(t,.))$ is not empty. Then there exists $T_0 \geq 0$ such that $\forall t \geq T_0$ we have*

$$N(h(t,\cdot)) \supset \{(t_0,x_0) \in N_\epsilon(a(t_0,.)) /\, d(x_0, S_\epsilon(a(t,.)) \geq R\}.$$

Proof. The proof is based on an original idea of Evans and Knerr [22] which applies when $n = 1$ and $a(t,x) \equiv 0$. See also Díaz and Hernández [16], for its adaptation to the case $n > 1$. In our multivalued case, with $n > 1$ but $a(t,x) \not\equiv 0$, we argue as follows. We consider the set $N_\epsilon(a(t,.))$ and define the function

$$\tilde{h}(t,x) = \psi^{-1}\left(\eta(|x - x_0|) + \psi(U(t))\right)$$

where

$$\eta(r) = c\, r^{\frac{p}{p-1}}, \qquad c = \frac{p-1}{p}\left(\frac{\epsilon}{2}\right)^{\frac{1}{p-1}}, \tag{4.3}$$

and $U(t)$ is the (unique) solution of the initial value problem

$$\left\{\begin{array}{rcl} U_t + \frac{1}{2}\beta(U) & \ni & -\frac{\epsilon}{2}, \\ U(0) & = & ||h_0||_{L^\infty(\Omega)}. \end{array}\right. \tag{4.4}$$

It is easy to see that $U(t) = [-\frac{\epsilon}{2}t + ||h_0||_{L^\infty}]^+$ whence

$$U(t) \equiv 0, \qquad \forall t \geq T_0 = \frac{2}{\epsilon}||h_0||_{L^\infty(\Omega)}$$

On the other hand, as by construction $\phi(\psi(\tilde{h})_x)_x = \phi(\eta_x)_x = \epsilon/2$, we have (in $N_\epsilon(a(t,.))$)

$$\tilde{h}_t - \mu\phi(\psi(\tilde{h})_x)_x + \beta(\tilde{h}) \equiv$$

$$\equiv \frac{d}{dt}\left[\psi^{-1}\left(\eta(|x-x_0|) + \psi(U(t))\right)\right] - \mu\phi(\eta_x(|x-x_0|))_x + \beta\left(\psi^{-1}\left(\eta(|x-x_0|) + \psi(U(t))\right)\right) \supseteq$$

$$\supseteq \frac{\psi'(U)}{\psi'\left(\psi^{-1}\left(\eta(|x-x_0|) + \psi(U(t))\right)\right)}\frac{dU}{dt} - \mu\phi(\eta_x(|x-x_0|))_x + \frac{1}{2}\beta(\psi^{-1}(\eta(|x-x_0|))) + \frac{1}{2}\beta(U) \supseteq$$

$$\supseteq U_t + \frac{1}{2}\beta(U) - \mu\phi(\eta_x(|x-x_0|))_x + \frac{1}{2}\beta\left(\psi^{-1}(\eta(|x-x_0|))\right) \ni -\epsilon \geq a(t,x).$$

Using the Comparison Principle written in terms of the original variable h, the following estimate holds (see Benilan and Wittbold [3])

$$||h||_{L^\infty(Q)} \leq ||h_0||_{L^\infty(\Omega)} + \int_0^t ||a||_{L^\infty(\Omega)} = M(t).$$

Then

$$||h||_{L^\infty(Q)} \leq M(t) \leq \tilde{h}(t,\cdot), \qquad \text{on } N_\epsilon(a(t,\cdot))$$

iff $\psi^{-1}\left(\eta((|x-x_0|)) + \psi(U(t))\right) \geq M(t)$, i.e.; $\eta(|x-x_0|) + \psi(U(t)) \geq \psi(M(t))$. In particular this is true if $c\,|x-x_0|^{\frac{p}{p-1}} \geq \psi(M(t))$; by (4.3) the above reads

$$|x-x_0| \geq \frac{\psi(M(T))^{\frac{p-1}{p}}}{\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{\epsilon}{2}\right)^{\frac{1}{p}}} = R \tag{4.5}$$

and (4.5) implies that $\tilde{h} \geq h$ on $\partial N_\epsilon(a(t,.))$. At $t = 0$ we use the monotonicity of ψ^{-1}:

$$\tilde{h}(0,x) = \psi^{-1}\left(\eta(|x-x_0|) + \psi(U(0))\right) = \psi^{-1}\left(\eta(|x-x_0|) + \psi(||h_0||_{L^\infty})\right) \geq$$

$$\geq \psi^{-1}\left(\psi(||h_0||_{L^\infty})\right) = ||h_0||_{L^\infty} \geq h_0(x) \geq 0.$$

Summarizing we have, that if $(t,x) \in N_\epsilon(a(t,\cdot))$ is such that $|x-x_0| \geq R$ then

$$h_t - \mu\phi(\psi(h)_x)_x + \beta(h) \ni a \le \inf\left(\tilde{h}_t - \mu\phi(\psi(\tilde{h})_x)_x + \beta(\tilde{h})\right) \qquad \text{in} \quad N_\epsilon(a(t,\cdot)),$$

$$h(t,x) \le \tilde{h}(t,x) \qquad \text{on} \quad \partial N_\epsilon(a(t,\cdot)),$$

$$h_0(x) \le \tilde{h}(0,x) \qquad \text{on} \quad N_\epsilon(a(0,\cdot)).$$

It follows from comparison result (theorem 3.2) that

$$0 \le h(t,x) \le \tilde{h}(t,x), \qquad \text{in} \quad N_\epsilon(a(t,.)),$$

and we end up observing that $h(t,x_0) = 0 \ \forall t \ge T_0 = \frac{2}{\epsilon}||h_0||_{L^\infty}$ and x_0 satisfies inequality (4.5), i.e., $(t,x_0) \in \{N_\epsilon(a(t,.))/|x-x_0| \ge R\}$.□

We shall now analyze the so called waiting time property. As discussed in Fowler [24] p. 95, the slope of the surface is singular in advance but finite in retreat. This distinction causes the degenerate diffusion equation above to have waiting-time behaviour, because following a retreat, the margin slope must rebuild itself before another advance it possible. The following property applies if the initial data is sufficently *flat* in the *ablation region.*

THEOREM 4.2. *let $h \in C(\bar{Q})$, $h \ge 0$ be a solution of problem (4.1). Define $\delta = \eta^{-1}(\psi(M))$ and $B_\delta^+(x_0) = \{x \in \Omega/ \ x \in [x_0, x_0+\delta)\}$ being $M = ||h||_{L^\infty(Q)}$, $x_0 = S_+(0)$, $\tilde{c} = (\frac{p-1}{p})\epsilon^{\frac{1}{p-1}}$ and $\eta(|x-x_0|) = \tilde{c}\,|x-x_0|^{\frac{p}{p-1}}$. Assume that there exists $T^* > 0$ such that $a(t,x) \le -\epsilon$ a.e. $x \in B_\delta^+(x_0)$ and $t \in (0,T^*)$. If $x_0 \in \Omega$ satisfies $0 \le h_0(x_0) \le \psi^{-1}(\eta(|x-x_0|))$ then*

$$\exists\, t^*, \quad 0 < t^* \le T^*, \quad \textit{such that} \quad S_+(0) = S_+(t), \quad \forall t \in (0,t^*).$$

Proof. We define the function

$$\tilde{h}(x) := \psi^{-1}(\eta(|x-x_0|)), \qquad \text{in} \quad B_\delta^+(x_0) \times (0,T^*).$$

Then

$$h_t - \mu\phi(\psi(h)_x)_x + \beta(h) \ni a \le -\epsilon \le \inf\left(\tilde{h}_t - \mu\phi(\psi(\tilde{h})_x)_x + \beta(\tilde{h})\right), \quad \text{in } B_\delta^+(x_0)\times(0,T^*).$$

On $\partial B_\delta^+(x_0) \times [0,t^*]$ we have to verify that $h \le M \le \tilde{h} = \psi^{-1}(\eta)$ and this is true if and only if $\psi(M) \le \eta = \tilde{c}|x-x_0|^{\frac{p}{p-1}}$. On ∂B_δ^+ this reads $\psi(M) \le \tilde{c}\delta^{\frac{p}{p-1}}$. Using that $\delta = \eta^{-1}(\psi(M))$ then

$$h \le M \le \tilde{h} \quad \Longleftrightarrow \quad \psi(M) \le \tilde{c}[\eta^{-1}(\psi(M))]^{\frac{p}{p-1}} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \left[\frac{\psi(M)}{\tilde{c}}\right]^{\frac{p-1}{p}} \le \eta^{-1}(\psi(M)) \quad \Longleftrightarrow \quad \eta\left(\left[\frac{\psi(M)}{\tilde{c}}\right]^{\frac{p-1}{p}}\right) \le \psi(M),$$

and this is always verified as can be deduced by applying the definition of the function η. In conclusion we have

$$h_t - \mu\phi(\psi(h)_x)_x + \beta(h) \ni a \le \inf\left(\tilde{h}_t - \mu\phi(\psi(\tilde{h})_x)_x + \beta(\tilde{h})\right), \text{ in } B_\delta^+(x_0) \times (0, t^*)$$

$$h(x_0, 0) = h_0(x_0) \le \tilde{h}(x) = \psi^{-1}(\eta(|x - x_0|)), \quad \text{on } B_\delta^+(x_0)$$

$$h(t, x) \le M \le \tilde{h}(x), \quad \text{on } \partial B_\delta^+(x_0) \times (0, t^*).$$

Then the comparison result gives that $0 \le h(t,x) \le \tilde{h}(x)$ and so $h(t, x_0) \equiv 0$, $\forall t \in (0, t^*)$.□

4.2. The slip condition (diffusive-convective case): Existence of the free boundary and the waiting time property. In this section we shall consider the general formulation (GF) assuming that $u_b \not\equiv 0$. The presence of the convection term makes the method of super and subsolutions very hard to apply, except for very special data. Thus, in order to prove the existence of the free boundary we shall use a different technique, called the *energy method*. It has been developed by different authors in the last twenty years for the study of nonlinear problems lacking the maximum principle (see, for instance, the monograph of Antontsev, Díaz and Shmarev [2]). In fact, although this energy method can be applied in different ways, we shall follows the ideas introduced in Díaz and Galiano [15] in order to apply the method to some Fluid Dynamics problems. We start by pointing out that the equation of problem (3.5) can be written in terms of a non-conservative transport multivalued equation in form

$$b(u)_t + u_b b(u)_x - \mu\phi(u_x)_x + (u_b)_x b(u) + \beta(u) \ni a(t,x), \qquad \text{in} \quad Q. \tag{4.6}$$

In this way, the equation involves the material derivative $b(u)_t + u_b b(u)_x$ which can be associate to a *virtual non-Newtonian fluid with a reactive term* $(u_b)_x b(u) + \beta(u)$. We shall prove the existence of the free boundary in terms of the, so called, *finite speed of propagation* near a given point x_0.

In the next results we shall assume that u_b is a globally Lipschitz continuous function. So, we can define the characteristics of the associate flow by

$$\begin{cases} \dfrac{d}{dt} X(t,x) &= u_b(t, X(t,x)), \qquad \text{on} \quad (0,T), \\ X(0,x) &= x. \end{cases} \tag{4.7}$$

As usual in Continuum Mechanics, given a ball $B_\rho(x_0) = \{x \in I\!R : |x - x_0| \le \rho\}$ we denote the transformed set by

$$B_\rho(x_0)_t = \{\, y \in I\!R : y = X(t,x) \quad \text{for some} \quad x \in B_\rho(x_0)\}.$$

THEOREM 4.3. *Assume b, ϕ, β, a and u_0 as in Section 3. Let u_b be a globally Lipschitz continuous function on Q. For $\epsilon \ge 0$ let $N_\epsilon(a(t,\cdot)) := \{(t,x) \in \{t\} \times \Omega / \, a(t,x) \le -\epsilon\}$. Assume also that $\epsilon = 0$ if $m(p-1) > 1$ and $\epsilon > 0$ if $m(p-1) \le 1$. Let $u_0 = 0$ on a ball $B_{\rho_0}(x_0)$ for some x_0 such that $(t, B_{\rho_T}(x_0)) \subset N_\epsilon(a(t,\cdot))$ for any $t \in [0,T]$ and some $L \ge \rho_0$. Then there exists a $T_\epsilon \in (0,T]$ and a function $\rho : [0, T_\epsilon] \to [0, \rho_0]$ such that $u(t,x) = 0$ a.e. $x \in B_{\rho(t)}(x_0)$ for any $t \in [0, T_\epsilon]$.*

Proof. We introduce the change of variable $b(w(t,x)) = b(u(t,x))e^{\lambda t}$. Then, it is easy to see that w satisfies the equation

$$\begin{aligned}(4.8)\qquad & b(w)_t + u_b b(w)_x - \mu e^{\lambda t(1-(p-1)m)}\phi(w_x)_x + [(u_b)_x + \lambda]b(w) \\ & +\beta(w) \ni a(t,x)e^{\lambda t}, \ \text{ in } \ Q.\end{aligned}$$

So, by taking $\lambda > 2C$ with $C = \|(u_b)_x\|_{L^\infty(Q)}$ (which is finite since u_b is a globally Lipschitz continuous function) we have that $[(u_b)_x + \lambda] \geq C > 0$. Now, we shall argue as in Díaz and Galiano [15]. By multiplying (formally) by w (i.e. by some arguments of regularization, localization and passing to the limit as in Díaz and Veron [19]) we get that if $\rho \leq L$ then

$$\int_{B_\rho(x_0)_t} \frac{\partial}{\partial t}\Psi(w)dx + \int_{B_\rho(x_0)_t} u_b\Psi(w)_x dx + \mu e^{\lambda t(1-(p-1)m)} \int_{B_\rho(x_0)_t} |w_x|^p \, dx \leq$$

$$\leq \mu e^{\lambda t(1-(p-1)m)} w(t,\cdot)\,|w_x(t,\cdot)|^{p-1}\, w_x(t,\cdot)\,|_{\partial B_\rho(x_0)_t} - \epsilon \int_{B_\rho(x_0)_t} w dx$$

where

$$(4.9)\qquad \Psi(w) := wb(w) - \int_0^w b(s)ds$$

and we used that $w \geq 0$ and that $\beta(w)w = \{0\}$. Now, by using the Reynolds Transport Lemma

$$\int_{B_\rho(x_0)_t} \frac{\partial}{\partial t}\Psi(w) + \int_{B_\rho(x_0)_t} u_b\Psi(w)_x = \frac{d}{dt}\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy.$$

Thus, integrating in $(0,t)$ and using the information on u_0 we get that

$$\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy + C_1 \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds \leq$$

$$(4.10)\qquad \leq C_2 \int_0^t w(s,\cdot) \left| |w_x(s,\cdot)|^{p-1} w_x(s,\cdot)\,|_{\partial B_\rho(x_0)_t} \right| ds - \epsilon \int_0^t \int_{B_\rho(x_0)_t} wdxds$$

with

$$C_1 = \mu \min_{t\in[0,T]} e^{\lambda t(1-(p-1)m)}, \qquad C_2 = \mu \max_{t\in[0,T]} e^{\lambda t(1-(p-1)m)}.$$

Assume now, for the moment, that $1 < (p-1)m$ and $\epsilon = 0$. Then we define the energies

$$(4.11)\qquad B(t,\rho) = \sup_{0\leq s\leq t} \int_{B_\rho(x_0)s} \Psi(w(s,y))dy, \quad E(t,\rho) = \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds.$$

Using Hölder inequality and the interpolation-trace inequality of Díaz and Veron [19]) we get that

$$B+E \leq K\left(\frac{\partial E}{\partial \rho}\right)^{\omega} \tag{4.12}$$

for some positive constant K and some $\omega > 1$ and the result follows in a standard way (see, e.g. Díaz and Veron [19], or Antontsev, Díaz and Shmarev [2]). In the case $1 \geq (p-1)m$ and $\epsilon > 0$ we pass the term $\epsilon \int_0^t \int_{B_\rho(x_0)_t} w dx ds$ to the other side of the inequality (4.10) and introduce another additional energy fuction defined as

$$C(t,\rho) = \int_0^t \int_{B_\rho(x_0)_t} |w|\, dyds$$

(remember that $|w| = w$). Then we can apply Theorem 1 of Antontsev, Díaz and Shmarev [1], with $\lambda = 0$ since the interpolation-trace inequality (2.6) of that paper applies also to the limit case $\lambda = 0$. Then we arrive to an inequality of the form

$$E+C \leq K\left(\frac{\partial (E+C)}{\partial \rho}\right)^{\omega} \tag{4.13}$$

for some positive constant K and some $\omega > 1$ and the conclusion holds. □

Remark 4.1. *We point out that due to the presence of the convective term and concrete exponents involved in equation (4.6) the statement of the parabolic part of Díaz and Veron [19], is not directly applicable and so the reason of the characteristic transformation argument. Notice, also, that in contrast with the case $u_b = 0$ now it may occurs that $T_\epsilon < T$ for any $\epsilon \geq 0$ and that the energy method allows the consideration of the case $\epsilon = 0$ when $m(p-1) > 1$. Moreover, any estimate of the function $\rho(t)$ gives, automatically, an estimate on the location of the free boundary. Finally, we indicate that it is possible to get global consequences of the above result by estimating (globally) the energies introduced in (4.11) (for some related arguments see, e.g. Díaz and Veron [19] or Antontsev, Díaz and Shmarev [2]).*

The waiting time property can be also studied by energy methods once it is reformulated in terms of the characteristics associate to u_b.

Theorem 4.4. *Assume b, ϕ, β, a, u_b, ϵ, $N_\epsilon(a(t,\cdot))$ and x_0 as in the previous Theorem but now with $L > \rho_0$. Let $u_0(x) = 0$ such that $B_{\rho_0}(x_0)$ and satisfies also that*

$$\int_{B_\rho(x_0)_t} \Psi(u_0(y))dy \leq \theta[(\rho-\rho_0)^+]^{\omega/(\omega-1)}, \quad \textit{for any} \quad \rho_0 \leq \rho \leq L \tag{4.14}$$

for some $\theta > 0$ small enough and some $L > \rho_0$, where Ψ is defined by (4.9) and $\omega > 1$ is the exponent given in (4.12) or (4.13). Then, there exists a $T_0 \in (0,T]$ such that $u(t,x) = 0$ a.e. $x \in B_{\rho_0}(x_0)_t$ for any $t \in [0,T_0]$, where $B_{\rho_0}(x_0)_t = \{y \in I\!R : y = X(t,x)$ for some $x \in B_{\rho_0}(x_0)\}$ with $X(t,x)$ the characteristics defined by (4.7).

Proof. It follows the same arguments than Antontsev, Díaz and Shmarev [1], but adapted to our framework. So, the integration by parts formulae (4.10) must be replaced by

$$
\begin{aligned}
&\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy + C_1 \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds \leq \\
&\leq \int_0^t w(s,\cdot) \left| |w_x(s,\cdot)|^{p-1} w_x(s,\cdot) \,|_{\partial B_\rho(x_0)_t} \right| ds - \\
&\epsilon \int_{B_\rho(x_0)_t} w dx + \int_{B_\rho(x_0)_t} \Psi(u_0(y))dy.
\end{aligned} \tag{4.15}
$$

In particular, inequality (4.12) becomes non homogeneous

$$
B + E \leq K \left(\frac{\partial E}{\partial \rho} \right)^\omega + \theta(\rho - \rho_0)_+^{\omega/(\omega-1)}
$$

and the conclusion holds thanks to a technical lemma (see, e.g., Lemma 1 of Antontsev, Díaz and Shmarev [1]). □

REMARK 4.2. *We point out that if $u_b = 0$ then the characteristics are vertical lines and the conclusion of the above result coincides with the waiting time property.*

5. Numerical solution. This section is devoted to the numerical resolution of the ice sheet moving boundary problem stated in the multivalued formulation (MF) given in (3.2). We first introduce the total derivative notation in conservative form

$$
\frac{D^* h}{Dt} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_b h),
$$

so that the complementary formulation (CF) given en (3.3) can be written in the conservative form as:

$$
\begin{cases}
\dfrac{D^* h}{Dt} - \left(\dfrac{h^{n+2}}{n+2} |h_x|^{n-1} h_x \right)_x - a \geq 0 & \text{in} \quad Q, \\
\left[\dfrac{D^* h}{Dt} - \left(\dfrac{h^{n+2}}{n+2} |h_x|^{n-1} h_x \right)_x - a \right] h = 0 & \text{in} \quad Q, \\
h \geq 0 & \text{in} \quad Q, \\
h = 0 & \text{on} \quad \Sigma, \\
h = h_0(x) & \text{on} \quad \Omega.
\end{cases} \tag{5.1}
$$

In a previous paper a first attempt to solve the previous problem has been performed by using a fixed point method to deal with the nonlinear diffusion term. In Calvo, Durany and Vázquez [11] the linearization process was based on freezing the nonlinear diffusion term at each step of the algorithm. This method has been already combined with the algorithm proposed in [10] to approximate ice sheet temperature distribution in order to solve a temperature profile coupled problem (see Calvo [9] , for details). This approach to the profile problem requires extremely small time steps and consequently leads to high computing times to obtain the stationary solution.

In this paper to overcome the drawbacks of the previous numerical approach we use the complementary formulation (CF) introduced in section 3 to formulate the nonlinear diffusion term by means a monotone operator. Let $n = 3$ (hence $m = 2(n+1)/n = 8/3 > 1$ and $p = n+1 = 4$). Using (3.4) we introduce the new variable u defined as

$$u(t,x) = h^{8/3}(t,x) \tag{5.2}$$

so that the problem (5.1) can be written in terms of u as

$$\left\{ \begin{array}{lll} \dfrac{D^*}{Dt}\left(u^{3/8}\right) - \mu\left(|u_x|^2 u_x\right)_x - a \geq 0 & \text{in} & Q, \\ \left[\dfrac{D^*}{Dt}\left(u^{3/8}\right) - \mu\left(|u_x|^2 u_x\right)_x - a\right] u = 0 & \text{in} & Q, \\ u \geq 0 & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u = u_0(x) = h_0^{8/3}(x) & \text{on} & \Omega, \end{array} \right. \tag{5.3}$$

where the constant μ takes the value $\mu = \frac{(3/8)^3}{5}$. The previous change of unknown in the equations allows to introduce a maximal monotone formulation of the new nonlinear diffusion term but gives rise to a new nonlinear convection term.

5.1. Time semidiscretization. Let T, M and Δt fixed, positive real numbers such that $T = M\Delta t$. Problem (5.3) has been discretized in time using the scheme of characteristics with time step Δt (as in previous works of Calvo, Durany and Vázquez, [10] and [11] in the glaciology setting). In short, this upwinded scheme is based on the approximation of the total derivative (see Pironneau [29] , for linear convection-diffusion equations). Thus, in our particular case with nonlinear convection, for $m = 0, 1, \ldots, M$, $(M = T/\Delta t)$, we consider the approximation:

$$\frac{D^*}{Dt}(u^{3/8})((m+1)\Delta t, x) \approx \frac{(u^{m+1})^{3/8}(x) - J^m(x)\,(u^m)^{3/8}(\chi^m(x))}{\Delta t} \tag{5.4}$$

where

$$u^{m+1}(x) = u((m+1)\Delta t, x), \qquad \text{in} \quad \Omega, \tag{5.5}$$

and $J^m(x)$ is obtained by numerical quadrature in the expression

$$J^m(x) = J(t^{m+1}, x; t^m) = 1 - \int_{t^m}^{t^{m+1}} (u_b(\tau, \chi(x, t^{m+1}; \tau)))_x \, d\tau,$$

where J is the jacobian associated to the change of variable defined by the map $x \to \chi(t, x; \tau)$. The presence of J arises from the application of characteristics method when the convection is written in conservative form (see Bercovier, Pironneau and Sastri [4] for details). The value $\chi^m(x)$ is given by $\chi^m(x) = \chi((m+1)\Delta t, x; m\Delta t)$ being χ the solution of the final value problem

$$\left\{ \begin{array}{rcl} \dfrac{d\chi(t,x;s)}{ds} & = & u_b(s,\chi(x,t;s)), \\ \chi(t,x;t) & = & x. \end{array} \right. \tag{5.6}$$

The next step consists of the substitution of the approximation (5.4) in (5.3) to obtain the following sequence of nonlinear elliptic complementarity problems:

For $m = 0,1,2,\ldots,M$, *find* u^{m+1} *such that:*

$$\left\{ \begin{array}{r} \dfrac{(u^{m+1})^{3/8} - J^m((u^m)^{3/8} \circ \chi^m)}{\Delta t} - \mu \dfrac{\partial}{\partial x}\left(|u_x^{m+1}|^2 u_x^{m+1}\right) - a^{m+1} \geq 0 \quad \text{in } \Omega \\ u^{m+1} \geq 0 \quad \text{in } \Omega \\ \left[\dfrac{(u^{m+1})^{3/8} - J^m((u^m)^{3/8} \circ \chi^m)}{\Delta t} - \mu \dfrac{\partial}{\partial x}\left(|u_x^{m+1}|^2 u_x^{m+1}\right) - a^{m+1}\right] u^{m+1} = 0 \quad \text{in } \Omega \\ u^{m+1} = 0 \quad \text{in } \partial\Omega \\ u^0(x) = (h_0)^{8/3}(x) \quad \text{in } \Omega \end{array} \right. \tag{5.7}$$

where $a^{m+1}(x) = a((m+1)\Delta t, x)$ and ` $\circ$ ´ denotes the composition symbol.

5.2. Spatial discretization. In order to solve the nonlinear complementarity problems (5.7), for each value of m we initialize $u^{m+1,0} \in K = \{\varphi \in W_0^{1,4}(\Omega) \,/\, \varphi \geq 0 \text{ a.e. in } \Omega\}$ so that in the step $k+1$ we solve the variational inequality:

Find $u^{m+1,k+1} \in K$ *such that*

$$\begin{array}{l} \dfrac{1}{\Delta t}\displaystyle\int_\Omega (u^{m+1,k+1})^{3/8}\,(\varphi - u^{m+1,k+1})\,dx + \\ \mu \displaystyle\int_\Omega |\, u_x^{m+1,k+1}\,|^2\, u_x^{m+1,k+1}(\varphi - u^{m+1,k+1})_x dx \geq \\ \dfrac{1}{\Delta t}\displaystyle\int_\Omega J^m\,((u^m)^{3/8} \circ \chi^m)(\varphi - u^{m+1,k+1})\,dx + \\ \displaystyle\int_\Omega a^{m+1}\,(\varphi - u^{m+1,k+1})dx, \qquad \forall \varphi \in K. \end{array} \tag{5.8}$$

In order to solve the nonlinear problem associated to the inequality constraint on the solution, the algorithm proposed in Bermúdez and Moreno [6], is applied to the variational inequality (5.8) expressed in terms of the indicatrix function, I_K, of the convex K in the form:

Find $u^{m+1,k+1} \in W_0^{1,4}(\Omega)$ *such that*

$$
\begin{aligned}
&\frac{1}{\Delta t}\int_\Omega (u^{m+1,k+1})^{3/8}(\varphi - u^{m+1,k+1}))dx+\\
&\mu\int_\Omega \mid u_x^{m+1,k+1}\mid^2 u_x^{m+1,k+1}(\varphi - u^{m+1,k+1})_x dx\\
&+I_K(\varphi) - I_K(u^{m+1,k+1}) \geq \frac{1}{\Delta t}\int_\Omega J^m\left((u^m)^{3/8}\circ\chi^m\right)(\varphi - u^{m+1,k+1})dx+\\
&\int_\Omega a^{m+1}(\varphi - u^{m+1,k+1})dx, \qquad \forall \varphi \in W_0^{1,4}(\Omega).
\end{aligned} \tag{5.9}
$$

Moreover, the use of subdifferential calculus for the convex function I_K leads to the equivalent formulation

$$
\xi_1^{m+1,k+1} = -(\mathcal{A}(u^{m+1,k+1}) - f^m) \in \partial I_K(u^{m+1,k+1}) \tag{5.10}
$$

where $\partial I_K(u)$ denotes the subdifferential of I_K at the point u (see Brezis [8], for more details), the operator $\mathcal{A} : W_0^{1,4}(\Omega) \to W^{-1,4/3}(\Omega)$ is defined by

$$
<\mathcal{A}(\varphi), \psi>= \frac{1}{\Delta t}\int_\Omega \varphi^{3/8}\psi dx + \mu\int_\Omega \mid \varphi_x\mid^2 \varphi_x\psi_x dx,
$$

and the element $f^m \in W^{-1,4/3}(\Omega)$ is defined by

$$
< f^m, \psi >= \int_\Omega a^{m+1}\psi dx + \frac{1}{\Delta t}\int_\Omega J^m((u^m)^{3/8}\circ\chi^m)\psi dx.
$$

Therefore equation (5.10) is equivalent to the problem:

Find $u^{m+1,k+1} \in W_0^{1,4}(\Omega)$ *such that*

$$
\begin{aligned}
&\frac{1}{\Delta t}\int_\Omega (u^{m+1,k+1})^{3/8}\ \psi\ dx \int_\Omega \xi_1^{m+1,k+1}\ \psi\ dx + \mu\int_\Omega \xi_2^{m+1,k+1}\psi_x dx -\\
&\frac{1}{\Delta t}\int_\Omega J^m((u^m)^{3/8}\circ\chi^m)\psi dx = \int_\Omega a^{m+1}\ \psi\ dx, \qquad \forall\psi \in W_0^{1,4}(\Omega)
\end{aligned} \tag{5.11}
$$

$$
\xi_1^{m+1,k+1} \in \partial I_K\left[u^{m+1,k+1}\right] \tag{5.12}
$$

$$
\xi_2^{m+1,k+1} = \Lambda\left(\frac{\partial u^{m+1,k+1}}{\partial x}\right) \tag{5.13}
$$

where $\Lambda(v) = \mid v\mid^2 v = v^3$.

The application of method proposed in Bermúdez and Moreno [6], to solve the nonlinear problem (5.11)-(5.13) introduces the new unknowns $q_1^{m+1,k+1}$ and $q_2^{m+1,k+1}$ defined by

$$q_1^{m+1,k+1} \in \partial I_K \left[u^{m+1,k+1}\right] - \omega_1 u^{m+1,k+1} \tag{5.14}$$

$$q_2^{m+1,k+1} = \Lambda \left(\frac{\partial u^{m+1,k+1}}{\partial x}\right) - \omega_2 \frac{\partial u^{m+1,k+1}}{\partial x} \tag{5.15}$$

in terms of the positive parameters ω_1 and ω_2. The equation (5.11) is then formulated as

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega (u^{m+1,k+1})^{3/8}\psi dx + \int_\Omega \left(q_1^{m+1,k+1} + \omega_1 u^{m+1,k+1}\right) \psi \, dx + \\ &\mu \int_\Omega \left(q_2^{m+1,k+1} + \omega_2 \frac{\partial u^{m+1,k+1}}{\partial x}\right) \frac{\partial \psi}{\partial x}\, dx \; = \\ &\int_\Omega a^{m+1} \; \psi \; dx \; + \frac{1}{\Delta t}\int_\Omega J^m \left((u^m)^{3/8} \circ \chi^m\right)\psi dx, \qquad \forall \psi \in W_0^{1,4}(\Omega). \end{aligned} \tag{5.16}$$

Since ∂I_K and Λ are maximal monotone operators, from Bermúdez and Moreno [6], the definitions (5.14) and (5.15) are characterized by the respective identities:

$$q_1^{m+1,k+1} = (\partial I_K)_{\lambda_1}^{\omega_1}\left[u^{m+1,k+1} + \lambda_1 q_1^{m+1,k+1}\right] \tag{5.17}$$

$$q_2^{m+1,k+1} = \Lambda_{\lambda_2}^{\omega_2}\left[\frac{\partial u^{m+1,k+1}}{\partial x} + \lambda_2 q_2^{m+1,k+1}\right] \tag{5.18}$$

where $(\partial I_K)_{\lambda_1}^{\omega_1}$ and $\Lambda_{\lambda_2}^{\omega_2}$ denote the Yosida approximations (see Brezis [8], for example) for the operators $(\partial I_K - \omega_1 I)$ and $(\Lambda - \omega_2 I)$ with positive parameters λ_1 and λ_2, respectively.

To discretize in space the equations (5.16)-(5.18) we consider piecewise linear Lagrange finite elements. Thus, for a given positive parameter h we build a uniform finite element mesh τ_h for Ω whose nodes are $x_i = (i-1)h$, $i = 1, \ldots, N+1$. Next we introduce the classical finite elements spaces and sets:

$$\begin{aligned} V_h &= \{\varphi_h \in C^0(\Omega) \; / \; \varphi_h \mid_E \in P_1, \quad \forall E \in \tau_h\} \\ V_{0h} &= \{\varphi_h \in V_h \; / \; \varphi_h \mid_{\partial\Omega} = 0\} \\ K_h &= \{\varphi_h \in V_{0h} \; / \; \varphi_h \geq 0 \quad \text{a.e. in}\, \Omega\} \end{aligned} \tag{5.19}$$

where E denotes a standard finite element interval. The resulting discretized problem can be written as follows:

Find $u_h^{m+1,k+1} \in K_h$ *such that*

$$\frac{1}{\Delta t}\int_\Omega (u_h^{m+1,k+1})^{3/8}\; \psi_h \; dx \; + \; \omega_1 \int_\Omega u_h^{m+1,k+1} \; \psi_h \; dx \; +$$

$$
\begin{aligned}
&\mu\omega_2 \int_\Omega \frac{\partial u_h^{m+1,k+1}}{\partial x}\,\frac{\partial \psi_h}{\partial x}\,dx \;=\; \int_\Omega a_h^{m+1}\,\psi_h\,dx\;+\\
&\frac{1}{\Delta t}\int_\Omega J^m\left((u_h^m)^{3/8}\circ\chi^m\right)\psi_h\,dx - \int_\Omega q_{1,h}^{m+1,k+1}\,\psi_h\,dx\,-\\
&\mu\int_\Omega q_{2,h}^{m+1,k+1}\,\frac{\partial\psi_h}{\partial x}\,dx, \qquad \forall \psi_h\in V_{0h}.
\end{aligned} \tag{5.20}
$$

Thus, if we freeze the first term in (5.20) at each step of the inner loop in multipliers, the duality method for solving the discretized problem (5.20), (5.18) and (5.17) gives rise to the following algorithm :

Step 0 : Initialize $(u_h^{m+1,k+1})_0$ (equal to $u_h^{m+1,k}$, for example)

Step j : For a given$(u_h^{m+1,k+1})_j$, compute $(u_h^{m+1,k+1})_{j+1}\in V_{0h}$ by solving the linear problem:

$$
\begin{aligned}
&\omega_1\int_\Omega (u_h^{m+1,k+1})_{j+1}\psi_h\,dx + \mu\omega_2\int_\Omega \frac{\partial (u_h^{m+1,k+1})_{j+1}}{\partial x}\frac{\partial\psi_h}{\partial x}\,dx \;=\\
&-\frac{1}{\Delta t}\int_\Omega (u_h^{m+1,k+1})_j^{3/8}\,\psi_h\,dx \;-\int_\Omega (q_{1,h}^{m+1,k+1})_j\;\psi_h\,dx\\
&-\mu\int_\Omega (q_{2,h}^{m+1,k+1})_j\frac{\partial\psi_h}{\partial x}dx + \int_\Omega a_h^{m+1}\psi_h\,dx\\
&+\frac{1}{\Delta t}\int_\Omega J^m\left(((u_h^m)_j)^{3/8}\circ\chi^m\right)\psi_h\,dx, \qquad \forall\psi\in V_{0h}.
\end{aligned} \tag{5.21}
$$

The multipliers updating is performed by means of the equations:

$$
(q_{1,h}^{m+1,k+1})_{j+1} = (\partial I_K)_{\lambda_1}^{\omega_1}\left[(u_h^{m+1,k+1})_{j+1} + \lambda_1(q_{1,h}^{m+1,k+1})_j\right] \tag{5.22}
$$

$$
(q_{2,h}^{m+1,k+1})_{j+1} = \Lambda_{\lambda_2}^{\omega_2}\left[\frac{\partial}{\partial x}(u_h^{m+1,k+1})_{j+1} + \lambda_2(q_{2,h}^{m+1,k+1})_j\right] \tag{5.23}
$$

The convergence of the duality method is established in Bermúdez and Moreno [6] and Bermúdez [5] under the technical constraint $\lambda_i\omega_i = 0.5$, for i=1,2. For this choice of the parameters the Yosida approximation can be easily computed by

$$
(\partial I_K)_{\frac{1}{2\omega_1}}^{\omega_1}(r) = -2\omega_1\mid r\mid,
$$

$$
\Lambda_{\frac{1}{2\omega_2}}^{\omega_2}(r) = 2\,\Lambda_{\frac{1}{\omega_2}}(2r) - 2\,\omega_2\,r\,,
$$

where $\Lambda_\lambda(r) = (r-s)/\lambda$ being s the solution of the nonlinear equation $\lambda s^3 + s = r$, which has been solved for each r by using Cardano' s formulae.

5.3. Numerical results: comparison tests. One goal of this work is to propose a numerical solution method for the mathematical moving boundary model which governs the ice sheet profile in terms of a given accumulation-ablation ice rate and of a given sliding velocity. A further goal is to illustrate the qualitative properties which have been analysed in Section 4. For this, in the Section 5.2 we propose a numerical algorithm based on upwinding techniques for the material derivative, a maximal monotone operator for the non-linear diffusion term, finite elements and duality methods for the maximal monotone operator involved in several aspects of the problem.

As in previous work of Calvo, Durany and Vázquez [11], in order to validate the correct performance of our numerical approach, we have considered the Test 1 with closed form stationary solution which corresponds to a no sliding case (i.e. $u_b = 0$) and that is adapted from Paterson [28]. For a sufficiently large time interval $(0, T)$, the open set $\Omega = (-L, L)$ and the following piecewise constant accumulation-ablation function are defined:

$$a(x) = \begin{cases} a_1 & \text{if } 0 \leq |x| < R, \\ -a_2 & \text{if } R \leq |x| \leq L, \end{cases} \tag{5.24}$$

where $L > 1$, $a_1 > 0$, $a_2 > 0$ and $R \in (0,1)$ and the identity

$$a_1 R = a_2(1 - R) \tag{5.25}$$

holds. Thus, for $a_1 = 0.01$ and $a_2 = 0.03$, we have a steady state solution

$$\overline{\eta}(x) = \begin{cases} H\left[1 - \left(1 + \dfrac{a_1}{a_2}\right)^{1/3} \left(\dfrac{|x|}{L}\right)^{4/3}\right]^{3/8} & \text{if } |x| \leq R, \\ H\left(1 + \dfrac{a_2}{a_1}\right)^{1/8} \left(1 - \dfrac{|x|}{L}\right)^{1/2} & \text{if } R \leq |x| \leq 1, \\ 0 & \text{if } 1 \leq |x| \leq L, \end{cases} \tag{5.26}$$

where $H = (40\, a_1\, R)^{1/8}$ represents the thickness at $x = 0$ (*divide*).

In the Test 1 presented in this work the values $L = 2$ and $R = 0.75$ have been chosen so that $H = 0.86$. As initial condition for the evolutive problem we have considered

$$\eta_0(x) = \begin{cases} c\,(1 - |x|^{4/3})^{3/8} & \text{if } \ |x| \leq 1 \\ 0 & \text{if } \ 1 \leq |x| \leq 2\,, \end{cases} \tag{5.27}$$

with $c = 0.5$.

For the numerical solution a uniform finite element mesh with $N = 4001$ nodes and a time step equal to $\Delta t = 1$ has been taken.

In Fig. 5.1 the initial profile ($t = 0$), the computed solutions for $t = 5$ and $t = 75$, and the stationary exact solution for Test 1 (which matches the numerical solution for $t = 125$) are presented. Fig. 5.1 is obtained with the described duality method with $\omega_1 = 15$ and $\omega_2 = 30$. The computed results agree with the same test example solved with another numerical approach in Calvo, Durany and Vázquez [11]. Notice that in Fig. 5.1 for $t = 5$ the ice sheet is retreating. The retreat process occurs until $t = 25$ and then it expands with time until reaching the stationary solution given by expression (5.26). The initial contraction is mainly due to the fact that accumulation taking place at the center cannot balance the initial effect of ablation near the margins.

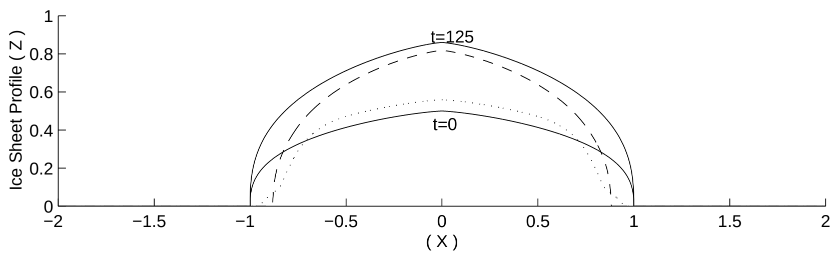

FIG. 5.1. *Computed numerical solution of Test 1.* $t = 0(-)$, $t = 5(\cdots)$, $t = 75(--)$, *Stationary* $(-)$.

Test 2 is proposed to verify the behaviour of the concave shape in the absence of convection when increasing the sliding velocity. In this case, we take $L = 2$ and $R = 1$ in (5.24), so that equation (5.25) is not verified. Thus, in Fig. 5.2 , 5.3 and 5.4 several examples are presented by considering the velocity field

$$u_b(t, x) = \begin{cases} C\,x^2 & \text{if} \quad x \geq 0, \\ -C\,x^2 & \text{if} \quad x < 0. \end{cases} \tag{5.28}$$

and the initial condition (5.27).

More precisely, Fig. 5.2 shows the results obtained for $t = 5$, $t = 50$ and $t = 90$ in the case $C = 0.005$, Fig. 5.3 shows the numerical solution for $t = 1$, $t = 5$ and $t = 9$ in the case $C = 0.05$, and Fig. 5.4 presents the computed profiles for $t = 1$, $t = 3$ and $t = 5$ when $C = 0.1$. The set of these figures illustrates the lack of concave profiles in the presence of enough convection ($C = 0.05$ or $C = 0.1$). The time instants have been chosen to present the profiles that most emphazise this phenomenon. The time step, the number of nodes and the parameters in Bermúdez and Moreno [6] algorithm are the same as those taken in Test 1.

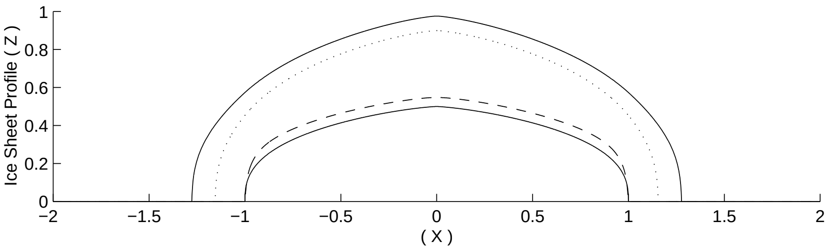

FIG. 5.2. *Numerical solution of Test 2 in the case* $C = 0.005$. $t = 0(-), t = 5(--)$, $t = 50(\cdots), t = 90(-)$.

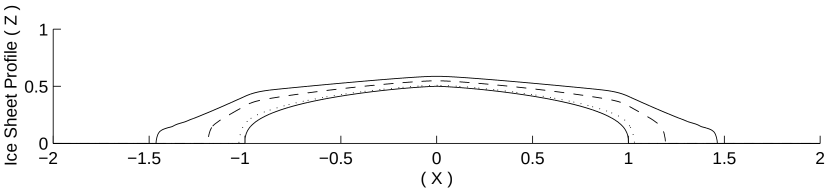

FIG. 5.3. *Numerical solution of Test 2 in the case* $C = 0.05$. $t = 0(-), t = 1(\cdots)$, $t = 5(--), t = 9(-)$.

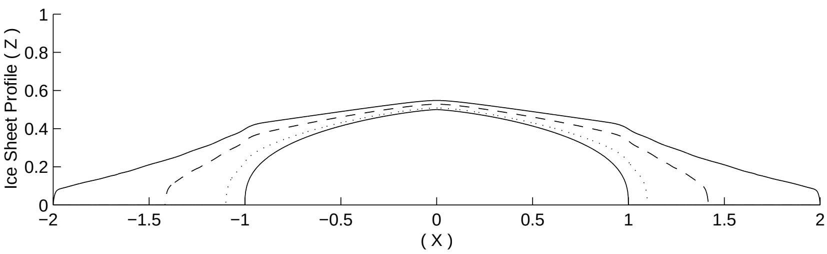

FIG. 5.4. *Numerical solution of Test 2 in the case* $C = 0.1$. $t = 0(-, \ t = 1(\cdots)$, $t = 3(--), t = 5(-)$.

Test 3 has been designed to illustrate the *waiting time property* discussed in Section 4. The idea is to show that when the initial condition of the problem has a sufficiently flat convex-concave shape (see Fig. 5.5) then the displacement of the initial free-boundary ($S_+(t_0)$, for example) starts after a certain time while for a concave initial condition (as (5.27), for example) this displacement occurs instantaniously.

To illustrate this fact the numerical solutions obtained from (5.27) and the alternative initial condition

$$\overline{\eta}_0(x) = \begin{cases} c\,(1-|x|^{4/3})^{3/8} & \text{if} \quad -0.75 \le x \le 0.75 \\ 16.77\,c\left(\dfrac{a_2}{2}\right)^{1/3}|x-1|^{4/3} & \text{if} \quad 0.75 \le x \le 1 \\ 16.77\,c\left(\dfrac{a_2}{2}\right)^{1/3}|x+1|^{4/3} & \text{if} \quad -1 \le x \le -0.75 \\ 0 & \text{otherwise} \end{cases} \tag{5.29}$$

are compared for different values of c. Thus, in Fig. 5.6 and Fig. 5.7 the moving boundaries are presented for the initial conditions (5.27) and (5.29) for $c = 0.5$ and $c = 0.75$, respectively.

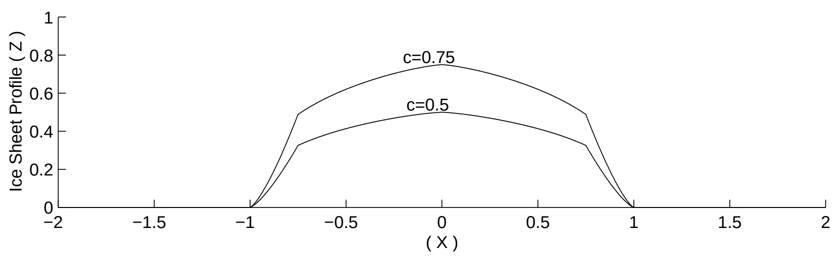

FIG. 5.5. *Convex-concave initial condition* $\overline{\eta}_0$ *for* $c = 0.5$ *and* $c = 0.75$.

In view that the ice mass associated to an initial condition such as (5.29) depends linearly on the parameter c, in Fig. 5.8 we compare the moving boundary evolution for different values of c in the absence of convection. In Fig. 5.9 the influence of convection is illustrated for the value $c = 0.75$.

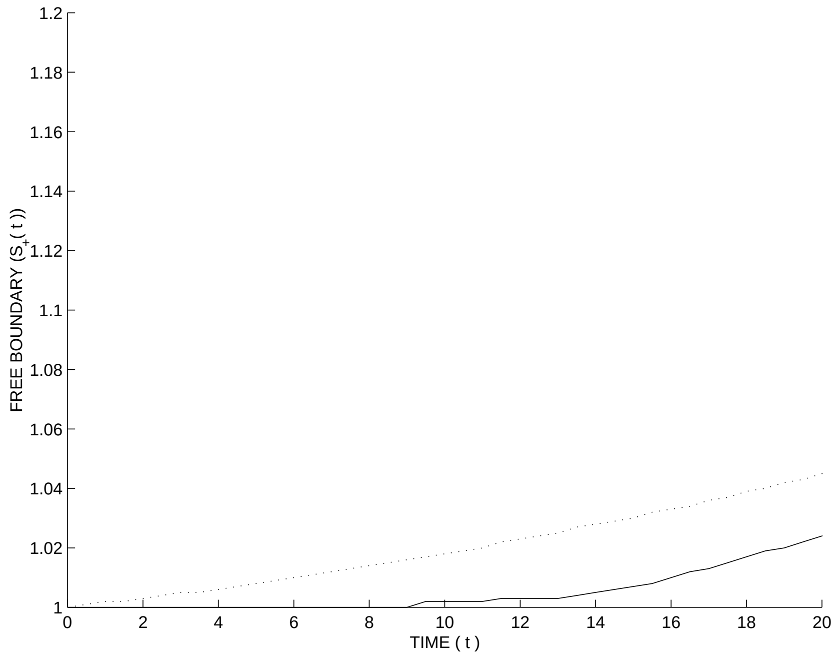

FIG. 5.6. *Moving boundary $S_+(t)$ in Test 3 with convex-concave $\overline{\eta}_0$ $(-)$ and purely concave η_0 $(\cdots)$ initial conditions for $c = 0.5$.*

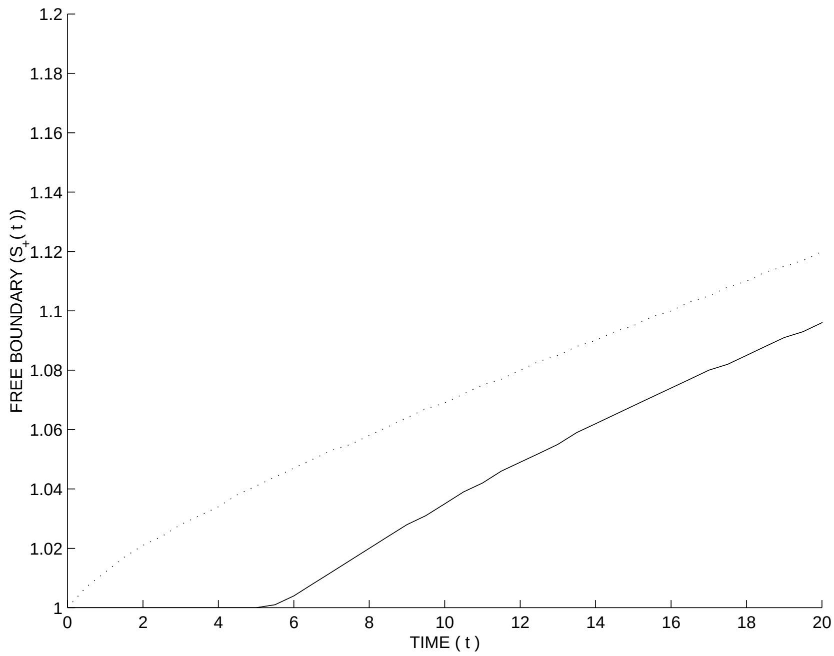

FIG. 5.7. *Moving boundary $S_+(t)$ in Test 3 with convex-concave $\overline{\eta}_0$ $(-)$ and purely concave η_0 $(\cdots)$ initial conditions for $c = 0.75$.*

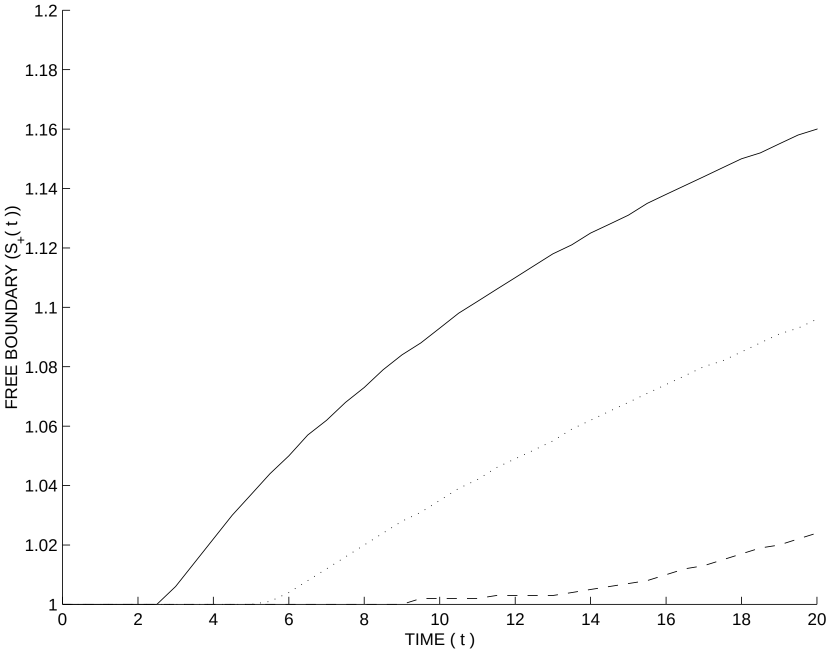

FIG. 5.8. *Moving boundary $S_+(t)$ in Test 3 with convex-concave $\overline{\eta}_0$ function and $C = 0$ for $c = 0.5(--)$, $c = 0.75(\cdots)$, $c = 0.9(-)$.*

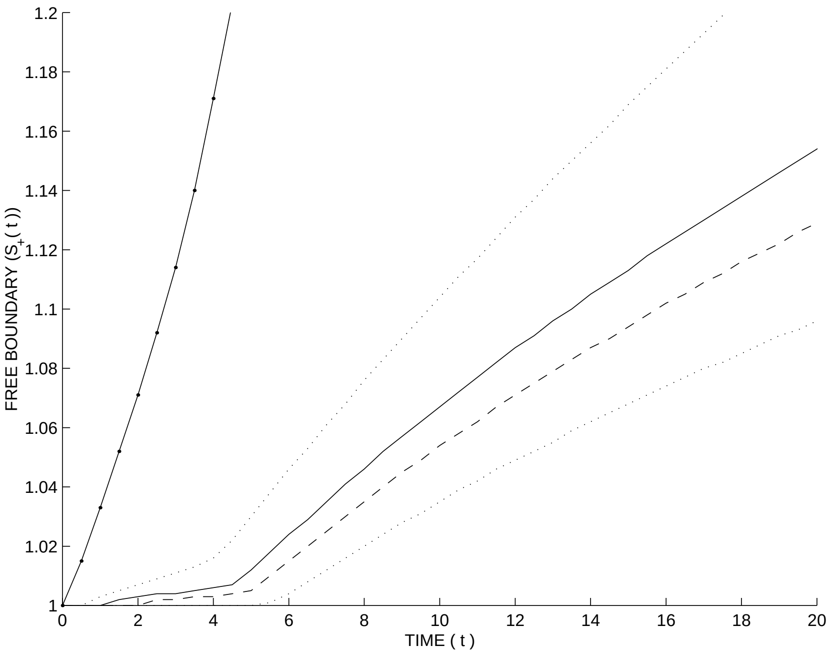

FIG. 5.9. *Moving boundary $S_+(t)$ in Test 3 with $\overline{\eta}_0$ and $c = 0.75$.* $C = 0(\cdots)$, $C = 0.003(--)$, $C = 0.005(-)$, $C = 0.01(\cdots)$, $C = 0.05(.-.-)$.

6. Discussion. In this paper we used different but equivalent weak formulations, expressed in terms of multivalued equations or of variational inequalities when the complementary formulation is considered for numerical purposes. This approach makes more precise the original doubly nonlinear formulation of Fowler [23] converting it in an obstacle problem for the associated operator. Assuming some extra regularity properties of the solution we give sufficent conditions (in terms of a, the accumulation rate and h_0, the initial thickness) for the existence of the free moving boundary and its spatial location. For this, we employed two different methods: a comparison principle combined with the construction of suitable barrier functions in the case $u_b \equiv 0$ and a local energy method if $u_b \neq 0$. In both cases, we prove rigorously the possible existence of a waiting time in the dynamics of the free boundary, whose location and evolution can be qualitatively described as long as suitable and physically admisible hypothesis on the data of the problem hold.

From the numerical point of view, the main advantage of the proposed new approach follows from the introduction of a maximal monotone operator for the nonlinear diffusion term which had already been treated in explicit form [11]. Thus, a duality method can also be applied now to greatly improve the speed of convergence with respect to the previous work. In order to verify the good performance of the new algorithm as well as the computational cost reduction, a problem with a closed form solution has been tested. Moreover, to complete the theoretical results, several test examples illustrate some qualitative properties of the ice profile and the free boundary associated.

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