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Well-formed scales, non-well-formed words and the Christoffel duality

Escalas bien formadas, palabras no bien formadasa y la dualidad de Christoffel

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Manuel Domínguez Romero

Director

Marco Castrillón and Thomas Noll

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WELL-FORMED SCALES, NON-WELL-FORMED WORDS AND THE CHRISTOFFEL DUALITY

ESCALAS BIEN FORMADAS, PALABRAS NO BIEN FORMADAS Y LA DUALIDAD DE CHRISTOFFEL

BY
MANUEL DOMÍNGUEZ ROMERO
PhD Directors: Marco Castrillón and Thomas Noll

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Introducción

La presente tesis analiza las escalas musicales generadas desde la perspectiva y las técnicas que ofrece la combinatoria algebraica de palabras.

La noción de escala musical es una de las más primitivas: intuitivamente se puede reducir a un conjunto de notas ordenadas según la frecuencia de su fundamental (altura del sonido). Ya desde tiempos de la Escuela Pitagórica se vio que al pulsar una cuerda tensa, los sonidos que mejor suenan juntos, *los más consonantes*, están determinados por unas longitudes de cuerda cuyas proporciones son números fraccionarios sencillos. El más consonante de ellos, la octava, tiene una relación de longitudes 2:1. Este intervalo es tan *consonante*, que muchas veces los sonidos cuyas frecuencias están separadas en una octava suenan indistinguibles. Es por ello por lo que al estudiar las escalas se suelen identificar las notas cuya distancia es de una o varias octavas. Como resultado, suele entenderse por escala un conjunto de notas dentro de un rango de una octava, *transportando* dicha secuencia al resto de octavas en caso de necesidad. La definición formal de escala se llevará a cabo en la sección 2.2, donde se mostrará cómo cada octava puede representarse geométricamente mediante una circunferencia unitaria o, aritméticamente, como el conjunto cociente $\mathbb{R}/\mathbb{Z}$, es decir, como el intervalo (0,1]. De esta forma, una escala queda determinada por un conjunto de números ordenados entre el 0 y el 1 o bien, geométricamente, por un polígono inscrito en el círculo unidad.

Dentro de toda la infinidad de escalas en las que se puede pensar, hay una que ha sido particularmente significativa en la llamada música occidental: la *escala diatónica*. No es otra que la formada por las siete teclas blancas consecutivas de un piano (empezando, normalmente desde el do):

Do-Re-Mi-Fa-Sol-La-Si y vuelta al Do

A lo largo de la segunda mitad del siglo XX, numerosos teóricos de la música trataron de buscar las propiedades estructurales de la escala diatónica en términos matemáticos, de modo que pudiesen estudiarse otras escalas con comportamientos *diatónicos*. A la postre, esta nueva teoría sería conocia como *teoría diatónica de conjuntos* ("diatonic set theory" en inglés). Muchas de estas propiedades ya eran conocidas en teoría de la música, pero se estudiaban por primera vez desde un punto de vista estrictamente formal. Tal es el caso de la *propiedad de Myhill* –Myhill's property– introducida por J. Clough y G. Myerson en 1985 ([27]): la escala diatónica tiene exactamente dos tipos de segundas (segunda mayor y segunda menor), de terceras (tercera mayor y tercera menor), de cuartas (cuarta justa y cuarta disminuida)... y dos tipos de séptimas (séptima mayor y séptima menor). También era de sobra conocido que la escala es una escala *generada*: hay un intervalo (la quinta justa) a partir del cual puede generarse toda la escala –según su ciclo de quintas– de forma que las notas se pueden recorrer en orden de generación:

Fa-Do-Sol-Re-La-Mi-Si,

La escala diatónica verifica también una propiedad curiosa. Las distancias entre notas consecutivas pueden ser de dos tipos distintos: tonos y semitonos (un tono si entre las teclas correspondientes del piano hay otra tecla negra y semitono en caso contrario). Resulta que en la escala diatónica los tonos y semitonos están distribuidos de tal forma que los semitonos están separados *lo máximo posible* entre sí (las teclas negras del piano guardan una distancia máxima entre sí). Se dice que la escala diatónica tiene *distribución máxima* (en inglés, "maximally even"). Esta noción fue introducida por J Clough y J Douthett en 1991 [25].

N. Carey y D. Clampitt introdujeron en 1989 [16] la noción de *escala bien formada* ("well-formed scale") como aquellas escalas para las que se puede pasar del orden de generación al orden natural a través de la permutación determinada por una homotecia en $\mathbb{Z}/N$, siendo N el número de notas de la escala. Resulta que la noción de escala bien formada no sólo conectaba muchas de las nociones de la teoría diatónica de conjuntos que habían sido estudiadas hasta la fecha, sino que enlazaba también con problemas y nociones de la teoría de números: con el *teorema de los tres pasos* ("three gaps theorem") y con los convergentes de un número irracional.

El teorema de los tres pasos (conocido originalmente como *conjetura de Steinhaus*) fue demostrado simultaneamente en 1958 por V. T. Sòs ([48]) y S. Świerckowski ([49]) y asegura que los puntos del círculo unidad dados por la secuencia

$$\{\{n\theta\}, n = 0, 1, \ldots, N-1\},$$

donde $\{x\}$ es la parte decimal de x, dividen al círculo unidad en pasos que son, a lo sumo, de tres distancias distintas. En términos musicales, las escalas generadas tienen, como mucho, tres tipos de pasos (o segundas). D. Clampitt y N. Carey lograron identificar las escalas bien formadas de generador irracional con las escalas generadas con dos pasos distintos. También lograron demostrar que esto ocurría precisamente si N (el número de notas de la escala) es el denominador de un convergente o semi-convergente de θ (el generador de la escala). Resulta notable observar que una noción análoga a la de escala bien formada había sido introducida de manera totalmente independiente por J. Navarro y M.J. Garmendia en 1996 ([42]) tomando como generador el caso particular de la quinta pura $\log_2\left(\frac{3}{2}\right)$. Estos autores llamaron *escalas pitagóricas* a las escalas generadas por quintas que tienen exactamente dos pasos.

Dado que las escalas bien formadas son escalas con dos tipos de pasos (en la escala diatónica, tono y semitono), una cuestión interesante es estudiar cómo están distribuidos estos pasos. Esta cuestión fue resuelta en [32] al demostrarse que el patrón de una escala bien formada es una secuencia w de 0 y 1 dada por una fórmula del tipo

$$w(k) = \left[\frac{M \cdot (k+1)}{N}\right] - \left[\frac{M \cdot k}{N}\right] \quad \text{con } k = 0, 1, \ldots, N-1,$$

donde $[x]$ representa la parte entera de x. Estas palabras reciben el nombre de *palabras de Christoffel* de pendiente $\frac{M}{N}$ y son una de las nociones centrales de la *combinatoria algebraica de palabras*.

La combinatoria algebraica de palabras es un campo de la matemática discreta que se centra en las propiedades de las secuencias de símbolos (letras). Tiene sus orígenes en los trabajos de A. Thue ([51] y [50]) quien, a principios del siglo XX, se interesó por la existencia de palabras infinitas que fuesen libres de cuadrados, esto es, que no contuviesen factores idénticos consecutivos. Desde entonces, y especialmente a partir del último cuarto del siglo XX, la combinatoria de palabras ha crecido de forma exponencial. Los principales resultados en este campo se han recopilado en dos volúmenes bajo el pseudónimo de Lothaire ([37] y [38]) y un tercero ([39]) dedicado a las aplicaciones de la combinatoria a distintos campos: lingüística, biología, computación y física teórica.

D. Clampitt y N. Carey definieron en 1996 ([17]) una noción de dualidad sobre las escalas bien formadas: dos escalas bien formadas son duales si las homotecias (que transforman orden natural en orden de generación) asociadas a ambas escalas son inversas. Paralelamente, en el campo de las palabras, V. Berthé, A. de Luca y C. Reutenauer introdujeron en 2006 ([12]) una involución sobre las palabras de Christoffel: dos palabras de Christoffel de longitud N son duales si sus pendientes $\frac{M}{N}$ y $\frac{M^*}{N}$ verifican que $M^* = M^{-1} \bmod N$. Se puede ver con sencillez que ambas nociones de dualidad son equivalentes.

La caracterización de los patrones de las escalas bien formadas como palabras de Christoffel y la identificación entre la noción de dualidad de escalas bien formadas y la del automorfismo de palabras de Christoffel supone el punto de partida de la presente tesis. A partir de aquí caben plantearse dos cuestiones: por un lado ¿podemos aprovechar la maquinaria de la combinatoria de palabras para analizar nuevas propiedades de las escalas generadas? Pero por otro lado ¿podemos sacar partido de este nuevo punto de vista con el que estamos viendo las palabras de Christoffel (como patrones de escalas) para trasladar preguntas naturales en el campo de las escalas al campo de la combinatoria de palabras? Dicho de otra forma, ¿se puede beneficiar también la combinatoria de palabras de este punto de encuentro escalas-palabras? Veremos que la respuesta a ambas preguntas es afirmativa.

Objetivos

La presente tesis tiene principalmente tres objetivos. En primer lugar, se ofrecerá al lector una visión global de la teoría de escalas bien formadas desde un punto de vista de la combinatoria de palabras. Desde esta perspectiva se desarrollarán las propiedades de las escalas generadas, del teorema de los tres pasos, la definición y las distintas caracterizaciones de las escalas bien formadas y, por último, la noción de dualidad. Esto será llevado a cabo en el Capítulo 3.

Según el *Teorema de los tres pasos*, si una escala está generada por un número irracional, puede tener dos o tres pasos distintos. Si tiene dos pasos, la escala es bien formada y sus propiedades están plenamente descritas en el Capítulo 3. Los patrones de estas escalas coinciden con las palabras de Christoffel del alfabeto $\{0, 1\}$. Sin embargo, queda pendiente de estudiar en profundidad el caso *malo* del teorema de los tres pasos: hasta la fecha no se han estudiado las propiedades matemáticas que verifican las escalas generadas que no son bien formadas. Puesto que sus patrones son palabras ternarias, el objetivo es intentar relacionar estas palabras con alguna de las generalizaciones de las palabras de Christoffel a alfabetos de n letras que existen en la literatura de la combinatoria algebraica de palabras. Esta idea centrará los contenidos del Capítulo 4.

Por último, se pretende estudiar si la dualidad sobre las escalas bien formadas puede extenderse a otros ámbitos. El primero de ellos es el de los modos de una escala. Dado que matemáticamente los patrones de los modos de una escala son rotaciones del patrón original, la idea es extender la involución de palabras de Christoffel sobre sus rotaciones (conjugados). Pero estas extensiones se deben comportar bien cuando consideramos la involución sobre los morfismos de palabras. Este problema centrará los contenidos del Capítulo 5. Una segunda forma de extender la dualidad de Christoffel es la siguiente. Si fijamos un número irracional θ, cada convergente (o semiconvergente) de θ es una fracción que es, a su vez, la pendiente de una palabra de Christoffel. ¿Es posible encontrar un *generador dual* θ^* de forma que las palabras de Christoffel cuyas pendientes sean (semi-)convergentes, respectivamente de θ y de θ^*, sean duales dos a dos? El Capítulo 6 está dedicado a responder a esta pregunta.

Resumen y resultados

En el Capítulo 2 introducimos las nociones de la teoría de escalas y de la teoría de palabras que serán necesarias en el resto de capítulos. En primer lugar (Sección 2.1) se introducen las nociones básicas sobre aproximación de irracionales por racionales: los convergentes y semiconvergentes, las fracciones de Farey, el árbol de Stern-Brocot y la relación entre estas nociones y las matrices del monoide $SL(\mathbb{N}, 2)$. En segundo lugar (Sección 2.2) se establecen las bases del estudio matemático de las escalas musicales al describirlas como polígonos inscritos en el círculo unidad. Por último (Sección 2.3) se presentan algunos conceptos elementales de la combinatoria algebraica de palabras, introduciendo las palabras de Christoffel, su análogo infinito; las palabras de Sturm y, por último, los morfismos de Sturm que se pueden usar para dar otra descripción de las palabras de Christoffel y de Sturm.

En el Capítulo 3 establecemos las bases de la teoría de las escalas bien formadas. En primer lugar (Sección 3.1) se presenta la noción de escala generada y se demuestra el teorema de los tres pasos. Posteriormente, en la Sección 3.2 se definen las escalas bien formadas siguiendo las ideas que fueron usadas por N. Carey y D. Clampitt en [16] pero con un lenguaje acorde al contexto. Posteriormente se exponen seis caracterizaciones de la noción de escala bien formada: según que su generador sea uniforme (Sección 3.3), vía el desarrollo en fracciones continuas del generador (Sección 3.4), mediante palabras de Christoffel (Sección 3.5), a través de las matrices de acumulación asociadas con la escala (Sección 3.6), mediante la propiedad de uniformidad máxima (Sección 3.7) y la propiedad de Myhill (Sección 3.8). La siguiente sección (Sección 3.9) recopila todos los resultados en el teorema resumen del capítulo (Teorema 12). Para concluir el capítulo, se describe la noción de dualidad entre palabras de Christoffel y entre escalas bien formadas, caracterizando la relación entre las matrices de incidencia de palabras duales, y la situación en el árbol de Christoffel de sus correspondientes pendientes (Sección 3.10). En este Capítulo aparecen los primeros resultados originales de la tesis: la Proposición 16 (publicada en [32]) y la Proposición 17, que relacionan las escalas bien formadas con las palabras de Christoffel. Se ofrece también una nueva caracterización de las escalas WF, dada por las matrices de acumulación (Proposición 19), que será demostrada en el Capítulo 5. A pesar de que los resultados que relacionan a las escalas WF con los conjuntos de uniformidad máxima y con la propiedad de Myhill ya son conocidos, se presentan por primera vez en un contexto netamente de combinatoria algebraica de palabras al relacionar los patrones de las escalas WF con las palabras equilibradas (*balanced words*) y con las palabras Lyndon. Se incluye también un resultado nuevo referente a los conjuntos de uniformidad máxima: la Proposición 22 determina la relación entre los conjuntos J (que se utilizaron en el artículo original [25] para definir los conjuntos ME) con la afinidad asociada al conjunto ME. Los resultados de la última sección del capítulo sobre la dualidad de Christoffel son también originales y fueron publicados en [32].

El Capítulo 4 describe las escalas generadas que no son bien formadas. Los resultados que recoge, originales todos ellos, han sido publicados en [22] y [21]. Después de ofrecer una descripción global del capítulo, se construyen formalmente las palabras NWF (*non well-formed words*) y se integran en el árbol de Christoffel (Sección 4.1). Posteriormente (Sección 4.2) se demuestran algunas propiedades recurrentes de las palabras NWF que serán claves para los principales resultados del capítulo. A continuación (Sección 4.3) se demuestra que las palabras NWF factorizan de forma natural como producto de palabras WF y NWF y, como consecuencia, son palabras Lyndon. Seguidamente (Sección 4.4) se estudian las factorizaciones Lyndon por la izquierda y por la derecha de las palabras NWF. Resulta que la factorización Lyndon por la derecha queda determinada por el generador de la escala (tal y como ocurre en el caso de las escalas bien formadas). Sin embargo, la factorización Lyndon por la izquierda, en general no coincide con ésta. Cuando ambas factorizaciones sí que coinciden, se dice que la palabra verifica la condición LR. La sección termina caracterizando las palabras que verifican la propiedad LR. En la última sección del capítulo (Sección 4.5) se demuestra que las palabras que verifican la propiedad LR están distribuidas de manera simétrica en el árbol de Christoffel y son, por tanto, exactamente la mitad. También se estudia la distribución de palabras que verifican y que no verifican la propiedad LRbajo una única palabra de Christoffel w: se prueba que esta distribución queda determinada por la palabra de Christoffel dual de w. Por último se relacionan las palabras NWF con las palabras de Lyndon-Christoffel.

El Capítulo 5 estudia la extensión de la dualidad de Christoffel sobre el conjunto de rotaciones de una palabra de Christoffel (patrones de los modos de la escala bien formada asociada). Los resultados del capítulo son todos (excepto el Teorema 17) originales y están recogidos en [23]. En primer lugar (Sección 5.1) se introducen las aplicaciones *balance map* y *accumulation map* y se justifica su definición tanto geométricamente como a través de las matrices de acumulación. Estas aplicaciones sirven para caracterizar las rotaciones de una escala bien formada. El resto del capítulo (Sección 5.2) está dedicado a definir una noción de dualidad sobre el conjunto de rotaciones de una palabra de Christoffel (que preserve la noción de dualidad introducida en la Sección 3.10). Para ello se le asociará a cada rotación una afinidad, de modo que dos palabras se dirán duales si sus respectivas afinidades son inversas. La asignación de afinidades asociadas a cada rotación se va a hacer de dos formas distintas. La más intuitiva dará pie al llamado "*plain adjoint*". Dado que esta asignación no respeta al comportamiento de la dualidad sobre los morphismos de Sturm, buscamos una segunda asignación que sí que respete este comportamiento. Esto dará pie a una segunda noción de dualidad, llamada "*twisted adjoint*". Se finaliza el capítulo con los Teoremas Jónicos (*Ionian Theorems*)–Teoremas 17 (original de T. Noll [43]) y 18–que permiten dar una interpretación musical a los *plain adjoint* y *twisted adjoint*.

El penúltimo capítulo de la tesis (Capítulo 6) trata de extender la dualidad de Christoffel (que se define sobre fracciones irreducibles) a una involución sobre los números irracionales. Los resultados contenidos en este capítulo (salvo los de la sección final) fueron publicados en [31]. Las palabras de Christoffel codifican los caminos en zig-zag que resultan de unir los puntos de coordenadas enteras más próximos bajo una recta de pendiente racional. Las palabras de Sturm codifican los caminos *de longitud infinita* que aproximan de igual forma a las rectas de pendiente irracional. Una palabra de Sturm de pendiente θ admite como prefijos palabras de Christoffel (que tienen por pendientes los (semi-)convergentes de θ). La idea es estudiar la posible definición de una dualidad sobre el conjunto de las palabras de Sturm, de modo que dos palabras sean duales si sus respectivos prefijos de Christoffel son duales. Para ello, en la Sección 6.1 se definen dos funciones, llamadas *left and right companion functions* $\mathfrak{L}$ y $\mathfrak{R}$. Seguidamente se analizan las propiedades de ambas funciones en términos de desarrollos en fracciones continuas y matrices de $SL(\mathbb{N},2)$. En la Sección 6.2 se justifican geométricamente las definiciones de estas dos funciones. Las funciones $\mathfrak{L}$ y $\mathfrak{R}$ permiten definir funciones $\mathfrak{L}_{\frac{M}{N}}$ y $\mathfrak{R}_{\frac{M}{N}}$, contractivas sobre el intervalo (0,1], para cada fracción $\frac{M}{N}$. Según la motivación inicial, si una palabra de Sturm tiene pendiente θ y $\frac{M}{N}$ es un semiconvergente suyo, la pendiente dual de θ debería de ser $\mathfrak{L}_{\frac{M}{N}}(\theta)$ o $\mathfrak{R}_{\frac{M}{N}}(\theta)$. En la Sección 6.3 se muestra como estas asignaciones no definen una verdadera dualidad, dado que, en general, las funciones $\mathfrak{L}_{\frac{M}{N}} \circ \mathfrak{L}_{\frac{M^*}{N}}$ y $\mathfrak{R}_{\frac{M}{N}} \circ \mathfrak{R}_{\frac{M^*}{N}}$ con $M^{-1} = M^* \bmod N$ no son involutivas. Sin embargo, sí que dan lugar a una interesante cuestión: estudiar sus puntos fijos. Se demuestra que estos puntos fijos son *números de Sturm* cuyos desarrollos en fracciones simples son periódicos y su periodo es palindrómico. A estos números se les llama *números espejo* (*"mirror numbers"*). Se calculan a continuación los morfismos de Sturm que fijan las palabras características cuya pendiente es un número espejo. En la siguiente sección (6.4) se relacionan los números espejo con las palabras harmónicas (*harmonic words*), introducidas en [18]: se demuestra que las palabras características cuya pendiente es un número espejo son harmónicas. El artículo [31] concluía con unas conjeturas sobre la relación entre las palabra características cuyas pendientes son números espejos duales. La última sección del capítulo (Sección 6.5) demuestra estas conjeturas en términos de las conocidas como *palabras retorno* (*"return words"*), que fueron presentadas y estudiadas en [33] y [54].

El último capítulo (Capítulo 7) está dedicado a recopilar algunas preguntas sin resolver, conjeturas y líneas de trabajo que puedan desarrollar los resultados de la presente tesis en un futuro.

Conclusiones

Como primer objetivo nos habíamos propuesto unificar las bases de la teoría de escalas bien formadas. Este objetivo se ha alcanzado, al lograr ofrecer una visión global de los fundamentos de las escalas bien formadas y de sus principales caracterizaciones, aportando dos nuevas (mediante palabras de Christoffel y mediante matrices de acumulación) y reformulando gran parte de las demostraciones existentes hasta la fecha del resto en un contexto de combinatoria algebraica de palabras.

Como segundo objetivo, nos propusimos analizar las escalas generadas que no son bien formadas (por tener tres pasos distintos). Se ha logrado ofrecer propiedades interesantes de los patrones de estas escalas (llamados palabras NWF). Resulta sugerente cómo estas palabras son palabras Lyndon y admiten una factorización Lyndon por la derecha similar al caso de las palabras de Christoffel. La asimetría se produce al analizar la factorización Lyndon por la izquierda. Resulta que para la mitad de las palabras NWF ambas factorizaciones coinciden. Se ha encontrado una sorprendente relación entre la distribución de estas palabras y la dualidad de Christoffel. También se han encontrado conexiones con las palabras de Lyndon-Christoffel: generalización a alfabetos arbitrarios de las palabras de Christoffel. Se plantean dos cuestiones abiertas interesantes: caracterizar la factorización Lyndon por la izquierda de palabras NWF, y ofrecer propiedades que caractericen las palabras NWF en $\{0, 1, 2\}^*$.

El último objetivo se centraba en la extensión de la dualidad de Christoffel en dos ámbitos: sobre las rotaciones de palabras de Christoffel y sobre las palabras de Sturm. El primer intento por definir una dualidad sobre el conjunto de modos de una escala bien formada ha producido distintos frutos: en primer lugar se ofrece una nueva caracterización de las rotaciones de palabras de Christoffel, a través de las matrices de acumulación y, como caso particular, una nueva caracterización de las escalas bien formadas. Por otro lado, las extensiones tienen un estrecho vínculo con la interpretación musical, como atestiguan los Teoremas Jónicos (Teoremas 17 y 18). Además, se sugieren posibles vías de aplicar la maquinaria desarrollada en el ámbito de los poliominós.

El segundo ámbito sobre el que se buscó extender la dualidad de Christoffel fue el de las palabras de Sturm. Este intento por definir la extensión de modo que prefijos de Christoffel de palabras de Sturm duales resultaran también duales fue fallido: la extensión no es involutiva, con lo que no resulta una veradera dualidad. Sin embargo, el intento sí dio sus frutos, aunque no fueran los buscandos en un principio: los puntos fijos del *doble dual* resultan ser números irracionales muy particulares: su desarrollo en fracciones simples es periódico y el periódo es palindrómico. De manera indirecta, por tanto, se han obtenido resultados dentro del campo de la teoría de números y de la combinatoria algebraica de palabras.

Los contenidos que se derivan de la presente tesis han sido publicados, por orden cronológico, en [32], [23], [31], [20], [22] y [21].

Introduction

The present PhD studies generated scales from the perspective and the methods of algebraic combinatorics on words.

The notion of musical scale is a primitive notion in music theory: intuitively it is a (finite) set of tones ordered by their pitch. It was known since the time of the Pythagorean School that if one plucks a tight string, the tones that sound better together –the more consonant ones– are determined by lengths of string the proportions of which are simple fractional numbers. The most consonant one, the octave, has a ratio 2:1. This interval is so *consonant* that two pitches with frequencies at a distance of an octave may sound almost indistinguishable. Thus, when we study scales, pitches at a distance of one ore more octaves are usually identified. As a result, a scale can be thought of as a set of pitches within a range of an octave, that can be moved up and down by one or more octaves if needed. The formal definition of scale will be made in section 2.2, where it will be proven how each octave can be represented geometrically with a circle of length 1 or, arithmetically, by the quotient set $\mathbb{R}/\mathbb{Z}$, that is, with the interval $(0, 1]$. In this way, a scale is determined by an ordered set of numbers between 0 and 1 or, geometrically, by a polygon inscribed in the unit circle.

Among all the possible scales one might think of, there is one that is particularly significant in western music: the *diatonic scale*. It is the scale that is played with the seven consecutive white keys of a piano (starting from C):

C-D-E-F-G-A-B- and again C

Throughout the second half of the 20th century, many musicologists tried to find the structural properties of the diatonic scale in mathematical terms, so that one could find *diatonic behaviors* in other scales. In the longer term, this new theory would be known as *diatonic set theory*. Many of the properties that were studied were enough known in music theory, but they were studied for the first time with a purely formal perspective. Such is the case of Myhill's property, introduced by J. Clough and G. Myerson in 1985 ([27]): the diatonic scale has exactly two kind of seconds (major and minor second), two kind of thirds (major and minor thirds), two fourths (perfect and augmented fourth)... and two kind of sevenths (major and minor seventh). It was enough known as well that the diatonic scale is a generated scale: following the cycle of fifths every note of the scale is reached:

F-C-G-D-A-E-B

The diatonic scale fulfills a intriguing property: *maximal evenness*. The distance between two consecutive pitches of the scale can be a tone or a semitone. It turns out that the semitones are placed as evenly as possible in the scale. This notion was firstly introduced first by J Clough and J Douthett in 1991 [25] and it has been deeply studied ever since.

N. Carey and D. Clampitt presented in 1989 ([16]) the notion of well-formed scale as that scale for which one can switch from generation order into natural order via a permutation that is determined by a homothetic transformation in $\mathbb{Z}/N$, where N is the number of notes of the scale. It turns out that the notion of well-formed scale not only connects many other notions in the diatonic set theory, but it is connected with important notions and problems in the classical theory of numbers: *the three gap theorem* and the convergents of an irrational number.

The three gap theorem (originally known as the *Steinhaus conjecture*) was proven simultaneously in 1958 by V. T. Sòs ([48]) and S. Świerckowski ([49]). It asserts that the points in the unit circle given by the sequence

$$\{\{n\theta\}, n = 0, 1, \ldots, N-1\},$$

where $\{x\}$ is the fractional part of x, divide the circle in gaps that are, at most, of three different lengths. In musical terms, a generated scale has, at most, three different steps (seconds). D.Clampitt and N. Carey managed to identify the well-formed scales of irrational generator with the generated scales that have two different steps. They also prove that this happened precisely when the number of notes N of the scale is the denominator of a convergent or semi-convergent of the generator θ. It is remarkable that an analogue notion of scale was introduced by J. Navarro and M. J. Garmendia in 1996 ([42]) in a completely independent way. These late authors used the particular case of pure fifth $\log_2\left(\frac{3}{2}\right)$ as a generator and name the scales with two different steps *pythagorean scales*.

As we commented, well-formed scales with irrational generator have two kind of steps (in the diatonic scale these are tone and semitone). Thus, it is interesting to study how these steps are distributed. This question was answered in [32] where it was proven that the step-pattern of a well-formed scale can be encoded with sequence w of 0's and 1's given by the formula:

$$w(k) = \left[\frac{M \cdot (k+1)}{N}\right] - \left[\frac{M \cdot k}{N}\right] \quad \text{with } k = 0, 1, \ldots, N-1,$$

where $[x]$ denotes the integer part of x. These words are called *Christoffel words* of slope $\frac{M}{N}$ and they are one of the central notions within *algebraic combinatorics on words*.

Algebraic combinatorics on words is a field of discrete mathematics that studies the properties of sequences of symbols (letters). It is rooted in the works of A. Thue ([51] and [50]) who, in the first quarter of the XX century studied the existence of infinite words that were square free (that is, infinite words that did not include two identical and consecutive factors. Since those works, and specially from the last quarter of the XX century, the combiantorics on words has grown exponentially. Main results on this field were compiled in two volumes under the pseudonym of Lothaire in 1983 ([37]) and 2002 ([38]). A third volume was published devoted to the applications of this field ([39]): linguistics, biology, computation and theoretical physics.

D. Clampitt and N. Carey defined in 1996 ([17]) a notion of duality between well-formed scales: two well-formed scales are dual if their associated homothetic transformations (that transform natural order into generation order) are inverses of one another. In a parallel way, V. Berthé, A. de Luca and C. Reutenauer introduced in 2006 ([12]) an involution over the set of Christoffel words: two words of length N are dual if their slopes $\frac{M}{N}$ and $\frac{M^*}{N}$ verify que $M^* = M^{-1}$ mod N. One can see easily that both duality notions are equivalent.

The characterization of the step patterns of well-formed scales as Christoffel words and the identification of the duality notions in both domains (the duality of well-formed scales and the automorphism of Christoffel words) are the starting points of the present PhD dissertation. Based on these, some intriguing questions arise: can we take advantage of the combinatorics on words to analyze new properties of generated scales? In the other hand, can we take advantage of this new dimension of the Christoffel words (as step pattern of well-formed scales) to translate into the field of combinatorics on words questions that are naturally presented in the domain of musical scales? In other words, can the combinatorics on words take advantage of this meeting point between scale and words? As we will see, the answer to both questions is affirmative.

Objectives

The present PhD has three different objectives. Firstly, we try to present a unified version of the well-formed scale theory using the language of the combinatorics on words. The main notions of scale theory will be introduced within this perspective: generated scales, the three gap theorem, the definition and some characterizations of well-formed scales and the duality notion. This will be made in the Chapter 3.

Following the aforementioned three gap theorem, if a scale is generated by an irrational number, it can have two or three different steps. In the first case, the scale is well-formed and its properties are fully described in Chapter 3. The latter case –generated scales with three different steps– has not been studied in depth so far. We intend to study the mathematical properties of the ternary words that are the step patterns of the generated scales that are not well-formed. We will try to connect these ternary words with some of the generalizations of Christoffel words into words of alphabets of more than two letters that exist in the literature of combinatorics on words. This idea will be undertaken in Chapter 4.

Finally, we intend to study if the duality of well-formed scales can be extended in other spheres. The first attempt will use the modes of a scale. Mathematically, the step patterns of the modes of an scale are rotations of the original pattern. We will try hence to extend the involution of Christoffel words over the set of their rotations (the conjugation classes). But this extension must fulfill some properties when we translate it to the set of Christoffel morphisms. This problem will be studied in Chapter 5. We will examine a second way to extend the Christoffel duality: if we consider an irrational number θ, each one of its convergents (or semiconvergents) is a fraction, the slope of a certain Christoffel word. Is it possible to find a *dual generator* θ^* in such a way that the Christoffel words the slope of which are (semi-)convergents of θ and θ^* are successively dual words? The Chapter 6 is devoted to give an answer to the preceding question.

Summary and results

In Chapter 2 we introduce the necessary notions of theory of scales and theory of words that will be needed in the rest of the chapters. First of all (Section 2.1) the basic notions concerning the approximation of irrational numbers with fractions are introduced: convergents, semiconvergents, Farey fractions, the Stern-Brocot tree and the relationship between all these notions and the monoid $SL(\mathbb{N}, 2)$. Secondly (Section 2.2), we build the foundations of the mathematical study of a musical scale: they are described as polygons inscribed in the unit circle. Finally (Section 2.3), the basics of algebraic combinatorics on words are presented: Christoffel words, their infinite analogues Sturm words, and finally Sturm morphisms that can be used to give a different description of Christoffel and Sturm words.

In Chapter 3 we present the theory of well-formed scales. We begin (Section 3.1) by introducing the notion of generated scale and the three gaps theorem. Later on, in the Section 3.2 we define the well-formed scales following the original ideas of N. Carey and D. Clampitt in [16] but with a language that is adapted to our domain: the combinatorics on words. Subsequently, we describe six different characterizations of the notion of well-formed scale: a scale with uniform generator (Section 3.3), via the expansion in continued fractions of the generator (Section 3.4), by means of the Christoffel words (Section 3.5), using the accumulation matrices associated with the scale (Section 3.6), using the maximally evenness property (Section 3.7) and the Myhill's property (Section 3.8). Next section (Section 3.9) compiles all these characterizations in the fundamental Theorem that summarizes the chapter (Theorem 12). We conclude the chapter with the description of duality between Christoffel words and well-formed scales. We characterize the accumulation matrices of dual words and the way that dual words are displayed in the Christoffel tree (Section 3.10). In this Chapter we find the first original results of the PhD dissertation: the Proposition 16 (see [32]) and the Proposition 17, that relate the well-formed scales with the Christoffel words. We also offer a new characterization of well-formedness using the accumulation matrices (Proposition 19), that will be proven in Chapter 5. The results that connect the well-formed scales with maximally even sets and with Myhill's property are already known. We present them nevertheless in a completely original way: they are described in a pure context of algebraic combinatorics on words for the first time as we relate them with *balanced words* and Lyndon words. A new result concerning maximal evenness is included: the Proposition 22 determines the relationship between the J sets (that where used in the original paper [25] to define the maximally even sets) with the affinity associated to the maximally even set. The material of the last section concerning the Christoffel duality is original as well and most of it was published in [32].

Chapter 4 covers the study of the generated scales that are not well-formed. The results presented in this Chapter, all of the original, were published in [22] and [21]. First we build formally the NWF words (*non well-formed words*) and then we insert them into the Christoffel tree (Section 4.1). Later on (Secton 4.2), we prove some of their first properties, based on their recursive nature. These properties will be essential for the main results of the chapter. Next (Section 4.3), it is proven that NWF words factorize in a natural way as a product of WF and NWF words and, as a consequence, they are Lyndon words. Then (Section 4.4), left and right Lyndon factorizations of NWF words are studied. It turns out that the right Lyndon factorization is determined by the generator of the scale (as in the well-formed case). However, the left Lyndon factorization does not coincide, in general, with the right one. When both factorizations coincide we say that the word verifies LR property. The section ends with a characterization of the NWF words that satisfy LR property. In the last section of Chapter (Section 4.5) it is proven that the words that verify the LR property are symmetrically displayed in the Christoffel tree. The distribution of the NWF words that lay just below a fixed Christoffel word and that verify LR property is studied as well. It will be proven that this distribution is established by the Christoffel duality. We end up by connecting NWF words with Lyndon-Christoffel words.

Chapter 5 studies the extension of the Christoffel duality over the set of rotations of a Christoffel word (over the step patterns of the modes of a well-formed scale). All the results (except Theorem 17) are original and they are included in [23]. Firstly (Section 5.1), we introduce the applications *balance map* and *accumulation map* and we justify their definition in geometrical terms and using the accumulation matrices. These maps characterize the rotations of a well-formed scale. Secondly (Section 5.2), we define the notion of duality over the set of rotations of Christoffel words (preserving the Christoffel duality, introduced in 3.10). For that purpose, we will associate to each rotation an affinity, in such a way that two modes will be dual if their respective affinities are each ones inverses. The assignment of an affinity to each mode will be made in two different ways. The first one, more intuitive, will give rise to the "*plain adjoint*". This assignment does not behave properly over the set of Sturm morphisms. Hence we look for a second one to correct this fact: the "*twisted adjoint*". We end the chapter with the Ionian Theorems (*Ionian Theorems*)– Theorems 17 and 18– the first one due to Thomas Noll (see [43]). These results offer a musical interpretation of both *plain adjoint* and *twisted adjoint*.

The last but one chapter of the PhD (Chapter 6) tries to extend the Christoffel duality (that is defined over irreducible fractions) to an involution of the set of irrational numbers in the interval (0,1]. Its content, except for the last section, was published in [31]. Christoffel words encode the zig-zag paths that are determined by the points of the Gauss grid $\mathbb{Z}^2$ laying closest under a line of rational slope. In a similar way, the Sturm words encode the *infinite* paths that approximate the lines with irrational slope. Some of the prefixes of a Sturm word of slope θ are Christoffel words (their slopes are precisely the (semi-)convergents of θ). The objective is to study the possible definition of a duality over the set of Sturm words, in such a way that two Sturm words are dual words if their respective Christoffel prefixes are dual words. For that purpose, in Section 6.1 the *left and right companion functions* $\mathfrak{L}$ and $\mathfrak{R}$ are defined. Next, we analyze their properties in terms of continued fraction expansions and matrices of $SL(\mathbb{N},2)$. In the Section 6.2 we justify geometrically their definitions. The functions $\mathfrak{L}$ y $\mathfrak{R}$ enable us to introduce a couple of contractive functions $\mathfrak{L}_{\frac{M}{N}}$ and $\mathfrak{R}_{\frac{M}{N}}$ within the interval (0,1], for each irreducible fraction $\frac{M}{N}$. Following our original motivation, if the slope of a Sturm word is θ and $\frac{M}{N}$ is a semi-convergent of θ, then the dual slope of θ should be $\mathfrak{L}_{\frac{M}{N}}(\theta)$ or $\mathfrak{R}_{\frac{M}{N}}(\theta)$. In the Section 6.3 we show how these assignments do not define a proper duality since, in general, the maps $\mathfrak{L}_{\frac{M}{N}} \circ \mathfrak{L}_{\frac{M^*}{N}}$ and $\mathfrak{R}_{\frac{M}{N}} \circ \mathfrak{R}_{\frac{M^*}{N}}$ with $M^{-1} = M^* \bmod N$ are not involutive. Nevertheless, they do give rise to an interesting question: the study of their fixed points. We show how these fixed points are *Sturm numbers* and their expansion in continued fractions are periodic with palindromic period. Thus we call these numbers *mirror numbers*. We compute the Sturm morphisms that fix the characteristic words with slope a mirror number. In the next section (6.4) we connect the mirror numbers with the *harmonic words*, introduced in [18]. We show that the characteristic words with slope a mirror number are harmonic words. The paper [31] concludes with some conjectures about the relationship between two characteristic words the slope of which are dual mirror numbers.

The last section of the chapter (Section 6.5) proves these conjectures in terms of the *"return words"* (see [33] and [54]).

The last chapter (Chapter 7) is devoted to enclose some open questions, conjectures and guidelines that can enhance the results of this PhD in a future.

Conclusions

As a first target, we wanted to unify the grounds of well-formed scales. This objective has been accomplished, since we managed to offer a global overview of the main characterizations of well-formedness using the unified language of algebraic combinatorics of words. Even more, we proved two new characterizations: using Christoffel words and by means of accumulation matrices.

As a second objective, we focused our attention into the generated scales that are not well-formed. We found interesting properties of the step patterns of theses scales (NWF words). It is very stimulating that these words share more properties with the well-formed case than one could think at first glance: they are Lyndon words and their right Lyndon factorization is similar to the case of Christoffel words. The asymmetry arises with the left-Lyndon factorization. It turns out that exactly for half of the NWF words both factorizations coincide and, even more surprisingly, there is a tight connection between the distribution of these words in the *extended* Christoffel tree and the Christoffel duality. We found some connections with the Lyndon-Christoffel words. Some intriguing open tasks can be stated at this point: finding a characterization of the left Lyndon factorization of NWF words, but especially some characteristic properties of those words within $\{0, 1, 2\}^*$ that are NWF.

Our last objective was to extend the Christoffel duality in a double direction: over the rotations of Christoffel words and over the Sturm words. The first attempt to define a duality over the set of modes of a well-formed scale produced different results: in the one hand it offers a new characterization of Christoffel words by mean of the accumulation matrices and, as a particular case, a new characterization of well-formed scales. In the other hand, the extensions given by the *the plain and twisted adjoints* have a tight connection with the musical interpretation, as it is proven in the Ionian Theorems (Theorems 17 and 18).

The second attempt tried to extend the Christoffel duality over the set of Sturm words in such a way that Christoffel prefixes of Sturm dual words were dual words as well. This attempt failed: the transformation of slopes obtained with our intuition is not involutive. However, the search did generate results, although different than expected: the fix points of the *double dual* are a family of very interesting irrational numbers: their expansion in continued fractions is periodic and their period is palindromic. In an indirect way, we obtained results within the realm of number theory and algebraic combinatorics on words.

The contents of the present PhD dissertation have been published, chronologically, in [32], [23], [31], [20], [22] and [21].

Chapter 2

Preliminaries: numbers, words and scales

The main contents of the present work link concepts from three different fields: number theory, musical scale theory and algebraic combinatorics on words. In this first chapter we compile the basic notions that will be needed from each one of them.

2.1 Continued fractions and Farey numbers

Definition. *We call simple continued fraction to any expression of the following type:*

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all $i \geq 1$.

It is well-known that any irrational number can be expressed in just one way as an infinite simple continued fraction (see [35, Theorem 170]). The case of rational numbers corresponds to finite continued fractions: a rational number can be expressed as a finite continued fraction in just two ways, since

$$[a_0; a_1, \ldots, a_n + 1] = [a_0; a_1, \ldots, a_n, 1].$$

We call k-th **convergent** associated with the continued fraction $[a_0; a_1, a_2, \ldots]$ to the rational number

$$\frac{p_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k]$$

and **semiconvergents** (or *intermediate fractions*) to the rational numbers

$$[a_0; a_1, \ldots, a_{k-1}, j], \text{ with } 0 < j < a_k \text{ and } a_k > 1.$$

The (semi-)convergents (convergents or semiconvergents) of a number can be computed in a recursive way (see [36, Theorem 1]):

Proposition 1. *Let $\theta = [a_0; a_1, a_2, \ldots]$ and let $\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k]$ be the $k-th$ convergent of θ, $k \geq 0$, where $\frac{p_k}{q_k}$ is a fraction written in lower terms. It holds*

$$[a_0; a_1, \ldots, a_{k-1}, r] = \frac{p_{k-2} + p_{k-1}r}{q_{k-2} + q_{k-1}r} \quad \textit{with } 0 < r \leq a_k \textit{ and } k \geq 2.$$

There is a tight connection between the coefficients of a continued fraction and the Euclid's algorithm, as the following statement asserts (see [35, Theorem 161]).

Proposition 2. *Let $\theta = [a_0; a_1, a_2, a_3, \ldots]$ be a irrational number, and let I_1 and I_2 be two segments of lengths 1 and θ. Then I_2 fits a_0 times in I_1 and there remains a smaller segment I_3. Now, I_3 fits a_1 times in I_2 and there remains a smaller segment I_4, then I_4 fits a_2 times in I_3 and so on.*

Theorem 1. *Even-order convergents form an increasing and odd-order convergents a decreaing sequence. Also, every odd-order convergent is greater than any even-order convergent.*

Proof. See [36, Theorem 4] □

A similar result holds for (semi-)convergents (see [36, Section 1.4]):

Proposition 3. *The (semi-)convergents sequence*

$$\{[0; a_1, \ldots, a_{k-2}], [0; a_1, \ldots, a_{k-2}, a_{k-1}, 1], \ldots, [0; a_1, \ldots, a_{k-2}, a_{k-1}, a_k]\}$$

is increasing for even k and decreasing for odd k.

The (semi-)convergents of an irrational number are the fractions that *approximate best* that number, in the following way (see [36], again, for a proof):

Theorem 2. *If $\frac{M}{N}$ is a (semi-)convergent of an number α, then it holds that*

$$0 \leq N' < N \textit{ and } \frac{M'}{N'} \neq \frac{M}{N} \Longrightarrow \left|\alpha - \frac{M}{N}\right| < \left|\alpha - \frac{M'}{N'}\right|$$

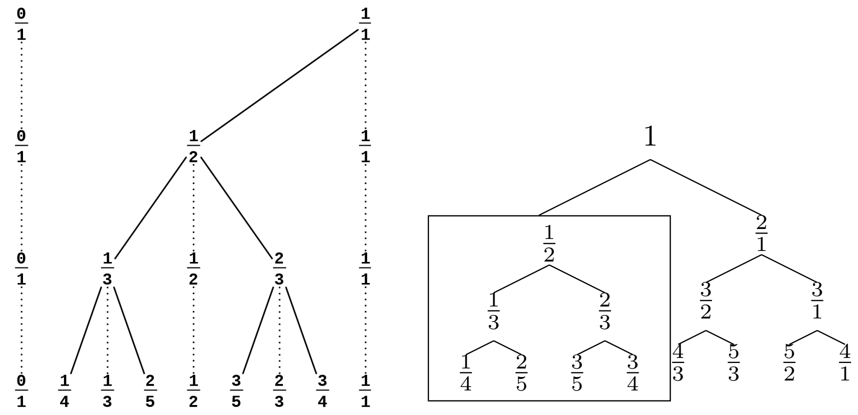
A second notion that will be of great interest as we study the properties of scales is the Farey sequence. Given a natural number $N \geq 1$, the sequence of irreducible fractions between 0 and 1 with denominator q not bigger than N is called Farey sequence $\mathfrak{F}_N$ of order N. For example, the Farey sequence of order 5 is the following ordered set of irreducible fractions:

$$\mathfrak{F}_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}.$$

The main properties of consecutive Farey numbers are given by the following couple of propositions (see [35, Theorem 28 and 30]):

Proposition 4. *If two irreducible fractions $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ are consecutive in a Farey sequence $\mathfrak{F}_N$ ($N \geq \max(q_1, q_2)$), then*

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = -1.$$



(a) Left half of the Stern-Brocot tree, comprising the Farey sequences.

(b) Stern-Brocot tree.

Figure 2.1: The mediant ratio determines a tree structure over the set of positive rational numbers $\mathbb{Q}^+$. In the left half (right half) of the tree are displayed the fractions of the interval (0,1) (resp. the fractions greater than 1).

Proposition 5. *Given three fractions $\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_2}{q_2}$ in lower terms that are consecutive in a Farey sequence $\mathfrak{F}_N$ ($N \geq \max(q_1, q_2, q_3)$), then it holds that*

$$\frac{p_3}{q_3} = \frac{p_1 + p_2}{q_1 + q_2}.$$

The **mediant ratio** of two fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ is defined to be the fraction

$$\boxminus\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \frac{a_1 + a_2}{b_1 + b_2}.$$

We can display the set of positive irreducible fractions in the nodes of a binary tree, called the Stern Brocot tree (see Figure 2.1) by means of the mediant ratio. In this tree each fraction is the mediant ratio of the pair of fractions above it, one to the left and one to the right. In the $N-th$ level of the left half of the Stern-Brocot tree one can find the whole Farey sequence $\mathfrak{F}_N$, together with some fractions with denominator greater than N (see Figure 2.1(a)).

Let us use G and D to stand for going down to the left or right branch as we proceed from the root of the Stern-Brocot tree to a particular fraction $\frac{M}{N}$. Let us, furthermore, write the composition of branches from right to left, in a similar way as the composition of morphisms. For example, the fraction associated with the path GD is $\frac{3}{2}$.

Recall that $SL_2(\mathbb{N})$ denotes the monoid of 2×2-matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with natural number entries $a, b, c, d \in \mathbb{N}$ and determinant $ad - bc = 1$. As it is well known (see for example [45]) the monoid $SL_2(\mathbb{N})$ is freely generated by the matrices $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The tight connection between matrices in $SL_2(\mathbb{N})$ and continued fraction expansions is revealed clearly by means of with **the Möbius transform** :

With every matrix $\beta \in GL_2(\mathbb{N}) \subset GL_2(\mathbb{C})$ (set of 2x2 invertible matrices with coefficients in $\mathbb{C}$) we associate the linear fractional (or Möbius) transform of the extended complex plane $\mathbb{C} \cup \{\infty\} \overset{\textbf{def}}{=} \widehat{\mathbb{C}}$

$$\mu[\beta] : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \quad \text{with} \quad \mu\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right](z) = \frac{az+b}{cz+d}.$$

The mediant ratio establishes a bijection between the set of positive rational numbers $\mathbb{Q}^+$ and the monoid $SL_2(\mathbb{N})$:

Proposition 6. *For every irreducible fraction $\frac{p}{q} \in \mathbb{Q}^+$ there is a unique matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{N})$ such that*

$$\mu[M](1) = \boxminus\left(\frac{a}{c}, \frac{b}{d}\right) = \frac{p}{q}.$$

Proof. See [9, Proposition 6.2]. □

Remark 1. *Notice that the matrices $M \in SL_2(\mathbb{N})$ can be written as the nodes of a binary tree which left and right branches are labeled respectively with L and R. Previous Proposition determines a bijection between matrices of $SL_2(\mathbb{N})$ and the nodes of the Stern-Brocot tree: one has that $\mu[M](1) = \frac{p}{q}$ if and only if the decomposition of M in $\langle L, R\rangle$ also serves to enconde the path that leads from 1 to $\frac{p}{q}$ in the Stern-Brocot tree.*

The following proposition provides more details of the strong relationship between the continued fraction expansion of an irreducible fraction $\frac{M}{N}$, the path that leads to it in the Stern-Brocot tree and the matrix of $SL_2(\mathbb{N})$ that encodes the fractions, the mediant ratio of which is $\frac{M}{N}$.

Proposition 7. *The following three facts are equivalent:*

1. $\frac{M}{N} = [0; a_1, \ldots, a_k + 1]$.
2. $\frac{M}{N}$ *is the node associated with the path $G^{a_1}D^{a_2}\cdots A^{a_k}$ in the Stern-Brocot tree, where $A = G$ if k is odd and $A = D$ otherwise.*
3. $L^{a_1}R^{a_2}\cdots A^{a_k} = \begin{pmatrix} M_2 & M_1 \\ N_2 & N_1 \end{pmatrix}$ *with* $\frac{M}{N} = \frac{M_1+M_2}{N_1+N_2}$ *and* $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ *are three consecutive N-Farey numbers.*

Proof. See [34][Section 6.7]. □

Example 1. *Let us consider the 7-Farey number $\frac{4}{7}$, the continued fraction expansion of which is $[0; 1, 1, 3] = [0; 1, 1, 2+1]$. The path on the Stern-Brocot tree leading to $\frac{4}{7}$ is GDG^2. The corresponding $SL_2(\mathbb{N})$-matrix is $\alpha_{\frac{4}{7}} = LRL^2 = \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$. $\frac{1}{2} < \frac{4}{7} < \frac{3}{5}$ are three consecutive 7−Farey numbers.*

Let $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ denote the matrix whose associated linear fractional transform is the reciprocal map: $\mu[J](z) = \frac{1}{z}$. Let furthermore $Aut\left(\widehat{\mathbb{C}}\right)$ denote the group of automorphisms of the extended complex plane $\widehat{\mathbb{C}}$. The following facts are easily checked:

1. $\mu : GL_2(\mathbb{C}) \to Aut\left(\widehat{\mathbb{C}}\right)$ is a group homomorphism, and its restriction $\mu : SL_2(\mathbb{N}) \to Aut\left(\widehat{\mathbb{C}}\right)$ is a monoid homomorphism.

2. The formula $\mu[L](g) = \frac{g}{1+g} = \frac{1}{1+\frac{1}{g}}$ with $g \in \mathbb{R}$ determines the following couple of equations:

$$\begin{aligned} \mu[L]([0; a_1, a_2, \ldots]) &= [0; a_1 + 1, a_2, \ldots] \\ \mu[L]([a_0; a_1, a_2, \ldots]) &= [0; 1, a_0, a_1, \ldots] \quad \forall a_0 > 0. \end{aligned}$$

3. From facts 1 and 2 one can easily deduce that

$$\begin{aligned} \mu[L^k]([0; a_1, a_2, \ldots]) &= [0; a_1 + k, a_2, \ldots] \\ \mu[L^k]([a_0; a_1, a_2, \ldots]) &= [0; k, a_0, a_1, a_2, \ldots] \quad \text{if } a_0 > 0. \end{aligned}$$

4. $\mu[R](g) = 1 + g$ implies

$$\mu[R^k]([a_0; a_1, a_2, \ldots]) = [a_0 + k; a_1, a_2, \ldots] \quad a_o \geq 0.$$

5. Finally, it holds that $\mu[J](g) = \frac{1}{g}$ for every $g \in \mathbb{R}$ and thus

$$\begin{aligned} \mu[J]([0; a_1, a_2, \ldots]) &= [a_1; a_2, \ldots] \\ \mu[J]([a_0; a_1, a_2, \ldots]) &= [0; a_0, a_1, \ldots] \quad \forall a_0 > 0. \end{aligned}$$

The following proposition relates Farey numbers with continued fractions:

Proposition 8. *Let $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ be three consecutive fractions in $\mathfrak{F}_N$, and let moreover $\frac{M}{N} = [0; a_1, \ldots, a_{i-1}, k+1]$ with $k \geq 1$ be the continued fraction expansion of the central fraction. Then it holds:*

- *If i is even, $\frac{M_1}{N_1} = [0; a_1, \ldots, a_{i-1}, k]$ and $\frac{M_2}{N_2} = [0; a_1, \ldots, a_{i-1}]$.*
- *If i is odd, $\frac{M_1}{N_1} = [0; a_1, \ldots, a_{i-1}]$ and $\frac{M_2}{N_2} = [0; a_1, \ldots, a_{i-1}, k]$.*

Proof. It is a consequence of Propositions 1, 3 and 5. □

2.2 A mathematical model for musical scales

In the world around us, every sound (every vibration of air) can be decomposed as a combination of "pure sounds". In these pure sounds, the air pressure can be determined by a periodical function of time, e.g. $p(t) = A\sin(2\pi f t)$, where A is the maximum pressure of the wave, and f is the *frequency* or number of cycles per second. The maximum pressure of the wave determines the *intensity* of the sound, meanwhile the frequency, which is usually measured in *Herzs* (Hz), or cycles per second, establishes its pitch. Higher pitches correspond to higher sounds, while lower pitches correspond to deeper sounds. Another essential property of the sound is its *timbre*: two different musical instruments *playing the same note* with frequency f will generate very different sounds. This is because the combination of "pure sounds" that both instruments produce while playing the same note is very different.

Among all the possible *continuous* ranges of frequency, we often use only a discrete set within every *tessitura*, range of pitches of a given voice or instrument. For example, in a piano, there are twelve keys, that is, twelve possible sounds in each octave. We can define a scale *grosso modo* as a fixed set of frequencies that can be transposed to different tessituras. Due to the fact that the present work analyzes mathematical properties of a certain kind of musical scale, among the four fundamental aspects of a sound —intensity, timbre, duration and pitch— from now on we are going to focus solely on pitch.

The frequency $2f$ determines a tone which is said to be *one octave above* the one determined by the frequency f. In general, it is difficult to distinguish two tones that are at a *distance* of an octave. In fact, if the same melody is sung by a woman, a child and a grown man, the voices will melt almost completely into the same melodic line. In musical terms we say that the interval $2 \in \mathbb{R}^+$ is the most *consonant* one (see Remark 2 for a less naive aproach to the notion of musical consonance). The subgroup $\langle 2 \rangle \subset \mathbb{R}^+$ of the octaves, $\langle 2 \rangle = \{2^k;\, k \in \mathbb{Z}\}$, sets an equivalence relationship over the set of frequencies: $\mathcal{F}$: $f \sim f' \Leftrightarrow f = 2^k f'$ for some $k \in \mathbb{Z}$ (f and f' determine the same pitch modulo octaves).

Definition. *We call* ***octave*** $\mathcal{O}$ *to the quotient set* $\mathcal{F}/\langle 2 \rangle$*. The equivalence classes of* $\mathcal{O}$ *are called musical* ***pitches****.*

If we fix a frequency $f_0 \in \mathcal{F}$, the set $\mathcal{F}$ can be identified with the set of intervals $\mathbb{R}^+$ by means of the morphism

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi_{f_0}} & \mathbb{R}^+ \\ g & \longmapsto & g/f_0 \end{array}$$

which assigns to each frequency $g \in \mathcal{F}$ the distance —the **interval**— between g and f_0.

The orbit of each frequency f with respect to the action of the group $\langle 2 \rangle$ over the set $\mathbb{R}^+$ can be identified with:

$$\{2^k \cdot f \quad \text{with} \quad k \in \mathbb{Z}\}.$$

Conversely, the set of octaves that one can construct from a given frequency f are represented in $\mathbb{R}^+$ as intervals of length an integer power of 2. If we want to represent every octave with a segment of *length* 1, we will have to take $\log_2$. There is a second reason why one should consider to work with the logarithm of the frequencies: the ear detects the distance between sounds as the ratio of their frequencies. In other words: the relationship between stimulus and perception is logarithmic. This fact is known as the *Weber-Fechner* law. For example, the distance between two pitches of frequencies $200Hz$ and $500Hz$ will be perceived to be similar as the distance between two tones of frequencies $500Hz$ and $1250Hz$.

The previous discussion is resumed in the following commutative diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi_{f_0}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^+ \arrow[r, "\log_2"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\pi"] \\ \mathcal{F}/\langle 2 \rangle \arrow[r] & \mathbb{R}^+/\langle 2 \rangle \arrow[r, "\log_2"] & \mathbb{R}/\mathbb{Z} \end{tikzcd} \tag{2.1}$$

Example 2. *Given a frequency $f_0 = 440Hz$ (which corresponds to the standard pitch **A**), let us see how the diagram works in one and the other way. The frequency $f_0 \in \mathcal{F}$ corresponds with $0 \in \mathbb{R}/\mathbb{Z}$ via $\pi \circ \log_2 \circ \phi_{f_0}$. If we take another frequency $g = 500$, it will be associated with $\log_2(\frac{500}{440}) \simeq 0.184\,42$ in $\mathbb{R}/\mathbb{Z}$. Conversely, for every number $a \in \mathbb{R}/\mathbb{Z}$ there is a unique frequency g within the octave determine by $440Hz$ and $880Hz$, such that $\log_2(\frac{g}{440}) = a$. For example, the pitch associated with $0.25 \in \mathbb{R}/\mathbb{Z}$ is $2^{0.25} \cdot 440 \simeq 523.25Hz$.*

In short, once we fix a frequency, the set of intervals of the octave $\mathcal{O}$ is identified with the circumference $\mathbb{R}/\mathbb{Z}$ via $\log_2$. We conclude that

> *...the study of the octave is fully equivalent to the study of the intrinsic geometry of an oriented circle.* [42]

We give, finally, the definition of scale.

Definition. *A **scale of** N **notes** is an ordered subset Σ of N elements of the circumference $\mathbb{R}/\mathbb{Z}$:*

$$\Sigma = \{0 \leq x_0 < x_1 < \cdots < x_{N-1} < 1\} \subset \mathbb{R}/\mathbb{Z}.$$

*We will call **step** to the distance $x_{i+1} - x_i$ of two consecutive notes x_i, x_{i+1} in the scale.*

We can assume that the first note of a scale is $(x_0 = 0)$. Otherwise, it would be enough to rotate the circumference so that the first note of the scale matches the 0 in $\mathbb{R}/\mathbb{Z}$.

Example 3. *We call **equal tempered scale of** N **notes** to scale $\Sigma = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$. These scales are represented geometrically by the vertices of regular polygons circumscribed in $\mathbb{R}/\mathbb{Z}$. In Figure 2.2 (on the left) one can see the equal temperament scale of 12 notes, which is the scale produced by 12 consecutive black and white keys in a keyboard.*

Example 4. *The diatonic scale of C Mayor (Figure 2.2, to the right), produced by the 7 white keys of a keyboard starting from a C, is associated with the following subset of the equal tempered scale of 12 notes:*

$$\left\{0, \frac{2}{12}, \frac{4}{12}, \frac{6}{12}, \frac{7}{12}, \frac{9}{12}, \frac{11}{12}\right\}.$$

Remark 2 (Consonance and Spectrum)**.** *Earlier we claimed that the interval $2 \in \mathbb{R}^+$ was the most consonant one. This fact is a simplification of a far more complicated situation. A sound is determined by a complex wave that can be decomposed —via the Fourier transform— into a family of simple sine waves that are called **partials** or **overtones** of the sound. The set of this partials is called the **spectrum** of the sound and basically determines its timbre. The spectrum of a sound produced by an instrument with an acoustic behaviour close to that of the ideal string or the ideal pipe, is composed just of integer multiples of some audible fundamental. Hence, the first two partials of such a sound have 2:1 ratios (the octave $2 \in \mathbb{R}^+$). This is the case for many of the string and wind instruments in western music.*

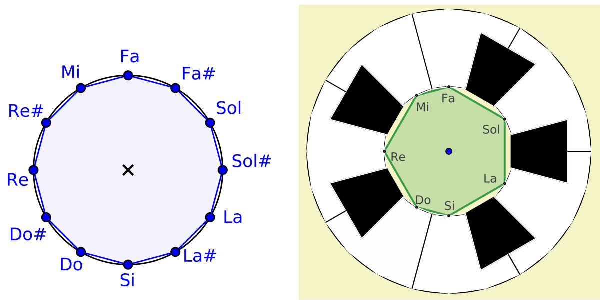


Figure 2.2: The chromatic and the diatonic scales.

In a parallel way, the consonance of an interval formed by two general sounds depends, not only on the frequencies of the fundamental of each sound, but on their respective partials, that is on their timbre or spectrum. In the case of the ideal string or the ideal pipe, two sounds with a ratio 2:1 will have very similar spectra, and hence will be very consonant. For instruments that do not fit this acoustic classical model, like percussion instruments, this is not the case. In general, it would be possible to choose an appropriate timbre so that, for example, the octave between f *and* $2f$ *is dissonant, but the interval* f *to* $2.1f$ *is consonant. This example is the starting point in the fabulous work of W. Sethares [46], where he manages to introduce a whole new perspective to the old problem of consonance-dissonance and establishes the basis of the acoustic in arbitrary scales-temperaments. A short overview of the topic "dissonance and scales" can be found in [19]. Dave Benson offers in [7] a far more extensive survey on this topic (and many others).*

Nevertheless, in our definition of scale, one could fix any interval α *that would work as octave, and then reproduce the diagram 2.1 (replacing 2 by* α*). In any case, the scale is determined by a set of points in the circle* $\mathbb{R}/\mathbb{Z}$ *no matter which interval, the octave or any other, is chosen as the "most consonant" one.*

2.3 Combinatorics on words

A word of length $N \in \mathbb{N}$ over the alphabet $\mathcal{A} = \{x_0, x_1, \ldots, x_{n-1}\}$ is a mapping

$$w : \{0, 1, \ldots, N-1\} \to \mathcal{A}.$$

The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. The empty word is denoted by ϵ. The set $\mathcal{A}^*$ with the composition of words is a free monoid with identity element $\epsilon \in \mathcal{A}^*$. We may consider (right) infinite words. The set of infinite words over the alphabet $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathbb{N}}$. If the alphabet $\mathcal{A}$ is a binary set, a word over $\mathcal{A}$ is called binary word. In this case, $\mathcal{A}$ will be denoted usually by $\{0, 1\}$.

Given a finite word $w \in \mathcal{A}^*$, then $|w|_{x_i}$ denotes the number of occurrences of the letter $i \in \{0, \ldots, N-1\}$ in w and $|w|$ the length of the word. If w is a binary word, $|w|_1$ is called **weight** of w.

We say that a word $w \in \mathcal{A}^*$ is a **factor** of another word u if $u = xwy$ for some words $x, y \in \mathcal{A}^*$. If $x = \epsilon$ $(y = \epsilon)$, w is called prefix (resp. suffix) of u.

The words x, y are **conjugated** if there are factors u, v such that $x = uv$, $y = vu$. If we write the letters of x and y around a circle, the words x, y are

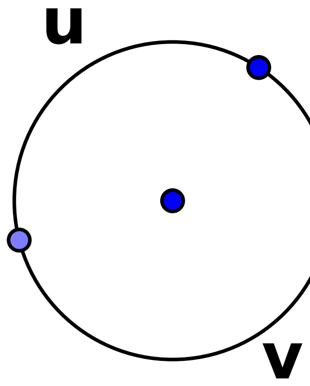


Figure 2.3: Two words are conjugated if and only if they are equivalent modulo rotations.

conjugated if and only if they are equivalent modulo rotations (see Figure 2.3). If we denote by γ the *rotation operator* defined by $\gamma(xa) = ax$, for every word $x \in \mathcal{A}^*$ and every letter $a \in \mathcal{A}$, then it is clear that two words x and y are conjugated words if and only if $x = \gamma^k(y)$ for a certain $k \geq 0$.

Hence, conjugation of words is an equivalent relationship, the classes of which are called **circular words**. The circular word determined by w is denoted by (w). The word u is a factor of (w) if u is a factor of a word conjugated with w.

We call step pattern of a scale Σ to the word w_Σ that *codifies* the sequence of steps of the scale. If the scale has two different steps, its step pattern is a binary word. The scale of an equal tempered scale is the word 0^k.

2.3.1 Christoffel words

In this and the nexts sections we will consider only binary words. The word w of length N given by

$$w(i) = \left[\frac{(i+1)M}{N}\right] - \left[\frac{iM}{N}\right]$$

is called **Christoffel word of slope** $\frac{M}{N}$. Here, $[x]$ denotes the integer part of x. The words 0^k1 and 01^k are called **trivial Christoffel words**.

Christoffel words can be interpreted from a geometrical point of view as a *discretization* of a line of rational slope. Indeed if one consider the line of equation $y = \frac{M}{N}x$ (see Figure 2.4) and one takes the points with natural coordinates closest and below the line, that is $P_n = \left(n, \left[\frac{M}{N}n\right]\right)$. Two consecutive points P_n and P_{n+1} are linked by an horizontal (resp. vertical) segment if $w(n) = 0$ (resp. if $w(n) = 1$).

A Christoffel word w of slope $0 < \frac{M}{N} < 1$ may be written as $w = 0u1$. Moreover, given the evident symmetry of Figure 2.4, the word u is a palindrome (a word that coincides with its retrograde word). The factor u is called **central word** associated with the Christoffel word w, and it plays a central role in the study of the properties of w.

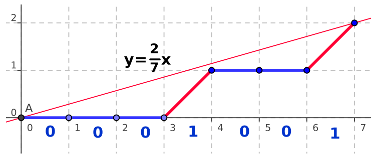


Figure 2.4: Christoffel word of slope $\frac{2}{7}$

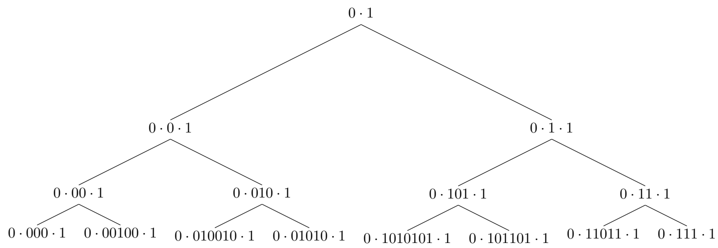


Figure 2.5: The set $0 \cdot PAL(v) \cdot 1$ coincides with the set of Christoffel words.

Now we recall a procedure to generate central words. Given an arbitrary word u, let $\tilde{u}$ denote the mirror image of u, where letters are written in reversed order. Let $w \in \{0,1\}^*$ be a word and let us write $w = uv$, where v is the longest suffix of w that is a palindrome. Then the **right palindromic closure** $w^+ = w\tilde{u}$ is the unique shortest palindrome having w as a prefix (see [29], Lemma 5).

We also define the **right iterated palindromic closure** operator, denoted by PAL, and defined recursively by $PAL(\epsilon) = \epsilon$ and $PAL(w) = (PAL(u)z)^+$, where $w = uz$ and z is the last letter of w.

Example 5. *We compute $PAL(00101)$ in a recursive way to illustrate the definition of right iterated palindromic closure:*

$$\begin{aligned}
PAL(0) &= (PAL(\epsilon)0)^+ = 0 \\
PAL(00) &= (PAL(0)0)^+ = 00 \\
PAL(001) &= (PAL(00)1)^+ = (001)^+ = 00100 \\
PAL(0010) &= (PAL(001)0)^+ = (001000)^+ = 001000100 \\
PAL(00101) &= (PAL(0010)1)^+ = (0010001001)^+ = 0010001001000100.
\end{aligned}$$

The image of the right iterated palindromic closure $PAL : \{0,1\}^* \to \{0,1\}^*$ coincides with the set of central words ([29], Proposition 8). If $PAL(v) = w$, then v is called the directive word of w. Figure 2.5 displays in a tree diagram the set of Christoffel words $0 \cdot PAL(v) \cdot 1$, where $v \in \{0,1\}^*$.

The following classical characterization of central words (see [38, Corollary 2.2.9]) will be useful in the Chapter 4:

Proposition 9. *The set of central words coincides with*

$$0^* \cup 1^* \cup (PAL \cap PAL10PAL)$$

where PAL is the set of palindrome words. The decomposition of a central word c as $c = p10q$ with p, q palindrome words is unique.

This characterization of Christoffel words and many others are covered in the monograph [10]. Throughout the present work we will make use of some of them: standard factorization and the Christoffel tree (Section 3.6), balanced primitive words (Section 3.7) and we will present a couple of new characterizations as well. The first one is based on the Myhill's property (see Corollary 7) and the second one in WF words and accumulation matrices (see Proposition 40).

2.3.2 Sturmian words

The natural generalization of Christoffel words to infinite words are the Sturmian words. A Sturmian word $w_{\rho,\alpha}$ is a mechanical infinite word of irrational slope θ and intercept $\rho \in (0, 1]$, that is, the infinite word $w_{\rho,\alpha} : \mathbb{N} \to \{0, 1\}$ given by the formula

$$w_{\rho,\alpha}(i) = [(i + 1)\theta + \rho] - [i\theta + \rho] .$$

There are plenty equivalent definitions of Sturmian words.

Let w be an infinite word. The complexity function of w is the function P_w:$\mathbb{N} \to \mathbb{N}$ defined by

$$P_w(n) = \text{number of different factors of length } n \text{ in } w.$$

It was proven in [38, Theorem 2.1.13] the following result:

Proposition 10. *A word w is Sturmian if and only if $P_w(n) = n+1, \quad \forall n \geq 1$.*

An infinite word w is said eventually periodic if it can be writen as

$$w = uvvv\ldots \text{ with } u, v \in \{0, 1\}^*.$$

Following Lothaire [38], a word is eventually periodic if and only if $P_w(n) = n$ for a certain $n \in \mathbb{N}$. Therefore, Sturmian words are non-eventually periodic words with minimal complexity.

If $\rho = 0$ and $0 < \alpha < 1$, the Sturmian word $w_{\rho,\alpha}$ can be written as

$$w_{0,\alpha} = 0c_\alpha$$

and the infinite word c_α is called *characteristic word*. The characteristic word can be computed as the limit of a sequence of finite prefixes called standard sequence. Let

$$\alpha = [0; a_1 + 1, a_2, a_3, \ldots] \quad a_1 \geq 0, \quad a_i \geq 1 \, \forall \, i \geq 1$$

be the continued fraction expansion of α. The standard sequence associated with $(a_1, a_2, \ldots)$ is the sequence $\{t_k\}_{k\geq -1}$ defined as:

$$\begin{aligned} t_{-1} &= 1 \\ t_0 &= 0 \\ t_1 &= t_0^{a_1} t_{-1} = 0^{a_1} 1 \\ t_k &= t_{k-1}^{a_k} t_{k-2} \quad \forall \, k \geq 2 \end{aligned}$$

Any term of a standard sequence is called *standard word*. Following [38, Proposition 2.2.24]:

Proposition 11. *Every word t_k of the standard sequence associated to $(a_1, a_2, \ldots)$ is a prefix of the characteristic word c_α of slope $\alpha = [0; a_1 + 1, a_2, a_3, \ldots]$, and one has that*

$$\lim_{k \to \infty} t_k = c_\alpha.$$

A word w is balanced if $||x|_1 - |y|_1| \leq 1$ for every pair of factors x, y of the same length $|x| = |y|$.

Following [38, Theorem 2.1.5], Sturmian words coincide with the aperiodic balanced words.

2.3.3 Sturmian morphisms

An endomorphism of $\{0,1\}^*$ is a set mapping $f : \{0,1\}^* \mapsto \{0,1\}^*$ such that

$$f(ww') = f(w)f(w') \quad \text{for all} \quad w, w' \in \{0,1\}^*.$$

An endomorphism f is determined by the images of 0 and 1, so we identify f with the ordered pair $(f(0), f(1))$. A morphism f is Sturmian if $f(w)$ is a Sturmian word for every Sturmian word w.

There is a close connection between Sturmian morphisms and conjugates of Christoffel words, as following result from [12, Lemma 4.1] shows:

Proposition 12. *A morphism is Sturmian if and only if it sends each Christoffel word onto the conjugate of a Christoffel word.*

The set of Sturmian morphisms is denoted by *St*. The following five endomorphisms are, as we will soon see, particularly important in the study of St:

$$G = (0, 01), \quad D = (10, 1), \quad E = (1, 0),$$
$$\tilde{G} = (0, 10), \quad \tilde{D} = (01, 1).$$

Theorem 3. *The monoid St coincides with $\left\langle E, G, \tilde{G} \right\rangle$.*

Proof. See [38, Section 2.3.1] □

Notice that following identities hold:

$$EGE = D \qquad E\tilde{G}E = \tilde{D}$$

and it follows that

$$St = \left\langle E, G, \tilde{G}, D, \tilde{D} \right\rangle.$$

Some submonoids of St are particularly important. A Sturmian morphism f is called *Christoffel morphism* if $f \in \left\langle G, \tilde{D} \right\rangle$. We will provide in Corollary 4 an alternative proof of a classical result concerning this monoid: every Christoffel word w can be written as $f(01)$ for a certain Christoffel morphism f. In other words, Christoffel morphims generate the set of Christoffel words. A Sturmian morphism f is called Standard morphism if $f \in \langle G, D, E \rangle$. Following [38, Proposition 2.3.11],

Proposition 13. *A morphism f is Standard if and only if one of the words $f(01)$ or $f(10)$ is a standard word.*

Chapter 3

Characterizations of well-formed scales

In this chapter we introduce the family of scales that will be center of our attention in the rest of the work. We present firstly the Three Gaps Theorem, a classical result from number theory that asserts, in scale language, that any generated scale has at most three different steps. Among all the generated scales, some of them have just two steps. These are, basically, the so called *well-formed scales*. Later on, we prove seven different characterizations of these kind of scales. We end up by introducing a notion of duality between well-formed scales. This notion will be of interest in each one of the chapters of this PhD.

3.1 Generated scales. Three gaps theorem

In section 2.2, the most consonant interval, the octave, was used to define an equivalece relationship on pitches and, thereafter, the set of pitches within a range of an octave was identified with the circumference $\mathbb{R}/\mathbb{Z}$. The second most consonant interval, the *pure fifth*, which corresponds with the ratio $\frac{3}{2}$, can be used to build a scale with *integer multiples* of $\log_2\left(\frac{3}{2}\right) = \{\log_2 3\}$:

$$\{0, \{\log_2 3\}, \{2\log_2 3\}, \ldots, \{(N-1)\log_2 3\}\},$$

where $\{x\}$ denotes the decimal part of x. This scale generated by fifths is called *pitagoric scale* (See Figure 3.5).

The notion of pythagoric scale can be generalized easily to the notion of scale generated by an arbitrary generator $\theta \in [0, 1)$ and number of notes $N > 1$:

Definition. *The scale*

$$\Gamma(\theta, N) = \{\{k\theta\}, k = 0, 1, \ldots, N-1\}$$

is called ***generated scale of*** N ***notes*** *and* θ *is called* ***generator*** *of the scale.*

Geometrically, a generated scale of N notes coincides with a star polygonal chain in the circumference with constant central angle $2\pi\theta$. In Figure 3.1 we can see how this kind of polygonal *regular* chain divides the circumference into

(a) $\Gamma(\{e\},4)$

Notes	Steps
$x_0 = 0$	$x_1 - x_0 \simeq 0.154$
$x_1 \simeq 0.154$	$x_2 - x_1 \simeq 0.282$
$x_2 \simeq 0.436$	$x_3 - x_2 \simeq 0.282$
$x_3 \simeq 0.718$	$1 - x_3 \simeq 0.282$

(b) $\Gamma(\{e\},5)$

Notes	Steps
$x_0 = 0$	$x_1 - x_0 \simeq 0.154$
$x_1 \simeq 0.154$	$x_2 - x_1 \simeq 0.282$
$x_2 \simeq 0.436$	$x_3 - x_2 \simeq 0.282$
$x_3 \simeq 0.718$	$x_4 - x_3 \simeq 0.154$
$x_4 \simeq 0.872$	$1 - x_4 \simeq 0.128$

Table 3.1: Two scales generated by $\{e\}$ that have 2 and 3 different steps.

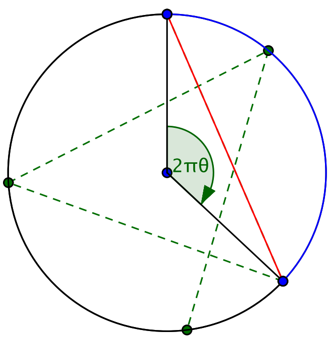


Figure 3.1: The scale generated by θ is represented in the circumference as a star polygonal chain which arches have a fixed amplitude $2\pi\theta$.

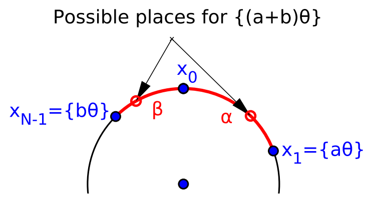


Figure 3.2: First, last note of $\Gamma(\theta, N)$ and the $(N+1)$-th note $x_N = \{(a+b)\theta\}$.

N arches. If one repeats this construction with distinct values of θ and N, one obtains arches of at most three different lengths. In musical terms *every generated scale has, at most, three different steps.* For example, table 3.1 shows the notes and steps of the scales of 4 and 5 notes generated by $\{e\} = 0.718\ldots$. They have, respectively, 2 and 3 different steps.

This fact, called *Steinhaus conjecture* or *three gaps theorem*, was proved simultaneously by Sós [48] and Świerczkowski [49] in 1958. Many other equivalent formulations of the three gaps problem have been made. Some of them from an arithmetic point of view ([47], [53]), some using the language of the combinatorics on words (see [3]).

We present a proof based on the ideas of N. B. Slater in [47], but using a terminology an notation of the language of musical scales.

Theorem 4 (Three gaps theorem)**.** *The number of different steps of a scale* $\Gamma(\theta, N)$ *is* 1, 2 *or* 3.

Proof. Let a and $b \in \{1, \ldots, N-1\}$ such that

$$x_1 = \{a\theta\} = \min_{1 \le k \le N-1} \{k\theta\}, \qquad x_{N-1} = \{b\theta\} = \max_{1 \le k \le N-1} \{k\theta\},$$

so that $\alpha = \{a\theta\}$ and $\beta = 1 - \{b\theta\}$ are the first and the last steps of the scale (see Figure 3.2). Depending on whether $\{a\theta\} + \{b\theta\}$ is ≤ 1 or ≥ 1, it holds:

$$\{b\theta\} < \{(a+b)\theta\} \leq 1 \quad \text{if} \quad \{(a+b)\theta\} = \{a\theta\} + \{b\theta\} \leq 1 \tag{3.1}$$
$$0 \leq \{(a+b)\theta\} < \{a\theta\} \quad \text{if} \quad \{(a+b)\theta\} = \{a\theta\} + \{b\theta\} - 1 \geq 0. \tag{3.2}$$

It follows that $\{(a+b)\theta\}$ yields in the arch determined by x_{N-1} and x_1, as one can see in Figure 3.2.

Let us compute now the interval between two arbitrary notes $\{r\theta\} > \{s\theta\}$ in the scale:

$$\{r\theta\} - \{s\theta\} = \begin{cases} \{(r-s)\theta\} \geq \min\{k\theta\} = \alpha & \text{if } r > s, \\ 1 - \{(s-r)\theta\} \geq 1 - \max\{k\theta\} = \beta & \text{if } s > r. \end{cases}$$

In the first case, if $\{(r-s)\theta\} = \alpha$, the notes $\{r\theta\}$ and $\{s\theta\}$ must be consecutive. Effectively, if there were a $t \in \{1, \ldots, N-1\}$ such that $\{r\theta\} > \{t\theta\} > \{s\theta\}$:

- if $t > s$, then $\{t\theta\} - \{s\theta\} = \{(t-s)\theta\} \geq \alpha$, impossible.
- if $t < s$, then $r > t$ so that $\{(r-t)\theta\} \geq \alpha$, which is again not possible.

In a similar way one could prove that, if $1 - \{(s-r)\theta\} = \beta$, $\{s\theta\}$ and $\{r\theta\}$ are consecutive notes. Therefore, we have $N-a$ steps of lengths α:

$$\{(r+a)\theta\} - \{r\theta\} = \alpha \quad \forall r = 0, 1, \ldots, N-a-1 \tag{3.3}$$

and $N-b$ steps of length β:

$$1 - \{b\theta\}, \{(r-b)\theta\} - \{r\theta\} = \beta \quad \forall r = b+1, \ldots, N-1. \tag{3.4}$$

It remains to compute the length of $N-(N-a)-(N-b) = a+b-N$ steps. Notice that, by definition of a and b, $\{(a+b)\theta\}$ cannot be a note in the scale, and therefore $a+b \geq N$. If $N = a+b$, the scale has just two different steps, α and β, or just one in case that $\alpha = \beta$ (if $\theta = \frac{p}{q}$ and $N \geq q$, the scale coincides with a regular star polygon of q vertices, the steps of which are $\frac{1}{q}$). Let us suppose that $a+b > N$. We will prove that the length of the remaining $N-(a+b)$ steps is $\alpha+\beta$. These steps start from the notes:

$$\{k\theta\}, \text{ con } k \in \{N-a, N-a+1, \ldots, b-1\},$$

so we have to determine when the length of the interval $\{\nu\theta\} - \{k\theta\}$, with $\{\nu\theta\} > \{k\theta\}$ and $k \in \{N-a, N-a+1, \ldots, b-1\}$ is minimum.

- If $\nu > k$, we have that $\nu - k < N-(N-a) = a$. It follows that,

  $$\{(\nu-k)\theta\} = \{a\theta - (a-\nu+k)\theta\} = \{a\theta\} + 1 - \{(a-\nu+k)\theta\} \geq \alpha + \beta$$

  and the equality holds if and only if $a - \nu + k = b$, that is, $\nu = a+k-b$.
- An analogous argumentation shows the same result in case $k > \nu$.

Summarizing, if $k \in \{N-a, N-a+1, \ldots, b-1\}$, the notes $\{(k+a-b)\theta\}$ and $\{k\theta\}$ are consecutive and they determine a step of length $\alpha+\beta$. $\square$

We have implicitly proven the following result:

Corollary 1. *If θ is irrational, the scale $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ has two different steps if and only if the next note to be generated, $\{N\theta\}$, lays between the first x_1 and the last note x_{N-1} ot the scale, that is, if one of the following conditions hold:*

- $0 < \{N\theta\} < x_1$.
- $x_{N-1} < \{N\theta\} < 1$.

Proof. Using the same notation as in previous proof, $x_1 = \{a\theta\}$ and $x_{N-1} = \{b\theta\}$. The scale has two intervals if and only if $N = a + b$ or, equivalently, if and only if equations 3.1 and 3.2 of previous theorem hold. □

The two-steps case will be of central interest in the following sections. The theorem below reveals a deeper characterization of the scales $\Gamma(\theta, N)$ that have two different steps. The Figure 3.15 illustrates the proof in the particular case of $\theta = [0; 3, 2, 3, 4, \ldots] \simeq 0.2912\ldots$ and $N = 4, 7, 10, 31$.

Theorem 5. *Let $0 < \theta < 1$ be an irrational number. The scale*

$$\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$$

has two different steps $\alpha = x_1$ and $\beta = 1 - x_{N-1}$ if and only if N is the denominator of a semi-convergent M/N of θ. In this case, $x_M = \{\theta\}$, that is, the first note to be generated is the $M-th$ in natural order. Moreover, if $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ are three consecutive fractions in $\mathfrak{F}_N$, the following statements hold:

1. *N_1 and N_2 are the frequencies of β and α respectively.*
2. *M_1 and M_2 are the frequencies of β and α in the interval delimited by $x_0 = 0$ and $x_M = \theta$.*

Proof. Let $\theta = [0; a_1, a_2, \ldots]$ the continued-fraction expansion of θ. We will prove that for every $i = 1, 2, \ldots$, the values of M and N for which $x_M = \{\theta\}$ and $\Gamma(N, \theta)$ is a two-steps scale, can be computed by the recurrent formulas given in Proposition 1 and hence, they coincide with the numerators and denominators of the convergents and semi-convergents of θ: $[0; a_1 \ldots, a_{i-1}, k+1]$ for $k = 1, 2, \ldots, a_i$. We will proceed by induction on i.

Let us check that the Theorem is true if $i = 1$. Following Proposition 2, the segment of length θ fits a_1 times in the segment of length 1. Thus the scales with $N = 2, \ldots, a_1 + 1$ steps have two different steps, namely $\alpha = \theta$ and $\beta = 1 - N\theta$ (see Figure 3.3(a)). In any case, $\{\theta\} = x_1$ (and thus $M = 1$). Notice that $\frac{1}{N}$ is a (semi-)convergent of θ for $N = 2, \ldots, a_1 + 1$ (the case $N = a_1$ corresponds with a full convergent). The three consecutive fractions in $\mathfrak{F}_N$ will be

$$\frac{0}{1} < \frac{1}{N} < \frac{1}{N-1}.$$

There are $N - 1$ steps of length α and 1 step of length β. The interval delimited by $x_0 = 0$ and $x_M = x_1 = \theta$ splits in 0 intervals of length β and 1 interval of length α. It follows that the proposition is true for $i = 1$ and the (semi-) convergents are $[0; k+1]$ with $k = 1, \ldots, a_1$.

Let us assume that the proposition holds for $j < i$: the (semi-)convergents $[0; a_1, \ldots, a_{j-1}, k+1]$ for all $k = 1, \ldots, a_j$ are precisely those fractions that satisfy the conditions of the Theorem. In particular, if we take

$$\frac{M}{N} = [0; a_1, \ldots, a_{i-2}, a_{i-1} + 1],$$

the scale $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ fulfils the conditions of the Theorem: it has two different steps $\alpha = x_1$ and $\beta = 1 - x_{N-1}$, $x_M = \theta$ and, if $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ are three consecutive fractions in $\mathfrak{F}_N$, N_1 y N_2 are the frequencies of β and α respectively and, finally, M_1 y M_2 are the frequencies of β and α in the interval delimited by $x_0 = 0$ and $x_M = \theta$.

Let us prove the i−th step of the induction. If i is odd, then $\beta > \alpha$ (see Figure 3.3(c)). This is an immediate consequence of the geometrical interpretation of the coefficients of the continued fraction expansion of θ (Proposition 2). The next note to be generated, x_N, will divide a step of length β into two steps of lengths $\alpha' = \alpha$ and $\beta' = \beta - \alpha'$. Therefore, if we want to construct the next two-steps scale, we have to divide all the steps of length β into two different steps of lengths α' and β'. For that purpose we need N_1 new notes. Thus, the scale

$$\Gamma(N + N_1, \theta) = \{0 = x'_0 < x'_1 < \cdots < x'_{N+N_1-1}\}$$

has two different steps, α' and β', which appear, respectively, $N_1 + N_2 = N$ and N_1 times in the scale. Moreover, between $x'_0 = 0$ and $\{\theta\} = x_M$ we have introduced M_1 new notes. It follows that $\theta = x'_{M+M_1}$.

As i is odd, in accordance with Propositions 8 and 1, $\frac{M_1}{N_1} = [0; a_1, \ldots, a_{i-1}]$, $\frac{M}{N} = [0; , \ldots, a_{i-1} + 1]$ and

$$\frac{M_1 + M}{N_1 + N} = \frac{M_2 + 2M_1}{N_2 + 2N_1} = [0; a_1, \ldots, a_{i-1}, 2].$$

In short, we have three Farey numbers $\frac{M_1}{N_1} < \frac{M+M_1}{N+N_1} < \frac{M}{N}$ that are consecutive in $\mathfrak{F}_{N+N_1}$ and satisfying all conditions of the Proposition. Moreover, the center fraction $\frac{M+M_1}{N+N_1}$ coincides with the next (semi-)convergent of θ.

Following a similar argumentation, the scales $\Gamma(N_2 + kN_1, \theta)$ have two distinct steps for all $k = 2, \ldots, a_i + 1$. For each scale it holds that $\theta = x_{M_2+kM_1}$ and the three fractions associated with each of those scales, namely

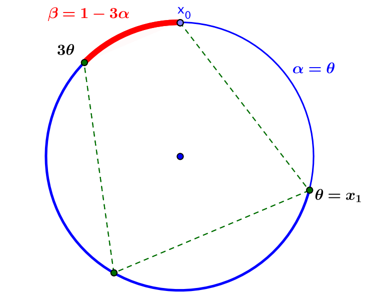
$$\frac{M_1}{N_1} < \frac{M_2 + kM_1}{N_2 + kN_1} < \frac{M_2 + (k-1)M_1}{N_2 + (k-1)N_1}$$

the numerator and denominator of which verify conditions 1 and 2 of our Theorem. The corresponding central fractions $\frac{M_2+kM_1}{N_2+kN_1}$ coincide with the successive (semi-)convergents $[0; a_1, \ldots, a_{i-1}, k]$ of θ, as we wanted to prove.

The case of $\beta < \alpha$ (or, equivalently, the case of i even) is symmetric (see Figure 3.3(b) and 3.3(d)). If we generate $N_1 + kN_2$ notes with the generator θ, the step α will split in two intervals of lengths β and $\alpha - k\beta$ for $k = 2, \ldots, a_i + 1$. Following the argumentation of previous paragraphs, the three fractions that carry all the information of the scale $\Gamma(N_1 + kN_2, \theta)$ are

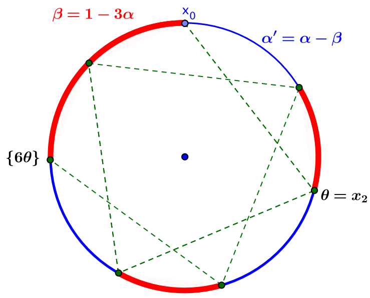
$$\frac{M_1 + (k-1)M_2}{N_1 + (k-1)N_2} < \frac{M_1 + kM_2}{N_1 + kN_2} < \frac{M_2}{N_2}.$$

□



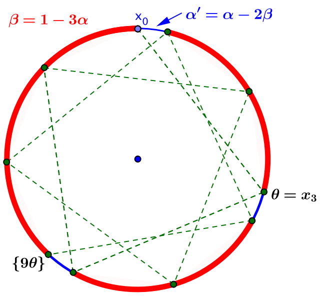
Semiconvergent of θ	$[0;3,1] = \frac{1}{4}$
Frequency of α	3
Frequency of β	1
Number of notes N	4
Position of the note θ	$x_1 = \theta$
Decomposition of θ	$\theta = 1\alpha + 0\beta$
Consecutive Farey numbers	$\frac{0}{1} < \frac{1}{4} < \frac{1}{3}$

(a) $\Gamma(\theta, 4)$ is a well-formed scale.



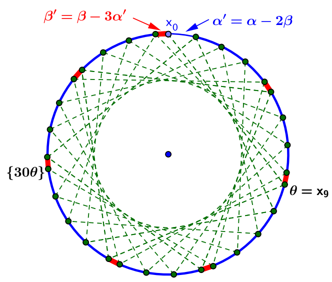
Semiconvergent of θ	$[0;3,2] = \frac{2}{7}$
Frequency of α	3
Frequency of β	4
Number of notes N	7
Position of the note θ	$x_2 = \theta$
Decomposition of θ	$\theta = 1\alpha + 1\beta$
Consecutive Farey numbers	$\frac{1}{4} < \frac{2}{7} < \frac{1}{3}$

(b) The scale $\Gamma(\theta, 7)$ is the following well-formed case.



Semiconvergent of θ	$[0;3,2,1] = \frac{3}{10}$
Frequency of α	3
Frequency of β	7
Number of notes N	10
Position of the note θ	$x_3 = \theta$
Decomposition of θ	$\theta = 1\alpha + 2\beta$
Consecutive Farey numbers	$\frac{2}{7} < \frac{3}{10} < \frac{1}{3}$

(c) The scale $\Gamma(\theta, 10)$, the following case of well-formedness.



Semiconvergent of θ	$[0;3,2,3,1] = \frac{9}{31}$
Frequency of α	24
Frequency of β	7
Number of notes N	31
Position of the note θ	$x_9 = \theta$
Decomposition of θ	$\theta = 7\alpha + 2\beta$
Consecutive Farey numbers	$\frac{2}{7} < \frac{9}{31} < \frac{7}{24}$

(d) The scale $\Gamma(\theta, 31)$ is, again, well-formed.

Figure 3.3: Four scales generated by $\theta = [0;3,2,3,4,\ldots]$ with two different steps

If the generator is rational, there could be just two steps and the number of notes N may not coincide with the denominator of a (semi-)convergent of θ:

Proposition 14. *If the scale $\Gamma(\theta, N)$ is generated by a rational number $\theta = \frac{p}{q} = [0; a_1, \ldots, a_k] \in \mathbb{Q}$ where $a_k > 1$ and $\frac{p'}{q'} = [0; a_1, \ldots, a_k - 1]$ is the penultimate (semi-)convergent of θ:*

1. *Γ has just one step if $N \geq q$.*
2. *Γ has two different steps if one of the following conditions holds:*
   (a) *N is the denominator of a (semi-)convergent of θ.*
   (b) *$q' < N < q$.*
3. *Γ has three different steps in other case.*

Proof. Notice that, if $N \geq q$ the scale $\Gamma(\theta, N)$ coincides with an equal temperament scale. On the other hand, if $N = q'$, the steps of the scale are α y $2\alpha = \beta$, and therefore every scale $\Gamma(\theta, N')$ with $q' < N' < q$ will have two different steps. □

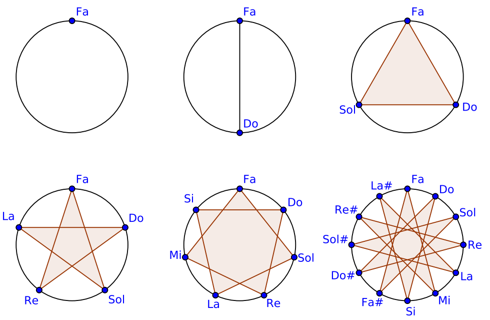
3.2 Well-formed scales

Throughout the last third of XX century, structural properties of musical scales have become the focus of a field of music theory called the "Diatonic Set Theory". The "diatonic" aspect of the title indicates the fact that the diatonic scale (see Figure 3.11) has served as a starting point for more general scale theories. Among others, we can cite some of the goals of the Diatonic Set Theory:

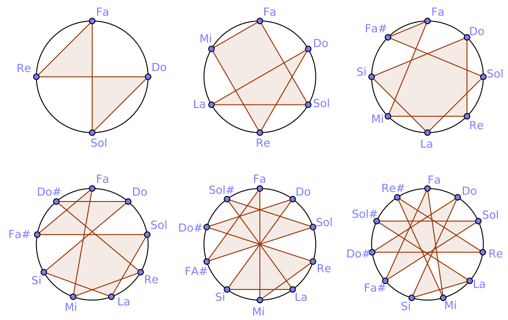
1. To identify and generalize the structural properties of the most frequent scales.
2. To propose new "microtonal" scales (scales with steps smaller than $\frac{1}{12}$) based upon the above generalizations.
3. To analyse, from a structural point of view, the existence of certain scales over time and across different cultures.

The core of the Diatonic Set Theory was formed by the works of Gerald Balzano ([6]), Eytan Agmon ([1] and [2]) and John Clough ([24] and [27]). N. Carey and D. Clampitt ([15], [16], [17]) managed to bring together a great proportion of the theory when they introduced the notion of **well-formed scale**. These scales were first defined ([16]) as the scales generated by fifths ($\theta = \log_2\left(\frac{3}{2}\right)$) that verify a couple of conditions: the *closure condition* and the *symmetry condition*.

A generated scale $\Gamma(\theta, N)$ verifies the *closure condition* if the note $\{N\theta\}$ yields in the arch of circumferences delimited by the last note x_{N-1} of Γ, and the first one x_1. As it was seen in Corollary 1 and Theorem 5, the closure condition holds whenever N is the denominator of a (semi-)convergent of the scale Γ.



(a) Pitagoric scales of $\{1, 2, 3, 5, 7, 12\}$ notes verify the symmetry condition.



(b) The pitagoric scales of $\{4, 6, 8, 9, 10, 11\}$ do not.

Figure 3.4: Scales generated by fifths of $\{1, 2, \ldots, 12\}$ notes.

N. Carey y D. Clampitt described the *symmetry condition* in the following way. They placed the first N notes of the scale generated by fifths, (Fa-Do-Sol-...), in N equidistant points of the circumference, and then they drew the polygonal chain that joins these points, starting from the lower note (Fa) and ending in the higher one. If the resulting star polygon is regular, the scale exhibits the symmetry condition (see Figures 3.4(a) and 3.4(b)). As it is well-known, in a regular star polygon of N vertexes, there is a M, prime with N such that....

Given $k \in \mathbb{Z}_N^* = \{x \in \mathbb{Z}_N \text{ such that } (x, N) = 1\}$, the automorphism τ_k of the additive group $\mathbb{Z}_N$ is defined by

$$\begin{array}{rccl} \tau_k : & \mathbb{Z}_N & \longrightarrow & \mathbb{Z}_N \\ & i & \longmapsto & ki \mod N. \end{array}$$

We now give a formal definition of well-formed scale.

Definition. *Let $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ be a scale of N notes generated by θ. We say that Γ is* ***well-formed****, or Γ is a* ***WF scale****, if there is a $M \in \mathbb{Z}_N^*$, called* ***multiplier of the scale****, such that:*

$$\{k\theta\} = x_{\tau_M(k)} \quad \forall k = 0, 1, \ldots, N-1.$$

Previous condition can be reformulated as follows:

$$\{k\theta\} = x_{\tau_M(k)} \quad \forall k \in \mathbb{Z}_N \;\Leftrightarrow\; \{\tau_{M^*}(k)\theta\} = x_k \quad \forall k \in \mathbb{Z}_N,$$

where $M^* = M^{-1} \mod N$. Thus, we can state an equivalent definition of WF scale as follows:

Definition. *A scale $\Gamma(\theta, N)$ is WF with multiplier $M \in \mathbb{Z}_N$ if it coincides with the scale:*

$$\Gamma = \{0 = \{0M^*\theta\} < \{M^*\theta\} < \{(2M^*)_N\theta\} < \cdots < \{((N-1)M^*)_N\theta\} < 1\}$$

where $M^* = M^{-1} \mod N$, and we have use $(k)_N$ to denote $(k \mod N)$ (in other words, the coefficients of the expresion above are reduced modulo N). The family of Well-formed scales of N notes and multiplier M is denoted by $WF(N, M)$.

The simplest example of WF scale is the equal tempered scales. These scales are called **degenerate WF scales**:

Example 6. *Let $\theta = \frac{p}{N} < 1$ a fraction in lower terms. The corresponding equal temperament scale $\Gamma(\theta, N)$ is well-formed. Notice that $x_1 = \{p^* \frac{p}{N}\} = \frac{1}{N}$ and, thus,*

$$\{k\theta\} = \left\{\frac{kp}{N}\right\} = x_{kp} = x_{\tau_p(k)}.$$

It follows that $\Gamma(\frac{p}{N}, N)$ is a WF scale with multiplier p and it has just one step of length $\frac{1}{N}$.

A second example of WF scale is the pitagoric scale of 7 notes:

Example 7. *$\Gamma(\{\log_2 3\}, 7)$ is a WF scale with multiplier 4 (see Figure 3.5).*

And a last important example is the diatonic scale of 7 notes (see Figure 2.2 to the right):

Example 8. *The scale $\Gamma\left(\frac{7}{12}, 7\right) = \left\{0, \frac{2}{12}, \frac{4}{12}, \frac{6}{12}, \frac{7}{12}, \frac{9}{12}, \frac{11}{12}\right\}$ is WF with multiplier 4.*

In the following sections we will give equivalent definitions of WF scale, that will be gathered in the final Theorem 12.

3.3 WF scales and scales with even generator

Let $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ be a generated scale and let us consider $(n_i)_{i \in \mathbb{Z}_N}$ such that $\{i\theta\} = x_{n_i}$. That is, $i \to n_i$ is the mapping that transforms generation order ($\{k\theta\}$) into natural order (x_i).

Definition. *The generator θ of the scale $\Gamma(\theta, N)$ is **even** if there is a constant $k \in \mathbb{Z}_N$, called **span** of the generator, such that $n_{i+1} = n_i + k \mod N$, $\forall i = 0, 1, \ldots, N-2$.*

Geometrically, the generator of $\Gamma(\theta, N)$ is even if the number of steps in each appearance of θ is constant. This notion corresponds to the fact that the star polygon that can be built in a similar way than those in Figure 3.4 is regular. Therefore, the symmetry condition can be thought of being equivalent to the evenness of the generator.

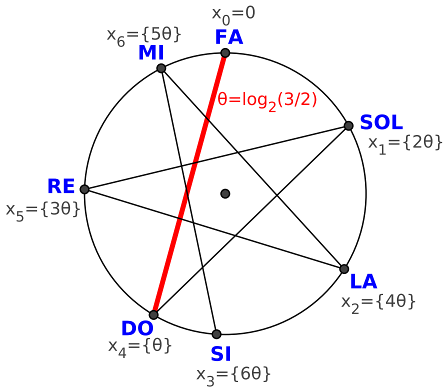
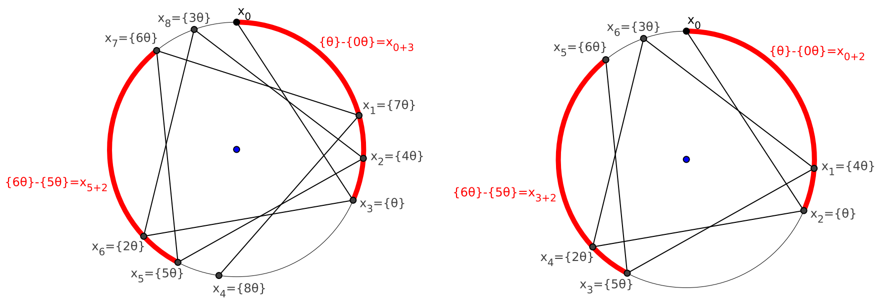


Figure 3.5: The scale of 7 notes generated by $\{\log_2(3)\}$ is WF with multiplier 4, that is, $x_{\tau_4(i)} = \{i\theta\}\ \forall i \in \mathbb{Z}_N$.



(a) The scale $\Gamma(\{\log_2 5\}, 9)$ does not have an even generator.

(b) The generator of scale $\Gamma(\{\log_2 5\}, 7)$ is even and its span is 2.

Figure 3.6: Two examples of scales generated by the just third $\frac{5}{4}$.

Example 9 (Figure 3.6)**.** *The generator of the scale* $\Gamma(\{\log_2 5\}, 9)$*, shown in Figure 3.6(a), decomposes in* 2 *or* 3 *steps. The generator of this scale is, therefore, not even. On the contrary, the generator of the scale* $\Gamma(\{\log_2 5\}, 7)$*, shown in Figure 3.6(b), is even of span 2.*

The following proposition shows the equivalence between the WF scales and the scales with even generator.

Proposition 15. *The scale* $\Gamma(\theta, N)$ *is WF with multiplier* M *if and only if* $\Gamma(\theta, N)$ *has an even generator of span* M*.*

Proof. By definition, $\Gamma(\theta, N)$ is WF with multiplier M if $x_{\tau_M(k)} = \{k\theta\}\ \forall k \in \mathbb{Z}_N$; but at the same time

$$\begin{aligned} x_{\tau_M(k)} = \{k\theta\} \quad \forall k \in \mathbb{Z}_N \ &\Leftrightarrow\ n_k = \tau_M(k) \quad \forall k \in \mathbb{Z}_N \\ &\Leftrightarrow\ n_{k+1} = Mk + M = n_k + M \quad \forall k \in \mathbb{Z}_N, \end{aligned}$$

and thus the scale has an even generator of span M. $\square$

In summary, in a well-formed scale generated by θ and with multiplier M, each appearance of θ decomposes in M steps.

3.4 WF scales and the continued fraction expansion of the generator

As a consequence of Theorem 5, we prove now that a scale is WF if the number of notes is the denominator of a (semi-)convergent of the generator. This result was proven by Norman Carey and David Clampitt in [?]. Here we present an alternative proof.

Theorem 6. *Let $0 < \theta < 1$ be an irrational number and $N > 1$ a natural number. The scale $\Gamma(\theta, N)$ is well-formed and its multiplier is $M \in \mathbb{Z}_N^*$ if and only if $\frac{M}{N}$ is a (semi-)convergent of the generator θ.*

Proof. Let $\Gamma(\theta, N)$ be a well-formed scale with multiplier $M \in \mathbb{Z}_N^*$ By Theorem 5, it will be enough to show that the WF scale $\Gamma(\theta, N)$ with multiplier M has two different steps. Notice that, as the scale is WF, $x_{i+1} - x_i = \{(M^*(i+1) \mod N) \cdot \theta\} - \{(M^* i \mod N) \cdot \theta\}$. This expression has two possible values, $\{M^*\theta\}$ and $1 - \{(N - M^*)\theta\}$, which are different, since θ is irrational and M^* is prime with N.

Let $\frac{M}{N}$ be, now, a fraction in lower terms that is a (semi-)convergent of θ and let us prove that that $\Gamma(\theta, N)$ is WF with multiplier M. I is evident, from Theorem 5, that the scale has two different steps. Following the notation of the proof of the three-gap theorem, $N = a + b$ with $x_1 = \{a\theta\}$ and $x_{N-1} = \{b\theta\}$ the first and the last note of the scale. If we use the equations (3.3) and (3.4) of that proof, it is possible to write x_{i+1} depending on $x_i = \{r\theta\}$ in the following way:

$$x_{i+1} = \begin{cases} \{(r+a)\theta\} \Leftrightarrow r + a \leq N - 1, \\ \{(r-b)\theta\} = \{(r+a-N)\theta\} \Leftrightarrow N \leq r + a < 2N. \end{cases}$$

This is equivalent to $x_j = \{\tau_a(j)\theta\}$ $\forall j = 1, \ldots, N-1$ and, by definition, the scale $\Gamma(\theta, N)$ if WF and its multiplier is $a^* = a^{-1} \mod N$. We conclude, since again following Theorem 5, $x_M = \{\theta\} \Rightarrow a^* = M$. □

It is now immediate, again following the cornerstone Theorem 5, that WF scales coincide with two-step scales, for irrational generators:

Corollary 2. *If θ is irrational, the generated scale $\Gamma(\theta, N)$ is WF if and only if it has 2 different steps.*

If the generator θ is rational, the scale is WF if it is degenerated (1 step) or it has two steps with the exception given in Proposition 14, case 2b:

Corollary 3. *If $\frac{p}{q}$ is a fraction in lower terms, the scale $\Gamma(\frac{p}{q}, N)$ is a non degenerated WF-scale if and only if it has 2 different steps and $N \leq q'$, where q' is the denominator of the last (semi-)convergent of $\frac{p}{q}$ different from $\frac{q}{p}$ itself.*

Proof. Notice that if $N = q$ the scale is an equal temperament scale of N notes and steps $\frac{1}{N}$ (regular star-polygon). If $N = q'$ the scale is WF and has two different steps with length $\frac{1}{N}$ and $\frac{2}{N}$. It follows that for every $N =$

$q'+1, q'+2, \ldots, q-1$ the scale has two different steps, namely $\frac{1}{N}$ and $\frac{2}{N}$. But in this cases the scale is not WF since the number of notes does not coincide with the denominator of a (semi-)convergent of the generator $\frac{q}{p}$. $\square$

Remark 3. *M. J. Garmendia and J.A. Navarro defined in [41] the Pythagorean scales as those scales generated by fifths which have two different steps. They showed that this is the case precisely when the number of notes generated in the scale coincides with the denominator of a (semi-convergent) of the generator* $\log_2\left(\frac{3}{2}\right)$. *It is remarkable how these authors were pointing at the idea of well-formed scale in a totally independent way of N. Carey and D. Clampitt.*

3.5 WF scales and Christoffel words

A scale with two steps α and β can be interpreted as an ordered sequence of two letters α and β, that is, a finite word in the alphabet $\{\alpha, \beta\}$. For example, the scale in Figure 3.5 is associated with the word $\alpha\alpha\alpha\beta\alpha\alpha\beta$, which coincides with the Christoffel word of slope $\frac{2}{7}$ (see Figure 2.4). In this section we will prove that this is not a coincidence: the *pattern* of a WF scale is always a Christoffel word.

This link between theory of words and theory of scales opens a new perspective: one can translate plenty of problems of one field into the other. This is the case of the last two chapters of the present work: the problem of extending Christoffel duality to the set of modes of a WF scale is considered and partially solved in mathematical terms (Chapter 5). On the other hand, the musical intuition of this problem leads to deal with a purely mathematical issue: the extension of Christoffel duality to the set of (infinite) Sturm sequences, which has an unexpected application to number theory (Chapter 6).

We start now to build the conceptual bridge that links algebraic combinatorics with musical scale theory. The cornerstone of this bridge is the notion of **scale pattern**:

Definition. *Let* $\Sigma = \{0 = x_0 < x_1 < \ldots < x_{N-1}\}$ *a scale of* N *notes with* k *different steps* $\{\alpha_0, \ldots, \alpha_{k-1}\}$, *labelled in clockwise order of appearance. We call* ***step pattern*** *of* Σ *to the word* w_Σ *in the alphabet* $\{0, 1, \ldots, k-1\}$ *defined by*

$$w_\Sigma(i) = j \Leftrightarrow x_{i+1} - x_i = \alpha_j \quad \forall i = 0, \ldots, N-1$$

where we took $x_N = 1$, *so that last equality also holds for* $i = N-1$.

In a similar way, one can define the step pattern of an interval delimited by two arbitrary notes $x_i < x_j$ of the scale Γ. This pattern is a factor of the circular word (w_Γ).

If Σ is a generated scale, following the three-gaps theorem, $w_\Sigma \in \{0, 1, 2\}^*$. The pattern of a WF scale of N notes is $0 \ldots 0 = 0^N$ if the scale is degenerated, or, on the contrary a binary word $w_\Gamma \in \{0, 1\}^*$. Notice that, by definition, the first interval $0 \to x_1$ will always be labelled with 0. The following result, taken from [32], identifies the set of patterns of WF non degenerate scales with the set of Christoffel words and establishes the basis of the link between WF scale theory and algebraic combinatorics on words.

Proposition 16. *The pattern of a non degenerated WF scale, of* N *notes and multiplier* M *is a Christoffel word of slope* $\frac{M^*}{N}$, *where* $M^* = M^{-1} \mod N$.

Proof. Let $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \ldots < x_{N-1}\}$ a non degenerate WF scale with multiplier M, and let α and β be its steps. Following the definition of multiplier of the scale, one has,

$$x_{k+1} - x_k = \{\tau_{M^*}(k+1)\theta\} - \{\tau_{M^*}(k)\theta\}.$$

Therefore:

$$x_{k+1}-x_k = \begin{cases} \alpha \Leftrightarrow \tau_{M^*}(k+1) - \tau_{M^*}(k) = M^* & \Leftrightarrow \left[\frac{(k+1)M^*}{N}\right] - \left[\frac{kM^*}{N}\right] = 0 \\ \\ \beta \Leftrightarrow \tau_{M^*}(k+1) - \tau_{M^*}(k) = M^* - N & \Leftrightarrow \left[\frac{(k+1)M^*}{N}\right] - \left[\frac{kM^*}{N}\right] = 1, \end{cases}$$

Thus we conclude that w_Γ is a Christoffel word of slope $\frac{M^*}{N}$.

□

As a nice first example of the interrelation between scales theory and Combinatorics on words, we show how the monoid $\left\langle G, \tilde{D} \right\rangle$ generated by the Sturmian morphisms G and $\tilde{D}$ relies on the three steps theorem. Let $\frac{M_1}{M_2} < \frac{M}{N} < \frac{M_2}{N_2}$ be three consecutive fractions in $\mathfrak{F}_N$. Let $\theta \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$ and let us focus on the WF scale $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$. Its step pattern w is, following previous result, the Christoffel word of slope $\frac{M^*}{N}$.

If $\frac{M}{N} < \theta < \frac{M_2}{N_2}$ then the first step $(x_1 - x_0 = \alpha)$ is bigger than the last one $(1 - x_{N-1} = \beta)$. If we generate the $N+1$-th note $\{N\theta\}$, it must lay between x_0 and x_1. In order to build the next WF scale, we must divide hence every step of length α into two steps of lengths $\alpha - \beta$ and β. In terms of the step pattern w, we must replace every 1 of w with 01. In other words, the new step pattern is $\tilde{D}(w)$. This process is pictured in Figure 3.7(b).

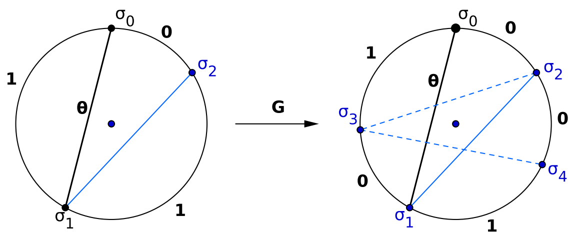
The symmetric case, for which $\frac{M_1}{N_1} < \theta < \frac{M}{N}$ is analogue. Now, $\alpha < \beta$ and the note $\{N\theta\}$ will be generated between x_{N-1} and x_0. The step pattern of the following WF scale will be $G(w)$, since we have to replace every 0 with 01. See Figure 3.7(a).

In conclusion, the three steps theorem, together with the characterization of WF scale patterns and the geometrical description of the sturmian morphisms G and $\tilde{D}$, give as a result that the image under G or $\tilde{D}$ of a Christoffel word is another Christoffel word. In particular, the following corollary, which is a classical result on Christoffel words, holds:

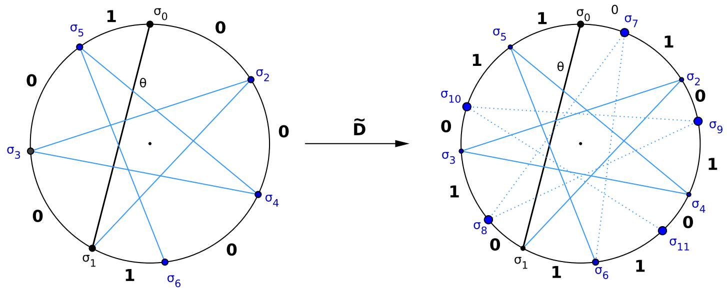
Corollary 4. *Every Christoffel word w can be written as the image of 01 under a sturmian morphism within the monoid $\left\langle G, \tilde{D} \right\rangle$.*

We end up the study of WF scales by introducing a geometrical representation of the family $WF(N, M)$ of well-formed scale of N notes and multiplier M.

Proposition 17. *Let w be a Christoffel word of slope $\frac{M^{-1}}{N}$ and let, moreover, $\frac{M_1}{N_1}$ and $\frac{M_2}{N_2}$ be two fractions in lowest terms such that $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ are three consecutive fractions in the Farey sequence $\mathfrak{F}_N$. The step pattern of a well-formed scale $\Gamma(\theta, N)$ coincides with w if and only if $\theta \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$ and $\theta \neq \frac{M}{N}$. In this case, the generator θ coincides with the $M-$th note of the scale $\Gamma(\theta, N)$.*



(a) $G(01|1) = (001|01)$



(b) $\tilde{D}(0001|001) = (0101011|01011)$

Figure 3.7: Geometrical interpretation of the Sturmian morphisms G and $\tilde{D}$. They connect consecutive cases of 2-step scales in the 3-steps Theorem.

Proof. Notice that $\theta \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$ if and only if $\frac{M}{N}$ is a (semi-)convergent of θ. If θ is irrational, we may use Corollary 2. If $\theta = \frac{M'}{N'}$ is a rational number, we apply Corollary 3 instead (notice that the denominator of the last semi-convergent of θ different from θ itself must be at least N and $N \leq N'$). In any case, the scale $\Gamma(\theta, N)$ is well-formed. Now, Theorem 5 describes entirely the scale. □

The previous Proposition may be formulated as follows: the *perforated* interval

$$\left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right) - \left\{\frac{M}{N}\right\}$$

parametrizes the set of WF scales which step pattern is the Christoffel word of slope $\frac{M^*}{N}$. In the last section of the present chapter (Section 3.9) this poken interval will be transformed into a *poken segment* in $\mathbb{R}^3$ (see Figure 3.16). The set of all those *poken segments* associated with every irreducible fraction $\frac{M}{N}$ as we run down the Stern-Brocot tree will built a *pseudo-fractal* (see Figure 3.17) that represents the set of all WF scales of any number of notes and generator.

Remark 4 (Three-steps words and generalization of Christoffel words)**.** *As it was shown in the three-steps theorem (Theorem 4), generated scales may have at most three different steps. The case of two steps corresponds with Christoffel words. What kind of words of three letters are the step patterns of generated scales with three different steps?*

In the last years, some generalizations of Christoffel words to alphabets with more than two letters have been studied. They all take as a starting point a characteristic properties of the Christoffel words in $\{0,1\}^*$*.*

The first generalization can be found in [44] and it is based on "episturmian words". Recall that a word is Sturmian if and only if it has exactly one right special factor for each length. Moreover, the set of factors of a Sturmian word is closed under reversal (see [38, Theorem 2.1.19]). The episturmian words generalize Sturmian words for an arbitrary alphabet $\mathcal{A}$ *as follows: an infinite sequence* $s \in \mathcal{A}^w$ *is episturmian if its set of factors is closed under reversal and* s *has at most one right special factor for each length. For a survey on Sturmian and episturmian words, see [8]. An episturmian morphism* f *is a morphism such that* $f(s)$ *is episturmian for any episturmian sequence* s*. Recall that a word* $w \in \{0,1\}^*$ *is conjugated to a Christoffel word if and only if it can be obtained as the application of a Sturmian morphim to a letter. Now, a word written on an arbitrary alphabet* $\mathcal{A}$ *is epichristoffel if it is a Lyndon word that can be written as the image of a letter of* $\mathcal{A}$ *by an episturmian morphism.*

In the aforementioned [44], Section 5, it is described an algorithm to determine if there is an epichristoffel word $w \in \mathcal{A} = \{x_0, x_1, \ldots, x_{n-1}\}$ *such that* $|w|_{x_i} = p_i$*, for a given* $n-$*tuple* $(p_0, p_2, \ldots, p_{n-1}) \in \mathbb{N}^n$*. The scale* $\Gamma\left(\frac{3}{11}, 5\right)$ *has three different steps. Its step pattern is 02112, with triplet of frequences (1,2,2). This triplet does not satisfies the necessary condition given in [44, Proposition 26]. It follows that 02112 cannot be an epichristoffel word. We conclude that the step-pattern of generated scales are not, in general, epichristoffel words.*

A second characterization, based on the Lyndon factorization of Christoffel words has been studied (see [40]). This characterization will be studied in the following Chapter, since it is quite close (but does not exactly fit) with the step patterns of the non well-formed scales.

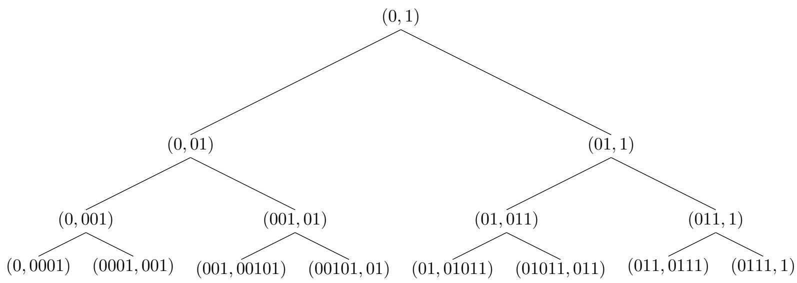


Figure 3.8: The Christoffel tree displays the set of Christoffel pairs.

3.6 WF scales, trees and the monoid $SL(\mathbb{N}, 2)$

The previous characterization of the pattern of WF-scales as Christoffel words enables us to formulate new results on WF-scales based on the theory of Christoffel words. In this section we will make use of the *standard factorization* of Christoffel words and the Christoffel tree. The following result can be found in [11, Chapter 3].

Theorem 7. *A non trivial Christoffel word w (that is, a Christoffel word different from 0^k1 or 01^k) has a unique factorization $w = w_1 \cdot w_2$ with w_1 and w_2 Christoffel words.*

This factorization is called *standard factorization* of w. The pair (w_1, w_2) is called **Christoffel pair** associated with w. The Christoffel pairs for the trivial Christoffel words 0^k1 and 01^k are defined as $(0, 0^{k-1}1)$ and $(01^{k-1}, 1)$ respectively. There is a nice geometrical interpretation of the standard factorization (w_1, w_2) of a Christoffel word w of slope $\frac{M}{N}$: if w represents the *discretization* of the line of equation $y = \frac{M}{N}x$ (see Figure 2.4), then w_1 encodes the portion of the Christoffel path from (0,0) to the closest point of $\mathbb{Z}^2$ to the line. The second factor, w_2, encodes the portion from this point to (N, M). In the example of Figure 2.4, the word 0001001 decomposes as $(0001, 001)$ and the point (4,1) is the closest point of the grid to the segment of line of equation $y = \frac{2}{7}$ between the points (0,0) and (7,2).

The set of Christoffel pairs can be displayed in the nodes of a binary tree, called **Christoffel tree** and shown in Figure 3.8. It has as starting node $(0, 1)$ and it is subjected to the branching rules Γ (left branch) and Δ (right branch) given by:

$$\Gamma(u, v) = (u, uv) \qquad \Delta(u, v) = (uv, v) \quad \text{for } u, v \in \{0, 1\}^*$$

The pairs of words (u, v) built with the transformations Δ and Γ are called Christoffel pairs.

There is an identification between Christoffel pairs and the set of Christoffel words:

Theorem 8. *The Christoffel tree contains exactly once the Christoffel pair associated with every Christoffel word.*

Proof. See [9] $\square$

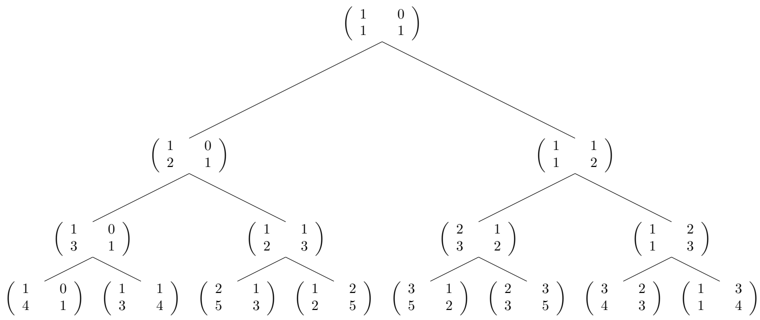


Figure 3.9: The monoid $SL_2(\mathbb{N})$ represented in a binary tree.

For every Christoffel word $w = x \cdot y$ there is a unique path that leads from $(0,1)$ to (x,y). Hence, there is a word $W \in \{\Gamma, \Delta\}^*$ such that $W(0,1) = (x,y)$. This word W is called the **generating word of** the Christoffel pair (x,y).

Remark 5. *Following an argumentation similar to the one in [38, Section 2.2], a Christoffel pair can be expressed as composition of the sturmian morphisms $G = (0,01)$ and $\tilde{D} = (01,1)$. If f is a morphism of words,*

$$\Gamma(f(0), f(1)) = (f(0), f(0)f(1)) = (f(0), f(01)) = (f \circ G(0), f \circ G(1))$$

and

$$\Delta(f(0), f(1)) = (f(0)f(1), f(1)) = (f(01), f(1)) = (f \circ \tilde{D}(0), f \circ \tilde{D}(1))$$

If $W = A_1 \circ \ldots A_n$ with $A_i \in \{\Gamma, \Delta\}$ is the generating word of a Christoffel pair (x,y), and we define $\overline{W} = \overline{A_n} \circ \ldots \overline{A_1}$, with $\overline{\Gamma} = G$ and $\overline{\Delta} = \tilde{D}$, then $W(0,1) = (\overline{W}(0), \overline{W}(1))$. It follows that morphism $\overline{W} \in \left\langle G, \tilde{D} \right\rangle$, thought as a word in $\{G, \tilde{D}\}^*$ is the retrograde of the path within the Christoffel tree that leads to the Christoffel word $\overline{W}(01)$.

We now link the Christoffel tree with the monoid $SL(\mathbb{N},2)$ of 2x2 matrices with integer entries and determinant equal to 1. It is well known that $SL(\mathbb{N},2)$ is freely generated by the matrices:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Therefore, one may display the elements of the monoid $SL_2(\mathbb{N})$ by means of the nodes of another binary tree. See Figure 3.9.

If (x,y) is a Christoffel pair and $x \cdot y = w$ is the corresponding Christoffel word, the matrix $A_w \in SL(\mathbb{N},2)$ that lays in the same node as (x,y) (that is, the one obtained exchanging Γ by L, Δ by R and the concatenation by the product of matrices) is called the **incidence matrix** of the word w. Following proposition computes the matrix A_w in terms of the corresponding Christoffel word (x,y).

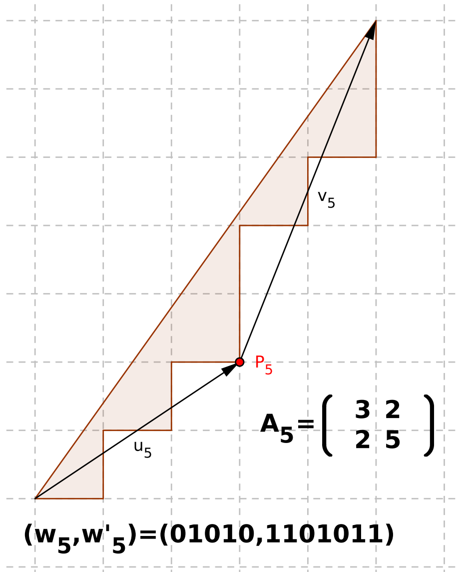


Figure 3.10: The decomposition of the word $w = 010101101011$ and the associated accumulation matrix A_5

Proposition 18. *The incidence matrix of a Christoffel word w with standard factorization $w = x \cdot y$ verifies:*

$$A_w = \begin{pmatrix} |x|_0 & |x|_1 \\ |y|_0 & |y|_1 \end{pmatrix}$$

Proof. The matrix $A_{(0,1)} = Id$ satisfies the formulation. We prove by induction that $A_{\Gamma(x,y)} = L \cdot A_{(x,y)}$ (the assertion $A_{\Delta(x,y)} = R \cdot A_{(x,y)}$ can be proven in a similar way).

$$\begin{aligned} L \cdot A_{(x,y)} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} |x|_0 & |x|_1 \\ |y|_0 & |y|_1 \end{pmatrix} = \begin{pmatrix} |x|_0 & |x|_1 \\ |x|_0 + |y|_0 & |x|_1 + |y|_1 \end{pmatrix} = \\ &= \begin{pmatrix} |x|_0 & |x|_1 \\ |xy|_0 & |xy|_1 \end{pmatrix} = A_{(x,xy)} = A_{\Gamma(x,y)} \end{aligned}$$

□

In a way that we will make precise just below, this matrix associated with the standard factorization characterizes the Christoffel words, and hence the step patterns of WF scales. Let Σ be a scale with step pattern w_Σ and n notes. Let w_k denote the prefix of length k of w_Σ and let w'_k be the suffix of length $n-k$ of w_Σ so we can write $w_\Sigma = w_k \cdot w'_k$. Let A_k denote the matrix associated with the previous factoritation of w, as in the previous proposition, that is

$$A_k = \begin{pmatrix} |w_k|_0 & |w_k|_1 \\ |w'_k|_0 & |w'_k|_1 \end{pmatrix}.$$

This matrix will be called *accumulation matrix* (its name will be justified in the following chapter). The proposition below establishes a new characterization of WF scales:

Proposition 19. *A scale $\Gamma = \{0 = x_0 < x_1 < \ldots < x_{N-1}\}$ with two different steps α and β is well-formed if and only if the following equality holds*

$$\{|A_k| \text{ where } k = 1, 2, \ldots N\} = \{0, 1, \ldots, N-1\}.$$

In such case, if $|A_k| = 1$, the multiplier of the scale is k and the scale coincides with $\Gamma(\theta, N)$, where

$$\theta = |w_k|_0 \cdot \alpha + |w_k|_1 \cdot \beta.$$

Proof. As we know, the two-steps scale is well-formed if and only if its step pattern is a Christoffel word. Now, we can close the proof following a result that will be presented in following chapter, namely Proposition 40. Notice that the matrix A_k with determinant 1 determines the position of the decomposition of the step pattern w in a Christoffel pair, and hence the value of the multiplier. □

Remark 6. *As a consequence of the preceding proposition, if w is a Christoffel word of slope $\frac{M^*}{N}$, which is the step-pattern of a WF scale*

$$\Gamma = \{0 = x_0 < x_1 < \ldots < x_{N-1}\},$$

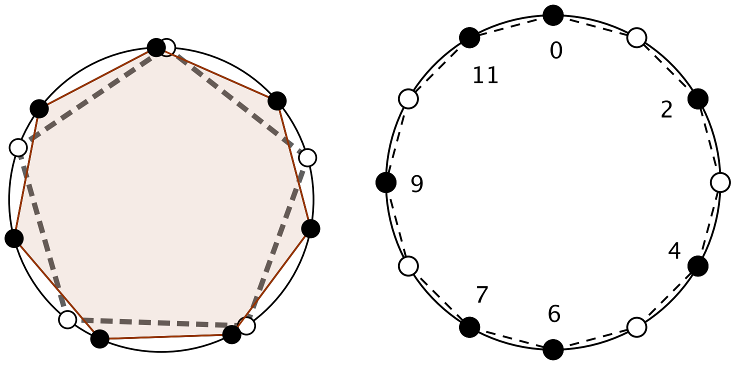
with generator θ, then its standard factorization $w = w_1 \cdot w_2$ verifies:

- *The length of the first factor w_1 is M.*
- *This factor w_1 codifies the portion of scale $0 = x_0 < x_1 < \cdots < x_M = \theta$ between the first note and the generator.*

Remark 7. *It is easy to present a geometrical interpretation of the accumulation matrix. If a word $w \in \{0,1\}^*$ encodes a zig-zag path in $\mathbb{Z}^2$ (a concatenation of horizontal and vertical segments of length 1) linking the sequence of points, $\{P_0, P_1, \ldots, P_n\}$, then the arrows of the matrix A_k are the vectors $\overrightarrow{u_k} = \overrightarrow{P_0P_k}$ and $\overrightarrow{v_k} = \overrightarrow{P_kP_n}$ (see Figure 3.10).*

3.7 Maximally even sets and maximally even words

The notion of maximally even (ME) sets was formalized in 1991 by John Clough and Jack Douthett in [26] with the aim, in some sense, of generalizing the way in which the white and black keys of an octave lay in a piano (Figure 2.2). In one octave there are 7 white keys (corresponding to the diatonic scale of C Major) and 5 black ones. The whole set of 12 keys is called *chromatic scale*. If we place, as in Figure 3.12, the sequence of white-black keys of the chromatic scale over 12 points located equidistantly on the circumference, we see that the white points (also the black points) are distributed as *evenly* as possible around the circle.



(a) A regular pentagon and a regular heptagon inscribed in a circle.

(b) Re-ordering of the 12 vertexes into a regular dodecagonon.

Figure 3.13: The set $\{0, 2, 4, 6, 7, 9, 11\}$ is ME.

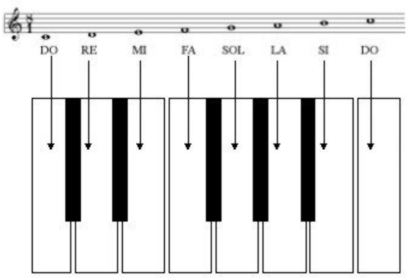


Figure 3.11: The white and black keys of the piano in the C Major scale.

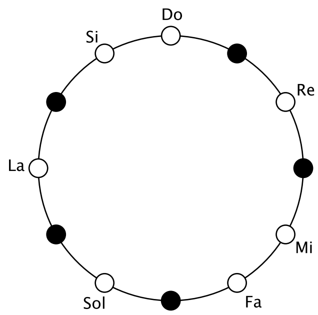


Figure 3.12: The distribution on the circle of the white and black keys.

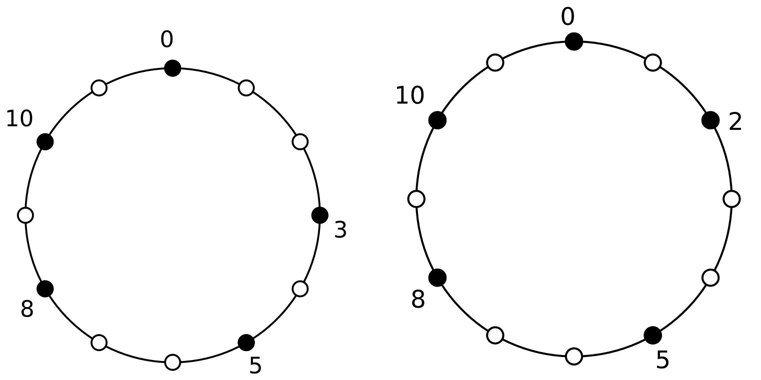
The authors also introduced the notion of maximally evenness with the following metaphor: if we want to place 5 electrons in 5 out of 12 points located equidistantly on the circumference, the distribution that minimizes the disturbance to the charge equilibrium coincides with the distribution of the black and white keys of the piano modulo rotations of the circle.

Remark 8 (ME generated by regular polygons)**.** *The ME sets were also presented in [26] with the following geometrical procedure: let us draw a regular M-gon with black vertexes and a regular $(N - M)$-gon with white vertexes inscribed in a circle in such a way that both polygons share no common point (see Figure 3.13 with $M = 7$ and $N = 12$). Let A_0 denote one of the vertexes and let $A_0, \ldots A_{N-1}$ label the N different vertexes clockwise. The set of positions of the black verteces*

$$\{i \text{ such that } A_i \text{ is a black vetex}\}$$

is a ME set.

The objective of this section is to formalize the notion of maximally evenness



(a) The set $\{0, 3, 5, 8, 10\}$ is ME. (b) The set $\{0, 2, 5, 8, 10\}$ is not ME.

Figure 3.14: Two examples of subsets of $\mathbb{Z}_{12}$.

in terms of combinatorics on words linking it with the Christoffel words. It will be proven that the step pattern of a WF scale is maximally even.

We take as starting point the definition of maximally even set given by John Clough and Jack Douthett in [26]. Let X_d be a subset of d elements of $\mathbb{Z}_c$, where $0 \leq d \leq c$. We can define the *intervals of length* k, I_k (called k-th spectrum in [26]):

$$I_k = \{x_{j+k} - x_j, \text{ such that } j \in \mathbb{Z}_d \text{ and } x_j \in X_d\}$$

Definition. *The set X_d is **maximally even** (ME) if I_k is either a singleton, or a pair of consecutive integer numbers, for every $k = 0, 1, \ldots, d-1$.*

The empty subset $X_0 = \{\emptyset\}$, the whole set $X_c = \mathbb{Z}_c$, the singleton $X_1 = \{a \in \mathbb{Z}_c\}$ and the complement of the singleton $\mathbb{Z}_c - X_1$ are called trivial ME sets.

The Figure 3.14 shows the ME set $\{0, 3, 5, 8, 10\}$ and the set $\{0, 2, 5, 8, 10\} \subset \mathbb{Z}_{12}$, which is not ME since $I_2 = \{x_{j+2} - x_j\} = \{4, 5, 6\}$ has three elements.

There is a natural bijection between subsets of $\mathbb{Z}_c$ and the binary words of length c:

Definition. *Given a subset $X \subset \mathbb{Z}_c$, we define the **characteristic word** w_X as:*

$$w_X = w_X(0)w_X(1)\cdots w_X(c-1) \quad \textit{where} \quad w_X(i) = 1 \Leftrightarrow i \in X$$

*Conversely, given a word $w \in \{0, 1\}^*$ with length $|w| = c$, we can define its **characteristic set** as*

$$X_w = \{i \in \mathbb{Z}_c \quad \textit{with} \quad w(i) = 1\}.$$

Now we can translate the notion of maximally evenness to the domain of combinatorics on words. The natural way of defining ME words is the following: a word is ME if its characteristic set is ME. Notice that the definition of ME sets is nevertheless invariant under rotations $X \to X + k$, with $k \in \mathbb{Z}_c$ and consequently the definition of ME must be invariant under conjugation.

Given a set $X = \{x_0 < \cdots < x_{d-1}\} \subset \mathbb{Z}_c$, an interval $\{x_j, x_{j+k}\}$ of length k within X determines a factor of (w_X), that must fit the shape $1u1$, with $|u|_1 = k-1$. Therefore, the definition of ME sets can be extended to the set of words as follows.

Definition. *A word w is ME if $|w|_1 \leq 1$, or for every pair of factors $1u1$, $1v1$ of (w), such that $|u|_1 = |v|_1$, one has that $||u| - |v|| \leq 1$.*

Remark 9. *Notice that the words 0^k, 1^k, 0^k1 and 1^k0 associated respectively to the subsets $X_0 = \emptyset$, $X_c = \mathbb{Z}_c$, the singleton $X_1 = \{1\}$ and its complement $X_{c-1} = \mathbb{Z}_c - X_1$ verify the previous definition. They are called trivial ME words.*

If we take a closer look to the example given in Figure 3.14(b) we can find the reason why $X = \{0, 2, 5, 8, 10\} \subset \mathbb{Z}_{12}$ is not ME in terms of the factors of its step pattern. Its characteristic word is $w_X = 101001001010$ and the words $10101 = 1u1$, $1001001 = 1v1$ are both factors of (w_X) verifying that $|v| - |u| = 2$.

We recall now the classical notions of *balanced set* and *balanced word*, which play a central roll in the description of sturmian words and Christoffel words:

Definition. *A set of words X is balanced if one has $||u|_1 - |v|_1| \leq 1$ for every pair of words $u, v \in X$ of the same length. A word w (resp. a circular word (w)) is balanced if its set of factors is balanced.*

Notice the similarity between the definitions of balanced and ME words. The following result makes this similarity explicit:

Proposition 20. *A circular word (w) is balanced if and only if the word $\left(\tilde{D}(w)\right)$ is ME.*

Proof. Let (w) be a balanced word, and let $1u1$ and $1v1$ be factors of $\left(\tilde{D}(w)\right)$ verifying $|u|_1 = |v|_1$. There is a factor $\overline{u}$ of (w) such that one of the following equalities hold:

$$\tilde{D}(1\overline{u}) = 1\tilde{D}(\overline{u}) = 1u1 \quad \text{or} \quad \tilde{D}(0\overline{u}) = 01\tilde{D}(\overline{u}) = 01u1$$

In a parallel way, there is another factor $\overline{v}$ of (w) such that

$$\tilde{D}(1\overline{v}) = 1\tilde{D}(\overline{v}) = 1v1 \quad \text{or} \quad \tilde{D}(0\overline{v}) = 01\tilde{D}(\overline{v}) = 01v1.$$

In any case, it holds that

$$\tilde{D}(\overline{v}) = v1 \quad \text{and} \quad \tilde{D}(\overline{u}) = u1$$

and it follows

$$|u|_1 = |v|_1 \Rightarrow |\overline{u}| = |\overline{v}| \Rightarrow ||\overline{u}|_0 - |\overline{v}|_0| \leq 1 \Rightarrow ||\tilde{D}(u)|_0 - |\tilde{D}(v)|_0| \leq 1.$$

We conclude that $\left(\tilde{D}(w)\right)$ is ME. The converse is similar. □

In order to prove the central result of this section, which characterizes ME words, we need some previous results.

Lemma 1. *If two words w and w' are conjugated and f is any morphism of words, then $f(w)$ and $f(w')$ are conjugated as well.*

Proof. Trivial. □

Lemma 2. *If w is a ME word and $|w|_1 \leq |w|_0$, then w is conjugated to a word in $\{0^k1, 0^{k+1}1\}^*$ for a given $k \geq 1$.*

Proof. If w is a trivial ME word there is nothing to prove. Let us suppose that $|w|_1 \geq 2$. The words 00 and 11 cannot be factors of (w) at the same time, since that would imply that 10^k1 and 11 would be factors of (w) for a certain $k \geq 2$, which impossible since w is ME. Since $|w|_1 \leq |w|_0$, it follows that 00 is a factor of (w) and 11 is not. We end up by arguing that if 10^k1 is a factor of (w), then $10^{k+s}1$ cannot be a factor of (w) $\forall s \geq 2$, by definition of ME word. □

Lemma 3. *If w is a ME word and $|w|_0 \leq |w|_1$, then w is conjugated to a word in $\{1^k0, 1^{k+1}0\}^*$ for a given $k \geq 1$.*

Proof. Again, as in previous proof, the assertion is trivial if w is a trivial ME word. Let therefore $|w|_0 \geq 2$ hold. It must hold that 11 is a factor of (w) and 00 is not. If 01^k0 and $01^{k+s}0$ with $k \geq 1$ and $s \geq 2$ where both factors of (w), then the words 101^k01 and 1^{k+2} would be factors of (w) at the same time. This is a contradiction, since w is ME. □

Lemma 4. *If w is a ME word and $|w|_1 = |w|_0$, then w is conjugated to $(01)^k$ for a given $k \geq 1$.*

Proof. It is a consequence of lemmas 2 and 3. □

Proposition 21. *The word w is ME if and only if $G(w)$ is ME.*

Proof. Let w be ME, and let $1u1$ and $1v1$ be two factors of $(G(w))$ such that $|u|_1 = |v|_1$. The words $01G(u')01$ and $01G(v')01$ are also factors of $(G(w))$, where $u = G(u')0$, $v = G(v')0$ and u', v' are factors of (w). It holds that

$$|u|_1 = |v|_1 \Leftrightarrow |G(u')0|_1 = |G(v')0|_1 \Leftrightarrow |u'|_1 = |v'|_1$$

One has therefore that $1u'1$ and $1v'1$ are factors of w with $|u'|_1 = |v'|_1$. The word w is ME, so it follows that $||u'| - |v'|| \leq 1$.

$$||u| - |v|| = ||G(u')0| - |G(v')0|| = |(|u'| + |u'|_1) - (|v'| + |v'|_1)| = ||u'| - |v'|| \leq 1$$

We conclude that $G(w)$ is ME. The converse is similar. □

The aforementioned characterization of ME words relies on previous lemmas and also on the following theorem, which is a classical result in Combinatorics of words (see [10][Theorem 6.9]):

Theorem 9. *A word $w \in \{0, 1\}^*$ is a conjugate of a Christoffel word if and only if w and all of its conjugates are primitive and balanced.*

We will need, in fact, its *non primitive* similar:

Theorem 10. *A word $w \in \{0, 1\}^*$ is conjugated to a power of a Christoffel word if and only if the circular word (w) is balanced.*

Proof. Let w be conjugated to $\overline{w}^k$, where $k \geq 1$ and $\overline{w}$ is a Christoffel word. We shall prove that if $w' \sim w$ then w' is balanced. If u, v are factors of w' with the same length, $|u| = |v| \leq |\overline{w}|$, then u and v are factors of a word conjugated to $\overline{w}$, which is a Christoffel word, and thus, following Theorem 9, it is primitive and balanced. Hence, $||u|_1 - |v|_1| \leq 1$. If $|u| = |v| > |\overline{w}|$, then it holds that

$$u = u_0\overline{w}^t u_1 \quad v = v_0\overline{w}^t v_1 \quad \text{ with } |u_0u_1| = |v_0v_1| < |\overline{w}|\,.$$

Notice that u_0u_1 and v_0v_1 are factors of a word conjugated to $\overline{w}$ and following Theorem 9,

$$||u|_1 - |v|_1| = ||u_0u_1|_1 - |v_0v_1|_1| \leq 1.$$

Let (w), conversely, be a balanced circular word. If w if primitive, so it is any of its congujates, that is, (w) is primitive. It follows that w is conjugated to a Christoffel word. In the other hand, if $w = \overline{w}^k$ with $\overline{w}$ primitive, then $(\overline{w})$ is a balanced circular word. To show that, let u, v be factors of $(\overline{w})$ with $|u| = |v|$. As any factor of $(\overline{w})$ is a factor of (w), which is balanced, then it must hold that $||u|_1 - |v|_1| \leq 1$. Again, following Theorem 9, $\overline{w}$ is conjugated to a Christoffel word and, finally, $\overline{w}^k$ is conjugated to a power of a Christoffel word. □

Finally, we show that the notions of maximally-evenness and circular balance coincide for circular words:

Theorem 11 (Characterization of ME words)**.** *A word is ME if and only if it is conjugated to a power of a Christoffel word.*

Proof. Let w be a ME word. If $|w|_1 = |w|_0$, following lemma 4, w is conjugated to a power $(01)^k$ of the Christoffel word 01.

Let us suppose now that $|w|_1 > |w|_0$. From lemma 3, w is conjugated to a word in $\{01^k, 01^{k+1}\}^* = \{\tilde{D}(01^{k-1}), \tilde{D}(01^k)\}^*$. Thus, there is a word $w_1 \in \{0,1\}^*$ such that w is a conjugated of $\tilde{D}(w_1)$. It follows that $\left(\tilde{D}(w_1)\right)$ is ME and hence, following Proposition 20, (w_1) is a balanced word. Now, by Theorem 10, there is a power of a Christoffel word w_2 such that $w_2 \sim w_1$. And one concludes that $\tilde{D}(w_2)$ is a power of a Christoffel word and so it is (w) since:

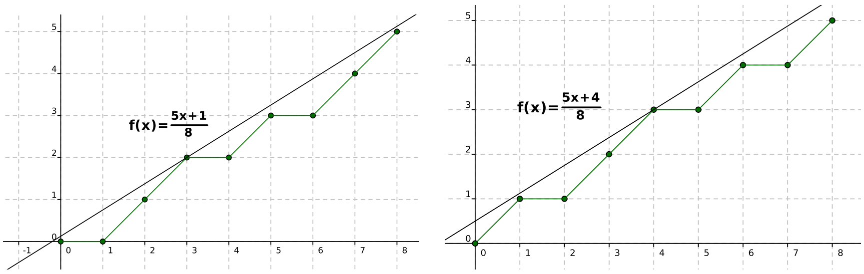
$$\left(\tilde{D}(w_2)\right) = \left(\tilde{D}(w_1)\right) = (w).$$

Let us suppose finally that $|w|_0 > |w|_1$. From lemma 2, w is conjugated to a word in $\{0^k1, 0^{k+1}1\}^* = \{G(0^{k-1}1), G(0^k1)\}^*$, hence $w \sim G(w_1)$ for a certain word $w_1 \in \{0,1\}^*$. It follows that $(G(w_1))$ is ME and by Proposition 21 (w_1) is ME as well. One can close the argument using induction over the length of w.

It remains to prove the converse. Since $(01)^k$ is a ME word, it is enough to prove that whenever w is ME, then $G(w)$ and $\tilde{D}(w)$ are both ME words. $G(w)$ is ME by Proposition 21. Now if w is ME, we have proven that w is conjugated to a power of Christoffel word. It follows that (w) is balanced (by Theorem 10) and and then, by Proposition 20, $\tilde{D}(w)$ is ME. □

Now it is immediate to prove some classical results of ME sets (words). The first one asserts that the complement of a ME set is a ME set as well:

Corollary 5. *A word (w) is ME if and only if the complement word $(E(w))$ is ME.*



(a) The word $w_1 = 01101011$ determined by the affinity of $\mathbb{Z}_8$ $f(x) = 5x + 1$. Its characteristic set is $\{J^6_{8,5}(\delta)\} = \{1, 2, 4, 6, 7\}$.

(b) The word $w_2 = 10110101$ determined by the affinity of $\mathbb{Z}_8$ $f(x) = 5x + 4$. Its characteristic set is $\{J^3_{8,5}(\delta)\} = \{0, 2, 3, 5, 7\}$.

Figure 3.15: Two words that are rotations of the Christoffel word of slope $\frac{5}{8}$ $w = 01011011$ and their associated $J-$functions.

Proof. It is enough to show that the $E(w)$ is conjugated to a power of a Christoffel word whenever w is so and, since $E(w^k) = E(w)^k$, we just need to prove the result in the primitive case. But E is a Christoffel morphism (see [10][Theorem 2.8], that is E sends Christoffel words onto conjugates of Christoffel words and, hence, $E(w)$ is conjugated to a Christoffel word for every w Christoffel word. □

Remark 10. *Every rotation w of a (power of a) Christoffel word of slope $\frac{M}{N}$ can be represented as the prefix of length N of the mechanical sequence of same slope, $\frac{M}{N}$, and intercept $\frac{k}{N}$ for a certain $k \in \{0, 1, \ldots, N-1\}$. That is, the word w can be obtained by the equation:*

$$w(i) = \left\lceil \frac{M(i+1)+k}{N} \right\rceil - \left\lceil \frac{Mi+k}{N} \right\rceil \quad \text{with } k = 0, 1, \ldots, N-1$$

and encodes the zig-zag under the line of equation $f(x) = \frac{Mx+k}{N}$. The function $f_w(x) = Mx + k \mod N$ is called affinity associated with the word w and will play a special important role in the study of Christoffel duality, as we will see in the following chapter.

In their original paper, J. Clough and J. Douthett [26] defined the ME sets by means of the following functions that they called $J-$ functions. For every $N > 0$, $1 \leq M < N$ and $0 \leq k < N$ the $J^k_{N,M}$ function is defined as:

$$J^k_{N,M}(x) = \left\lceil \frac{Nx+k}{M} \right\rceil \quad \text{with } x = 0, 1, \ldots, M-1$$

The following proposition asserts that the sets determined by the $J-$functions coincides with the characteristic sets of ME words:

Proposition 22. *If $f(x) = Mx + k$ is the affinity in $\mathbb{Z}_N$ associated with a ME word w of length N, then its characteristic set can be determined by the $J-$function $J^{N-1-k}_{N,M}$, and conversely.*

Proof. Notice first that $w(x) = 1$ for a ME word w of length N with associated affinity $f(x) = Mx + k$ if and only if it holds

$$\left\lceil \frac{M(x+1)+k}{N} \right\rceil - \left\lceil \frac{Mx+k}{N} \right\rceil = 1$$

and this is equivalent to

$$N - M \leq (Mx + k) \mod N \tag{3.5}$$

It is enough to prove that the equation 3.5 holds if and only if there is a $\delta_x \in \{0, 1, \ldots, M-1\}$ such that it holds

$$J_{N,M}^{N-1-k}(\delta_x) = \left\lceil \frac{N\delta_x + N - 1 - k}{M} \right\rceil = x. \tag{3.6}$$

For that purpose it will be useful the following identity, which is very easy to check:

$$x \left\lceil \frac{y}{x} \right\rceil = y - (y \mod x) \quad \forall\ x, y \geq 1.$$

Assuming that the condition 3.5 holds, let us check that 3.6 is satisfied for $\delta_x = \left\lceil \frac{Mx+k}{N} \right\rceil$:

$$\left\lceil \frac{N\left\lceil \frac{Mx+k}{N} \right\rceil + N - 1 - k}{M} \right\rceil = \left\lceil \frac{Mx + k - ((Mx+k) \mod N) + N - 1 - k}{M} \right\rceil =$$

$$= \left\lceil x + \frac{N - 1 - (Mx+k) \mod N}{M} \right\rceil = x,$$

where last equality holds since $N - M \leq (Mx + k) \mod N$.

Let us assume, conversely, that $\delta \in \{0, 1, \ldots, M-1\}$ verifies the equation of condition 3.6.

$$\left(M \left\lceil \frac{N\delta + N - 1 - k}{M} \right\rceil + k \right) \bmod N =$$

$$= (N\delta + N - 1 - k - (N\delta + N - 1 - k) \bmod M + k) \bmod N =$$

$$= (N - 1 - (N\delta + N - 1 - k) \bmod M) \bmod N \geq N - M$$

Hence, equation 3.6 holds, as we wanted to prove. □

Remark 11 (Maximally evenness and rhythms)**.** *Circular words can be used to code cyclic rhythms. If we fix a time interval as a unit time or pulse, every (circular) word can be identified with a (cyclic) rhythm: a 1 represents a played note (onset) and a 0 represents a silence. Rhythms determined by ME words have the property that their onset patterns are distributed as evenly as possible. They have been deeply studied (see [52] and [30]) in connection with the Euclidean algorithm. It turns out that plenty of timelines from traditional world music satisfy the ME property (see [30]). For example, the Bossa-Nova rhythm:*

[1001001000100100]

is encoded by the ME set $\{0, 3, 6, 10, 13\}$ *of* $\mathbb{Z}_{15}$*. We illustrate the connection between the Euclidean algorithm and the ME sets that is shown in these papers, with the recursive construction of a ME word* w *with* $|w|_1 = 3$ *and* $|w| = 8$*:*

11100000	*We divide 8 in 3 groups headed by the 1's*
[10] [10] [10] 00	$8 \mod 3 = 2$
[100] [100] [10]	$3 \mod 2 = 1$

We stop the algorithm in the moment that we get the $GCD(8,3) = 1$*, and the resulting word is the ME word* (10010010) *which encodes the cuban rhythm tresillo.*

Remark 12 (ME sets and the Discrete Fourier Transform)**.** *Emmanuel Amiot developed in [4] a equivalent notion of ME words with a language of complex numbers. Notice that any subset of $\mathbb{Z}_N$ can be seen as a subset of the Nth roots of the unity in the complex plane $\mathbb{C}$. Given a subset A of $\mathbb{Z}_N$, the Discrete Fourier Transform of A is the map $\mathcal{F}_A : \mathbb{Z}_N \to \mathbb{C}$ defined by*

$$\mathcal{F}_A(t) = \sum_{k \in A} \mathrm{e}^{\frac{-2\mathrm{i}\pi k t}{N}}.$$

Emmanuel Amiot shows that the maximum of the set

$$\{|\mathcal{F}_A(d)|, \text{ where } A \subset \mathbb{Z}_N \text{ and } card(A) = d\}$$

is reached precisely for those sets A which are ME.

3.8 Balanced words and Myhill's property

The objective of this section is to establish the relationship between the balanced words and scales satisfying the Myhill's property. This is the quality of musical scales that have two distinct intervals for each *generic length*. The paradigmatic example is the diatonic scale; it has two different intervals of each kind: two seconds (Major second and minor second), two thirds (Major third and minor third) and so on.

Let $\Sigma = \{0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}$ be a scale. All the subindexes of the notes of the scale in this section are considered in $\mathbb{Z}_N$. We say that the following set

$$I_k = \{x_{j+k} - x_j, \text{ such that } j \in \mathbb{Z}_N\} \tag{3.7}$$

is the set of intervals of *length* k of the scale. The steps of a scale are its intervals the length of which is 1.

Definition. *The scale Σ verifies the **Myhill's property** if $\#I_k = 2$ with $k = 1, 2, \ldots, N-1$. Equivalently, a scale with Myhill's property has two different intervals of each length.*

A scale $\Sigma = \{0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}$ satisfying Myhill's property must have two different steps, $\alpha = x_1 - x_0$ and β. Thus, its step pattern w_Σ is a word in $\{0,1\}^*$.

Example 10. *The well-formed scale of 7 notes generated by the pure fifth $\log_2\left(\frac{3}{2}\right)$ (figure 3.5) verifies Myhill's property. Table 3.2 shows the values of the notes of the scale and the value of the pair of intervals of each length.*

Example 11. *The diatonic scale $\Gamma(\frac{7}{12}, 7)$ (the scale of 7 notes generated by the fifth in equal temperament $\frac{7}{12}$) verifies Myhill's property as well. Table 3.3 shows the values of the two distinct intervals of each length.*

We translate now the Myhill's property into the language of Combinatorics of words.

Given a scale $\Sigma = \{0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}$, we can associate to every interval $x_{i+k} - x_i$ of length k of Σ the word

$$w_\Sigma(i) w_\Sigma(i+1) \cdots w_\Sigma(i+k),$$

k	x_k	I_k
1	0.1699	$\{0.1699, 0.0752\}$
2	0.3398	$\{0.3398, 0.2451\}$
3	0.5097	$\{0.5097, 0.4150\}$
4	0.5849	$\{0.5849, 0.4903\}$
5	0.7548	$\{0.7548, 0.6602\}$
6	0.9248	$\{0.9248, 0.8301\}$

Table 3.2: $\Gamma(\log_2\left(\frac{3}{2}\right), 7)$ is a WF scale verifying Myhill's property.

k	x_k	I_k	Usual name of the diatonic intervals
1	$\frac{2}{12}$	$\{\frac{1}{12}, \frac{2}{12}\}$	{Major Second, Minor Second}
2	$\frac{4}{12}$	$\{\frac{3}{12}, \frac{4}{12}\}$	{Major Third, Minor Third}
3	$\frac{6}{12}$	$\{\frac{5}{12}, \frac{6}{12}\}$	{Augmented Forth, Just Forth}
4	$\frac{7}{12}$	$\{\frac{6}{12}, \frac{7}{12}\}$	{Augmented Fifth, Just Fifth}
5	$\frac{9}{12}$	$\{\frac{8}{12}, \frac{9}{12}\}$	{Major Sixth, Minor Sixth}
6	$\frac{11}{12}$	$\{\frac{10}{12}, \frac{11}{12}\}$	{Major Seventh, Minor Seventh}

Table 3.3: Diatonic scale verifies Myhill's Property

which is a factor of length k of the circular word (w_Σ). This word is called *pattern* of the interval $x_i \to x_{i+1}$.

Lemma 5. *Let $\Sigma = \{0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}$ be a scale with two different steps. Two intervals of Σ of the same length coincide if and only if their patterns have the same weight.*

Proof. If w_i and w_j are the patterns of $x_{i+k} - x_i$ and $x_{j+k} - x_j$, and $\alpha = x_1, \beta$ are the two distinct steps of Σ, then one has that

$$x_{i+k} - x_i = |w_i|_0\alpha + |w_i|_1\beta = (k - |w_i|_1)\alpha + |w_i|_1\beta = |w_i|_1(\beta - \alpha) + k\alpha.$$

It follows that $x_{i+k} - x_i = x_{j+k} - x_j \iff |w_i|_1(\beta - \alpha) = |w_j|_1(\beta - \alpha) \iff |w_i|_1 = |w_j|_1$. ☐

Hence, we can define equivalently Myhill's property restrained to words as follows:

Definition. *A word w verifies Myhill's property if the set*

$$J_k = \{|u|_1 \quad \textit{where } |u| = k \textit{ and } u \textit{ is a factor of } (w)\}$$

has two different elements for every $k = 1, \ldots, |w| - 1$.

It follows from previous definition that Myhill's Property is invariant under conjugation of words, hence, w verifies Myhill's property if and only if (w) does. Recall that a word is primitive if it is not empty and not a proper power of another word and that Christoffel words are the patterns of well-formed scales.

Lemma 6. *A word $w \in \{0,1\}^*$ is not primitive if and only if there is $k \in \{1,\ldots,|w|-1\}$ such that $|u|_1 = |v|_1$ for every pair of factors u and v of (w) of length k.*

Proof. If $(w) = (x^t)$ for some $x \in \{0,1\}^*$, then it holds trivially that $|u|_1 = |v|_1$ whenever u and v are factors of (w) which length is $|u| = |v| = |x|$.

Let, conversely, $u_1u_2\ldots u_ku_{k+1}$ be a factor of length $k+1$ of w. Then, it holds that

$$|u_1u_2\ldots u_k|_1 = |u_2\ldots u_ku_{k+1}|_1 \Longrightarrow u_1 = u_{k+1}.$$

It follows that (w) has a period of length k, which means that (w) is not primitive. One concludes that w cannot be primitive. □

It follows from Theorem 9 and Lemma 6 the following characterization of the modes of a well-formed scale or, equivalently, of the conjugation class defined by a Christoffel word:

Proposition 23. *A word is conjugated to a Christoffel word if and only if it verifies Myhill's property.*

Proof. Let w be a word conjugated to a Christoffel word. Following Theorem 9, w and all of its conjugated words are primitive balanced words. Hence (w) is a balanced word and the set J_k of weights of the factors of length k of (w) has two different elements at most. If there were a $k \in \{1,\ldots|w_\Sigma|-1\}$ such that J_k was a singleton, we would be under the hypothesis of lemma 6 and therefore w_Σ would be conjugated of a non primitive word. This is impossible as every word conjugated to a Christoffel word is primitive. We deduce that J_k has two different elements for every $k \in \{1,\ldots|w|-1\}$ and, hence, that w verifies Myhill's property.

Let w be a word satisfying Myhill's property. One has that $||u|_1 - |v|_1| = 1$ for every pair of factors u and v of (w) having the same length. It follows that (w) is a balanced word and, according to Lemma 6, the word w and every other one conjugated to w are primitive words. Now we can assert, in basis of Theorem 9, that w_Σ is conjugated to a Christoffel word. □

Corollary 6. *The modes of any WF scale verify the Myhill's property.*

Definition. *A Lyndon word is a primitive word which is minimal for the lexicographic order.*

It holds that a balanced word is Lyndon if and only if it is a Christoffel word. (see [9]). The following characterization of Christoffel words in terms of Myhill's property thus holds.

Corollary 7. *$w \in \{0,1\}^*$ is a Christoffel word if and only if it is a Lyndon word satisfying Myhill's property.*

3.9 Overview of Characterizations of WF scales

We close up the chapter by giving an overview of all the characterizations of WF scales that have been proven so far.

Theorem 12 (Characterizations of WF scales)**.** *A scale of N notes is well-formed and its multiplier is $M \in \mathbb{Z}_N^*$ if and only if it holds one of the following equivalent conditions:*

1. *It is a generated scale and its prime permutation is τ_M (definition).*
2. *It is generated by an even generator that decomposes in M steps.*
3. *It is a generated scale and $\frac{M}{N}$ is a (semi-)convergent of the generator.*
4. *It is a degenerated WF scale or the set $\{|A_k|\}_{k=0,\ldots,N-1}$ of determinants of the accumulation matrices associated with the step pattern is a permutation of $\{0, 1, \ldots, N-1\}$ and, moreover, $|A_M| = 1$.*
5. *Its step pattern is either 0^N or a Christoffel word of slope $\frac{M^*}{N}$, where $M^* = M^{-1} \mod N$.*
6. *It is a degenerated WF scale or it verifies Myhill's property, its step pattern is a Lyndon word and the frequence of its first interval is $N - M$.*
7. *Its step pattern is either 0^N or a binary word that is a primitive, Lyndon and maximally even word and the frequence of its first interval is $N - M$.*

The scales that share the same number of notes N and the same multiplier M have therefore all discrete parameters in common. According to Theorem 5, the real lengths of both steps α, β and generator θ may vary within limits given by the following:

Proposition 24. *If $\Gamma \in WF(N, M)$ is a scale with steps $\alpha = x_1$, $\beta = 1 - x_{N-1}$ and generator θ, then α, β and θ satisfy the following restrictions:*

$$\left.\begin{array}{r} M_2\alpha + M_1\beta = \theta \\ \\ N_2\alpha + N_1\beta = 1 \end{array}\right\}\text{(1)} \qquad \left.\begin{array}{c} 0 < \alpha < \frac{1}{N_1} \\ \\ 0 < \beta < \frac{1}{N_2} \\ \\ \frac{M_1}{N_1} < \theta < \frac{M_2}{N_2} \end{array}\right\}\text{(2)}$$

In accordance with the previous proposition, the set of scales $WF(N, M)$ can be represented geometrically as the segment of the line given by the equations (1) and contained in the parallelepiped defined by the inequalities (2). This representation is shown in Figure 3.16 (where the particular case $WF(5, 3)$ is displayed).

If one represents the segments with ending points $\left(\frac{M_2}{N_2}, 0, \frac{1}{N_2}\right)$ and $\left(\frac{M_1}{N_1}, \frac{1}{N_1}, 0\right)$ and the inner point $\left(\frac{M}{N}, \frac{1}{N}, \frac{1}{N}\right)$, for every three consecutive $N-$Farey fractions $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$, one obtains a fractal-like concatenation of segments that is shown in Figure 3.16. In this representation, every point corresponds with a unique WF scale and every segment with a class of WF scales $WF(M, N)$. In the same way as every node of the Stern-Brocot tree is the father of two fractions that are one row lower in the tree, every segment in the pseudo-fractal representation is extended by two *son-segments*, one starting from the bottom end, the other from the upper end of the *father segment*.

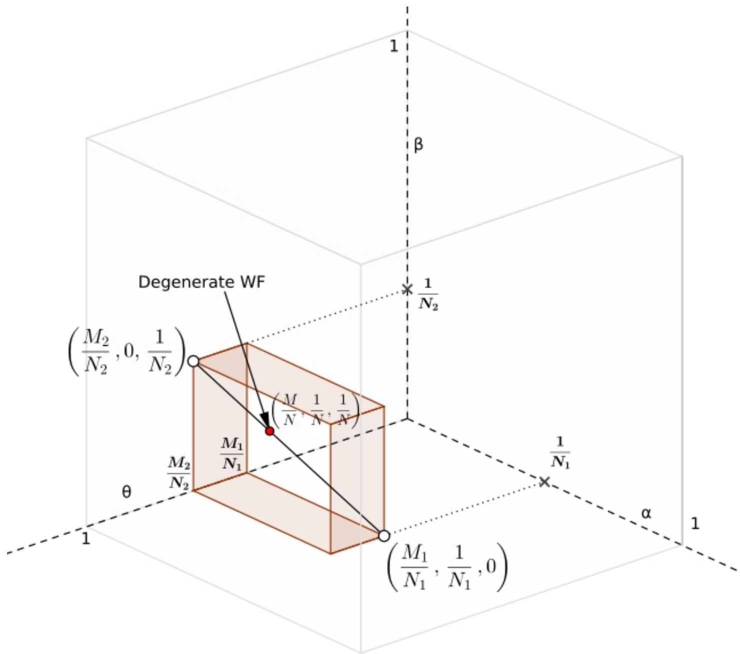


Figure 3.16: Representation of the set $WF(N, M)$ of well-formed scales of N notes and multiplier M.

3.10 Christoffel duality

The characterizations of WF-scales given in Theorem 12 establish a detailed description of this kind of scales. In particular, the fact that the number M of steps of the generator of a WF scale Σ and the frequency of its last interval M^* are inverse numbers modulo the number of notes of Σ enabled N. Carey and D. Clampitt in [17] to introduce a notion of duality between WF-scales families $WF(N, M)$ (the set of scales of N notes and multiplier M).

Definition. *Given the family of well-formed scales $WF(N, M)$, we defined its dual family as the set $WF(N, M^*)$, where $M^* = M^{-1} \bmod N$. If Σ is a well-formed scale in $WF(N, M)$, we will usually write Σ^* to denote a scale in $WF(N, M^*)$.*

Notice that if $\Sigma \in WF(N, M)$ is a WF scale generated by θ, there is not a unique θ^* such that $\Gamma(\theta^*, N) \in WF(N, M^*)$. The problem to find the appropriate *dual generator* θ^* associated with the generator of a given WF scale is discussed in Chapter 6.

Shortly after N. Carey and D. Clapitt introduced this notion of duality between WF notes, a notion of duality between Christoffel words was being studied. It formally first appeared in [12] by V. Berthé, A. de Luca and C. Reutenauer. We present here a version adapted to our purposes:

Definition. *If w is a Christoffel word of slope $\frac{M}{N}$, the Christoffel word w^* of slope $\frac{M^*}{N}$ is called Christoffel dual word of w.*

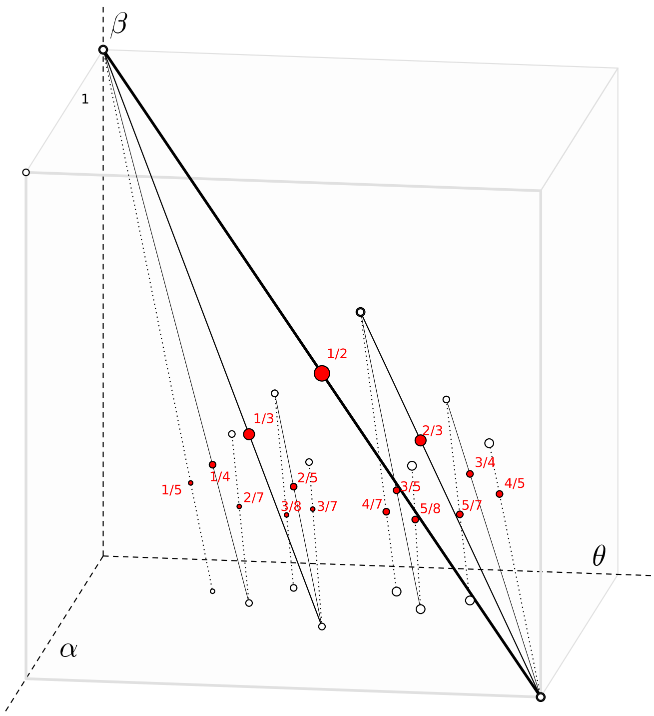


Figure 3.17: Three-dimensional representation of the set of WF scales. The red points that are labelled by $\frac{M}{N}$ represent the degenerated WF-scales of N notes corresponding with the generator $\frac{M}{N}$.

	$\Sigma \in WF(N,M)$	$\Sigma^* \in WF(N,M^*)$
M	Number of steps of the generator θ	Frequency of the last step β
	Length $\lvert w_1\rvert$ of the first factor of the standard factorization $w_\Sigma = w_1 \cdot w_2$	Number of 1's $\lvert w_{\Sigma^*}\rvert_1$ of the step pattern w_{Σ^*}
$N-M$	Number of steps within $1-\theta$	Frequency of the first step α
	Length $\lvert w_2\rvert$ of the second factor of the standard factorization $w_\Sigma = w_1 \cdot w_2$	Number of 0's $\lvert w_{\Sigma^*}\rvert_0$ of the step pattern w_{Σ^*}
$x \cdot M \bmod N$	Transforms generation order into natural order	Transforms natural order into generation order

Table 3.4: Notions of WF scale families connected via duality.

Recall that a Christoffel word w has a unique standard factorization $w = w_1 \cdot w_2$ where (w_1, w_2) is a Christoffel pair (see Theorem 7). The length of the first factor w_1 of the standard factorization coincides with the number of 1 $|w^*|_1$ of the dual word w^*. As it was proven in the Proposition 16, the step pattern of a WF scale within $WF(N,M)$ is the Christoffel word of slope $\frac{M^*}{N}$. It follows that the step pattern a non degenerated scale in $WF(N,M^*)$ is the Christoffel word of slope $\frac{M}{N}$.

It follows that the notions of duality between Christoffel words and WF scales coincide, as they identify each family with the corresponding step pattern. As it often happens with a duality notion in any field of mathematics, some properties of WF scales are cross-related via duality. Table 3.4 provides a list of notions connected by means of the duality between WF scales.

We compute now the incidence matrix of the dual word w^* in terms of the incidence matrix of w. The following lemma shows that to compute the incidence matrix of w^* one has just to *flip* the main diagonal of the incidence matrix of the original word w.

Lemma 7. *If $w = w_1 \cdot w_2$ is the standard factorization of a Christoffel word with incidence matrix*

$$A_w = \begin{pmatrix} |w_1|_0 & |w_1|_1 \\ |w_2|_0 & |w_2|_1 \end{pmatrix}$$

then the incidence matrix of the dual word w^* is

$$A_{w^*} = \begin{pmatrix} |w_2|_1 & |w_1|_1 \\ |w_2|_0 & |w_1|_0 \end{pmatrix}.$$

Proof. Notice first that A_w is the incidence matrix of w, then $\mu[A_w](1) = \frac{|w_1|}{|w_2|}$, were μ denotes the Möbius transform, defined in Section 2.1. Due to the bijection that establishes Proposition 6 between irreducible fractions and matrices in $SL(\mathbb{N}, 2)$, the incidence matrix of w^* is the unique matrix A_{w^*} in $SL(\mathbb{N}, 2)$ verifying

$$\mu[A_w](1) = \frac{|w_1|^{-1} \bmod |w|}{|w_2|^{-1} \bmod |w|} = \frac{|w|_1}{|w|_0}.$$

Last equality holds by Proposition 19 (see Remark 6). Now, the matrix

$$M = \begin{pmatrix} |w_2|_1 & |w_1|_1 \\ |w_2|_0 & |w_1|_0 \end{pmatrix}$$

verifies:

- $det\,(M) = det\,(A_w)$ and hence $M \in SL(\mathbb{N}, 2)$.
- $\mu[M](1) = \frac{|w|_1}{|w|_0}$.

It follows that M coincides with the incidence matrix A_{w^*} of w^*. □

It follows that dual Christoffel words have retrograde paths in the Christoffel tree:

Proposition 25. *If $w = w_1 \cdot w_2$ and $w^* = w_1^* \cdot w_2^*$ are two dual Christoffel words, and $W \in \{\Gamma, \Delta\}^*$ is the generating word of the Christoffel pair (w_1, w_2), then W^t is the generating word of (w_1^*, w_2^*).*

Proof. Let us consider the decomposition of the incidence matrix of w as a product of R's and L's, the generators of the monoid $SL(\mathbb{N}, 2)$:

$$A_w = \begin{pmatrix} |w_1|_0 & |w_1|_1 \\ |w_2|_0 & |w_2|_1 \end{pmatrix} = \Lambda_0 \cdot \Lambda_1 \cdot \cdots \cdot \Lambda_k \quad \text{with } \Lambda_i \in \{R, L\}.$$

Following Lemma 7, we know that

$$A_{w^*} = \begin{pmatrix} |w_2|_1 & |w_1|_1 \\ |w_2|_0 & |w_1|_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that

$$A_{w^*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot (\Lambda_0 \cdot \cdots \cdot \Lambda_k)^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda_k^t \cdot \cdots \cdot \Lambda_0^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we introduce in previous equality the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = Id$$

between each pair of matrices Λ_i:

$$\begin{aligned} A_{w^*} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda_k^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda_0^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \Lambda_k \cdot \dots \cdot \Lambda_0. \end{aligned}$$

In last equality we made use of the following identity:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda_i^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Lambda_i \quad \text{where } \Lambda_i \in \{R, L\}.$$

Finally notice that the path in the Christoffel tree that leads from 01 to a Christoffel word w is obtained from the characteristic matrix of w with the substitutions $L \to \Gamma$ and $R \to \Delta$. □

As a consequence, one can compute the continued fraction expansion of $\frac{M^*}{N} = \frac{M^{-1} \bmod N}{N}$ in terms of the continued fraction expansion of $\frac{M}{N}$:

Corollary 8. *Let* $\frac{M}{N} = [0, a_1, a_2, \dots, a_{k-1}, a_k + 1]$ *be a irreducible fraction. Then:*

$$\frac{M^*}{N} = \begin{cases} [0; 1, a_k, a_{k-1}, \dots, a_1] & \text{if } k \text{ is even} \\ [0; 1 + a_k, a_{k-1}, \dots, a_1] & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Following 7, the matrix A that verifies $\mu[A](1) = \frac{M}{N}$ decomposes as

$$A = L^{a_1} \cdot R^{a_2} \cdots B^{a_k} \text{ with } B = L \text{ if } k \text{ is odd, and } B = R \text{ otherwise.}$$

It follows that

$$\mu \left[L^{a_1 - 1} \cdot R^{a_2} \cdots B^{a_k} \right] (1) = \frac{M}{N - M}$$

and, by Proposition 25

$$\mu \left[B^{a_k} \cdots R^{a_2} \cdot L^{a_1 - 1} \right] (1) = \frac{M^*}{(N - M)^{-1} \bmod N} = \frac{M^*}{N - M^*}.$$

It follows that

$$\mu \left[L \cdot B^{a_k} \cdots R^{a_2} \cdot L^{a_1 - 1} \right] (1) = \frac{M^*}{N}$$

and one concludes, again using Proposition 7. □

Chapter 4

Non Well-formed Words

As we proved in previous chapter (Theorem 4), generated scales with irrational generator may have two or three different steps. We also showed (Theorem 5) that the scale has exactly two steps precisely if the number of notes coincides with the denominator of a (semi-)convergent of the generator. Moreover, the step-pattern is a Christoffel word (Proposition 16). In this chapter we investigate the *bad case*: generated scales with three different steps. We will deeply study their step-patterns, that are words in the alphabet $\{0, 1, 2\}$ and that will be called *non well-formed words* (NWF words). They share some properties with the Christoffel words: they are Lyndon words as well and their right Lyndon factorization is determined by the generator. Nevertheless, the similarity ends here: the left Lyndon factorization coincides with the right Lyndon factorization just in half of the cases. NWF can be written between two consecutive Christoffel words in the Christoffel tree building in this way the so-called *extended Christoffel tree*. We characterize the NWF words for which left and right Lyndon factorizations coincide (we say that they verify the LR property). This happens symmetrically in the extended Christoffel tree. Moreover, we find a surprising connection between the LR property and the Christoffel duality. Finally, it is proven that among the set of NWF words there are infinite many Christoffel-Lyndon words and thus there are infinite many generated scales having as step-pattern a Christoffel-Lyndon word.

4.1 Construction of non well-formed words

Let $\theta \in (0, 1)$ be an irrational number, and let us consider a generated scale

$$\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$$

and let $x_1 = \alpha$, $1 - x_{N-1} = \beta$ and $\alpha + \beta$ be its three possible steps. The step-pattern of Γ may be enconded with a word $w_\Gamma \in \{0, 1, 2\}^*$ as follows:

$$w_\Gamma = w[0]w[1]\dots w[N-1] \in \{0, 1, 2\}^* \quad \text{with} \quad w[k] = \begin{cases} 0, & \text{if } x_{k+1} - x_k = \alpha \\ 2, & \text{if } x_{k+1} - x_k = \beta \\ 1, & \text{if } x_{k+1} - x_k = \alpha + \beta \end{cases}$$

In this way, the first step of the scale will always be encoded with 0 and the last one with 2. As we saw in the previous chapter, if N is a (semi-)convergent of θ,

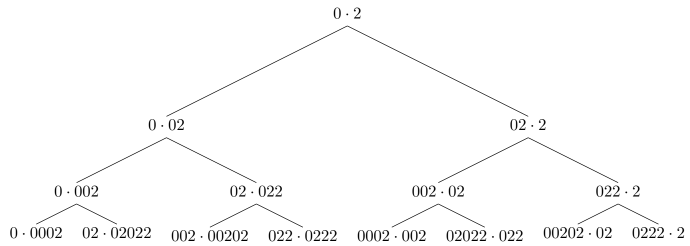


Figure 4.1: The Christoffel tree. The left and right sons of a word w are, respectively, $G(w)$ and $\tilde{D}(w)$.

then the third step $\alpha+\beta$ does not appear in the scale, the scale is well-formed and its step-pattern w_Γ is a Christoffel word within $\{0,2\}^*$. The following notion will center our attention in the whole chapter:

Definition. *A word $w \in \{0,1,2\}^*$ is a non well-formed word (a NWF-word) if it is the step-pattern of a generated scale that is not well-formed.*

Our first goal is to give an explicit construction of NWF words. As we proved in Corolary 4, every Christoffel word over $\{0,2\}$ can be written as the image of 02 under a morphism within $\left\langle G, \tilde{D}\right\rangle$. This enables us to display the set of Christoffel words in the nodes of a binary tree that is shown in Figure 4.1. Two consecutive Christoffel words w and w' in this tree correspond with two consecutive cases of generated scales Γ and Γ' that have just two steps. The step-patterns of the scales that are built in the process of the three steps theorem in-between Γ and Γ' are described in terms of w and w' in section 4.1. These NWF words can be inserted as new nodes between w and w'. The tree obtained in this way (see Figure 4.2) will be called extended Christoffel tree.

Let $\frac{M}{N} \in \mathfrak{F}$ and $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ be three consecutive $N-$ Farey numbers in $\mathfrak{F}_N$. As we proved in Proposition 17, given $\theta \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$, the scale

$$\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$$

is WF and its step-pattern is the Christoffel word w_Γ of slope $\frac{M^*}{N}$. Following propositions describe the NWF words corresponding with the step-patterns of the scales $\Gamma(\theta, N+k)$ for $k = 1, 2, \ldots$ We need first an auxiliary result that will be the cornerstone of the construction of NWF-words:

Lemma 8. *If $w \in \{0,2\}^*$ is a Christoffel word of slope $\frac{M^*}{N}$, then it holds:*

1. $w[k] = 0 \Leftrightarrow k = s * M \bmod N$ *for a certain* $s \in \{0, 1, \ldots, N_2 - 1\}$.

2. $w[k] = 2 \Leftrightarrow k = N - 1 + s * M \bmod N$ *for a certain* $s \in \{0, 1, \ldots, N_1 - 1\}$.

Proof. Notice first that $w = (w'|w'')$ with $|w'| = M$ and $|w''| = N - M$ is the standard factorization of w. It follows that

$$w[0] = 0 = w[M] \quad \text{and} \quad w[N-1] = 2 = w[N-M-1].$$

But $w[i] = w[i+M]$ for $i = 1, \dots, N-M-1$ and $w[i] = w[i+N-M]$ for $i = 1, \dots, M-1$ (M and $N-M$ are periods of w). Then $w[i] = [i + M \bmod N]$ for $i = 1, \dots, N-1$, which concludes the proof. $\square$

Let $\frac{M}{N} \in \mathfrak{F}$ and $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ three consecutive $N-$ Farey numbers in $\mathfrak{F}_N$. As we previously saw, given $\theta \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$, the scale $\Gamma(\theta, N) = \{x_i\}$ is WF and its step-pattern is the Christoffel word w_Γ of slope $\frac{M^*}{N}$. Let us denote by ψ and ξ the morphisms of words in $\{0,1,2\}^*$ given by $\psi(0,1,2) = (0,1,1)$ and $\xi(0,1,2) = (1,1,2)$. Furthermore, let $u_0 = \psi(w_\Gamma)$ and $v_0 = \xi(w_\Gamma)$. Following Propositions describe the NWF words corresponding with the step-patterns of the scales $\Gamma(\theta, N+k)$, being $\Gamma(\theta, N)$ a WF scale:

Proposition 26. *With the previous notation, let* $\theta \in \left(\frac{M_1}{N_1}, \frac{M}{N}\right)$, *let*

$$t_s = (N - 1 + s \cdot M) \bmod N \quad \textit{with } s = 0, 1, \dots N_1 - 1.$$

For $k = 1, \dots, N_1$, *the step-pattern of the scale* $\Gamma(\theta, N+k)$ *is given by*

$$u_k = u_{k,0} \cdot u_{k,1} \cdots u_{k,N-1},$$

with

$$u_{k,i} = \begin{cases} 02 & \textit{if } i \in \{t_0, t_1, \dots, t_{k-1}\} \\ u_0[i] & \textit{if } i \notin \{t_0, t_1, \dots, t_{k-1}\} \end{cases}$$

Proof. If $\theta \in \left(\frac{M_1}{N_1}, \frac{M}{N}\right)$, then the first step $\alpha = x_1$ is smaller than the last one $\beta = 1 - x_{N-1}$. When we generate the $(N+1)$-th note $\{N\theta\}$, it will lay between x_{N-1} and x_0 and the step of length β will have split into two steps: α and $\beta - \alpha$. Following Lemma 8, the order in which the steps of length β split in two of lengths α and $\beta - \alpha$ is given by the sequence $\{t_s\}$.

It follows that the step-pattern of $\Gamma(\theta, N+k)$ will be built from u_0 replacing $u_0[t_s] = 1$ with 02 for all $s = 0, 1, \dots, k-1$. $\square$

Notice that, since $|w_\Gamma|_2 = N_1$, the next WF scale will be $\Gamma(\theta, N+N_1)$ and its step-pattern is the Christoffel word $G(w_\Gamma)$, where G stands for the Christoffel morphism $G(0,2) = (0,02)$. The step-patterns of the scales $\Gamma(\theta, N+k)$ with $k = 1, \dots, N_1 - 1$ (given in previous Proposition) will be called *left*-NWF words.

Proposition 27. *Let* $\theta \in \left(\frac{M}{N}, \frac{M_2}{N_2}\right)$ *and let, moreover,*

$$t_s = s \cdot M \bmod N \quad \textit{with } s = 0, 1, \dots N_2 - 1.$$

For $k = 1, \dots, N_2$, *the step-pattern of the scale* $\Gamma(\theta, N+k)$ *is given by*

$$v_k = v_{k,0} \cdot v_{k,1} \cdots v_{k,N-1},$$

with

$$v_{k,i} = \begin{cases} 02 & \textit{if } i \in \{t_0, t_1, \dots, t_{k-1}\} \\ v_0[i] & \textit{if } i \notin \{t_0, t_1, \dots, t_{k-1}\} \end{cases}$$

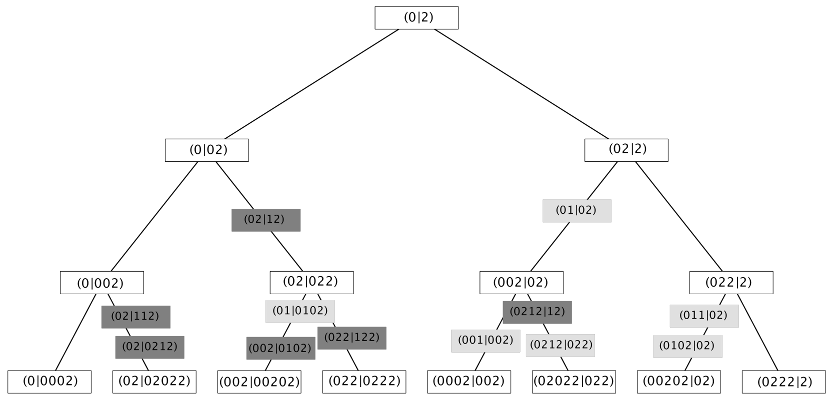


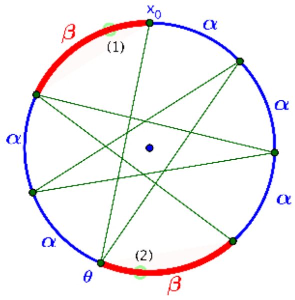
Figure 4.2: The Christoffel tree (white nodes) completed with the NWF words (dark and light grey nodes) and their standard factorization.

In this *right side* of the tree, the scale $\Gamma(\theta, N + N_2)$ has as step-pattern the Chrsitoffel word $\tilde{D}(w_\Gamma)$, where $\tilde{D}$ stands for the *right* Christoffel morphism $\tilde{D}(0,2) = (02,2)$. The NWF words berween w_Γ and $\tilde{D}(w_\Gamma)$ will be called *right-NWF* words. Following previous constructions of right and left NWF words, every NWF word w_i with $i \geq 1$ starts with 0 and ends with 2.

Example 12. *In Figure 4.3 we represent the two possible situations that appear when one considers a generator having $\frac{4}{7}$ as (semi-)convergent. Notice that $\frac{1}{2} < \frac{4}{7} < \frac{3}{5}$ are three consecutive 7−Farey numbers and hence the dichotomy arises when $\theta \in \left(\frac{4}{7}, \frac{3}{5}\right)$ (Figure 4.3(a)) or $\theta \in \left(\frac{1}{2}, \frac{3}{5}\right)$ (Figure 4.3(b)). On the left side it is represented the generated scale around the circle. The places and order in which the notes $x_7, x_8, \ldots$ appear are shown in parenthesis. The tables on the right depict the left and right-NWF words. In each step, from u_k to u_{k+1} (resp. from v_k to v_{k+1}) a 1–highlighted in boldface– is replaced by 02. The generator that we chose in Figure 4.3(b) is the perfect fifth $\log_2(\frac{3}{2})$. Hence, in this case the scale w represents the diatonic scale (with tones for 0 and semitones for 2) and $v_5 = \tilde{D}(w)$ represents the step-pattern of the chromatic scale of 12 notes.*

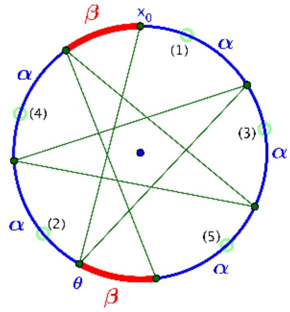
Example 13. *The Figure 4.4 shows a second example to illustrate the previous two propositions. The step pattern of the pentatonic scale $\Gamma(\log_2(\frac{3}{2}), 5)$ is $w = (002|02)$. Since the last step is greater than the first one (Figure 4.4(a)) the next WF scale will be the diatonic $\Gamma(\log_2(\frac{3}{2}), 7)$ with step pattern $G(w) = (0002|002)$ (each long step will split into two that are labeled with 02 (Figure 4.4(c)) and following the ordering given by the sequence $\{r_k\} = \{4 + 3 \cdot k \bmod 5\} = \{4, 2\}$). By Proposition 26, the sequence of left-NWF words is $u_0 = \psi(w)$, $u_1 = (001|002)$ (which corresponds with the step pattern of the Guidonian Hexachord) and at last $u_2 = G(w) = (0002|002)$.*

Between the Christoffel word w of slope $\frac{M^*}{N}$ and its left son $G(w)$ (resp. its right son $\tilde{D}(w)$) the $N_1 - 1$ NWF words $\{u_k\}_{k=1,\ldots,N_1-1}$ given in Proposition 26 (resp. the $N_2 - 1$ NWF words $\{v_k\}_{k=1,\ldots,N_2-1}$ given in Proposition 27) can be inserted. The tree obtained in that way will display the set of step-patterns of every generated scale (see Figure 4.2) and it will be called *extended Christoffel*



$w = 0002 \vert 002$
$u_0 = 0001 \vert 00\mathbf{1}$
$u_1 = 000\mathbf{1} \vert 00\mathit{02}$
$u_2 = 000\mathit{02} \vert 0002 = G(w)$

(a) The left-NWF words corresponding to the scales $\Gamma(\theta, 5+k)$ with $\theta \in \left(\frac{1}{2}, \frac{4}{7}\right)$ and $k = 0, 1, 2$.



$w = 0002 \vert 002$
$v_0 = \mathbf{1}112 \vert 112$
$v_1 = \mathit{02}112 \vert \mathbf{1}12$
$v_2 = 02\mathbf{1}12 \vert \mathit{02}12$
$v_3 = 02\mathit{02}12 \vert 02\mathbf{1}2$
$v_4 = 0202\mathbf{1}2 \vert 02\mathit{02}2$
$v_5 = 0202\mathit{02}2 \vert 02022 = \tilde{D}(w)$

(b) The right-NWF words corresponding to the scales $\Gamma(\theta, 5+k)$ with $\theta \in \left(\frac{4}{7}, \frac{3}{5}\right)$ and $k = 0, 1, \ldots, 5$.

Figure 4.3: The two halves of the interval $\left(\frac{1}{2}, \frac{3}{5}\right)$ determine the NWF words that can be built from the Christoffel word w of slope $\frac{4}{7} = \frac{1+3}{2+5}$.

tree.

It will be convenient to denote with $u_0 = \psi_{01}(w)$ and $v_0 = \psi_{21}(w)$, since many of the properties that are satisfied by the NWF words, fit for u_0 and v_0 as well.

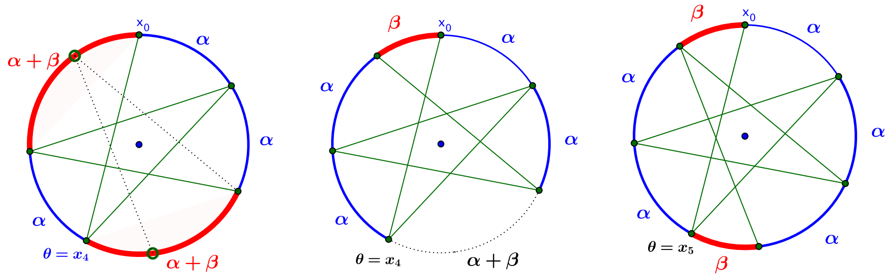
Every Christoffel word w of slope $\frac{M^*}{N}$ has a unique factorization, called *standard factorization*, as $w = w' \cdot w''$, where w' and w'' are Christoffel words (or a single letter) and $|w'| = M$ (see Remark 6). The standard factorization $w = w' \cdot w''$ determines a standard factorization of its NWF sons:

Definition. *If w is a Christoffel word of slope $\frac{M^*}{N}$ and u_k is a left or right NWF word, we call standard factorization of u_k to the decomposition given by:*

$$u_k = u'_k \cdot u''_k = (u_{k,0} \cdot u_{k,1} \cdots u_{k,M-1}) \cdot (u_{k,M} \cdots u_{k,N-1})$$

where $u_{k,j}$ are the factors of u_k given by Proposition 26 if u_k is a left-NWF word or by Proposition 27 if u_k is a right-NWF word.

We define slope $\mu(u)$ of a NWF word u as the slope of the first Christoffel word that precedes it in the extended Christoffel tree. The slope of a NWF word can be computed by means of the following:



(a) $u_0 = (001|01)$. (b) $u_1 = (001|002)$. (c) $u_2 = (0002|002)$.

Figure 4.4: Three scales generated by the pure fifth $\log_2\left(\frac{3}{2}\right)$ and their step patterns.

Proposition 28. *If u is a NWF word, its slope is given by:*

$$\mu(u) = \begin{cases} \frac{|u|-|u|_0}{|u|-|u|_2} & \text{if } |u|_0 > |u|_2, \\ \\ \frac{|u|_2-|u|_0}{|u|-|u|_0} & \text{if } |u|_0 < |u|_2. \end{cases}$$

Proof. Notice first that if u is a left-NWF word, then u proceeds from $u_0 = \phi(w) \in \{0,1\}^*$ replacing some 1's with 02. It follows that u is a left-NWF word if and only if $|u|_0 > |u|_2$. Let us compute $|w|_2$ and $|w|$. Since there is no factor 22 nor 12 in u it follows that

$$|w|_2 = |u|_1 + |u|_{02} = |u|_1 + |u|_2 = |u| - |u|_0 \quad \text{and} \quad |w| = |u| - |u|_2.$$

One concludes that $\mu(u) = \mu(w) = \frac{|u|-|u|_0}{|u|-|u|_2}$. For the right-NWF case, the NWF word u proceeds from $v_0 = \xi(w) \in \{1,2\}^*$. It follows that $|u|_0 < |u|_2$ and a argument similar to the one above proves the formula for the right-NWF case. □

4.2 Some properties of NWF words

We will present now some basic properties of non-well-formed words. Firstly, they can be computed recursively, as the following proposition asserts:

Proposition 29. *Let u_{i-1} and u_i be two consecutive NWF words with $1 < i \leq N_r$ ($r = 1$ for left-NWF words and $r = 2$ for right-NWF words). If the standard factorization of u_i is*

$$u_i = u_i' \cdot u_i'' = 0x \cdot y2 \quad \text{with } x, y \in \{0,1,2\}^*,$$

then the word u_{i-1} can be written as $u_{i-1} = y1x$.

Proof. If u_{i-1} and u_i are NWF of slope $\frac{M^*}{N}$ and $w = 0c2$ is the corresponding Christoffel word with $0p2 \cdot 0q2$ the standard factorization of w, with c, p, q palindrome words in $\{0,2\}^*$, it follows that $0q$ is a prefix of w of length $N - M$. Every time that $w[k] = 1$ is replaced by 02 in a NWF word u_s under w with

$k \geq M$, previously the letter $w[k-M] = 1$ had been replaced by 02 in the immediately preceding NWF word u_{s-1}. It follows that y must be a prefix of u_{i-1}. A similar argument shows that x is a suffix of u_{i-1}

To show that $y1$ is a prefix of u_{i-1} as well, let u_i be a NWF word between w and $\tilde{D}(w)$. The last 1 of $\psi_{01}(w)$ to be replaced by 02 is (see Proposition 27),

$$M \cdot (N_2 - 1) \bmod N = M \cdot (N - M^* - 1) \bmod N = N - M - 1 = |0q|.$$

As $y2$ is the word that lays below $w_2 = 0q2$ in the extended Christoffel tree, it follows this last 1 to be replaced immediately follows the prefix y in u_{i-1}. The same happens if u_i lays between w and $G(w)$, since in this case, the last 1 of v_0 to be replaced by 02 is:

$$N - 1 + M \cdot (N_1 - 1) \bmod N = N - 1 + M \cdot (M^* - 1) \bmod N = N - M$$

□

Corollary 9. *Let u_i and u_{i-1} be two consecutive NWF words, with standard factorizations*

$$u_{i-1} = u'_{i-1} \cdot u''_{i-1} = 0x \cdot y2 \qquad u_i = u'_i \cdot u''_i$$

where $x, y \in \{0,1,2\}^*$. The following two statements hold:

1. *$u''_{i-1} = u''_i$ if and only if $y1$ is a prefix of u_{i-1}.*
2. *$u''_{i-1} \neq u''_i$ if and only if $1x$ is a suffix of u_{i-1}.*

Proof. . We will prove the first statement, since the second one is symmetric.

If $u''_{i-1} = u''_i$, then $u_i = 0x' \cdot y2$ and following Proposition 29 it holds that $u_{i-1} = y1x'$ and $y1$ is a prefix of u_{i-1}.

If, on the contrary, $u''_{i-1} \neq u''_i$ we will check that $y1$ is not a prefix of u_{i-1}. In this case $u_i = 0x \cdot y'2$, with $|y'| = |y| + 1$. Following Proposition 29, $u_{i-1} = y'1x$. It follows that $y1$ is a prefix of u_{i-1} if and only if $y' = y1$. But this is impossible, since that would imply that $u_{i-1} = 0x \cdot y2$ and $u_i = 0x \cdot y12$ and these words cannot be consecutive NWF words (notice that in the transformation $u_{i-1} \to u_i$ one factor 1 is replaced by 02). □

Corollary 10. *Let u be a NWF word and $u = 0x \cdot y2$ be its standard factorization. If $|x| < |y|$ and $y1$ is a prefix of u, then $y2 = (0x)^k z2$ with $k \geq 1$ and $z \in \{0,1,2\}^*$ a certain prefix of $0x$.*

Proof. Let $u = 0x \cdot y2$ be a NWF word and $|x| < |y|$, with $y1$ a prefix of u. It follows that $0x$ is a prefix of $y1$, thus $u = 0x \cdot 0xa_1 2$. For a certain $a_1 \in \{0,1,2\}^*$. If a_1 is shorter than $0x$, since $y = 0xa_1$ is a prefix of $0x \cdot 0xa_1 2$, the factor a_1 must be a prefix of $0x$ and the claim is proven. Otherwise we may repeat the argumentation to show that the word u can be factorized as $u = 0x \cdot (0x)^2 a_2 2$ for a certain $a_2 \in \{0,1,2\}^*$. One closes the proof by recursion. □

A similar reasoning proves the symmetric result:

Corollary 11. *If a NWF word u has the standard factorization $u = 0x \cdot y2$ with $|y| < |x|$ and $1x$ is a suffix of u, then $0x$ decomposes as $0x = 0z(y2)^k 2$, for a certain suffix z of $y2$.*

4.3 NWF words are Lyndon words

If one take a closer look to the extended Christoffel tree of Figure 4.2, one can see that the factors u' and u'' of any NWF word $u = u' \cdot u''$ are themselves nodes of the tree (in case they are not a single letter or a Christoffel word). To prove this fact we need the following lemma:

Lemma 9. *Let $\frac{M}{N}$ be an irreducible fraction. Let moreover $\frac{M_1}{M_2} < \frac{M}{N} < \frac{M_2}{N_2}$ be three consecutive $N-$Farey fractions and $M^* = M^{-1} \pmod N$. The following equalities hold:*

1. $M^* - N \bmod M^* = M_1^{-1} \bmod N_1$.
2. $M^* \bmod (N - M^*) = M_2^{-1} \bmod N_2$.

Proof. In this whole proof we use general properties of continued fraction expansions (see [36]). We have to distinguish two cases, namely $M_2 > M_1$, $N_2 > N_1$ or $M_2 < M_1$,$N_2 < N_1$.

Let $M_2 > M_1$ and $N_2 > N_1$ hold. If $\lambda \in \mathbb{N}$ is such that

$$M_2 - (\lambda+1)M_1 < 0 < M_2 - \lambda M_1 = C \quad N_2 - (\lambda+1)N_1 < 0 < N_2 - \lambda N_1 = D,$$

then, we have the following sequence of (semi-)convergents of $\frac{M}{N}$:

$$\frac{M_1 - C}{N_1 - D} < \frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2} < \frac{M_2 - M_1}{N_2 - N_1} < \cdots < \frac{M_2 - \lambda M_1}{N_2 - \lambda N_1} = \frac{C}{D}.$$

If we change the notation, taking $A = M_1 - C$ and $B = N_1 - D$, we have:

$$\frac{A}{B} < \frac{M_1}{N_1} < \frac{C}{D} \qquad \text{are consecutive } N_1 - \text{Farey numbers,}$$

$$\frac{M_1}{N_1} < \frac{C+\lambda M_1}{D+\lambda N_1} = \frac{M_2}{N_2} < \frac{C+(\lambda-1)M_1}{D+(\lambda-1)N_1} \text{ are consecutive in } \mathfrak{F}_{N_2} \text{ and}$$

$$\frac{M_1}{N_1} < \frac{C+(\lambda+1)M_1}{D+(\lambda+1)N_1} = \frac{M}{N} < \frac{M_2}{N_2} \text{ are consecutive } N - \text{Farey numbers.}$$

Now, we can compute $M^* - N \bmod M^*, M_1^{-1} \bmod N_1, M^* \bmod (N - M^*)$ and $M_2^{-1} \bmod N_2$ via Lemma 7, that asserted that the characteristic matrices of dual slopes have flipped the main diagonal.

$$\begin{pmatrix} C & A \\ D & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} c & a \\ d & b \end{pmatrix} \Rightarrow \frac{M_1}{N_1} = \frac{a+c}{a+b+c+d}.$$

It follows that $M_1^{-1} \bmod N_1 = (a+b) \bmod (a+b+c+d)$.

$$\begin{pmatrix} C+(\lambda-1)M_1 & M_1 \\ D+(\lambda-1)N_1 & N_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c+(\lambda-1)(a+c) & a+c \\ d+(\lambda-1)(b+d) & b+d \end{pmatrix}.$$

Thus one has that,

$$\frac{M_2}{N_2} = \frac{c+\lambda(a+c)}{c+d+\lambda(a+b+c+d)} \qquad \text{and} \qquad M_2^{-1} \bmod N_2 = (a+b+c+d) \bmod N_2.$$

Finally we have:

$$\begin{pmatrix} M_2 & M_1 \\ N_2 & N_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c+\lambda(a+c) & a+c \\ d+\lambda(b+d) & b+d \end{pmatrix}.$$

Hence $N = c + d + (\lambda + 1)(a + b + c + d)$ and

$$M^* = M^{-1} \bmod N = (a + b + c + d) \bmod N.$$

Now we close the proof, since

$$M^*-N \bmod M^* = a+b+c+d-(c+d+(\lambda+1)(a+b+c+d)) \bmod (a+b+c+d) = a+b$$

and

$$M^* \bmod (N - M^*) = (a+b+c+d) \bmod (c+d+\lambda(a+b+c+d)) = a+b+c+d.$$

The second case ($M_2 < M_1$ and $N_2 < N_1$) is completely symmetric. $\square$

Proposition 30. *If $u = u' \cdot u''$ is a word in the extended Christoffel tree, each of the words u' and u'' verify one of the following three conditions:*

1. *It is a single letter.*
2. *It is a Christoffel word within $\{0, 1\}^*$, $\{0, 2\}^*$ or $\{1, 2\}^*$.*
3. *It is a NWF word.*

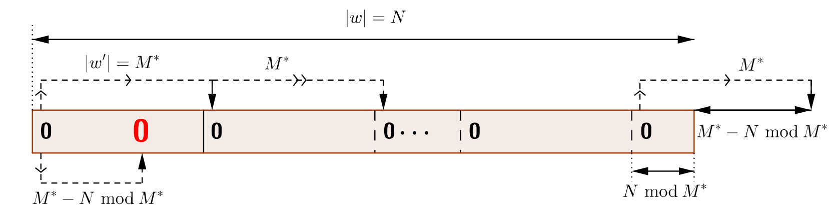
Proof. Notice first that every Christoffel word verifies the statement. Let $w = w' \cdot w''$ be a Christoffel word of slope $\frac{M}{N}$ and $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ are three consecutive N-Farey numbers. We need to prove that if $u = u' \cdot u''$ is a NWF word laying between w and $G(w)$ –resp. between w and $\tilde{D}(w)$– both u' and u'' meet the conditions of the Proposition. We will prove that u' is under the word w' and u'' is under the word w'' in the extended Christoffel tree.

If u is a NWF word that lays between w and $\tilde{D}(w)$, it is obtained from v_0 with the substitution $1 \to 02$ of some 1's following the ordering $i \cdot M^* \bmod N$, where $M^* = M^{-1} \bmod N$ (see Proposition 27). For $i = 1$ we obtain M^*, the place were w is divided in its standard factorization $(w'|w'')$. Notice that

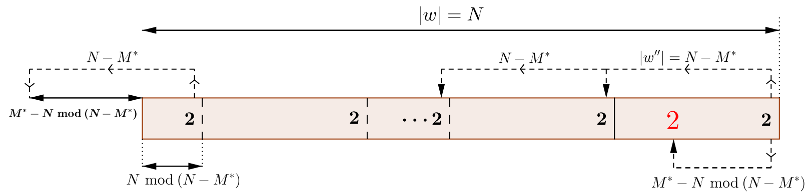
$$|w| = N = \left[\frac{N}{M^*}\right] \cdot M^* + N \bmod M^*.$$

It follows that the smaller i satisfying $i \cdot M^* \bmod M^* < M^*$, verifies the condition $i \cdot M^* \bmod M^* = M^* - N \bmod M^*$ (see Figure 4.5). The word u' will be a node of the extended Chrsitoffel tree if the Christoffel word w' has a standard factorization that splits as $(w')' \cdot (w')''$ with $|(w')'| = M^* - N \bmod M^*$. But we know that $|(w')'| = M_1^{-1} \bmod N_1$. We conclude due to Lemma 9.

Something similar (rather symmetrical) can be said if u is a NWF word that lays between w and $G(w)$. We will replace every 1 of v_0 by 02 following the ordering given by $(N - 1 + i \cdot M^*) \bmod N$. Again, for $i = 1$ we get the last 2 of w', and the next index that lays in w'' is $M^* - N \bmod (N - M^*)$ (see Figure 4.5(b)). But this is exactly the place where w'' splits as a product of two Christoffel words, due to Lemma 9. $\square$



(a) The ordering in which 0's are replaced by 02 in the right-NWF words.



(b) The ordering in which 2's are replaced by 02 in the left-NWF words.

Figure 4.5: Graphical motivation for the modular equalities of Lemma 9.

Proposition 31. *If $u = u' \cdot u''$ is a word in the extended Christoffel tree, then it holds that $u' < u''$.*

Proof. Notice first that the result holds for every Christoffel word. We proceed by induction on the depth of the Christoffel tree. Let $w = w' \cdot w''$ be a Christoffel word and let $\{u_s\}_{s=1,2,\ldots,N_1-1}$ and $\{v_t\}_{t=1,2,\ldots,N_2-1}$ the NWF words that lay between w and, respectively, $G(w)$ and $\tilde{D}(w)$.

We will prove that the result holds first for every NWF word u_i, with standard factorization $u_i' \cdot u_i''$, between w and $G(w)$ by induction over i. Let $u_0 = \psi_{21}(w' \cdot w'') = 0x1 \cdot 0y1$, with x and y palindrome words in $\{0,1\}^*$. By construction (Proposition 26),

$$u_1 = 0x1 \cdot 0y02 = u_1' \cdot u_1'' \quad \text{and} \quad u_2 = 0x02 \cdot 0y02 = u_2' \cdot u_2''.$$

Following Proposition 29, $0y01$ must be a prefix of u_1. If $|u_1''| < |u_1'|$, then $u_1' \cdot u_1'' = 0y01z \cdot 0y02$, with $z \in \{0,1\}^*$. If $|u_1''| > |u_1'|$, then u_1' is a prefix of u_1''. In both cases $u_1' < u_1''$. Thus, the statement holds for $i = 1$.

In each step of the induction $u_{i-1}' \cdot u_{i-1}'' \to u_i' \cdot u_i''$ a 1 is replaced by a 02. If the replaced 1 is in u_{i-1}', there is nothing to prove since, in that case, by induction:

$$u_i' < u_{i-1}' < u_{i-1}'' = u_i''.$$

We can suppose that we are in the following situation:

$$u_{i-1}' = u_i' \quad \text{and} \quad u_{i-1}'' = x1y2 \to x02y2 = u_i''$$

and $i < N_1$. Following Proposition 29, $x02y1$ is a prefix of u_{i-1}.

Now, if $|x02y1| > |u_{i-1}'|$, then $u_{i-1}' = u_i'$ is a prefix of u_i'' and it follows that $u_i' < u_i''$. If $|x02y1| < |u_{i-1}'| = |u_i'|$, then

$$u_i' = x02y1z \quad u_i'' = x02y2 \quad \text{for some } z \in \{0,1,2\}^*.$$

and $u'_i < u''_i$ holds as well.

Now, let v_i be a NWF word with standard factorization $v'_i \cdot v''_i$, that lays between w and $\tilde{D}(w)$. Again, we proceed by induction over i. The case $i = 1$ is trivial now, since

$$v'_1 = 0x \quad v''_1 = y \quad \text{with } x, y \in \{1,2\}^* \Rightarrow v'_1 < v''_1.$$

We use a similar reasoning as above. In each step of the induction

$$v'_{i-1} \cdot v''_{i-1} \to v'_i \cdot v''_i \qquad \text{a 1 is replaced by a 02.}$$

If the replaced 1 is in v'_{i-1}, there is nothing to prove since, by induction:

$$v'_i < v'_{i-1} < v''_{i-1} = v''_i.$$

In the other case,

$$v'_{i-1} = v'_i \quad \text{and} \quad v''_{i-1} = x1y2 \to x02y2 = u''_i,$$

following Proposition 29, the word $x02y1$ is a prefix of $v'_{i-1} \cdot v''_{i-1}$. It holds then necessarily that $v'_i < v''_i$.

□

Following [37, Proposition 5.1.3], a word w is Lyndon if and only if it is a single letter or it can be written as $w = w' \cdot w''$ with $w' < w''$ Lyndon words. This characterization will be used in the following result:

Theorem 13. *If u is a word of the extended Christoffel tree, it is a Lyndon word.*

Proof. It is enough to show the assertion for the NWF words since Christoffel words are Lyndon words. If $u = u' \cdot u''$ is a NWF word, following Proposition 30, u' is a single letter, a Christoffel word (in both cases, a Lyndon word) or it is a NWF word that appears upper in the extended Christoffel tree. The same holds for u''. By induction, u' and u'' are Lyndon words. Since $u' < u''$ (Proposition 31) u must be a Lyndon word. □

4.4 Lyndon factorizations of NWF words

In this section we study the right and left Lyndon factorization of NWF words. As we will prove, the right Lyndon factorization coincides with the standard factorization, as in the Christoffel case, but the left Lyndon factorization does not. We describe the nodes of the extended tree for which left and right Lyndon factorization coincide. These nodes are painted in light grey in Figure 4.2.

We begin by determining the longest proper prefix of a NWF word that is a Lyndon word, that is, the right Lyndon factorization of the NWF word. The following result [37, Proposition 5.1.4] states a characterization of this factorization:

Proposition 32. *Let w be a Lyndon word different from a single letter and let $w = u' \cdot u''$ be its right Lyndon factorization. For any Lyndon word v with $w < v$ the right Lyndon factorization of wv is $w \cdot v$ if and only if $u'' \geq v$.*

Theorem 14. *If u is a word of the extended Christoffel tree, its standard factorization $u' \cdot u''$ coincides with its right Lyndon factorization. That is, u'' is the longest proper suffix of u that is Lyndon.*

Proof. The assertion is already known for Christoffel words. Let us suppose, thus, that u_i is a NWF word, and let $u_i' \cdot u_i''$ be its standard factorization. By induction and following Proposition 32, if $u_i' = (u_i')' \cdot (u_i')''$ is the standard factorization of u_i', it is enough to show that $(u_i')'' > u_i''$. We will proceed by induction over i. If $i = 0$ the statement is clear, since u_0 is a Christoffel word.

Let us suppose that u_{i-1} is a NWF word, with $i \geq 1$ verifying the Theorem. Now we can distinguish two cases:

1. If the slope $\frac{M^*}{N}$ of w is greater than $\frac{1}{2}$, then w is on the right side of the Christoffel tree and $w = w_1 w_2 \cdot w_2$ is its standard factorization, with $|w_1 w_2| = M$. Let $((u_{i-1}')' \cdot (u_{i-1}')'') \cdot u_{i-1}''$ be the standard factorization of u_{i-1}. Each of the three factors lays below the corresponding factor of w. In the transformation $u_{i-1} \to u_i$ a 1 is replaced by 02. This 1 can come from three different factors of $w = w_1 w_2 \cdot w_2$. If the 1 comes from w_1, then
$$(u_i')'' = (u_{i-1}')'' \geq u_{i-1}'' = u_i''.$$
If the 1 comes from the first w_2 of $w_1 w_2 \cdot w_2$, then $(u_{i-1}')'' = a1b$ and $u_{i-1}'' = (u_i')'' = u_i'' = a02b$ for certain words $a, b \in \{0, 1, 2\}^*$ (notice that every time a 1 coming from the middle factor w_2 is replaced by a 02, in the immediate previous step, the same 1 but from the right factor w_2 had been replaced, due to the order of replacement that Propositions 26 and 27) determine. It follows that $(u_i')'' = u_i''$. Finally, if the 1 comes from the second factor w_2 of $w_1 w_2 \cdot w_2$, then
$$(u_i')'' = (u_{i-1}')'' \geq u_{i-1}'' > u_i''.$$

2. Let us suppose now that $\frac{M^*}{N} < \frac{1}{2}$ and thus the standard factorization of w is $w = w_1 \cdot (w_1 \cdot w_2)$. Let $w_1 = x_1 \cdot x_2$ be the standard factorization of w_1. Following [37, Propositon 5,1,3], $x_1 < x_1 x_2 < x_2$. If
$$u_{i-1} = (y_1 \cdot y_2 \,|\, y_3 \cdot y_4 \cdot z) \quad \text{with} \quad y_2 \geq y_3 \cdot y_4 \cdot z,$$
where the factors y_1, y_2, y_3, y_4 and z lay respectively below the factors of the decomposition of $w = (x_1 \cdot x_2 \,|\, x_1 \cdot x_2 \cdot w_2)$, the 1 that is replaced with 02 in the trasformation $u_{i-1} \to u_1$ can be in y_1, y_2, y_3, y_4 or z. If the replaced 1 comes from y_3, y_4 or z, there is nothing to prove, since
$$(u_i')'' = (u_{i-1}')'' = y_2 \geq y_3 \cdot y_4 \cdot z > u_i''.$$
If the replaced 1 comes from y_1 then
$$(u_i')'' = (u_{i-1}')'' = y_2 \geq y_3 \cdot y_4 \cdot z = u_i''.$$
It remains to check only the case when the 1 that is replaced comes from y_2. Let $u_{i-1}' = y_1 \cdot y_2 = 0a \cdot b1c2$, $u_i' = 0a \cdot b02c2$ with $y_1 = 0a$ and $y_2 = b1c2$. Since u_i' is a NWF word, following Proposition 29, one has that $u_{i-1}' = b02c1a$. Notice now that $u_i'' = u_{i-1}''$ and also that u_{i-1}' is a prefix of u_{i-1}''. One may conclude, since
$$(u_i')'' = b02c2 > b02c1a \geq u_{i-1}'' = u_i''.$$

□

Once the description of right Lyndon factorization is complete we analyze when it coincides with the left factorization:

Definition. *A word verifies the LR property if its left and right Lyndon factorizations coincide.*

To characterize the NWF words that verify LR property, we need a previous result:

Lemma 10. *If $u_i = a1b$ and $u_{i+1} = a02b$ with $i \geq 0$ are two consecutive NWF words with $a, b \in \{0,1,2\}^*$, then the word $a1$ is Lyndon.*

Proof. We proceed by induction over the length $|u_i|$ of u_i. The first cases are trivial. We finally suppose that the result is true for $|u_i| \leq n$ and let $u_i = a1b$ be a NWF of length n such that $u_{i+1} = a02b$. Let $u_i = u_i' \cdot u_i''$ be its standard factorization. If $|a1| \leq |u_i'|$, we may apply induction, since u_i' is a NWF word (see Proposition 30) shorter than u_i and it follows that $a1$ is Lyndon. If, on the contrary, $|a1| > |u_i'|$, then $u_i = a1b = u_i' \cdot a'1b$, with $a'1b = u_i''$. Again, following Proposition 30, u_i'' is a NWF word and by induction, $a'1$ is a Lyndon word. To prove that $a1$ is a Lyndon word as well, we just need to show that $u_i' < a'1$. Notice that $u_i'' = u_{i-1}'' = a'1b$ and following Corollary 9, if $b = b'2$, then $a'1b'1$ is a prefix of u_{i-1}. It follows that $a'02b'1$ is a prefix of u_i. We may distinguished now two cases: if $|u_i'| > |u_i''|$, then the word u_i can be written as

$$u_i = u_i' \cdot u_i'' = a'02b'1z \cdot a'1b'2.$$

As $a'02 < a'1$, it follows that $a'02b'1z = u_i' < a'1$. Let us finally suppose that $|u_i'| < |u_i''|$. Following Corollary 10,

$$u_{i-1} = 0x \cdot (0x)^k z2 = a'1b' \cdot (a'1b')^k z2 \quad \text{for certain } z, b' \in \{0,1,2\}^k$$

and $u_i = a'02b' \cdot (a'1b')^k z2$. It follows that $u_i' = a'02b' < a'1$. □

Notice that Lemma 10 ensures that for every NWF word u there is a Lyndon prefix longer than the left side of the standard factorization of u, but it does not determines the *longest* prefix that is Lyndon.

Theorem 15. *Let $u_{i-1} = u_{i-1}' \cdot u_{i-1}''$ and $u_i = u_i' \cdot u_i''$ be the standard factorization of two consecutive NWF words u_{i-1} and u_i with $i \geq 1$. The left and right Lyndon factorizations of u_{i-1} coincide if and only if $u_{i-1}'' = u_i''$.*

Proof. Let us suppose first that $u_{i-1}'' = u_i''$. If $u_{i-1} = 0x \cdot y2$, following Lemma 9, $y1$ is a prefix of u_{i-1}. We will distinguish two cases, namely $|u_{i-1}'| > |u_{i-1}''|$ or $|u_{i-1}'| < |u_{i-1}''|$. If $|u_{i-1}'| < |u_{i-1}''|$, then $u_{i-1} = y1a \cdot y2$ for a certain $a \in \{0,1,2\}*$. If b is a prefix of y, then $bc1a < c1ab$, as $bc1a$ is Lyndon. It follows that $bbc1a < bc1ab$ and $y1ab = bc1ab$ cannot be a Lyndon word. Something similar happens if $|u_{i-1}'| > |u_{i-1}''|$. In this case, by Lemma 10,

$$y2 = (0x)^k z2 \quad \text{with } z \in \{0,1,2\}^* \quad \text{a prefix of } x.$$

Given a prefix b of $y2$, b can be written as $b = (0x)^{k'} b'$, with b' a prefix of $0x$. But

$$(0x)^{k'} b' = (b'a')^{k'} b'.$$

But, as $0x$ is a Lyndon word, $b' < a'$ and thus $b'(b'a')^{k'+1} < (b'a')^{k'+1}b'$ and it follows that $0xb = (b'a')^{k'+1}b'$ cannot be a Lyndon word.

Let us suppose now that $u''_{i-1} \neq u''_i$. One has that $u''_{i-1} \cdot a'1b$ is a the prefix of u_{i-1} such that $u_i = u'_i \cdot a'02b$. Following Lemma 10, it holds that $u''_{i-1} \cdot a'1b$ is Lyndon. It is possible to find thus a prefix of u_{i-1} longer than u'_{i-1} that is a Lyndon word. It follows that u_{i-1} does not verify LR property. □

Corollary 12. *If $\{u_i\}_{i=0,\dots,N_1-1}$ and $\{v_j\}_{j=0,\dots,N_2-1}$ are the left and right-NWF words associated with a Christoffel word w, the following list of facts hold:*

1. *The left firstborn u_1 verifies LR property and the rights firstborn v_1, does not.*

2. *If w is on the left side of the Christoffel tree (if its slope is smaller than $\frac{1}{2}$), then the last NWF words u_{N_1-1} and v_{N_2-1} do not verify LR property.*

3. *On the contrary, if w is on the right side of the Christoffel tree, u_{N_1-1} and v_{N_2-1} do verify LR property.*

Proof.

1. If $u_0 = (0x1|0y1)$, then $u_1 = (0x1|0y02)$ and $u_2 = (0x02|0y02)$. It follows that u_1 verifies the LR property. Similarly, if $v_0 = (1x2|1y2)$, then $v_1 = (02x2|1y2)$ and $v_2 = (02x2|02y2)$. Thus v_1 does not verify the LR property.

2. It is enough to show that the last 1 to be replaced with 02 proceeds from a certain $w[k] = 1$ with $k > M - 1$. Notice that this 1 is determined by the sequences of Lemma 8, namely, $k = N_1 - 1$ for left-NWF words and $k = N_2 - 1$ for right-NWF words. We compute these indexes taking into account that $N_1 \cdot M \equiv 1 \bmod N$ and $N_2 \cdot N - M \equiv 1 \bmod N$:

$$N - 1 + M(N_1 - 1) \equiv N - M > M - 1 \Leftrightarrow \frac{M}{N} < \frac{1}{2} + \frac{1}{2N}.$$

$$M(N_2 - 1) \equiv N - M - 1 > M - 1 \Leftrightarrow \frac{M}{N} < \frac{1}{2}$$

and both conditions hold if the slope of w is smaller than $\frac{1}{2}$.

3. If the slope of w is grater than $\frac{1}{2}$ previous two conditions do not hold, and hence u_{N_1-1} and v_{N_2-1} verify LR property.

□

The last theorem can be translated into the terminology of generated scales as follows:

Proposition 33. *The step-pattern of a generated scale $\Gamma(\theta, N)$ verifies the LR property if and only if it holds that $0 < \{N\theta\} < \theta$.*

4.5 Distribution of the LR property in the extended Christoffel tree

If one takes a glance at the extended Christoffel tree, one can easily guess that the distribution of the NWF words with or without the LR property meets certain rules. The first one: the light and dark grey nodes are distributed symmetrically in the tree.

Proposition 34. *If $u = 0x2$ and $u' = 0x'2$ are two NWF words whose paths in the extended Christoffel tree are symmetric, then it holds that $x' = E(\tilde{x})$, where $\tilde{x}$ denotes the retrogade of x and E is the morphism of words given by $E(0, 1, 2) = (2, 1, 0)$.*

Proof. Notice first that if w and w' are the Christoffel words associated with u and u' and the slope of w is θ, then the slope of w' is $1 - \theta$. Hence, if c and c' are the central words of w and w', $c' = E(c)$. Since c and c' are palindrome words, we conclude that the Lemma holds in the case of Christoffel words:

$$0c'2 = 0E(c)2 = 0E(\tilde{c})2$$

Now, let us examine $\phi(w) = u_0$ and $\phi(w')$ in the one hand and $\xi(w')$ and $\xi(w) = v_0$ in the other in order to show that the places of the 1's of these words are in *opposite positions*. Two letters $w[i]$ and $w[j]$ of the word $w[0]w[1] \dots w[N-1]$ are in opposite positions if $i = N - 1 - j$. Now, the 1's of u_0 and v_0 are given in Propositions 26 and 27. The symmetric word w' has slope $\frac{N-M}{N}$ and thus the places of the 1's in the words $\xi(w')$ and $\phi(w')$ are given by:

$$\xi(w')[s \cdot (N - M) \bmod N] \quad \text{with } s = 0, 1, \dots, N_2 - 1$$

and

$$\phi(w')[N - 1 + s \cdot (N - M) \bmod N] \quad \text{with } s = 0, 1, \dots, N_1 - 1.$$

From the equalities

$$N-1-((N-M)+s\cdot M) \bmod N = (N-M)\cdot s \bmod N \quad \text{for } s = 0, 1, \dots, N_2-1$$

and

$$N - 1 - s \cdot M \bmod N = N - 1 + (N - M) \cdot s \bmod N$$

one concludes that the 1's of u_0 and $\phi(w')$ (resp. $\xi(w')$ and v_0) are in opposite positions and hence

$$u_0 = E\left(\widetilde{\phi(w')}\right) \quad \text{and} \quad \xi(w') = E\left(\widetilde{v_0}\right).$$

As in each step, from u_i to u_{i+1} in the extended Christoffel tree, a 1 is replaced by a 02 and one has that $E\left(\widetilde{02}\right) = 02$ we conclude the proof.

□

Corollary 13. . *If two NWF words x and y are symmetric in the extended Christoffel tree, then x verifies LR property if and only if y does not. As a consequence, there is the same number of NWF words satisfying LR property than those which do not satisfy it.*

Proof. Since $y = E(\tilde{x})$, the 1 that will we replaced by 02 in $0y2$ is in the left half of its standard factorization if and only if the same 1 is in the right half of the standard factorization of $0x2$. □

And the corresponding translation into the theory of scales can be made:

Corollary 14. *Let θ be an irrational number. The step pattern of a non-well-formed scale $\Gamma(\theta, N)$ verifies the LR property if and only if the step-pattern of the scale $\Gamma(1-\theta, N)$ does not verify it.*

Proof. Notice first that if $\Gamma(\theta, N)$ is well formed, its step-pattern and the one of $\Gamma(1-\theta, N)$ are symmetric in the Christoffel tree. Now, if $\Gamma(\theta, N)$ is not WF, its step-pattern is a left-NWF word if and only the step-pattern of $\Gamma(1-\theta, N)$ is a right-NWF word. Moreover, their paths in the extended Christoffel tree are symmetric. We close the proof due to Corollary 13. □

We study hereafter a second property of the layout of the NWF words satisfying the LR property in the extended Christoffel tree. Given a Christoffel word w the objective is to determine the distribution of the light and dark grey nodes of the left and right branches drawn from w. First we present a direct way to determine whether a left or right-NWF verifies the LR property or not. It can be deduced easily from the Theorem 15:

Proposition 35. *Let w be a Christoffel word of slope $\frac{M^*}{N}$ and $\{u_i\}_{i=0,1,\dots,N_1-1}$, $\{v_j\}_{j=0,1,\dots,N_2-1}$ the sequences of left and right-NWF words associated with w. The following statements hold:*

1. *The word u_i verifies the LR property if and only if $N-1+Mi \bmod N < M$.*
2. *The word v_j verifies the LR property if and only if $Mi \bmod N < M$.*

Proof. If $(u_0'|u_0'')$ is the standard factorization of $u_0 = \phi(w)$, then $|u_0'| = M$. Following Theorem 15, the word u_i verifies LR property if and only if the 1 that will be replaced by 02 in u_{i+1} corresponds with a 1 in u_0', but the place of this 1 in u_0 is given by $N-1+Mi \bmod N$. Second statement is proven in a similar way. □

With the notations of the previous proposition, let us define a couple of words $U, V \in \{0,1\}^*$ of lengths, respectively $N_1 - 1$ and $N_2 - 1$, by means of the formulas:

$$U[i] = \begin{cases} 1 & \text{if } u_{i+1} \text{ verifies the LR property} \\ 0 & \text{if } u_{i+1} \text{ does not verify the LR property} \end{cases} \qquad \text{for all } i = 0, 1, \dots, N_1-2$$

$$V[j] = \begin{cases} 1 & \text{if } v_{j+1} \text{ verifies the LR property} \\ 0 & \text{if } v_{j+1} \text{ does not verify the LR property} \end{cases} \qquad \text{for all } j = 0, 1, \dots, N_2-2$$

The word U (resp. V) encodes the sequence of light (1) and dark (0) grey nodes in a left (resp. right) branch of the extended Christoffel tree. The following result describes the words U and V as factors of a very particular Christoffel word:

Proposition 36. *With the previous notations, the word $V0U1$ is a Christoffel word of slope $\frac{M}{N}$.*

Proof. It is a consequence of the Proposition 35. Notice that

$$M(i+1) \bmod N < M \iff \left\lceil \frac{M(i+1)}{N} \right\rceil - \left\lceil \frac{Mi}{N} \right\rceil = 1.$$

It follows that $V[0]V[1]\cdots V[N_2-2]$ coincides with the prefix of length N_2-1 of the Christoffel word of slope $\frac{M}{N}$. The N_2-th of this Christoffel word is 0, since $M \cdot N_2 \equiv .1 \bmod N$. Now, this late modular equation also implies that:

$$\begin{aligned} N-1+M(i+1) \bmod N < M &\Leftrightarrow M(N_2+i+1) \bmod N < M \\ &\Leftrightarrow \left\lceil \frac{M(N_2+i+1)}{N} \right\rceil - \left\lceil \frac{(M(N_2+i)}{N} \right\rceil = 1. \end{aligned}$$

It follows that the next N_1-1 letters of the Christoffel word of slope $\frac{M}{N}$ coincide with the word U. □

The previous result presents an unexpected tight connection between the LR property and the duality of Christoffel words: the words w and $V0U1$ are Christoffel dual words.

Example 14. *We return to the Example 12. The sequences of indexes of u_0 and v_0 that determine which 1 will be replaced by 02 are*

$$\{N-1+Mi \bmod N\}_{i=0,\ldots,N_1-1} = \{6+4i \bmod 7\}_{i=0,1} = \{6,3\}$$

$$\{Mi \bmod N\}_{i=0,\ldots,N_2-1} = \{4i \bmod 7\}_{i=0,\ldots,4} = \{0,4,1,5,2\}.$$

Hence, the left (resp. right) NWF words $\{u_i\}_{i=1,\ldots,N_1-2}$ (resp. $\{v_i\}_{i=1,\ldots N_2-2}$) verify the LR property if the corresponding index from $\{3\}$ (resp. from $\{4,1,5,2\}$) is smaller than $M=4$. It follows that the words u_1, v_2 and v_4 verify the LR property; v_1 and v_3 do not. The words $U=1$ and $V=0101$ encode the sequence of NWF words verifying the LR property in each side and the word $V0U1 = 0101011$ is the Christoffel word of slope $\frac{4}{7}$.

Finally, we connect the NWF words with the Christoffel-Lyndon words, a certain type of finite word over an ordered alphabet that can be defined directly using the left and right Lyndon factorizations. Christoffel words verify the LR property recursively, that is, if $w = w_1 \cdot w_2$ is the standard factorization of the Christoffel word w, then this factorization coincide with the left and right Lyndon factorizations and, moreover, w_1 and w_1 verify LR property as well. With this fact in mind, G. Melançon and C. Reutenauer defined in [40] the notion of Christoffel-Lyndon words for an arbitrary, totally-ordered alphabet $\mathcal{A}$ in a recursive way: a letter is a Christoffel-Lyndon word; otherwise, a Lyndon word w is a Christoffel-Lyndon word if its left and right Lyndon factorizations coincide, say $w = u \cdot v$ and if, moreover, u and v are both Christoffel-Lyndon words.

The following proposition proves that there are infinite-many Christoffel-Lyndon words in the extended Christoffel tree:

Proposition 37. *If u_1 is a left-NWF word firstborn of a Christoffel word w, then u_1 is Christoffel-Lyndon. Moreover, if w is on the right side of the Christoffel tree (that is, if $|w''| > |w'|$ where $(w'|w'') = w$ is the standard factorization of w) then u_2 is a Christoffel-Lyndon as well.*

Proof. Notice first that, following Corollary 12, u_1 verifies LR property. Let $u_1 = (u_1'|u_1'')$ and $w = (w'|w'')$ denote the standard factorizations of u_1 and w. Notice that, by construction, $u_1 = x02$, with $x \in \{0,1\}^*$. It follows that u_1' is a Christoffel word in $\{0,1\}^*$ and also that u_1'' is the left-NWF word firstborn of w''. Thus, both u_1' and u_1'' verify LR property and the first part of the proof is closed by recursion.

For the second part, if $u_2 = (u_2'|u_2'')$ is the standard factorization of u_2, notice that $u_2' = y02$ and $u_2'' = z02$ with $y, z \in \{0,1\}^*$. The words u_2' and u_2'' are the firstborn left-NWF words of w' and w'' respectively. It follows that u_2' and u_2'' verify LR property. Notice finally that $|u_2''| > |u_2'|$ and hence $u_2'' = u_3''$, where $u_3 = (u_3'|u_3'')$ is the next left-NWF word. Following Theorem 15, u_2 satisfies LR property as well and we close again the proof by recursion. $\square$

A reformulation of the previous result in scale terminology can be made:

Corollary 15. *If the generated scale* $\Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ *is well-formed and* $x_{N-1} < \{N\theta\} < 1$*, then the step-pattern of* $\Gamma(\theta, N+1)$ *is a Christoffel-Lyndon word. Moreover, if* $\theta > \frac{1}{2}$*, then the step-pattern of* $\Gamma(\theta, N+2)$ *is a Christoffel-Lyndon word as well.*

Chapter 5

Extension of Christoffel duality over the modes of WF scales

In this chapter we formalize the musical notion of *mode of a scale*. The step patterns of the modes of a WF scale coincide with the rotations of a Christoffel word. The main objective of this chapter is to extend the Christoffel duality over the set of modes of WF scales. The first attempt to fulfill this is made by considering the geometrical representation of the rotations of Christoffel words as mechanical sequences of functions like $f(x) = \frac{Mx+k}{N}$ and will lead to define the *plain and twisted adjoints*. We analyze how these two adjoints behave over the standard morphisms generated by $E, G, \tilde{G}, D, \tilde{D}$. We end up by interpreting the results in musical terms.

5.1 Modes of WF-scales

In this section we characterize the modes of a WF scale, that is, the scales that are rotations of a given WF scale. For that purpose we will use the *accumulation matrices*, as defined in section 3.6. As we work with rotations of scales, we will focus on the conjugates of a given (Christoffel) word.

Let $\Sigma = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ be a scale of N notes and step pattern w_Σ. For every $k = 0, \ldots, N-1$, we call $k-th$ mode of the scale Σ to the scale given by

$$\Sigma_k = \Sigma - x_k = \{\{x_i - x_k\}, \quad \text{where } i = 0, \ldots, N-1\}$$

With this notation, the first mode coincides with the original scale, that is $\Sigma_0 = \Sigma$. The step pattern of each mode coincides with a word conjugated with w_Σ. It follows that a circular word (w_Σ) parametrizes the set of modes of a given scale Σ.

Example 15 (Diatonic Modes)**.** *The modes of the diatonic scale generated by fifths are called diatonic modes. Each of the scales can be seen as a rotation of the G minor scale. The seven different modes, together with their names, step*

Mode	Scale pattern	Starting note
Lydian	0001001	F
Mixolydian	0010010	G
Aeolian	0100100	A
Locrian	1001000	B
Ionian	0010001	C
Dorian	0100010	D
Phrygian	1000100	E

Table 5.1: Diatonic modes.

patterns and starting notes can be seen in table 5.1. Notice that the Christoffel word corresponds with the Lydian mode.

In the case of generated scales, if one generates some of the N notes counterclockwise, one obtains a mode of the scale:

Definition. *Given a real number $\theta \in (0,1)$, $N > 1$ and a number $k \in \mathbb{Z}_N$, the scale*

$$\Gamma_k(\theta, N) = \{\{-k \cdot \theta\}, \{(-k+1) \cdot \theta, \dots, 0, \theta, \dots, \{(N-1-k) \cdot \theta)\}\}$$

is called ***generalized generated scale****.*

Let $\Sigma = \Gamma(\theta, N)$ be a generated scale. Notice that

$$\Gamma_k(\theta, N) + (1 - \{-k \cdot \theta\}) = \Gamma_0(\theta, N) = \Sigma$$

and thus, the generalized generated scales are rotations of generated scales. It follows that the step patterns of the generalized generated scales associated with a generated scale coincides with the set of the step patterns of its modes. The following proposition describes the relationship between step patterns of modes and generalized generated scales in the case of WF scales. It is a trivial consequence of the geometrical interpretation of the multiplier of the scale:

Proposition 38. *Let $\Sigma = \Gamma(\theta, N) = \{0 = x_0 < x_1 < \cdots < x_{N-1}\}$ be a WF-scale with multiplier M and let $M^* = M^{-1} \mod N$. Then one has that*

$$\Gamma_t(\theta, N) = \Sigma_k = \Sigma - x_k$$

where $t = M^* \cdot k \mod N$.

Proof. It is enough to show that Σ_1 is the step pattern of Γ_{M^*}, but this is an immediate consequence of the second characterization of WF-scales given in Theorem 12. $\square$

In the following paragraphs we introduce a new characterization of the conjugation class of a given Christoffel word. It is based on the *balance* and *accumulation* maps:

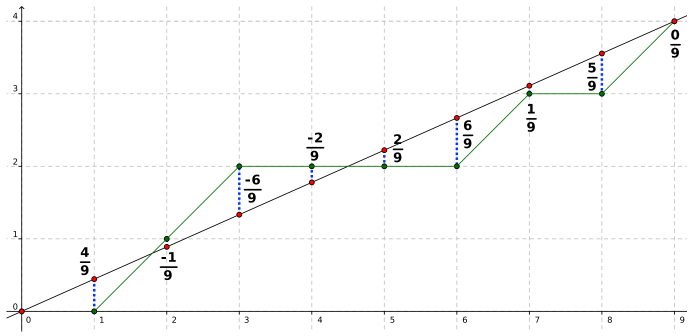


Figure 5.1: The accumulation map associated with the word $w = 011000101$ is $\{\alpha_w(k)\} = \{4, -1, -6, -2, 2, 6, 1, 5, 0\}$.

Definition. *Let $w \in \{0,1\}^*$ be a finite binary word.*

1. *The* ***balance map*** *of w is the map $\beta_w : \{1, \dots, |w|\} \longrightarrow \mathbb{Z}$ with*

$$\beta_w(k) = \begin{cases} |w|_1 & \textit{for} \quad w_k = 0 \\ -|w|_0 & \textit{for} \quad w_k = 1 \end{cases}$$

2. *The* ***accumulation map*** *of w is the map $\alpha_w : \mathbb{Z}_{|w|} \longrightarrow \mathbb{Z}$ with*

$$\alpha_w(0) := 0 \qquad \alpha_w(k) := \sum_{r=1}^{k} \beta_w(r)$$

A *geometrical interpretation* of the maps defined above can be given in terms of *mechanical sequences*. For a given binary word $w = w_1 w_2 \dots w_N \in \{0,1\}^*$ with $|w|_1 = M$ and $|w| = N$, let us consider the following two sequences of $N + 1$ points of the integer grid $\mathbb{Z}^2$:

$$\begin{array}{ll} P_0 = (0,0) & Q_i = (i, |w_1 w_2 \dots w_i|_1) \\ Q_0 = (0,0) & P_i = \left(i, \dfrac{M}{N} i\right) \quad \text{for all } i = 1, \dots N \end{array}$$

The sequence $\{P_i\}$ is on the line of slope $\frac{M}{N}$ and the sequence $\{Q_i\}$ represents the mechanical sequence associated with w (see Figure 5.1). The following lemma states that accumulation map labels the numerators of the distances between P_i and Q_i:

Lemma 11. *With the previous notations, it holds that:*

$$\frac{\beta(i)}{|w|} = dist(P_i, Q_i) - dist(P_{i-1}, Q_{i-1})$$

and

$$\frac{\alpha_w(i)}{|w|} = dist(P_i, Q_i) \qquad \forall\, i = 1, \dots, N.$$

Proof. Notice that

$$dist(P_i, Q_i) = \frac{M}{N} \cdot i - |w_1 \dots w_i|_1 = \frac{M \cdot i - |w_1 \dots w_i|_1 \cdot N}{N}$$

and thus it holds

$$dist(P_i, Q_i) - dist(P_{i-1}, Q_{i-1}) = \begin{cases} \frac{Mi - M(i-1)}{N} = \frac{M}{N} = \frac{|w|_1}{|w|} \text{ if } w_i = 0 \\ \frac{M-N}{N} = \frac{-|w|_0}{|w|} \text{ if } w_i = 1 \end{cases}$$

We conclude that $dist(P_i, Q_i) - dist(P_{i-1}, Q_{i-1}) = \frac{\beta_w(i)}{|w|}$ and hence

$$\frac{\alpha_w(i)}{|w|} = dist(P_i, Q_i) - dist(P_0, Q_0) = dist(P_i, Q_i).$$

□

We present a second interpretation of the accumulation map, in matricial terms. Let w_k denote the prefix of length k of a given word w of length n and let w'_k be the suffix of length $n-k$ of w. Thus we write $w = w_k \cdot w'_k$. Let A_k denote the accumulation matrix of previous factorisation of w, that is

$$A_k = \begin{pmatrix} |w_k|_0 & |w'_k|_0 \\ |w_k|_1 & |w'_k|_1 \end{pmatrix}.$$

The accumulation map coincides with the determinants of the accumulation matrices of all the possible decompositions $w = w_k \cdot w'_k$ with $k = 0, 1, \dots, N$:

Proposition 39. *For every finite word $w \in \{0,1\}^*$ and every $k = 1, \dots, |w|-1$, one has that*

$$\begin{aligned} |A_{k+1}| - |A_k| &= \beta_w(k+1) \\ |A_k| &= \alpha_w(k) \end{aligned}$$

.

Proof. Notice that

$$|A_k| = \begin{vmatrix} |w_k|_0 & |w_k|_0 + |w'_k|_0 \\ |w_k|_1 & |w_k|_1 + |w'_k|_1 \end{vmatrix} = \begin{vmatrix} |w_k|_0 & |w|_0 \\ |w_k|_1 & |w|_1 \end{vmatrix} = \begin{vmatrix} |w_k|_0 & |w|_0 \\ k & |w| \end{vmatrix}.$$

It follows that

$$|A_{k+1}| - |A_k| = \begin{vmatrix} |w_{k+1}|_0 & |w|_0 \\ k+1 & |w| \end{vmatrix} - \begin{vmatrix} |w_k|_0 & |w|_0 \\ k & |w| \end{vmatrix} = \begin{vmatrix} |w_{k+1}|_0 - |w_k|_0 & |w|_0 \\ 1 & |w| \end{vmatrix}.$$

And, finally

$$|A_{k+1}| - |A_k| = \beta_w(k+1) = \begin{cases} |w|_1 \text{ if } |w|_{k+1} = 0 \\ -|w|_0 \text{ if } |w|_{k+1} = 1. \end{cases}$$

The equality $|A_k| = \alpha_w(k)$ is now immediate, since $|A_0| = 0$. □

Now we connect the accumulation map with the conjugation class of Christoffel words.

Definition. *A word w is called **well-formed** if the set*

$$\{\alpha(k), \ \text{with } k = 1, \dots, |w|\}$$

coincides with a set of $|w|$ consecutive integer numbers including the number 0. In other words, if there exists an integer $\mu_w \in \{0, \dots, |w| - 1\}$ such that $\{\alpha_w(1) + \mu_w, \dots, \alpha_w(|w| - 1) + \mu_w\} = \{0, \dots, |w| - 1\}$. The number μ_w is called the mode of w.

Proposition 40. *Given a word w of length $|w| = N$, it is a well-formed word with mode 0 if and only if w is a Christoffel word.*

Proof. If w is a Christoffel word, on the basis of [10, Lemma 1.3], the different distances between P_i and Q_i take all possible values between $\frac{0}{N}$ and $\frac{N-1}{N}$, being the numerators all natural numbers. Hence w is WF with mode 0.

The converse is immediate as well. If w is ME with mode 0, the distances $dist(P_i, Q_i)$ are all smaller than 1. Then w encodes a mechanical sequence, starting from $(0, 0)$ and ending in $(|w|, |w|_1)$, for which there is point of the grid between P_i and Q_i $\forall\, i = 1, \dots, |w| - 1$. w must be a Christoffel word. $\square$

If we make the same geometrical procedure but with a rotation of the Christoffel word, we see that the set of corresponding signed distances

$$\{dist(Q_i, P_i)\}_{i=0,\dots,N-1}$$

coincides with a set of the type $\left\{\frac{-k}{N}, \frac{-k+1}{N}, \dots, \frac{N-k-1}{N}\right\}$, being the numerators N different consecutive integer numbers and k the mode of the rotation. The following result formalizes this idea:

Lemma 12. *If w is well-formed, $\gamma^k w$ is also well-formed. Furthermore, the modes of two conjugated WF words are linked by means of the following equation:*

$$\mu_{\gamma^k w} = \mu_w + k \cdot |w|_y \mod |w|$$

Proof. We just show that γw is well-formed whenever w is so. By definition of balance map one has:

$$\begin{aligned}&\beta_{\gamma w}(k) = \beta_w(k+1) \quad \forall k = 1, \dots, |w| - 1\\ &\beta_{\gamma w}(|w|) = \beta_w(1)\end{aligned}$$

Thus, we have that

$$\alpha_{\gamma w}(k) = \sum_{r=1}^{k} \beta_{\gamma w}(r) = \sum_{r=1}^{k} \beta_w(r+1) = \sum_{r=2}^{k+1} \beta_w(r) = \alpha_w(k+1) - \beta_w(1)$$

and finally:

$$\mu_{\gamma w} = -\inf \alpha_{\gamma w}(k) = -(\inf \alpha_w(k+1) - \beta_w(1)) = \mu_w + \beta_w(1) \equiv \mu_w + |w|_y$$

$\square$

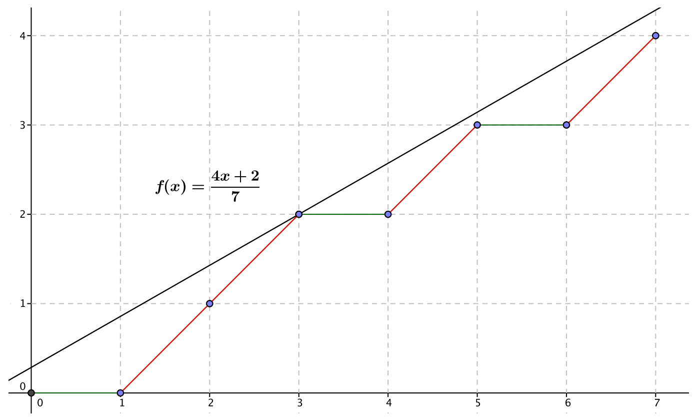


Figure 5.2: The word $w = 0110101$ is the WF word of slope $\frac{4}{7}$ and mode 2.

Corollary 16. *A word is well-formed if and only if it is conjugated with some Christoffel word.*

The following geometrical interpretation of the WF word is now immediate. The word encodes the zig-zag that lays under a line of rational slope:

Corollary 17. *A WF word w of slope $\frac{M}{N}$ and mode k is the mechanical sequence associated with the slope $\frac{M}{N}$ and the intercept $\frac{k}{N}$, in other words, it can be computed as:*

$$w[i] = \left[\frac{M \cdot (i+1) + k}{N}\right] - \left[\frac{M \cdot i + k}{N}\right]$$

Proof. It is a consequence of the geometrical interpretation of the accumulation map α_w. Notice that the sequence

$$\alpha_w(i) + k \quad \text{with } i = 0, 1 \ldots, N-1$$

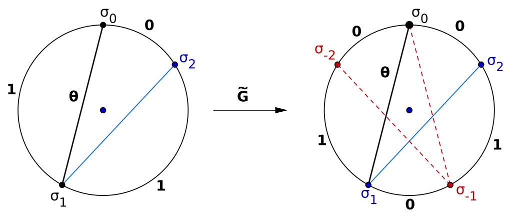
comprises all natural numbers between 0 and N exactly once. It follows that w encodes a mechanical sequence of slope $\frac{M}{N}$. Notice that the first point of the sequence is $(0, \frac{\alpha_w(0)+k}{N}) = (0, \frac{k}{N})$. Hence, the intercept of the line is $\frac{k}{N}$). $\square$

Corollary 18. *Let $\Gamma_0(\theta, N)$ be a non-degenerated WF scale. The step pattern $w(k)$ of the generalized generated scale $\Gamma_k(\theta, N)$ is a WF word with mode k.*

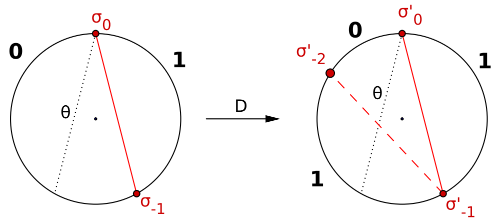
Remark 13 (The generators of the Sturmian morphisms $\tilde{G}$, D and E)**.** *As it was shown in the proof of Corollary 4, the sturmian morphism $\tilde{D}$ and G connect consecutive cases of generated scales having a Christoffel step pattern in the three gaps theorem. We now give a geometrical description of the sturmian morphisms*

$$E(0,1) = (1,0) \quad \tilde{G}(0,1) = (0,10) \quad \textit{and} \quad D(0,1) = (10,0)$$

in terms of generalized generated scales. The substitution θ by $1-\theta$ produces a symmetry on the scale $\Gamma(\theta, N)$ and thus steps are exchanged. This corresponds to the morphism E.



(a) The morphism $\tilde{G}$.



(b) The morphism D.

Figure 5.3: The morphisms $\tilde{G}$ and D connect cases of generated scales with just two steps but in *negative* direction.

If one generates notes in the "negative direction" (that is, if one uses the generator $1-\theta$ *instead of* θ*) and the interval* α*, labeled with 1, is greater than* β*, labeled with 0, then one has to subdivide every interval* α *into two intervals, with lengths* $\alpha-\beta$ *and* β*. Thus, we transform the step pattern via the morphism* $\tilde{G}$*. Notice that, since we generate in negative direction, the interval* α *will decompose as* $\beta \cdot (\alpha-\beta)$ *and not the other way around (see Figure 5.3(a)). Something similar will happen if the interval* α *that is labelled with 0 is shorter than the one* β *labelled with 1. This time, since* $\beta > \alpha$ *and we generate notes in negative direction, the interval* β *will be decomposed as* $(\beta-\alpha)\cdot\alpha$*, and the morphism used will be* D *(see Figure 5.3(b)).*

From the previous Remark and Corollary 4 one deduces the following result, which is a classical result in combinatorics of words:

Corollary 19. *The morphisms of the monoid* $\left\langle G, D, \tilde{G}, \tilde{D}, E \right\rangle$ *send conjugates of Christoffel words to conjugates of Christoffel words.*

5.2 Plain and Twisted Adjoints

The aim of this section is to extend the notion of duality between Christoffel words (Section 3.10) over the set of conjugates of a Christoffel word that, as it was explained in Section 5.1, are canonically associated to the modes of WF scales. As we saw in Corollary 16, to each word w of length N that is the conjugate of a Christoffel word, we can assign an affinity $f(x)$ in such a way that w is the prefix of length N of the mechanical sequence associated with the

Table 5.2: The conjugation class of the diatonic step pattern 0001001 together with the conjugation class of its *plain adjoint* 0101011.

w	0001001	0010010	0100100	1001000	0010001	0100010	1000100
$f_w(k)$	$2k$	$2k+2$	$2k+4$	$2k+6$	$2k+1$	$2k+3$	$2k+5$
$f_w^{\square}(k)$	$4k$	$4k+6$	$4k+5$	$4k+4$	$4k+3$	$4k+2$	$4k+1$
$w^{\square}$	0101011	1101010	1011010	1010110	1010101	0110101	0101101

line $y = f(x)$. In this way there is a canonical bijection between the rotations of Christoffel words of length N and the affinities of $\mathbb{Z}_N$:

Definition. *Given a WF word w of mode ν and length N, the plain affinity f_w is the affinity of $\mathbb{Z}_n$ given by the formula*

$$f_w(k) = |w|_1 \cdot k + \nu \mod N.$$

Notice that, if w and w^* are dual Christoffel words, it holds that both are WF words of mode 0. In the other hand, since $|w|_1 = |w^*|_1^{-1} \bmod |w|$, one has that $f_w = f_{w^*}^{-1}$. This facts lead to the following definition :

Definition. *Given a WF word w, we call* ***plain adjoint*** *of w, denoted by $w^{\square}$, the unique word, the affinity of which coincides with the inverse affinity of w. In other words, the plain adjoint $w^{\square}$ is defined by the equation:*

$$f_{w^{\square}} = f_w^{-1}$$

The word w and its plain dual $w^{\square}$ encode two zig-zags that connect the grid points in the mechanical sequence: the first one from left two right, the second one downwards, following the direction perpendicular with the slope of w (see Figure 5.4).

Recall that the standard monoid is the submonoid of St generated by:

$$Stand = < E, G, D > .$$

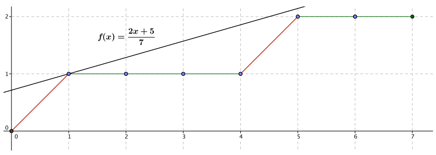
For every standard morphism f and every $0 \le i \le |f(01)| - 1$ we define the morphism f_i as the unique morphisms that satisfies the conditions

$$f_i(01) = \gamma^i(f(01)) \quad \text{and} \quad |f_i(0)| = |f(0)|$$

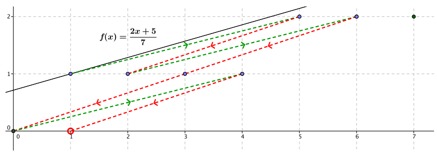
where γ denotes the rotation operator. Standard morphisms are essential in the study of Sturmian morphisms, since every Sturmian morphism can be seen as the conjugate of a Standard morphism, as the following result (see [38, Proposition 2.3.22]) asserts:

Theorem 16. *A morphism ψ is Sturmian if and only if there is a standard morphism f and a $i \in \{0, 1, \ldots, |\psi(01)| - 2\}$ such that $\psi = f_i$.*

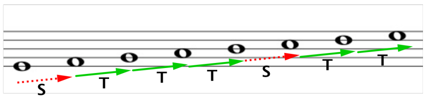
If f is a standard morphism, and $|f(01)| = N$, then the morphisms f_0, $f_1, \ldots, f_{N-2}$ are Sturmian and f_{N-1} is not (see [38, Propositioin 2.3.21]). Since standard words and Christoffel words are conjugated, every Sturmian morphism



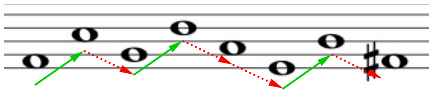
(a) The word $w_1 = 1000100$ and its associated affinity $f_1(x) = 2x + 5$.



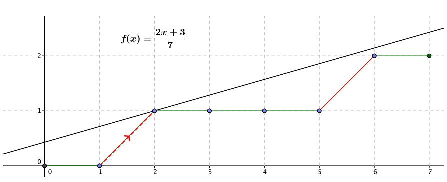
(b) The plain dual $w_1^{\square} = 1010010$.



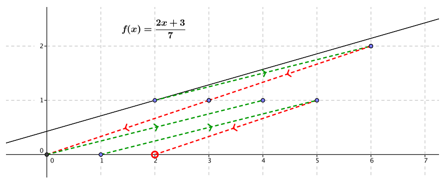
(c) The Phrygian mode and its step pattern, the word $w_1 = 1000100$.



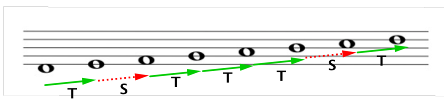
(d) The cycle of fifths corresponding with the Phrigyan mode and its step pattern, the plain dual $w_1^{\square} = 1010010$.



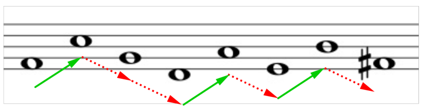
(e) The word $w_2 = 0100010$ and its associated affinity $f_2(x) = 2x + 3$.



(f) The plain dual $w_2^{\square} = 0110101$.



(g) The Dorian mode and its step pattern, the word $w_2 = 0100010$.



(h) The cycle of fifths corresponding with the Dorian mode and its step pattern, the plain dual $w_2^{\square} = 0110101$.

Figure 5.4: Two examples of WF words, their plain duals and their musical interpretations.

Table 5.3: The conjugation class of the diatonic step pattern 0001001 determines 6 Sturmian morphisms that are complied in the table. Together with them are shown the morphisms that correspond with each plain dual word

w	0001001	0010010	0100100	1001000	0010001	0100010	1000100
$f(01) = w$	$G^2\tilde{D}$	$G\tilde{G}D$	$\tilde{G}^2\tilde{D}$	not a morphism	G^2D	$G\tilde{G}D$	$\tilde{G}^2D$
$f(01) = w^{\square}$	$\tilde{D}G^2$	not a morphism	$D\tilde{G}^2$	$D\tilde{G}G$	DG^2	$\tilde{D}\tilde{G}^2$	$\tilde{D}G\tilde{G}$
$w^{\square}$	0101011	1101010	1011010	1010110	1010101	0110101	0101101

is determined by a word that is conjugated with a certain Christoffel word. The converse is not true: for every Christoffel word w of length N, there are $N-1$ words that can be written as $f(01)$ for a certain Sturmian word f and one that does not determine a Sturmian morphism.

The purpose of the following paragraphs is to study the behavior of plain adjoint over the Sturmian morphisms. Let us consier the canonical inyection $St_0 \hookrightarrow WF$ associates to every Sturmian morphism f the WF word $f(01)$. If we focus on the monoid of Christoffel morphisms $\left\langle G, \widetilde{D} \right\rangle$ we have the following result:

Proposition 41. *The diagram:*

$$\begin{tikzcd} \left\langle G, \widetilde{D} \right\rangle \arrow[r, hook] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & WF \arrow[d, "\square"] \\ \left\langle G, \widetilde{D} \right\rangle \arrow[r, hook] & WF \end{tikzcd}$$

is commutative or, in other words:

$$f(01)^{\square} = f^t(01)$$

where f^t denotes the retrogradation of f as a word in $\left\langle G, \widetilde{D} \right\rangle$.

Proof. Observe that $w^{\square}$ is the dual of w and the Christoffel morphisms related to dual words are retrogade (see [32, Proposition 10]). $\square$

The same result holds for morphisms in $< G, D >$ as well. To prove that, we need first a couple of auxiliary lemmas:

Lemma 13. *If $u\{0,1\}^*$ is a palindrome, then it holds that $1\tilde{D}(u) = D(u)1$.*

Proof. We proof it by induction over the length of u. The result is trivial for all the palindromes of lengths 1 or 2, namely $u \in \{0, 1, 00, 11\}$. Let us asume that the equality holds for a palindrome u and let us check that the analogue equalities hold for the palindromes $0u0$ and $1u1$. In the one hand

$$1\tilde{D}(0u0) = 101\tilde{D}(u)01 = 10D(u)101 = D(0u0)1$$

and in the other

$$1\tilde{D}(1u1) = 11\tilde{D}(u)1 = 1D(u)11 = D(1u1)1$$

as we wanted to prove. $\square$

Lemma 14. *If u is a central word and $f(01) = 0u1$ for a certain $f \in \left\langle G, \tilde{D} \right\rangle$, then $\rho(f)(01) = u01$, where ρ is the morphism of monoids*

$$\rho : f \in \left\langle G, \tilde{D} \right\rangle \longrightarrow \langle G, D \rangle \,,$$

where $\rho(G) = G$ and $\rho(\tilde{D}) = D$.

Proof. We prove it recursively. First, notice that

$$G^k(01) = 00^k1 \Rightarrow \rho(G^k)(01) = G^k(01) = 00^k1 = 0^k01.$$

And it holds as well that

$$\tilde{D}^k(01) = 01^k1 \Rightarrow \rho(\tilde{D}^k)(01) = D^k(01) = 1^k01.$$

Thus, the statement holds for these two first cases. Now, let us suppose that the equality $\rho(f)(01) = u01$ also holds, where f is the morphism $f \in \left\langle G, \tilde{D} \right\rangle$ such that $f(01) = 0u1$. We will prove that the corresponding equalities hold for the morphisms Gf and $\tilde{D}f$. The first case is trivial, since

$$Gf(01) = G(0u1) = 0G(u)01, \quad \text{where } G(u)0 \text{ is a central word.}$$

Then

$$\rho(Gf)(01) = G\rho(f)(01) = G(u01) = G(u)G(01) = G(u)001.$$

Let us consider, finally, the morphism $\tilde{D}f$. We have:

$$\tilde{D}f(01) = \tilde{D}(0u1) = 01\tilde{D}(u)1, \quad \text{where } 1\tilde{D}(u) \text{ is a central word,}$$

and now we compute $\rho(\tilde{D}f)(01)$:

$$\rho(\tilde{D}f)(01) = D\rho(f)(01) = D(u01) = D(u)101.$$

Since the central word u is a palindrome, following Lemma 13, it holds that $1\tilde{D}(u) = D(u)1$ and it follows that $\rho(\tilde{D}f)(01) = 1\tilde{D}(u)01$. □

Plain adjoint behaves *properly* for the special standard monoid as well:

Proposition 42. *The following diagram is commutative:*

$$\begin{tikzcd}
\langle G, D \rangle \arrow[r, hook] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowright"] & WF \arrow[d, "\square"] \\
\langle G, D \rangle \arrow[r, hook] & WF
\end{tikzcd}$$

Proof. The square in the proposition decomposes in the way that follows:

$$\begin{tikzcd}
\langle G, D \rangle \arrow[rrr, hook] \arrow[dr, "\rho"] \arrow[ddd] & & & WF \arrow[ddd, "\square"] \\
& \left\langle G, \tilde{D} \right\rangle \arrow[r] \arrow[d, "t"] & Ch \arrow[ur, "\phi"] \arrow[d, "\square"] & \\
& \left\langle G, \tilde{D} \right\rangle \arrow[r] & Ch \arrow[dr, "\phi"] & \\
\langle G, D \rangle \arrow[rrr, hook] \arrow[ur, "\rho"] & & & WF
\end{tikzcd}$$

Table 5.4: The conjugation class of the diatonic step pattern 0001001 in relation with the corresponding twisted adjoints and the associated sturmian morphisms.

w	$0001 \cdot 001$	$0010 \cdot 010$	$0100 \cdot 100$	$1001 \cdot 000$	$0010 \cdot 001$	$0100 \cdot 010$	$1000 \cdot 100$
$g_w(k)$	$5x+6$	$5x+4$	$5x+2$	$5x$	$5x+5$	$5x+3$	$5x+1$
$f(01)=w$	$G^2\tilde{D}$	$G\tilde{G}\tilde{D}$	$\tilde{G}^2\tilde{D}$	not morphic	$G^2 D$	$G\tilde{G}D$	$\tilde{G}^2 D$
$g_w^{\boxtimes}(k)$	$3x+3$	$3x+2$	$3x+1$	$3x$	$3x+6$	$3x+5$	$3x+4$
$w^{\boxtimes}$	$10 \cdot 10101$	$10 \cdot 10110$	$10 \cdot 11010$	$11 \cdot 01010$	$01 \cdot 01011$	$01 \cdot 01101$	$01 \cdot 10101$
$f(01)=w^{\boxtimes}$	DG^2	$D\tilde{G}G$	$D\tilde{G}^2$	not morphic	$\tilde{D}G^2$	$\tilde{D}\tilde{G}G$	$\tilde{D}\tilde{G}^2$

where ρ is the monoid morphism defined in the previous Lemma and $\phi(0u1) = u01$. One has just to show that every smaller square commutes. The left square trivially commutes and the square on the center is commutative by proposition 41. Following Lemma 14, upper and lower squares also commute. The commutativity of the square on the right, finally, is equivalent to the formula $u01^{\square} = u^{*}01$ where u^{*} is the central word of the dual of $0u1$. But this is true, since $0u1^{\square} = 0u^{*}1$ holds. $\square$

The problem is that this nice formula $f(01)^{\square} = f^{t}(01)$ does not extend to the whole special Sturmian monoid St_0. That is, the following diagram *is not* commutative:

$$\begin{tikzcd} St_0 \arrow[r, hook] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & WF \arrow[d, "\square"] \\ St_0 \arrow[r, hook] & WF \end{tikzcd}$$

Notice that $\tilde{G}(01)^{\square} = (1|00)$ is not a morphic WF word.

We can solve this problem by giving a parallel definition to the plain adjoint: the **twisted adjoint.**

Definition. *Given a well-formed word w, we define the **twisted map** as the affinity given by the formula*

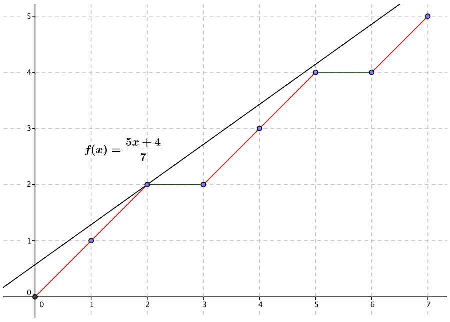
$$g_w^{\boxtimes}(k) = |w|_0 \cdot k + N - \nu_w - 1$$

where $\nu_w = \max\{\alpha_w(0), \ldots \alpha_w(|w|-1)\}$ and $N = |w|$. We define then the twisted adjoint of the word w by the same formula as the plain adjoint:

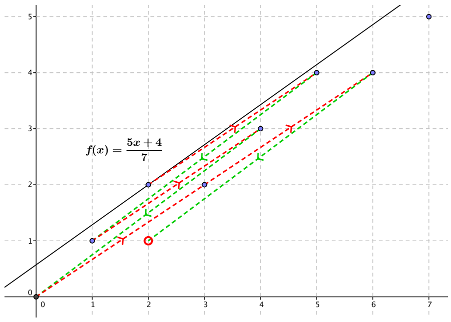
$$g_{w^{\boxtimes}} = g_w^{-1}$$

in other words, twisted maps associated to words which are twisted adjoints one of each other are are each other's inverses.

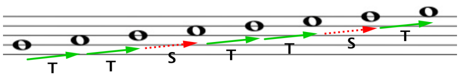
We define the Sturmian involution as the anti-automorphism $*$ over the monoid of special Sturmian morphisms $St_0 \to St_0$, which fixes G and $\tilde{G}$ and exchanges D and $\tilde{D}$. Sturmian involution extends Christoffel duality to the monoid of Sturmian morphisms (see [12, Proposition 4.1]). In a parallel way, the twisted adjoint extends the involution of special Sturmian morphisms to the set of WF words:



(a) The word $w = 0010010$ and its associated twisted affinity $g_w(x) = 5x + 5$.



(b) The twisted dual $w^{\boxtimes} = 1010110$.



(c) The Mixolydian mode with step pattern $w = 0010010$.

(d) The cycle of fiths associated with the Mixolydian mode and its step pattern $w^{\boxtimes} = 1010110$.

Figure 5.5: Twisted adjoints has a similar geometrical interpretation as plain adjoint.

Proposition 43. *The following diagram is commutative:*

$$\begin{tikzcd} St_0 \arrow[r, hook] \arrow[d, "*"'] \arrow[dr, phantom, "\circlearrowleft"] & WF \arrow[d, "\boxtimes"] \\ St_0 \arrow[r, hook] & WF \end{tikzcd}$$

Lemma 15. *Given a WF word w and let $f_1, f_2, \ldots, f_{N-1}$ be the set of special Sturmian morphisms related to w. Let $w_i = f_i(01)$ for $i = 1, \ldots, N-1$. Then, the twisted affinity g_{w_i} associated with the special Sturmian morphism f_i is given by $g_{w_i}(k) = |w|_0 \cdot k + |w|_0 \cdot i$.*

Proof. It is enough to show that $g_{w_1}(k) = |w|_0 \cdot k + |w|_0$. From [12, Lemma 4.2], we have that $f_{|w|_1^{-1}}(01) = u$ is a Christoffel word, and thus its twisted affinity is $g_u(k) = |w|_0 \cdot k + N - 1$. Following Lemma 12, the mode of the word γu is $\nu_{\gamma u} = \nu_u + |u|_1$. It follows that

$$g_{\gamma u}(k) = g_u(k) - |u|_1 = g_u(k) + |u|_0.$$

The twisted affinity associated with $w_1 = \gamma^{|w|_0^{-1}+1}(u)$ is thus given by

$$g_{w_1}(k) = |w|_0 \cdot k + N - 1 + (|w|_0^{-1} + 1) \cdot |w|_0 = |w|_0 \cdot k + N + |w|_0 = |w|_0 \cdot k + |w|_0.$$

□

Proof of Proposition 43. Notice that $f_w^{\boxtimes}$ is an homothecy $\Leftrightarrow \nu_w = N - 1$. This happens precisely for the word $w = f_N(01)$, which is the non morphic word.

Then the twisted adjoint sends amorphic words to amorphic words and therefore it sends morphic words to morphic words.

Following [12, Corollary 4.1]: if $f_1, f_2, \ldots, f_{N-1}$ (resp. $f'_1, f'_2, \ldots, f'_{N-1}$) is the succession of special Sturmian morphisms related to a Christoffel word w (resp. to $w^{\boxtimes}$) then one has $f_i^* = f'_{i|w|_1}$. It is enough to show that the twisted affinities associated to f_i and $f'_{i\cdot|w|_1}$ are inverses one of each other. By the previous lemma one has:

$$g_{f_i^*}(k) = g_{f'_{i\cdot|w|_1}}(k) = |w|_0^{-1}\cdot k + |w|_0^{-1}\cdot|w|_1\cdot i = |w|_0^{-1}\cdot k + |w|_0^{-1}\cdot(N-|w|_0)\cdot i =$$
$$= |w|_0^{-1}\cdot k - i = (|w|_0\cdot k + |w|_0\cdot i)^{-1} = g_{f_i}^{-1}(k),$$

which completes the proof. $\square$

We conclude the section with a result that analyzes the concatenation of plain and twisted adjoints. Let WF_N denote the set of WF words w of length N in $\{0,1\}^*$ and let $Aff^*(\mathbb{Z}_N)$ denote the group of affine automorphisms of the cyclic group $\mathbb{Z}_N$ of order N. Let furthermore $j_{\square}, j_{\boxtimes} : \mathbb{Z}_N \to \mathbb{Z}_N$ denote the affine automorphisms given by the formulas $j_{\square}(k) = -k+1 \bmod N$ and $j_{\boxtimes}(k) = -k-1 \bmod N$, respectively.

Proposition 44. *The concatenation $\square\circ\boxtimes$ is an involution over the set WF_N. Furthermore, we have:*

$$\square\circ\boxtimes = \boxtimes\circ\square : WF_N \to WF_N,$$

and the application induced over $Aff*(\mathbb{Z}_N)$ is an inner automorphism:

$$f_{w^{\boxtimes\square}} = j_{\square}\circ f_w \circ j_{\square}^{-1} \quad \textit{and} \quad g_{w^{\square\boxtimes}} = j_{\boxtimes}\circ g_w\circ j_{\boxtimes}^{-1}.$$

Proof. Notice that $\Delta(f_{w^{\boxtimes\square}}) = f_{w^{\boxtimes\square}}(k+1) - f_{w^{\boxtimes\square}}(k) = \Delta(j_{\boxtimes}\circ g_w\circ j_{\boxtimes}^{-1}) = |w|_1$ and therefore one has just to check that $j_{\boxtimes}\circ g_w\circ j_{\boxtimes}^{-1}(0) = f_{w^{\boxtimes\square}}(0)$ to conclude that the affinities $f_{w^{\boxtimes\square}}$ and $j_{\boxtimes}\circ g_w\circ j_{\boxtimes}^{-1}$ coincide. An analogous argument lieds to $g_{w^{\square\boxtimes}} = j_{\boxtimes}\circ g_w\circ j_{\boxtimes}^{-1}$. $\square$

5.2.1 Divider Incidence

Here we describe the Ionian Theorem that was presented in [43] for the plain adjoint and extended to the twisted adjoint in [23]. Plain adjoint verifies a very singular property that will lead to state the Ionian Theorem in the plain adjoint case. Recall that if a word w is the step pattern of a generated scale $\Gamma(\theta, N)$, the word $w^{\square}$ encodes the zig-zag that correspond with the scale when we cycle through the notes of the scale in generation order (fifth up-forth down in case that $\theta = \log_2(\frac{3}{2})$). Figure 5.6 shows the modes $\Gamma_k(\log_2(\frac{3}{2}, 7))$ for $k = 0, 1, \ldots, 6$, the fifth-forth foldings together with their step patterns w_k and $w_k^{\square}$. As one can see, the note that determines the factorization of w and the one that produces the factorization of $w^{\square}$ coincide in just one case: the Ionian mode. Moreover, the step pattern of the Ionian mode is the only word that is determined by a standard morphism (see Table 5.3 on page 93).

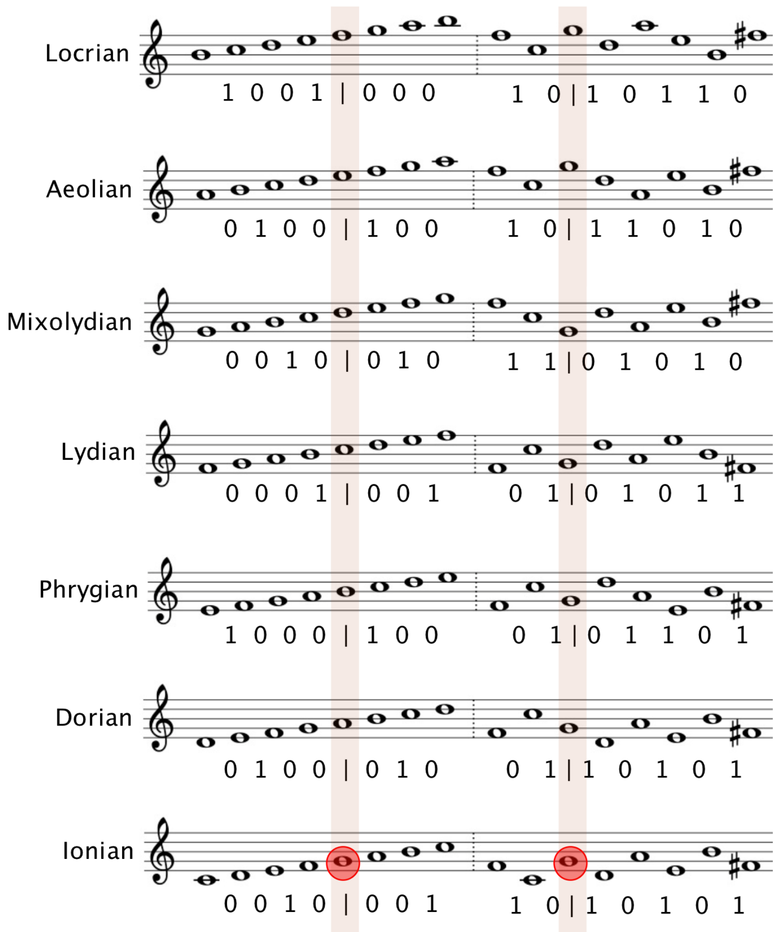


Figure 5.6: Musical motivation for the Ionian Theorem.

Theorem 17 (Ionian Theorem)**.** *Given a WF word w of length N, which is the step-pattern of a generated mode $\Sigma_r = \{\{-k\theta\}, \dots, \{(N-1-r)\theta\}\}$, then the following assertions are equivalent:*

1. *$w = \Phi(01)$ with $\Phi \in \langle G, D\rangle$ a special standard morphism.*
2. *The associated plain affinity f_w verifies $f_w(N-1) = N-1$.*
3. *The associated plain affinity f_w verifies $f_w(M^*) = M$, where $M = |w|_1$ and $M^* = M^{-1} \bmod N$.*
4. *The notes $\{-r\theta\}$ and $\{\theta\}$ are consecutive in scale order, that is, the origin of the scale zig-zag folding is the divider predecessor in scale order.*

Proof. The equivalence between (2) and (3) is clear, since

$$f_w(N-1) = M \cdot (N-1) + \nu \bmod N = N-1 \Longleftrightarrow \nu = M-1$$

and

$$f_w(M^*) = M \cdot M^* + \nu \bmod N = M \Longleftrightarrow \nu = M-1.$$

The equivalence between (2), (3) and (4) is also easy. Notice that the values $\{\alpha_w(k)\theta\}$ are the notes in scale order. Hence, the minimum of $\alpha_w(k)$ is r. But this minimum is $M-1$. It follows that $1-M$ is reached at

$$f_w(k) = 0 \Leftrightarrow k = M^* - 1$$

and the next value of α_w is $\alpha_w(M^*) = 1 - M + M = 1$.

To show the equivalence between (2), (3), (4) and (1) we need to prove that the mode of w is $M-1$ if and only if $\Phi(01)$ with $\Phi \in < G, D >$ or, equivalently, that

$$\min\{\alpha_w(k),\ \text{with } k = 0, 1, \dots, N-1\} = 1 - |w|_1 = 1 - M. \tag{5.1}$$

Firstly, notice that if $w = 01$, then $\min\{\alpha_w(k)\} = 1 - |01|_1 = 0$ and the assertion holds. We now have to prove that if equality 5.1 holds for the word w, then it also holds for the words $G(w)$ and $D(w)$. Let us denote with W_k (resp. with W'_k) the prefix of W of length $k+1$ (resp. the suffix of length $|W| - k - 1$ of W) for $W \in \{0,1\}^*$ and $k \in \{0, 1, \dots, |W|-1\}$. If $k \in \{0, 1, \dots, |G(w)|-1\}$ is such that $G(w)_k = G(w_{k'})$ for a certain $k' \in \{0, 1, \dots, |w|\}$, then, following Proposition 39:

$$\begin{aligned}\alpha_{G(w)}(k) = \det \begin{pmatrix} |G(w)_k|_0 & |G(w)'_k|_0 \\ |G(w)_k|_1 & |G(w)'_k|_1 \end{pmatrix} = \det \begin{pmatrix} |G(w_{k'})|_0 & |G(w'_{k'})|_0 \\ |G(w_{k'})|_1 & |G(w'_{k'})|_1 \end{pmatrix} = \\ = \det \begin{pmatrix} |w_{k'}| & |w'_{k'}| \\ |w_{k'}|_1 & |w'_{k'}|_1 \end{pmatrix} = \det \begin{pmatrix} |w_{k'}|_0 + |w_{k'}|_1 & |w'_{k'}|_0 + |w'_{k'}|_1 \\ |w_{k'}|_1 & |w'_{k'}|_1 \end{pmatrix} = \\ = \alpha_w(k').\end{aligned}$$

Since $G(0,1) = (0, 01)$, the condition $G(w)_k = G(w_{k'})$ does not hold whenever $G(w)[k+1] = 1$. Let us compute $\alpha_{G(w)}(k)$. $\alpha_{G(w)}(0) = 0$ by definition, thus, we can assume that $k \geq 1$. Notice that, in this case, there is a $k' \in \{0, 1, \dots, |w|-1\}$ such that

$$G(w)_{k+1} = G(w)_{k-1}01 = G(w'_k)01$$

Table 5.5: Balance map, accumulation map and plain affinity of the Ionian Mode $w = 0010|001$ and its folding pattern $w^{\square} = 10|10101$.

$w[k-1]$		0	0	1	0	0	0	1
$\beta_w(k)$		2	2	-5	2	2	2	-5
$\sigma_w(k)$	0	2	4	-1	1	3	5	0
$f_w(k)$	1	3	5	0	2	4	6	1
k	0	1	2	3	4	5	6	7

$w^{\square}[k-1]$		1	0	1	0	1	0	1
$\beta_{w^{\square}}(k)$		-3	4	-3	4	-3	4	-3
$\sigma_{w^{\square}}(k)$	0	-3	1	-2	2	-1	3	0
$f_{w^{\square}}(k)$	3	0	4	1	5	2	6	3
k	0	1	2	3	4	5	6	7

and therefor

$$\alpha_{G(w)}(k) = \alpha_{G(w)}(k-1)+\beta_{G(w)}(k) = \alpha_w(k')+|G(w)|_1 = \alpha_w(k')+|w|_1 > \alpha_w(k').$$

We conclude that

$$\begin{aligned}\min_{k=0,1,\ldots,|G(w)|}\{\alpha_{G(w)}(k)\} &= \min_{k=0,1,\ldots,|w|}\{\alpha_w(k)\} = \\ &= 1-|w|_1 = 1-|G(w)|_1.\end{aligned}$$

In a similar way, one can prove that if $D(w)_k = D(w_{k'})$ holds for certain subscripts $k \in \{0,1,\ldots,|D(w)|\}$ and $k' \in \{0,1,\ldots,|w|\}$, then $\alpha_{D(w)}(k) = \alpha_w(k')$. Now, since $D(0,1) = (10,1)$, then $D(w)_k = D(w_{k'})$ holds if and only if $D(w)[k+1] = 0$ and we have to compute $\alpha_{D(w)}(k)$. Again, $\alpha_{D(w)}(0) = 0$ and we can take $k \geq 1$. If we write

$$D(w)_{k+1} = D(w)_k 10 = D(w_{k'})10 = D(w_{k'}0)$$

then

$$\alpha_{D(w)}(k) = \alpha_{D(w)}(k-1)+\beta_{D(w)}(k) = \alpha_w(k') - |D(w)|_0 = \alpha_w(k') - |w|_0.$$

Therefor it holds that

$$\begin{aligned}\min_{k=0,1,\ldots,|D(w)|}\{\alpha_{D(w)}(k)\} &= \min_{k=0,1,\ldots,|w|}\{\alpha_w(k)\} - |w|_0 = \\ &= 1-|w|_1-|w|_0 = 1-|w| = 1-|D(w)|_1.\end{aligned}$$

□

Table 5.5 shows the values of the accumulation map and the plain affinities of the word $w = 0010|001$ and its plain adjoint $w^{\square} = 10|10101$. Notice that $f_w(k) = 2k+1$, $f_{w^{\square}}(k) = 4k+3$ and thus $f_w(4) = 2$ and the conditions of the theorem are satisfied.

An analogue version of Ionian Theorem for twisted adjoint can be stated:

Theorem 18 (Divider Incidence for twisted adjoint). *Given a WF word w of length N, which is the step-pattern of a generated mode $\Sigma_k = \{\sigma_{-k},\ldots,\sigma_{N-1-k}\}$, then the following assertions are equivalent:*

1. *$w = \Phi(01)$ with $\Phi \in \left\langle \tilde{G}, \tilde{D} \right\rangle \cdot G \cdot \left\langle \tilde{G}, D \right\rangle$.*

2. *The associated twisted affinity g_w verifies $g_w(1) = 1$.*

3. *The associated twisted affinity g_w verifies $g_w(M^*) = M$, where $M = |w|_1$ and $M^* = M^{-1} \bmod N$.*

Proof. As in the proof of Divider Incidence for the twisted adjoint, conditions (2) and (3) are trivially equivalent. If a word w verifies any one of them, then its mode is $\nu = |w|_0 + N - 2$, as one can easily check. Now we prove that both conditions are equivalent with (1). Notice first that the equations

$$\tilde{G}\tilde{D}^k G = G D^k \tilde{G} \quad \text{for all } k = 0, 1, \ldots$$

imply the equality of monoids

$$\left\langle \tilde{G}, \tilde{D} \right\rangle \cdot G \cdot \left\langle \tilde{G}, D \right\rangle = \left\langle \tilde{D} \right\rangle \cdot G \cdot \left\langle \tilde{G}, D \right\rangle$$

Hence, we need to compute the mode of any word $F(01)$ with

$$F \in \left\langle \tilde{D} \right\rangle \cdot G \cdot \left\langle \tilde{G}, D \right\rangle.$$

To accomplish that, first we analyze the monoid $G \cdot \left\langle \tilde{G}, D \right\rangle$. We need to prove that the minimum of $\{\alpha_{G\Phi(01)}(k)\}$ with $\Phi \in \left\langle \tilde{G}, D \right\rangle$ is $2 - |G\Phi(01)|_0 = 2 - |\Phi(01)|$.

We use a similar reasoning as in the proof of the Ionian Theorem and proceed by recursion. Notice first that $2 - |01| = 0$ and the mode of the word 01 is 0. Let $w = \Phi(0, 1)$ with $\Phi \in \left\langle \tilde{G}, D \right\rangle$ be a word such that

$$\min_{k=0,\ldots,|w|-1} \alpha_w(k) = 2 - |w|.$$

Let us first prove that $\min \alpha_{\tilde{G}(w)}(k) = 2 - |\tilde{G}(w)|$. Again, W_k will denote the prefix of length k of a word W. If k is such that $\tilde{G}(w)_k = \tilde{G}(w_{k'})$ for a certain k', as we proved before, $\alpha_{G(w)}(k) = \alpha_w(k')$. It follows that $\alpha_{\tilde{G}(w)}(k) = \alpha_w(k')$ for all k such that $\tilde{G}(w)[k] = 0$. Let $s \in \{0, 1, \ldots, |w|-1\}$ such that $\alpha_w(s) = 2 - |w|$. Since 11 is not a factor of $\tilde{G}(w)$, the minimum of $\alpha_{\tilde{G}(w)}(k)$ is reached at the subscript $|\tilde{G}(w_s)| - 1 = s'$. Notice that $\tilde{G}(w)[s'] = 1$ and $\tilde{G}(w)[s' + 1] = 0$ and hence one can compute now the minimum:

$$\begin{aligned}\alpha_{\tilde{G}(w)}(s') &= \alpha_{\tilde{G}(w)}(s'+1) - \beta_{\tilde{G}(w)}(s'+1) = \alpha_{\tilde{G}(w)}(|\tilde{G}(w_s)|) - |\tilde{G}(w)|_1 = \\ &= \alpha_w(s) - |w|_1 = 2 - |w| - |w|_1 = 2 - |\tilde{G}(w)|.\end{aligned}$$

Now we have to prove that $\min \alpha_{D(w)} = 2 - |D(w)|$. As it was stated in the proof of Ionian Theorem,

$$\min_{k=0,1,\ldots,|D(w)|} \{\alpha_{D(w)}(k)\} = \min_{k=0,1,\ldots,|w|} \{\alpha_w(k)\} - |w|_0 = 2 - |w| - |w|_0 = 2 - |D(w)|.$$

Finally, we just have to prove that

$$\min \alpha_{\tilde{D}(w)}(k) = 2 - |\tilde{D}(u)|_0 \text{ whenever } \min \alpha_w(k) = 2 - |w|_0.$$

As above, $\alpha_{\tilde{D}(w)}(k) = \alpha_w(k')$ if $\tilde{D}(w)_k = \tilde{D}(w_{k'})$ for a certain k'. We have to compute $\alpha_{\tilde{D}(w)}(k)$ for the rest of the cases, namely those k such that $\tilde{D}(w)[k] = 0$ holds. Notice that

$$\tilde{D}(w)_{k+1} = \tilde{D}(w)_{k-1}01 = \tilde{D}(w_{k'}0),$$

thus

$$\alpha_{\tilde{D}(w)}(k) = \alpha_w(k') + \beta_{\tilde{D}(w)}(k) = \alpha_w(k') + |\tilde{D}(w)|_1$$

and finally

$$\min \alpha_{\tilde{D}(w)}(k) = \min \alpha_w(k) = 2 - |w|_0 = 2 - |\tilde{D}(w)|_0.$$

$\square$

Chapter 6

An Extension of Christoffel-Duality to a Subset of Sturm Numbers and their Characteristic Words

This chapter investigates an extension of Christoffel duality to a certain family of sturmian words and their (irrational) slopes. Given a characteristic word c_θ of slope θ, and a standard prefix w of length N of c_θ, we associate a N-*companion slope* θ_N^* such that the characteristic word $c_{\theta_N^*}$ has a prefix w^* of length N which is the corresponding Christoffel dual of w. Although this condition is satisfied by infinitely many slopes, we show that the left-companion slope θ_N^* and the correlated right-companion slope $\overline{\theta_N^*}$ are interesting and somewhat natural choices, as we provide geometrical and music-theoretical motivations for their definition.

6.1 The $N-$companion slopes

We define first the *left-companion* function $\mathfrak{L}_{\frac{M}{N}}$ for every $\frac{M}{N} \in \mathfrak{F}_N$, in such a way that $\frac{M^*}{N} = \frac{M^{-1} \mod N}{N}$ is a semiconvergent of $\mathfrak{L}_{\frac{M}{N}}(g)$ for every $g \in [0,1)$. Simultaneously, the right-companion function is defined as $\mathfrak{R}_{\frac{M}{N}} = 1 - \mathfrak{L}_{\frac{M}{N}}$.

Definition. *For every irreducible fraction $\frac{M}{N} \in \mathfrak{F}$, we define a couple of functions,* ***left companion*** *and the* ***right companion*** *functions, as follows:*

$$\begin{aligned} \mathfrak{L}_{\frac{M}{N}} : [0,1) &\longrightarrow [0,1) \\ h &\longmapsto \mathfrak{L}_{\frac{M}{N}}(h) = \frac{M_1 h + N_1}{Mh + N} \end{aligned}$$

and

$$\begin{aligned}\mathfrak{R}_{\frac{M}{N}} : [0,1) &\longrightarrow [0,1) \\ h &\longmapsto \mathfrak{R}_{\frac{M}{N}}(h) = \frac{M_2 h + N_2}{Mh + N}\end{aligned}$$

where $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ *are three consecutive* $N-$*Farey numbers.*
When $\frac{M}{N}$ *is a semi-convergent of* g, *we write*

$$g^*_N := \mathfrak{L}_{\frac{M}{N}}(g) \quad \text{and} \quad \overline{g^*}_N := \mathfrak{R}_{\frac{M}{N}}(g)$$

and call them, respectivelly, the N-***left*** *and* N-***right-companions*** *of* g.

Remark 14. *Since* $M = M_1 + M_2$ *and* $N = N_1 + N_2$, *it holds that*

$$\mathfrak{L}_{\frac{M}{N}}(h) = 1 - \mathfrak{R}_{\frac{M}{N}}(h).$$

If $\alpha = \left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ is the matrix of $SL_2(\mathbb{N})$ associated with $\frac{M}{N}$ (see Proposition 7), then it is immediate to check that

$$(\alpha \cdot L \cdot J)^t = \begin{pmatrix} M_1 & N_1 \\ M & N \end{pmatrix} \Rightarrow \mathfrak{L}_{\frac{M}{N}}(h) = \mu[(\alpha \cdot L \cdot J)^t](h)$$

and

$$(\alpha \cdot R)^t = \begin{pmatrix} M_2 & N_2 \\ M & N \end{pmatrix} \Rightarrow \mathfrak{R}_{\frac{M}{N}}(h) = \mu[(\alpha \cdot R)^t](h),$$

where L, R and J are the matrices of $SL(\mathbb{N}, 2)$ that where defined in Section 2.1.

Proposition 45. *Let* $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ *be three consecutive* N-*Farey fractions, let* $\frac{M^*}{N} = \frac{M^{-1} \mod N}{N}$ *and* $\frac{M_1^*}{N_1^*} < \frac{M^*}{N} < \frac{M_2^*}{N_2^*}$ *the corresponding* $N-$*Farey consecutive numbers. It holds that*

1. $\mathfrak{L}_{\frac{M}{N}}(x) \in \left(\frac{M_1^*}{N_1^*}, \frac{M^*}{N}\right) \forall x \in (0,1)$.

2. $\mathfrak{R}_{\frac{M}{N}}(x) \in \left(1 - \frac{M^*}{N}, 1 - \frac{M_1^*}{N_1^*}\right) \forall x \in (0,1)$.

Proof. Recall that the matrices associated with $\frac{M}{N}$, $\frac{M^*}{N}$ and $1 - \frac{M^*}{N}$ are, respectivelly

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} M_2 & M_1 \\ N_2 & N_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} d & b \\ c+d & b+a \end{pmatrix} = \begin{pmatrix} M_2^* & M_1^* \\ N_2^* & N_1^* \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} c & a \\ c+d & b+a \end{pmatrix} = \begin{pmatrix} N_2^* - M_2^* & N_1^* - M_1^* \\ N_2^* & N_1^* \end{pmatrix}$$

As one can easyly check,

$$\frac{\partial}{\partial x}\mathfrak{L}_{\frac{M}{N}}(x)=\frac{-1}{(Mx+N)^2}$$

and thus, $\mathfrak{L}_{\frac{M}{N}}(x)$ is decreasing and $\mathfrak{R}_{\frac{M}{N}}(x)$ is increasing. Finally

$$\begin{aligned}
\mathfrak{L}_{\frac{M}{N}}(0) &= \frac{N_1}{N}=\frac{b+d}{N}=\frac{M^*}{N}\\
\mathfrak{L}_{\frac{M}{N}}(1) &= \frac{M_1+N_1}{M+N}=\frac{b+b+d}{a+b+N}=\frac{M^*+M_1^*}{N+N_1^*}>\frac{M_1^*}{N_1^*}\\
\mathfrak{R}_{\frac{M}{N}}(0) &= \frac{a+c}{N}=\frac{N-M^*}{N}=1-\frac{M^*}{N}\\
\mathfrak{R}_{\frac{M}{N}}(1) &= \frac{a+a+c}{N+a+b}=\frac{N-M^*+N_1^*-M_1^*}{N+N_1^*}<1-\frac{M_1^*}{N_1^*}
\end{aligned}$$

□

Now we relate the values of $\mathfrak{L}_{\frac{M}{N}}(h)$ and $\mathfrak{R}_{\frac{M}{N}}(h)$ with the continued fraction expansion of h and $\frac{M}{N}$. For that purpose we need the following lemma, which is an immediate consequence of the properties of the Möbius transform that were seen in section 2.1:

Lemma 16. *Let P and Q be the matrices*

$$\begin{aligned}
P &= L^{a_1}R^{a_2}\cdots R^{a_{2k}}L^{a_{2k+1}}J\\
Q &= L^{a_1}R^{a_2}\cdots R^{a_{2k}} \quad \textit{with } a_i>0 \quad \forall i=1,\dots 2k+1,
\end{aligned}$$

and let $g=[0;b_1,b_2,\dots]$. Then one finds

$$\begin{aligned}
\mu[P](g) &= [0;a_1,a_2,\dots,a_{2k+1}+b_1,b_2,\dots]\\
\mu[Q](g) &= [0;a_1,a_2,\dots,a_{2k},b_1,b_2,\dots].
\end{aligned}$$

Proposition 46. *If $\frac{M}{N}=[0;a_1,\dots a_k+1]$ and $h=[0;b_1,b_2,\dots]$ with $a_k\geq 1$ and $b_i\geq 1$, it holds:*

$$\mathfrak{L}_{\frac{M}{N}}(h)=\begin{cases}[0;a_k+1,a_{k-1},\dots,a_1,b_1,b_2,\dots] \textit{ if } k \textit{ is odd}\\ [0;1,a_k,a_{k-1},\dots,a_1,b_1,b_2,\dots] \textit{ if } k \textit{ is even,}\end{cases}$$

and

$$\mathfrak{R}_{\frac{M}{N}}(h)=\begin{cases}[0;a_k+1,a_{k-1},\dots,a_1,b_1,b_2,\dots] \textit{ if } k \textit{ is even}\\ [0;1,a_k,a_{k-1},\dots,a_1,b_1,b_2,\dots] \textit{ if } k \textit{ is odd.}\end{cases}$$

Proof. Let α be the matrix of $SL_2(\mathbb{N})$ associated with $\frac{M}{N}$. Then $\mathfrak{L}_{\frac{M}{N}}(h)=\mu[(\alpha\cdot L\cdot J)^t](h)$. If k is odd,

$$(\alpha\cdot L\cdot J)^t=\left(L^{a_1}R^{a_2}\cdots L^{a_k+1}J\right)^t=JR^{a_k+1}\cdots L^{a_2}R^{a_1}=L^{a_k+1}\cdots R^{a_2}L^{a_1}J.$$

If k is even,

$$(\alpha\cdot L\cdot J)^t=(L^{a_1}R^{a_2}\cdots R^{a_k}LJ)^t=JRL^{a_k}\cdots L^{a_2}R^{a_1}=LR^{a_k}\cdots R^{a_2}L^{a_1}J,$$

and we close the proof on the basis of lemma 16. The computation of $\mathfrak{R}_{\frac{M}{N}}(h)$ is similar. □

As a consequence of the previous description of $\mathfrak{L}_{\frac{M}{N}}$ and $\mathfrak{R}_{\frac{M}{N}}$, it holds:

Proposition 47. *For every $\frac{M}{N} \in \mathfrak{F}$ and every $g \in (0,1]$, it holds that $\frac{M^*}{N}$ is a semiconvergent of $\mathfrak{L}_{\frac{M}{N}}(g)$ and $1 - \frac{M^*}{N}$ is a semiconvergent of $\mathfrak{R}_{\frac{M}{N}}(g)$.*

Proof. It is an immediate consequence of Proposition 45. We give, nevertheless, an alternative proof attending to the continued fraction expansion of $\mathfrak{L}_{\frac{M}{N}}(g)$ and $\mathfrak{R}_{\frac{M}{N}}(g)$. Let $\frac{M}{N} = [0, a_1, \ldots, a_k + 1]$. As it is well known, it holds:

$$1 - \frac{M}{N} = \begin{cases} [0; 1, a_1 - 1, a_2, \ldots, a_k + 1] \text{ if } a_1 > 1 \\ [0; 1 + a_2, \ldots, a_k + 1] \text{ if } a_1 = 1 \end{cases}$$

and

$$\frac{M^*}{N} = \begin{cases} [0; 1, a_k, a_{k-1}, \ldots, a_1] \text{ if } k \text{ is even} \\ [0; 1 + a_k, a_{k-1}, \ldots, a_1] \text{ if } k \text{ is odd.} \end{cases}$$

(see Corollary 8).

It follows that

$$1 - \frac{M^*}{N} = \begin{cases} [0; 1 + a_k, \ldots, a_2, a_1] \text{ if } k \text{ is even} \\ [0; 1, a_k, \ldots, a_2, a_1] \text{ if } k \text{ is odd} \end{cases}$$

and one may conclude, based on Proposition 46. □

As a consequence, the prefixes of length N of the characteristic words of slopes g and $\mathfrak{L}_{\frac{M}{N}}(g)$ are standard dual words, whenever $\frac{M}{N}$ is a semiconvergent of g.

Example 16. *The fractions $\frac{1}{2} < \frac{4}{7} < \frac{3}{5}$ are three consecutive 7-Farey numbers. The 7-companions of $\frac{4}{7} = [0; 1, 1, 3]$ are*

$$\mathfrak{L}_{\frac{4}{7}}\left(\frac{4}{7}\right) = [0; 3, 1, 1, 1, 1, 3] = \frac{18}{65} \quad \mathfrak{R}_{\frac{4}{7}}\left(\frac{4}{7}\right) = [0; 1, 2, 1, 1, 1, 1, 3] = \frac{47}{65}.$$

These values can be computed directly from the definition of $\mathfrak{L}$ and $\mathfrak{R}$:

$$\mathfrak{L}_{\frac{4}{7}}(h) = \frac{h+2}{4h+7} \Longrightarrow \mathfrak{L}_{\frac{4}{7}}\left(\frac{4}{7}\right) = \frac{\frac{4}{7}+2}{4\frac{4}{7}+7} = \frac{18}{65}.$$
$$\mathfrak{R}_{\frac{4}{7}}(h) = \frac{3h+5}{4h+7} \Longrightarrow \mathfrak{R}_{\frac{4}{7}}\left(\frac{4}{7}\right) = \frac{3\frac{4}{7}+5}{4\frac{4}{7}+7} = \frac{47}{65}.$$

Notice that $\frac{2}{7}$ is a semiconvergent of $\frac{18}{65}$ and

$$1 - \frac{2}{7} = \frac{5}{7} = [0; 1, 2, 2]$$

is a semiconvergent of $\frac{47}{65}$.

6.2 A geometric description

In this section the geometrical argumentation that motivate the definitions of $\mathfrak{L}_{\frac{M}{N}}$ and $\mathfrak{R}_{\frac{M}{N}}$ is gathered. Let us consider the line r satisfying the equation $y = gx$ and let

$$S(N, g) = \{P_k = (k, \lfloor k \cdot g \rfloor)\}_{k=0,1,\ldots,N-1}$$

be the first N points of the corresponding lower mechanical sequence. In this Subsection we will see how the well-formed sets $\Gamma(g, N)$ and $\Gamma(g_N^*, N)$ can be generated from the prefix of length N of the mechanical word s_g by means of a pair of projections with direction parallel and perpendicular to r. For that purpose, the well-formed sets $\Gamma(g, N)$ will be displayed as sets of points within certain intervals $[a, b)$, rather than the unit circle, as they were defined in Chapter 2.

Let s be a second line through the origin $O = (0, 0)$, different from $y = gx$ and from the perpendicular $y = -\frac{1}{g}x$, and let finally Π_s and Π'_s be the parallel and perpendicular projections with respect to the line $y = gx$ onto the line s:

$$\begin{array}{rccc} \Pi_s : & \mathbb{R}^2 & \longrightarrow & s \\ & P & \longmapsto & r_P \cap s \end{array} \qquad \begin{array}{rccc} \Pi'_s : & \mathbb{R}^2 & \longrightarrow & s \\ & P & \longmapsto & r'_P \cap s \end{array},$$

where r_P (resp. r'_P) denotes the line parallel to r (resp. perpendicular to r) through a given point P.

Now, let us study the behavior of the set of points $S(N, g)$ when we project it onto s by means of Π_s and Π'_s. For that purpose N will be the denominator of a semi-convergent $\frac{M}{N}$ of g, such that $\Gamma(g, N)$ is a well-formed set and the prefix of length N of s_g is a lower Christoffel word.

Proposition 48. *With the previous notation the following assertions are true:*

1. *$\Pi_s S(g, N)$ and the well-formed set $\Gamma(g, N)$ are homothetic.*

2. *$\Pi'_s S(g, N)$ and the well-formed set $\Gamma(g_N^*, N)$ are homothetic.*

Proof. 1. The mechanical sequence of slope g is in the region of the plane delimited by the lines $y = gx$ and $y = gx - 1$. Therefore the projection of every point of that sequence is in the segment of r determined by $\Pi_s(O)$ and $\Pi_s(Q)$. Moreover, one has by Thales that

$$\frac{\overline{O\Pi_s(P_1)}}{\overline{O\Pi_s(Q)}} = g \qquad \text{See figure 6.1}$$

Notice that $\Pi_s S(g, N)$ is congruent with the projection of $S(g, N)$ onto the ordinate axis. This late set coincides with the set

$$-\Gamma(g, N) = \{-\{k * g\}, k = 0, 1, \ldots, N\} \qquad \text{See figure 6.1}$$

It follows that the set $\Pi_s S(g, N)$ and the set $\Gamma(N, g)$ are congruent.

2. Now, let us consider the projection $\Pi'_s S(g, N)$ onto the segment $\overline{O\Pi'_s(N, M)}$ of r. Notice how Π'_s preserves the pattern of $S(g, N)$, so one has that the step pattern of $\Pi'_s S(g, N)$ coincides with the Christoffel word of slope $\frac{M^*}{N}$.

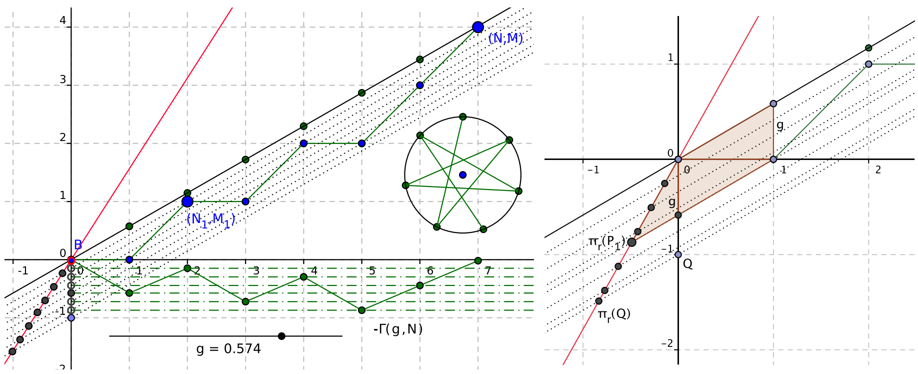


Figure 6.1: Parallel projection of a mechanical sequence

If we project by means of Π_s the first step of $S(g, N)$, that is $\overline{OP_1}$, onto s we get the *generator* $\overline{O\Pi_s P_1} = g$. Therefore, if we want to compute the new generator, we need to project by means of Π'_s the new *first step*, which coincides with the segment that joins the origin O with the point closest to the line $y = gx$. From [10] Lemma 1.3, we know that this point is the one which divides the Christoffel word of slope $\frac{M}{N}$ into a Christoffel pair (w_1, w_2) and its coordinates are $(|w_1|_0, |w_1|_1) = (N_1, M_1) = P_{N_1}$.

Let r_{N_1} and r_N be the lines of slope $\frac{-1}{g}$ passing through P_{N_1} and P_N respectively. Their equations are given by:

$$r_{N_1} \equiv y = \frac{-1}{g}x + M_1 + \frac{N_1}{g} \qquad r_N \equiv y = \frac{-1}{g}x + M + \frac{N}{g}$$

We may compute *the new generator* with the projections of P_{N_2} and (N, M) over the horizontal axis, by Thales (see Figure 6.2):

$$g^* = \frac{\overline{P_0\Pi'_s(P_{N_1})}}{\overline{P_0\Pi'_s(P_N)}} = \frac{\overline{OA}}{\overline{OC}} = \frac{M_1 + \frac{N_1}{g}}{M + \frac{N}{g}} = \frac{M_1 g + N_1}{Mg + N} = \mathfrak{L}_{\frac{M}{N}}(g).$$

□

6.3 Duality: mirror numbers

In general, the *pseudo-duality* defined by means of $\mathfrak{L}_{\frac{M}{N}}$ is not a real duality since, in general

$$\mathfrak{L}_{\frac{M^*}{N}}\left(\mathfrak{L}_{\frac{M}{N}}(\theta)\right) \neq \theta.$$

In this section the fixed-points under the maps

$$\mathfrak{L}_{\frac{M*}{N}} \circ \mathfrak{L}_{\frac{M}{N}} \qquad 1 - \mathfrak{R}_{1-\frac{M^*}{N}} \circ \mathfrak{R}_{\frac{M}{N}}$$

are studied. It turns out that these are Sturm numbers (denoted respectively by $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$) and they have a very particular continued fraction expansion: purely periodic with palindromic period. They will be called hence *mirror*

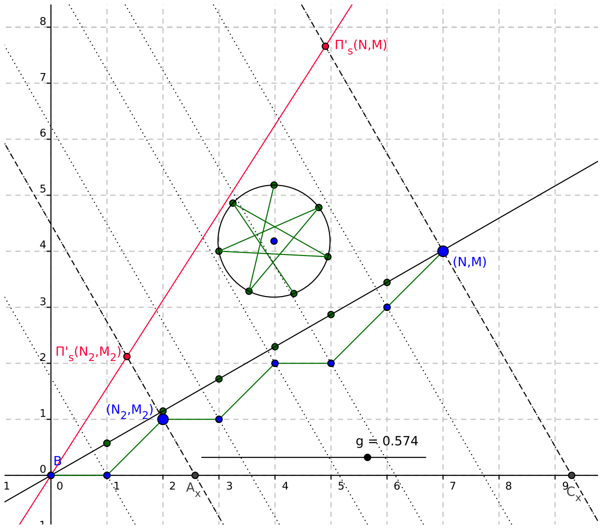


Figure 6.2: Perpendicular projection of a mechanical sequence

numbers. We introduce explicit formulas for computing the mirror numbers as quadratic irrational numbers. As it is well known, every characteristic word c_θ, the slope θ of which is a sturm number, is fixed by a non-trivial standard sturmian morphism. A description of the morphism associated to every mirror number in terms of its semi-period closes the section.

Let $\frac{M}{N}$ be a semi-convergent of $g \in (0,1]$. If the N-left and N-right companion functions

$$g \mapsto g_N^* = \mathfrak{L}_{\frac{M}{N}}(g) \qquad g \mapsto \overline{g_N^*} = \mathfrak{R}_{\frac{M}{N}}(g)$$

were proper dualities, they would be *involutive*, that is g would have to coincide with g_N^{**} or $\overline{g_N^{**}}$. In fact, in order to be precise, we first need to give a formal definition of g_N^{**} and $\overline{g_N^{**}}$. For that purpose, we make use of Proposition 47:

Definition. *The functions $\mathfrak{L}^{**}_{\frac{M}{N}} : (0,1) \to (0,1)$ and $\mathfrak{R}^{**}_{\frac{M}{N}} : (0,1) \to (0,1)$ given by :*

$$\mathfrak{L}^{**}_{\frac{M}{N}}(x) = \mathfrak{L}_{\frac{M^*}{N}}\left(\mathfrak{L}_{\frac{M}{N}}(x)\right)$$
$$\mathfrak{R}^{**}_{\frac{M}{N}}(x) = 1 - \mathfrak{R}_{1-\frac{M^*}{N}}\left(\mathfrak{R}_{\frac{M}{N}}(x,)\right)$$

are called, respectivelly, left and right double companion associated with $\frac{M}{N}$.

Definition. *A real number $\mathfrak{a} \in (0,1)$ is called mirror number if its continued fraction expansion is purelly periodic and its period is palindromic. In other words, if it can be written as*

$$\mathfrak{a} = [0; \overline{a_1, a_2, \ldots, a_k, a_k, \ldots, a_2, a_1}]\,, \text{ with } a_i > 0.$$

If k is odd (even), ϕ is called odd (even) mirror number. $(a_1, a_2, \ldots, a_k)$ is called semi-period of $\mathfrak{a}$.

$\frac{M}{N}$	$\alpha_{\frac{M}{N}}$	$\mathfrak{l}_{\frac{M}{N}}$		$\mathfrak{r}_{\frac{M}{N}}$	
$\frac{1}{2}$	$L \cdot Id = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\frac{-1+\sqrt{8}}{2}$	$[0, \overline{22}]$	$\frac{-1+\sqrt{5}}{2}$	$[0, \overline{1111}]$
$\frac{1}{3}$	$L \cdot L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\frac{-3+\sqrt{13}}{2}$	$[0, \overline{33}]$	$\frac{-11+\sqrt{221}}{10}$	$[0, \overline{2112}]$
$\frac{1}{4}$	$L \cdot LL = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$	$\frac{-4+\sqrt{20}}{2}$	$[0, \overline{44}]$	$\frac{-23+\sqrt{725}}{14}$	$[0, \overline{3113}]$
$\frac{2}{3}$	$L \cdot R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\frac{-1+\sqrt{5}}{2}$	$[0, \overline{111111}]$	$\frac{-5+\sqrt{221}}{14}$	$[0, \overline{1221}]$
$\frac{3}{4}$	$L \cdot RR = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$	$\frac{-1+\sqrt{10}}{3}$	$[0, \overline{121121}]$	$\frac{-7+\sqrt{725}}{26}$	$[0, \overline{1331}]$
$\frac{2}{5}$	$L \cdot LR = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$	$\frac{-29+\sqrt{1517}}{26}$	$[0, \overline{211112}]$	$\frac{-1+\sqrt{8}}{2}$	$[0, \overline{2222}]$
$\frac{5}{7}$	$L \cdot RL = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$	$\frac{-19+\sqrt{1517}}{34}$	$[0, \overline{112211}]$	$\frac{-1+\sqrt{5}}{2}$	$[0, \overline{11111111}]$

Table 6.1: Mirror numbers.

With every Farey number $\frac{M}{N} = [0; a_1, \ldots, a_k + 1]$ one can associate an odd mirror number $\mathfrak{l}_{\frac{M}{N}}$ and an even mirror number $\mathfrak{r}_{\frac{M}{N}}$ as

$$\mathfrak{l}_{\frac{M}{N}} = \begin{cases} [0; \overline{a_1, \ldots, a_k + 1, 1 + a_k, \ldots, a_1}] & \text{if } k \text{ is odd} \\ [0; \overline{a_1, \ldots, a_k, 1, 1, a_k, \ldots, a_1}] & \text{if } k \text{ is even.} \end{cases}$$

$$\mathfrak{r}_{\frac{M}{N}} = \begin{cases} [0; \overline{a_1, \ldots, a_k + 1, 1 + a_k, \ldots, a_1}] & \text{if } k \text{ is even} \\ [0; \overline{a_1, \ldots, a_k, 1, 1, a_k, \ldots, a_1}] & \text{if } k \text{ is odd.} \end{cases}$$

Remark 15. *Every odd-mirror number is also an even-mirror number. If*

$$\frac{M}{N} = [0, a_1, \ldots, a_{2k+1}] \text{ with } a_{2k+1} \geq 1,$$

then

$$\mathfrak{l}_{\frac{M}{N}} = \mathfrak{r}_{\frac{M'}{N'}} \quad \text{where } \frac{M'}{N'} = [0; a_1, \ldots, a_{2k+1}, a_{2k+1}, \ldots, a_1].$$

Despite this fact, it will be more effective to consider the sets of even and odd-mirror numbers separately.

As the following proposition asserts, the odd (resp. even) mirror numbers are the numbers for which the $N-$ left companion (resp. $N-$right-companion) is involutive:

Proposition 49 (Characterization of mirror numbers)**.** *The set of fixed points of the double left-companion function $\mathfrak{L}^{**}_{\frac{M}{N}}$ (resp. $\mathfrak{R}^{**}_{\frac{M}{N}}$) coincides with the set of left (resp. right)-mirror numbers.*

The proof of this assertion is based on the following lemma:

Lemma 17. *Let α be the matrix associated with the Farey number $\frac{M}{N}$. Then*

1. $\mathfrak{L}_{\frac{M^*}{N}}(g) = \mu\left[\alpha L J\right](g)$.

2. $1 - \mathfrak{R}_{1-\frac{M^*}{N}}(g) = \mu\left[\alpha R\right](g)$.

Proof. Notice that dual Farey numbers have retrograde paths in the left half of the Stern-Brocot tree. Moreover, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix of $SL_2(\mathbb{N})$ and $\tilde{\gamma} = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ (main diagonal flipped), then γ and $\tilde{\gamma}$ can be written as retrogradations words in $\{L, R\}^*$ (see [34]). One can check also that $J\tilde{\gamma}^t J = \gamma$ and finally, if we take γ such that $\alpha = L\gamma$, we have that $L\gamma$ and $L\tilde{\gamma}$ are the matrices associated, respectively with $\frac{M}{N}$ and $\frac{M^*}{N}$. Thus, we have:

$$\mathfrak{L}_{\frac{M^*}{N}}(g) = \mu\left[(L\tilde{\gamma}LJ)^t\right](g) = \mu\left[JR\tilde{\gamma}^t R\right](g) = \mu\left[L\gamma LJ\right](g) = \mu\left[\alpha LJ\right](g).$$

To prove the second assertion, notice first that

$$1 - x = \frac{-x+1}{1} = \mu\left[\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right](x) = \mu[LR^{-1}J](x)$$

and thus

$$\begin{aligned} 1 - \mathfrak{R}_{1-\frac{M^*}{N}}(g) &= \mu[LR^{-1}J]\left(\mathfrak{R}_{1-\frac{M^*}{N}}(g)\right) = \\ &= \mu[LR^{-1}L(LR^{-1}JL\tilde{\gamma}R)^t](g) = \mu[LR^{-1}L\tilde{\gamma}^t RJL^{-1}R](g) \\ &= \mu[LJ\tilde{\gamma}^t JR](g) = \mu[L\gamma R](g) = \mu[\alpha R](g) \end{aligned}$$

□

Now we can prove the characterization of mirror numbers, condensed in Proposition 49:

Proof. Let $\frac{M}{N} = [0; a_1, a_2, \ldots, a_k + 1]$ and $x = [0; b_1, b_2, \ldots]$. Let us suppose that k is even. Then, follwing Lemma 17, one has that:

$$\begin{aligned} \mathfrak{L}^{**}_{\frac{M}{N}}(x) = x &\Leftrightarrow \mu[(\alpha LJ)(\alpha LJ)^t](x) = x \\ \mathfrak{R}^{**}_{\frac{M}{N}}(x) = x &\Leftrightarrow \mu[(\alpha R)(\alpha R)^t](x) = x \end{aligned}$$

We compute $\mu[(\alpha LJ)(\alpha LJ)^t](x)$ and $\mu[(\alpha R)(\alpha R)^t](x)$:

$$\begin{aligned} \mu[(\alpha LJ)(\alpha LJ)^t](x) &= \mu[\alpha LJ]\mu[(\alpha LJ)^t](x) = \mu[\alpha LJ]([0; 1, a_k, \ldots, a_1, b_1, b_2, \ldots]) = \\ &= [0; a_1, \ldots, a_k, 1, 1, a_k, \ldots, a_1, b_1, b_2, \ldots] \\ \mu[(\alpha R)(\alpha R)^t](x) &= \mu[\alpha R]\mu[(\alpha R)^t](x) = \mu[\alpha R]([0; 1 + a_k, \ldots, a_1, b_1, b_2, \ldots]) = \\ &= [0; a_1, \ldots, a_k + 1, 1 + a_k, \ldots, a_1, b_1, b_2, \ldots] \end{aligned}$$

The case of k odd is similar. □

Remark 16. *Following Proposition 45, the maps $\mathfrak{L}^{**}_{\frac{M}{N}}$ and $\mathfrak{R}^{**}_{\frac{M}{N}}$ are contractive (with respect to the euclidean distance), therefor one may interpret the mirror numbers $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ as the limits*

$$\begin{aligned} \lim_{n\to\infty} \left(\mathfrak{L}^{**}_{\frac{M}{N}}\right)^n(g) = \mathfrak{l}_{\frac{M}{N}} \quad \textit{for every } g \in (0,1). \\ \lim_{n\to\infty} \left(\mathfrak{R}^{**}_{\frac{M}{N}}\right)^n(g) = \mathfrak{r}_{\frac{M}{N}} \quad \textit{for every } g \in (0,1). \end{aligned}$$

Remark 17. *Recall that the incidence matrices of dual Christoffel words decompose as retrograde words in $\langle L, R\rangle$ generators. Therefore an odd-mirror number $\mathfrak{l}_{\frac{M}{N}}$ and its N-dual $\left(\mathfrak{l}_{\frac{M}{N}}\right)^*_N = \mathfrak{l}_{\frac{M^*}{N}}$ have retrograde semi-periods. The same fact holds for even mirror numbers, since $\overline{\left(\mathfrak{r}_{\frac{M}{N}}\right)^*_N} = \mathfrak{r}_{\frac{M^*}{N}}$. In this sense, the extension of duality over the set of mirror numbers has a similar behavior as the duality of Christoffel words.*

Corollary 20. *There is no number g that verifies $\mathfrak{L}^{**}_{\frac{M}{N}}(g) = g$ or $\mathfrak{R}^{**}_{\frac{M}{N}}(g) = g$ for every (semi-)convergent $\frac{M}{N}$ of g.*

Proof. Let $g = [0; a_1, a_2, \ldots]$ and let $\frac{1}{2} = [0, 2]$ be the first (semi-)convergent of g. The equation $\mathfrak{R}^{**}_{\frac{1}{2}}(g) = g$ implies that

$$\mathfrak{R}^{**}_{\frac{1}{2}}(g) = [0; 1, 1, 1, 1, a_1, a_2, \ldots] = [a_1, a_2, \ldots]$$

and it follows that $a_1 = a_2 = a_3 = a_4 = 1$. Hence, $[0, 1, 1, 1, 2] = \frac{5}{8}$ is another (semi-)convergent of g. Thus, we have that $\mathfrak{R}^{**}_{\frac{5}{8}}(g) = g$ if and only if

$$\mathfrak{R}^{**}_{\frac{5}{8}}(g) = [0; 1, 1, 1, 2, 2, 1, 1, 1, a_1, a_2, \ldots] = [a_1, a_2, \ldots]$$

and then $a_4 = 2$. This is impossible.

The case of $\mathfrak{L}^{**}_{\frac{M}{N}}(g) = g$ is similar. □

Recall that a *Sturm number* is a quadratic irrational number ξ if $\xi \in (0, 1)$ and its algebraic conjugate $\xi' \notin (0, 1)$.

Proposition 50. *The mirror numbers $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ are Sturm numbers for every $\frac{M}{N} \in \mathfrak{F}$. If $\left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ is the matrix of $SL_2(\mathbb{N})$ associated with $\frac{M}{N}$, then they can be computed as:*

$$\mathfrak{l}_{\frac{M}{N}} = \frac{M_1^2 + M^2 - N_1^2 - N^2 + \sqrt{(M_1^2 + M^2 - N_1^2 - N^2)^2 + 4(N_1 M_1 + NM)^2}}{2(N_1 M_1 + MN)}.$$

$$\mathfrak{r}_{\frac{M}{N}} = \frac{M_2^2 + M^2 - N_2^2 - N^2 + \sqrt{(M_2^2 + M^2 - N_2^2 - N^2)^2 + 4(N_2 M_2 + NM)^2}}{2(N_2 M_2 + MN)}.$$

Proof. We prove the sesult just for the odd mirror numbers, since the even case is similar. Following Lemma 17, $\mathfrak{l}_{\frac{M}{N}}$ is a solution of the equation

$$\mu[(\alpha LJ)(\alpha LJ)^t](g) = g.$$

The matrix $(\alpha LJ)(\alpha LJ)^t$ is symmetric. Thus, previous equation is equivalent to:

$$\mu\left[\begin{pmatrix} A & B \\ B & C \end{pmatrix}\right](g) = g \Leftrightarrow \frac{A \cdot g + B}{B \cdot g + C} = g \Leftrightarrow Bg^2 + (C - A)g - B = 0,$$

where $A = M_1^2 + M^2$, $B = M_1N_1 + NM$ and $C = N_1^2 + N^2$. Let $\mathfrak{l}_{\frac{M}{N}}$ and γ be the solutions of previous equation. Then we have

$$\gamma \cdot \mathfrak{l}_{\frac{M}{N}} = -1 \Leftrightarrow \gamma = \frac{-1}{\mathfrak{l}_{\frac{M}{N}}} < 1.$$

It follows that $\mathfrak{l}_{\frac{M}{N}} \in (0,1)$ and $\gamma \notin (0,1)$. One may conclude that $\mathfrak{l}_{\frac{M}{N}}$ is a Sturm number (see [38], Theorem 2.3.26). The explicit formula for computing $\mathfrak{l}_{\frac{M}{N}}$ can be obtained by computing the positive solution of the equation

$$(M_1N_1 + NM)g^2 + (N_1^2 + N^2 - M_1^2 - M^2)g - M_1N_1 - NM = 0.$$

□

Following [38][Theorem 2.3.25], the characteristic words of slope Sturm numbers are fixed by some no trivial standard sturmian morphisms. Next propositions compute the morphisms that fix the characteristic words of slopes $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ in terms of G, D and E (the generators of the standard monoid).

Proposition 51. *Let $\frac{M}{N} = [0; a_1, \ldots, a_k]$ with $a_i \geq 1$ be a Farey number, $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ the associated mirror numbers. Let*

$$i_{\frac{M}{N}} = G^{a_1-1} \cdot D^{a_2} \cdots A^{a_k}$$

$$j_{\frac{M}{N}} = G^{a_k-1} \cdot D^{a_{k-1}} \cdots A^{a_k}$$

with $A = G$ if k is odd and $A = D$ otherwise. Then the standard morphisms $f^{\mathfrak{l}}_{\frac{M}{N}}$ and $f^{\mathfrak{r}}_{\frac{M}{N}}$ that fix respectivelly the characteristic words of slope $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ are:

$$f^{\mathfrak{l}}_{\frac{M}{N}} = i_{\frac{M}{N}} DE \cdot j_{\frac{M}{N}} DE$$

$$f^{\mathfrak{r}}_{\frac{M}{N}} = i_{\frac{M}{N}} G \cdot j_{\frac{M}{N}} G.$$

Proof. Notice that any morphism f that fixes that characteristic word of slope $\mathfrak{l}_{\frac{M}{N}}$ or $\mathfrak{r}_{\frac{M}{N}}$ is locally characteristic, and thus (see [38][Theorem 2.3.12]) standard. Since $D = E \cdot G \cdot E$, we can write f as:

$$f = G^{n_1} \cdot E \cdot G^{n_2} \cdot E \cdots E \cdot G^{n_{2s+1}} \text{ where } \begin{cases} n_1, n_{2s+1} \geq 0 \\ n_2, \ldots, n_{2s} \geq 1 \\ s \geq 0 \end{cases}$$

Depending on the subindex n_i, there are three possibilities for the continued fraction expansion of the slope x of the fixed characteristic word (see [38] theorem 2.3.25):

1. $x = [0; 1 + n_1, \overline{n_2, \ldots, n_{2s+1} + n_1}]$ if $n_{2s+1} > 0$.
2. $x = [0; 1, n_2, \overline{n_3, \ldots, n_{2s}, n_{2s+1} + n_1}]$ if $n_{2s+1} = 0$ and $n_1 = 0$.
3. $x = [0; 1 + n_1, \overline{n_2, \ldots, n_{2s}, n_1}]$ if $n_{2s+1} = 0$ and $n_1 > 0$.

Now, x must coincide with either $\mathfrak{l}_{\frac{M}{N}}$ or $\mathfrak{r}_{\frac{M}{N}}$. The cases 2 and 3 are incompatible with the purely periodic, palindromic, continued expansion of $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ (the second case would imply that whether $n_2 = 0$ or $n_k = 0$, and in the third case we would necessarily have that $n_1 = 1 + n_1$). Therefore the first case is the only possible. Hence we can compute the exponents n_i depending on the coefficients a_i of the continued fraction expansion:

$$\begin{aligned} n_1 &= a_1 - 1 \\ n_i &= a_i && \forall i = 2, \ldots, k \\ n_{k+i} &= a_{k+1-i} && \forall i = 1, \ldots, k \\ n_{2s+1} &= 1. \end{aligned}$$

Now we can compute the morphisms $f^{\mathfrak{l}}_{\frac{M}{N}}$ (case k is odd) and $f^{\mathfrak{l}}_{\frac{M}{N}}$ (case k even). If k is odd one has:

$$\begin{aligned} f^{\mathfrak{l}}_{\frac{M}{N}} &= G^{a_1-1} \cdot E \cdot G^{a_2} \cdot E \cdots G^{a_k} \cdot E \cdot G^{a_k} \cdot E \cdots G^{a_1} \cdot E \cdot G = \\ &= G^{a_1-1} \cdot E \cdot G^{a_2} \cdots G^{a_k} \cdot D \cdot E \cdot G^{a_k-1} \cdots G^{a_1} \cdot D \cdot E = i_{\frac{M}{N}} DE \cdot j_{\frac{M}{N}} DE. \end{aligned}$$

If k is even, on the contrary :

$$\begin{aligned} f^{\mathfrak{r}}_{\frac{M}{N}} &= G^{a_1-1} \cdot E \cdot G^{a_2} \cdot E \cdots G^{a_k} \cdot E \cdot G^{a_k} \cdot E \cdots G^{a_1} \cdot E \cdot G = \\ &= G^{a_1-1} \cdot D^{a_2} \cdots D^{a_k} \cdot G \cdot G^{a_k-1} \cdot D^{a_{k-1}} \cdots D^{a_1} \cdot G = i_{\frac{M}{N}} G \cdot j_{\frac{M}{N}} G. \end{aligned}$$

□

It is straightforward now to compute the standard morphisms $f^{\mathfrak{l}}_{\frac{M*}{N}}$ and $f^{\mathfrak{r}}_{\frac{M*}{N}}$ fixing the characteristic words of slope, respectively $\mathfrak{l}_{\frac{M*}{N}}$ and $\mathfrak{r}_{\frac{M*}{N}}$:

Corollary 21. *Let $\frac{M}{N}$ be a Farey number, and $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$ the corresponding odd and even mirror numbers. If $f^{\mathfrak{l}}_{\frac{M}{N}} = i_{\frac{M}{N}} DE \cdot j_{\frac{M}{N}} DE$ and $f^{\mathfrak{r}}_{\frac{M}{N}} = i_{\frac{M}{N}} G \cdot j_{\frac{M}{N}} G$ are the standard morphisms that fix the characteristic words of slopes $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$, then it holds:*

$$f^{\mathfrak{l}}_{\frac{M*}{N}} = j_{\frac{M}{N}} DE \cdot i_{\frac{M}{N}} DE$$

$$f^{\mathfrak{r}}_{\frac{M*}{N}} = j_{\frac{M}{N}} G \cdot i_{\frac{M}{N}} G.$$

In other words,

$$i_{\frac{M}{N}} DE(c_{\mathfrak{l}_{\frac{M*}{N}}}) = c_{\mathfrak{l}_{\frac{M}{N}}} \quad \textit{and} \quad j_{\frac{M}{N}} DE(c_{\mathfrak{l}_{\frac{M}{N}}}) = c_{\mathfrak{l}_{\frac{M*}{N}}},$$

$$i_{\frac{M}{N}} G(c_{\mathfrak{r}_{\frac{M*}{N}}}) = c_{\mathfrak{r}_{\frac{M}{N}}} \quad \textit{and} \quad j_{\frac{M}{N}} G(c_{\mathfrak{r}_{\frac{M}{N}}}) = c_{\mathfrak{r}_{\frac{M*}{N}}}.$$

Proof. It is an immediate consequence of previous proposition and Corollary 8. □

Example 17 (Continuation)**.** *We compute the standard morphisms* $f^{\mathfrak{l}}_{\frac{4}{7}}$ *and* $f^{\mathfrak{r}}_{\frac{4}{7}}$ *that fix the characteristic words which slope are the mirror numbers that associated with* $\frac{4}{7}$*. Recall that* $\frac{4}{7} = [0;1,1,3]$ *which means that* $\mathfrak{l}_{\frac{4}{7}} = \left[0;\overline{1,1,3,3,1,1}\right]$ *and* $\mathfrak{r}_{\frac{4}{7}} = \left[0;\overline{1,1,2,1,1,2,1,1}\right]$*. It follows that*

$$f^{\mathfrak{l}}_{\frac{4}{7}} = G^0DG^3 \cdot DE \cdot G^2DG \cdot DE = DG^3D \cdot D^2GDG$$

is the standard morphism that fixes the characteristic word of slope $\mathfrak{l}_{\frac{4}{7}}$*, and*

$$f^{\mathfrak{r}}_{\frac{4}{7}} = G^0DG^2D \cdot G \cdot G^0D^2GD \cdot G = DG^2DG \cdot D^2GDG.$$

The decompositions above enable us to determine the following equalities that connect the characteristic words of slopes $\mathfrak{l}_{\frac{4}{7}}$ *and* $\mathfrak{r}_{\frac{4}{7}}$ *with the characteristic words of slopes* $\mathfrak{l}_{\frac{2}{7}}$ *and* $\mathfrak{r}_{\frac{2}{7}}$*:*

$$DG^3D(c_{\mathfrak{l}_{\frac{2}{7}}}) = c_{\mathfrak{l}_{\frac{4}{7}}} \quad \textit{and} \quad D^2GDG(c_{\mathfrak{l}_{\frac{4}{7}}}) = c_{\mathfrak{l}_{\frac{2}{7}}},$$

$$DG^2DG(c_{\mathfrak{r}_{\frac{2}{7}}}) = c_{\mathfrak{r}_{\frac{4}{7}}} \quad \textit{and} \quad D^2GDG(c_{\mathfrak{r}_{\frac{4}{7}}}) = c_{\mathfrak{r}_{\frac{2}{7}}}.$$

6.4 Mirror numbers, Harmonic and Gold Words

Now we can relate the characteristic words of odd mirror number slope with *harmonic words* as introduced in [18]. A central word u beginning with 0 (resp. beginning with 1) and having the periods p, q is harmonic if the following equality holds:

$$\frac{|u|_1 + 1}{|u|_0 + 1} = \frac{p}{q} \quad \text{(resp. if } \frac{|u|_0 + 1}{|u|_1 + 1} = \frac{p}{q} \text{ holds.)}$$

Equivalently, u is harmonic if and only if the ratio of the frequencies of the letters 0 and 1 in the Christoffel word $0u1$ coincides with the ratio of the periods of the central word u. Moreover, an infinite word $w = a_1a_2\dots$ is said to be harmonic, if it has infinitely many harmonic prefixes.

Proposition 52. *Every characteristic word with slope a mirror number is a harmonic word.*

Proof. Let $\frac{M}{N} = [0; a_1, a_2, \dots, a_{k+1},]$ and let k be even. Then the directive words of $c_{\mathfrak{l}_{\frac{M}{N}}}$ and $c_{\mathfrak{r}_{\frac{M}{N}}}$, following 11, are

$$0^{a_1-1}1^{a_2}0^{a_3}\cdots 1^{a_k+1}0^{a_k+1}\cdots 1^{a_1}\cdots$$

This directive word has infinite many sesqui-palindromic prefixes Proposition 3.5 in [18] completes the proof. A similar argumentation proves the result if k is odd. □

Example 18 (Continuation)**.** *The odd mirror number* $\mathfrak{l}_{\frac{4}{7}} = \left[0;\overline{1,1,3,3,1,1}\right] = \frac{-3+\sqrt{34}}{5}$ *has period* $(1,1,3,3,1,1)$ *with positive integers as coefficients. It can be associated with a binary path segment of length* $1+1+3+3+1+1 = 10$ *on the Stern-Brocot tree, and can be represented by the sesqui-palindrome* 0100011101*.*

This serves as the directive word for an harmonic word of length 128, that can be computed with the right iterated palindromic closure PAL (see Section 2.3.1:

$PAL(0100011101) =$

0100100100101001001001010010010010100100100100101001001001010010

0100101001001001010010010010010100100100101001001001010010010010.

Let us denote with $MIR_{\mathfrak{l}}$ and $MIR_{\mathfrak{r}}$ the sets of characteristic words which slope is a mirror number, that is:

$$MIR_{\mathfrak{l}} = \left\{ c_{\mathfrak{l}_{\frac{M}{N}}}, \text{ where } \frac{M}{N} \in \mathfrak{F} \right\} \quad MIR_{\mathfrak{r}} = \left\{ c_{\mathfrak{r}_{\frac{M}{N}}}, \text{ where } \frac{M}{N} \in \mathfrak{F} \right\}.$$

Proposition 52, together with Remark 15, implies the following chain of inclusions:

$$MIR_{\mathfrak{l}} \subset MIR_{\mathfrak{r}} \subset HARM$$

where $HARM$ denotes the set of harmonic words. It is not dificult to find a continued fraction expansion with infinite many palindromic prefixes that is not a mirror number. Hence, $MIR_{\mathfrak{r}}$ is a proper subset of $HARM$.

Both sets $MIR_{\mathfrak{l}}$ and $MIR_{\mathfrak{r}}$ share a property with $HARM$: they encode the set St of finite factors of all standard words (see Section 2.3.1):

Proposition 53. $Fact(MIR_{\mathfrak{l}}) = Fact(MIR_{\mathfrak{r}}) = St$

Proof. Any factor of an infinite standard word is a factor of a finite standard one. Hence $St = Fact(Stand)$. But any finite standard word can be extended to the right to a characteristic word of a certain mirror number slope. This proves our assertion. □

6.5 Dual mirror numbers: return words

This final section provides a proper word-theoretic meaning to the extended duality for mirror number slopes: given a characteristic word $c_{\mathfrak{l}_{\frac{M}{N}}}$, the succession of those letters which immediately precede the occurrences of the left special factor of length N coincides -up to a letter exchange- with the $G-$image of the left-dual word of slope $\mathfrak{L}_{\frac{M}{N}}\left(\mathfrak{l}_{\frac{M}{N}}\right) = \mathfrak{l}_{\frac{M*}{N}}$. Something similar happens with the right-dual word of slope $\mathfrak{R}_{\frac{M}{N}}\left(\mathfrak{r}_{\frac{M}{N}}\right)$. This fact is directly connected with the decomposition of a characteristic word in *return words*, as it will be shown.

Given a Sturmian word w, a factor u of w such that both $0u$ and $1u$ are also factors of w is called left special factor of w. Recall that a Sturmian word w has $N+1$ distinct factors of length N, for every $N > 0$. Hence, for every N, there is exactly one left special factor, u_N of w of length N. Let u_N denote the left special factor of a sturmian word $w = a_1 a_2 \ldots$, and let i_k be the place of the first letter of the $k-$th occurrence of u_N in w. That is, let $a_{i_k} a_{i_k+1} \ldots a_{i_k+N-1} = u_N$ be the $k-$th occurrence of the left special factor u_N in w.

Definition. *We call $N-$ derivative of the sturmian word $w = a_1 a_2 \ldots$ to the word $\Delta_N(w)$ defined by:*

$$\begin{array}{rccl} \Delta_N(w): & \mathbb{N} & \longrightarrow & \{0,1\} \\ & k & \longmapsto & a_{i_k - 1} \end{array}$$

The $N-$ derivative of a sturmian word w encodes the sequence of those letters which immediately precede the left special factor of length N in w.

The following example was presented in [31] as a conjecture that linked the characteristic words of slopes $\iota_{\frac{M}{N}}$ and $\iota^*_{\frac{M}{N}}$ by means of the $N-$derivative:

Example 19. *The characteristic word $c_{\iota_{\frac{4}{7}}}$ of mechanical slope $\iota_{\frac{4}{7}}$ is:*

$$c_{\iota_{\frac{4}{7}}} = 101010\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}1\boldsymbol{0}101010\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}1\boldsymbol{0}101010$$
$$\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}1\boldsymbol{0}101010\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}101010\boldsymbol{1}1\ldots$$

The left special factor of length 7 is 1010101*, and the letters which immediately precede it are highlighted in boldface. Thus, the 7-derivative of $c_{\iota_{\frac{4}{7}}}$ is:*

$$\Delta_7\left(c_{\iota_{\frac{4}{7}}}\right) = 11101111011101111\ldots$$

Now, if we compute $G\left(c_{\iota^*_{\frac{4}{7}}}\right) = G\left(c_{\iota_{\frac{2}{7}}}\right)$ we obtain:

$$G\left(c_{\iota_{\frac{2}{7}}}\right) = 00010000100010000\ldots$$

One may conjecture that

$$E\left(\Delta_7\left(c_{\iota_{\frac{4}{7}}}\right)\right) = G\left(c_{\iota_{\frac{2}{7}}}\right).$$

In order to prove previous conjecture, and the analogue one for right-mirror numbers, it will by essential the return words, a notion that was introduced by Durand in [33]. If w is a Sturmian word and x is a finite factor of w, a return word of x in w is a word that starts at an occurrence of x in w and ends exactly before the next occurrence of x. More precisely:

Definition. *If $w = w_1w_2\ldots$ with $w_i \in \{0,1\}$ is a Sturmian word, and*

$$w_iw_{i+1}\ldots w_{i+|x|-1} = x = w_jw_{j+1}\ldots w_{j+|x|-1} \quad \textit{with } i < j$$

are two consecutive occurrences of the factor x in w, then the word $w_iw_{i+1}\ldots w_{j-1}$ is called return word of x in w.

Vullion used the notion of return words in [54] to show that a word is Sturmian if and only if each non-empty factor x of w has exactly two distinct return words. In [5] Araújo and Bruyère computed explicitly the return words of the prefixes of a characteristic word c_α:

Proposition 54. *Let $\alpha = [0; a_1, a_2, \ldots] \in (0,1]$ be an irrational number and*

$$\frac{M}{N} = [a_1, a_2, \ldots, a_k, j] \quad \textit{with } k \geq 1 \textit{ and } 0 \leq j < a_{k+1}$$

be a semi-convergent of α. Let, moreover, $\{t_k\}_{k\geq -1}$ be the standard sequence associated with $(a_1 - 1, a_2, a_3, \ldots)$. Then the return words of the prefix on length N in the characteristic word c_α are

$$t_k \quad \textit{and} \quad t_k^j t_{k-1}.$$

The $N-$derivative is a transformation of Sturmina words very similar to the transformation that Araújo and Bruyère studied in [5] to codify the decomposition of a characteristic word in return words. Let w be a Sturmian word, x a prefix of w and u, v be the two return words of x. By definition of return words, w can be writen as a concatenation of the words u and v. Suppose that u appears before v in this concatenation. Let us consider de bijection:

$$\begin{aligned} \delta : \{u, v\} &\to \{0, 1\} \\ u &\mapsto u_1 \\ v &\mapsto 1 - u_1 \end{aligned}$$

where u_1 denotes the first letter of u. If $w = x_1 x_2 \dots$ with $x_i \in \{u, v\}$, then we call derivated word of w with respect to x to the word $D_x(w)$ defined by

$$D_x(w) = \delta(x_1)\delta(x_2)\dots$$

The return word codifies the occurrencies of u and v in the decomposition of w in return words. The following result characterizes the derivated word of a characteristic word c_α with respect to a prefix x of length N, where N is the denominator of a semi-convergent of the slope α. It is a particular case of the Proposition 15 in [5].

Proposition 55. *Let $\alpha = [0; a_1, a_2, \dots] \in (0, 1]$ be an irrational number and*

$$\frac{M}{N} = [0; a_1, a_2, \dots, a_k, j] \quad \textit{with } k \geq 1 \textit{ and } 0 \leq j < a_{k+1}$$

be a semi-convergent of α. Let moreover x be the prefix of length N of the characteristic word c_α. Then $D_x(c_\alpha)$ is the characteristic word of slope α', where α' is given by:

$$\alpha' = \begin{cases} [0; a_{k+1} + 1 - j, a_{k+2}, a_{k+3}, \dots] \textit{ if } a_1 > 1 \\ [0; 1, a_{k+1} - j, a_{k+2}, a_{k+3}, \dots] \textit{ if } a_1 = 1. \end{cases}$$

Now we can easily prove the equalities that follow, which relate the characteristic words of dual mirror numbers:

Proposition 56. *If $\frac{M}{N}$ is a full-convergent of the mirror numbers $\mathfrak{l}_{\frac{M}{N}}$ and $\mathfrak{r}_{\frac{M}{N}}$, it holds that*

$$\Delta_N \left(0c_{\mathfrak{l}_{\frac{M}{N}}} \right) = EG \left(0c_{\mathfrak{l}^*_{\frac{M}{N}}} \right)$$
$$\Delta_N \left(0c_{\mathfrak{r}_{\frac{M}{N}}} \right) = G \left(0c_{\mathfrak{r}^*_{\frac{M}{N}}} \right)$$

Proof. Let $\theta = \left[0; \overline{a_1, a_2, \dots, a_k + 1, a_k + 1, \dots, a_2, a_1}\right]$ with $a_k \geq 0$. Notice that $\theta = \mathfrak{l}_{\frac{M}{N}}$ if k is even and $\theta = \mathfrak{r}_{\frac{M}{N}}$ otherwise. Let u be the prefix of length N of the characteristic word c_θ. Let u_1 and u_{-1} denote the first and last letter of the word u. By definition of the $N-$derivative transformation Δ_N and the derivated transformation D_u with respect to u one has that

$$\Delta_N \left(c_\theta \right) = D_u \left(c_\theta \right) \Leftrightarrow u_1 = u_{-1}$$

and, in parallel,

$$\Delta_N\left(c_\theta\right) = ED_u\left(c_\theta\right) \Leftrightarrow u_1 \neq u_{-1}.$$

Notice that $u_1 = 0 \iff a_1 > 1$ and $u_{-1} = 0 \iff k$ is even. In this case, following Proposition 55,

$$\begin{aligned} D_u\left(c_{\mathfrak{r}_{\frac{M}{N}}}\right) &= D_u\left(c_{[0;\overline{a_1,a_2,\ldots,a_k+1,a_k+1,\ldots,a_2,a_1}]}\right) \\ &= c_{[0;2+a_k,a_{k-1},\ldots,a_1,a_1,\ldots a_{k-1},a_k+1,\ldots]} \\ &= G\left(c_{\mathfrak{r}^*_{\frac{M}{N}}}\right) \end{aligned}$$

In a similar way, $u_1 = 1 \iff a_1 = 1$ and $u_{-1} = 1 \iff k$ is odd. Again by Proposition 55, it holds

$$\begin{aligned} D_u\left(c_{\mathfrak{l}_{\frac{M}{N}}}\right) &= D_u\left(c_{[0;\overline{a_1,a_2,\ldots,a_k+1,a_k+1,\ldots,a_2,a_1}]}\right) \\ &= c_{[0;1,1+a_k,a_{k-1},\ldots,a_1,a_1,\ldots a_{k-1},a_k+1,\ldots]} \\ &= E\left(c_{[0;2+a_k,a_{k-1},\ldots,a_1,a_1,\ldots a_{k-1},a_k+1,\ldots]}\right) \\ &= EG\left(c_{\mathfrak{l}^*_{\frac{M}{N}}}\right) \end{aligned}$$

$\square$

Chapter 7

Further investigations

In this chapter we present some guidelines or lines of actions that can enhance and extend the contents of the present work.

7.1 On the non WF-words

We have not offered a characterization of the ternary words within $\{0, 1, 2\}^*$ that are non WF-words. Neither have we given a characterization of the non-WF-words that are Christoffel-Lyndon. Proposition 37 does not characterize the NWF words that are Christoffel-Lyndon. For example, the word $(0102|010202)$ is Christoffel-Lyndon but does not satisfy the conditions of Proposition 37: it is the third left-son of the Christoffel word $(022|0222)$. A natural, open question arises at this point: is there a characterization of the NWF words that are Christoffel-Lyndon?

Definition. *The word w verifies the quasi-LR condition if its left and right Lyndon factorizations do not coincide, but they determine a factorization of w as $w = u_1 \cdot u_2 \cdot u_3$ and it holds that the words $u_2 \cdot u_3 \cdot u_1$ and $u_3 \cdot u_1 \cdot u_2$ verify LR property.*

We define now in a recursive way *weak-Christoffel-Lyndon words* for an arbitrary, totally-ordered alphabet $\mathcal{A}$: a letter is a weak-Christoffel-Lyndon word; otherwise, a word is weak-Christoffel-Lyndon if one of the following assertions hold:

- w verifies LR condition, that is, its left and right Lyndon factorizations coincide, say $w = u \cdot v$ and if, moreover, u and v are both weak-Christoffel-Lyndon words.

- w verifies quasi-LR condition and left and right Lyndon factorizations determine a decomposition of w as $w = u_1 \cdot u_2 \cdot u_3$ where all the four words u_1, u_2u_3, u_1u_2 and u_3 are week-Christoffel-Lyndon.

It has been tested with computer experimentation that the following conjecture may hold:

Conjecture 1. *Every NWF word is a week-Christoffel-Lyndon word.*

Another open task is to determine the left Lyndon factorization of NWF words. Theorem 15 determines just when left and right Lyndon factorizations coincide, but does not compute explicitly the left Lyndon factorization for half of the NWF words (those for which both factorizations do not coincide). This explicit computation does not seem to be trivial.

7.2 Accumulation matrices, the free group and polyominoes

Let us consider the finite words of the alphabet $\{0, 1, \overline{0}, \overline{1}\}$. This monoid can be identified with the free group of two elements if one introduces the relations

$$\overline{0}0 = \epsilon \qquad 0\overline{0} = \epsilon \qquad \overline{1}1 = \epsilon \qquad 1\overline{1} = \epsilon$$

The notions of balance map, accumulation map and accumulation matrix that were studied in the Chapter 5 can be easily extended to the domain of words in $\{0, 1, \overline{0}, \overline{1}\}^*$:

Definition. *Let $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ be a finite word.*

1. *The **balance map** of w is the map $\beta_w : \{1, \ldots, |w|\} \longrightarrow \mathbb{Z}$ given by*

$$\beta_w(k) = \begin{cases} |w|_1 - |w|_{\overline{1}} & \textit{for} \quad w_k = 0 \\ -(|w|_1 - |w|_{\overline{1}}) & \textit{for} \quad w_k = \overline{0} \\ -|w|_0 - |w|_{\overline{0}} & \textit{for} \quad w_k = 1 \\ (|w|_0 - |w|_{\overline{0}}) & \textit{for} \quad w_k = \overline{1} \end{cases}$$

2. *The **accumulation map** of w is the map $\alpha_w : \mathbb{Z}_{|w|} \longrightarrow \mathbb{Z}$ with*

$$\alpha_w(0) = 0 \qquad \alpha_w(k) = \sum_{r=1}^{k} \beta_w(r)$$

3. *For $k = 1, \ldots, |w|$, the k−th **accumulation matrix** is the $2x2$ matrix A_k given by*

$$A_k = \begin{pmatrix} |w_k|_0 - |w_k|_{\overline{0}} & |w'_k|_0 - |w'_k|_{\overline{0}} \\ |w_k|_1 - |w_k|_{\overline{1}} & |w'_k|_1 - |w'_k|_{\overline{1}} \end{pmatrix},$$

where w_k denotes the prefix of w of length k and w'_k the suffix of w of length $|w| - k$.

The link between accumulation map and the set of accumulation matrices is exactly the same for words in $\{0, 1, \overline{0}, \overline{1}\}^*$ as the one for binary words (see Proposition 39):

Proposition 57. *For every word $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ and every $k = 1, \ldots, |w| - 1$, one has that*

$$\begin{aligned} |A_{k+1}| - |A_k| &= \beta_w(k+1) \\ |A_k| &= \alpha_w(k) \end{aligned}$$

.

Proof. Notice that the value $|A_{k+1}|$ depends on the $k+1-$th letter of w, that is, on $w(k+1) = w'_k(1)$. If $w'_k(1) = 0$, it holds that

$$A_{k+1} = A_k + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and now we compute the determinant:

$$\begin{aligned}
|A_{k+1}| = \left| A_k + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right| &= \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right| = \\
&= \begin{vmatrix} a+1 & b-1 \\ c & d \end{vmatrix} = ad - cb + c + d = \\
&= |A_k| + |w_k|_1 - |w_k|_{\overline{1}} + |w'_k|_1 - |w'_k|_{\overline{1}} = |A_k| + |w|_1 - |w|_{\overline{1}} = \\
&= |A_k| + \beta_w(k+1).
\end{aligned}$$

In a similar way one can prove that $|A_{k+1}|$ coincides with $|A_k| + \beta_w(k+1)$ for $w(k+1) = 1, \overline{0}, \overline{1}$. ☐

Following previous extensions of the definitions of the balance and accumulation maps and the accumulation matrices, it could be possible to extend the plain and twisted adjoints studied in Chapter 5 to the set of words $\{0, 1, \overline{0}, \overline{1}\}^*$. It could be interesting to study possible implications of this extension to the domains of theory of scales and the theory of words.

The monoid $\{0, 1, \overline{0}, \overline{1}\}^*$ can be used, in the other hand, to study polyominoes in $\mathbb{Z}^2$. These are the interior of a closed non-intersecting grid path of $\mathbb{Z}^2$, such as the one in Figure 7.1(a). The border of a polyomino (the border of any closed grid path in $\mathbb{Z}^2$) can be coded as a (circular) word in $\{0, 1, \overline{0}, \overline{1}\}^*$, where $0, 1, \overline{0}$ and $\overline{1}$ stand, respectively, for the vectors $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

Can we use the accumulation map and matrices of words in $\{0, 1, \overline{0}, \overline{1}\}^*$ to study the properties of polyominoes (or general pahts in the grid $\mathbb{Z}^2$)? A simple example of the use of these new tools is the following characterization of close paths in $\mathbb{Z}^2$:

Definition. *A word $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ is* ***closed*** *if $|w|_0 = |w|_{\overline{0}}$ and $|w|_1 = |w|_{\overline{1}}$.*

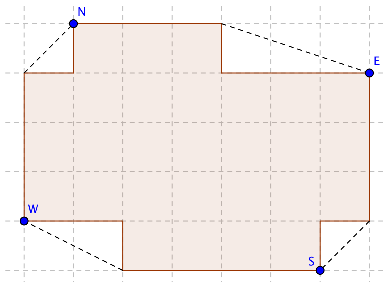
Corollary 22. *A word $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ is closed if and only if $|A_k| = 0$ for all $k = 1, 2, \ldots, |w|$.*

Proof. If w is closed, then, for $k = 1, \ldots, |w|$:

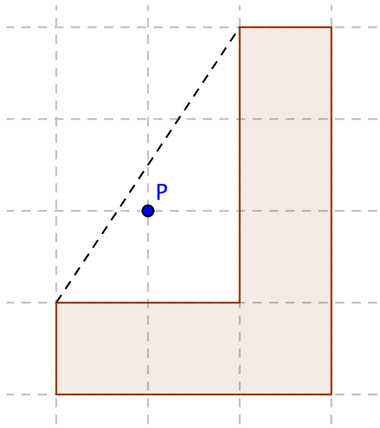
$$|w_k|_0 + |w'_k|_0 = |w_k|_{\overline{0}} + |w'_k|_{\overline{0}} \Leftrightarrow |w_k|_0 - |w_k|_{\overline{0}} = -(|w'_k|_0 - |w'_k|_{\overline{0}})$$

and the same equality holds if we change 0 by 1. It follows that

$$\begin{aligned}
|A_k| &= \begin{vmatrix} |w_k|_0 - |w_k|_{\overline{0}} & |w'_k|_0 - |w'_k|_{\overline{0}} \\ |w_k|_1 - |w_k|_{\overline{1}} & |w'_k|_1 - |w'_k|_{\overline{1}} \end{vmatrix} = \\
&= (|w_k|_0 - |w_k|_{\overline{0}})(|w_k|_1 - |w_k|_{\overline{1}}) \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0.
\end{aligned}$$



(a) A digitally convex polyomino in $\mathbb{Z}^2$.



(b) A polyomino that is not digitally convex.

Let now $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ be a word such that $|A|_k = 0 \, \forall k = 1, \ldots, |w|$. Following Proposition 57, one has that $\beta_w(k) = 0 \, \forall k = 1, \ldots, |w|$. Thus $|w|_0 = |w|_{\overline{0}}$ and $|w|_1 = |w|_{\overline{1}}$.

□

The notion of Christoffel words was used in [14] to study the digitally convexity of discrete subsets in $\mathbb{Z}^2$ (polyominoes). Recall that the convex hull of a set $X \subset \mathbb{R}^2$ is the intersection of the convex sets that contain X. Now, we can give a definition of convexity for polyominos:

Definition. *A polyomino P is called convex if it is the Gauss digitization of a convex subset X of the plane, that is, if $P = Conv(X) \cap \mathbb{Z}^2$, where $Conv(X)$ denotes the convex hull of X.*

The Figure 7.1(a) shows a polyomino that is digitally convex, and the Figure 7.1(b) shows a non convex polyomino. Notice how the point of the integer grid P yields between the convex hull and the polyomino.

Let us consider a polyomino in $\mathbb{Z}^2$. By convention we will start the coding from the lowest point on the left side (the point W in Figure 7.1(a)) and in clockwise order. We can define another three extremal points on the border: N (the leftmost on the top side), E (the highest on the right side) and S (the rightmost on the bottom side). In this way the border w of a polyomino decomposes as $w = w_1 w_2 w_3 w_4$. In the example of Figure 7.1(a) one has the following decomposition:

$$w = 11101000\overline{1}000\overline{1110100001}\overline{00} = 11101 \cdot 000\overline{1}000 \cdot \overline{11101} \cdot \overline{0000}1\overline{00}$$

This decomposition is called the standard decomposition of w. It was proven in [14] , essentially, that a word in $w \in \{0, 1, \overline{0}, \overline{1}\}^*$ is the contour word of a polyomino that is digitally convex if and only if the four words of the standard decomposition $w_1 w_2 w_3 w_4$ of w verify the following condition: their Lyndon factorization decomposes as a product of powers of Christoffel words. Recall that we gave a characterization of Christoffel words using accumulation matrices (see Proposition 40). Can we use accumulation matrices to show a similar characterization for digital convexity of the contour word of a polyomino?

A word w in $\{0,1\}^*$ (in $\{0,\overline{1}\}^*$, $\{\overline{0},1\}^*$, $\{\overline{0},\overline{1}\}^*$) is called NE (resp. SE, NW or SW). We say that a SE-word (resp. a NW-word or a SW-word) is a Christoffel word if the associated NW-word (via the morphism $\overline{0} \to 0$ and $\overline{1} \to 1$) is a Christoffel word. Can we extend the characterization of Christoffel words via accumulation matrices as well? Can we do the same for rotations of Christoffel words (that is, what we called in Chapter 5 well-formed words)?

Recall that any (NW-)Christoffel word w decomposes as Christoffel pair (w_1, w_2) and the corresponding matrix $A_{|w_1|}$ can be decomposed in $\langle L, R\rangle$, where $L = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Moreover, this decomposition determines the path in the Christoffel tree that leads from $(0,1)$ to (w_1, w_2). Can we find a similar situation for NW, SE and SW-Christoffel words? Let $\overline{L}$ and $\overline{R}$ be the matrices in $SL(\mathbb{Z},2)$ defined by

$$\overline{L} = \left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right) \qquad \overline{R} = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right).$$

How do NE (resp. SE, NW or SW) words decompose in Christoffel pairs and how are the factorizations in $< L, R, \overline{L}, \overline{R} >$ of the corresponding accumulation matrices?

Accumulation matrices might be used for interpreting another property of the borders of polyominoes: inertia of digitally sets. S. Brlek, G. Labelle and A. Lacasse studied in [13] a version of the winding number for discrete curves in the integer grid $\mathbb{Z}^2$. The *left-turns*, right turns and forward steps of a word consist, respectively of the factors of length two within the sets

$$V_L = \{01, 1\overline{0}, \overline{01}, \overline{1}0\} \quad V_R = \{10, 0\overline{1}, \overline{10}, \overline{0}1\} \quad V_F = \{00, 11, \overline{00}, \overline{11}\}.$$

The winding number is a weight P over the set of finite words $w \in \{0,1,\overline{0},\overline{1}\}^*$, defined as follows. For words u of length 2,

$$P(u) = \begin{cases} 1 \text{ if } u \in V_L \\ -1 \text{ if } u \in V_R \\ 0 \text{ if } u \in V_F. \end{cases}$$

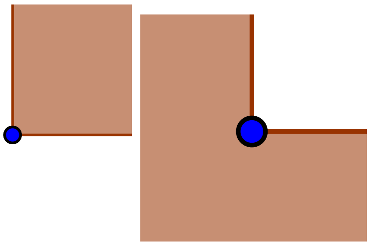
For a word w of length $|w| \geq 2$, $P(w)$ is given by:

$$\sum_{i=1}^{|w|-1} P(w_i w_{i+1})$$

and represents the number of direction changes counter-clockwise by $\frac{\pi}{2}$ units. In [13] it is shown that if w is a non-self-intersecting closed path in $\mathbb{Z}^2$, then $P(w) = 3$. The authors connect the winding number with the notion of salient and reentrant points from [28] (see Figure 7.1) to offer a proof of a result from [28] using the winding number: in a closed, non self-intersecting path, the number of salient S points and the number of reentrant points R verifies the formula $S = R + 4$.

It can be checked that the value of P for a factor $w_i w_{i+1}$ of length 2 coincides with the determinant of the accumulation matrix of the corresponding factor. That is,

$$P(w_i w_{i+1}) = k \quad \Longleftrightarrow \quad \left|A_{w_i w_{i+1}}\right| = k \quad \text{for all } k = -1, 0, 1$$



(c) A salient point (d) A reentrant point

Figure 7.1: Salient and reentrant points of closed paths in $\mathbb{Z}^2$.

and hence, the winding number can be computed directly via accumulation matrices (accumulation maps) of the factors of length 2 of the word w:

$$P(w) = \sum_{i=1}^{|w|-1} \left| A_{w_i w_{i+1}} \right| = \sum_{i=1}^{|w|-1} \alpha_{w_i w_{i+1}}(1).$$

This brings out a natural question: is there a formula to compute the winding number $P(w)$ of a word directly with the values of its accumulation map $\{\alpha_w(k), k = 0, \ldots, |w|\}$?

7.3 Mirror numbers and gold words

A notion introduced in [18] and with plenty of connections with Number Theory is that of *Gold Word*. A finite word w is a gold word if $0w1$ is a Christoffel word (that is, w is central) and the frequencies of the letters 0 and 1 in $0w1$ are two prime numbers. An infinite word is said to be gold if it has infinite many prefixes which are gold.

It is clear that any finite gold word can be extended to the right to a characteristic word which slope is a mirror number (a finite continued fraction can be seen as a semi-convergent of infinite many mirror numbers).

The characteristic word determined by a mirror number may not be a gold word. In fact, the characteristic word of slope

$$[0; \overline{1}] = \frac{-1 + \sqrt{5}}{2} = \{\phi\}$$

is not. This is due to a fundamental property of the Fibonacci sequence

$$\{f_n\}_{n \geq 1} = \{1, 1, 2, 3, 5, 8, \ldots\}.$$

Following [35, Theorem 179],

$$f_r \text{ divides } f_{pr} \quad p \geq 1 \text{ and } r \geq 2.$$

It follows that two consecutive numbers in the Fibonacci sequence cannot be simultaneously prime numbers. But the set of convergents of the Fibonacci number is $\{\frac{f_n}{f_{n+1}}\}_n$, as it is well known. One concludes that the frequencies of the letters 0 and 1 in any standard prefix of $c_{\frac{-1+\sqrt{5}}{2}}$ are not simultaneously two prime numbers. Thus, $c_{\{\phi\}}$ is not gold.

A natural question arises at this point: will something similar happen for every mirror number? In other words, will there exist a mirror number that determines a gold word? The following conjecture, tested with computer experimentation, points out that the answer to previous question might be negative:

Conjecture 2. *If $x = \left[0; a_1, a_2, \ldots, a_t, \overline{b_1, b_2, \ldots, b_s}\right]$ is a quadratic irrational number, and*

$$\left\{\frac{p_k}{q_k}\right\}_{k\geq 1}$$

is the sequence of convergents of x, then there is a $k \in \mathbb{Z}$ such that $p_{k'}$ and $q_{k'}$ are not both primes whenever $k' \geq k$.

As a consequence, no quadratic slope (in particular no mirror number) would determine a characteristic word that is a gold word. If *GOLD* denotes the set of infinite gold Sturmian words, then last assertion could be written as

$$GOLD \cap MIR_{\mathfrak{r}} = \emptyset.$$

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