



On the effects of the Chern-Simons term in an Abelian gauged Skyrme model in $d = 4 + 1$ dimensions

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ABSTRACT

We study an Abelian gauged $O(5)$ Skyrme model in $4 + 1$ dimensions, featuring the F^4 Maxwell and the Chern-Simons terms. Our aim is to expose the mechanism, discovered in the analogous Abelian gauged $O(3)$ Skyrme model in $2 + 1$ dimensions, which leads to the *unusual* relation of the mass-energy E to the electric charge Q_e and angular momentum J , and, to the change in the value of the “baryon number” q due to the influence of the Abelian field on the Skyrmion. Chern-Simons dynamics together with the dynamics of the gauged Skyrme scalar, allows for solutions with varying asymptotic values of the *magnetic* field, resulting in these *unusual* properties listed. Numerical work is carried out on an effective one dimensional subsystem resulting from imposition of an enhanced radial symmetry on \mathbb{R}^4 .

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1. Introduction

The Skyrme model has started almost sixty years ago [1,2], providing the very first explicit example of solitons in a relativistic non-linear field theory in $d = 3 + 1$ spacetime dimensions. This model has been generalised subsequently to all¹ spacetime dimensions. Concrete Skyrmions were constructed in [4] for the $O(3)$ sigma model on \mathbb{R}^2 , and for the $O(5)$ model on \mathbb{R}^4 in [5]. These are finite energy solutions (Skyrmions) whose energy is bounded from below by a topologically invariant charge n , which is the baryon number.

The situation changes in the presence of gauge fields. The gauging prescription given in [3,6] provides for a lower bound q for the energy of the gauged system, q being the volume integral of a density ϱ which, as required, is *Lorentz* and *gauge invariant*. Like the baryon number density prior to gauging, say ϱ_0 , its gauge-deformed counterpart ϱ is by construction *essentially total divergence*, meaning that in a constraint-compliant parametrisation of the Skyrme scalar it is *total divergence*. This property enables

the evaluation of the lower bound of the energy as a surface integral determined only by the asymptotic values of the solution, provided it is regular at the origin. The energy lower bound q coincides with the topologically invariant baryon number n in the gauge decoupling limit,² and can be seen as a deformation of the baryon number n . However, unlike the baryon number n , q is not topologically invariant. It may nonetheless be reasonable to call q a “gauge-deformed baryon number”.

Since this gauging prescription does not preserve the topological invariance of the energy lower bound, the gauge group in question is not constrained by the topology and hence one can gauge the $O(D + 1)$ Skyrme scalar on \mathbb{R}^D with $SO(N)$, $2 \leq N \leq D$ gauge group.³

It is because q is not topologically invariant that there exist finite energy solutions for varying values of q , and this is what gives rise to the features of gauged Skyrmions discussed below.

The new features in question were first discovered in the recent work [11,12], for the Abelian gauged $O(3)$ Skyrme model supple-

² This applies to any gauged system that in the gauge decoupling limit supports topologically stable solutions, notably the Goldstone model [9] on \mathbb{R}^D described by the real D -component scalar, whose solitons are stabilised by the topological *winding number* n . After gauging with $SO(N)$, $2 \leq N \leq D$ using the prescription given for the Skyrme systems [6,3], the energy lower bound cannot be saturated [7] and is topologically invariant only for gauge group $SO(D)$.

³ In the particular case of $SO(2)$ gauging of the $O(4)$ model on \mathbb{R}^3 , this prescription coincides with that employed in Ref. [8] much earlier.

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¹ These are direct extensions of the usual Skyrme model [1,2], namely of the $O(4)$ sigma model on \mathbb{R}^3 . $O(D + 1)$ Skyrme models on \mathbb{R}^D consists of all possible kinetic terms and potential term, consistent, with the Derrick scaling requirement. Such models are analysed in Appendix B of Ref. [3], and in references therein.

mented with a Chern-Simons term in $d = 2 + 1$ dimensions. The main new features reported there are:

- the change, or deformation, of the baryon number q of the Skyrme due to the influence of the gauge field through Chern-Simons dynamics, with

$$q = n + a_\infty, \quad (1)$$

where a_∞ is the asymptotic value of the magnetic gauge potential and $n = 1, 2, \dots$ is the winding number of the Skyrme scalars.

- the *unusual* dependence of the mass-energy E on the (global) electric charge Q_e and angular momentum J of the Skyrme, with E taking both negative and positive gradients, in contrast with the *usual* positive slope of monotonically increasing E vs. (Q_e, J) . The values of Q_e and J are also determined by the magnetic flux at infinity, with

$$Q_e = 8\pi(n - a_\infty), \quad J = 4\pi(a_\infty^2 - n^2). \quad (2)$$

The properties (1) and (2) of $SO(2)$ gauged Skyrmions depend on the existence of solutions with $a_\infty \neq 0$, which occurs exclusively by virtue of Chern-Simons (CS) dynamics. This limits such phenomena to odd dimensional spacetimes, where CS densities can be defined. Thus for example these phenomena are absent in an $SO(2)$ gauged Skyrme model in $3 + 1$ dimensions, where there is no CS term, with the mass-energy of the solutions increasing monotonically with increasing electric charge, and, the topological charge q remaining fixed [11], in which case $a_\infty = 0$ and $J \sim Q_e$.

The purpose of the present work is to demonstrate that the mechanism giving rise to the effects (1) and (2), of the Chern-Simons dynamics in the Abelian gauged $O(3)$ Skyrme model in $2 + 1$ dimensions, are also present in the Abelian gauged $O(5)$ Skyrme model in $4 + 1$ dimensions. Moreover, this mechanism is present in all Abelian gauged $O(D)$ Skyrme models in $D + 1$ space-time dimensions (with $D = 2p$).

The Skyrme model we employ here is the Abelian gauged $O(5)$ model in $4 + 1$ dimensions, which is a variant of the model studied in Ref. [13], with the (usual) Maxwell F^2 term in the action replaced here with the F^4 ‘‘Maxwell-like’’ term. This change allows for a slower asymptotic decay of the gauge field, resulting in a nonvanishing magnetic field at infinity. This is what enables the construction of solutions exhibiting the promised effects of Chern-Simons dynamics in $4 + 1$ dimensions, replicating qualitatively those found in $2 + 1$ dimensions. Also, following Ref. [13], we shall impose an enhanced symmetry on the system that renders the residual system one-dimensional, depending only on the radial variable, bypassing the more general bi-azimuthal symmetry.⁴

The paper is structured as follows. In Section 2, we present the model and impose symmetry, while in Section 3, we present our results. In Section 4, we summarize our results and make some remarks, including to point out that the mechanism described here can be extended to all $d = 2p + 1$ dimensions.

2. The model and symmetry imposition

2.1. The matter content

The only difference between the $SO(2)$ gauged Skyrme model in this work and the model studied in [13] is that the quadratic

⁴ This short-cut is justified since we have verified in [13] that qualitative features of the more general solutions do not differ from those of the enhanced symmetry solutions.

($p = 1$) Maxwell term F^2 is replaced⁵ by the quartic ($p = 2$) ‘‘Maxwell-like’’ term F^4 .

Therefore, for a $d = 5$ flat spacetime geometry $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ (with $x^0 = t$ -the time coordinate), the model’s Lagrangian reads

$$\mathcal{L} = \frac{1}{4!}\lambda_M F_{\mu\nu\rho\sigma}^2 + \kappa \Omega_{CS}^{(5)} - \lambda_1 |\phi_\mu^a|^2 + \frac{3}{2}\lambda_2 |\phi_{\mu\nu}^{ab}|^2 - \frac{1}{(3!)^2}\lambda_3 |\phi_{\mu\nu\lambda}^{abc}|^2 + \lambda_0 V[\phi^5], \quad (3)$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the Maxwell curvature and $F_{\mu\nu\rho\sigma} = F_{[\mu\nu}F_{\rho\sigma]}$ is a four form (totally antisymmetrised product of two two-forms $F(2)$). Also $\Omega_{CS}^{(5)}$ is the Chern-Simons (CS) density

$$\Omega_{CS}^{(5)} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma\lambda}A_\lambda F_{\mu\nu}F_{\rho\sigma}, \quad (4)$$

and κ is the CS coupling constant. λ_i ($i = 0, \dots, 3$) and λ_M are the rest of coupling constants that parameterise the model. To simplify a number of relations in what follows, we set $\lambda_M = 1$ without any loss of generality.

It is useful the record that the Chern-Simons term (4), which mixes electric and magnetic fields, reduces, apart from an irrelevant total divergence term, to

$$\Omega_{static}^{(5)} = -\varepsilon^{ijkl}A_0 F_{ij}F_{kl}, \quad i, j, \dots = 1, 2, 3, 4, \quad (5)$$

in the static limit.

It is further relevant to point out that while the term F_{ijkl}^2 in (3) also consists of both electric and magnetic fields, nonetheless differs qualitatively from the Chern-Simons density in that its dependence on A_0 is through F_{ijk0}^2 , namely on the second power of F_{i0} , while (5) is linearly dependent on A_0 .

The notation used in (3) is formally $\phi_\mu^a = D_\mu\phi^a$, the covariant derivative of the Skyrme scalar ϕ^a ; $a = 1, 2, 3, 4, 5$, which satisfies the constraint $|\phi^a|^2 = 1$. Then $\phi_{\mu\nu}^{ab}$ and $\phi_{\mu\nu\lambda}^{abc}$ are defined as the totally antisymmetrised products of ϕ_μ^a , in terms of which the Skyrme kinetic terms⁶ in the Lagrangian (3) are expressed. Also, $V = 1 - \phi^5$ is the usual Skyrme potential.

Labelling the $O(5)$ Skyrme scalar as $\phi^a = (\phi^\alpha, \phi^A, \phi^5)$, with $\alpha = 1, 2$; $A = 3, 4$, the covariant derivatives in (3) are defined by

$$\begin{aligned} \phi_\mu^\alpha &= D_\mu\phi^\alpha = \partial_\mu\phi^\alpha + A_\mu(\varepsilon\phi)^\alpha, \\ \phi_\mu^A &= D_\mu\phi^A = \partial_\mu\phi^A + A_\mu(\varepsilon\phi)^A, \\ \phi_\mu^5 &= D_\mu\phi^5 = \partial_\mu\phi^5, \end{aligned} \quad (6)$$

subject to the constraint

$$|\phi^\alpha|^2 + |\phi^A|^2 + (\phi^5)^2 = 1.$$

Here ε denotes the Levi-Civita symbol in each of the two-dimensional subsets of internal indices, (1, 2) and (3, 4), respectively. More specifically, $(\varepsilon\phi)^1 = \phi^2$, $(\varepsilon\phi)^2 = -\phi^1$ and similar for indices (3, 4).

The Maxwell equations resulting from the variations *w.r.t.* A_λ are,

⁵ Models with such higher-order terms have been extensively studied in the literature (although mainly for the non-Abelian case) leading to a variety of new features (see Ref. [9] for a review). Moreover, such $F(2p)^2$ terms occur in Born-Infeld theory [17] or in the higher loop corrections to the $d = 10$ heterotic string low energy effective action [18]. Here, however, we adopt a ‘phenomenological’ viewpoint and replace F^2 with F^4 primarily for the purpose of reproducing the results in [11,12].

⁶ We have omitted the octic kinetic term $|\phi_{\mu\nu\rho\sigma}^{abcd}|^2$ in (3) as we limit our considerations up to the sextic one.

$$F_{\rho\sigma}\partial_\mu F^{\mu\lambda\rho\sigma} - \frac{3}{2}\kappa\varepsilon^{\lambda\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = j^\lambda[A_\mu, \phi^a], \quad (7)$$

where $j^\lambda[A_\mu, \phi^a]$ is the Skyrme current resulting from the variations w.r.t. A_λ of the Skyrme kinetic terms in (3). Restricting, for example, to the quadratic term with λ_1 in (3), one finds

$$j_\lambda = -2\lambda_1 \left[(\varepsilon\phi)^\alpha D_\lambda\phi^\alpha + (\varepsilon\phi)^A D_\lambda\phi^A \right]. \quad (8)$$

The Gauss Law equation, namely the time component of (7), is

$$F_{jk}\partial_i F^{i0jk} - \frac{3}{2}\kappa\varepsilon^{0ijkl}F_{ij}F_{kl} = j^0[A_\mu, \phi^a], \quad (9)$$

from which follows the definition⁷ of the electric charge as the volume integral in \mathbb{R}^4

$$\begin{aligned} Q_e &\stackrel{\text{def.}}{=} -\frac{1}{8\pi^2} \int j_0 d^4x \\ &= -\frac{1}{8\pi^2} \int \left(F_{jk}\partial_i F^{i0jk} - \frac{3}{2}\kappa\varepsilon^{ijkl}F_{ij}F_{kl} \right) d^4x. \end{aligned} \quad (10)$$

In this work we are interested in time-independent, rotating configurations, in which case the energy density is the T_{00} component of the stress tensor $T_{\mu\nu}$ pertaining to the Lagrangian (3), while the angular momentum densities here are those in the two planes $x_\alpha = (x_1, x_2)$ and $x_A = (x_3, x_4)$ in \mathbb{R}^4 . These are defined in terms of the components $T_{0\alpha}$ and T_{0A} components of $T_{\mu\nu}$ respectively.

The definition of the gauge-deformed ‘‘baryon number’’ density [6,3] is given by the generic expression

$$\varrho = \varrho_0 + \varrho_1, \quad \varrho_1 \stackrel{\text{def.}}{=} \partial_i \Omega_i[A, \phi], \quad (11)$$

with $\varrho_0 = \varepsilon_{ijkl}\varepsilon^{abcde}\partial_i\phi^a\partial_j\phi^b\partial_k\phi^c\partial_l\phi^d\phi^e$, being the (topological) baryon number density, which is deformed, or modified, by the addition of the total divergence term $\varrho_1 = \partial_i \Omega_i$, yielding the total ‘‘baryon number’’

$$q = -\frac{1}{64\pi^2} \int \varrho d^4x. \quad (12)$$

For the system at hand, Ω_i is derived in detail in Ref. [13] so we just quote it here

$$\begin{aligned} \Omega_i &= 2 \cdot 3! \varepsilon_{ijkl} A_l \left\{ \frac{1}{3}(\phi^5)^3 F_{jk} \right. \\ &\quad \left. + \phi^5 \left(\varepsilon^{\alpha\beta} \partial_j \phi^\alpha \partial_k \phi^\beta + \varepsilon^{AB} \partial_j \phi^A \partial_k \phi^B \right) \right\}. \end{aligned} \quad (13)$$

Also, we point out that the definition (13) is more general than the example given in [3], where only one pair of the five components $O(5)$ Skyrme scalar on \mathbb{R}^4 were gauged with $SO(2)$. In [13] and here by contrast, two pairs of the Skyrme scalar are gauged. While in [13] this was done for a technical reason, namely to enable the imposition of (enhanced) radial symmetry on the bi-azimuthal system, here, the explicit expression (13) is essential for achieving a gauge-deformed ‘‘baryon number’’ that differs from the (topological) baryon number.

While the gauge-deformed ‘‘baryon number’’ density given in [3] features only the electric component A_0 of the Abelian gauge connection, the more general prescription resulting from (13) can

be seen to feature both the electric component A_0 and the magnetic component A_i of the Abelian gauge connection. The presence of A_i is crucial for achieving our aim, to change the baryon number.

2.2. Imposition of symmetry and quantities of interest

We consider the following parametrization of the spatial \mathbb{R}^4 part of the background metric

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi_1^2 + \cos^2\theta d\varphi_2^2), \quad (14)$$

where $0 \leq r < \infty$ is the radial coordinate, and $\theta, \varphi_{1,2}$ are coordinates on S^3 , with $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi_{1,2} < 2\pi$.

The dependence on the angular variables $\varphi_{1,2}$ can be factorized for stationary, axially symmetric configurations, the generic configuration possessing a dependence on both r and θ . However, the recent work [13] has proposed a special Maxwell-Skyrme (axially symmetric) Ansatz which results in an enhanced symmetry on the system that renders the residual system one-dimensional, depending only on the radial variable. Moreover, this Ansatz provides the natural generalization of that employed in Ref. [11,12] in the $d = 2 + 1$ case, with

$$\phi^1 = \sin f(r) \sin \theta \cos \varphi_1, \quad \phi^2 = \sin f(r) \sin \theta \sin \varphi_1, \quad (15)$$

$$\phi^3 = \sin f(r) \cos \theta \cos \varphi_2, \quad \phi^4 = \sin f(r) \cos \theta \sin \varphi_2,$$

$$\phi^5 = \cos f(r),$$

for the Skyrme scalars, and

$$A = A_\mu dx^\mu = a(r) \sin^2 \theta d\varphi_1 + a(r) \cos^2 \theta d\varphi_2 + b(r) dt, \quad (16)$$

for the Maxwell field (with a magnetic potential $a(r)$ and an electric one $b(r)$), the angular dependence being factorized.

This restrictive Ansatz greatly reduces the complexity of the system and simplifies the numerical construction of solutions. For example, one finds the following effective Lagrangian of the system

$$\begin{aligned} r^{-3}L_{\text{eff}} &= \frac{4a^2}{r^4} \left(\frac{a'^2}{r^2} - b'^2 \right) + \frac{8\kappa}{r^3} (ba' - ab')a \\ &\quad + \lambda_1 \left[f'^2 + \frac{\sin^2 f}{r^2} \left[2 + (1-a)^2 - r^2 b^2 \right] \right] \\ &\quad + 6\lambda_2 \frac{\sin^2 f}{r^2} \left[f'^2 \left[2 + (1-a)^2 - r^2 b^2 \right] \right. \\ &\quad \left. + \frac{\sin^2 f}{r^2} \left[1 + 2(1-a)^2 - 2r^2 b^2 \right] \right] \\ &\quad + \lambda_3 \frac{\sin^4 f}{r^4} \left[f'^2 \left[1 + 2(1-a)^2 - 2r^2 b^2 \right] \right. \\ &\quad \left. + \frac{\sin^2 f}{r^2} \left[(1-a)^2 - r^2 b^2 \right] \right] \\ &\quad + \lambda_0 (1 - \cos f), \end{aligned} \quad (17)$$

the contribution of various terms being transparent.

Next, we evaluate the one dimensional integral resulting from the imposition of (enhanced) radial symmetry on the integral (10) defining the global charge Q_e ,

$$Q_e = \int_0^\infty dr \frac{d}{dr} \left(2 \frac{a^2 b'}{r} + 3\kappa a^2 \right). \quad (18)$$

⁷ It should be stressed that this definition of the electric (global) charge is the Paul-Khare [14,15] definition, which differs from the usual one where the electric charge is read directly from the asymptotic decay of the electric potential. Refs. [14,15] pertain to the Abelian gauged Higgs model in 2 + 1 dimensions. The corresponding formulation for the $O(3)$ sigma model is given in Refs. [16].

The total mass-energy, E , and angular momenta $J_{1,2}$ in the two sub-planes of \mathbb{R}^4 , of a solution are defined as the volume integral of T_t^t and $T_{\varphi_{1,2}}^t$, respectively, with

$$E = \int d^4x \sqrt{g} T_t^t, \quad J_1 = \int d^4x \sqrt{g} T_{\varphi_1}^t, \quad J_2 = \int d^4x \sqrt{g} T_{\varphi_2}^t, \quad (19)$$

where $\sqrt{g} = r^3 \sin \theta \cos \theta$ and $J_1 = J_2 = J$ for the considered Ansatz. The T_t^t component of the stress tensor can be read off (17) by changing the appropriate signs and setting $\kappa = 0$, so it is not necessary to display it again, while the explicit form of the angular momenta densities is

$$\begin{aligned} \frac{T_{\varphi_1}^t}{\sin^2 \theta} &= \frac{T_{\varphi_2}^t}{\cos^2 \theta} = \frac{8}{r^4} a^2 a' b' \\ &- 2 \sin^2 f (1-a) b \left[\lambda_1 + 6\lambda_2 \left(f'^2 + \frac{2 \sin^2 f}{r^2} \right) \right. \\ &\left. + 2\lambda_3 \frac{\sin^2 f}{r^2} \left(f'^2 + \frac{\sin^2 f}{2r^2} \right) \right]. \end{aligned}$$

Using the field equations, one can show that

$$\sqrt{g} T_{\varphi_1}^t = \sin^3 \theta \cos \theta S', \quad \sqrt{g} T_{\varphi_2}^t = \cos^3 \theta \sin \theta S', \quad (20)$$

where

$$S = -\frac{8}{r} a^2 (1-a) b' - 4\kappa a^2 (3-2a), \quad (21)$$

which makes manifest the total derivative structure of $T_{\varphi_i}^t$.

Finally, the one dimensional baryon number density ϱ_0 , and $\varrho_1 = \partial_i \Omega_i$ given by the choice (13), reduce to

$$\begin{aligned} \varrho_0 &= -\frac{4!}{r^3} \frac{d}{dr} \left[\cos f - \frac{1}{3} \cos^3 f \right] \quad \text{and} \\ \varrho_1 &= \frac{4!}{r^3} \frac{d}{dr} \left[\frac{2}{3} a^2 \cos^3 f + a \sin^2 f \cos f \right], \end{aligned} \quad (22)$$

yielding $\varrho = \varrho_0 + \varrho_1$ as per (11), from which q defined by (12) can be calculated.

3. Results

3.1. The boundary conditions and global charges

The resulting system of three ODEs for the functions a, b and f is solved numerically⁸ subject to the following set of boundary conditions which are compatible with the finite global charges and regularity requirements:

$$\begin{aligned} f(0) &= \pi, \quad a(0) = 0, \quad b(0) = b_0, \quad f(\infty) = 0, \\ a(\infty) &= a_\infty, \quad b(\infty) = b_\infty. \end{aligned} \quad (23)$$

These boundary conditions are similar to those in [13], except for $a(\infty) \neq 0$. That is, for a model with an F^2 term in the action, the contribution of the Maxwell term to the total mass-energy diverges unless the magnetic potential vanishes asymptotically. This feature is not present when replacing F^2 with a F^4 (quartic) Maxwell

⁸ The numerical treatment of the problem was similar to that employed in Ref. [13].

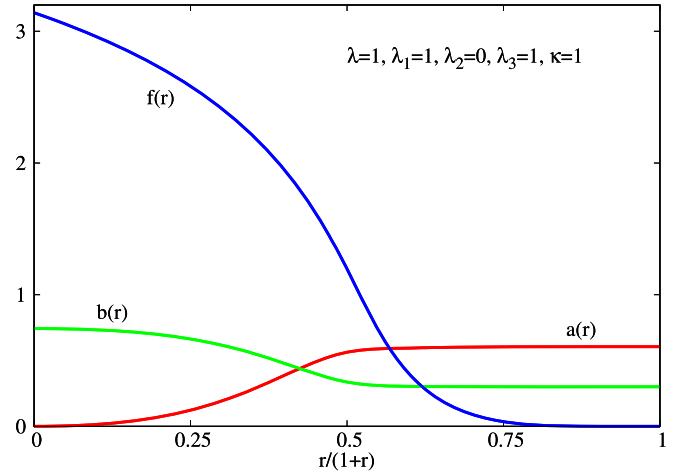


Fig. 1. The profile of a typical SO(2) gauged Skyrme solution is shown as a function of the (compactified) radial coordinate.

term, whose faster asymptotic decay allows $a(\infty)$ to take an arbitrary value.⁹

When replacing these asymptotic behaviour in (18), (19) and (22), one finds the simple relations

$$Q_e = 3\kappa a_\infty^2, \quad J = 4\pi^2 \kappa a_\infty^2 (2a_\infty - 3), \quad q = 1 - \frac{1}{2} a_\infty^2, \quad (24)$$

which, as seen from (1), (2), are qualitatively similar with those found in the 2+1 dimensional model.

3.2. Numerical results

We have constructed numerical solutions for several choices of the parameters in the theory, the profile of a typical solution with $a_\infty = 0.605$, $b_\infty = 0.3$ being shown in Fig. 1.

In what follows, we shall concentrate on the case $\lambda = \lambda_1 = \lambda_3 = 1$, $\lambda_2 = 0$, and several values of κ . In Fig. 2 (left panel) we represent the asymptotic value b_∞ of the electric potential $b(r)$ as a function of the asymptotic value a_∞ of the magnetic function $a(r)$. In contrast to the case in Ref. [11], a_∞ can only take positive values. On the other hand, b_∞ is bounded by the inequality¹⁰

$$b_\infty^2 < \frac{\lambda}{2\lambda_1}. \quad (25)$$

The maximum value of b_∞ coincides with the maximum value of a_∞ .

The dependence of the mass-energy E on the topological charge q is exhibited in Fig. 2 (right panel). While in the SO(2) gauged model in Ref. [11] featuring a standard F^2 term, the topological charge is always $q = 1$, the situation in this case is different. While the solutions with $a_\infty = 0$ still have a unit topological charge, a varying a_∞ gives rise to solutions with $q < 1$, as implied by (12). Moreover, while for large values of the CS constant one finds solutions with q still positive, decreasing κ allows for configurations with vanishing topological charge, negative values being also permitted. Notice that the allowed range of q decreases as κ increases.

⁹ Note that, following [19], the parameter a_∞ can be identified with the magnetic flux at infinity through the base space S^2 of the S^1 fibration of S^3 , with the sphere written in term of Euler angles. Also, it is interesting to remark that several F^2 -models with $a(\infty) \neq 0$ were considered in the literature [19], [20], they requiring anti-de Sitter asymptotics of the spacetime geometry.

¹⁰ This is true for the branches considered in this work. However, there are some numerical indications that other branches might exist for which this bound is exceeded.

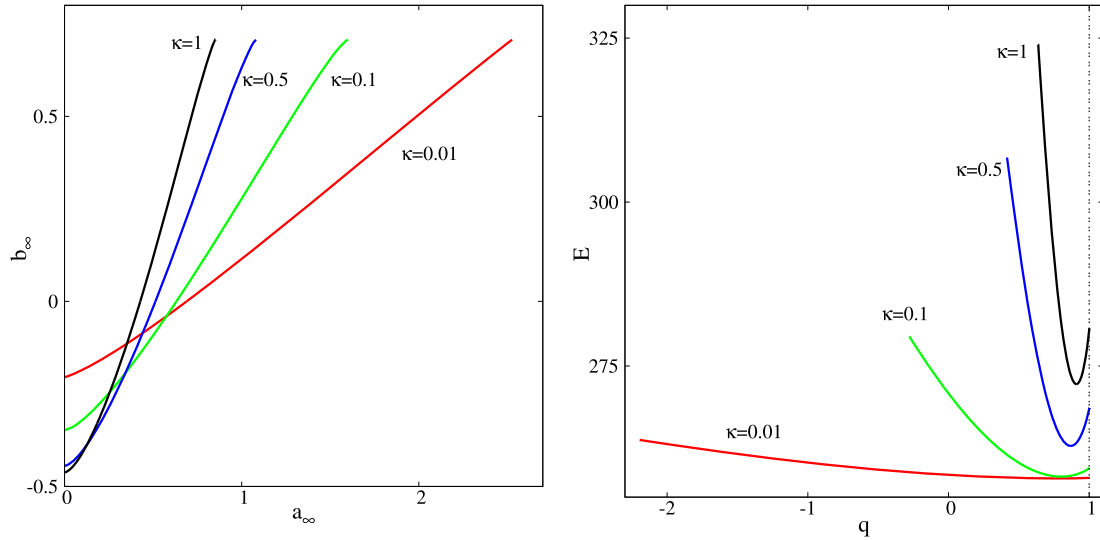


Fig. 2. Left panel: The asymptotic value of the electric potential b_∞ is shown vs. the value a_∞ of the magnetic potential at infinity, for solutions with $\lambda = \lambda_1 = \lambda_3 = 1$, $\lambda_2 = 0$, and $\kappa = 0.01, 0.1, 0.5, 1.0$. Right panel: Energy E vs. topological charge q for the same solutions.

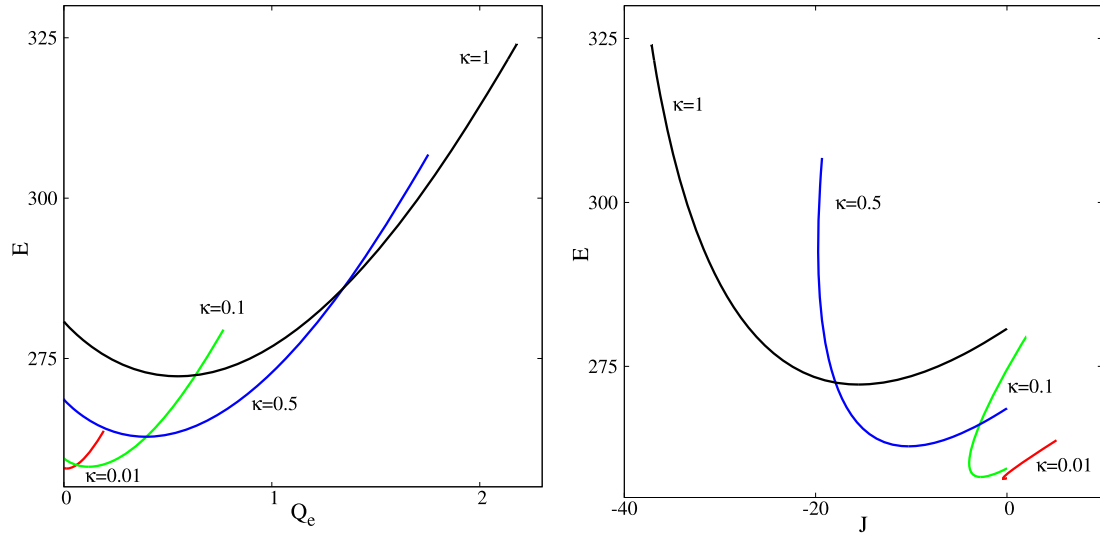


Fig. 3. The total mass-energy E vs. is shown vs. electric charge Q_e (left panel) and angular momentum J (right panel) for same solutions as in Fig. 2.

Also, solutions in the limit $\kappa \rightarrow 0$ are either static and uncharged or singular.

The dependence of the total mass-energy E on the electric charge Q_e and the angular momentum J is addressed in Figs. 3 (left and right panels, respectively). It is clearly seen that the solution with the least energy does not correspond to the uncharged solution, in general. That fact can be clearly seen for $\kappa = 1$ curve in the plot. On the other hand, the relation between the energy and the angular momentum is also quite unconventional. The least energetic solution corresponds usually to a rotating (with $J \neq 0$) solution. Moreover, for small values of κ there exist rotating solutions (in the sense of solutions having a non-zero angular momenta density, $T_{\varphi_{12}}^t \neq 0$) with vanishing total angular momenta ($J = 0$).

We would like to point out that most of these behaviours also happen in $2 + 1$ dimensions [11], so one can infer that they are not inherent to a concrete spacetime odd dimension but to the present of a CS term in the Lagrangian, together with the possibility of existence of solutions with a non-vanishing a_∞ .

4. Summary and outlook

The main purpose of this work was to expose the mechanism leading to several new unusual features discovered in the Abelian gauged $O(3)$ Skyrme model with a Chern-Simons term in $2 + 1$ dimensions, which were reported in Ref. [10]. The features in question are: (i) the value of the topologically invariant baryon number n are altered by the gauge dynamics as expressed by Eq. (1), and (ii) the (global) electric charge and angular momentum of the (gauged) Skyrmions exhibit the non-standard feature that their slopes vs. the mass-energy are not only positive, but also negative, resulting from (2).

Specifically, this mechanism hinges on the possibility of a_∞ , the asymptotic value of the magnetic field $a(r)$ of the gauged Skyrmion, not vanishing.¹¹ A necessary ingredient of this mech-

¹¹ This is exclusively a feature of gauged Skyrmions in contrast to gauged Higgs solitons. In the latter case $a_\infty = 0$ always, by virtue of the covariant constancy of the Higgs field resulting from symmetry breaking dynamics.

anism is the Chern-Simons dynamics. The Chern-Simons density (5) intertwines the magnetic field F_{ij} and the electric potential A_0 , the latter appearing linearly. Since the asymptotic value b_∞ of A_0 is not fixed, then the dependence of a_∞ on b_∞ provides the leverage on the variation of the former.

Our strategy here was to replicate the features in question for an Abelian gauged $O(5)$ Skyrme model featuring a F^4 Maxwell-like term in $d = 4 + 1$ dimensions, endowed with the Chern-Simons density. While this model is considerably more complex than the one [10] in $d = 2 + 1$ dimensions, the said properties are qualitatively reproduced. This analysis exposes the studied mechanism. Moreover, it can be seen that it can be reproduced in all $d = 2p + 1$ dimensions for a $O(2p + 1)$ Abelian gauged Skyrme model with a F^{2p} Maxwell-like term. The choice of F^{2p} Maxwell-like terms is made to ensure the appropriate asymptotic decay of the Abelian field, such that the surface integrals giving the electric charge and angular momentum receive their contributions from the p -th Pontryagin density as formulated by Paul and Khare [14]. (Of course, in all but $2 + 1$ dimensions the Pontryagin index vanishes.)

In the $2 + 1$ dimensional case, it was shown in [10] that the dependence of the mass-energy of the solution on the electric charge and angular momentum Q_e and J was a *non standard* case, in the sense that E did not increase monotonically as in the *standard* case, with increasing (Q_e, J) . The effect of the Chern-Simons dynamics resulted in negative gradients of E vs. (Q_e, J) in some regions of the parameter space. In [12], it was found that the same mechanism resulted in the departure of the “baryon number” q of the gauged solutions from the value of the baryon number n of the Skyrminion prior to gauging. Also, (Q_e, J, q) , are evaluated as “surface integrals” encoded by a_∞ , the asymptotic value of the magnetic potential $a(r)$ (see Eqs. (1), (2)).

Similarly, in the $4 + 1$ dimensional case, the solutions are characterised by a_∞ and b_∞ – the asymptotic value of the electric function $b(r)$. The Chern-Simons dynamics results in the dependence of a_∞ on b_∞ , and the studied family of solutions result from the (free) choice of b_∞ .

The results of our numerical analysis are displayed in Figs. 1, 2 and 3:

(i) The relation of a_∞ with b_∞ , displayed in Fig. 1. In this case there is a detail in which the $2 + 1$ and $4 + 1$ cases differ, namely that in the former case a_∞ can take both positive and negative values, while in the case at hand it takes only positive values. This is not surprising since in $2 + 1$ dimensions $a(r)$ appears in the imposition of azimuthal symmetry in \mathbb{R}^2 , while the function $a(r)$ here has a different origin, resulting from the imposition of an enhanced symmetry on a system featuring two distinct magnetic potentials.

(ii) The change (deformation) of the baryon number of the Skyrminion due to the influence of the gauge field. This is displayed in Fig. 2, by plotting the deformed “baryon number” vs. the energy E of the continuum of solutions.

(iii) The *unusual* dependence of E on Q_e and J involving both negative and positive gradients, displayed in *left panel* of Fig. 3 and in *right panel* of Fig. 3, respectively.

It may be instructive to display the Lagrangians in $2p + 1$ dimensions, generalising the $p = 2$ Lagrangian (3). This can be expressed formally as

$$\mathcal{L} = \frac{1}{(2p)!} \lambda_M F(2p)^2 + \kappa \Omega_{CS}^{(2p+1)} - \lambda_1 |\phi(1)|^2 + \lambda_2 |\phi(2)|^2 - \dots + \lambda_{2p} |\phi(2p)|^2 + \lambda V[\phi^{2p+1}] \quad (26)$$

where $F(2p)$ is the p -fold totally antisymmetrised product of the 2-form Maxwell curvature, $\Omega_{CS}^{(2p+1)}$ is the Abelian Chern-Simons density

$$\Omega_{CS}^{(2p+1)} = -\frac{1}{2} \varepsilon^{\lambda\mu_1\mu_2\dots\mu_{2p-1}\mu_{2p}} A_\lambda F_{\mu_1\mu_2} \dots F_{\mu_{2p-1}\mu_{2p}} \quad (27)$$

$$\Omega_{static}^{(2p+1)} = -\varepsilon^{i_1 i_2 \dots i_{2p-1} i_{2p}} A_0 F_{i_1 i_2} \dots F_{\mu_{2p-1} \mu_{2p}}, \quad i_1, i_2, \dots = 1, 2, \dots, 2p - 1, 2p, \quad (28)$$

where, again, in the static limit (28), the linear dependence on A_0 implies the dependence of a_∞ on b_∞ . The $|\phi(p)|^2$ are the Skyrme kinetic terms consistent with Derrick scaling, expressed in terms of $\phi(p)$ which is the p -fold totally antisymmetrised product of the 2-form $D_\mu \phi^a$. $V[\phi^{2p+1}]$ is the (optional) potential term.

To keep the full analogy with the considered $p = 1, 2$ models, we impose first a p -azimuthal symmetry (*i.e.* the solutions possess an azimuthal symmetry in each \mathbb{R}^2 sub-plane of \mathbb{R}^{2p}), and then an enhanced symmetry, by equating the angular momenta in each of the p sub-planes, $J_1 = J_2 = \dots = J_p \equiv J$ (see Ref. [13] for a discussion of this procedure in $d = 5$ dimensions). In other words, a suitable Maxwell-Skyrme Ansatz factorizing the angular dependence exists in the generic $d = 2p + 1$ case, which generalizes the expressions (15), (16). As such, the p -model reduces to an effectively one dimensional radial system, in terms of three functions $f(r)$, $a(r)$ and $b(r)$.

The following speculative comparisons of the quantities $Q_e(p)$, $J(p)$ and $q(p)$, in dimensions $d = 2p + 1$ may be illustrative:

$$\begin{aligned} Q_e(1) &\sim n - a_\infty, \quad Q_e(2) \sim a_\infty^2 \quad \Rightarrow \quad Q_e(p) \sim a_\infty^p \\ J(1) &\sim \pi(a_\infty^2 - n^2), \quad J(2) \sim \pi^2(3a_\infty^3 - 2a_\infty) \Rightarrow \\ J(p) &\sim \pi^p(\alpha_1 a_\infty^{p+1} + \alpha_2 a_\infty^{p-1} + \dots) \\ q(1) &\sim n + a_\infty, \quad q(2) \sim 1 - \frac{1}{2} a_\infty^2 \quad \Rightarrow \quad q(p) \sim 1 - \beta a_\infty^p \end{aligned} \quad (29)$$

where the constants $\alpha_1, \alpha_2, \dots$ and β have only symbolic meaning.

What distinguishes the $p = 1$ members in (29) is that they feature the *winding number* n , which is absent in the $p > 1$ case. The winding number n is in fact the 1st Pontryagin number in the expression of the electric charge for the $p = 1$ counterpart of (10) above,

$$Q_e(1) = \frac{1}{4\pi} \int j_0 d^2x = \frac{1}{8\pi} \int \left(\partial_i F_{i0} - \frac{1}{2} \kappa \varepsilon^{ij} F_{ij} \right) d^2x, \quad (30)$$

n resulting from the integral of the second term in the integrand of (30). For $a_\infty = 0$, $Q_e(1)$ is “quantised” with n , as in Refs. [14,15]. Clearly, in $4 + 1$ dimensions the 2nd Pontryagin integral in (10) vanishes for the Abelian curvature, unless if $a_\infty \neq 0$.

The electric charge for the general system (26) is

$$\begin{aligned} Q_e(p) = \frac{1}{2^{p+1}\pi} \int &\left(F_{j_1 j_2 \dots j_{2p-2}} \partial_i F_{i0 j_1 j_2 \dots j_{2p-2}} \right. \\ &\left. - \frac{1}{2} \kappa \varepsilon^{j_1 j_2 \dots j_{2p}} F_{j_1 j_2 \dots j_{2p}} \right) d^2p x. \end{aligned} \quad (31)$$

The first term in the integrand will decay sufficiently fast that it will not contribute to the integral. The contribution of the second term will also vanish unless $a_\infty \neq 0$.

For $Q_e(p)$ to receive a contribution from the p -th Pontryagin integral ($p \geq 2$) the Chern-Simons density must feature a non-Abelian field. The natural choice for such a system in $2p + 1$ dimensions is a $SO_\pm(2p) \times SO(2)$ gauge theory.

In the $p = 2$ case at hand, such a candidate is a $SO_\pm(4) \times SO(2)$ (or $SU(2) \times U(1)$) model which may support solutions with nonvanishing 2nd Pontryagin integral. For such putative solutions $Q_e(2)$ cannot be “quantised” by the 2nd Pontryagin charge as is the case in $2 + 1$ dimensions [14] where $Q_e(1)$ is quantised by the 1st Pontryagin charge. This is because in the $p = 2$ case the $SU(2)$ Yang-Mills equation is sourced by the Abelian field, *i.e.*, it is not the vacuum Yang-Mills equation solved by the self-dual (BPST) field. Such a $4 + 1$ dimensional model is under study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- [1] T.H.R. Skyrme, *Proc. R. Soc. Lond. A* 260 (1961) 127.
- [2] T.H.R. Skyrme, *Nucl. Phys.* 31 (1962) 556.
- [3] D.H. Tchrakian, *J. Phys. A* 48 (37) (2015) 375401, arXiv:1505.05344 [hep-th].
- [4] B.M.A.G. Piette, W.J. Zakrzewski, H.J.W. Mueller-Kirsten, D.H. Tchrakian, *Phys. Lett. B* 320 (1994) 294–298.
- [5] Y. Brihaye, C. Herdeiro, E. Radu, D.H. Tchrakian, *J. High Energy Phys.* 11 (2017) 037, arXiv:1710.03833 [gr-qc].
- [6] D.H. Tchrakian, *Lett. Math. Phys.* 40 (1997) 191–201.
- [7] D.H. Tchrakian, Winding number versus Chern-Pontryagin charge, arXiv:hep-th/0204040 [hep-th].
- [8] C.G. Callan Jr., E. Witten, *Nucl. Phys. B* 239 (1984) 161–176.
- [9] T. Tchrakian, *J. Phys. A* 44 (2011) 343001, arXiv:1009.3790 [hep-th].
- [10] F. Navarro-Lérida, E. Radu, D.H. Tchrakian, *Phys. Rev. D* 95 (8) (2017) 085016, arXiv:1612.05835 [hep-th].
- [11] F. Navarro-Lérida, E. Radu, D. Tchrakian, *Phys. Lett. B* 791 (2019) 287–292, arXiv:1811.09535 [hep-th].
- [12] F. Navarro-Lérida, D.H. Tchrakian, *Phys. Rev. D* 99 (4) (2019) 045007, arXiv:1812.03147 [hep-th].
- [13] F. Navarro-Lérida, E. Radu, D. Tchrakian, *Phys. Rev. D* 101 (12) (2020) 125014, arXiv:2003.05899 [hep-th].
- [14] S.K. Paul, A. Khare, *Phys. Lett. B* 174 (1986) 420; S.K. Paul, A. Khare, *Phys. Lett. B* 177 (1986) 453 (Erratum).
- [15] J. Hong, Y. Kim, P.Y. Pac, *Phys. Rev. Lett.* 64 (1990) 2230; R. Jackiw, E.J. Weinberg, *Phys. Rev. Lett.* 64 (1990) 2234.
- [16] P.K. Ghosh, S.K. Ghosh, *Phys. Lett. B* 366 (1996) 199, arXiv:hep-th/9507015; K. Kimm, K.-M. Lee, T. Lee, *Phys. Rev. D* 53 (1996) 4436, arXiv:hep-th/9510141; K. Arthur, D.H. Tchrakian, Y.-s. Yang, *Phys. Rev. D* 54 (1996) 5245.
- [17] A.A. Tseytlin, *Nucl. Phys. B* 501 (1997) 41, arXiv:hep-th/9701125.
- [18] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond*, Cambridge University Press, Cambridge, 1998.
- [19] J.L. Blázquez-Salcedo, J. Kunz, F. Navarro-Lérida, E. Radu, *Phys. Lett. B* 771 (2017) 52–58, arXiv:1703.04163 [gr-qc].
- [20] J.L. Blázquez-Salcedo, J. Kunz, F. Navarro-Lérida, E. Radu, *Phys. Rev. D* 97 (8) (2018) 081502, arXiv:1711.08292 [gr-qc]; J.L. Blázquez-Salcedo, J. Kunz, F. Navarro-Lérida, E. Radu, *J. High Energy Phys.* 02 (2018) 061, arXiv:1711.10483 [gr-qc].