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Another paraconsistent algebraic semantics for Lukasiewicz–Pavelka logic

J. Tinguaro Rodríguez [a,*], Esko Turunen [b,1], Da Ruan [c,2], Javier Montero [a]

[a] *Faculty of Mathematics, Complutense University of Madrid, Madrid, Spain*
[b] *Department of Mathematics, Tampere University of Technology, Tampere, Finland*
[c] *Belgian Nuclear Research Centre (SCK CEN), Mol, Belgium*

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Abstract

As recently proved in a previous work of Turunen, Tsoukiàs and Öztürk, starting from an evidence pair (a, b) on the real unit square and associated with a propositional statement α, we can construct evidence matrices expressed in terms of four values t, f, k, u that respectively represent the logical valuations `true`, `false`, `contradiction` (both true and false) and `unknown` (neither true nor false) regarding the statement α. The components of the evidence pair (a, b) are to be understood as evidence for and against α, respectively. Moreover, the set of all evidence matrices can be equipped with an injective MV-algebra structure. Thus, the set of evidence matrices can play the role of truth-values of a Lukasiewicz–Pavelka fuzzy logic, a rich and applicable mathematical foundation for fuzzy reasoning, and in such a way that the obtained new logic is paraconsistent. In this paper we show that a similar result can be also obtained when the evidence pair (a, b) is given on the real unit triangle. Since the real unit triangle does not admit a natural MV-structure, we introduce some mathematical results to show how this shortcoming can be overcome, and another injective MV-algebra structure in the corresponding set of evidence matrices is obtained. Also, we derive several formulas to explicitly calculate the evidence matrices for the operations associated to the usual connectives.

Keywords: Mathematical fuzzy logic; Paraconsistency; MV-algebras

1. Introduction

Logic and knowledge representation models are closely and necessarily related fields. Since logic studies both the expressive power of formal systems and the principles of valid reasoning, it constitutes a main tool for knowledge representation in order to *translate* the relevant information and knowledge about a reality into a suitable formal language *in an operative way*. Hence, the resulting model enables both an adequate representation of such knowledge (with its uncertainty) and an inference framework powerful enough to transform the available information into relevant

* Corresponding author. Tel.: +34913944679; fax: +34913944406.
E-mail address: jtrodrig@mat.ucm.es (J.T. Rodríguez).

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and reliable conclusions. Similarly, the study and development of the properties of some knowledge representation formalisms often pose challenging problems for logic (e.g. in a fuzzy framework, see [27]).

In this sense, paraconsistent logics provide a significant tool for those areas (e.g. decision theory [21], data mining [24], etc.) in which information inconsistencies can appear. Recall that a relation of logical consequence $\vDash$ is called *paraconsistent* if it is not *explosive*, i.e. if it does not validate $\{A, \neg A\} \models B$ for every A and B. Thus, if a logic is explosive, anything can be inferred from contradictory premises, as it is usual in classical logic as well as in some other explosive non-classical logics, as intuitionistic logic. On the contrary, in a paraconsistent logic the inference relation does not explode into triviality even in circumstances where the available information is inconsistent. In other words, "*paraconsistent logic accommodates inconsistency in a sensible manner that treats inconsistent information as informative*" (cite extracted from the Stanford Encyclopedia of Philosophy [9]).

In [20] Tsoukiàs introduced DDT paraconsistent logic with the aim of providing a first-order language for preference and decision analysis able to cope with information inconsistencies. As these are common in decision problems, for example in the presence of various underlying criteria, the introduction of a paraconsistent formalism allowed Tsoukiàs to generalize Partial Comparability Theory (see [21]), particularly enabling to distinguish different kinds of *incomparability* or *indifference* between alternatives – associated to certain kinds of *inconsistencies* in the preference information.

In [25] a continuous valued paraconsistent Pavelka logic on $[0, 1]^2$ was built upon the DDT language. This was done by showing that, given a pair (a, b) on the unit square $[0, 1]^2$, one can construct (what are called) evidence matrices such that the set of evidence matrices constitute an injective MV-algebra, thus an algebraic semantics for Lukasiewicz–Pavelka-style fuzzy logic. In this paper we show that similar evidence matrices can be obtained also by starting from elements (a, b) on the unit triangle. This results again in an injective MV-algebra structure on the set of evidence matrices; however, the approach in this paper differs from the Turunen et al. approach [25].

Let us briefly recall that Pavelka-style fuzzy sentential logic [15] is a generalization of classical logic in which axioms, theorems and tautologies can be not only fully true but also true to a degree. In this way, concepts such as *fuzzy set of axioms*, *provability degree*, *degree of theoremhood* and *evaluated proof* are allowed. In this sense, Pavelka's logic has been named *fuzzy logic with evaluated syntax* by Novak [12], who also extended Pavelka's ideas on $[0, 1]$-valued zero-order fuzzy logic to first-order fuzzy logic with evaluated syntax. Turunen [22] generalized Pavelka's ideas to another direction by introducing a Pavelka-style fuzzy sentential logic with truth-values on an injective MV-algebra, thus generalizing $[0, 1]$-valued logic. In particular it was proven (see [22]) that Pavelka-style fuzzy sentential logic is complete, in the sense that if the truth-value set L forms an injective MV-algebra, then for all $a \in L$ the set of a-tautologies and the set of a-provable formulae coincide. Furthermore, the usage of Pavelka logic in the context of data mining to provide a suitable framework for the treatment of information inconsistencies has been analyzed in [24]. For further details about Pavelka logic, please refer to [23].

On the other hand, it is necessary to note that DDT is an extension of Belnap's logic [2] (see also [7]), in which four values t, f, k, u are associated to an atomic proposition α attending to both the presence and lack of evidence for and against α: the value t (`told only true` or, for simplicity, `true`) is assigned if there is evidence for α and there is no evidence against it; if there is no evidence for α but there is evidence against it, α obtains the value f (`told only false` or `false`); in case of a simultaneous presence of evidence for and against α, it is assigned the value k (`both told true and false` or just `contradiction`); and finally, when no evidence for nor against α is present, then α obtains the value u (`told neither true nor false` or simply `unknown`). Therefore, in this framework evidence is assessed in a crisp way (i.e. it is measured in the $\{0, 1\}$ binary scale), and thus it is possible to represent these four states by means of the pairs $t = (1, 0)$, $f = (0, 1)$, $k = (1, 1)$ and $u = (0, 0)$.

By allowing the basic evidence assessments to be gradable, a continuous extension of the DDT language was developed in [16] and [14], in such a way that formulae α are assessed through *evidence couples* or pairs $(a(\alpha), b(\alpha)) \in [0, 1]^2$, in which the first element a represents the degree of evidence for α and the second element b represents the degree of evidence against α. To extend the definition of the four states to a continuous setting, the authors devised a set of reasonable conditions to be imposed to such an extension, for example the usage of t-norms as conjunction operators in the computation of the four graded values, or the constraint $t + f + k + u = 1$. In fact, in [14] it is shown that the graded values obtained as

$$t(\alpha) = \min(a, 1-b), \tag{1}$$

$$f(\alpha) = \min(1-a, b), \tag{2}$$

$$k(\alpha) = \max(a+b-1, 0), \tag{3}$$

$$u(\alpha) = \max(1-a-b, 0), \tag{4}$$

constitute the unique solution of the system of functional equations they imposed, subject to the boundary conditions $t(1,0) = 1$, $f(1,0) = k(1,0) = u(1,0) = 0$; $f(0,1) = 1$, $t(0,1) = k(0,1) = u(0,1) = 0$; etc. (in fact, notice the resemblance between formulae (1)–(4) and some fuzzy preference structures in [11] and [5]). In the role of truth-values of the continuous DDT logic are then the *evidence matrices* of the form

$$E(a,b) = \begin{bmatrix} f(\alpha) & k(\alpha) \\ u(\alpha) & t(\alpha) \end{bmatrix}, \tag{5}$$

and thus the set of truth-values is given by the set of evidence matrices

$$M_{[0,1]^2} = \{E(a,b) \mid (a,b) \in [0,1]^2\}. \tag{6}$$

Other approaches that extend the ideas of Belnap to the unit square can be found in [6,13,17–19].

It should be noticed that the above mentioned association [25] of DDT logic and Pavelka sentential logic crucially depends on the fact that the product of two injective MV-algebras (and thus the set of evidence pairs) can also be naturally given an injective MV-algebra structure. In this paper, we show how to overcome this constraint and extend the work in [25] by developing a continuous paraconsistent logic on truth-scales that cannot be viewed as the product of MV-algebras. In this sense, we will focus on the study of possible MV-algebra structures over the unit triangle

$$\mathcal{T} = \{(a,b) \in [0,1]^2 \mid a+b \leqslant 1\},$$

thus introducing an approach to the study of the logical properties of this scale different from but complementary to that of triangle algebras considered in [26].

At this point, it is relevant to remind that $\mathcal{T}$ is usually equipped with a lattice structure through the partial order $\leqslant_t$ defined by

$$x \leqslant_t y \quad \Leftrightarrow \quad x_1 \leqslant y_1 \text{ and } x_2 \geqslant y_2,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{T}$. Furthermore, in [3] it is proven that the lattice $(\mathcal{T}, \leqslant_t)$ cannot be given an MV-algebra structure if we use t-norms (and thus also t-conorms) as logical operators. Nevertheless, it does not mean that the set $\mathcal{T}$ cannot be endowed with an MV-algebra structure. In fact, as we shall see in next section, different MV-algebra structures can be achieved for $\mathcal{T}$, though then the canonical order of such an MV-algebra cannot coincide with $\leqslant_t$ (see Remarks 2 and 3 and Section 3.3 for a further discussion on this issue).

Before to proceed, let us observe that formulae (1)–(4) can also be applied on $\mathcal{T}$. That is, the functions $t, f, k, u : [0,1]^2 \to [0,1]$ are well defined when restricted to $\mathcal{T} \subset [0,1]^2$, and naturally produce a continuous extension of the crisp DDT logic, based on the unit triangle $\mathcal{T}$ instead of $[0,1]^2$. Note that the boundary conditions for t, f, u in the vertices of $\mathcal{T}$ as well as the equation $t + f + k + u = 1$ trivially hold since $\mathcal{T} \subset [0,1]^2$, but also that the value `contradiction` is excluded since $k \equiv 0$ in $\mathcal{T}$, as it follows from expression (3) and the constraint $x + y \leqslant 1$ introduced in the definition of $\mathcal{T}$. Therefore, though a formula α could still be assigned the label `unknown`, it cannot obtain the value `contradiction` for evidence pairs on $\mathcal{T}$. This has even lead some authors (see [1,4]) to define $\mathcal{T}$ as the subset of non-inconsistent evidence pairs in $[0,1]^2$. Evidence matrices associated with this extension on $\mathcal{T}$ are then given by

$$E(a,b) = \begin{bmatrix} f(\alpha) & k(\alpha) \equiv 0 \\ u(\alpha) & t(\alpha) \end{bmatrix}, \tag{7}$$

and the correspondent set of evidence matrices is denoted by

$$M_{\mathcal{T}} = \{E(a,b) \mid (a,b) \in \mathcal{T}\}. \tag{8}$$

To conclude this introduction, let us briefly describe another extension on T, which will be useful later on this paper and which is closely related to the previous extension. It is straightforward to check that the graded values $T, F, U, K : \mathcal{T} \to [0,1]$, obtained as

$$T(\alpha) = \max(a - b, 0), \tag{9}$$
$$F(\alpha) = \max(b - a, 0), \tag{10}$$
$$K(\alpha) = 2\min(a, b), \tag{11}$$
$$U(\alpha) = 1 - (a + b), \tag{12}$$

can be easily derived from the extension (1)–(4) when applied on $\mathcal{T}$ through the relationships

$$T = \max(t - f, 0), \tag{13}$$
$$F = \max(f - t, 0), \tag{14}$$
$$K = 2\min(t, f), \tag{15}$$
$$U = u. \tag{16}$$

In fact, expressions (13)–(16) define a bijection between the extensions (1)–(4) and (9)–(12) when applied on $\mathcal{T}$, i.e. a one-to-one correspondence between the sets of evidence matrices $M_{\mathcal{T}}$ and

$$\mathbf{M} = \left\{ M(a,b) = \begin{bmatrix} F(\alpha) & K(\alpha) \\ U(\alpha) & T(\alpha) \end{bmatrix},\ (a,b) \in \mathcal{T} \right\}. \tag{17}$$

In other words, when restricted to $\mathcal{T}$, both extensions, that on (1)–(4) and the one in (9)–(12), are basically the same thing, only representing different (though equivalent) ways of organizing the information contained in the evidence pairs $(a, b) \in \mathcal{T}$.

Effectively, since $u' = u = 1 - (a + b)$ in $\mathcal{T}$, both extensions only differ in the way the remaining amount of intensity $a + b$ is distributed between the other three values (t, f, k and/or T, F, K), so the equations $t + f + k + u = 1$ and $T + F + U + K = 1$ hold for all (a, b) in $\mathcal{T}$. In fact, in expressions (1)–(4) k is set to 0 and the amount $a + b$ is distributed among t and f (in fact $t = a$ and $f = b$ in $\mathcal{T}$), while in (9)–(12) T and F are associated to the excesses of evidence for and against, i.e. to the differences $a - b$ and $b - a$ truncated by 0 (thus T and F are mutually exclusive, i.e. $T > 0 \Rightarrow F = 0$ and conversely). Then, K is the total amount of intensity of evidence for and against that does not contribute to the differences T and F, i.e. the remaining amount up to $a + b$, which is equal to $2a$ if $a \leqslant b$ and to $2b$ if $b \leqslant a$. Hence, K is twice the minimum of a and b. Therefore, the extension (9)–(12) verifies the same boundary conditions as (1)–(4) in the vertices of the triangle, but in this case it also holds that $K(\frac{1}{2}, \frac{1}{2}) = 1$, $T(\frac{1}{2}, \frac{1}{2}) = F(\frac{1}{2}, \frac{1}{2}) = U(\frac{1}{2}, \frac{1}{2}) = 0$. As exposed above, expressions (9)–(12) are just another way of looking on the extension (1)–(4), providing an equivalent extension (with a closely related semantics) of DDT logic on $\mathcal{T}$, in which the value `contradiction` is not formally excluded. For this reason, we will use $\mathbf{M}$ as the valuation space for a new formal paraconsistent logic on $\mathcal{T}$.

The remainder of this paper is organized as follows: in Section 2, after some preliminaries to introduce concepts and notation of MV-algebras, we provide the main mathematical results allowing to consider $\mathbf{M}$ (or equivalently $M_{\mathcal{T}}$) as the valuation space of a paraconsistent Pavelka-style fuzzy sentential logic (i.e. as an injective MV-algebra). Section 3 is devoted to describing more in depth the MV-algebra structures obtained in $\mathcal{T}$ and $\mathbf{M}$, particularly the operations representing the different connectives are studied as well as the lattice structure arising from the canonical order of such MV-algebras. Finally, some concluding remarks are exposed in Section 4.

2. Mathematical results

This section is devoted to present the results that allow developing a paraconsistent Pavelka-style fuzzy sentential logic based on the unit triangle T. As commented above, this entails proving that the set $\mathbf{M}$ of evidence matrices obtained through a one-to-one extension function $M : \mathcal{T} \to \mathbf{M}$ (as the given by expressions (1)–(4) or (9)–(12)) has an injective MV-algebra structure.

2.1. Preliminaries

We start by giving the correspondent definitions and some basics facts about MV-algebras. A more detailed introduction to formal fuzzy logics can be found in [8,23].

Definition 1. Given a set L and an element $\mathbf{0} \in L$, an *MV-algebra* $(L, \oplus, *, \mathbf{0})$ is a structure such that $(L, \oplus, \mathbf{0})$ is a commutative monoid, i.e., such that for all $x, y, z \in L$ it holds that

$$x \oplus y = y \oplus x, \tag{18}$$

$$(x \oplus y) \oplus z = x \oplus (y \oplus z), \tag{19}$$

$$x \oplus \mathbf{0} = x, \tag{20}$$

and also verifying

$$x^{**} = x, \tag{21}$$

$$x \oplus \mathbf{0}^* = \mathbf{0}^*, \tag{22}$$

$$\left(x^* \oplus y\right)^* \oplus y = \left(y^* \oplus x\right)^* \oplus x. \tag{23}$$

By denoting $x \odot y = (x^* \oplus y^*)^*$ and $\mathbf{1} = \mathbf{0}^*$, it follows that $(L, \odot, *, \mathbf{1})$ is also an MV-algebra, which is known as the *dual* MV-algebra of $(L, \oplus, *, \mathbf{0})$. Clearly, it also holds that $x \oplus y = (x^* \odot y^*)^*$, so the triple $(\oplus, *, \odot)$ verifies the De Morgan laws. Notice also that it is possible to derive a *lattice* structure from that of MV-algebra, since

$$x \leqslant_L y \quad \text{iff} \quad x^* \oplus y = \mathbf{1} \quad \text{iff} \quad x \odot y^* = \mathbf{0} \tag{24}$$

defines a partial order on L, in such a way that $(L, \leqslant_L, \wedge, \vee)$ is a lattice with the meet and join operations respectively given, for all $x, y \in L$, by

$$x \wedge y = \left(x^* \odot y\right)^* \odot y, \tag{25}$$

$$x \vee y = \left(x^* \oplus y\right)^* \oplus y. \tag{26}$$

Also, by stipulating

$$x \rightarrow y = x^* \oplus y, \tag{27}$$

the structure $(L, \leqslant_L, \wedge, \vee, \odot, \rightarrow, \mathbf{0}, \mathbf{1})$ is a residuated lattice with bottom and top elements $\mathbf{0}$ and $\mathbf{1}$, respectively.

The most typical instance of MV-algebra is the *standard Lukasiewicz structure* $([0, 1], \oplus, *, 0)$, in which the valuation space is the unit interval $[0, 1]$, and for all $x, y \in [0, 1]$ the operations $\oplus$ and $*$ are given by $x \oplus y = \min\{x + y, 1\}$ (and thus $x \odot y = \max\{x + y - 1, 0\}$) and $x^* = 1 - x$. Furthermore, it is $\wedge = \min$ and $\vee = \max$.

Given an MV-algebra $(L, \oplus, *, \mathbf{0})$, it is immediate to extend its structure to the product $L_{EC} = L \times L$ through the operations

$$(x_1, x_2) \oplus_{EC} (y_1, y_2) = (x_1 \oplus y_1, x_2 \odot y_2), \tag{28}$$

$$(x_1, x_2)^{*_{EC}} = \left(x_1^*, x_2^*\right), \tag{29}$$

$$\mathbf{0}_{EC} = (\mathbf{0}, \mathbf{1}). \tag{30}$$

Thus, $(L_{EC}, \oplus_{EC}, *_{EC}, \mathbf{0}_{EC})$ is also an MV-algebra, in fact it is known as the *MV-algebra of evidence couples induced by L* [25]. Therefore, the unit square $[0, 1]^2$ is easily given an MV-algebra structure as the MV-algebra of evidence pairs of the Lukasiewicz structure $\Gamma = ([0, 1], \oplus, *, 0)$.

Moreover, given a pair $x = (x_1, x_2) \in L_{EC}$, the authors of [25] defined the *evidence matrix* associated to such a pair as the matrix

$$m(x) = \begin{bmatrix} x_1^* \wedge x_2 & x_1 \odot x_2 \\ x_1^* \odot x_2^* & x_1 \wedge x_2^* \end{bmatrix}, \tag{31}$$

and the *set of evidence matrices* as

$$M_{EC} = \left\{m(x) \mid x = (x_1, x_2) \in L_{EC}\right\}. \tag{32}$$

Furthermore, in [25] the existence of a one-to-one correspondence between evidence pairs and evidence matrices, i.e. between L_{EC} and M_{EC}, is proved. Finally, the same authors proved that it is possible to transfer the MV-algebra structure from L_{EC} to M_{EC} by means of the following definitions:

$$m(x) \oplus_m m(y) = m(x \oplus_{EC} y), \tag{33}$$

$$m(x)^{*_m} = m\left(x^{*_{EC}}\right), \tag{34}$$

$$\mathbf{0}_m = m(\mathbf{0}_{EC}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{35}$$

Therefore, the MV-algebra structure descends from the base valuations set $(L, \oplus, *, \mathbf{0})$ to the evidence matrix set $(M_{EC}, \oplus_m, *_m, \mathbf{0}_m)$ through the set of evidence pairs $(L_{EC} = L \times L, \oplus_{EC}, *_{EC}, \mathbf{0}_{EC})$.

Remark 1. Note that if the base MV-algebra L is the Lukasiewicz structure Γ, then M_{EC} coincides with the set

$$M_{[0,1]^2} = \left\{E(a,b) \,\middle|\, (a,b) \in [0,1]^2\right\} \tag{36}$$

obtained through expressions (1)–(4). That is, the continuous extension of DDT logic in (1)–(4) is expressible in terms of the MV-algebra operations of the Lukasiewicz structure $([0,1], \oplus, *, 0)$. Moreover, note that the extension in (9)–(12) is also expressible in terms of Lukasiewicz connectives, since it holds that

$$M(a,b) = \begin{bmatrix} a^* \odot b & (a \wedge b) \oplus (a \wedge b) \\ a^* \odot b^* & a \odot b^* \end{bmatrix}. \tag{37}$$

Definition 2. An MV-algebra $(L, \oplus, *, \mathbf{0})$ is *complete* whenever the infimum and supremum of every subset of L belong to L, i.e., if $\wedge\{a_i \mid i \in I\}, \vee\{a_i \mid i \in I\} \in L$ for any subset $\{a_i \mid i \in I\}$ of L.

Definition 3. An MV-algebra $(L, \oplus, *, \mathbf{0})$ is *divisible* if for all $x \in L$ and any natural number n, there exists an element $a \in L$, known as the N-*divisor* of x, such that $n \cdot a = \underbrace{a \oplus \cdots \oplus a}_{n \text{ times}} = x$ and $(x^* \oplus (n-1)a)^* = a$. Furthermore, an MV-algebra $(L, \oplus, *, \mathbf{0})$ is *injective* if it is complete and divisible.

Note that n-divisors, if they exist, are always unique [10]. The Lukasiewicz structure is an example of injective MV-algebra. Moreover, any finite product of injective MV-algebras is also injective. Therefore, the set of evidence pairs of an injective MV-algebra is also an injective MV-algebra, and in [25] it is proven that this property can also be transferred to the MV-algebra of evidence matrices. As a consequence, it is possible to associate the continuous extension of DDT logic $M_{[0,1]^2}$ proposed in [14,16] with a Pavelka-style fuzzy sentential logic exhibiting the paraconsistent semantics of the DDT logic.

Therefore, in order to build an injective MV-algebra structure on the set of evidence matrices M_{EC} it is crucial to verify that the set of evidence pairs L_{EC} also has such a structure. The reasoning in [25] was based on such an assumption. In the case of the extension $M_{[0,1]^2}$, this step is straightforward as its set of evidence couples is the unit square $[0,1]^2$, i.e., the product MV-algebra of the Lukasiewicz structure.

However, this is not the case for the unit triangle $\mathcal{T}$, since it cannot be regarded as the product of any MV-algebra L and its dual. Moreover, the negation $*_{EC}$ in $[0,1]^2$, given by $(x_1, x_2)^{*_{EC}} = (x_1^*, x_2^*) = (1 - x_1, 1 - x_2)$, is not well defined on $\mathcal{T}$ as, for example, $(0,0)^{*_{EC}} = (1,1) \notin \mathcal{T}$. This forces not only to search for a different complementation $*$, but also for different operations $\oplus$ and $\odot$, since the Lukasiewicz ones only fulfill (23) with $*_{EC}$. Furthermore, in [3] it is shown that a t-conorm cannot be used as the operation $\oplus$ in order to build an MV-algebra structure over the lattice $(\mathcal{T}, \leqslant_t)$. Particularly, either the order $\leqslant_t$ or the connection between the operation $\oplus$ and such an order (i.e. the $\leqslant_t$-increasing condition of t-norms and t-conorms) has to be discarded in order to obtain a potential MV-algebra $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$ on the unit triangle.

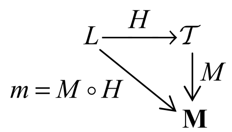

Fig. 1. The set of evidence matrices $\mathbf{M}$ can be regarded as an extension on L if a bijective mapping H is provided.

2.2. Main results

To overcome these difficulties and obtain the desired injective MV-algebra structure, in this paper we follow a different approach, based on showing that set $\mathbf{M}$ of evidence matrices of a continuous extension of DDT logic on $\mathcal{T}$ can also be regarded as an extension on $[0,1]^2$. In this way, it is possible to take advantage of the results in [25] to assure that the set $\mathbf{M}$ has the desired structure. Implicitly, this will also allow to obtain an MV-algebra structure on $\mathcal{T}$ as well as a new extension on $[0,1]^2$, which will be discussed later on. Our basic observation is that, given the extension function $M:\mathcal{T}\to\mathbf{M}$ that assigns an evidence matrix $M(x)\in\mathbf{M}$ to each $x\in\mathcal{T}$ and a mapping $H:L\to\mathcal{T}$ from an MV-algebra $(L,\oplus,*,\mathbf{0})$ into $\mathcal{T}$, it is also possible to assign evidence matrices in $\mathbf{M}$ to the elements of L through the composition $m=M\circ H$, as it is shown in Fig. 1. If it is possible to construct this mapping m to be bijective, then a one-to-one correspondence between elements in L and matrices in $\mathbf{M}$ would be available, which would be the same as to say that $\mathbf{M}$ also constitutes a continuous extension over L. However, the bijectivity of m will follow from that of H, as it is easy to proof that the extension function M is one-to-one. In fact, this is direct in the case of the extension (1)–(4), as it is $t(\alpha)=a$, $f(\alpha)=b$ for all $(a,b)\in\mathcal{T}$. The next results show that the extension function M arising from expressions (9)–(12) is also bijective.

Proposition 1. *Let $v=(a,b)$ and $s=(c,d)$ be two evidence pairs in $\mathcal{T}$ such that*

$$M(a,b)=\begin{bmatrix} a^*\odot b & 2(a\wedge b)\\ a^*\odot b^* & a\odot b^*\end{bmatrix}=\begin{bmatrix} f & k\\ u & t\end{bmatrix},\qquad M(c,d)=\begin{bmatrix} c^*\odot d & 2(c\wedge d)\\ c^*\odot d^* & c\odot d^*\end{bmatrix}=\begin{bmatrix} F & K\\ U & T\end{bmatrix}.$$

If $M(a,b)=M(c,d)$, then $v=s$, that is, $a=c$ and $b=d$.

Proof. Suppose first that $a\geqslant b$. If $M(a,b)=M(c,d)$, then $t=a\odot b^*=a-b=T=c\odot d^*\geqslant 0$, hence $c\geqslant d$ and $a-b=c-d$. Moreover, $\frac{1}{2}k=a\wedge b=b=\frac{1}{2}K=c\wedge d=d$, and thus $a=c$ and $b=d$. If $b>a$, using $f=F$ it follows that $d>c$ and $b-a=d-c$, and then, taking into account that $k=K$, it holds again that $a=c$ and $b=d$. □

Proposition 2. *If an evidence matrix*

$$\begin{bmatrix} f & k\\ u & t\end{bmatrix}=\begin{bmatrix} a^*\odot b & 2(a\wedge b)\\ a^*\odot b^* & a\odot b^*\end{bmatrix}$$

is given, then $(a,b)=(t\oplus\frac{1}{2}k,\ f\oplus\frac{1}{2}k)\in\mathcal{T}$ is the evidence pair associated to such a matrix.

Proof. Firstly, note that $t\oplus\frac{1}{2}k=\max(a-b,0)+\min(a,b)$, since $\max(a-b,0)+\min(a,b)\leqslant 1$ for any $(a,b)\in\mathcal{T}$, as it holds that $a+b\leqslant 1$. Thus, if $a\leqslant b$, it holds that $t\oplus\frac{1}{2}k=0+a=a$, and if $a>b$ it then follows that $t\oplus\frac{1}{2}k=a-b+b=a$. Hence, in either case it is $t\oplus\frac{1}{2}k=a$. The proof of $f\oplus\frac{1}{2}k=b$ can be carried out in a symmetric way. □

Hence, m is bijective if and only if H is bijective. Therefore, if an MV-algebra $(L,\oplus,*,\mathbf{0})$ and a bijective mapping $H:L\to\mathcal{T}$ are given, then it is possible to define on $\mathbf{M}$ operations $\oplus_{\mathbf{M}}$, $*_{\mathbf{M}}$ and neutral element $\mathbf{0}_{\mathbf{M}}$ in the following way:

$$m(x)\oplus_{\mathbf{M}} m(y)=m(x\oplus y), \tag{38}$$

$$\big(m(x)\big)^{*_{\mathbf{M}}}=m\big(x^*\big), \tag{39}$$

$$\mathbf{0}_{\mathbf{M}}=m(\mathbf{0}), \tag{40}$$

with $x, y \in L$. Furthermore, the set $\mathbf{M}$ equipped with these operations has the desired injective MV-algebra structure if L also has such a structure:

Proposition 3. *Let $(L, \oplus, *, \mathbf{0})$ be an MV-algebra, and let $H : L \to \mathcal{T}$ be a bijective mapping. Then, $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is an MV-algebra with the operations $\oplus_\mathbf{M}, *_\mathbf{M}$ and the neutral element $\mathbf{0_M}$ defined as shown in* (38)–(40).

Proof. Since m is a bijection between L and $\mathbf{M}$, the operations $\oplus_\mathbf{M}, *_\mathbf{M}$ and the neutral element $\mathbf{0_M}$ are well defined. Furthermore, it is obvious that the structure $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ fulfills properties (18)–(23) if they also hold for the structure $(L, \oplus, *, \mathbf{0})$. For example, for all $M_1, M_2 \in \mathbf{M}$ it holds that

$$\begin{aligned}\left(M_1^{*\mathbf{M}} \oplus_\mathbf{M} M_2\right)^{*\mathbf{M}} \oplus_\mathbf{M} M_2 &= \left(m\left(x^*\right) \oplus_\mathbf{M} m(y)\right)^{*\mathbf{M}} \oplus_\mathbf{M} m(y)\\ &= m\left(\left[x^* \oplus y\right]^*\right) \oplus_\mathbf{M} m(y) = m\left(\left[x^* \oplus y\right]^* \oplus y\right) = m\left(\left[y^* \oplus x\right]^* \oplus x\right)\\ &= m\left(\left[y^* \oplus x\right]^*\right) \oplus_\mathbf{M} m(x) = \left(m\left(y^*\right) \oplus_\mathbf{M} m(x)\right)^{*\mathbf{M}} \oplus_\mathbf{M} m(x)\\ &= \left(M_2^{*\mathbf{M}} \oplus_\mathbf{M} M_1\right)^{*\mathbf{M}} \oplus_\mathbf{M} M_1 \quad \left(\text{condition (23)}\right).\end{aligned}$$

Thus, $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is an MV-algebra. □

Therefore, any bijection from an MV-algebra $(L, \oplus, *, \mathbf{0})$ into $\mathcal{T}$ will lead to obtain an MV-algebra structure for $\mathbf{M}$. Note that the construction above could be repeated in the opposite direction, and thus $(L, \oplus, *, \mathbf{0})$ is an MV-algebra if $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is. Notice also that in this last structure the order relation obtained through formula (24) is given by

$$M_1 \leqslant_\mathbf{M} M_2 \quad \Leftrightarrow \quad M_1^{*\mathbf{M}} \oplus_\mathbf{M} M_2 = \mathbf{1_M} \quad \Leftrightarrow \quad m\left(x^* \oplus y\right) = m(\mathbf{1}) \quad \Leftrightarrow \quad x^* \oplus y = \mathbf{1} \quad \Leftrightarrow \quad x \leqslant_L y,$$

where $M_1 = m(x)$, $M_2 = m(y)$ for some $x, y \in L$. Moreover, the meet and join operations $\wedge$ and $\vee$ of L can be easily transferred to $\mathbf{M}$ by means of the expressions

$$M_1 \wedge_\mathbf{M} M_2 = m(x \wedge y) \quad \text{and} \quad M_1 \vee_\mathbf{M} M_2 = m(x \vee y).$$

Furthermore, a similar reasoning holds for the property of *divisibility* of the MV-algebra $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$, since the n-divisor of a matrix $M = m(x) \in \mathbf{M}$ is the matrix given by the n-divisor a of x, i.e., the matrix $A = m(a)$, as it is shown in the next proposition:

Proposition 4. *In the hypothesis of Proposition* 3 *above, $(L, \oplus, *, \mathbf{0})$ is complete if and only if $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is complete. Moreover, $(L, \oplus, *, \mathbf{0})$ is divisible if and only if $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is divisible.*

Proof. To see that $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is complete, it is enough to define the infimum and supremum of an arbitrary family of elements of $\mathbf{M}$ by means of the corresponding operations on $(L, \oplus, *, \mathbf{0})$, i.e.,

$$\wedge_\mathbf{M}\{M_i \mid i \in I\} = m\left(\wedge\{x_i \mid i \in I\}\right) \quad \text{and} \quad \vee_\mathbf{M}\{M_i \mid i \in I\} = m\left(\vee\{x_i \mid i \in I\}\right),$$

whenever $M_i = m(x_i)$, $x_i \in L$ for all $i \in I$. Then, $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is complete if and only if $(L, \oplus, *, \mathbf{0})$ is complete.

To see that $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is divisible, let M be an element of $\mathbf{M}$. Then there is $x \in L$ such that $M = m(x)$. If $(L, \oplus, *, \mathbf{0})$ is divisible, then there exists a (unique) $a \in L$ such that $n \cdot a = x$ and $(x^* \oplus (n-1)a)^* = a$. Therefore, it is $M = m(x) = m(n \cdot a) = n \cdot m(a) = n \cdot A$, and $(M^{*\mathbf{M}} \oplus_\mathbf{M} (n-1)A)^{*\mathbf{M}} = m([x^* \oplus (n-1)a]^*) = m(a) = A$, so $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is divisible. As this construction can be repeated in the opposite sense, it clearly holds that $(L, \oplus, *, \mathbf{0})$ is divisible if and only if $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is divisible. □

The following consequence is then immediate:

Corollary 1. *Let $H : L \to \mathcal{T}$ be a bijective mapping. Then, $(\mathbf{M}, \oplus_\mathbf{M}, *_\mathbf{M}, \mathbf{0_M})$ is an injective MV-algebra if and only if $(L, \oplus, *, \mathbf{0})$ is an injective MV-algebra.*

Remark 2. It is important to notice that through Propositions 3 and 4 we have implicitly proved that $\mathcal{T}$ can also be endowed with an injective MV-algebra structure. In fact, to prove that $(\mathbf{M}, \oplus_{\mathbf{M}}, *_{\mathbf{M}}, \mathbf{0}_{\mathbf{M}})$ is an injective MV-algebra we have only used the assumption that a bijective mapping $m : L \to \mathbf{M}$ between an MV-algebra $(L, \oplus, *, \mathbf{0})$ and $\mathbf{M}$ can be obtained. Effectively, once m is assured to be bijective, the fact that $H : L \to \mathcal{T}$ is also bijective is never used again, but all the reasoning in the proofs of those results is based only on the bijectivity of m. Therefore, if a bijective mapping $H : L \to \mathcal{T}$ is available, it is then possible to repeat the steps in Propositions 3 and 4 in order to build an injective MV-algebra structure on $\mathcal{T}$. Particularly, given an injective MV-algebra $(L, \oplus, *, \mathbf{0})$ and a bijective mapping $H : L \to \mathcal{T}$, it is possible to show that the operations $\oplus_{\mathcal{T}}$, $*_{\mathcal{T}}$ and the element $\mathbf{0}_{\mathcal{T}}$ given by

$$v \oplus_{\mathcal{T}} w = H\big(H^{-1}(v) \oplus H^{-1}(w)\big) \quad \forall v, w \in \mathcal{T}, \tag{41}$$

$$v^{*_{\mathcal{T}}} = H\big(\big[H^{-1}(v)\big]^*\big) \quad \forall v \in \mathcal{T}, \tag{42}$$

$$\mathbf{0}_{\mathcal{T}} = H(\mathbf{0}), \tag{43}$$

define an injective MV-algebra structure $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$.

Remark 3. Is $\oplus_{\mathcal{T}}$ (or the correspondent product $\odot_{\mathcal{T}}$) a t-conorm (resp. a t-norm)? With respect to the order $\leqslant_{\mathcal{T}}$ obtained through (24), $\oplus_{\mathcal{T}}$ ($\odot_{\mathcal{T}}$) is increasing and thus is a t-conorm (a t-norm) on the correspondent lattice $(\mathcal{T}, \leqslant_{\mathcal{T}})$. However, this order cannot coincide with the usual order $\leqslant_t$ of $\mathcal{T}$, as follows from the results in [3] previously referred. Therefore, $\oplus_{\mathcal{T}}$ ($\odot_{\mathcal{T}}$) is not a t-conorm (a t-norm) with respect to the order $\leqslant_t$. Or, equivalently, though the lattice $(\mathcal{T}, \leqslant_t)$ cannot be associated with an MV-algebra, such a structure is indeed possible on other lattices $(\mathcal{T}, \leqslant_{\mathcal{T}})$. In this respect, some authors have defined and studied the notion of *triangle algebra* (see [26]), which discards the MV-algebra structure and analyzes the properties of the structure resulting from using t-norms as logical operations on the lattice $(\mathcal{T}, \leqslant_t)$. In this sense, in [26] it is shown that these triangle algebras retain some of the main properties of MV-algebras, and therefore the former constitute interesting structures for the development of a formal logic on $(\mathcal{T}, \leqslant_t)$. However, instead to study possible structures on the lattice $(\mathcal{T}, \leqslant_t)$, in this paper we implicitly study possible lattices $(\mathcal{T}, \leqslant_{\mathcal{T}})$ compatible with an MV-algebra structure, and therefore a different approach to the study of the unit triangle $\mathcal{T}$, complementary to that of triangle algebras, is presented here.

Remark 4. Now we show an example of such a bijective mapping H proving the existence of MV-algebras on $\mathbf{M}$ and $\mathcal{T}$. Therefore, in the subsequent we will consider $L = [0,1]^2$ equipped with the MV-algebra structure of the set of evidence pairs of the Lukasiewicz structure $([0,1], \oplus, *, 0)$. That is, the components of the MV-algebra $(L = [0,1]^2, \oplus_{[0,1]^2}, *_{[0,1]^2}, \mathbf{0}_{[0,1]^2})$, obtained through formulae (28)–(30), are given by

$$x \oplus_{[0,1]^2} y = (x_1 \oplus y_1, x_2 \odot y_2) = \big(\min(x_1 + y_1, 1), \max(x_2 + y_2 - 1, 0)\big), \tag{44}$$

$$x^{*_{[0,1]^2}} = \big(x_1^*, x_2^*\big) = (1 - x_1, 1 - x_2), \tag{45}$$

$$\mathbf{0}_{[0,1]^2} = (0, 1), \tag{46}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in [0,1]^2$. Notice that the order $\leqslant_{[0,1]^2}$ defined through (24) coincides with the usual order $\leqslant_t$ on $[0,1]^2$. The standard Lukasiewicz product $\odot_{[0,1]^2}$ and implication $\to_{[0,1]^2}$ on $[0,1]^2$ are then given by

$$x \odot_{[0,1]^2} y = \big(x^* \oplus_{[0,1]^2} y^*\big)^* = (x_1 \odot y_1, x_2 \oplus y_2), \tag{47}$$

$$x \to y = x^{*_{[0,1]^2}} \oplus_{[0,1]^2} y = \big(x_1^* \oplus y_1, x_2^* \odot y_2\big). \tag{48}$$

As explained above, under these conditions $([0,1]^2, \oplus_{[0,1]^2}, *_{[0,1]^2}, \mathbf{0}_{[0,1]^2})$ is an injective MV-algebra. Consider now the mapping $H : [0,1]^2 \to \mathcal{T}$ given by the expression

$$H(x) = \left(x_1 - \frac{1}{2}(x_1 \wedge x_2), x_2 - \frac{1}{2}(x_1 \wedge x_2)\right) \tag{49}$$

with inverse $H^{-1} : \mathcal{T} \to [0,1]^2$ given by

$$H^{-1}(v) = \big(v_1 + (v_1 \wedge v_2), v_2 + (v_1 \wedge v_2)\big), \tag{50}$$

for all $x = (x_1, x_2) \in [0, 1]^2$ and $v = (v_1, v_2) \in \mathcal{T}$. It is straightforward to see that H is bijective. Therefore, we can apply Corollary 1 to conclude that both $(\mathbf{M}, \oplus_{\mathbf{M}}, *_{\mathbf{M}}, \mathbf{0}_{\mathbf{M}})$, as defined through formulae (38)–(40), and $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$, as defined by (41)–(43), are also injective MV-algebras.

3. On the triangle and its evidence matrices MV-algebra structures

This section is devoted to describe more in depth the MV-algebra structures obtained through the results previously exposed. Particularly, we show explicitly the relevant operations of the MV-algebras $(\mathbf{M}, \oplus_{\mathbf{M}}, *_{\mathbf{M}}, \mathbf{0}_{\mathbf{M}})$ and $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$ implicitly defined in last section. The main tool for this task will be the commutative diagram of Fig. 1, that allow to transfer the operations from the MV-algebra $([0, 1]^2, \oplus_{[0,1]^2}, *_{[0,1]^2}, \mathbf{0}_{[0,1]^2})$ of evidence couples of the standard Lukasiewicz structure $([0, 1], \oplus, *, 0)$ to the unit triangle $\mathcal{T}$ and its set of evidence matrices $\mathbf{M}$. Furthermore, we also describe the lattice structure obtained on these sets by means of expression (24).

3.1. Operations on M

In last section, an injective MV-algebra structure was defined on the set of evidence matrices $\mathbf{M}$ through the mapping m, which was constructed as the composition of the bijection H from $[0, 1]^2$ into $\mathcal{T}$ and the extension function M. The following result describes m explicitly:

Proposition 5. *Let $(a, b) \in \mathcal{T}$ be an evidence pair, and denote $x = a + (a \wedge b)$ and $y = b + (a \wedge b)$. Then, $(x, y) \in [0, 1]^2$ and the following equalities hold*:

(i) $x^* \odot y = a^* \odot b$,
(ii) $x \wedge y = 2(a \wedge b)$,
(iii) $x^* \wedge y^* = a^* \odot b^*$,
(iv) $x \odot y^* = a \odot b^*$.

Proof. As $a + b \leqslant 1$, it is obvious that $(x, y) \in [0, 1]^2$. Point (i) follows by noting that $x^* \odot y = \max(1 - a - (a \wedge b) + b + (a \wedge b) - 1, 0) = \max(b - a, 0) = a^* \odot b$. The reasoning for point (iv) is symmetrical. To see points (ii) and (iii), assume first that $a \leqslant b$. Then $x \wedge y = 2a \wedge (a + b) = 2a = 2(a \wedge b)$, and $x^* \wedge y^* = 1 - a - b = a^* \odot b^*$. If $a > b$, an identical reasoning can be carried out, which completes the proof. □

Therefore, each pair $(a, b) \in \mathcal{T}$ has the same associated evidence matrix as the pair $(x, y) \in [0, 1]^2$. That is, the evidence matrix defined for (x, y) by the extension function m in Fig. 1 of Section 2.2 and the evidence matrix defined for (a, b) through expressions (9)–(12) (or equivalently through the extension function M) are the same matrix, which we will denote by

$$M_1 = M(a, b) = m(x, y) = \begin{bmatrix} f & k \\ u & t \end{bmatrix}.$$

It is important to notice that, as defined in Proposition 5, the extension function m is bijective, and thus it is possible to obtain its inverse m^{-1}. In fact, note that it is $x = t + k$ and $y = f + k$. This result then enables a simple method to get explicit expressions for the operations of the MV-algebra $(\mathbf{M}, \oplus_{\mathbf{M}}, *_{\mathbf{M}}, \mathbf{0}_{\mathbf{M}})$ in terms of the components of the evidence matrices. Particularly, from expressions (38)–(40) it is obtained

$$M_1^{*\mathbf{M}} = \begin{bmatrix} f & k \\ u & t \end{bmatrix}^{*\mathbf{M}} = m\big((t + k)^*, (f + k)^*\big) = \begin{bmatrix} (t + k) \odot (f + k)^* & (t + k)^* \wedge (f + k)^* \\ (t + k) \wedge (f + k) & (t + k)^* \odot (f + k) \end{bmatrix} = \begin{bmatrix} t & u \\ k & f \end{bmatrix}, \quad (51)$$

as it holds that

$$(t + k) \odot (f + k)^* = \max(t + k + 1 - f - k - 1, 0) = \max(t - f, 0) = t,$$
$$(t + k) \wedge (f + k) = k,$$
$$(t + k)^* \wedge (f + k)^* = 1 - t - k - f = u,$$
$$(t + k)^* \odot (f + k) = \max(f - t, 0) = f,$$

since t and f are mutually exclusive (i.e., either $f=0$, $t \geqslant 0$ or $t=0$, $f \geqslant 0$ hold), and thus, for example, it is $\min(t+k, f+k) = t+k = k$ if $t=0$ and $\min(t+k, f+k) = f+k = k$ if $f=0$. Similarly, $\min(1-t-k, 1-f-k) = 1-t-f-k = u$ since $t+f+k+u=1$.

Also, if we consider another evidence pair $(c,d) \in \mathcal{T}$, for which we can denote $w = c + (c \wedge d)$, $z = d + (c \wedge d)$ and

$$M_2 = M(c,d) = m(w,z) = \begin{bmatrix} F & K \\ U & T \end{bmatrix},$$

it is then straightforward to obtain

$$M_1 \oplus_{\mathbf{M}} M_2 = \begin{bmatrix} f & k \\ u & t \end{bmatrix} \oplus_{\mathbf{M}} \begin{bmatrix} F & K \\ U & T \end{bmatrix} = m\big((t+k) \oplus (T+K), (f+k) \odot (F+K)\big)$$
$$= \begin{bmatrix} [(t+k) \oplus (T+K)]^* \odot [(f+k) \odot (F+K)] & [(t+k) \oplus (T+K)] \wedge [(f+k) \odot (F+K)] \\ [(t+k) \oplus (T+K)]^* \wedge [(f+k) \odot (F+K)]^* & [(t+k) \oplus (T+K)] \odot [(f+k) \odot (F+K)]^* \end{bmatrix}, \quad (52)$$

$$M_1 \odot_{\mathbf{M}} M_2 = \begin{bmatrix} f & k \\ u & t \end{bmatrix} \odot_{\mathbf{M}} \begin{bmatrix} F & K \\ U & T \end{bmatrix} = m\big((t+k) \odot (T+K), (f+k) \oplus (F+K)\big)$$
$$= \begin{bmatrix} [(t+k) \odot (T+K)]^* \odot [(f+k) \oplus (F+K)] & [(t+k) \odot (T+K)] \wedge [(f+k) \oplus (F+K)] \\ [(t+k) \odot (T+K)]^* \wedge [(f+k) \oplus (F+K)]^* & [(t+k) \odot (T+K)] \odot [(f+k) \oplus (F+K)]^* \end{bmatrix}, \quad (53)$$

$$M_1 \rightarrow_{\mathbf{M}} M_2 = \begin{bmatrix} f & k \\ u & t \end{bmatrix} \rightarrow_{\mathbf{M}} \begin{bmatrix} F & K \\ U & T \end{bmatrix} = m\big((t+k)^* \oplus (T+K), (f+k)^* \odot (F+K)\big)$$
$$= \begin{bmatrix} [(t+k)^* \oplus (T+K)]^* \odot [(f+k)^* \odot (F+K)] & [(t+k)^* \oplus (T+K)] \wedge [(f+k)^* \odot (F+K)] \\ [(t+k)^* \oplus (T+K)]^* \wedge [(f+k)^* \odot (F+K)]^* & [(t+k)^* \oplus (T+K)] \odot [(f+k)^* \odot (F+K)]^* \end{bmatrix}. \quad (54)$$

Furthermore, it is possible to express the operations (51)–(54) in terms of the evidence pairs $(a,b), (c,d) \in \mathcal{T}$. By using the abbreviations x, y, w, z in order to get shorter formulae and substituting the values $t+k=x$, $f+k=y$, $T+K=w$ and $F+K=z$, we obtain the following equivalent formulation of expressions (51)–(54):

$$M_1^{*\mathbf{M}} = m\big(x^*, y^*\big) = \begin{bmatrix} x \odot y^* & x^* \wedge y^* \\ x \wedge y & x^* \odot y \end{bmatrix}, \quad (55)$$

$$M_1 \oplus_{\mathbf{M}} M_2 = m(x \oplus w, y \odot z) = \begin{bmatrix} (x \oplus w)^* \odot (y \odot z) & (x \oplus w) \wedge (y \odot z) \\ (x \oplus w)^* \wedge (y \odot z)^* & (x \oplus w) \odot (y \odot z)^* \end{bmatrix}, \quad (56)$$

$$M_1 \odot_{\mathbf{M}} M_2 = m(x \odot w, y \oplus z) = \begin{bmatrix} (x \odot w)^* \odot (y \oplus z) & (x \odot w) \wedge (y \oplus z) \\ (x \odot w)^* \wedge (y \oplus z)^* & (x \odot w) \odot (y \oplus z)^* \end{bmatrix}, \quad (57)$$

$$M_1 \rightarrow_{\mathbf{M}} M_2 = m\big(x^* \oplus w, y^* \odot z\big) = \begin{bmatrix} (x^* \oplus w)^* \odot (y^* \odot z) & (x^* \oplus w) \wedge (y^* \odot z) \\ (x^* \oplus w)^* \wedge (y^* \odot z)^* & (x^* \oplus w) \odot (y^* \odot z)^* \end{bmatrix}. \quad (58)$$

3.2. Operations on $\mathcal{T}$

In Section 2.2, operations on the unit triangle $\mathcal{T}$ were implicitly defined through a bijective mapping H from $[0,1]^2$ into $\mathcal{T}$ and its inverse (see expressions (41)–(43)), in such a way that the corresponding algebra has an injective MV-algebra structure. In this way, it was shown that with each bijective mapping from $[0,1]^2$ into $\mathcal{T}$ there exists an associated MV-algebra structure on $\mathcal{T}$. Now, our aim is to explicitly show the obtained operations on the triangle for the particular bijective mapping H provided in Remark 4. The main tool for this task will be the direct application of H and its inverse H^{-1}, in such a way that the results of the operations are expressed in terms of the corresponding evidence pairs in $\mathcal{T}$, but notice that equivalent results (expressed in terms of the components of the associated evidence matrices) can be obtained from those in Section 3.1 through the use of the inverse of M (see Fig. 1 again).

Thus, let $v = (a, b) \in \mathcal{T}$ be an evidence pair. Then $H^{-1}(a, b) = (a + (a \wedge b), b + (a \wedge b)) = (x, y)$ and $H(x^*, y^*) = (x^* - \frac{1}{2}(x^* \wedge y^*), y^* - \frac{1}{2}(x^* \wedge y^*))$. Therefore it holds that

$$v^{*\mathcal{T}} = \left(x^* - \frac{1}{2}\left(x^* \wedge y^*\right), y^* - \frac{1}{2}\left(x^* \wedge y^*\right)\right) = \left(a^* \odot b + \frac{1}{2}(a+b)^*, a \odot b^* + \frac{1}{2}(a+b)^*\right). \tag{59}$$

To see the last equality, note that if we denote again by

$$M_1(v) = \begin{bmatrix} f & k \\ u & t \end{bmatrix},$$

the evidence matrix corresponding to v, by Proposition 2 it is $(a, b) = (t + \frac{1}{2}k, f + \frac{1}{2}k)$, and applying the inverse of M to (51) it then results

$$v^{*\mathcal{T}} = \left(t + \frac{1}{2}k, f + \frac{1}{2}k\right)^{*\mathcal{T}} = \left(f + \frac{1}{2}u, t + \frac{1}{2}u\right) = \left(a^* \odot b + \frac{1}{2}(a+b)^*, a \odot b^* + \frac{1}{2}(a+b)^*\right). \tag{60}$$

This is a meaningful result for a complementation $*$ in $\mathcal{T}$, since it switches the favorable and unfavorable evidences contained in v as well as the values u and k. Particularly, note that $(1, 0)^{*\mathcal{T}} = (0, 1)$, $(0, 1)^{*\mathcal{T}} = (1, 0)$, $(0, 0)^{*\mathcal{T}} = (\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})^{*\mathcal{T}} = (0, 0)$, in consonance with (51).

In order to describe the binary operations on $\mathcal{T}$, let $s = (c, d) \in \mathcal{T}$ be another evidence pair, such that $H^{-1}(c, d) = (c + (c \wedge d), d + (c \wedge d)) = (w, z)$. Then, it is straightforward to proof that the following results hold through the application of H and its inverse (the abbreviations x, y, w, z are used instead of a, b, c, d for simplicity):

$$v \oplus_{\mathcal{T}} s = \left\langle (x \oplus w) - \frac{1}{2}\big[(x \oplus w) \wedge (y \odot z)\big], (y \odot z) - \frac{1}{2}\big[(x \oplus w) \wedge (y \odot z)\big]\right\rangle, \tag{61}$$

$$v \odot_{\mathcal{T}} s = \left\langle (x \odot w) - \frac{1}{2}\big[(x \odot w) \wedge (y \oplus z)\big], (y \oplus z) - \frac{1}{2}\big[(x \odot w) \wedge (y \oplus z)\big]\right\rangle, \tag{62}$$

$$v \rightarrow_{\mathcal{T}} s = \left\langle \left(x^* \oplus w\right) - \frac{1}{2}\big[\left(x^* \oplus w\right) \wedge \left(y^* \odot z\right)\big], \left(y^* \odot z\right) - \frac{1}{2}\big[\left(x^* \oplus w\right) \wedge \left(y^* \odot z\right)\big]\right\rangle. \tag{63}$$

Also, it is possible to define meet and join operations on $\mathcal{T}$ via

$$v \wedge_{\mathcal{T}} s = \left\langle (x \wedge w) - \frac{1}{2}\big[(x \wedge w) \wedge (y \vee z)\big], (y \vee z) - \frac{1}{2}\big[(x \wedge w) \wedge (y \vee z)\big]\right\rangle, \tag{64}$$

$$v \vee_{\mathcal{T}} s = \left\langle (x \vee w) - \frac{1}{2}\big[(x \vee w) \wedge (y \wedge z)\big], (y \wedge z) - \frac{1}{2}\big[(x \vee w) \wedge (y \wedge z)\big]\right\rangle. \tag{65}$$

To illustrate the usage of the previous formulae, let us consider that, for example, it is $v = (a, b) = (0.2, 0.6)$ and $s = (c, d) = (0.4, 0.1)$. Then we obtain $x = a + (a \wedge b) = 0.2 + (0.2 \wedge 0.6) = 0.4$, $y = b + (a \wedge b) = 0.6 + (0.2 \wedge 0.6) = 0.8$, $w = c + (c \wedge d) = 0.4 + (0.4 \wedge 0.1) = 0.5$ and $z = d + (c \wedge d) = 0.1 + (0.4 \wedge 0.1) = 0.2$. Thus, using Eqs. (60)–(65), it follows that

$$\begin{aligned}
v^{*\mathcal{T}} &= \left(0.2^* \odot 0.6 + \frac{1}{2}(0.2 + 0.6)^*, 0.2 \odot 0.6^* + \frac{1}{2}(0.2 + 0.6)^*\right) \\
&= \left(0.4 + \frac{1}{2}0.2, 0 + \frac{1}{2}0.2\right) = (0.5, 0.1), \\
s^{*\mathcal{T}} &= \left(0.4^* \odot 0.1 + \frac{1}{2}(0.4 + 0.1)^*, 0.4 \odot 0.1^* + \frac{1}{2}(0.4 + 0.1)^*\right) \\
&= \left(0 + \frac{1}{2}0.5, 0.3 + \frac{1}{2}0.5\right) = (0.25, 0.55), \\
v \oplus_{\mathcal{T}} s &= \left\langle (0.4 \oplus 0.5) - \frac{1}{2}\big[(0.4 \oplus 0.5) \wedge (0.8 \odot 0.2)\big], (0.8 \odot 0.2) - \frac{1}{2}\big[(0.4 \oplus 0.5) \wedge (0.8 \odot 0.2)\big]\right\rangle \\
&= \left\langle 0.9 - \frac{1}{2}(0.9 \wedge 0), 0 - \frac{1}{2}(0.9 \wedge 0)\right\rangle = (0.9, 0),
\end{aligned}$$

$$v \odot_{\mathcal{T}} s = \left\langle (0.4 \odot 0.5) - \frac{1}{2}\big[(0.4 \odot 0.5) \wedge (0.8 \oplus 0.2)\big], (0.8 \oplus 0.2) - \frac{1}{2}\big[(0.4 \odot 0.5) \wedge (0.8 \oplus 0.2)\big]\right\rangle$$
$$= \left\langle 0 - \frac{1}{2}(0 \wedge 1), 1 - \frac{1}{2}(0 \wedge 1)\right\rangle = (0, 1) = \mathbf{0}_{\mathcal{T}},$$
$$v \to_{\mathcal{T}} s = \left\langle \big(0.4^* \oplus 0.5\big) - \frac{1}{2}\big[\big(0.4^* \oplus 0.5\big) \wedge \big(0.8^* \odot 0.2\big)\big], \big(0.8^* \odot 0.2\big) - \frac{1}{2}\big[\big(0.4^* \oplus 0.5\big) \wedge \big(0.8^* \odot 0.2\big)\big]\right\rangle$$
$$= \left\langle 1 - \frac{1}{2}(1 \wedge 0), 0 - \frac{1}{2}(1 \wedge 0)\right\rangle = (1, 0) = \mathbf{1}_{\mathcal{T}},$$
$$v \wedge_{\mathcal{T}} s = \left\langle (0.4 \wedge 0.5) - \frac{1}{2}\big[(0.4 \wedge 0.5) \wedge (0.8 \vee 0.2)\big], (0.8 \vee 0.2) - \frac{1}{2}\big[(0.4 \wedge 0.5) \wedge (0.8 \vee 0.2)\big]\right\rangle$$
$$= \left\langle 0.4 - \frac{1}{2}(0.4 \wedge 0.8), 0.8 - \frac{1}{2}(0.4 \wedge 0.8)\right\rangle = (0.2, 0.6) = v,$$
$$v \vee_{\mathcal{T}} s = \left\langle (0.4 \vee 0.5) - \frac{1}{2}\big[(0.4 \vee 0.5) \wedge (0.8 \wedge 0.2)\big], (0.8 \wedge 0.2) - \frac{1}{2}\big[(0.4 \vee 0.5) \wedge (0.8 \wedge 0.2)\big]\right\rangle$$
$$= \left\langle 0.5 - \frac{1}{2}(0.5 \wedge 0.2), 0.2 - \frac{1}{2}(0.5 \wedge 0.2)\right\rangle = (0.4, 0.1) = s.$$

3.3. The lattice $(\mathcal{T}, \leqslant_{\mathcal{T}})$

As exposed in Section 2.1, from the MV-algebra $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$, expression (24) allows to define an order relation on $\mathcal{T}$, given by

$$v \leqslant_{\mathcal{T}} s \quad \Leftrightarrow \quad v^{*\mathcal{T}} \oplus_{\mathcal{T}} s = \mathbf{1}_{\mathcal{T}}, \quad v, s \in \mathcal{T}, \tag{66}$$

in such a way that $(\mathcal{T}, \leqslant_{\mathcal{T}}, \wedge_{\mathcal{T}}, \vee_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}})$ is a bounded lattice. Thus, $\leqslant_{\mathcal{T}}$ is a partial order on $\mathcal{T}$, which, as discussed in Remark 3, cannot coincide with the usual order $\leqslant_t$. However, as we will show next, $\leqslant_{\mathcal{T}}$ is closely related to $\leqslant_t$, and can be given a simple formulation through expressions (9)–(12).

Firstly, note that, due to the bijectivity of H, for all $v, s \in \mathcal{T}$ it holds that $v^{*\mathcal{T}} \oplus_{\mathcal{T}} s = \mathbf{1}_{\mathcal{T}} \Leftrightarrow H(H^{-1}(v^{*\mathcal{T}}) \oplus_{[0,1]^2} H^{-1}(s)) = H(\mathbf{1}_{[0,1]^2}) \Leftrightarrow [H^{-1}(v)]^* \oplus_{[0,1]^2} H^{-1}(s) = \mathbf{1}_{[0,1]^2}$, and then it follows that $v \leqslant_{\mathcal{T}} s \Leftrightarrow H^{-1}(v) \leqslant_{[0,1]^2} H^{-1}(s)$, i.e.

$$v \leqslant_{\mathcal{T}} s \quad \Leftrightarrow \quad H^{-1}(v) \leqslant_t H(s), \tag{67}$$

since the MV-algebra order of $[0, 1]^2$ as the product of the Lukasiewicz structure coincides with $\leqslant_t$. Therefore, the lattice $(\mathcal{T}, \leqslant_{\mathcal{T}})$ is just the image by H of the lattice $([0, 1]^2, \leqslant_t)$.

Comparing now with the lattice $(\mathcal{T}, \leqslant_t)$, it is easy to see that for all $v = (a, b)$, $s = (c, d) \in \mathcal{T}$

$$v \leqslant_{\mathcal{T}} s \quad \Rightarrow \quad v \leqslant_t s. \tag{68}$$

For example, if $a \leqslant b$ and $c \leqslant d$, from (67) we obtain that

$$v \leqslant_{\mathcal{T}} s \quad \Leftrightarrow \quad (2a, a + b) \leqslant_t (2c, c + d), \tag{69}$$

so it is $a \leqslant c$ and $a + b \geqslant c + d$, from which $b \geqslant d$ follows, and then $v \leqslant_t s$ holds. The same conclusion is also easily obtained in the remaining cases regarding the relative order of a, b and c, d. Therefore, the partial order $\leqslant_{\mathcal{T}}$ is contained in $\leqslant_t$, which is a partial order on $\mathcal{T}$ as well. As shown above, $\leqslant_{\mathcal{T}}$ admits an MV-algebra structure, in the sense of it being the order obtained through (24) for an MV-algebra on $\mathcal{T}$. However, $\leqslant_t$ as a partial order on $\mathcal{T}$ does not admit an MV-algebra structure, since if it is the case then the operation $\oplus_{\mathcal{T}}$ of such a structure would be a t-conorm, something that was proved to be impossible in [3].

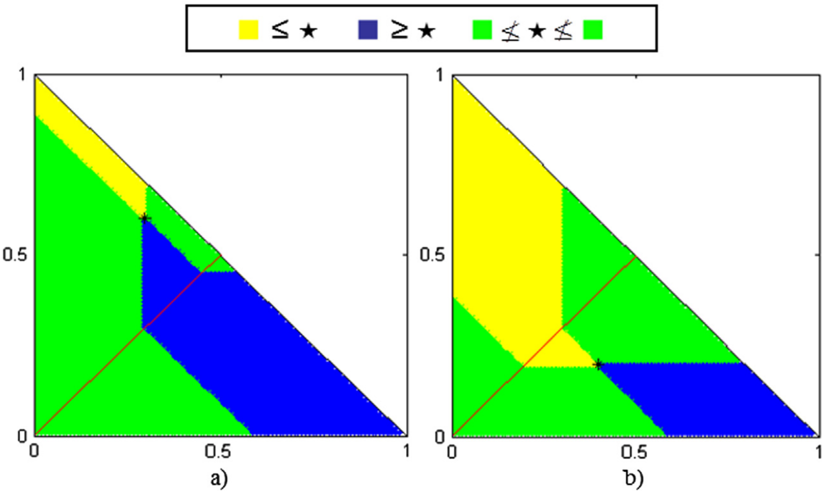

Fig. 2. Behavior of the partial order $\leqslant_{\mathcal{T}}$: a) in the subset of evidence couples with $t = 0$ (N); b) in the subset of evidence couples with $f = 0$ (P). (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

Apart from enabling an MV-algebra structure on $\mathcal{T}$, perhaps the most interesting property of the partial order $\leqslant_{\mathcal{T}}$ is the possibility of obtaining it directly from the extension (9)–(12), i.e. without the use of a bijective mapping H. To see this, it is enough to fully develop (67) in a similar way as done in (69), and have another look to the formulae (9)–(12). Effectively, notice that, if

$$M_1 = M(v) = \begin{bmatrix} f & k \\ u & t \end{bmatrix} \quad \text{and} \quad M_2 = M(s) = \begin{bmatrix} F & K \\ U & T \end{bmatrix},$$

the conditions $a \leqslant b$ and $c \leqslant d$ can be translated as $t = T = 0$, and thus in this case the right side of (69) is equal to $(k, 1-u) \leqslant_t (K, 1-U)$, so (67) can be restated as $v \leqslant_{\mathcal{T}} s \Leftrightarrow k \leqslant K$ and $u \leqslant U$ if $t = T = 0$. Applying a similar reasoning for the remaining orderings of a, b and c, d, it is then possible to re-establish (67) as

$$v \leqslant_{\mathcal{T}} s \quad \Leftrightarrow \quad M_1 \leqslant_{\mathbf{M}} M_2 \quad \Leftrightarrow \quad \begin{cases} k \leqslant K \text{ and } u \leqslant U, & \text{if } t = T = 0, \\ k \geqslant K \text{ and } u \geqslant U, & \text{if } f = F = 0, \\ 1 - u \geqslant K \text{ and } 1 - k \leqslant U, & \text{if } t = F = 0. \end{cases} \tag{70}$$

Note that, due to the commutativity of the diagram in Fig. 1, a similar reasoning to that of (67) leads to conclude that $\leqslant_{\mathcal{T}}$ is equivalent to the partial order $\leqslant_{\mathbf{M}}$ on the MV-algebra $(\mathbf{M}, \oplus_{\mathbf{M}}, *_{\mathbf{M}}, \mathbf{0}_{\mathbf{M}})$ as defined through expression (24).

A picture of the partial order $\leqslant_{\mathcal{T}}$ is shown in Fig. 2. Given an evidence pair $v \in \mathcal{T}$, notice that the shape of the sets $G(v) = \{s \in \mathcal{T} \mid v \leqslant_{\mathcal{T}} s\}$ (in blue in Fig. 2) vary depending on whether $v \in N = \{v \in \mathcal{T} \mid t = 0\}$ or $v \in P = \{v \in \mathcal{T} \mid f = 0\}$, see the difference between Figs. 2a) and 2b). In fact, if $v \in N$, the set $G(v) \cap N$ has a triangular shape, going downwards from v to the line $P \cap N = \{v \in \mathcal{T} \mid v_1 = v_2\}$ (in red in Fig. 2). If $v \in P$, $G(v) \subset P$ has a trapezoidal shape, going downwards from v to the vertex $t = (1, 0)$. If $v \in N$ and $s \in P$, these two behaviors are combined through the third condition in (70) in such a way that the transitivity of $\leqslant_{\mathcal{T}}$ is conserved.

4. Concluding remarks

(1) It is worth to note that the operations $\oplus_{\mathcal{T}}, \odot_{\mathcal{T}}, \rightarrow_{\mathcal{T}}$, etc. (or equivalently $\oplus_{\mathbf{M}}, \odot_{\mathbf{M}}, \rightarrow_{\mathbf{M}}$, etc.) resemble and behave in a similar way to their correlates in $[0, 1]^2$. In this sense, for all $v = (a, b) \in \mathcal{T}$ it is $v \oplus_{\mathcal{T}} \mathbf{0}_{\mathcal{T}} = (a, b) \oplus_{\mathcal{T}} (0, 1) = (a, b) = v$, since the evidence pair $(0, 1)$ is the neutral element for the operation $\oplus_{\mathcal{T}}$ (as it is for the Lukasiewicz operation $\oplus_{[0,1]^2}$). Also, for example $(0.6, 0) \oplus_{\mathcal{T}} (0.1, 0) = (0.7, 0)$, which is the same result that would be obtained through $\oplus_{[0,1]^2}$. Or $(0.3, 0.5) \oplus_{\mathcal{T}} (0.1, 0.7) = (0.45, 0.35)$, while $(0.3, 0.5) \oplus_{[0,1]^2} (0.1, 0.7) = (0.4, 0.2)$. That is, when they exist, the differences between $\oplus_{\mathcal{T}}$ and $\oplus_{[0,1]^2}$ are rather small, and in any case the behavior of both operations with respect to the evidences for and against contained in the evidence pairs is quite similar.

(2) Also, it is important to stress that the set of usual MV-tautologies is trivially verified in the paraconsistent logic introduced in this paper. For example, the formula α and not $-\alpha$ receives the truth-value `false` in any valuation of α, as it is not difficult to see that

$$M(\alpha \text{ and not } -\alpha) = M(\alpha) \odot_{\mathbf{M}} M(\text{not } -\alpha) = \begin{bmatrix} f & k \\ u & t \end{bmatrix} \odot_{\mathbf{M}} \begin{bmatrix} t & u \\ k & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \texttt{false},$$

for all valuations of α, i.e. for any evidence matrix $M(\alpha)$.

(3) To the extent of our knowledge, no other MV-algebra structure has been previously defined nor studied on the real unit triangle $\mathcal{T}$. Therefore, this paper shows the possibility of developing a Pavelka-style fuzzy sentential logic grounded on the real unit triangle through the described MV-algebra $(\mathcal{T}, \oplus_{\mathcal{T}}, *_{\mathcal{T}}, \mathbf{0}_{\mathcal{T}})$. Moreover, since the corresponding set of Lukasiewicz-based evidence matrices can also be endowed with an MV-algebra structure, a new paraconsistent algebraic semantics for this logic can be obtained. In consequence, this paper complements the usual approach of studying the logical properties of the lattice $(\mathcal{T}, \leqslant_t)$, showing that it could be worth to study different lattices $(\mathcal{T}, \leqslant_{\mathcal{T}})$ compatible with an MV-algebra structure, and proposing an alternative approach that enables such study to be carried out. At this respect, in forthcoming works we pretend to analyze more deeply the possibility of obtaining lattice structures (i.e. partial orders) from evidence matrices in more general scales, and in relation with their compatibility with an MV-algebra structure. Also, the possible usefulness for applications of the reasoning framework provided by the proposed Pavelka-style logic will be further studied.

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