Function spaces of Lorentz-Sobolev type: Atomic decompositions, characterizations in terms of wavelets, interpolation and multiplications

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Abstract

We establish atomic decompositions and characterizations in terms of wavelets for Besov-Lorentz spaces $B^s_qL_{p,r}(\mathbb{R}^n)$ and for Triebel-Lizorkin-Lorentz spaces $F^s_qL_{p,r}(\mathbb{R}^n)$ in the whole range of parameters. As application we obtain new interpolation formulae between spaces of Lorentz-Sobolev type. We also remove the restrictions on the parameters in a result of Peetre on optimal embeddings of Besov spaces. Moreover, we derive results on diffeomorphisms, extension operators and multipliers for $B^s_qL_{p,\infty}(\mathbb{R}^n)$. Finally, we describe $B^s_qL_{p,r}(\mathbb{R}^n)$ as an approximation space, which allows us to show new sufficient conditions on parameters for $B^s_qL_{p,r}(\mathbb{R}^n)$ to be a multiplication algebra.

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1. Introduction

The scales of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ include many important spaces of functions and distributions as (fractional) Sobolev spaces, classical Besov spaces, Hölder-Zygmund spaces or Lebesgue spaces. These scales are the spinal column of the theory of function spaces (see, for example, the books by Triebel [39, 40, 41, 46]).

Spaces $A^s_{p,q}(\mathbb{R}^n), A \in \{B, F\}$, have many useful properties but they do not contain the spaces that arise by real interpolation of the couple

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$(A^s_{p_1,q}(\mathbb{R}^n), A^s_{p_2,q}(\mathbb{R}^n))$ with $p_1 \neq p_2$ (see [30, Theorem 6, p. 106] or [38, Section 2.4, p. 181]). To include them one needs to generalize the scales. This leads to function spaces $A^s_q L_{p,r}(\mathbb{R}^n), A \in \{B, F\}$, with Lorentz smoothness. They are defined by changing in the Fourier-analytical definition of $A^s_{p,q}(\mathbb{R}^n)$ the space $L_p(\mathbb{R}^n)$ by the more general Lorentz space $L_{p,r}(\mathbb{R}^n)$. These spaces with Lorentz smoothness have been used during the last 50 years by a number of authors in different contexts and recently they are receiving increasing interest. See, for example, the contributions by Peetre [29, 30], Fefferman, Riviere and Sagher [17], Stein [37], Caetano [10], Cianchi and Pick [11], Xiang and Yan [47, 48], Almeida and Caetano [2, 3] and the very recent papers by Grafakos and Slavíková [23], Seeger and Trebels [36], Hobus and Saal [26], by Triebel and the present authors [7] and by Haroske, Triebel and one of the present authors [8].

The characterization of $F^s_q L_{p,r}(\mathbb{R}^n)$ in terms of wavelets was done by Yang, Cheng and Peng [49, Theorems 3 and 4]. The case of Besov-Lorentz spaces was considered by Almeida [1, Corollary 3.2] but only for spaces $B^s_q L_{p,q}(\mathbb{R}^n)$. The restriction $q = r$ is due to the techniques used in [1] which are based on interpolation properties of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. The first goal of the present paper is to eliminate this restriction, giving the wavelet characterization of $B^s_q L_{p,r}(\mathbb{R}^n)$ for $q \neq r$. For this we follow an approach recently developed by Haroske, Skandera and Triebel [25], calling first for establishing atomic decompositions of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$. This is done in Sections 3 and 4 where we also use this approach to derive the wavelet description for spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ which is of some interest because the paper of Yang, Cheng and Peng [49] being in Chinese is not very accessible.

Then, in Section 5, with the help of the wavelets, we establish new interpolation formulae between Lorentz-Sobolev spaces. Among other results, we eliminate the restrictions on the parameters in a result by Peetre [30, Theorem 6/(ii), p. 106] on optimal embeddings of Besov spaces. Furthermore, we derive results on spaces $B^s_q L_{p,\infty}(\mathbb{R}^n)$ which complement those of Triebel and the present authors in [7], where this limit case was left over due to the techniques used there. We show the invariance of $B^s_q L_{p,\infty}(\mathbb{R}^n)$ with respect to diffeomorphisms of $\mathbb{R}^n$ onto itself, and we give results on extension operators and multipliers for $B^s_q L_{p,\infty}(\mathbb{R}^n)$. We also describe $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space in the sense of [33] (see also [15, 32]).

Finally, in Section 6, we give another description of $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space. This time we base the arguments on the characterization of $B^s_q L_{p,r}(\mathbb{R}^n)$ by means of best approximation by entire analytic functions of exponential type. As a first consequence of this description we generalize Hölder inequality to Besov-Lorentz spaces. Then we study the question whether the product of two elements of $B^s_q L_{p,r}(\mathbb{R}^n)$ is again in $B^s_q L_{p,r}(\mathbb{R}^n)$. That is to say, whether $B^s_q L_{p,r}(\mathbb{R}^n)$ is multiplication algebra. This property is of some use in connection with Cauchy problems for non-linear PDEs, including non-linear heat equations and Navier-Stokes equations among others (see [35, 43, 45] and [46, Remark 2.38]). We complement [7, Theorem 7.2] eliminating the restrictions on q and r in order to $B^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra for $s > n/p$. A result for spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ is also established.

2. Preliminaries

Let $(A, \|\cdot|A\|)$ be a quasi-Banach space with constant $C_A \geq 1$ in the quasi-triangle inequality. By the Aoki-Rolewicz theorem (see [4], [34]), if $0 < p \leq 1$ satisfies that $2^{1/p-1} = C_A$, then there is another quasi-norm $|||\cdot|||$ on A, equivalent to $\|\cdot|A\|$, such that $|||\cdot|||^p$ satisfies the triangle inequality. We say that $|||\cdot|||$ is a p-norm and that A is a p-Banach space. Note that if A is a p-Banach space, then it is also r-Banach for any $0 < r \leq p \leq 1$.

In what follows, if X and Y are quantities depending on certain parameters some of them being the significant ones in our reasoning, we write $X \lesssim Y$ if $X \leq CY$ where C is a constant independent of theses significant parameters (but which might depend on other quantities indicated between brackets when we want them to be explicit). We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Moreover, given any real number a, we put $a_+ = \max\{a, 0\}$.

If $(A, \|\cdot|A\|)$ and $(B, \|\cdot|B\|)$ are two quasi-Banach spaces, we put $A \hookrightarrow B$ if A is included in B as a subset and there is $M > 0$ such that $\|a|B\| \leq M\|a|A\|$ for every $a \in A$.

Let (A_1, A_2) be a quasi-Banach couple, that is to say, two quasi-Banach spaces A_1, A_2 continuously embedded in the same Hausdorff topological vector space. For $0 < \theta < 1$ and $0 < r \leq \infty$ the *real interpolation space* $(A_1, A_2)_{\theta,r}$ is the set of all $a \in A_1 + A_2$ having a finite quasi-norm

$$\|a|(A_1, A_2)_{\theta,r}\| = \begin{cases} \left(\int_0^\infty [t^{-\theta} K(t,a)]^r \frac{dt}{t} \right)^{1/r} & \text{if } 0 < r < \infty, \\ \sup_{0<t<\infty} \{t^{-\theta} K(t,a)\} & \text{if } r = \infty. \end{cases}$$

Here $K(t, a)$ is the *Peetre's K-functional* defined by

$$\begin{aligned} K(t,a) &= K(t, a; A_1, A_2) \\ &= \inf\{\|a_1|A_1\| + t\|a_2|A_2\| : a = a_1 + a_2, a_k \in A_k, k = 1, 2\}, \end{aligned}$$

for any $a \in A_1 + A_2$.

One of the main properties of these spaces is that if (A_1, A_2) and (B_1, B_2) are two quasi-Banach couples and T is a linear operator which is bounded from A_k into B_k, $k = 1, 2$, with norm $\|T : A_k \to B_k\|$, then T is also bounded from $(A_1, A_2)_{\theta,r}$ into $(B_1, B_2)_{\theta,r}$ and

$$\|T : (A_1, A_2)_{\theta,r} \longrightarrow (B_1, B_2)_{\theta,r}\| \leq \|T : A_1 \longrightarrow B_1\|^{1-\theta} \|T : A_2 \longrightarrow B_2\|^{\theta}. \tag{2.1}$$

This results still holds for sublinear operators if the couples are formed by quasi-Banach spaces of measurable functions with the lattice property (that is, if $|f|$ is smaller that $|g|$ almost everywhere, then the quasi-norm of f is smaller than the quasi-norm of g). For further details about the real interpolation method see, for example, the books by Bergh and Löfström [6], Triebel [38] and Bennett and Sharpley [5].

We are going to need also the complex interpolation method that we recall next. Let $\bar{A} = (A_1, A_2)$ be a Banach couple. By $\mathfrak{F}(\bar{A})$ we designate the space of all functions f from the closed strip $D = \{z \in \mathbb{C} : 0 \leq Rez \leq 1\}$ into $A_1 + A_2$ such that f is bounded and continuous on D and analytic on the interior of D, and the functions $t \to f(j-1+it), j = 1, 2$ are continuous from $\mathbb{R}$ into A_j and tend to zero as $|t| \to \infty$. We write

$$\|f|\mathfrak{F}(\bar{A})\| = \max\{\sup \|f(j-1+it)|A_j\| : j = 1, 2\}.$$

For $0 < \theta < 1$, *the complex interpolation space* $[A_1, A_2]_\theta$ is formed by all $a \in A_1 + A_2$ such that $a = f(\theta)$ for some $f \in \mathfrak{F}(\bar{A})$. We endow $[A_1, A_2]_\theta$ with the norm

$$\|a|[A_1, A_2]_\theta\| = \inf\{\|f|\mathfrak{F}(\bar{A})\| : f(\theta) = a, f \in \mathfrak{F}(\bar{A})\}\,.$$

This construction also has the interpolation property for bounded linear operators. For bilinear operators the following holds (see [6, Theorem 4.4.1]).

Theorem 2.1. Let $(A_1, A_2), (B_1, B_2), (E_1, E_2)$ be Banach couples and let $0 < \theta < 1$. Assume that T is a bilinear operator defined on $(A_1 \cap A_2) \times (B_1 \cap B_2)$ with values in $E_1 \cap E_2$ such that

$$\|T(a, b)|E_j\| \leq M_j \|a|A_j\| \|b|B_j\|,\ a \in A_1 \cap A_2,\ b \in B_1 \cap B_2,\ j = 1, 2\,.$$

Then T may be uniquely extended to a bounded bilinear operator from $[A_1, A_2]_\theta \times [B_1, B_2]_\theta$ to $[E_1, E_2]_\theta$.

The following reiteration result between real and complex interpolation methods will be useful for our computations: If (A_1, A_2) is a Banach couple, $0 < \theta_1 \neq \theta_2 < 1, 0 < \eta < 1, \theta = (1-\eta)\theta_1 + \eta\theta_2, 1 \leq q_1, q_2 \leq \infty$ and $1/q = (1-\theta)/q_1 + \theta/q_2$, we have with equivalent norms

$$[(A_1, A_2)_{\theta_1, q_1}, (A_1, A_2)_{\theta_2, q_2}]_\eta = (A_1, A_2)_{\theta, q} \tag{2.2}$$

(see [6, Theorem 4.7.2]).

Let (Ω, μ) be a σ-finite measure space, $0 < p < \infty$ and $0 < r \leq \infty$. The *Lorentz space* $L_{p,r}(\Omega)$ is defined as the space of all (equivalence classes of) μ-measurable functions from Ω into $\mathbb{C}$ such that

$$\|f|L_{p,r}(\Omega)\| = \Big(\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\Big)^{1/r} < \infty$$

(the integral should be replaced by the supremum if $r = \infty$). Here f^* denotes the *decreasing rearrangement* of f defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > s\}) \le t\}, \quad t \ge 0.$$

Lorentz spaces are quasi-Banach spaces and if $0 < b < p$ and $b \le \min\{1, r\}$, then $L_{p,r}(\Omega)$ admits an equivalent b-norm (see [27, Section 2]). If $p = r$ the space $L_{p,p}(\Omega)$ coincides with the Lebesgue space $L_p(\Omega)$ and

$$\|f|L_p(\Omega)\| = \Big(\int_\Omega |f(\omega)|^p d\mu\Big)^{1/p} = \|f|L_{p,p}(\Omega)\|.$$

Lorentz spaces arise by applying the real interpolation method to a couple of Lebesgue spaces: If $0 < p_1 < p < p_2 \le \infty$ and $\theta = \frac{p_2(p-p_1)}{p(p_2-p_1)}$, then

$$(L_{p_1}(\Omega), L_{p_2}(\Omega))_{\theta,r} = L_{p,r}(\Omega) \tag{2.3}$$

(see, for example, [6, Theorem 5.2.1] or [38, (16), p. 134 and Remark 5, p. 135]). Combining this formula with (2.2) we obtain the following equality with equivalent norms

$$[L_{p_1,q_1}(\Omega), L_{p_2,q_2}(\Omega)]_\eta = L_{p,q}(\Omega) \tag{2.4}$$

provided that $1 < p_1, p_2 < \infty, 1 \le q_1, q_2 \le \infty, 1/p = (1-\eta)/p_1 + \eta/p_2$ and $1/q = (1-\eta)/q_1 + \eta/q_2$.

Fix $n \in \mathbb{N}$. In what follows two measure spaces will be of special interest for us: the Euclidean n-space $\mathbb{R}^n$ endowed with the Lebesgue measure and $\mathbb{Z}^n$ with the counting measure. Lorentz spaces $L_{p,r}(\mathbb{R}^n)$ and $L_{p,r}(\mathbb{Z}^n)$ will appear repeatedly. From now on, we put $\ell_{p,r}(\mathbb{Z}^n) := L_{p,r}(\mathbb{Z}^n)$ and $\ell_p(\mathbb{Z}^n) := L_p(\mathbb{Z}^n)$. It can be easily shown that $\ell_{p,r}(\mathbb{Z}^n)$ admits the following equivalent quasi-norm

$$\|(x_m)_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\| \sim \Big(\sum_{k=0}^\infty |x_k^*|^r (k+1)^{r/p-1}\Big)^{1/r},$$

where (x_k^*) is the decreasing rearrangement of $(x_m)_{m\in\mathbb{Z}^n}$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We write $\mathcal{S}'(\mathbb{R}^n)$ for the space of all tempered distributions. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we put $\hat{f}$ for its Fourier transform and $\check{f}$ for its inverse Fourier transform.

Take $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} &\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x| < 3/2\} \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if } |x| \le 1. \\ &\text{For } k \in \mathbb{N}, \text{ put } \varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x). \end{aligned} \tag{2.5}$$

Write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Since $\sum_{k\in\mathbb{N}_0} \varphi_k(x) = 1$ for every $x \in \mathbb{R}^n$, the sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ is a *smooth dyadic resolution of unity.*

Definition 2.2. Let $0 < q, r \leq \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity.

1. The *Besov-Lorentz space* $B^s_q L_{p,r}(\mathbb{R}^n)$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| = \Big(\sum_{k=0}^{\infty} 2^{ksq}\|(\varphi_k \hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\|^q\Big)^{1/q}$$

   is finite.

2. The *Triebel-Lizorkin-Lorentz space* $F^s_q L_{p,r}(\mathbb{R}^n)$ is defined as the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f|F^s_q L_{p,r}(\mathbb{R}^n)\| = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ksq}|(\varphi_k \hat{f})^\vee|^q\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|$$

   is finite.

Observe that for $p = r$, the space $B^s_q L_{p,p}(\mathbb{R}^n)$ (respectively, $F^s_q L_{p,p}(\mathbb{R}^n)$) coincides with the classical Besov (respectively, Triebel-Lizorkin) space $B^s_{p,q}(\mathbb{R}^n)$ (respectively, $F^s_{p,q}(\mathbb{R}^n)$).

Proposition 2.3. Let $0 < q, r \leq \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$, then

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \text{and}$$
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proof. For $p = r$, the result corresponds to [39, Theorem 2.3.3]. This together with [36, Theorem 1.5] implies that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s'}_{r,q}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

if $r \leq p$, $s' > s + n(1/r - 1/p)$ and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s''}_{r,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

if $p \leq r$, $s'' < s - n(1/p - 1/r)$. As for Triebel-Lizorkin-Lorentz spaces, let $s'' < s < s'$, then it follows from [36, Theorem 1.1 and 1.2] that

$$B^{s'}_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s''}_q L_{p,r}(\mathbb{R}^n),$$

and the proof is reduced to the first case. □

3. Atomic decomposition of Lorentz smoothness function spaces

Let $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$. Consider the *dyadic cubes*

$$Q_{jm} = \prod_{\ell=1}^{n} (2^{-j} m_\ell - 2^{-j-1}, 2^{-j} m_\ell + 2^{-j-1}).$$

with center in $x_{jm} = (2^{-j} m_1, ..., 2^{-j} m_n)$ and sides of length 2^{-j}. Let $\chi_{j,m}$ be the characteristic function of Q_{jm}.

For $L \in \mathbb{N}$, we designate by $C^L(\mathbb{R}^n, \mathbb{C})$ the space of functions from $\mathbb{R}^n$ to $\mathbb{C}$ with continuous derivatives up to order L (included).

Definition 3.1. Let $L \in \mathbb{N}$ and $d > 1$. If $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$ the function $a_{jm} \in C^L(\mathbb{R}^n, \mathbb{C})$ is said to be an (L, d)-*atom* (or simply L-atom) if

1. supp $a_{jm} \subset dQ_{jm} := \prod_{\ell=1}^{n} (2^{-j} m_\ell - d2^{-j-1}, 2^{-j} m_\ell + d2^{-j-1})$.
2. $|\partial^\alpha a_{jm}(x)| \leq 2^{j|\alpha|}$ for $x \in \mathbb{R}^n$ and $|\alpha| \leq L$.
3. $\int_{\mathbb{R}^n} x^\beta a_{jm}(x) dx = 0$ for $|\beta| \leq L - 1$.

When $j = 0$ we simply ask that a_{0m} verifies (1) and (2), no moment conditions are required.

Classical Besov and Triebel-Lizorkin spaces admit atomic representations (see, for example, [42, Theorem 1.7]). In this section we study the case of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$. We start with some auxiliary results.

Lemma 3.2. Let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a smooth dyadic resolution of unity (2.5), let $(a_{jm})_{\substack{j \in \mathbb{N}_0,\\ m \in \mathbb{Z}^n}}$ be a sequence of (L, d)-atoms on $\mathbb{R}^n$ and $\lambda \geq L$. Then

$$|(\varphi_k \hat{a}_{jm})^\vee(x)| \leq C \frac{2^{-L|k-j|} 2^{n \min\{k-j, 0\}}}{(1 + 2^{\min\{k,j\}} |x - x_{jm}|)^\lambda},$$

for every $x \in \mathbb{R}^n$ with a constant C independent of j, k and m.

Proof. Let $N > 0$, then for any multi-index α with $|\alpha| \leq L$ we have

$$\sup_{y \in \mathbb{R}^n} |\partial^\alpha a_{jm}(y)| (1 + 2^j |y - x_{jm}|)^N \leq C(d, n, N) 2^{j|\alpha|}. \tag{3.1}$$

Indeed, from properties (1) and (2) in Definition 3.1, we obtain

$$\begin{aligned} |\partial^\alpha a_{jm}(y)| &\leq 2^{j|\alpha|} \frac{(1 + 2^j |y - x_{jm}|)^N}{(1 + 2^j |y - x_{jm}|)^N} \chi_{dQ_{jm}}(y) \\ &\leq 2^{j|\alpha|} \Big(1 + \frac{d}{2}\sqrt{n}\Big)^N (1 + 2^j |y - x_{jm}|)^{-N}. \end{aligned}$$

Besides, by construction of the partition of unity, if $k \in \mathbb{N}$, we have

$$\varphi_k^\vee(x) = \varphi_0(2^{-k} \cdot)^\vee(x) - \varphi_0(2^{-k+1} \cdot)^\vee(x) = 2^{kn} \varphi_0^\vee(2^k x) - 2^{(k-1)n} \varphi_0^\vee(2^{k-1} x)\,.$$

Hence,

$$\partial^\gamma \varphi_k^\vee(x) = 2^{k(n+|\gamma|)}\partial^\gamma \varphi_0^\vee(2^k x) - 2^{(k-1)(n+|\gamma|)}\partial^\gamma \varphi_0^\vee(2^{k-1}x)\,.$$

Let $M > 0$, for any multi-index γ with $|\gamma| \le M$ we have

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^n} |\partial^\gamma \varphi_k^\vee(x-y)|(1+2^k|x-y|)^M \\
&= 2^{k(n+|\gamma|)} \sup_{y\in\mathbb{R}^n} \big\{ |\partial^\gamma \varphi_0^\vee(2^k(x-y)) - 2^{-(n+|\gamma|)}\partial^\gamma \varphi_0^\vee(2^{k-1}(x-y))| \\
&\qquad\qquad \times (1+2^k|x-y|)^M \big\} \\
&\le 2^{k(n+|\gamma|)} C(M,n,\varphi_0). \qquad (3.2)
\end{aligned}$$

Assume first that $k \le j$ and $(j,k) \ne (0,0)$, then a_{jm} has vanishing moments up to order $L-1$ (included). Put $\alpha = 0$, $N = L+\lambda+n+1$, $M = \lambda$ and $|\gamma| \le L$. Then (3.1) and (3.2) allow us to use the results of [22, Appendix B.2, p. 596] based on Taylor's theorem, obtaining that for every $x \in \mathbb{R}^n$

$$\begin{aligned}
|(\varphi_k \hat{a}_{jm})^\vee(x)| &= \Big| \int_{\mathbb{R}^n} \varphi_k^\vee(x-y) a_{jm}(y)dy \Big| \\
&\le C(L,d,n,\lambda,\varphi_0) \frac{2^{-L(j-k)}2^{-n(j-k)}}{(1+2^k|x-x_{jm}|)^\lambda}.
\end{aligned}$$

Let now $j \le k$ and $(j,k) \ne (0,0)$. We claim that for any $x \in \mathbb{R}^n$ the Schwartz function $y \to \varphi_k^\vee(x-y)$ has vanishing moment conditions of any order. Indeed, let $\beta \in \mathbb{N}_0^n$ be any multi-index, for any $k \in \mathbb{N}$, we get $\partial^\beta \varphi_k(0) = 2^{-k|\beta|}\partial^\beta \varphi_0(0) - 2^{(-k+1)|\beta|}\partial^\beta\varphi_0(0) = 0$ because $\varphi_0 \equiv 1$ in a neighborhood of 0. This implies that $\int_{\mathbb{R}^n} y^\beta \varphi_k^\vee(y)dy = 0$ for any $\beta \in \mathbb{N}_0^n$ and so

$$\begin{aligned}
\int_{\mathbb{R}^n} y^\beta \varphi_k^\vee(x-y)dy &= \int_{\mathbb{R}^n} (y-x+x)^\beta \varphi_k^\vee(y)dy \\
&= \sum_{\eta\le\beta} \binom{\beta}{\eta} x^\eta \int_{\mathbb{R}^n} (y-x)^{\beta-\eta}\varphi_k^\vee(x-y)dy = 0. \qquad (3.3)
\end{aligned}$$

For $|\alpha| \le L$, $N = \lambda$, $M = L+\lambda+n+1$ and $\gamma = 0$, using the results in [22, Appendix B.2] we get that for every $x \in \mathbb{R}^n$

$$\begin{aligned}
|(\varphi_k \hat{a}_{jm})^\vee(x)| &= \Big| \int_{\mathbb{R}^n} \varphi_k^\vee(x-y) a_{jm}(y)dy \Big| \\
&\le C(L,d,n,\lambda,\varphi_0) \frac{2^{-L(k-j)}}{(1+2^j|x-x_{jm}|)^\lambda}.
\end{aligned}$$

Finally we deal with the case $k = j = 0$. From (3.1) with $\alpha = 0$, $N = \lambda$ and (3.2) with $\gamma = 0$ and $M = L + \lambda + n + 1$ we derive that

$$\begin{aligned} |a_{0m}(y)| &\le C(d,n,\lambda)(1+|y-x_{0m}|)^{-\lambda}, \quad y \in \mathbb{R}^n, \\ |\varphi_0^\vee(x-y)| &\le C(L,n,\lambda,\varphi_0)(1+|x-y|)^{-L-\lambda-n-1}, \quad x,y \in \mathbb{R}^n. \end{aligned}$$

Thus,

$$\begin{aligned} |(\varphi_0 \hat{a}_{0m})^\vee(x)| &\le \int_{\mathbb{R}^n} |\varphi_0^\vee(x-y)||a_{0m}(y)|dy \\ &\le C(L,d,n,\lambda,\varphi_0) \int_{\mathbb{R}^n} (1+|x-y|)^{-L-\lambda-n-1}(1+|y-x_{0m}|)^{-\lambda} dy \\ &\le C(L,d,n,\lambda,\varphi_0)(1+|x-x_{0m}|)^{-\lambda} \int_{\mathbb{R}^n} (1+|x-y|)^{-n-L-1} dy \\ &= C(L,d,n,\lambda,\varphi_0)(1+|x-x_{0m}|)^{-\lambda}. \end{aligned}$$

□

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a Lebesgue measurable function. Then

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy, \quad x \in \mathbb{R}^n,$$

is the *Hardy-Littlewood maximal operator*. Here Q is a cube containing x and $|Q|$ stands for its Lebesgue measure.

According to the Fefferman-Stein maximal inequality for vector valued Lebesgue spaces (see [18]) and interpolation properties of vector-valued Lebesgue spaces (see [38, Theorem 1.18.6.1]), for any sequence of Lebesgue measurable functions $(f_j)_{j\in\mathbb{N}_0}$, $1 < p < \infty$, $1 < q \le \infty$ and $0 < r \le \infty$, we have that

$$\Big\| \Big(\sum_{j=0}^{\infty} (Mf_j)^q \Big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\| \le C \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\|. \tag{3.4}$$

Lemma 3.3. Let $(\eta_{jm})_{\substack{j\in\mathbb{N}_0 \\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$, $0 < b \le 1$ and $\lambda > \frac{n}{b}$. Then

$$\sum_{m\in\mathbb{Z}^n} \frac{|\eta_{jm}|}{(1+2^{\min\{j,k\}}|x-x_{jm}|)^\lambda} \le C 2^{-\frac{n}{b}\min\{k-j,0\}} \Big\{ M\Big(\sum_{m\in\mathbb{Z}^n} |\eta_{jm}|^b \chi_{j,m} \Big)(x) \Big\}^{1/b},$$

for every $x \in \mathbb{R}^n$ with a constant C independent of $j,k \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Proof. For any $j,k \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, set

$$\begin{aligned} F_0 &= \{m \in \mathbb{Z}^n : |x_{jm} - x| 2^{\min\{k,j\}} \le 1\}, \\ F_u &= \{m \in \mathbb{Z}^n : 2^{u-1} \le |x_{jm} - x| 2^{\min\{k,j\}} \le 2^u\}, \quad u \in \mathbb{N}. \end{aligned}$$

We have

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n}\frac{|\eta_{jm}|}{(1+2^{\min\{j,k\}}|x-x_{jm}|)^\lambda} &= \sum_{u=0}^{\infty}\sum_{F_u}\frac{|\eta_{jm}|}{(1+2^{\min\{j,k\}}|x-x_{jm}|)^\lambda}\\ &\le \sum_{u=0}^{\infty}2^{-\lambda(u-1)}\sum_{F_u}|\eta_{jm}|\\ &\le \sum_{u=0}^{\infty}2^{-\lambda(u-1)}\big(\sum_{F_u}|\eta_{jm}|^b\big)^{1/b}.\end{aligned}$$

Note that $\big(\sum_{F_u}|\eta_{jm}|^b\big)^{1/b} = 2^{jn/b}\Big(\int_{\mathbb{R}^n}\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}(y)dy\Big)^{1/b}$. Furthermore,

$$\bigcup_{F_u}Q_{jm}\subset Q:=\prod_{\ell=1}^{n}[x_\ell-2^{u-\min\{k,j\}+1},x_\ell+2^{u-\min\{k,j\}+1}].$$

Indeed, if $y\in Q_{jm}$ and $m\in F_u$ then for every $\ell\in\{1,...,n\}$

$$|y_\ell-x_\ell|\le|y_\ell-(x_{jm})_\ell|+|(x_{jm})_\ell-x_\ell|\le 2^{-j-1}+2^{u-\min\{j,k\}}\le 2^{u-\min\{j,k\}+1}.$$

Collecting the previous estimates we have that

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^n}\frac{|\eta_{jm}|}{(1+2^{\min\{j,k\}}|x-x_{jm}|)^\lambda}\\ &\quad\le 2^{jn/b}\sum_{u=0}^{\infty}2^{-\lambda(u-1)}\Big(\int_{\mathbb{R}^n}\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}(y)dy\Big)^{1/b}\\ &\quad= 2^{jn/b}\sum_{u=0}^{\infty}2^{-\lambda(u-1)}\Big(\int_{Q}\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}(y)dy\Big)^{1/b}\\ &\quad\le C(n)2^{jn/b-\min\{j,k\}n/b}\sum_{u=0}^{\infty}2^{-\lambda(u-1)}2^{un/b}\Big\{M\Big(\sum_{F_u}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\\ &\quad\le C(n)2^{jn/b-\min\{j,k\}n/b}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\sum_{u=0}^{\infty}2^{-\lambda(u-1)}2^{un/b}\\ &\quad\le C(n,\lambda)2^{-\frac{n}{b}\min\{k-j,0\}}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n}|\eta_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b},\end{aligned}$$

where we have used that $\lambda>n/b$ in the last inequality. □

Definition 3.4. Let $a>0$ and $(f_k)_{k=0}^{\infty}$ be a sequence of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$. We define the *Peetre's maximal function* by

$$(f_k)^*_a=\sup_{y\in\mathbb{R}^n}\frac{|f_k(x-y)|}{(1+2^k|y|)^a}=\sup_{y\in\mathbb{R}^n}\frac{|f_k(y)|}{(1+2^k|x-y|)^a}.$$

We recall here an important property of Peetre's maximal function whose proof can be found in [49, Lemma 6] or [22, Lemma 2.2.3].

Proposition 3.5. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity, $f \in \mathcal{S}'(\mathbb{R}^n)$, $0 < r \le 1$ and $a = n/r$. Then

$$((\varphi_k \hat{f})^\vee)^*_a(x) \le C(n, \varphi_0, r)(M|(\varphi_k \hat{f})^\vee|^r)^{1/r}(x), \quad x \in \mathbb{R}^n.$$

Now we introduce some sequence spaces.

Definition 3.6. Let $0 < q, r \le \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$.

1. The space $\mathbf{b^s_q L_{p,r}}$ is the collection of all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$ having a finite quasi-norm
$$\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big\|\sum_{m\in\mathbb{Z}^n} |\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}.$$

2. The space $\mathbf{f^s_q L_{p,r}}$ is formed by all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$ having a finite quasi-norm
$$\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\mu_{jm}|^q\chi_{j,m}(\cdot)\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|.$$

Spaces $\mathbf{b^s_q L_{p,r}}$ can be also defined by using the Lorentz sequence spaces:

Lemma 3.7. For any $0 < q, r \le \infty, 0 < p < \infty$ and $s \in \mathbb{R}$, let $\mathbf{b^s_q}\boldsymbol{\ell}_{\mathbf{p,r}}$ be the collection of all sequences $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu_{jm})|\mathbf{b^s_q}\boldsymbol{\ell}_{\mathbf{p,r}}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|^q\Big)^{1/q}.$$

Then we have with equivalent quasi-norms

$$\mathbf{b^s_q L_{p,r}} = \mathbf{b^s_q}\boldsymbol{\ell}_{\mathbf{p,r}}\,.$$

Proof. Take any sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}} \subset \mathbb{C}$. For any $j \in \mathbb{N}_0$ put $f_j(x) = \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|\chi_{j,m}(x)$, $x \in \mathbb{R}^n$. If $\mu^j := (\mu_{jm})_{m\in\mathbb{Z}^n}$, since $|Q_{jm}| = 2^{-jn}$, we have that

$$f_j^*(t) = \sum_{k=0}^{\infty} (\mu^j)^*_k \chi_{[2^{-jn}k, 2^{-jn}(k+1))}(t)\,.$$

Therefore

$$\Big\| \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|\chi_{j,m}(x)|L_{p,r}(\mathbb{R}^n)\Big\| = \Big(\int_0^\infty [t^{1/p} f_j^*(t)]^r \frac{dt}{t}\Big)^{1/r}$$

$$= \Big(\sum_{k=0}^\infty [(\mu^j)_k^*]^r \int_{2^{-jn}k}^{2^{-jn}(k+1)} t^{r/p} \frac{dt}{t}\Big)^{1/r}$$

$$= \Big(\frac{p}{r} 2^{-jnr/p} \sum_{k=0}^\infty [(\mu^j)_k^*]^r \big((k+1)^{r/p} - k^{r/p}\big)\Big)^{1/r}$$

$$\sim 2^{-jn/p} \|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|. \tag{3.5}$$

This yields the result. □

Next we establish the atomic decomposition for function spaces with Lorentz smoothness.

Theorem 3.8. Let $0 < q, r \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$, $n, L \in \mathbb{N}$.

1. Let $L > \max\{n(\frac{1}{\min\{p,r\}} - 1)_+ - s, s\}$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, there exists a sequence of L-atoms $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and a sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ in $\mathbf{b^s_q L_{p,r}}$ such that

$$f(x) = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad \text{(convergence in } \mathcal{S}'(\mathbb{R}^n)).$$

   Moreover, $\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| \sim \inf\{\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|\}$.
2. Let $L > \max\{n(\frac{1}{\min\{p,q\}} - 1)_+ - s, s\}$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, there exists a sequence of L-atoms $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and a sequence $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ in $\mathbf{f^s_q L_{p,r}}$, such that

$$f(x) = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x) \quad \text{(convergence in } \mathcal{S}'(\mathbb{R}^n)).$$

   Moreover, $\|f|F^s_q L_{p,r}(\mathbb{R}^n)\| \sim \inf\{\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|\}$.

Proof. Assume that $f(x) = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \mu_{jm} a_{jm}(x)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$) with $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ satisfying the assumptions of (1). Let $0 < b < \min\{p, r, 1\}$ such that $L > n(\frac{1}{b} - 1) - s > n(\frac{1}{\min\{p,r\}} - 1)_+ - s$. Take $\lambda > \max\{L, n/b\}$. It follows from Lemmas 3.2 and 3.3 that for any

$k \in \mathbb{N}_0$,

$$
\begin{aligned}
|(\varphi_k \hat{f})^\vee(x)| &\le \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| |(\varphi_k \hat{a}_{jm})^\vee|(x) \\
&\lesssim \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| \frac{2^{-L|k-j|} 2^{n \min\{k-j,0\}}}{(1+2^{\min\{k,j\}}|x-x_{jm}|)^\lambda} \\
&\lesssim \sum_{j=0}^{\infty} \Big\{ M\Big(\sum_{m \in \mathbb{Z}^n} 2^{-L|k-j|b} 2^{-n \min\{k-j,0\}(1-b)} |\mu_{jm}|^b \chi_{j,m} \Big)(x) \Big\}^{1/b}.
\end{aligned}
\tag{3.6}
$$

Set $g_j(x) = \sum_{m \in \mathbb{Z}^n} 2^{-L|k-j|b} 2^{-n \min\{k-j,0\}(1-b)} |\mu_{jm}|^b \chi_{j,m}(x)$, $j \in \mathbb{N}_0$. Using (3.4) and the fact that $L_{p,r}(\mathbb{R}^n)$ is b-Banach, we derive that

$$
\begin{aligned}
&\|(\varphi_k \hat{f})^\vee(x) | L_{p,r}(\mathbb{R}^n)\| \lesssim \Big\| \Big(\sum_{j=0}^{\infty} (M g_j)^{1/b} \Big)^b | L_{p/b,r/b}(\mathbb{R}^n) \Big\|^{1/b} \\
&\lesssim \Big\| \Big(\sum_{j=0}^{\infty} |g_j|^{1/b} \Big)^b | L_{p/b,r/b}(\mathbb{R}^n) \Big\|^{1/b} = \Big\| \sum_{j=0}^{\infty} |g_j|^{1/b} | L_{p,r}(\mathbb{R}^n) \Big\| \\
&\lesssim \Big(\sum_{j=0}^{\infty} \Big\| |g_j|^{1/b} | L_{p,r}(\mathbb{R}^n) \Big\|^b \Big)^{1/b} \\
&= \Big(\sum_{j=0}^{\infty} 2^{-L|k-j|b} 2^{-n \min\{k-j,0\}(1-b)} \Big\| \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| \chi_{j,m}(\cdot) | L_{p,r}(\mathbb{R}^n) \Big\|^b \Big)^{1/b}.
\end{aligned}
\tag{3.7}
$$

Consequently, if $q^* = \min\{q/b, 1\}$ we obtain that

$$
\begin{aligned}
\|f | B^s_q L_{p,r}(\mathbb{R}^n)\| &= \Big(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^\vee | L_{p,r}(\mathbb{R}^n)\|^q \Big)^{1/q} \\
&\lesssim \Big(\sum_{k=0}^{\infty} \Big(\sum_{j=0}^{\infty} 2^{(k-j)sb} 2^{-L|k-j|b} 2^{-n \min\{k-j,0\}(1-b)} 2^{jsb} \\
&\qquad \times \Big\| \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| \chi_{j,m}(\cdot) | L_{p,r}(\mathbb{R}^n) \Big\|^b \Big)^{q/b} \Big)^{1/q} \\
&\lesssim \Big(\sum_{j=-\infty}^{\infty} 2^{jsbq*} 2^{-L|j|bq^*} 2^{-n \min\{j,0\}(1-b)q^*} \Big)^{1/bq^*} \\
&\qquad \times \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| \chi_{j,m}(\cdot) | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{1/q}
\end{aligned}
\tag{3.8}
$$

where we have used Young's inequality for convolution in the last inequality. Taking into consideration that $L > s$ and $L > n(\frac{1}{b} - 1) - s$, it follows that

$$\begin{aligned}\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| &\lesssim \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big\| \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ &= \|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|.\end{aligned}$$

Assume now that $f \in B^s_q L_{p,r}(\mathbb{R}^n)$. According to [20, Lemma 5.12], [19, p. 783] and [24, Lemma 3.11], there exist Θ_0, Θ, Φ_0, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with the following properties:

- supp $\Theta_0, \Theta \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} x^\beta \Theta(x)dx = 0$, for every $|\beta| \leq L - 1$.
- supp $\Phi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$ and supp $\Phi \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$.
- $\hat{\Theta}_0(x)\Phi_0(x) + \sum_{j=1}^{\infty} \hat{\Theta}(2^{-j}x)\Phi(2^{-j}x) = 1$, for all $x \in \mathbb{R}^n$.

Then, it follows from [22, Proposition 1.1.6/(b)] that

$$\begin{aligned} f(x) &= (\hat{\Theta}_0\Phi_0\hat{f})^\vee(x) + \sum_{j=1}^{\infty}(\hat{\Theta}(2^{-j}\cdot)\Phi(2^{-j}\cdot)\hat{f})^\vee(x) \\ &= \Theta_0 * (\Phi_0\hat{f})^\vee(x) + \sum_{j=1}^{\infty} 2^{jn}\Theta(2^j\cdot) * (\Phi(2^{-j}\cdot)\hat{f})^\vee(x), \end{aligned}$$

being the convergence in $\mathcal{S}'(\mathbb{R}^n)$. For every $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, set

$$\begin{aligned} &\mu_{jm} = C \sup_{y\in Q_{jm}} |(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)| \quad \text{with} \\ &C = \max\Big\{ \sup_{|\gamma|\leq L}\sup_{|x|\leq 1} |\partial^\gamma\Theta(x)|, \sup_{|\gamma|\leq L}\sup_{|x|\leq 1} |\partial^\gamma\Theta_0(x)|\Big\} \end{aligned} \tag{3.9}$$

and

$$a_{jm}(x) = \begin{cases} \frac{1}{\mu_{0m}} \int_{Q_{0m}} \Theta_0(x-y)(\Phi_0\hat{f})^\vee(y)dy & \text{if } j = 0, \\ \frac{2^{jn}}{\mu_{jm}} \int_{Q_{jm}} \Theta(2^j(x-y))(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)dy & \text{if } j \in \mathbb{N}. \end{cases} \tag{3.10}$$

Notice that

$$\begin{aligned} \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \mu_{jm}a_{jm}(x) &= \sum_{m\in\mathbb{Z}^n}\int_{Q_{0m}} \Theta_0(x-y)(\Phi_0\hat{f})^\vee(y)dy \\ &\quad + \sum_{j=1}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jn}\int_{Q_{jm}} \Theta(2^j(x-y))(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)dy \\ &= \Theta_0 * (\Phi_0\hat{f})^\vee(x) + \sum_{j=1}^{\infty} 2^{jn}\Theta(2^j\cdot) * (\Phi(2^{-j}\cdot)\hat{f})^\vee(x) = f(x), \end{aligned}$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, $(a_{jm})_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ is a sequence of L-atoms. Indeed, for every $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, the support of a_{jm} satisfies

$$\text{supp } a_{jm} \subset Q_{jm} + \text{supp } \Theta(2^j\cdot) = Q_{jm} + \{x \in \mathbb{R}^n : |x| \le 2^{-j}\} \subset 3Q_{jm}.$$

Furthermore, if $j \in \mathbb{N}$ we have that for any multi-index α with $|\alpha| \le L$

$$\begin{aligned}|\partial^\alpha a_{jm}(x)| &\le \frac{2^{jn}}{\mu_{jm}} \int_{Q_{jm}} 2^{j|\alpha|} |\partial^\alpha \Theta(2^j(x-y))| |(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)| dy \\ &\le \frac{2^{j(n+|\alpha|)}}{\mu_{jm}} \sup_{y\in Q_{jm}} |(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)| \sup_{x\in\mathbb{R}^n} |\partial^\alpha \Theta(x)| |Q_{jm}| \le 2^{j|\alpha|}.\end{aligned}$$

Analogously for $j = 0$. Finally, for any multi-index β with $|\beta| \le L-1$ and $j \in \mathbb{N}$ it holds that

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x) dx = \frac{2^{jn}}{\mu_{jm}} \int_{\mathbb{R}^n} x^\beta \Big(\int_{Q_{jm}} \Theta(2^j(x-y)) (\Phi(2^{-j}\cdot)\hat{f})^\vee(y) dy \Big) dx = 0,$$

due to the moment conditions on Θ.

Let now $(\varphi_j)_{j\in\mathbb{N}_0}$ be the dyadic resolution of unity in (2.5). For any $x, y \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and $a > \frac{n}{\min\{1,p\}}$ we have that

$$\begin{aligned}&|(\Phi(2^{-j}\cdot)\hat{f})^\vee(y)| \\ &= \Big| \sum_{\ell=-1}^{1} (\varphi_{j+\ell}\Phi(2^{-j}\cdot)\hat{f})^\vee(y) \Big| \le \sum_{\ell=-1}^{1} 2^{jn} \int_{\mathbb{R}^n} |\Phi^\vee(2^j(y-z))| |(\varphi_{j+\ell}\hat{f})^\vee(z)| dz \\ &\lesssim 2^{jn}(1+2^j|y-x|)^a \\ &\qquad \times \sum_{\ell=-1}^{1} \int_{\mathbb{R}^n} (1+2^j|y-z|)^{-n-1} (1+2^j|z-x|)^{-a} |(\varphi_{j+\ell}\hat{f})^\vee(z)| dz \\ &\le 2^{jn} \Big(\int_{\mathbb{R}^n} (1+2^j|y-z|)^{-n-1} dz \Big) (1+2^j|y-x|)^a \sum_{\ell=-1}^{1} ((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x) \\ &\lesssim (1+2^j|y-x|)^a \sum_{\ell=-1}^{1} ((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x).\end{aligned}$$

Whence, for $b = n/a < \min\{1,p\}$, using Proposition 3.5 we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\mu_{jm}| \chi_{j,m}(x) &\lesssim \sum_{m\in\mathbb{Z}^n} \sup_{y\in Q_{jm}} (1+2^j|y-x|)^a \sum_{\ell=-1}^{1} ((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x) \chi_{j,m}(x) \\ &\lesssim \sum_{\ell=-1}^{1} ((\varphi_{j+\ell}\hat{f})^\vee)^*_a(x) \lesssim \sum_{\ell=-1}^{1} (M|(\varphi_{j+\ell}\hat{f})^\vee|^b(x))^{1/b}.\end{aligned} \tag{3.11}$$

Thus, from (3.11) and (3.4)

$$
\begin{aligned}
\|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\| &\lesssim \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big\| \sum_{\ell=-1}^{1} (M|(\varphi_{j+\ell}\hat{f})^\vee|^b(x))^{1/b}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\
&\lesssim \sum_{\ell=-1}^{1}\Big(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_{j+\ell}\hat{f})^\vee|L_{p,r}(\mathbb{R}^n)\|^q\Big)^{1/q} \\
&\lesssim \|f|B^s_qL_{p,r}(\mathbb{R}^n)\|.
\end{aligned}
$$

The proof of (2) follows the same pattern. Assume that

$$
f(x) = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \mu_{jm}a_{jm}(x), \quad \text{(convergence in } \mathcal{S}'(\mathbb{R}^n)),
$$

with $(a_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ and $(\mu_{jm})_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}$ satisfying the assumptions in (2). Let $0 < b < \min\{1,p,q\}$ such that $L > n(1/b-1)-s > n(\frac{1}{\min\{p,q\}}-1)_+-s$. Following the same ideas as in (3.6), (3.7) and (3.8), now with $q^* = \min\{1,q\}$, we derive that for any $x\in\mathbb{R}^n$

$$
\begin{aligned}
&\Big(\sum_{k=0}^{\infty}[2^{ks}|(\varphi_k\hat{f})^\vee|(x)]^q\Big)^{1/q} \\
&\quad\lesssim \Big(\sum_{k=0}^{\infty}\Big[\sum_{j=0}^{\infty} 2^{(k-j)s}2^{-L|k-j|}2^{-n\min\{k-j,0\}(1/b-1)} \\
&\qquad\qquad \times \Big\{M\Big(\sum_{m\in\mathbb{Z}^n} 2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{1/b}\Big]^q\Big)^{1/q} \\
&\quad\lesssim \Big(\sum_{j=-\infty}^{\infty}[2^{js}2^{-L|j|}2^{-n\min\{j,0\}(1/b-1)}]^{q^*}\Big)^{1/q^*} \\
&\qquad\qquad \times \Big(\sum_{j=0}^{\infty}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n} 2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{q/b}\Big)^{1/q}.
\end{aligned}
$$

The first sum in the last inequality is finite since $L > s$ and $L > n(1/b-1)-s$. Consequently using (3.4), we finally get that

$$
\begin{aligned}
\|f|F^s_qL_{p,r}(\mathbb{R}^n)\| &\lesssim \Big\|\Big(\sum_{j=0}^{\infty}\Big\{M\Big(\sum_{m\in\mathbb{Z}^n} 2^{jsb}|\mu_{jm}|^b\chi_{j,m}\Big)(x)\Big\}^{q/b}\Big)^{b/q}|L_{p/b,r/b}(\mathbb{R}^n)\Big\|^{1/b} \\
&\lesssim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\mu_{jm}|^q\chi_{j,m}(x)\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| = \|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|.
\end{aligned}
$$

Conversely, assume that $f\in F^s_qL_{p,r}(\mathbb{R}^n)$. Take $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ and $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ as in (3.9) and (3.10), then (a_{jm}) is a sequence of L-atoms and $f(x) =$

$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$). Take any $0<b<\min\{1,p,q\}$. Proceeding as in (3.11), we have that

$$\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{j,m}(x)\lesssim\sum_{\ell=-1}^{1}(M|(\varphi_{j+\ell}\hat{f})^\vee|^b(x))^{q/b},\quad x\in\mathbb{R}^n.$$

Consequently,

$$\begin{aligned}\|(\mu_{jm})|\mathbf{f^s_qL_{p,r}}\| &\lesssim\sum_{\ell=-1}^{1}\Big\|\Big(\sum_{j=0}^{\infty}\Big(2^{jsb}M|(\varphi_{j+\ell}\hat{f})^\vee|^b(x)\Big)^{q/b}\Big)^{b/q}|L_{p/b,r/b}(\mathbb{R}^n)\Big\|^{1/b}\\ &\lesssim\sum_{\ell=-1}^{1}\Big\|\Big(\sum_{j=0}^{\infty}\Big(2^{js}|(\varphi_{j+\ell}\hat{f})^\vee|(x)\Big)^{q}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|\\ &\lesssim\|f|F^s_qL_{p,r}(\mathbb{R}^n)\|.\end{aligned}$$

This completes the proof. □

We close this section by showing that in the statement of Theorem 3.8 we may change convergence in $\mathcal{S}'(\mathbb{R}^n)$ by unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$. As usual, for any $1\le r\le\infty$, we put $1/r'=1-1/r$.

Proposition 3.9. Let $0<q,r\le\infty$, $0<p<\infty$, $s\in\mathbb{R}$, $n,L\in\mathbb{N}$.

1. Let $L>\max\{n(\frac{1}{\min\{p,r\}}-1)_+-s,s\}$, let $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\in\mathbf{b^s_qL_{p,r}}$ and assume that $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ is a sequence of L-atoms. Put

$$f(x)=\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x)\quad\text{(convergence in }\mathcal{S}'(\mathbb{R}^n)).$$

   Then the convergence of the series is unconditional on $\mathcal{S}'(\mathbb{R}^n)$.

2. Let $L>\max\{n(\frac{1}{\min\{p,q\}}-1)_+-s,s\}$, let $(\mu_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}\in\mathbf{f^s_qL_{p,r}}$ and assume that $(a_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ is a sequence of L-atoms. Put

$$f(x)=\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mu_{jm}a_{jm}(x)\quad\text{(convergence in }\mathcal{S}'(\mathbb{R}^n)).$$

   Then the convergence of the series is unconditional on $\mathcal{S}'(\mathbb{R}^n)$.

Proof. It suffices to show that the series $\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}(\mu_{jm}a_{jm},\varphi)$ converges absolutely for every $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Assume first that $1<p<\infty$ and let $1<p_1<p<p_2<\infty$, $M>n/p_2'$, $N>L+n$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Then

$$\sup_{|\beta|=L}\sup_{x\in\mathbb{R}^n}|\partial^\beta\varphi(x)|(1+|x|)^M<\infty,$$

and according to (3.1) $\sup_{x\in\mathbb{R}^n}|a_{jm}(x)|(1+2^j|x-x_{jm}|)^N$ is uniformly bounded on j and m. Since a_{jm} has vanishing moments up to order $L-1$, using Taylor's theorem (see [22, Appendix B.2, p. 596]), we derive that

$$\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| = \sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\Big|\int_{\mathbb{R}^n}a_{jm}(x)\varphi(x)dx\Big| \lesssim \sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\frac{2^{-j(L+n)}}{(1+|x_{jm}|)^M}, \tag{3.12}$$

where the constant in the last inequality is independent of $j\in\mathbb{N}_0$.

Let $T_j((\mu_{jm})_{m\in\mathbb{Z}^n}) = \Big(2^{-j(L+n)}\frac{\mu_{jm}}{(1+|x_{jm}|)^M}\Big)_{m\in\mathbb{Z}^n}$. Then $T_j:\ell_{p_k}(\mathbb{Z}^n)\longrightarrow \ell_1(\mathbb{Z}^n)$ is bounded for $k=1,2$, because

$$\begin{aligned}\|T_j(\mu_{jm})|\ell_1(\mathbb{Z}^n)\| &= 2^{-j(L+n)}\sum_{m\in\mathbb{Z}^n}\frac{|\mu_{jm}|}{(1+|x_{jm}|)^M}\\ &\le 2^{-j(L+n)}\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^{p_k}\Big)^{1/p_k}\Big(\sum_{m\in\mathbb{Z}^n}(1+|x_{jm}|)^{-Mp_k'}\Big)^{1/p_k'}\\ &\lesssim 2^{-j(L+n/p_k)}\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^{p_k}\Big)^{1/p_k}\Big(\int_{\mathbb{R}^n}(1+|x|)^{-Mp_k'}dx\Big)^{1/p_k'}.\end{aligned}$$

The last expression is finite since $M>n/p_2'>n/p_1'$. By the interpolation property (2.1) and (2.3) we obtain that T_j is bounded from $\ell_{p,r}(\mathbb{Z}^n)$ to $\ell_1(\mathbb{Z}^n)$ and

$$\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\frac{2^{-j(L+n)}}{(1+|x_{jm}|)^M}\lesssim 2^{-j(L+n/p)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|, \tag{3.13}$$

where the constant in the last inequality is independent of $j\in\mathbb{N}_0$. Now (3.12), (3.13) together with (3.5) imply that

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| &\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/p)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|\\ &\lesssim \sum_{j=0}^{\infty}2^{-j(L+s)}\|(\mu_{jm})|\mathbf{b^s_qL_{p,r}}\|<\infty,\end{aligned}$$

since $(\mu_{jm})\in\mathbf{b^s_qL_{p,r}}$ and $L>\max\{n(\frac{1}{\min\{p,r\}}-1)_+-s,s\}>-s$.

If $0<p\le 1$, as $L>n(1/p-1)-s$, there exists $P>1$ such that $L>n(1/p-1/P)-s$. Proceeding as before and noting that $\ell_{p,r}(\mathbb{Z}^n)\hookrightarrow\ell_{P,r}(\mathbb{Z}^n)$

we have that

$$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| \lesssim \sum_{j=0}^{\infty}2^{-j(L+n/P)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{P,r}(\mathbb{Z}^n)\|$$
$$\lesssim \sum_{j=0}^{\infty}2^{-j(L+n/P)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|$$
$$\lesssim \sum_{j=0}^{\infty}2^{-j(L+n(1/P-1/p)+s)}\|(\mu_{jm})|\mathbf{b_q^sL_{p,r}}\| < \infty,$$

since $(\mu_{jm}) \in \mathbf{b_q^sL_{p,r}}$ and $L > n(\frac{1}{p}-\frac{1}{P})-s$. This establishes (1).

In order to prove (2) first observe that

$$\Big\|\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|\chi_{jm}|L_{p,r}(\mathbb{R}^n)\Big\| = \Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{jm}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\| \tag{3.14}$$

since for any fixed $j \in \mathbb{N}_0$ the dyadic cubes $(Q_{jm})_{m\in\mathbb{Z}^n}$ are pairwise disjoints. If $1 < p < \infty$, proceeding as for the part (1) we obtain

$$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|(\mu_{jm}a_{jm},\varphi)| \lesssim \sum_{j=0}^{\infty}2^{-j(L+n/p)}\|(\mu_{jm})_{m\in\mathbb{Z}^n}|\ell_{p,r}(\mathbb{Z}^n)\|$$
$$\sim \sum_{j=0}^{\infty}2^{-jL}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^q\chi_{j,m}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|$$
$$\sim \sum_{j=0}^{\infty}2^{-j(L+s)}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}2^{jsq}|\mu_{jm}|^q\chi_{j,m}\Big)^{1/q}|L_{p,r}(\mathbb{R}^n)\Big\|$$
$$\leq \sum_{j=0}^{\infty}2^{-j(L+s)}\|(\mu_{jm})|\mathbf{f_q^sL_{p,r}}\| < \infty,$$

because now $(\mu_{jm}) \in \mathbf{f_q^sL_{p,r}}$ and $L > \max\{n(\frac{1}{\min\{p,q\}}-1)_+ - s, s\} > -s$. For the case $0 < p \leq 1$, taking into consideration (3.14), we can proceed again as in (1). □

4. Wavelet characterization of Lorentz smoothness function spaces

For $L \in \mathbb{N}$, let $\psi_F, \psi_M \in C^L(\mathbb{R})$ be real-valued compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}}\psi_F(t)^2dt = 1, \quad \int_{\mathbb{R}}\psi_M(t)^2dt = 1 \quad \text{and} \quad \int_{\mathbb{R}}\psi_M(t)t^\ell dt = 0, \quad \ell < L.$$

Let $G = (G_1, ..., G_n) \in G^0 = \{F, M\}^n$ which means that each G_ℓ is either F or M. For $j \in \mathbb{N}$, we define $G^j = \{F, M\}^{n*}$ as the collection of all

$G = (G_1, ..., G_n)$ such that each G_ℓ is either F or M but at least one of the components of G must be M. Put

$$\psi^j_{G,m}(x) = 2^{jn/2} \prod_{\ell=1}^{n} \psi_{G_\ell}(2^j x_\ell - m_\ell), \quad G \in G^j,\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0. \tag{4.1}$$

Then $\{\psi^j_{G,m} : G \in G^j, m \in \mathbb{Z}^n, j \in \mathbb{N}_0\}$ is called a *wavelet system.*

It is well-known that classical Besov and Triebel-Lizorkin spaces can be represented via wavelets (see, for example, [42, Theorem 1.20]). For Triebel-Lizorkin-Lorentz spaces the characterization was established by Yang, Cheng and Peng [49, Theorems 3 and 4] and for Besov-Lorentz spaces by Almeida [1, Corollary 3.2] but only when $q = r$, that is, for the spaces $B^s_q L_{p,q}(\mathbb{R}^n)$. Restriction $q = r$ is motivated by the interpolation techniques used in [1]. Next we are going to eliminate this restriction by using the atomic decomposition of Lorentz smoothness spaces (Theorem 3.8) and the approach developed by Haroske, Skandera and Triebel [25]. The same technique also applies for Triebel-Lizorkin-Lorentz spaces giving an alternative proof for their characterization in terms of wavelets.

We start by introducing the relevant sequence spaces.

Definition 4.1. Let $0 < q, r \le \infty$, $0 < p < \infty$ and $s \in \mathbb{R}$.

1. The space $b^s_q L_{p,r}$ is the collection of all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm
$$\|(\mu^{j,G}_m)|b^s_q L_{p,r}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G \in G^j} \Big\| \sum_{m \in \mathbb{Z}^n} |\mu^{j,G}_m| \chi_{j,m}(\cdot) | L_{p,r}(\mathbb{R}^n) \Big\|^q \Big)^{1/q}.$$

2. The space $f^s_q L_{p,r}$ is formed by all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm
$$\|(\mu^{j,G}_m)|f^s_q L_{p,r}\| = \Big\| \big(\sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} 2^{jsq} |\mu^{j,G}_m|^q \chi_{j,m}(\cdot) \big)^{1/q} | L_{p,r}(\mathbb{R}^n) \Big\|.$$

Space $b^s_q L_{p,r}$ can be also defined by using Lorentz sequence space $\ell_{p,r}$: Consider the space $b^s_q \ell_{p,r}$ formed by all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu^{j,G}_m)|b^s_q \ell_{p,r}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G \in G^j} \|\mu^{j,G} | \ell_{p,r}(\mathbb{Z}^n)\|^q \Big)^{1/q}$$

where $\mu^{j,G} = (\mu^{j,G}_m)_{m \in \mathbb{Z}^n}$. With the same argument as in Lemma 3.7 but putting now $f_{j,G}(x) = \sum_{m \in \mathbb{Z}} |\lambda^{j,G}_m| \chi_{j,m}(x)$, we derive that

$$b^s_q L_{p,r} = b^s_q \ell_{p,r} \quad \text{(equivalent quasi-norms).} \tag{4.2}$$

Remark 4.2. There is a useful connection between these sequence spaces and those of Definition 3.6. To describe it, let $G^* = \{F, M\}^{n*}$ and $G_0 = \{F\}^n$. If $(\mu_m^{j,G}) \in b_q^s L_{p,r}$ we have

$$\begin{aligned}
&\|(\mu_m^{j,G})|b_q^s L_{p,r}\| \\
&= \Big(\Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q \\
&\qquad + \sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^*} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{j,G}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q \Big)^{1/q} \\
&\sim \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| \\
&\qquad + \sum_{G\in G^*} \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{j,G}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q \Big)^{1/q} \\
&= \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| + \sum_{G\in G^*} \|(\mu_m^{j,G})|\mathbf{b_q^s L_{p,r}}\|.
\end{aligned}$$

Analogously, if $(\mu_m^{j,G}) \in f_q^s L_{p,r}$ then

$$\|(\mu_m^{j,G})|f_q^s L_{p,r}\| \sim \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_m^{0,G_0}|\chi_{0,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\| + \sum_{G\in G^*} \|(\mu_m^{j,G})|\mathbf{f_q^s L_{p,r}}\|.$$

Note that the above computations give equivalent quasi-norms in $b_q^s L_{p,r}$ and $f_q^s L_{p,r}$, respectively.

Let $m \in \mathbb{Z}^n$, $j, J \in \mathbb{N}_0$, $d > 1$ and $C_1 > 0$. From now on put

$$I_J^j(m) = \{M \in \mathbb{Z}^n : dQ_{JM} \cap C_1 Q_{jm} \neq \emptyset\}.$$

The following result shows that spaces $\mathbf{b_q^s L_{p,r}}$ and $\mathbf{f_q^s L_{p,r}}$ are κ-sequence spaces in the sense of [25, Definition 4.1].

Lemma 4.3. Let $0 < p < \infty$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$, $d > 1$ and $C_1 > 0$.

1. If $\kappa > \max\{s, n(\frac{1}{\min\{p,r\}} - 1)_+ - s, \frac{n}{p} - s\}$, then $\mathbf{b_q^s L_{p,r}}$ has the following properties.
   a) If $(\mu_{jm}) \in \mathbf{b_q^s L_{p,r}}$ and for every $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, $(\lambda_{jm}) \subset \mathbb{C}$ satisfy
   $$|\lambda_{jm}| \leq C_1 \sum_{J\in\mathbb{N}_0} 2^{-\kappa|J-j|} \sum_{M\in I_J^j(m)} 2^{-n(J-j)_+}|\mu_{JM}|, \tag{4.3}$$
   then $(\lambda_{jm}) \in \mathbf{b_q^s L_{p,r}}$ and $\|(\lambda_{jm})|\mathbf{b_q^s L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{b_q^s L_{p,r}}\|$.

b) For any cube Q there exists a constant $C_Q > 0$ such that for any $(\mu_{jm}) \in \mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}(\mathbb{R}^n)$, we have that

$$|\mu_{JM}| \leq C_Q 2^{J\kappa} \|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\|,$$

for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$ such that $Q_{J,M} \subset Q$.

2. If $\kappa > \max\{s, n(\frac{1}{\min\{p,q\}} - 1)_+ - s, \frac{n}{p} - s\}$, then $\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ has the following properties.

a) If $(\mu_{jm}) \in \mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfy (4.3), then $(\lambda_{jm}) \in \mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and

$$\|(\lambda_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| \lesssim \|(\mu_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\|.$$

b) For any cube Q there exists a constant $C_Q > 0$ such that for any $(\mu_{jm}) \in \mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$, we have that

$$|\mu_{JM}| \leq C_Q 2^{J\kappa} \|(\mu_{jm})|\mathbf{f}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\|,$$

for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$ such that $Q_{J,M} \subset Q$.

Proof. Let $(\mu_{jm}) \in \mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfying (4.3). Then for any $x \in \mathbb{R}^n$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}(x) &\lesssim \sum_{m\in\mathbb{Z}^n}\sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{M\in I^j_J(m)} |\mu_{JM}|\chi_{j,m}(x) \\
&= \sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{m\in\mathbb{Z}^n}\sum_{M\in I^j_J(m)} |\mu_{JM}|\chi_{j,m}(x) \\
&= \sum_{J=0}^{\infty} 2^{-\kappa|J-j|}2^{-n(J-j)_+} \sum_{M\in\mathbb{Z}^n} |\mu_{JM}| \sum_{\substack{m\in\mathbb{Z}^n \\ dQ_{JM}\cap C_1Q_{jm}\neq\emptyset}} \chi_{j,m}(x).
\end{aligned}$$

It is straightforward that if $dQ_{JM} \cap C_1Q_{jm} \neq \emptyset$, then $Q_{jm} \subset Q_{j,J,M}$, where

$$Q_{j,J,M} = \prod_{\ell=1}^{n}[2^{-J}M_\ell - D2^{-\min\{j,J\}-1}, 2^{-J}M_\ell + D2^{-\min\{j,J\}-1}]$$

and $D = 1 + d + C_1$. We have

$$\begin{aligned}
M\chi_{J,M}(x) \geq \frac{1}{|Q_{j,J,M}|}\int_{Q_{j,J,M}} \chi_{J,M}(y)dy &= \frac{|Q_{JM} \cap Q_{j,J,M}|}{|Q_{j,J,M}|} \\
&= \frac{|Q_{JM}|}{|Q_{j,J,M}|} = \frac{1}{D^n}2^{-n(J-j)_+},
\end{aligned}$$

for any $x \in Q_{j,J,M}$. This implies that

$$\sum_{\substack{m\in\mathbb{Z}^n \\ dQ_{JM}\cap C_1Q_{jm}\neq\emptyset}} \chi_{j,m}(x) \leq \chi_{Q_{j,J,M}}(x) \lesssim 2^{n(J-j)_+}M\chi_{J,M}(x), \quad x \in \mathbb{R}^n.$$

Let now $0 < b < \min\{1, p, r\}$ such that $\kappa > n(1/b - 1) - s > n(\frac{1}{\min\{p,r\}} - 1)_+ - s$. Then

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}(x) &\lesssim \sum_{J=0}^{\infty} 2^{-\kappa|J-j|} 2^{-n(J-j)_+} \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{Q_{j,J,M}}(x) \\ &\lesssim \sum_{J=0}^{\infty} \sum_{M\in\mathbb{Z}^n} \Big(|\mu_{JM}|^b 2^{-\kappa|J-j|b} 2^{-n(J-j)_+(b-1)} M\chi_{J,M}(x)\Big)^{1/b} \\ &= \sum_{J=0}^{\infty} \sum_{M\in\mathbb{Z}^n} \Big\{M(|\mu_{JM}|^b 2^{-\kappa|J-j|b} 2^{-n(J-j)_+(b-1)} \chi_{J,M})(x)\Big\}^{1/b}.\end{aligned}$$

Using now (3.4), as we did in (3.7), we derive that

$$\begin{aligned}&\Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}(x)|L_{p,r}\Big\| \\ &\quad\lesssim \Big\|\Big(\sum_{J=0}^{\infty} 2^{-\kappa|J-j|} 2^{-n(J-j)_+(1-1/b)} \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{J,M})(x)\Big|L_{p,r}(\mathbb{R}^n)\Big\| \\ &\quad\lesssim \Big(\sum_{J=0}^{\infty} 2^{-\kappa|J-j|b} 2^{-n(J-j)_+(b-1)} \Big\| \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{J,M}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^b\Big)^{1/b},\end{aligned}$$

where in the last inequality we have used that $L_{p,r}$ is b-Banach. Now proceeding as in (3.8) we have that for $q^* = \min\{q/b, 1\}$

$$\begin{aligned}\|(\lambda_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| &= \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\chi_{j,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ &\lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{J=0}^{\infty} 2^{(j-J)sb} 2^{-\kappa|j-J|b} 2^{-n\min\{j-J,0\}(1-b)} 2^{Jsb} \\ &\qquad\times \Big\| \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{J,M}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^b\Big)^{q/b}\Big)^{1/q} \\ &\lesssim \Big(\sum_{j=-\infty}^{\infty} 2^{[js-\kappa|j|-n\min\{j,0\}(1/b-1)]bq^*}\Big)^{1/(q^*b)} \\ &\qquad\times \Big(\sum_{J=0}^{\infty} 2^{Jsq} \Big\| \sum_{M\in\mathbb{Z}^n} |\mu_{JM}|\chi_{J,M}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}.\end{aligned}$$

Since $\kappa > n(1/b-1) - s$ and $\kappa > s$, the first series is finite and we get that $\|(\lambda_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| \lesssim \|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\|$.

On the other hand, we have

$$\begin{aligned}\|(\mu_{jm})|\mathbf{b}^{\mathbf{s}}_{\mathbf{q}}\mathbf{L}_{\mathbf{p},\mathbf{r}}\| &= \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|\chi_{j,m}(\cdot)|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\ &\ge 2^{Js}|\mu_{JM}|2^{-Jn/p},\end{aligned}$$

for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$. As $\kappa > n/p - s$, we have that for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$

$$|\mu_{JM}| \leq 2^{\kappa J} \|(\mu_{jm})|\mathbf{b^s_q L_{p,r}}\|.$$

For the sequence space $\mathbf{f^s_q L_{p,r}}$, the second property can be proven in the same way.

$$\begin{aligned}\|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| &= \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} |\mu_{jm}|^q \chi_{j,m} \Big)^{1/q} |L_{p,r}(\mathbb{R}^n) \Big\| \\ &\geq 2^{Js} |\mu_{JM}| 2^{-Jn/p} = 2^{J(s-n/p)} |\mu_{JM}|,\end{aligned}$$

for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$. As $\kappa > n/p - s$, we get that for every $J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$

$$|\mu_{JM}| \leq 2^{\kappa J} \|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|.$$

The first property for $\mathbf{f^s_q L_{p,r}}$ can be also derived with similar arguments to those we have used for $\mathbf{b^s_q L_{p,r}}$, but we prefer to check it with the help of interpolation. Let $0 < p_1 < p < p_2$ such that $\kappa > \max\{s, n(\frac{1}{\min\{p_1,q\}} - 1)_+ - s\}$. Proceeding as in [7, Theorem 3.5], we have that $(\mathbf{f^s_q L_{p_1,p_1}}, \mathbf{f^s_q L_{p_2,p_2}})_{\theta,r} = \mathbf{f^s_q L_{p,r}}$ for $\theta = \frac{p_2(p-p_1)}{p(p_2-p_1)}$. Consider the operator

$$T(\mu_{jm}) = \Big(\sum_{J\in\mathbb{N}_0} 2^{-\kappa|J-j|} \sum_{M\in I^j_J(m)} 2^{-n(J-j)_+} |\mu_{JM}| \Big)_{\substack{j\in\mathbb{N}_0,\\ m\in\mathbb{Z}^n}}.$$

T is sublinear and $T : \mathbf{f^s_q L_{p_k,p_k}} \longrightarrow \mathbf{f^s_q L_{p_k,p_k}}$ is bounded for $k = 1, 2$ (see [25, Proposition 6.5]). Then $T : \mathbf{f^s_q L_{p,r}} \longrightarrow \mathbf{f^s_q L_{p,r}}$ is bounded, which means that if $(\mu_{jm}) \in \mathbf{f^s_q L_{p,r}}$ and $(\lambda_{jm}) \subset \mathbb{C}$ satisfy (4.3), then

$$\|(\lambda_{jm})|\mathbf{f^s_q L_{p,r}}\| \lesssim \|T(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\| \lesssim \|(\mu_{jm})|\mathbf{f^s_q L_{p,r}}\|.$$

□

Now we are ready to establish the characterization by means of wavelets for spaces with Lorentz smoothness.

Theorem 4.4. Let $0 < p < \infty$, $0 < q, r \leq \infty$, $s \in \mathbb{R}$ and let $\psi^j_{G,m}$ be the wavelets in (4.1).

1. Assume that

$$L > \max\{s, n(\frac{1}{\min\{p,r\}} - 1)_+ - s, \frac{n}{p} - s\}.$$

   Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_q L_{p,r}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m 2^{-jn/2} \psi^j_{G,m}(x), \quad (\lambda^{j,G}_m) \in b^s_q L_{p,r} \tag{4.4}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (4.4) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}(f, \psi_{G,m}^j)$$

and

$$I : f \longrightarrow (2^{jn/2}(f, \psi_{G,m}^j))_{j\in\mathbb{N}_0, G\in G^j, m\in\mathbb{Z}^n}$$

is an isomorphism from $B_q^s L_{p,r}(\mathbb{R}^n)$ onto $b_q^s L_{p,r}$.

2. Assume that

$$L > \max\{s, n(\frac{1}{\min\{p,q\}} - 1)_+ - s, \frac{n}{p} - s\}.$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F_q^s L_{p,r}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} \psi_{G,m}^j(x), \quad (\lambda_m^{j,G}) \in f_q^s L_{p,r} \tag{4.5}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (4.5) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}(f, \psi_{G,m}^j)$$

and

$$I : f \longrightarrow (2^{jn/2}(f, \psi_{G,m}^j))_{j\in\mathbb{N}_0, G\in G^j, m\in\mathbb{Z}^n}$$

is an isomorphism from $F_q^s L_{p,r}(\mathbb{R}^n)$ onto $f_q^s L_{p,r}$.

Proof. Since we have Proposition 2.3, Theorem 3.8, Proposition 3.9 and Lemma 4.3, we can apply [25, Theorem 5.1] to get the wanted characterizations. □

5. Interpolation. Besov-Lorentz spaces as approximation spaces

Besides the classical results of Peetre [30, Chapter 5] and Triebel [38, Theorem 2.4.2/1, p. 185], other interpolation properties of Besov-Lorentz spaces have been established in [7, Section 6]. Next, with the help of the characterization by wavelets, we are going to extend and complement these results. First we show that [7, Theorem 6.5] holds if replace the couple of Besov spaces by a couple of Besov-Lorentz spaces.

Let $s \in \mathbb{R}, 0 < q \leq \infty$ and let X be a quasi-Banach space. We write $\ell_q^s(X) = \ell_q(2^{ks}X)$ for the collection of all sequences $(x_k)_{k\in\mathbb{N}} \subseteq X$ having a finite quasi-norm

$$\|(x_k)|\ell_q^s(X)\| = \|(x_k)|\ell_q(2^{ks}X)\| = \big(\sum_{k=1}^{\infty} 2^{ksq}\|x_k|X\|^q\big)^{1/q}$$

(the sum should be replaced by the supremum if $q = \infty$). When $A = \mathbb{C}$ we simply write ℓ_q^s.

For $0 < p < \infty$ and $0 < r \le \infty$, it is useful to write $\langle \ell_{p,r}\rangle^0$ for the direct sum of 2^n copies of $\ell_{p,r}(\mathbb{Z}^n)$ and $\langle \ell_{p,r}\rangle = \langle \ell_{p,r}\rangle^1$ for the direct sum of $2^n - 1$ copies of $\ell_{p,r}$. The number of copies for $\langle \ell_{p,r}\rangle^j$ corresponds to the number of elements of G^j. According to Theorem 4.4, if $-\infty < s < \infty$ and $0 < q \le \infty$, then $B_q^s L_{p,r}(\mathbb{R}^n)$ is isomorphic to $b_q^s \ell_{p,r} = \langle \ell_{p,r}\rangle^0 \oplus \ell_q^{s-\frac{n}{p}}(\langle \ell_{p,r}\rangle)$ where the last equality holds with equivalence of quasi-norms.

Note also that if (A_1, A_2), (B_1, B_2) are quasi-Banach couples, $0 < q \le \infty$ and $0 < \theta < 1$ then we have with equivalent quasi-norms

$$(A_1 \oplus B_1, A_2 \oplus B_2)_{\theta,q} = (A_1, A_2)_{\theta,q} \oplus (B_1, B_2)_{\theta,q}$$

because

$$K(t, (a, b); A_1 \oplus B_1, A_2 \oplus B_2) = K(t, a; A_1, A_2) + K(t, b; B_1, B_2).$$

Theorem 5.1. Let $0 < \theta < 1$, $-\infty < s_1, s_2 < \infty$, $s = (1-\theta)s_1 + \theta s_2$, $0 < p_1 \ne p_2 < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $0 < q_1, q_2 < \infty$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$ and $0 < r_1, r_2 \le \infty$. Then we have with equivalent quasi-norms

$$(B_{q_1}^{s_1} L_{p_1,r_1}(\mathbb{R}^n), B_{q_2}^{s_2} L_{p_2,r_2}(\mathbb{R}^n))_{\theta,q} = B_q^s L_{p,q}(\mathbb{R}^n).$$

Proof. Using Theorem 4.4 with L sufficiently large and (4.2), we have that $If = \big(2^{kn/2}(f, \psi_{G,m}^k)\big)$ is an isomorphism from $B_{q_j}^{s_j} L_{p_j,r_j}(\mathbb{R}^n)$ onto $b_{q_j}^{s_j} \ell_{p_j,r_j} = \langle \ell_{p_j,r_j}\rangle^0 \oplus \ell_{q_j}(2^{k(s_j - \frac{n}{p_j})}\langle \ell_{p_j,r_j}\rangle)$, $j = 1, 2$. According to [38, Theorem 1.18.1 and Remark 1.18.1/4, pp. 120-123], we have

$$\begin{aligned}
&(b_{q_1}^{s_1} \ell_{p_1,r_1}, b_{q_2}^{s_2} \ell_{p_2,r_2})_{\theta,q} \\
&\quad = (\langle \ell_{p_1,r_1}\rangle^0 \oplus \ell_{q_1}(2^{k(s_1 - \frac{n}{p_1})}\langle \ell_{p_1,r_1}\rangle), \langle \ell_{p_2,r_2}\rangle^0 \oplus \ell_{q_2}(2^{k(s_2 - \frac{n}{p_2})}\langle \ell_{p_2,r_2}\rangle))_{\theta,q} \\
&\quad = \langle \ell_{p,q}\rangle^0 \oplus \ell_q(2^{k(s - \frac{n}{p})}\langle \ell_{p,q}\rangle) = b_q^s \ell_{p,q}.
\end{aligned}$$

Consequently, interpolating the operator I and using Theorem 4.4, we derive that

$$(B_{q_1}^{s_1} L_{p_1,r_1}(\mathbb{R}^n), B_{q_2}^{s_2} L_{p_2,r_2}(\mathbb{R}^n))_{\theta,q} = B_q^s L_{p,q}(\mathbb{R}^n).$$

□

Next we extend [30, Theorem 6/(ii), p. 106] to the whole range of parameters. As usual we put $[\,\cdot\,]$ for the greatest integer function.

Theorem 5.2. Let $0 < p_1, p_2, q_1, q_2 \le \infty$, $0 < p < \infty$, $-\infty < s_1, s_2 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, $s = (1-\theta)s_1 + \theta s_2$ and $0 < r \le \infty$. Then the following continuous embeddings hold

$$B_{\min\{q,r\}}^s L_{p,r}(\mathbb{R}^n) \hookrightarrow (B_{p_1,q_1}^{s_1}(\mathbb{R}^n), B_{p_2,q_2}^{s_2}(\mathbb{R}^n))_{\theta,r} \hookrightarrow B_{\max\{q,r\}}^s L_{p,r}(\mathbb{R}^n). \tag{5.1}$$

Furthermore, the exponents $\min\{q,r\}$ and $\max\{q,r\}$ in (5.1) are the best possible, at least if

$$\eta = \frac{s_1 - s_2}{\frac{n}{p_1} - \frac{n}{p_2}} < 1. \tag{5.2}$$

Proof. We know that for L sufficiently large $If = (2^{kn/2}(f, \psi^k_{G,m}))$ is an isomorphism from $B^{s_j}_{p_j,q_j}(\mathbb{R}^n)$ onto $b^{s_j}_{q_j}\ell_{p_j} = \langle \ell_{p_j}\rangle^0 \oplus \ell^{s_j - n/p_j}_{q_j}(\langle \ell_{p_j}\rangle)$. Applying the interpolation formula for vector valued sequence spaces established by Peetre in [30, Theorem 4, p. 98] and Theorem 4.4, we obtain

$$B^s_{\min\{q,r\}}L_{p,r}(\mathbb{R}^n) \hookrightarrow (B^{s_1}_{p_1,q_1}(\mathbb{R}^n), B^{s_2}_{p_2,q_2}(\mathbb{R}^n))_{\theta,r} \hookrightarrow B^s_{\max\{q,r\}}L_{p,r}(\mathbb{R}^n).$$

In view of isomorphism I, in order to show that the exponent $\max\{q,r\}$ is the best possible, it suffices to check that if for some $0 < v \le \infty$ we have

$$(\ell^{s_1 - \frac{n}{p_1}}_{q_1}(\langle \ell_{p_1}\rangle), \ell^{s_2 - \frac{n}{p_2}}_{q_2}(\langle \ell_{p_2}\rangle))_{\theta,r} \hookrightarrow \ell^{s - \frac{n}{p}}_{v}(\langle \ell_{p,r}\rangle) \tag{5.3}$$

then $\max\{q,r\} \le v$.

For $k \in \mathbb{N}$, we designate by $\overline{e}_k$ the sequence $(e^{d,k}_m)_{\substack{m\in\mathbb{Z}^n \\ 1\le d\le 2^n-1}}$ with

$$e^{d,k}_m = \begin{cases} 1 & \text{if } d = 1 \text{ and } m = (k, ..., k), \\ 0 & \text{otherwise.} \end{cases}$$

So, $\|\overline{e}_k|\langle \ell_{p,r}\rangle\| \sim 1$. Let $\pi : \ell^{s_j - \frac{n}{p_j}}_{q_j} \longrightarrow \ell^{s_j - \frac{n}{p_j}}_{q_j}(\langle \ell_{p_j}\rangle)$ be the linear operator defined by $\pi(\lambda_k) = (\lambda_k \overline{e}_k)$. We have

$$\begin{aligned}\Big\|\pi(\lambda_k)|\ell^{s_j - \frac{n}{p_j}}_{q_j}(\langle \ell_{p_j}\rangle)\Big\| &= \Big(\sum_{k=1}^{\infty} 2^{k(s_j - \frac{n}{p_j})q_j} \|\lambda_k \overline{e}_k|\langle \ell_{p_j}\rangle\|^{q_j}\Big)^{1/q_j} \\ &= \Big(\sum_{k=1}^{\infty} 2^{k(s_j - \frac{n}{p_j})q_j} |\lambda_k|^{q_j}\Big)^{1/q_j} = \|(\lambda_k)|\ell^{s_j - \frac{n}{p_j}}_{q_j}\|.\end{aligned}$$

Moreover, since $s_1 - \frac{n}{p_1} \neq s_2 - \frac{n}{p_2}$, according to [30, Theorem 4/(ii), pp. 98-99], we have that $(\ell^{s_1 - \frac{n}{p_1}}_{q_1}, \ell^{s_2 - \frac{n}{p_2}}_{q_2})_{\theta,r} = \ell^{s - \frac{n}{p}}_{r}$. Hence, it follows from the interpolation property and (5.3) that $\pi : \ell^{s - \frac{n}{p}}_{r} \longrightarrow \ell^{s - \frac{n}{p}}_{v}(\langle \ell_{p,r}\rangle)$ boundedly. This yields the continuous embedding $\ell^{s - \frac{n}{p}}_{r} \hookrightarrow \ell^{s - \frac{n}{p}}_{v}$, which implies that $r \le v$. To prove that $q \le v$, pick a sequence $(J_k)_{k\in\mathbb{N}}$ of subsets of $\mathbb{Z}^n$, pairwise disjoints and such that $[2^{n(1-\mu)k}]$ is the number of elements of J_k. Let $\overline{u}_k = (u^{d,k}_m)_{\substack{m\in\mathbb{Z}^n \\ 1\le d\le 2^n-1}}$ with

$$u^{d,k}_m = \begin{cases} 2^{-\xi k} & \text{if } d = 1 \text{ and } m \in J_k, \\ 0 & \text{otherwise.} \end{cases}$$

Here ξ and $\mu < 1$ are real numbers that will be fixed later. We have

$$\|\overline{u}_k|\langle \ell_{p,r}\rangle\| \sim 2^{-\xi k}\Big(\int_0^{2^{n(1-\mu)k}} x^{\frac{r}{p}-1}dx\Big)^{1/r} \sim 2^{-\xi k}2^{(-\mu\frac{n}{p}+\frac{n}{p})k}$$
$$= 2^{((-\xi-\mu\frac{n}{p})+\frac{n}{p})k},$$

where the constants in the equivalence are independent of ξ and μ. We want that

$$\xi + \mu\frac{n}{p_j} = s_j \text{ for } j = 1, 2.$$

For this we need that $\mu\Big(\frac{n}{p_1} - \frac{n}{p_2}\Big) = s_1 - s_2$. Therefore we choose

$$\mu = \eta = \frac{s_1 - s_2}{\frac{n}{p_1} - \frac{n}{p_2}},$$

which is less than 1 according to (5.2). Then take $\xi = s_1 - \mu\frac{n}{p_1} = s_2 - \mu\frac{n}{p_2}$. With these values for μ and ξ, we have that $\omega(\lambda_k) = (\lambda_k\overline{u}_k)$ is a linear operator from ℓ_{q_j} to $\ell_{q_j}^{s_j-\frac{n}{p_j}}(\langle \ell_{p_j}\rangle)$ with

$$\|\omega(\lambda_k)|\ell_{q_j}^{s_j-\frac{n}{p_j}}(\langle \ell_{p_j}\rangle)\| = \Big(\sum_{k=1}^{\infty}(2^{k(s_j-\frac{n}{p_j})}\|\lambda_k\overline{u}_k|\langle \ell_{p_j}\rangle\|)^{q_j}\Big)^{1/q_j}$$
$$\sim \Big(\sum_{k=1}^{\infty}|\lambda_k|^{q_j}\Big)^{1/q_j} = \|(\lambda_k)|\ell_{q_j}\|.$$

The choice we have done for ξ and μ also gives that

$$-s + \mu\frac{n}{p} = (1-\theta)(-s_1 + \mu\frac{n}{p_1}) + \theta(-s_2 + \mu\frac{n}{p_2})$$
$$= (1-\theta)(-\xi) + \theta(-\xi) = -\xi.$$

Then $\omega : \ell_v \longrightarrow \ell_v^{s-\frac{n}{p}}(\langle \ell_{p,r}\rangle)$ is bounded with

$$\|\omega(\lambda_k)|\ell_v^{s-\frac{n}{p}}(\langle \ell_{p,r}\rangle)\| \sim \|(\lambda_k)|\ell_v\|.$$

Using the interpolation property and (5.3) we get

$$\ell_{q,r} = (\ell_{q_1}, \ell_{q_2})_{\theta,r} \xrightarrow{\ \omega\ } (\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle \ell_{p_1}\rangle), \ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle \ell_{p_2}\rangle))_{\theta,r} \hookrightarrow \ell_v^{s-\frac{n}{p}}(\langle \ell_{p,r}\rangle).$$

Whence, the estimates for the operator ω yield the continuous embedding $\ell_{q,r} \hookrightarrow \ell_v$, which implies that $q \leq v$. Consequently, $\max\{q, r\} \leq v$.

Now we show that the exponent $\min\{q, r\}$ is the best possible. Suppose that $0 < u \leq \infty$ satisfies

$$\ell_u^{s-\frac{n}{p}}(\langle \ell_{p,r}\rangle) \hookrightarrow (\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle \ell_{p_1}\rangle), \ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle \ell_{p_2}\rangle))_{\theta,r}. \tag{5.4}$$

Given any $\lambda = (\lambda_m^{d,k})_{\substack{m\in\mathbb{Z}^n \\ 1\le d\le 2^n-1}}$, let $\lambda_k = \lambda^{1,k}_{(k,\dots,k)}$, $k\in\mathbb{N}$ and put $\nu(\lambda) = (\lambda_k)$. Then the operator $\nu : \ell_{q_j}^{s_j-\frac{n}{p_j}}(\langle \ell_{p_j}\rangle) \longrightarrow \ell_{q_j}^{s_j-\frac{n}{p_j}}$ is bounded because

$$\begin{aligned}
\|\nu(\lambda)|\ell_{q_j}^{s_j-\frac{n}{p_j}}\| &= \Big(\sum_{k=1}^{\infty}(2^{k(s_j-\frac{n}{p_j})}|\lambda_k|)^{q_j}\Big)^{1/q_j} \\
&= \Big(\sum_{k=1}^{\infty}(2^{k(s_j-\frac{n}{p_j})}\|\lambda_k\overline{e}_k|\ell_{p_j}\|)^{q_j}\Big)^{1/q_j} \\
&\le \Big(\sum_{k=1}^{\infty}(2^{k(s_j-\frac{n}{p_j})}\|(\lambda_m^{d,k})|\langle\ell_{p_j}\rangle\|)^{q_j}\Big)^{1/q_j} = \|\lambda|\ell_{q_j}^{s_j-\frac{n}{p_j}}(\langle\ell_{p_j}\rangle)\|.
\end{aligned}$$

Therefore,

$$\nu : (\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle), \ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle))_{\theta,r} \longrightarrow (\ell_{q_1}^{s_1-\frac{n}{p_1}}, \ell_{q_2}^{s_2-\frac{n}{p_2}})_{\theta,r} = \ell_r^{s-\frac{n}{p}}$$

is also bounded. Using the diagram

$$\ell_u^{s-\frac{n}{p}} \xrightarrow{\ \pi\ } \ell_u^{s-\frac{n}{p}}(\langle\ell_{p,r}\rangle) \hookrightarrow (\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle), \ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle))_{\theta,r} \xrightarrow{\ \nu\ } \ell_r^{s-\frac{n}{p}}$$

and the fact that $\nu(\pi(\lambda_k)) = (\lambda_k)$, we conclude that $\ell_u^{s-\frac{n}{p}} \hookrightarrow \ell_r^{s-\frac{n}{p}}$. Therefore, $u \le r$. To establish the relationship between u and q, let $\gamma = \min\{1, p_1, p_2\}$ and put $\sigma(\lambda_m^{d,k}) = \Big(2^{\xi k - \frac{n(1-\mu)k}{\gamma}}\Big(\sum_{m\in J_k}|\lambda_m^{1,k}|^\gamma\Big)^{\frac{1}{\gamma}}\Big)$. We have

$$\begin{aligned}
\Big(\sum_{m\in J_k}|\lambda_m^{1,k}|^\gamma\Big)^{1/\gamma} &\le \Big(\sum_{m\in J_k}|\lambda_m^{1,k}|^{p_j}\Big)^{1/p_j}\Big(\sum_{m\in J_k}1\Big)^{(1-\frac{\gamma}{p_j})\frac{1}{\gamma}} \\
&\le 2^{n(1-\mu)(\frac{1}{\gamma}-\frac{1}{p_j})k}\Big(\sum_{m\in J_k}|\lambda_m^{1,k}|^{p_j}\Big)^{1/p_j}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Big|2^{\xi k - \frac{n(1-\mu)k}{\gamma}}\Big(\sum_{m\in J_k}|\lambda_m^{1,k}|^\gamma\Big)^{1/\gamma}\Big| &\le 2^{\xi k - \frac{n(1-\mu)k}{p_j}}\|(\lambda_m^{d,k})|\langle\ell_{p_j}\rangle\| \\
&= 2^{(s_j-\frac{n}{p_j})k}\|\lambda|\langle\ell_{p_j}\rangle\|.
\end{aligned}$$

This implies that $\|\sigma(\lambda)|\ell_{q_j}\| \le \|\lambda|\ell_{q_j}^{s_j-\frac{n}{pj}}(\langle\ell_{p_j}\rangle)\|$. Let $|\lambda| = (|\lambda_m^{d,k}|)$. The operator σ is not linear, but if $\lambda = \rho_1 + \rho_2$, we have $|\sigma(\lambda)| = \sigma(\lambda) \le C(\sigma(\rho_1) + \sigma(\rho_2))$. Using the lattice property of ℓ_{q_j}, we get that

$$\begin{aligned}
K(t,\sigma(\lambda);\ell_{q_1},\ell_{q_2}) &= K(t,|\sigma(\lambda)|;\ell_{q_1},\ell_{q_2}) \le CK(t,\sigma(|\rho_1|)+\sigma(|\rho_2|);\ell_{q_1},\ell_{q_2}) \\
&\le C(\|\sigma(|\rho_1|))|\ell_{q_1}\| + t\|\sigma(|\rho_2|)|\ell_{q_2}\|) \\
&\le C(\|\rho_1|\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle)\| + t\|\rho_2|\ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle)\|).
\end{aligned}$$

Taking the infimum over all possible decompositions of λ we obtain that

$$K(t,\sigma(\lambda);\ell_{q_1},\ell_{q_2}) \lesssim K(t,\lambda;\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle),\ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle)).$$

Consequently,

$$\|\sigma(\lambda)|\ell_{q,r}\| \sim \|\sigma(\lambda)|(\ell_{q_1},\ell_{q_2})_{\theta,r}\| \lesssim \|\lambda|(\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle),\ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle))_{\theta,r}\|.$$

The diagram

$$\ell_v \xrightarrow{\ \omega\ } \ell_v^{s-\frac{n}{p}}(\langle\ell_{p,r}\rangle) \hookrightarrow (\ell_{q_1}^{s_1-\frac{n}{p_1}}(\langle\ell_{p_1}\rangle),\ell_{q_2}^{s_2-\frac{n}{p_2}}(\langle\ell_{p_2}\rangle))_{\theta,r} \xrightarrow{\ \sigma\ } \ell_{q,r}$$

and the fact that $(\sigma\circ\omega)(\lambda_k)=(\beta_k)$ with $|\beta_k|\sim|\lambda_k|$, imply that

$$\|(\lambda_k)|\ell_{q,r}\| \sim \|\sigma(\omega(\lambda_k))|\ell_{q,r}\| \lesssim \|(\lambda_k)|\ell_v\|.$$

Therefore, $\ell_v \hookrightarrow \ell_{q,r}$ which yields that $v\le q$. The proof is completed. $\square$

Recall that the Hardy-Lorentz space $h_{p,r}$ is defined by

$$h_{p,r}=\{f\in\mathcal{S}'(\mathbb{R}^n):\|f|h_{p,r}\|=\|\sup_{0<t<1}|(\psi(t\cdot)\hat{f})^\vee||L_{p,r}(\mathbb{R}^n)\|<\infty\},$$

where $\psi\in\mathcal{S}(\mathbb{R}^n)$ is compactly supported and $\psi(x)=1$ if $|x|\le 1$ (see [3]). For $p=r$, $h_{p,p}$ coincides with the usual local Hardy space h_p (see [21, 39]).

It was shown in [3, Theorems 4.3 and 4.7] that we have with equivalence of quasi-norms $(h_{p_1},h_{p_2})_{\theta,r}=h_{p,r}$ provided that $0<p_1\ne p_2<\infty, 0<r<\infty, 0<\theta<1$ and $1/p=(1-\theta)/p_1+\theta/p_2$. This formula was use in [7] in the proofs of several results and for this reason the restriction $r<\infty$ appears in them. Next we show that the characterization in terms of wavelets of $B^s_qL_{p,\infty}$ allows to establish the corresponding results for this limit case.

Theorem 5.3. Let $-\infty<s_1\ne s_2<+\infty$, $0<p<\infty$ and $0<q_1,q_2,q\le\infty$. Let $0<\theta<1$ and $s=(1-\theta)s_0+\theta s_1$. Then we have with equivalent quasi-norms

$$(B^{s_1}_{q_1}L_{p,\infty}(\mathbb{R}^n),B^{s_2}_{q_2}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}=B^s_qL_{p,\infty}(\mathbb{R}^n).$$

Proof. We proceed as in the proof of Theorem 5.1. For L sufficiently large, it follows from Theorem 4.4 that $If=(2^{kn/2}(f,\psi^k_{G,m}))$ is an isomorphism from $B^{s_j}_{q_j}L_{p,\infty}(\mathbb{R}^n)$ onto $b^{s_j}_{q_j}\ell_{p,\infty}=\langle\ell_{p,\infty}\rangle^0\oplus\ell_{q_j}^{s_j-\frac{n}{p}}(\langle\ell_{p,\infty}\rangle)$ for $j=1,2$. By [30, Theorem 4/(ii), p. 98] we have with equivalence of quasi-norms

$$\begin{aligned}&(b^{s_1}_{q_1}\ell_{p,\infty},b^{s_2}_{q_2}\ell_{p,\infty})_{\theta,q}\\&\quad=\langle\ell_{p,\infty}\rangle^0\oplus(\ell_{q_1}^{s_1-\frac{n}{p}}(\langle\ell_{p,\infty}\rangle),\ell_{q_2}^{s_2-\frac{n}{p}}(\langle\ell_{p,\infty}\rangle))_{\theta,q}\\&\quad=\langle\ell_{p,\infty}\rangle^0\oplus\ell_q^{s-\frac{n}{p}}(\langle\ell_{p,\infty}\rangle)=b^s_q\ell_{p,\infty}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f|(B^{s_1}_{q_1}L_{p,\infty}(\mathbb{R}^n),B^{s_2}_{q_2}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}\| &\sim \|If|(b^{s_1}_{q_1}\ell_{p,\infty},b^{s_2}_{q_2}\ell_{p,\infty})_{\theta,q}\| \\ &\sim \|If|b^s_q\ell_{p,\infty}\| \sim \|f|B^s_qL_{p,\infty}(\mathbb{R}^n)\|.\end{aligned}$$

□

This result covers the case $r=\infty$ that was left over in [7, Theorem 6.6].

Now we can establish the corresponding result to [7, Theorem 6.7] for the case $r=\infty$.

Theorem 5.4. Let $-\infty<s_1\neq s_2<\infty$, $0<p<\infty$ and $0<q_1,q_2,q\leq\infty$. Let $0<\theta<1$ and $s=(1-\theta)s_1+\theta s_2$. Then we have with equivalent quasi-norms

$$(F^{s_1}_{q_1}L_{p,\infty}(\mathbb{R}^n),F^{s_2}_{q_2}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}=B^s_qL_{p,\infty}(\mathbb{R}^n).$$

Proof. For $j=1,2$ pick $0<\tau_j<\infty$ such that

$$\begin{cases}\tau_j\leq\min\{p,q_j\} & \text{for } p\neq q_j,\\ \tau_j<p & \text{for } p=q_j.\end{cases}$$

According to [36, Theorems 1.1 and 1.2] we have

$$B^{s_j}_{\tau_j}L_{p,\infty}(\mathbb{R}^n)\hookrightarrow F^{s_j}_{q_j}L_{p,\infty}(\mathbb{R}^n)\hookrightarrow B^{s_j}_{\infty}L_{p,\infty}(\mathbb{R}^n),\quad j=1,2.$$

Whence, applying Theorem 5.3, we have

$$\begin{aligned}B^s_qL_{p,\infty}(\mathbb{R}^n)&=(B^{s_1}_{\tau_1}L_{p,\infty}(\mathbb{R}^n),B^{s_2}_{\tau_2}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}\\ &\hookrightarrow(F^{s_1}_{q_1}L_{p,\infty}(\mathbb{R}^n),F^{s_2}_{q_2}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}\\ &\hookrightarrow(B^{s_1}_{\infty}L_{p,\infty}(\mathbb{R}^n),B^{s_2}_{\infty}L_{p,\infty}(\mathbb{R}^n))_{\theta,q}=B^s_qL_{p,\infty}(\mathbb{R}^n).\end{aligned}$$

□

Now we can proceed as in [7, Theorem 7.1] but using Theorem 5.4 instead of [7, Theorem 6.7] with the effect that we can incorporate the case $r=\infty$ to [7, Theorem 7.1]. We obtain the following result.

Theorem 5.5. Let $s\in\mathbb{R}$, $0<p<\infty$ and $0<q\leq\infty$.

(i) If ψ is a diffeomorphism in $\mathbb{R}^n$ then

$$D_\psi:B^s_qL_{p,\infty}(\mathbb{R}^n)\longrightarrow B^s_qL_{p,\infty}(\mathbb{R}^n),\ D_\psi f=f\circ\psi,$$

is an isomorphic map in $B^s_qL_{p,\infty}(\mathbb{R}^n)$.

(ii) Let $\rho > \max\big(s, n\big(\max\big(\frac{1}{p}, 1\big) - 1\big) - s\big)$. Then

$$\|gf|B^s_q L_{p,\infty}(\mathbb{R}^n)\| \le c\|g|C^\rho(\mathbb{R}^n)\| \cdot \|f|B^s_q L_{p,\infty}(\mathbb{R}^n)\|$$

for all $g \in C^\rho(\mathbb{R}^n)$ and all $f \in B^s_q L_{p,\infty}(\mathbb{R}^n)$.

(iii) There is a bounded linear extension operator ext,

$$\mathrm{ext} : B^s_q L_{p,\infty}(\mathbb{R}^n_+) \longrightarrow B^s_q L_{p,\infty}(\mathbb{R}^n).$$

Furthermore,

$$\mathrm{re} \circ \mathrm{ext} = \mathrm{id}, \text{ identity in } B^s_q L_{p,\infty}(\mathbb{R}^n_+)$$

(iv) If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, there is a bounded linear extension operator

$$\mathrm{ext} : B^s_q L_{p,\infty}(\Omega) \longrightarrow B^s_q L_{p,\infty}(\mathbb{R}^n).$$

Furthermore,

$$\mathrm{re} \circ \mathrm{ext} = \mathrm{id}, \text{ identity in } B^s_q L_{p,\infty}(\Omega).$$

Next we characterize Besov-Lorentz spaces as approximation spaces using wavelets. This result is of independent interest but, in addition, it will allow us to derive another interpolation formula. We start by recalling the definition of abstract approximation spaces (see [9, 13, 15, 32, 33] and the references given there).

Let $(X, \|\cdot|X\|)$ be a quasi-Banach space and let $(A_k)_{k\in\mathbb{N}_0}$ be a sequence of subsets of X satisfying the following conditions:

$$A_0 = \{0\} \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq \cdots \subseteq X,$$

$$\lambda A_k \subseteq A_k \text{ for any } \lambda \in \mathbb{C} \text{ and any } k \in \mathbb{N}_0,$$

$$A_k + A_m \subseteq A_{k+m} \text{ for any } k, m \in \mathbb{N}_0.$$

For $f \in X$, we put $E_0(f) = \|f|X\|$ and

$$E_k(f) = E^A_k(f)_X = \inf\{\|f - g|X\| : g \in A_k\},\ k \in \mathbb{N}.$$

Let $\alpha > 0$ and $0 < q \le \infty$. The *approximation space* $X^\alpha_q = (X; A_k)^\alpha_q$ is formed by all those $f \in X$ having a finite quasi-norm

$$\|f|X^\alpha_q\| = \|(E_k(f))|\ell_{1/\alpha,q}\| = \begin{cases} \Big(\sum_{k=1}^\infty [k^\alpha E_{k-1}(f)]^q k^{-1}\Big)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{k\ge1}\{k^\alpha E_{k-1}(f)\} & \text{if } q = \infty. \end{cases}$$

According to [33, Proposition 2], an equivalent quasi-norm in X^α_q is

$$\|f|X^\alpha_q\|^\diamond = \Big(\|f|X\|^q + \sum_{k=1}^\infty 2^{k\alpha q} E_{2^k}(f)^q\Big)^{1/q} \tag{5.5}$$

(with the usual modification if $q = \infty$).

Let $s > 0, 0 < p < \infty$ and $0 < r, q \leq \infty$. We know by [36, Theorem 1.1] that

$$B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^0_2 L_{p,r}(\mathbb{R}^n). \tag{5.6}$$

For $1 < p < \infty$, the space $F^0_2 L_{p,r}(\mathbb{R}^n)$ coincides with the Lorentz space $L_{p,r}(\mathbb{R}^n)$ with equivalence of quasi-norms (see [44, Theorem 3.15]). To describe $B^s_q L_{p,r}(\mathbb{R}^n)$ as an approximation space, we choose $X = F^0_2 L_{p,r}(\mathbb{R}^n)$ and as sequence of subsets (A_u), following an idea of [14, Lemma 5.4] for the case of logarithmic Besov spaces, we take $A_0 = \{0\}$ and for $u = 1, 2, \ldots$ with $2^k \leq u < 2^{k+1}, k \in \mathbb{N}_0$, choose

$$A_u = \{g \in F^0_2 L_{p,r}(\mathbb{R}^n) : g = \sum_{\nu=0}^{k} \sum_{G \in G^\nu} \sum_{m \in \mathbb{Z}^n} c_m^{\nu,G} 2^{-\nu n/2} \psi^\nu_{G,m} \ , \ c_m^{\nu,G} \in \mathbb{C}\}.$$

Consider also the operators $P_{2^k} : F^0_2 L_{p,r}(\mathbb{R}^n) \longrightarrow F^0_2 L_{p,r}(\mathbb{R}^n)$ defined by

$$P_{2^k} f = \sum_{\nu=0}^{k} \sum_{G \in G^\nu} \sum_{m \in \mathbb{Z}^n} \lambda_m^{\nu,G}(f) 2^{-\nu n/2} \psi^\nu_{G,m}.$$

Clearly

$$A_u = P_{2^k}(F^0_2 L_{p,r}(\mathbb{R}^n)) \, , \quad 2^k \leq u < 2^{k+1} \, , \quad k \in \mathbb{N}_0 \, . \tag{5.7}$$

By the characterization in terms of wavelets for $F^0_2 L_{p,r}(\mathbb{R}^n)$ (see [49] or Theorem 4.4/2) and the Banach-Steinhaus theorem [28, p. 169], we get that

$$\sup\{\|P_{2^k}\|_{F^0_2 L_{p,r}(\mathbb{R}^n), F^0_2 L_{p,r}(\mathbb{R}^n)} : k \in \mathbb{N}\} < \infty. \tag{5.8}$$

From (5.7) and (5.8), it is not hard to check that

$$E_{2^k}(f) = E^A_{2^k}(f)_{F^0_2 L_{p,r}(\mathbb{R}^n)} \sim \|f - P_{2^k} f | F^0_2 L_{p,r}(\mathbb{R}^n)\| \, , \, k \in \mathbb{N}_0. \tag{5.9}$$

Theorem 5.6. Let $s > 0, 0 < p < \infty$ and $0 < q, r \leq \infty$. Then we have with equivalence of quasi-norms

$$B^s_q L_{p,r}(\mathbb{R}^n) = (F^0_2 L_{p,r}(\mathbb{R}^n); A_k)^s_q \, .$$

Proof. Using (5.9), the quasi-norm (5.5) and the description of $F^0_2 L_{p,r}(\mathbb{R}^n)$

in terms of wavelets we obtain

$$
\begin{aligned}
\|f|(F^0_2L_{p,r}(\mathbb{R}^n))^s_q\|^q &\sim \|f|F^0_2L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^\infty 2^{ksq}E_{2^k}(f)^q \\
&\sim \|f|F^0_2L_{p,r}(\mathbb{R}^n)\|^q \\
&\qquad + \sum_{k=1}^\infty 2^{ksq}\Big\| \sum_{\nu=k+1}^\infty \sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n} \lambda^{\nu,G}_m(f)2^{-\nu n/2}\psi^\nu_{G,m}|F^0_2L_{p,r}(\mathbb{R}^n)\Big\|^q \\
&\sim \|f|F^0_2L_{p,r}(\mathbb{R}^n)\|^q \\
&\qquad + \sum_{k=1}^\infty 2^{ksq}\Big\|\Big(\sum_{\nu=k+1}^\infty \sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n} |\lambda^{\nu,G}_m(f)\chi_{\nu,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|^q \\
&= \|f|F^0_2L_{p,r}(\mathbb{R}^n)\|^q \\
&\qquad + \sum_{k=1}^\infty 2^{ksq}\Big\|\Big(\sum_{\nu=1}^\infty \sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f)\chi_{k+\nu,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|^q.
\end{aligned}
\tag{5.10}
$$

Take $0<\rho<\min\{1,q,p,r\}$. Since $\ell_1\hookrightarrow\ell_2$ and $L_{p,r}$ is ρ-Banach, we derive

$$
\begin{aligned}
&\Big(\sum_{k=1}^\infty 2^{ksq}\Big\|\Big(\sum_{\nu=1}^\infty \sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f)\chi_{k+\nu,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \\
&\le \Big\{\Big(\sum_{k=1}^\infty\Big(2^{ks\rho}\Big\|\sum_{\nu=1}^\infty \sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f)\chi_{k+\nu,m}(\cdot)||L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{\rho/q}\Big\}^{1/\rho} \\
&\le \Big\{\sum_{\nu=1}^\infty \sum_{G\in G^\nu}\Big(\sum_{k=1}^\infty\Big(2^{ks\rho}\Big\|\sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f)\chi_{k+\nu,m}(\cdot)||L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{\rho/q}\Big\}^{1/\rho} \\
&= \Big\{\sum_{\nu=1}^\infty 2^{-\nu s\rho}\sum_{G\in G^\nu}\Big(\sum_{k=1}^\infty 2^{(k+\nu)sq}\Big\|\sum_{m\in\mathbb{Z}^n} |\lambda^{k+\nu,G}_m(f)\chi_{k+\nu,m}(\cdot)||L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{\rho/q}\Big\}^{1/\rho} \\
&\lesssim \Big\{\sum_{\nu=1}^\infty 2^{-\nu s\rho}\Big\}^{1/\rho}\|f|B^s_qL_{p,r}(\mathbb{R}^n)\|,
\end{aligned}
$$

where the last inequality follows from Theorem 4.4/1. Having in mind (5.6), we conclude that $B^s_qL_{p,r}(\mathbb{R}^n)\hookrightarrow(F^0_2L_{p,r}(\mathbb{R}^n))^s_q$.

In order to establish the converse embedding we start from

$$
\|f|B^s_qL_{p,r}(\mathbb{R}^n)\| \sim \Big(\sum_{k=0}^\infty\sum_{G\in G^k} 2^{ksq}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda^{k,G}_m(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}
\tag{5.11}
$$

(first part of Theorem 4.4). In this sum, the part with $k=0$ and $k=1$ can be bounded by

$$
\|f|F^0_2L_{p,r}(\mathbb{R}^n)\| \sim \Big\|\Big(\sum_{k=0}^\infty\sum_{G\in G^k}\sum_{m\in\mathbb{Z}^n}|\lambda^{k,G}_m(f)\chi_{k,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|
$$

because

$$
\begin{aligned}
\Big(\sum_{k=0}^{1}\sum_{G\in G^k} 2^{ksq}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}
&\lesssim \sum_{k=0}^{1}\sum_{G\in G^k}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&= \sum_{k=0}^{1}\sum_{G\in G^k}\Big\|\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k,G}(f)\chi_{k,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&\lesssim \Big\|\Big(\sum_{k=0}^{1}\sum_{G\in G^k}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k,G}(f)\chi_{k,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|\\
&\lesssim \|f|F_2^0L_{p,r}(\mathbb{R}^n)\|.
\end{aligned}
$$

For the remaining part of (5.11) we have

$$
\begin{aligned}
\sum_{k=2}^{\infty}\sum_{G\in G^k} 2^{ksq}\Big\|\sum_{m\in\mathbb{Z}^n}\lambda_m^{k,G}(f)\chi_{k,m}|L_{p,r}(\mathbb{R}^n)\Big\|^q
&\lesssim \sum_{k=1}^{\infty}2^{(k+1)sq}\Big\|\sum_{G\in G^k}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k+1,G}(f)\chi_{k+1,m}||L_{p,r}(\mathbb{R}^n)\Big\|^q\\
&\lesssim \sum_{k=1}^{\infty}2^{ksq}\Big\|\Big(\sum_{\nu=1}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}|\lambda_m^{k+\nu,G}(f)\chi_{k+\nu,m}(\cdot)|^2\Big)^{1/2}|L_{p,r}(\mathbb{R}^n)\Big\|^q\\
&\lesssim \|f|(F_2^0L_{p,r}(\mathbb{R}^n))_q^s\|^q,
\end{aligned}
$$

where we have used (5.10) for the last inequality. Consequently,

$$\|f|B_q^sL_{p,r}(\mathbb{R}^n)\| \lesssim \|f|(F_2^0L_{p,r}(\mathbb{R}^n))_q^s\|.$$

This completes the proof. □

As it is pointed out in [7, (3.1) and (6.3)] (see also [44, Theorem 3.15] for the case $1<p<\infty$), we have with equivalence of quasi-norms

$$F_2^0L_{p,r}(\mathbb{R}^n) = \begin{cases} L_{p,r}(\mathbb{R}^n) & \text{if } 1<p<\infty\,,\ 0<r\le\infty,\\ h_{p,r} & \text{if } 0<p<\infty\,,\ 0<r<\infty. \end{cases} \tag{5.12}$$

Hence, $B_q^sL_{p,r}(\mathbb{R}^n)$ can be realized as an approximation space taking as X any of these two spaces in the suitable range of parameters.

For couples of approximation spaces, the following interpolation formula was established by Peetre and Sparr [31] (see also [9]): Let $0<\alpha_0\neq\alpha_1<\infty$, $0<p_0,p_1,q\le\infty$, $0<\theta<1$ and $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, then we have

$$(X_{p_0}^{\alpha_0},X_{p_1}^{\alpha_1})_{\theta,q} = X_q^{\alpha}\,. \tag{5.13}$$

Furthermore, see for example [12, Proposition 2.7],

$$(X, X_{p_1}^{\alpha_1})_{\theta,q} = X_q^{\theta\alpha_1} \,. \tag{5.14}$$

Combining Theorem 5.6 with (5.13) and using the lift operator $I_\delta f = \big((1+|x|^2)^{-\delta/2}\hat{f}\big)^\vee$, $\delta \in \mathbb{R}$, we recover [7, Theorem 6.6]. We close this section with the following formula.

Theorem 5.7. Let $0 < \theta < 1$, $0 < s, p < \infty$ and $0 < q, r, u \le \infty$. Then we have with equivalence of quasi-norms

$$(F_2^0 L_{p,r}(\mathbb{R}^n), B_q^s L_{p,r}(\mathbb{R}^n))_{\theta,u} = B_u^{\theta s} L_{p,r}(\mathbb{R}^n) \,.$$

Proof. Theorem 5.6 and (5.14) yield

$$\begin{aligned}
(F_2^0 L_{p,r}(\mathbb{R}^n), B_q^s L_{p,r}(\mathbb{R}^n))_{\theta,u} &= (F_2^0 L_{p,r}(\mathbb{R}^n), (F_2^0 L_{p,r}(\mathbb{R}^n))_q^s)_{\theta,u} \\
&= (F_2^0 L_{p,r}(\mathbb{R}^n))_u^{\theta s} \\
&= B_u^{\theta s} L_{p,r}(\mathbb{R}^n) \,.
\end{aligned}$$

□

6. Another description of Besov-Lorentz spaces as approximation spaces. Multiplications

The following realization of spaces $B_q^s L_{p,q}(\mathbb{R}^n)$ as approximation spaces is based on entire functions of exponential type (see [38, Section 2.5.4]). The arguments relay on definition of $B_q^s L_{p,q}(\mathbb{R}^n)$ by means of the smooth resolution of unity (φ_k) of (2.5).

By the construction of (φ_k), for $N \in \mathbb{N}$, there exists $M < \infty$ independent of k, such that

$$\sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{|\alpha|/2} |\partial^\alpha \varphi_k(x)| \le M, \quad k \in \mathbb{N}_0.$$

It follows from [39, Theorem 2.3.7] that for any $0 < p < \infty$ and $0 < r \le \infty$ there is $c > 0$ such that

$$\|(\varphi_k \hat{f})^\vee | F_{p,r}^0(\mathbb{R}^n)\| \le c \|f | F_{p,r}^0(\mathbb{R}^n)\|, \quad f \in F_{p,r}^0(\mathbb{R}^n), \quad k \in \mathbb{N}_0.$$

Whence, applying [7, Theorem 3.6], we derive that for any $0 < p < \infty$ and $0 < r \le \infty$ there exists $C > 0$ such that

$$\|(\varphi_k \hat{f})^\vee | F_2^0 L_{p,r}(\mathbb{R}^n)\| \le C \|f | F_2^0 L_{p,r}(\mathbb{R}^n)\|, \quad f \in F_2^0 L_{p,r}(\mathbb{R}^n), \quad k \in \mathbb{N}_0. \tag{6.1}$$

This estimate will be useful later.

Let $D_0 = \{0\}$ and for $k \in \mathbb{N}$ put

$$D_k = \{g \in F_2^0 L_{p,r}(\mathbb{R}^n) : \operatorname{supp} \hat{g} \subseteq \{x : |x| \le k\}\}.$$

In what follows we work with approximation spaces $(X; D_k)^s_q$ generated by the sequence of subset (D_k) with X being $F^0_2 L_{p,r}(\mathbb{R}^n)$ and $B^s_q L_{p,r}(\mathbb{R}^n)$. For simplicity, we designate them by $[X]^s_q$.

Theorem 6.1. Let $0 < q \leq \infty$, $s > 0$, $0 < p < \infty$ and $0 < r < \infty$, allowing also $r = \infty$ if $1 < p < \infty$. Then we have with equivalence of quasi-norms

$$[F^0_2 L_{p,r}(\mathbb{R}^n)]^s_q = B^s_q L_{p,r}(\mathbb{R}^n)\,.$$

Proof. Take any $f \in B^s_q L_{p,r}(\mathbb{R}^n)$. Using (5.6), we get

$$E_2(f) \leq E_1(f) \leq E_0(f) = \|f|F^0_2 L_{p,r}(\mathbb{R}^n)\| \leq c\|f|B^s_q L_{p,r}(\mathbb{R}^n)\|. \tag{6.2}$$

Moreover, since

$$\operatorname{supp}\Big(\sum_{m=0}^{k} \varphi_m \hat{f}\Big) \subseteq \{x : |x| \leq 2^{k+2}\}, \quad k = 0, 1, 2, \dots\,,$$

we obtain for $k \geq 2$ that

$$E_{2^k}(f) \leq \Big\| f - \big(\sum_{m=0}^{k-2} \varphi_m \hat{f}\big)^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\| = \Big\| \sum_{m=k-1}^{\infty} (\varphi_m \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|.$$

Using the quasi-norm (5.5), we have

$$\begin{aligned}
\|f|[F^0_2 L_{p,r}(\mathbb{R}^n)]^s_q\|^\diamond &= \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{m=1}^{\infty} 2^{msq} E_{2^m}(f)^q\Big)^{1/q} \\
&\lesssim \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{m=2}^{\infty} 2^{msq}\Big\| \sum_{k=m-1}^{\infty} (\varphi_k \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^q\Big)^{1/q}.
\end{aligned}$$

By the Aoki-Rolewicz theorem (see [4, 34]), we can take $0 < \rho < \min\{1, q\}$ such that $F^0_2 L_{p,r}(\mathbb{R}^n)$ is ρ-Banach. For the second term in the last expression, we get

$$\begin{aligned}
&\Big(\sum_{m=2}^{\infty} \Big(2^{ms\rho}\Big\| \sum_{k=m-1}^{\infty} (\varphi_k \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{1/q} \\
&= \Big[\Big(\sum_{m=2}^{\infty} \Big(2^{ms\rho}\Big\| \sum_{k=0}^{\infty} (\varphi_{m-1+k} \hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\Big\|^\rho\Big)^{q/\rho}\Big)^{\rho/q}\Big]^{1/\rho} \\
&\leq \Big[\sum_{k=0}^{\infty} 2^{(1-k)s\rho}\Big(\sum_{m=1}^{\infty} 2^{(m-1+k)sq}\|(\varphi_{m-1+k}\hat{f})^\vee |F^0_2 L_{p,r}(\mathbb{R}^n)\|^q\Big)^{\rho/q}\Big]^{1/\rho} \\
&\lesssim [\sum_{k=0}^{\infty} 2^{(1-k)s\rho}]^{1/\rho}\|f|B^s_q L_{p,r}(\mathbb{R}^n)\|
\end{aligned}$$

where in the last inequality we have used (5.12) and the fact that $B^s_q L_{p,r}(\mathbb{R}^n) = B^s_q h_{p,r}(\mathbb{R}^n)$ (see [7, Theorem 6.4]). This together with (6.2) yield that $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow [F^0_2 L_{p,r}(\mathbb{R}^n)]^s_q$.

In order to establish the converse embedding, first note that given any $f \in F^0_2 L_{p,r}(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in D_{2^k}$, since supp $\hat{g} \cap$supp $\varphi_{k+1} = \emptyset$, it follows from (6.1) that

$$\begin{aligned}\|(\varphi_{k+1}\hat{f})^\vee | F^0_2 L_{p,r}(\mathbb{R}^n)\| &= \|(\varphi_{k+1}(\hat{f} - \hat{g}))^\vee | F^0_2 L_{p,r}(\mathbb{R}^n)\| \\ &\leq C\|f - g|F^0_2 L_{p,r}(\mathbb{R}^n)\| \,.\end{aligned}$$

This yields that

$$\|(\varphi_{k+1}\hat{f})^\vee | F^0_2 L_{p,r}(\mathbb{R}^n)\| \lesssim E_{2^k}(f) \quad , \quad k = 1, 2, \ldots$$

Moreover, using again (6.1), we get

$$\|(\varphi_m\hat{f})^\vee | F^0_2 L_{p,r}(\mathbb{R}^n)\| \leq c\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\| \quad \text{for} \quad m = 0, 1\,.$$

Consequently,

$$\begin{aligned}\|f|B^s_q L_{p,r}(\mathbb{R}^n)\| &\sim \Big(\sum_{k=0}^{\infty} 2^{ksq}\|(\varphi_k\hat{f})^\vee | F^0_2 L_{p,r}\|^q\Big)^{1/q} \\ &\lesssim \Big(\|f|F^0_2 L_{p,r}(\mathbb{R}^n)\|^q + \sum_{k=1}^{\infty} 2^{ksq} E_{2^k}(f)^q\Big)^{1/q} \\ &\sim \|f|[F^0_2 L_{p,r}(\mathbb{R}^n)]^s_q\| \,.\end{aligned}$$

□

Next we study multiplication properties of Besov-Lorentz spaces. Assume $0 < p < \infty, s > n/p$ and $0 < q, r \leq \infty$. Since the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is continuously embedded in $L_\infty(\mathbb{R}^n)$ (see, for example, [46, Theorem 2.3]), using Theorem 5.1 it is not hard to derive that $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Hence, if $f_j \in B^{s_j}_{q_j} L_{p_j,r_j}(\mathbb{R}^n)$ with $s_j > n/p_j$, $j = 1, 2$, then the product $f_1 f_2$ makes sense pointwise almost everywhere.

For $m, k \in \mathbb{N}$, $f \in D_m$ and $g \in D_k$, since $fg = (\hat{f} * \hat{g})^\vee$, we have that supp $\widehat{fg} \subseteq \{x : |x| \leq m + k\}$. Let $T(f, g) = fg$. According to the previous observation, we get

$$T(D_{2^m}, D_{2^k}) \subseteq D_{2^m+2^k}\,, \quad m, k = 1, 2, \ldots \tag{6.3}$$

Lemma 6.2. Let X, Y, Z be quasi-Banach function spaces on $\mathbb{R}^n$ containing the subsets (D_k). We assume that X and Y are formed by regular distributions and we put $T(f, g) = fg$. If the bilinear operator $T : X \times Y \longrightarrow Z$ is bounded then there is a constant $c > 0$ such that for any $k \in \mathbb{N}$ and any $f \in X$ and $g \in Y$ we have

$$E_{2^{k+1}}(fg)_Z \leq c\Big(E_{2^k}(f)_X\|g|Y\| + \|f|X\|E_{2^k}(g)_Y\Big)\,.$$

Proof. Let M be the norm of $T : X \times Y \longrightarrow Z$. Given any $f \in X$, $g \in Y$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there are $f_0, g_0 \in D_{2^k}$ such that

$$\|f - f_0|X\| \leq E_{2^k}(f)_X + \varepsilon\,,\ \|g - g_0|Y\| \leq E_{2^k}(g)_Y + \varepsilon\,.$$

Having in mind (6.3) and writing C_Z for the constant in the quasi-triangle inequality in Z, we obtain

$$\begin{aligned}
&E_{2^{k+1}}(fg)_Z \leq \|fg - f_0g_0|Z\| \\
&\quad\leq C_Z\big(\|fg - f_0g|Z\| + \|f_0g - f_0g_0|Z\|\big) \\
&\quad\leq C_ZM\big(\|f - f_0|X\|\|g|Y\| + \|f_0 - f + f|X\|\|g - g_0|Y\|\big) \\
&\quad\leq C_ZM\big[(E_{2^k}(f)_X + \varepsilon)\|g|Y\| \\
&\qquad\qquad + C_X\big((E_{2^k}(f)_X + \varepsilon + \|f|X\|)(E_{2^k}(g)_Y + \varepsilon)\big)\big] \\
&\quad\leq C_ZM\big[\big(E_{2^k}(f)_X + \varepsilon\big)\|g|Y\| + C_X\big(2\|f|X\| + \varepsilon\big)\big(E_{2^k}(g)_Y + \varepsilon\big)\big]\,.
\end{aligned}$$

Passing to the limit when $\varepsilon \to 0$ the wanted result follows with $c = 2C_XC_ZM$. □

Generalized Hölder inequalities for Besov and Triebel-Lizorkin spaces are important in the spectral theory of degenerate elliptic operators (see [16, Chapters 2 and 5] and [35, §4.8]). As a first consequence of Theorem 6.1 and Lemma 6.2 we extend Hölder inequality to Besov-Lorentz spaces.

Theorem 6.3. Let $s > 0$, $1 < p_1, p_2, p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $0 < q \leq \infty$ and $0 < r_1, r_2, r \leq \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then we have

$$B^s_qL_{p_1,r_1}(\mathbb{R}^n) \cdot B^s_qL_{p_2,r_2}(\mathbb{R}^n) \hookrightarrow B^s_qL_{p,r}(\mathbb{R}^n)\,.$$

Proof. Since $1 < p_j < \infty$, as we pointed out in (5.12), we have that $F^0_2L_{p_j,r_j}(\mathbb{R}^n) = L_{p_j,r_j}(\mathbb{R}^n)$, $j = 1, 2$. So, Hölder inequality for Lorentz spaces yields that

$$F^0_2L_{p_1,r_1}(\mathbb{R}^n) \cdot F^0_2L_{p_2,r_2}(\mathbb{R}^n) \hookrightarrow F^0_2L_{p,r}(\mathbb{R}^n)\,,$$

(see, for example, [7, (2.7)]). Let again $T(f, g) = fg$ and write for simplicity $X = F^0_2L_{p_1,r_1}(\mathbb{R}^n)$, $Y = F^0_2L_{p_2,r_2}(\mathbb{R}^n)$ and $Z = F^0_2L_{p,r}(\mathbb{R}^n)$. So, $T : X \times Y \longrightarrow Z$ is bounded. Using Lemma 6.2 we derive

$$\begin{aligned}
&\|T(f,g)|Z^s_q\| \sim \Big[\|T(f,g)|Z\|^q + \sum_{k=2}^{\infty} 2^{ksq}E_{2^k}(T(f,g))^q_Z\Big]^{1/q} \\
&\quad\lesssim \Big[\|f|X\|^q\|g|Y\|^q + \sum_{k=1}^{\infty} 2^{ksq}\big(E_{2^k}(f)_X\|g|Y\| + \|f|X\|E_{2^k}(g)_Y\big)^q\Big]^{1/q} \\
&\quad\leq \Big[\|f|X\|^q + \sum_{k=1}^{\infty} 2^{ksq}E_{2^k}(f)^q_X\Big]^{1/q}\Big[\|g|Y\|^q + \sum_{k=1}^{\infty} 2^{ksq}E_{2^k}(g)^q_Y\Big]^{1/q} \\
&\quad\lesssim \|f|X^s_q\|\|g|Y^s_q\|.
\end{aligned}$$

This yields the result having in mind that, by Theorem 6.1, we have

$$[F_2^0 L_{p,r}(\mathbb{R}^n)]_q^s = B_q^s L_{p,r}(\mathbb{R}^n) \text{ and } [F_2^0 L_{p_j,r_j}(\mathbb{R}^n)]_q^s = B_q^s L_{p_j,r_j}(\mathbb{R}^n), j = 1, 2.$$

□

In what follows we are interested in multiplication algebras. That is to say, Besov-Lorentz spaces such that

$$B_q^s L_{p,r}(\mathbb{R}^n) \cdot B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow B_q^s L_{p,r}(\mathbb{R}^n)\,.$$

In [7, Theorem 7.2] it is shown that $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra provided that $0 < p < \infty$, $s > n/p$, $0 < q \leq r \leq 1$ and $r < p$. Subsequently, we continue this research with the help of Theorem 6.1. Our first aim is to eliminate the restrictions on q.

Theorem 6.4. Let $0 < p < \infty$, $s > n/p$ and $0 < r < \infty$, allowing also $r = \infty$ if $1 < p < \infty$. If there are $0 < q_1 < \infty$ and $n/p < s_1 < s$ such that $B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$ is an algebra for multiplication, then for any $0 < q \leq \infty$ the space $B_q^s L_{p,r}(\mathbb{R}^n)$ is an algebra for multiplication.

Proof. Let $X = B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$, $T(f,g) = fg$ and $\tau = s - s_1$. Then $T : X \times X \longrightarrow X$ is bounded. Moreover, by Theorem 6.1 and the reiteration formula for approximation spaces [33, p. 123], we obtain

$$X_q^\tau = \big[[F_2^0 L_{p,r}(\mathbb{R}^n)]_{q_1}^{s_1}\big]_q^\tau = [F_2^0 L_{p,r}(\mathbb{R}^n)]_q^s = B_q^s L_{p,r}(\mathbb{R}^n)\,.$$

To complete the proof, it suffices to show that $T : X_q^\tau \times X_q^\tau \longrightarrow X_q^\tau$ is bounded and this follows by using Lemma 6.2 and proceeding as in the proof of Theorem 6.3. □

Now we are ready to get rid of the restrictions on q in [7, Theorem 7.2].

Theorem 6.5. Let $0 < p < \infty$, $s > n/p$, $0 < q \leq \infty$, $0 < r \leq 1$ and $r < p$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Take $n/p < s_1 < s$ and $0 < q_1 \leq r$. By [7, Theorem 7.2], the space $B_{q_1}^{s_1} L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra. Then, Theorem 6.4 yields that $B_q^s L_{p,r}(\mathbb{R}^n)$ is also a multiplication algebra. □

Next we consider the case $1 < r < p < \infty$. Besides the previous results, this time our arguments rely on complex interpolation.

Theorem 6.6. Let $1 < r < p < \infty$, $s > n/p$ and $0 < q \leq \infty$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Assume first $1 \le q < \infty$. We start by describing $B_q^s L_{p,r}(\mathbb{R}^n)$ by complex interpolation from a suitable couple. We take $0 < \delta < 1$ with

$$1 - (1/r - 1/p) < \delta \tag{6.4}$$

and such that δ is so close to 1 that $u = \delta p$ satisfies that $1 < u$, $s > n/u$ and

$$1/u < 1/r. \tag{6.5}$$

Put

$$\theta = \frac{1 - 1/r}{1 - 1/u}\,.$$

Then we have $0 < \theta < 1$, where the second inequality follows from (6.5). Moreover, we have

$$1/r = (1-\theta) + \theta/u. \tag{6.6}$$

Write

$$1/p_1 = (1-\theta)^{-1}(1/p - \theta/u).$$

By (6.4), it follows that $0 < 1/p_1 < 1/p$. Hence $s > n/p_1$. Furthermore,

$$1/p = (1-\theta)/p_1 + \theta/u. \tag{6.7}$$

Now using (2.4), (6.6) and (6.7) we obtain that

$$[L_{p_1,1}(\mathbb{R}^n), L_u(\mathbb{R}^n)]_\theta = L_{p,r}(\mathbb{R}^n).$$

Hence, applying [38, Theorem 1.18.1], we derive that

$$[\ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n)), \ell_q(2^{ks}L_u(\mathbb{R}^n))]_\theta = \ell_q(2^{ks}L_{p,r}(\mathbb{R}^n)). \tag{6.8}$$

Next consider the operators $Jf = \big((\varphi_k \hat{f})^\vee\big)$ and $R(f_k) = \sum_{k=0}^\infty (\tilde{\varphi}_k \hat{f}_k)^\vee$ where $\tilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ and $\varphi_{-1} = 0$. Since restrictions

$$J : B_q^s L_{p_1,1}(\mathbb{R}^n) \longrightarrow \ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n))\,,$$

$$J : B_{u,q}^s(\mathbb{R}^n) \longrightarrow \ell_q(2^{ks}L_u(\mathbb{R}^n))\,,$$

$$R : \ell_q(2^{ks}L_{p_1,1}(\mathbb{R}^n)) \longrightarrow B_q^s L_{p_1,1}(\mathbb{R}^n)\,,$$

$$R : \ell_q(2^{ks}L_u(\mathbb{R}^n)) \longrightarrow B_{u,q}^s(\mathbb{R}^n)$$

are bounded and $R(Jf) = f$, applying [38, Theorem 1.2.4] and using (6.8), we get

$$[B_q^s L_{p_1,1}(\mathbb{R}^n), B_{u,q}^s(\mathbb{R}^n)]_\theta = B_q^s L_{p,r}(\mathbb{R}^n). \tag{6.9}$$

By Theorem 6.5, the space $B_q^s L_{p_1,1}(\mathbb{R}^n)$ is a multiplication algebra and, according to [46, Theorem 2.41] or [35, Theorem 4.6.4/1], $B_{u,q}^s(\mathbb{R}^n)$ is also a multiplication algebra. Hence, applying the bilinear interpolation theorem for the complex method (Theorem 2.1) to the operator $T(f,g) = fg$ and using (6.9) we conclude that $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Finally, to cover the whole range for q it suffices to apply Theorem 6.4.

□

In [7, Conjecture 7.3] it is conjectured that spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ are multiplication algebras for $0 < p < \infty$, $s > n/p$ and $0 < q, r \leq \infty$. Theorems 6.5 and 6.6 give support to this conjecture. It remains open the case $p < r$.

We end the paper with the counterpart of Theorem 6.6 for Triebel-Lizorkin-Lorentz spaces with $1 \leq q \leq \infty$. In what follows, we consider the sequence space ℓ^s_q with indices on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 6.7. Let $1 < r < p < \infty$, $s > n/p$ and $1 \leq q \leq \infty$. Then $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Proof. Choose δ, u, θ and p_1 as in the proof of Theorem 6.6. So

$$1 < p_1, u < \infty,\ s > \max\{n/p_1, n/u\},\ \frac{1}{r} = (1-\theta) + \frac{\theta}{u} \text{ and } \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{u}\,.$$

Consider again the operators $Jf = \big((\varphi_k \hat{f})^\vee\big)$ and $R(f_k) = \sum_{k=0}^\infty (\tilde{\varphi}_k \hat{f}_k)^\vee$. This time the restrictions

$$\begin{aligned} J : F^s_q L_{p_1,1}(\mathbb{R}^n) &\longrightarrow L_{p_1,1}(\ell^s_q)\,, \\ J : F^s_{u,q}(\mathbb{R}^n) &\longrightarrow L_u(\ell^s_q)\,, \\ R : L_{p_1,1}(\ell^s_q) &\longrightarrow F^s_q L_{p_1,1}(\mathbb{R}^n)\,, \\ R : L_u(\ell^s_q) &\longrightarrow F^s_{u,q}(\mathbb{R}^n) \end{aligned}$$

are bounded. Moreover, by (2.2) and [38, Theorem 1.18.6/2], we have

$$[L_{p_1,1}(\ell^s_q), L_u(\ell^s_q)]_\theta = L_{p,r}(\ell^s_q) \text{ (equivalent norms).}$$

Since $R(Jf) = f$, it follows from [38, Theorem 1.2.4] that

$$[F^s_q L_{p_1,1}(\mathbb{R}^n), F^s_{u,q}(\mathbb{R}^n)]_\theta = F^s_q L_{p,r}(\mathbb{R}^n) \text{ (equivalent norms).} \tag{6.10}$$

The spaces $F^s_q L_{p_1,1}(\mathbb{R}^n)$ and $F^s_{u,q}(\mathbb{R}^n)$ are multiplication algebras by [7, Theorem 5.5] and [46, Theorem 2.41] or [35, Theorem 4.6.4/1]. Therefore, applying Theorem 2.1 to the bilinear operator $T(f,g) = fg$ and using (6.10) we obtain that $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra. □

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