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Differentiable approximation and extension of convex functions

Aproximación y extensión diferenciable de funciones convexas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Director

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DIFFERENTIABLE APPROXIMATION AND EXTENSION OF CONVEX FUNCTIONS

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Resumen

Las funciones convexas y, en particular, las funciones convexas diferenciables juegan un papel fundamental dentro del Análisis Matemático y tienen una gran cantidad de aplicaciones en otras disciplinas como la Geometría Diferencial, la teoría de EDP'S (por ejemplo, las ecuaciones de Monge-Ampère), las Dinámicas No Nineales, la Computación Cuántica o la Economía. Por tanto, es sin duda útil cualquier herramienta que permita aproximar o extender por funciones convexas diferenciables en distintos espacios de Banach.

Si tenemos una función convexa y acotada en conjuntos acotados definida en un espacio de Banach con dual LUR, se sabe por los resultados de D. Azagra que esta función puede aproximarse por funciones convexas diferenciables uniformemente en todo el espacio. Sin embargo, puesto que existen ejemplos de funciones convexas continuas no acotadas en conjuntos acotados, es deseable eliminar esta restricción de la función que deseamos aproximar. En esta tesis se elimina esta restricción y se demuestra que las funciones convexas y continuas definidas en abiertos convexos de espacios de Banach con dual LUR, pueden aproximarse uniformemente por funciones convexas diferenciables uniformemente en todo el espacio. Esto es consecuencia de un resultado más general que muestra que el problema de aproximación uniforme de funciones convexas por funciones convexas de una cierta clase de diferenciabilidad puede reducirse al caso en el que las funciones a aproximar son, además, Lipschitzianas.

En el resto de la tesis se investiga cuándo una función definida en un subconjunto arbitrario de un espacio de Hilbert (finito o infinito dimensional) junto con una familia de derivadas putativas indexada en este subconjunto puede ser extendida a una función diferenciable y convexa en todo el espacio cuyas derivadas extiendan a las derivadas putativas iniciales. Es decir, se estudia el problema de encontrar una versión del Teorema Clásico de Extensión de Whitney para funciones convexas, así como sus principales consecuencias y aplicaciones.

Se obtiene una solución completa al problema en espacios de Hilbert para funciones convexas de clase C^1 con gradiente uniformemente continuo por medio de fórmulas de extensión explícitas garantizando un control prácticamente óptimo del módulo de continuidad asociado al gradiente de la extensión, obteniéndose extensiones con la mejor constante de Lipschitz posible para el gradiente en el caso $C^{1,1}$. Como consecuencia se resuelve el correspondiente problema para funciones de clase $C^{1,1}$ no necesariamente convexas, por medio de fórmulas de extensión explícitas y sencillas. También se obtiene una caracterización de aquellos subconjuntos de un espacio de Hilbert que pueden ser interpolados por hipersuperficies convexas de clase $C^{1,1}$ con hiperplanos tangentes prescritos. El Teorema de Kirszbraun se obtiene también como corolario, mediante una demostración constructiva. Por último, vemos como estos resultados se pueden extender a espacios de Banach superreflexivos para cierta clase de módulos de continuidad Hölderianos asociados a la derivada.

Se obtiene una solución completa al problema en dimensión finita para funciones convexas de clase C^1 con derivada no necesariamente uniformemente continua. Cuando el dominio es compacto, resolvemos el problema por medio de condiciones geométricas naturales, asegurando que la extensión puede tomarse Lipschitziana, con un control absoluto de la constante de Lipschitz. Esto permite caracterizar los subconjuntos compactos que pueden ser interpolados por hipersuperficies convexas compactas de clase C^1 prescribiendo hiperplanos tangentes. Cuando el dominio es no acotado, el problema requiere un estudio detallado del comportamiento global de las funciones convexas diferenciables en $\mathbb{R}^n$; lo cual

nos lleva a desarrollar nuevos conceptos como el de esquina en el infinito a lo largo de subespacios. Lo que se obtiene entonces es un Teorema de Extensión de Whitney para funciones convexas de clase C^1 con comportamiento global geométrico prescrito. Se resuelve un problema similar para funciones convexas Lipschitz y diferenciables con un control prácticamente óptimo de la constante de Lipschitz de la extensión en términos de los valores de las derivadas putativas iniciales. Como consecuencia se caracterizan los subconjuntos de $\mathbb{R}^n$ que pueden ser interpolados por hipersuperficies convexas de clase C^1 con hiperplanos tangentes y subespacios de direcciones de normal exterior unitaria prescritos. Aplicaciones de estos resultados a distintas especialidades como la caracterización de las curvas fuertemente auto-contractantes en $\mathbb{R}^n$ y las propiedades de Lusin para funciones convexas diferenciables han sido obtenidas por otros autores.

Se estudia también el problema para funciones convexas en $\mathbb{R}^n$ de clase C^m, con $m \geq 2$. Se presentan resultados parciales para dominios convexos compactos generales y otros prácticamente óptimos para dominios que son intersección de una familia finita de ovaloides con cierto orden de diferenciabilidad. Por último, se da la solución completa al problema para funciones convexas de clase C^∞ en $\mathbb{R}^n$ prescribiendo las infinitas derivadas putativas desde dominios convexos compactos.

Abstract

The class of convex functions and, in particular, the class of differentiable convex functions play a very important role in the field of Mathematical Analysis and they have plenty of applications in other disciplines such as Differential Geometry, PDE theory (for instance, Monge-Ampère equations), Nonlinear Dynamics, Quantum Computing or Economics. Therefore, it is no doubt useful to be able to approximate or to extend by differentiable convex functions in various Banach spaces.

If we are given a convex function bounded on bounded sets defined on a Banach space whose dual norm is LUR, recent results by D. Azagra ensure that this function can be approximated by differentiable convex functions uniformly on the whole space. However, since there are example of continuous convex functions which are not bounded on bounded subsets, it is desirable to improve these results in such a way that this restriction on the function to be approximated can be removed. In this thesis, we drop this assumption and show that every continuous convex function (not necessarily bounded on bounded subsets) defined on an open convex subset of a Banach space whose dual norm is LUR, can be uniformly approximated by differentiable convex functions. This result is a consequence of a more general result which shows that the problem of approximating continuous convex functions uniformly by convex functions of a certain differentiability class can be reduced to the case when the original functions are, in addition, Lipschitz.

In the rest of the thesis we investigate the problem of extending a function defined on an arbitrary subset of a Hilbert space (finite or infinite dimensional) along with a family of putative derivatives indexed by this subset can be extended to a differentiable convex function on the whole space whose derivatives coincide with the original putative derivatives. In other words, we look for a version of the Classical Whitney Extension Theorem for convex functions, as well as its main consequences and applications.

We obtain a full solution to this problem in Hilbert spaces for convex functions of class C^1 with uniformly continuous gradient by means of an explicit extension formula, which guarantees an almost optimal control of the modulus of continuity associated to the gradient of the extension, providing extensions with the best possible Lipschitz constant of the gradient in the $C^{1,1}$ case. As a consequence, we solve the problem for general functions (not necessarily convex) of class $C^{1,1}$ by means of simple explicit extension formulas. Also, we obtain a characterization of those subsets of a Hilbert space which can be interpolated by hypersurfaces of class $C^{1,1}$ with prescribed tangent hyperplanes. Moreover, Kirszbraun's Theorem is recovered as a corollary, by means of a constructive proof. Finally, we show how these results can be extended to the setting of superreflexive Banach spaces for some Hölderian moduli of continuity associated to the derivatives.

We obtain the full solution to the problem on $\mathbb{R}^n$ for convex functions of the class C^1 with not necessarily uniformly continuous derivatives. If the domain is compact, we solve the problem by means of natural geometric conditions, ensuring that the extension can be taken to be Lipschitz, with an absolute control of the Lipschitz constant. This allows us to characterize those compact subsets of $\mathbb{R}^n$ which can be interpolated by compact convex hypersurfaces of class C^1 with prescribed tangent hyperplanes. If the domain is unbounded, our problem requires a deep understanding of the global geometric behaviour of differentiable convex functions on $\mathbb{R}^n$; which leads us to introduce new concepts such as corners at infinity directed by subspaces. We then obtain a Whitney Extension Theorem for convex functions of class C^1 with prescribed global geometric behaviour. We also solve a similar problem for convex

Lipschitz functions of class C^1 with an absolute control of the Lipschitz constant of the extension. As a consequence, we characterize those subsets of $\mathbb{R}^n$ which can be interpolated by convex hypersurfaces of class C^1 prescribing tangent hyperplanes and subspaces generated by the directions of the outer unit normal. Applications to other research lines such as the theory of self-contracted curves on $\mathbb{R}^n$ and Lusin properties for differentiable convex functions have been found by other authors.

We also study the same problem for convex functions of class C^m, $m \geq 2$ on $\mathbb{R}^n$. We present partial results por general compact convex domains and provide almost optimal results for domains which are the intersection of a finite number of ovaloids of a certain differentiability class. Finally, we obtain the full solution to our problem for convex functions of class C^∞ on $\mathbb{R}^n$ extending the infinite given family of putative derivatives from compact convex domains.

Index

Introducción

El tema principal de esta tesis es la aproximación y extensión diferenciable de funciones convexas en diversos espacios de Banach.

En el Capítulo 1 tratamos el problema de aproximar funciones convexas por funciones de clase C^1 y convexas uniformemente en espacios de Banach. En $\mathbb{R}^n$, es bien sabido que la convolución integral con *mollifiers* nos permite aproximar funciones convexas por funciones de clase C^∞ y convexas uniformemente en conjuntos compactos. En espacios de Banach cuyo dual admite una norma equivalente LUR (localmente uniformemente redonda), podemos aproximar funciones convexas por funciones de clase C^1 y convexas uniformemente en subconjuntos acotados mediante las técnicas de convolución infimal. En [1] se descubrió una nueva técnica de aproximación que nos permite aproximar funciones convexas (no necesariamente uniformemente continuas) en $\mathbb{R}^n$ por funciones convexas de clase C^∞ (o incluso real analíticas), uniformemente en todo $\mathbb{R}^n$. Combinando esta nueva técnica con el método de aproximación por convolución infimal mencionado se sigue que las funciones convexas f que están acotadas en subconjuntos acotados de un espacio de Banach E cuyo dual admite una norma equivalente LUR pueden aproximarse por funciones convexas g de clase C^1, uniformemente en todo E. Sin embargo, existen ejemplos de funciones convexas y continuas (diferenciables o no) que no son acotadas en conjuntos acotados. En esta tesis, refinando las técnicas introducidas en [1], vemos como podemos quitar la hipótesis de que la función f sea acotada en acotados y demostramos el siguiente teorema.

Teorema 1. *Sea X un espacio de Banach cuyo espacio dual X^* admite una norma equivalente LUR. Sea $f : U \to \mathbb{R}$ una función convexa y continua definida en un subconjunto abierto U de X. Dado $\varepsilon > 0$, existe una función convexa $g : U \to \mathbb{R}$ de clase $C^1(U)$ de manera que $f - \varepsilon \leq g \leq f$ en U.*

El Teorema 1 se sigue de un resultado más general que demuestra que el problema de aproximar funciones convexas continuas por funciones de clase C^m y convexas en subconjuntos abiertos U de X puede reducirse al problema de aproximar funciones Lipschitz y convexas.

Además, como consecuencia de estos resultados, establecemos una nueva caracterización de los espacios de Banach cuyo dual es separable, a saber, un espacio de Banach separable X tiene dual X^* separable si y solo si toda función convexa y continua definida en un subconjunto abierto de X puede aproximarse uniformemente por funciones convexas de clase C^1.

Estos resultados han sido publicados en [9].

En el resto de la tesis, abordamos el problema de encontrar una versión del Teorema Clásico de Extensión de Whitney [70] para funciones convexas. Este famoso resultado, para la clase C^m, proporciona condiciones necesarias y suficientes sobre una familia de funciones real valuadas $\{f_\alpha\}_{|\alpha| \leq m}$ (que llamaremos m-jet) definidas en un subconjunto cerrado $E \subset \mathbb{R}^n$ para la existencia de una función F de clase $C^m(\mathbb{R}^n)$ tal que $\partial^\alpha F = f_\alpha$ en E para todo multi-índice $|\alpha| \leq m$. Estas condiciones son relaciones entre las funciones f_α y el polinomio de Taylor putativo P_y de grado menor o igual que m centrado en $y \in E$ y cuyos coeficientes son justamente los números $(f_\alpha(y))_\alpha$, y la extensión F se define mediante una formula explícita que involucra una partición de la unidad adecuada subordinada a una familia de cubos cuidadosamente elegidos que descompone el complementario del conjunto E. Algunos años después, G. Glaeser [46] estableció una versión del Teorema de Extensión de Whitney para funciones de clase $C^{1,\omega}$ en $\mathbb{R}^n$, mediante una construcción similar a la de Whitney, que además permite obtener un buen control del módulo de continuidad de las derivadas de la extensión en términos de la familia $\{f_\alpha\}_{|\alpha|\leq m}$ dada. Por otro lado, J. C. Wells [69] demostró un análogo del Teorema de Extensión de Whitney para funciones de clase $C^{1,1}$ en espacios de Hilbert, obteniendo además un control óptimo de la constante Lipschitz del gradiente de la extensión, en términos del 1-jet inicial. La demostración de Wells se basa en una construcción geométrica muy complicada cuando el dominio es finito y después un proceso de paso al límite para conjuntos arbitrarios. Más recientemente, E. Le Gruyer [53] probó el mismo teorema para la clase $C^{1,1}$ simplificando considerablemente la demostración de Wells pero utilizando el Lema de Zorn. Debe también mencionarse que M. Jiménez y L. Sánchez [50] demostraron una versión del Teorema de Extensión de Whitney para funciones de clase C^1 en espacios de Banach separables que satisfacen una cierta propiedad relacionada con la aproximación de funciones Lipschitz por funciones diferenciables Lipschitz, a saber, que toda función 1-Lipschitz puede aproximarse uniformemente por funciones Lipschitz y de clase C^1 con constante de Lipschitz menor o igual que una constante absoluta que solo depende del espacio ambiente. Esta clase de espacios incluye, por ejemplo, el espacio de Hilbert separable. Esta construcción es un refinamiento de una técnica de extensión introducida por D. Azagra, R. Fry y L. Keener [6] para resolver el mismo problema cuando el dominio es un subespacio vectorial cerrado de un espacio de Banach separable, y en última estancia depende de una técnica de extensión inspirada por el Teorema de Extensión de Tietze y las *sup particiones de la unidad* introducidas por R. Fry en [42]. Finalmente, cabe destacar que J. Ferrera y J. Gómez Gil en [31] probaron una versión del Teorema de Extensión de Whitney para funciones subdiferenciables.

Otro problema relacionado es el *Problema de Extensión de Whitney para funciones* (en contraposición al problema para jets): dado un subconjunto arbitrario E de $\mathbb{R}^n$, y una función $f : E \to \mathbb{R}$ (pero sin candidatos a derivadas) ¿qué condiciones sobre f son necesarias y suficientes para la existencia de una función $F : \mathbb{R}^n \to \mathbb{R}$ de clase C^m o $C^{m-1,1}$ tal que $F = f$ en E? Además, ¿qué podemos decir de la norma de la extensión F en caso de existir? Este tipo de problemas son mucho más difíciles de resolver. El caso $C^{1,1}$ fue resuelto por Y. Brudnyi y P. Shvartsman en [20], y la solución completa al problema fue dada por C. L. Fefferman en [36] y [37]. Véanse también los trabajos de C. L. Fefferman, A. Israel, G. K. Luli and P. Shvartsman citados en la Bibliografía para resultados similares en ciertos espacios de Sobolev.

El problema general sobre el que vamos a trabajar es el siguiente.

Problema. Dado un entero positivo m, un subconjunto arbitrario E de $\mathbb{R}^n$ y un m-jet $\{f_\alpha\}_{|\alpha|\leq m}$ definido en E, ¿qué condiciones necesarias y suficientes sobre $\{f_\alpha\}_{|\alpha|\leq m}$ garantizarían la existencia de una función **convexa** y de clase $C^m(\mathbb{R}^n)$ de manera que $\partial^\alpha F = f_\alpha$ en E para todo $|\alpha| \leq m$?

Un problema similar a este ha sido considerado por M. Ghomi [44] y M. Yan [72], y como consecuencia de sus resultados se sabe que, si E es compacto y convexo y tenemos una función $f : E \to \mathbb{R}$ que admite una extensión C^m (no necesariamente convexa) a todo $\mathbb{R}^n$ cuya segunda derivada es definida positiva en ∂E, entonces existe una función convexa F y de clase C^m tal que F coincide con f en E. Por supuesto, este resultado es solo una solución parcial a nuestro problema porque el hecho de que nuestra función de partida tenga Hessiano estrictamente positivo es una hipótesis muy fuerte, que dista mucho de ser necesaria. Por otro lado, K. Schulz y B. Schwartz [59] caracterizaron las funciones propias y convexas en $\mathbb{R}^n$ definidas en dominios convexos que admiten extensiones convexas (no necesariamente diferenciables) a todo $\mathbb{R}^n$. También, B. Mulansky y M. Neamtu [55] probaron que cualquier conjunto finito de datos en $\mathbb{R}$ o en $\mathbb{R}^2$ que sea *estrictamente convexo* en un sentido apropiado puede ser interpolado por un polinomio convexo. Finalmente, mencionemos que O. Bucicovschi y J. Lebl [21] trabajaron el problema de extender funciones convexas a la envoltura convexa de sus dominios y que J. M. Borwein, V. Montesinos y J. Vanderwerff [18], y L. Veselý y L. Zajícek [66] demostraron que existen espacios de Banach X de dimensión infinita, *subespacios* cerrados $E \subset X$ y funciones continuas y convexas $f : E \to \mathbb{R}$ que no admiten ninguna extensión continua y convexa a todo X.

A continuación describimos el progreso realizado en la solución al problema en cuestión así como las principales consecuencias y aplicaciones de nuestros resultados.

En el Capítulo 2, damos la solución completa al problema anterior para funciones convexas de clase $C^{1,\omega}$, incluso en espacios de Hilbert, es decir, dado un subconjunto arbitrario E de un espacio de Hilbert X y dos funciones $f : E \to \mathbb{R}$ and $G : E \to X$, damos condiciones necesarias y suficientes sobre (f, G) para la existencia de una función convexa F de clase $C^{1,\omega}$ tal que $F_{|E} = f$ y $\nabla F_{|E} = G$. La solución pasa por definir una nueva condición necesaria y suficiente $(CW^{1,\omega})$, una sencilla desigualdad que solo involucra el módulo de continuidad ω, una constante $M > 0$ y los valores de f y G en E. Esta condición nos permite dar una formula explícita sencilla para la extensión F garantizando además un *control prácticamente óptimo* del módulo de continuidad ∇F en términos de (f, G). De hecho, en el caso $C^{1,1}$ se puede asegurar un *control óptimo* de la constante de Lipschitz de ∇F. El enunciado preciso del resultado es el siguiente.

Teorema 2. *Dado un subconjunto arbitrario $E \subset X$ de un espacio de Hilbert X, y dos funciones $f : E \to \mathbb{R}$, $G : E \to X$, de manera que (f, G) satisface la desigualdad $(CW^{1,1})$ con constante $M > 0$ en E, la formula*

$$F = conv(g), \quad g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2\}, \quad x \in X,$$

define una función convexa de clase $C^{1,1}$ en X tal que $F_{|E} = f$, $(\nabla F)_{|E} = G$ y $\mathrm{Lip}(\nabla F) \leq M$.

Esto indica que si definimos M como el menor número positivo tal que (f, G) satisface la condición $(CW^{1,1})$ con constante M, entonces la extensión F anterior tiene la propiedad de que

$$\mathrm{Lip}(\nabla F) = \inf\{\mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}_{\text{conv}}(X), \ H_{|E} = f, \ (\nabla H)_{|E} = G\},$$

y entonces podemos afirmar que ∇F tiene la mejor constante de Lipschitz posible. La formula para la clase $C^{1,\omega}$ es similar, y este caso obtenemos el mismo tipo de control salvo un factor 8.

La principal consecuencia de la formula anterior para funciones convexas $C^{1,1}$ es que nos permite dar una solución sencilla y explícita no solo para el problema de extensión $C^{1,1}_{\text{conv}}$ de jets sino también para el problema general de extensión $C^{1,1}$ de jets en espacios de Hilbert, y con la mejor constante de Lipschitz posible del gradiente de la extensión. En [46] se demuestra que puede obtenerse un control del tipo

$$\mathrm{Lip}(\nabla F) = k(n) \inf\{\mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X), \ H_{|E} = f, \ (\nabla H)_{|E} = G\},$$

para extensiones $C^{1,1}$ en $\mathbb{R}^n$, siendo $k(n)$ una constante que depende únicamente de la dimensión y tiende a ∞ cuando n tiende a infinito. Por otro lado, la solución que se da en [69] y en [53] son óptimas en el sentido anterior y son válidas para espacios de Hilbert de dimensión infinita, pero la demostración de [69] se basa en una construcción geométrica extremadamente complicada y la de [53] usa el Lema de Zorn y por tanto no es constructiva. Con la ayuda de nuestra solución al problema de extensión $C^{1,1}_{\text{conv}}$ de jets, podemos recuperar los resultados de [46], [69] y [53] para funciones $C^{1,1}$ mediante una formula explícita sencilla que proporciona una extensión con un control óptimo de la constante de Lipschitz del gradiente. Para hacer esto, consideramos la condición necesaria $(W^{1,1})$, que es una sencilla desigualdad involucrando sólamente los valores de f y G y una constante $M > 0$ y es equivalente a las condiciones consideradas anteriormente en [69] y [53]. En enunciado exacto de nuestra solución al problema de extensión $C^{1,1}$ de jets es el siguiente.

Teorema 3. *Dado un subconjunto arbitrario $E \subset X$ de un espacio de Hilbert X, y dos funciones $f : E \to \mathbb{R}$, $G : E \to X$, de manera que (f, G) satisface la desigualdad $(W^{1,1})$ con constante $M > 0$ en E, la formula*

$$\begin{aligned} &F = conv(g) - \tfrac{M}{2}\|\cdot\|^2, \\ &g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2\} + \tfrac{M}{2}\|x\|^2, \quad x \in X, \end{aligned}$$

define una función de clase $C^{1,1}(X)$ tal que $F_{|E} = f$, $(\nabla F)_{|E} = G$, y $\mathrm{Lip}(\nabla F) \leq M$.

Además, F puede tomarse con la propiedad de que

$$\mathrm{Lip}(\nabla F) = \inf\{\mathrm{Lip}(\nabla H) \, : \, H \in C^{1,1}(X), \; H_{|E} = f, \; (\nabla H)_{|E} = G\}.$$

Por otro lado, toda función $F \in C^{1,1}(X)$ satisface la condición $(W^{1,1})$ con constante $M = Lip(\nabla F)$ en todo subconjunto E de X.

Una consecuencia de nuestra solución al problema general de extensión $C^{1,1}$ para jets (que, como hemos dicho, es una consecuencia de nuestra solución al problema de extensión $C^{1,1}_{\mathrm{conv}}$ para jets) es que nos permite dar una breve demostración del Teorema de Extensión de Kirszbraun [52] para funciones Lipschitz entre dos espacios de Hilbert proporcionando además una sencilla formula constructiva para la extensión. Véase el Corolario 2.28 en el Capítulo 2.

Por último, hemos encontrado una aplicación de nuestro teorema de extensión $C^{1,1}_{\mathrm{conv}}$ en relación con la siguiente pregunta natural: dado un subconjunto arbitrario C de un espacio de Hilbert X y una colección $\mathcal{H}$ de hiperplanos afines de X tal que cada $H \in \mathcal{H}$ pasa por un punto $x_H \in C$, ¿qué condiciones necesarias y suficientes sobre esta familia nos garantizarían la existencia de una hipersuperficie convexa S en X de clase $C^{1,1}$ tal que H es tangente a S en x_H para todo $H \in \mathcal{H}$? Equivalentemente, dado un subconjunto C de X y una aplicación $N : C \to S_X$, ¿qué condiciones necesarias y suficientes garantizarían la existencia de cuerpos convexos V en X de clase $C^{1,1}$ tal que $C \subseteq \partial V$ y la normal exterior unitaria a ∂V coincide con la aplicación N en C? La condición pertinente es una desigualdad sencilla $(\mathcal{KW}^{1,1})$ para la aplicación N; véase el Teorema 2.20 en el Capítulo 2.

En el Capítulo 3, consideramos el mismo problema que en el Capítulo 2 para la clase de funciones convexas $C^{1,\alpha}$ en espacios superreflexivos, para cierto $\alpha \in (0,1)$ adecuado. El Teorema de Renormamiento de Pisier [56] nos dice que todo espacio superreflexivo posee una norma equivalente uniformemente suave con módulo de suavidad de tipo potencia $p = 1+\alpha$ para algún $\alpha \in (0,1)$. Demostramos que la condición $(CW^{1,\alpha})$ es necesaria y suficiente sobre un par de funciones $(f, G) : E \to \mathbb{R} \times X$ para la existencia de una función convexa F de clase $C^{1,\alpha}$ con $(F, \nabla F) = (f, G)$ en E. La expresión que define F es similar a las que obtuvimos en espacios de Hilbert y, nuevamente, podemos conseguir un *control prácticamente óptimo* del α-Hölder módulo de continuidad de ∇F en términos de (f, G) y de una constante absoluta que solo depende del espacio.

Los resultados de los Capítulos 2 y 3 están publicados en [8], [11], y [12].

En el Capítulo 4 resolvemos el problema de extender dos funciones convexas $f : E \to \mathbb{R}$ y $G : E \to X$, definidas en un subconjunto arbitrario E de $\mathbb{R}^n$ a una función convexa F de clase C^1 tal que $F_{|E} = f$ y $\nabla F_{|E} = G$, es decir, establecemos una versión C^1_{conv} del Teorema de Extensión de Whitney para jets. Empezamos resolviendo el problema cuando el dominio E es compacto por medio de dos condiciones nuevas (C) y (CW^1). La condición (C) significa que la función f debe quedar por encima de los *tangentes putativos* $f(y) + \langle G(y), \cdot - y\rangle$ y la condición (CW^1) dice que si dos puntos de la gráfica de f en E están en un segmento contenido en un hiperplano que queremos que sea tangente a la gráfica de la extensión en alguno de esos puntos, entonces los hiperplanos tangentes putativos en ambos puntos deben coincidir. El enunciado exacto de nuesto resultado para dominios compactos es el siguiente.

Teorema 4. *Sea E un subconjunto compacto de $\mathbb{R}^n$ y sean $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ dos funciones tal que G es continua. Existe una función convexa F de clase $C^1(\mathbb{R}^n)$ con $F_{|E} = f$ y $\nabla F_{|E} = G$ si y solo si (f, G) satisface las condiciones (C) y (CW^1) en E. Además, la extensión F puede tomarse con la propiedad de que* $\mathrm{Lip}(F) \leq \kappa \sup_E |G|$, *donde κ es una constante absoluta.*

Es importante observar que este tipo de control de la constante de Lipschitz de la extensión F únicamente en términos de G no puede obtenerse para funciones generales (no necesariamente convexas) de clase C^1, en cuyo caso lo mejor que se puede obtener es una estimación de $\mathrm{Lip}(F)$ en términos de $\mathrm{Lip}(f)$ y de $\sup_E |G|$.

Al igual que en el caso $C^{1,1}$, una aplicación geométrica de nuestra solución al problema de extensión C^1_{conv} para jets definidos en conjuntos compactos es la caracterización de aquellos conjuntos compactos

C de $\mathbb{R}^n$ equipados con una aplicación $N : C \to \mathbb{S}^{n-1}$ que pueden interpolarse por fronteras de cuerpos compactos convexos V de clase C^1 que contienen al origen en su interior y tales que la normal exterior unitaria a ∂V coincide con la aplicación N en C. Las condiciones pertinentes surgen de traducir de manera natural las condiciones (C) y (CW^1) al contexto de cuerpos convexos C^1 mediante el Teorema de la Función Implícita. Este resultado puede compararse con los de [43], donde M. Ghomi muestra cómo construir cuerpos fuertemente convexos de clase C^m prescribiendo subvariedades fuertemente convexas y planos tangentes. La caracterización que probamos nos permite tratar con conjuntos compactos arbitrarios en lugar de variedades y también eliminar la hipótesis de convexidad fuerte.

Destacamos también que, muy recientemente, E. Durand-Cartagena y A. Lemenant [29] han utilizado el Teorema 4 para demostrar que las curvas fuertemente auto-contractantes de clase $C^{1,\alpha}$ se caracterizan por ser soluciones de ecuaciones de flujo de gradiente de funciones convexas de clase C^1.

Consideremos ahora el problema cuando nuestro dominio $E \subset \mathbb{R}^n$ es arbitrario, y en particular no necesariamente acotado. Al contrario que el caso compacto, dadas dos funciones $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ con G continua, la hipótesis de que (f, G) satisface las condiciones (C) y (CW^1) no garantiza que existan funciones convexas (no necesariamente diferenciables) extendiendo (f, G) a todo $\mathbb{R}^n$, como demostraremos con ejemplos. No es difícil resolver este inconveniente y basta con añadir una condición necesaria sobre (f, G), que en este resumen denotaremos por (EX). Sin embargo, todavía tenemos que resolver un problema mucho más profundo que está relacionado con el comportamiento global que pueden tener las funciones convexas diferenciables en $\mathbb{R}^n$: la posible presencia de *esquinas en el infinito*. Diremos que el par de funciones (f, G) tiene un esquina en el infinito cuando (f, G) no satisface la generalización natural de la condición (CW^1) que introducimos en el caso compacto (sustituyendo puntos por sucesiones no acotadas). Por ejemplo, en $\mathbb{R}^2$, las funciones de clase C^1 y convexas definidas en todo $\mathbb{R}^n$ que tienen esquinas en el infinito pueden verse como funciones convexas diferenciables cuya gráfica es *tangente en el infinito* a la gráfica de funciones convexas que no son diferenciables a lo largo de alguna recta de $\mathbb{R}^2$. Y en dimensiones superiores pueden encontrarse más ejemplos patológicos de funciones convexas diferenciables que tienen *esquinas en el infinito en direcciones de subespacios* de dimensión k para todo $k \leq n$. Por otra parte, en [1] se demostró que las funciones convexas en $\mathbb{R}^n$ tienen subespacios unívocamente determinados que caracterizan su comportamiento global; lo que significa que a cada función convexa g podemos asociarle un único subespacio vectorial X_g de $\mathbb{R}^n$ de manera que g puede escribirse, salvo restar un funcional lineal, como una función convexa y coerciva definida en X_g compuesta con la proyección ortogonal sobre X_g. Pues bien, resulta que la presencia de esquinas del infinito en una función diferenciable y convexa g fuerza la coercividad esencial de g en las direcciones de la esquina, es decir, el subespacio X_g contiene dichas direcciones. Por estas razones, vamos a enunciar nuestros resultados de extensión no solo para 1-jets (f, G) sino también para un subespacio dado X de $\mathbb{R}^n$ que queremos que represente el comportamiento global de nuestra función convexa F de clase C^1 extendiendo el jet (f, G) a todo $\mathbb{R}^n$. Lo que necesitaremos será una variante de la condición (CW^1), a saber, una *condición* (CW^1) *para* (f, G) *con subespacio* Z, en la que se consideran sucesiones que, aunque puedan ser no acotadas, sus proyecciones sobre el subespacio Z sí están acotadas. También necesitamos suponer que el subespacio $Y = \operatorname{span}\{G(E) - G(E)\}$ de las diferencias de las derivadas putativas en E está contenido en el subespacio de partida X. La condición más técnica que utilizaremos podría resumirse informalmente diciendo que, en el caso en que el jet (f, G) no nos proporcione suficientes datos de diferenciabilidad para que (f, G) cumpla la condición (CW^1) con el subespacio Y de direcciones putativas, debe haber suficiente espacio en $\mathbb{R}^n \setminus \overline{E}$ para que podamos añadir nuevos datos $(\beta_1, w_1), \ldots (\beta_d, w_d)$ asociados a puntos $p_1, \ldots, p_d \in \mathbb{R}^n \setminus \overline{E}$ de tal manera que este nuevo jet es compatible con el problema de extensión convexo (es decir, que se sigue cumpliendo la condición (C)) y además se satisface la condición (CW^1) con subespacio X. Esta condición de compatibilidad de nuestro jet de partida (f, G) en términos de los subespacios X e Y será denotada en este resumen por (CO_X^Y). Veremos que estas condiciones son necesarias y suficientes para la existencia de funciones convexas F de clase C^1 que extienden el jet (f, G) definido en E a todo $\mathbb{R}^n$ y tienen un comportamiento global determinado por el subespacio X. En consecuencia, nuestra solución al problema de extensión convexa de clase C^1 para jets definidos en conjuntos arbitrarios tiene el siguiente enunciado (véase el Capítulo 4 para la definición exacta de todas estas condiciones necesarias y suficientes).

Teorema 5. *Sea E un subconjunto arbitrario de $\mathbb{R}^n$, $X \subset \mathbb{R}^n$ un subespacio vectorial y $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ dos funciones tales que G es continua. Denotemos $Y = span\{G(E) - G(E)\}$. Existe una función convexa $F : \mathbb{R}^n \to \mathbb{R}$ de clase C^1 tal que $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, y $X_F = X$, si y solamente si (f, G) satisface las condiciones (C), (EX), $Y \subseteq X$, (CO_X^Y) y (CW^1) con subespacio X.*

El Teorema 5 establece lo que podríamos llamar un *Teorema de Extensión de Whitney para funciones convexas de clase C^1 con comportamiento global prescrito*. Esto significa que podemos caracterizar el comportamiento global de las extensiones diferenciables convexas F de (f, G) en términos del subespacio generado por las diferencias de derivadas putativas G en E y del *comportamiento diferencial* del jet (f, G) en E.

También vamos a estudiar la situación particular en la que la función G es acotada en E. En este caso siempre existen extensiones convexas (no necesariamente diferenciables) y no necesitamos añadir la condición (EX) mencionada anteriormente. Por otra parte, la condición de compatibilidad (CO_X^Y) puede reescribirse, en este caso particular, de tal manera que solo es necesario verificar que la adherencia $\overline{E}$ de E es disjunta con la unión de una cierta familia finita de conos. Además, podemos garantizar un *control prácticamente óptimo* de la constante de Lipschitz de la extensión en términos de $\sup_E |G|$.

Teorema 6. *Sea E un subconjunto arbitrario de $\mathbb{R}^n$, $X \subset \mathbb{R}^n$ un subespacio vectorial y $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ dos funciones tales que G es continua y acotada. Denotemos $Y = span\{G(E) - G(E)\}$. Existe una función Lipschitz y convexa $F : \mathbb{R}^n \to \mathbb{R}$ de clase C^1 tal que $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, y $X_F = X$, si y solo si (f, G) satisface las condiciones $(C), Y \subseteq X, (CW^1)$ con subespacio X y, si $Y \neq X$ y denotamos $k = \dim Y$ y $d = \dim X$, existen puntos $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, un número $\varepsilon \in (0, 1)$, y vectores $w_1, \ldots, w_{d-k} \in X \cap Y^\perp$ normalizados y linealmente independientes tales que*

$$\overline{E} \cap \left(\bigcup_{j=1}^{d-k} V_j \right) = \emptyset,$$

donde, para cada $j = 1, \ldots, d - k$, se denota $V_j := \{x \in \mathbb{R}^n \ : \ \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x - p_j)|\}$, y $P_Y : \mathbb{R}^n \to Y$ es la proyección ortogonal sobre Y.

Además, existe una constante absoluta $\kappa > 0$ tal que, si se satisfacen las condiciones anteriores, la extensión F puede tomarse de modo que

$$\mathrm{Lip}(F) \leq \kappa \sup_{y \in E} |G(y)|.$$

Al igual que en caso compacto, el Teorema 6 puede usarse para dar respuesta a la siguiente pregunta: dado un subconjunto arbitrario E de $\mathbb{R}^n$ y una colección $\mathcal{H}$ de hiperplanos afines de $\mathbb{R}^n$ tal que cada $H \in \mathcal{H}$ pasa por un punto $x_H \in E$, ¿qué condiciones son necesarias y suficientes para la existencia de una hipersuperficie convexa S de clase C^1 en $\mathbb{R}^n$ tal que H es tangente a S en x_H para cada $H \in \mathcal{H}$? Equivalentemente, dado un subconjunto E de $\mathbb{R}^n$ equipado con una aplicación $N : E \to \mathbb{S}^{n-1}$, ¿qué condiciones sobre E y N son necesarias y suficientes para la existencia de un cuerpo convexo V (no necesariamente acotado) de clase C^1 tal que $E \subseteq \partial V$ y la normal exterior unitaria a ∂V coincida en E con la aplicación N dada? Para responder a esta pregunta proporcionamos una caracterización que también prescribe el subespacio de direcciones que queremos que tenga la normal exterior unitaria de nuestro cuerpo convexo. Más precisamente, dada $N : E \to \mathbb{S}^{n-1}$ y el subespacio X de $\mathbb{R}^n$, encontramos condiciones necesarias y suficientes en términos de N, $\mathrm{span}(N(E))$ y X, para la existencia de un cuerpo convexo V de clase C^1 con $0 \in \mathrm{int}(V)$ y $E \subseteq \partial V$, con normal exterior unitaria n_V a ∂V igual a N en E y cumpliendo que $X = \mathrm{span}(n_V(\partial V))$; véase el Teorema 4.69 en el Capítulo 4.

Finalmente, destacamos que, muy recientemente, D. Azagra y P. Hajłasz [7] han encontrado una aplicación de nuestro teorema de extensión de funciones convexas de clase C^1 para jets definidos en conjuntos arbitrarios al problema de caracterizar la clase de funciones convexas que tienen la propiedad de Lusin de clase C^1_{conv}. Más concretamente, demuestran que una función convexa $f : \mathbb{R}^n \to \mathbb{R}$ que no es de clase C^1 tiene una propiedad de Lusin de tipo C^1_{conv} (es decir, que para todo $\varepsilon > 0$ existe una función convexa $g : \mathbb{R}^n \to \mathbb{R}$ de clase C^1 tal que $\mathcal{L}^n\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) \leq \varepsilon$) si y solo si f es esencialmente coerciva, i.e., coerciva salvo la resta de un funcional lineal.

Los resultados del Capítulo 4 están publicados en [12] y [13].

En el Capítulo 5 consideramos el problema de extensión de funciones convexas de clase C^m para jets en $\mathbb{R}^n$, cuando $m \geq 2$. Veremos por medio de ejemplos sencillos que, si el dominio no se supone convexo, el problema es mucho más complicado que en los casos C^1 o $C^{1,\omega}$. Por este motivo, centraremos nuestra atención al caso en el que el dominio E de definición de nuestro jet es un convexo compacto. Como hemos dicho anteriormente, siempre existen extensiones convexas de clase C^m si suponemos que nuestro jet tienen segunda derivada putativa estrictamente positiva en ∂E, gracias a los resultados de [44] y [72]. Esta condición es claramente no necesaria y debemos considerar otras condiciones en su lugar. Introduciremos una condición nueva (CW^m) para m-jets definidos en compactos convexos, que esencialmente dice que la expresión del Hessiano putativo del jet en cada dirección $v \in \mathbb{S}^{n-1}$ en términos de los polinomios de Taylor putativos centrados en puntos $y \in \partial E$ tiene límite inferior mayor o igual que 0 uniformemente en $y \in \partial E$ y $v \in \mathbb{S}^{n-1}$. Aunque esta condición (CW^m) es necesaria sobre un m-jet para la existencia de extensiones convexas de clase C^m, daremos ejemplos que demuestran que no es suficiente, al menos en el caso en el que el dominio E tiene interior vacío. De hecho, esos ejemplos mostrarán que existen jets de clase C^∞ satisfaciendo la condición (CW^3) en E y sin embargo no tienen ninguna extensión convexa de clase C^2. No obstante, se puede demostrar que si nuestro m-jet cumple la condición (CW^m) entonces posee una extensión convexa de clase C^{m-n-1} en $\mathbb{R}^n$. Además, añadiendo algunas condiciones geométricas adicionales sobre el conjunto E (a saber, que E es la intersección de una cantidad finita de ovaloides de clase C^m), entonces la condición (CW^m) garantiza la existencia de extensiones convexas de clase $C^{m-1}(\mathbb{R}^n)$.

En el Capítulo 6, resolvemos el problema de extensión de funciones convexas de clase C^∞ para *jets infinitos* definidos en dominios compactos y convexos de $\mathbb{R}^n$. Por un *jet infinito* definido en E simplemente entendemos una familia infinita de funciones real valuadas $\{f_\alpha\}_{\alpha \in (\mathbb{N} \cup \{0\})^n}$ definidas en E. Introducimos una nueva condición (CW^∞) sobre el jet infinito $\{f_\alpha\}_{\alpha \in (\mathbb{N} \cup \{0\})^n}$, que dice que, para todo entero $m \geq 2$, la familia finita $\{f_\alpha\}_{|\alpha| \leq m}$ satisface la condición (CW^m) para el problema C^m mencionada anteriormente. Afortunadamente, y al contrario que en el caso C^m con m finito, esta condición (CW^∞) es necesaria y suficiente para que haya extensiones convexas C^∞, y nuestro principal resultado para la clase C^∞ es el siguiente.

Teorema 7. *Sea $E \subset \mathbb{R}^n$ un subconjunto convexo y compacto y sea $\{f_\alpha\}_{\alpha \in (\mathbb{N} \cup \{0\})^n}$ un jet infinito en E. Existe una función convexa F de clase $C^\infty(\mathbb{R}^n)$ tal que $\partial^\alpha F = f_\alpha$ en E para todo multi-índice α si y solo si el jet $\{f_\alpha\}_{\alpha \in (\mathbb{N} \cup \{0\})^n}$ satisface la condición (CW^∞) en E.*

Los resultados de los Capítulos 5 y 6 están publicados en [10].

Introduction

The main topic of this thesis is the differentiable approximation and extension of convex functions in different Banach spaces.

In Chapter 1 we deal with the problem of approximating convex functions by C^1 convex functions uniformly on Banach Spaces. In $\mathbb{R}^n$, it is well known that convolution with mollifiers provides approximation of convex functions by smooth convex functions uniformly on compact sets. In Banach spaces whose dual space has an equivalent LUR (locally uniformly rotund) norm, it is also well known that infimal convolution techniques provide approximations of convex functions by C^1 convex functions uniformly on bounded subsets. In [1] a new approximation technique was found which allows to approximate (not necessarily uniformly continuous) convex functions on $\mathbb{R}^n$ by smooth convex functions, uniformly on all of $\mathbb{R}^n$. By combining this new technique with the mentioned infimal convolution technique it also follows that convex functions f which are bounded on bounded subsets of a Banach space E whose dual has an equivalent LUR norm can be approximated by C^1 convex functions g, uniformly on all of E. However, there are examples of (smooth or nonsmooth) convex continuous functions which are not bounded on bounded subsets. In this thesis, by means of a refinement of the techniques introduced in [1], we show how to drop the hypothesis that the function f is bounded on bounded subsets, and we prove the following result.

Theorem 1. *Let X be a Banach space whose dual space X^* admits an equivalent LUR norm. Let $f : U \to \mathbb{R}$ be a convex continuous function defined on an open subset U of X. Given $\varepsilon > 0$, there exists a convex function $g : U \to \mathbb{R}$ of class $C^1(U)$ such that $f - \varepsilon \leq g \leq f$ on U.*

Theorem 1 actually follows from a more general result which shows that the problem of approximating convex continuous functions by C^m convex functions defined on open subsets U of X can be reduced to the problem of approximating Lipschitz convex functions.

Also, as a consequence of these results we establish a new characterization of those Banach spaces whose dual is separable, namely, a separable Banach space X has dual X^* separable if and only every continuous convex function defined on an open subset of X can be uniformly approximated by C^1 convex functions.

These results have been published in [9].

In the rest of the thesis, we deal with the problem of finding a version for convex functions of the classical Whitney Extension Theorem [70]. This famous result, for the class C^m, provides necessary and sufficient conditions on a family of real valued functions $\{f_\alpha\}_{|\alpha|\leq m}$ (which we call an m-jet) defined on a closed subset $E \subset \mathbb{R}^n$ for the existence of a function F of class $C^m(\mathbb{R}^n)$ such that $\partial^\alpha F = f_\alpha$ on E for every $|\alpha| \leq m$. The mentioned conditions are relations between the functions f_α and the putative Taylor polynomial P_y of order m centered at $y \in E$ whose coefficients are precisely $(f_\alpha(y))_\alpha$, and the extension F is defined by means of a formula involving a suitable partition of unity subordinated to a carefully chosen family of cubes decomposing the complement of E. A few years later, G. Glaeser [46] established a version of the Whitney Extension Theorem for functions of class $C^{1,\omega}$ on $\mathbb{R}^n$, by means of a construction similar to Whitney's, which also permits to obtain a good control on the modulus of continuity of the derivatives of the extension in terms of the given family $\{f_\alpha\}_{|\alpha|\leq m}$. On the other hand, J. C. Wells [69] provided an analogue of the Whitney Extension Theorem for functions of class $C^{1,1}$ in

Hilbert spaces in which he managed to get an optimal control of the Lipschitz constant of the gradient of the extension, in terms of the given 1-jet. Wells's proof involved an intricate geometrical construction for finite domains and then a limiting process for arbitrary domains. More recently, E. Le Gruyer [53] proved the same $C^{1,1}$ theorem simplifying considerably Wells' proof but making use of Zorn's Lemma. Let us also mention that M. Jiménez and L. Sánchez [50] proved a version of the Whitney Extension Theorem for C^1 in separable Banach spaces which satisfy a certain property related to the approximation of Lipschitz functions by smooth Lipschitz functions, namely, that every 1-Lipschitz function can be uniformly approximated by C^1 Lipschitz functions with Lipschitz constant smaller than an absolute constant which only depends on the space. This class of spaces includes, for instance, the separable Hilbert space. This construction refines an extension technique introduced by D. Azagra, R. Fry and L. Keener [6] in order to solve the same problem when the domain is a closed subspace of a Banach separable space, and ultimately relies on an extension technique inspired by Tietze's extension theorem and the *sup partitions of unity* which were discovered by R. Fry in [42]. Finally, let us also mention that a version of Whitney's Extension Theorem for subdifferentiable functions has been established by J. Ferrera and J. Gómez Gil in [31].

A related issue is the *Whitney Extension Problem for functions* (as opposed to jets): if we are given an arbitrary subset E of $\mathbb{R}^n$, and a function $f : E \to \mathbb{R}$ (but no candidates for derivatives) what conditions on f are necessary and sufficient to guarantee the existence of a C^m or a $C^{m-1,1}$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on E? And what can be said about the norm of the extension F when it exists? These questions are much more difficult to deal with. The $C^{1,1}$ case was solved by Y. Brudnyi and P. Shvartsman in [20], and the problem was solved in full generality by C. L. Fefferman in [36] and [37]. See also the papers by C. L. Fefferman, A. Israel, G. K. Luli and P. Shvartsman listed in the Bibliography for similar results for some Sobolev spaces.

The general problem we will be dealing with is the following.

Problem. Given a positive integer m, an arbitrary subset E of $\mathbb{R}^n$ and a m-jet $\{f_\alpha\}_{|\alpha|\leq m}$ defined on E, what necessary and sufficient conditions on $\{f_\alpha\}_{|\alpha|\leq m}$ would guarantee the existence of a **convex** function F of class $C^m(\mathbb{R}^n)$ or $C^{m,\omega}(\mathbb{R}^n)$ such that $\partial^\alpha F = f_\alpha$ on E for every $|\alpha| \leq m$?

A similar problem was considered by M. Ghomi [44] and M. Yan [72], and a consequence of their results is that, if E is compact and convex and we are given a function $f : E \to \mathbb{R}$ which admits a C^m (not necessarily convex) extension to the whole $\mathbb{R}^n$ whose second derivative is positive definite on ∂E, then there exists a C^m convex function F which extends f from E. Of course, this is only a partial solution to our problem, as strictly positiveness of the Hessian is a very strong assumption, which is far from being necessary. On the other hand, K. Schulz and B. Schwartz [59] provided a characterization of those proper convex functions on $\mathbb{R}^n$ defined on convex domains which admit convex (not necessarily differentiable) extensions to all of $\mathbb{R}^n$. Also, B. Mulansky and M. Neamtu [55] proved that any finite subset of data $\mathbb{R}$ or $\mathbb{R}^2$ which is *strictly convex* in an appropriate sense can be interpolated by a convex polynomial. Finally, let us mention that O. Bucicovschi and J. Lebl [21] studied the problem of extending convex functions to the convex hull of their domain, and that J. M. Borwein, V. Montesinos and J. Vanderwerff [18], and L. Veselý and L. Zajícek [66] showed that there are infinite-dimensional Banach spaces X, closed *subspaces* $E \subset X$ and continuous convex functions $f : E \to \mathbb{R}$ which have no continuous convex extensions to X.

Let us now describe our progress in the solution to the mentioned problem as well as the main consequences and applications of our results.

In Chapter 2, we give a full solution to the above problem for convex functions of class $C^{1,\omega}$, even in the setting of Hilbert spaces, that is, given an arbitrary subset E of a Hilbert space X and two functions $f : E \to \mathbb{R}$ and $G : E \to X$, we give necessary and sufficient conditions on (f, G) for the existence of a convex function F of class $C^{1,\omega}$ so that $F_{|_E} = f$ and $\nabla F_{|_E} = G$. In order to do this, we define a new necessary and sufficient condition $(CW^{1,\omega})$, a simple inequality which only involves the function ω, a constant $M > 0$ and the values of f and G on E. This condition allows us to provide a simple and explicit formula for the extension F and obtain an *almost optimal control* on the modulus of continuity of ∇F in terms of (f, G). In the $C^{1,1}$ case our result provides optimal control of the Lipschitz constant of ∇F. More precisely we have the following.

Theorem 2. *Given $E \subset X$ an arbitrary subset of a Hilbert space X, and two function $f : E \to \mathbb{R}$, $G : E \to X$, such that (f, G) satisfies the inequality $(CW^{1,1})$ with constant $M > 0$ on E, then the formula*

$$F = conv(g), \quad g(x) = \inf_{y \in E} \{ f(y) + \langle G(y), x - y \rangle + \tfrac{M}{2} \|x - y\|^2 \}, \quad x \in X,$$

defines a $C^{1,1}$ convex function with $F_{|E} = f$, $(\nabla F)_{|E} = G$ and $\mathrm{Lip}(\nabla F) \leq M$.

This indicates that if we define M as the smallest positive number for which (f, G) satisfies condition $(CW^{1,1})$ with constant M, the extension F above has the property that

$$\mathrm{Lip}(\nabla F) = \inf\{ \mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}_{\text{conv}}(X), \ H_{|E} = f, \ (\nabla H)_{|E} = G \},$$

and then we one can say that ∇F has the best possible Lipschitz constant. The formula for $C^{1,\omega}$ is similar, and in this case we obtain the same kind of control up to a factor 8.

The main consequence of the above formula for $C^{1,1}_{\text{conv}}$ functions is that it allows us to give simple and explicit solution not only for the $C^{1,1}_{\text{conv}}$ extension problem for jets but also for the general $C^{1,1}$ extension problem for jets in Hilbert spaces, and with the best possible Lipschitz constant of the gradient of the extension. In [46] it is proved that a control of the type

$$\mathrm{Lip}(\nabla F) = k(n) \inf\{ \mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X), \ H_{|E} = f, \ (\nabla H)_{|E} = G \},$$

can be obtained for $C^{1,1}$ extensions on $\mathbb{R}^n$, where $k(n)$ depends on the dimension and tends to ∞ as n grows large. On the other hand, the solutions given in [69] and [53] are optimal in the above sense and are valid for infinite dimensional Hilbert spaces, but the proof in [69] relies on a extremely complicated geometrical construction and the proof in [53] is not constructive as it relies on Zorn's Lemma. With the help of our solution to $C^{1,1}_{\text{conv}}$ extension problem for jets, we can recover the results in [46], [69] and [53] for $C^{1,1}$ functions by means of a simple and explicit formula which provides an extension with an optimal control of the Lipschitz constant of the gradient. In order to do this, we consider a necessary condition $(W^{1,1})$, which is a simple inequality only involving the values of f and G and a constant $M > 0$ and is equivalent to the conditions considered before by [69] and [53]. Our result for the $C^{1,1}$ extension problem for jets reads as follows.

Theorem 3. *Given $E \subset X$ an arbitrary subset of a Hilbert space X, and two functions $f : E \to \mathbb{R}$, $G : E \to X$, such that (f, G) satisfies the inequality $(W^{1,1})$ with constant $M > 0$ on E, then the formula*

$$\begin{aligned} &F = conv(g) - \tfrac{M}{2} \| \cdot \|^2, \\ &g(x) = \inf_{y \in E} \{ f(y) + \langle G(y), x - y \rangle + \tfrac{M}{2} \|x - y\|^2 \} + \tfrac{M}{2} \|x\|^2, \quad x \in X, \end{aligned}$$

defines a $C^{1,1}(X)$ function with $F_{|E} = f$, $(\nabla F)_{|E} = G$, and $\mathrm{Lip}(\nabla F) \leq M$.

Moreover, F can be taken so as to satisfy

$$\mathrm{Lip}(\nabla F) = \inf\{ \mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X), \ H_{|E} = f, \ (\nabla H)_{|E} = G \}.$$

On the other hand, every $F \in C^{1,1}(X)$ satisfies $(W^{1,1})$ with $M = Lip(\nabla F)$ on every subset E of X.

As a consequence of our solution to the $C^{1,1}$ extension problem for general jets (which, in turn, is a consequence of our solution to the $C^{1,1}_{\text{conv}}$ extension problem for jets) we can give a short proof for the Kirszbraun Extension Theorem [52] for Lipschitz mappings between two Hilberts spaces providing, in addition, a constructive and simple formula for the extension; see Corollary 2.28 in Chapter 2.

Finally, we have found a geometrical application of our $C^{1,1}_{\text{conv}}$ extension theorem concerning the following natural question: given an arbitrary subset C of a Hilbert space X and a collection $\mathcal{H}$ of affine hyperplanes of X such that every $H \in \mathcal{H}$ passes through a point $x_H \in C$, what conditions are necessary and sufficient for the existence of a $C^{1,1}$ convex hypersurface S in X such that H is tangent to S at x_H for every $H \in \mathcal{H}$? Equivalently, given a subset C of X and a mapping $N : C \to S_X$, what conditions are necessary and sufficient to ensure the existence of $C^{1,1}$ convex bodies V such that $C \subseteq \partial V$ and the outer unit normal to ∂V coincides with the given mapping N on C? The pertinent condition is a simple inequality $(\mathcal{KW}^{1,1})$ for the mapping N; see Theorem 2.20 in Chapter 2.

In Chapter 3, we consider the same problem as in Chapter 2 for the class of $C^{1,\alpha}$ convex functions in superreflexive spaces, for suitable $\alpha \in (0,1)$. By Pisier's renorming Theorem [56], every superreflexive space has an equivalent norm which is uniformly smooth with modulus of smoothness of power type $p = 1+\alpha$ for some $\alpha \in (0,1)$. It is proved that condition $(CW^{1,\alpha})$ is necessary and sufficient on a pair of functions $(f,G) : E \to \mathbb{R} \times X$ for the existence of a convex function F of class $C^{1,\alpha}$ with $(F, \nabla F) = (f,G)$ on E. The formula defining F is similar to the ones we obtained in Hilbert spaces and, again, we can arrange an *almost optimal control* on the α-Hölder modulus of continuity of ∇F in terms of (f,G) and of an absolute constant depending only on the space.

The results of Chapters 2 and 3 are contained in the papers [8], [11], and [12].

In Chapter 4 we solve the problem of extending two functions $f : E \to \mathbb{R}$ and $G : E \to X$, defined on an arbitrary subset of E of $\mathbb{R}^n$ to a C^1 convex function F such that $F_{|_E} = f$ and $\nabla F_{|_E} = G$, that is, we establish a C^1_{conv} Whitney Extension Theorem for jets. We first solve this problem when E is a compact subset by introducting two new conditions (C) and (CW^1). Condition (C) tells us that f must lie above the *putative tangents* $f(y) + \langle G(y), \cdot - y\rangle$ and condition (CW^1) tells us that if two points of the graph of f lie on a line segment contained in a hyperplane which we want to be tangent to the graph of an extension at one of the points, then our putative tangent hyperplanes at both points must be the same. Our main result for compact domains reads as follows.

Theorem 4. *Let E be a compact subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be two mappings such that G is continuous. There exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F_{|_E} = f$ and $\nabla F_{|_E} = G$ if and only if (f,G) satisfies the conditions (C) and (CW^1) on E. In addition, we can arrange that* $\text{Lip}(F) \leq \kappa \sup_E |G|$, *where κ is an absolute constant.*

It is worth mention that this kind of control of the Lipschitz constant of the extension F solely in terms of G cannot be obtained for general (not necessarily convex) C^1 functions, in which the best possible estimation of $\text{Lip}(F)$ is in terms of $\text{Lip}(f)$ and $\sup_E |G|$.

As in the $C^{1,1}$ case, we have found a geometrical application of our solution to the C^1_{conv} extension problem concerning characterizations of those compact subsets C of $\mathbb{R}^n$ equipped with a mapping $N : C \to \mathbb{S}^{n-1}$ which can be interpolated by boundaries of C^1 compact convex bodies V which contains the origin as an interior point and such that the outer unit normal to ∂V coincides with the given mapping N on C. The pertinents conditions are the natural translation of conditions (C) and (CW^1) to the setting of C^1 convex bodies via the Implicit Function Theorem. This result may be compared to [43], where M. Ghomi showed how to construct C^m smooth strongly convex bodies with prescribed strongly convex submanifolds and tangent planes. Our characterization allows us to deal with arbitrary compacta instead of manifolds, and to drop the strong convexity assumption.

Let us also mention that, very recently, E. Durand-Cartagena and A. Lemenant [29] have used the Theorem 4 in order to prove that strongly self contracted curves of class $C^{1,\alpha}$ can be characterized as being solutions to gradients flow equations of C^1 convex functions.

Let us now consider the case when our domain $E \subset \mathbb{R}^n$ is arbitrary, and in particular not necessarily bounded. Unlike the compact case, if we are given two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ with G continuous, the assumption that (f,G) satisfies conditions (C) and (CW^1) does not ensure the existence of convex (not necessarily differentiable) extensions to all of $\mathbb{R}^n$, as we will see in examples. It is not difficult to deal with this inconvenient and it is enough to assume an extra necessary condition on (f, G) which we denote momentarily by (EX). However we still need to deal with a much harder problem which is related to the global behaviour of differentiable convex functions on $\mathbb{R}^n$: the possible presence of *corners at infinity*. We will say that the pair (f, G) has a corner at infinity if the natural generalization of condition (CW^1) (replacing points with unbounded sequences) fails to be satisfied for (f, G). For instance, in $\mathbb{R}^2$, the C^1 convex functions defined everywhere which have corners at infinity can be seen as differentiable convex functions whose graphs are *tangent at infinity* to graphs of convex functions which are not differentiable along some lines in $\mathbb{R}^2$. More pathological examples of functions which have *corners at infinity in directions of subspaces* of dimension k for every $k \leq n$ can be given in higher dimensions. On the other hand, in [1] it was proved that convex functions on $\mathbb{R}^n$ have uniquely determined subspaces that characterize their global behaviour; this means that one can associate to each convex function g a unique subspace X_g of $\mathbb{R}^n$ in such a way that g can be written, up to substracting a linear function, as the composition of a coercive convex function on X_g with the orthogonal projection onto X_g. The presence of corners at infinity for a differentiable convex function g forces the essential coercivity of g in the directions of the corner, that is, the subspace X_g contains those directions. For these reasons we formulate our extension theorem not only for single 1-jets (f, G) but also for a given subspace X of $\mathbb{R}^n$ which we want to represent the global behaviour of our C^1 convex extension F of (f, G). We will need to define a variant of the condition (CW^1), namely condition (CW^1) for (f, G) with subspace Z, in which we consider (possibly unbounded) sequences with bounded projection onto Z. We will need to assume that the subspace $Y = \text{span}\{G(E) - G(E)\}$ of the differences of the putative derivates on E is contained in the given X. The most technical condition that we will need to assume could be informally summed up by saying that whenever the jet (f, G) does not provide us with enough differential data so that condition (CW^1) is satisfied for (f, G) with subspace Y, then there has to be enough room on $\mathbb{R}^n \setminus \overline{E}$ so we can define new jets $(\beta_1, w_1), \ldots (\beta_d, w_d)$ associated to points $p_1, \ldots, p_d \in \mathbb{R}^n \setminus \overline{E}$ in such a way that this new jet is compatible with the convex extension problem (that is, satisfies condition (C)) and does satisfy condition (CW^1) with subspace X. This condition of compatibility of our given jet (f, G) in terms of the subspaces X and Y will be denoted here by (CO_X^Y). All these conditions happen to be necessary for the existence of C^1 convex functions F which extend the jet (f, G) from E and have global behaviour determined by X. In conclusion, the solution to the C^1 convex extension problem for jets defined on arbitrary subsets reads as follows (see Chapter 4 for the precise definitions of these necessary and sufficient conditions).

Theorem 5. *Let E be an arbitrary subset of $\mathbb{R}^n$, $X \subset \mathbb{R}^n$ a linear subspace and $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ two functions such that G is continuous. Let us denote $Y = span\{G(E) - G(E)\}$. There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if (f, G) satisfies the conditions (C), (EX), $Y \subseteq X$, (CO_X^Y) and (CW^1) with subspace X.*

The above theorem establishes what one could call a C^1 *Whitney Extension Theorem for convex functions with prescribed global behaviour.* This means that we can characterize the global behaviour of the C^1 convex extension F in terms of the subspace generated by the putative derivatives G on E and of the *differential behaviour* of the pair (f, G) on E.

We will also study the particular case when the function G is bounded on E. In this case there always exist convex (not necessarily differentiable) extensions and we do not need to assume the mentioned condition (EX). On the other hand, the condition of compatibily (CO_X^Y) can be reformulated in such a way that we only need to check that the closure $\overline{E}$ of E does not intersects a union of a certain finite family of cones. In addition, we can provide an *almost optimal control* of the Lipschitz constant of the extension in terms of $\sup_E |G|$.

Theorem 6. *Let E be an arbitrary subset of $\mathbb{R}^n$, $X \subset \mathbb{R}^n$ a linear subspace and $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ two functions such that G is continuous and bounded. Let us denote $Y = span\{G(E) - G(E)\}$. There exists a Lipschitz convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if (f, G) satisfies conditions $(C), Y \subseteq X, (CW^1)$ with subspace X and, if*

$Y \neq X$ and we denote $k = \dim Y$ and $d = \dim X$, there exist points $p_1, \dots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, a number $\varepsilon \in (0,1)$, and linearly independent normalized vectors $w_1, \dots, w_{d-k} \in X \cap Y^{\perp}$ such that

$$\overline{E} \cap \left(\bigcup_{j=1}^{d-k} V_j \right) = \emptyset,$$

where, for every $j = 1, \dots, d-k$, we denote $V_j := \{ x \in \mathbb{R}^n \ : \ \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x-p_j)| \}$, where $P_Y : \mathbb{R}^n \to Y$ is the orthogonal projection onto Y.

Moreover, there exists an absolute constant $\kappa > 0$ such that, whenever these conditions are satisfied, the extension F can be taken so that

$$\mathrm{Lip}(F) \leq \kappa \sup_{y \in E} |G(y)|.$$

As in the compact case, we can use the above result to answer the following question: given an arbitrary subset E of $\mathbb{R}^n$ and a collection $\mathcal{H}$ of affine hyperplanes of $\mathbb{R}^n$ such that every $H \in \mathcal{H}$ passes through a point $x_H \in E$, what conditions are necessary and sufficient for the existence of a C^1 convex hypersurface S in $\mathbb{R}^n$ such that H is tangent to S at x_H for every $H \in \mathcal{H}$? Equivalently, given a subset E of $\mathbb{R}^n$ equipped with a mapping $N : E \to \mathbb{S}^{n-1}$, what conditions on E and N are necessary and sufficient for the existence of a C^1 convex body V (not necessarily bounded) such that $E \subseteq \partial V$ and the outer unit normal to ∂V coincides with the given N on E? As a matter of fact we will answer to these questions by providing a characterization which also takes into account the directions that we want the outer unit normal of our convex body to have. That is, given $N : E \to \mathbb{S}^{n-1}$ and a subspace X we find necessary and sufficient conditions, in terms of $N, \mathrm{span}(N(E))$ and X, for the existence of a C^1 convex body V with $0 \in \mathrm{int}(V)$ and $E \subseteq \partial V$, with outer unit normal n_V to ∂V equal to N on E and $X = \mathrm{span}(n_V(\partial V))$; see Theorem 4.69 in Chapter 4.

Finally, let us mention that an application of our solution to the C^1 convex extension problem for jets defined on arbitrary domains to the question of characterizing the class of convex function which have the Lusin property of class C^1_{conv} has been found by D. Azagra and P. Hajłasz [7]. They have proved that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ which is not of class C^1 has a Lusin property of type C^1_{conv} (meaning that for every $\varepsilon > 0$ there exists a convex function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $\mathcal{L}^n \left(\{ x \in \mathbb{R}^n : f(x) \neq g(x) \} \right) \leq \varepsilon$) if and only if f is essentially coercive, i.e., coercive up to substracting a linear function.

The results of Chapter 4 are contained in the papers [12] and [13].

In Chapter 5 we consider the C^m convex extension problem for jets on $\mathbb{R}^n$, when $m \geq 2$. We will see, by means of easy examples, that if the domain is not assumed to be convex, the situation gets much more complicated than in the C^1 or $C^{1,\omega}$ case. For this reason we will restrict our attention to the case when the domain E of definition of our jet is convex and compact. As we have said before, there always exist convex extensions of class C^m if we assume that our jet has a putative second derivative which is strictly positive on ∂E, thanks to the works in [44] and [72]. This condition is far from begin necessary and some other assumptions must be made in its place. We define new conditions (CW^m) for m-jets defined on compact convex subsets, which essentially tell us that the expression of the putative Hessian at every direction $v \in \mathbb{S}^{n-1}$ in terms of the putative Taylor polynomial centered at points $y \in \partial E$ has $\liminf$ greater than or equal 0, uniformly on $y \in \partial E$ and $v \in \mathbb{S}^{n-1}$. Although this condition is necessary for the existence of a C^m convex extension of the m-jet, we show, by means of examples, that this condition is no longer sufficient, at least when the set E has empty interior. In fact, we will see examples of functions of class C^∞ which satisfy the condition (CW^3) on E and yet has no convex extension of class C^2. Nevertheless, we show that if our m-jet satisfies (CW^m) then it has a convex extension of class C^{m-n-1} to $\mathbb{R}^n$. Moreover, if we make further geometrical assumptions on the set E (namely, that E is the intersection of a finite number of ovaloids of class C^m), then condition (CW^m) guarantees the existence of a convex extension of class $C^{m-1}(\mathbb{R}^n)$.

In Chapter 6, we solve the C^∞ convex extension problem for *infinite jets* on compact convex subsets of $\mathbb{R}^n$. Here, by a *infinite jet* on E we merely understand an infinite family of real valued functions $\{f_\alpha\}_{\alpha\in(\mathbb{N}\cup\{0\})^n}$ defined on E. We introduce a new condition (CW^∞) on the infinite jet $\{f_\alpha\}_{\alpha\in(\mathbb{N}\cup\{0\})^n}$, which says that, for every integer $m\geq 2$, the finite family $\{f_\alpha\}_{|\alpha|\leq m}$ satisfies the condition (CW^m) for the C^m problem mentioned above. Unlike the C^m case, for m finite, this condition (CW^∞) is necessary and sufficient for the existence of C^∞ convex extensions, and our main result for the C^∞ class reads us follows.

Theorem 7. *Let $E\subset\mathbb{R}^n$ be a convex compact subset and $\{f_\alpha\}_{\alpha\in(\mathbb{N}\cup\{0\})^n}$ be a infinite jet on E. There exists a convex function F of class $C^\infty(\mathbb{R}^n)$ with $\partial^\alpha F=f_\alpha$ on E for every multi-index α if and only if $\{f_\alpha\}_{\alpha\in(\mathbb{N}\cup\{0\})^n}$ satisfies condition (CW^∞) on E.*

The results of Chapters 5 and 6 are contained in the paper [10].

Chapter 1

C^1 approximation of convex functions on Banach spaces

1.1 Introduction and main results

It is no doubt useful to be able to approximate convex functions by smooth convex functions. In $\mathbb{R}^n$, standard techniques (integral convolutions with mollifiers) enable us to approximate a given convex function by C^∞ convex functions, uniformly on compact sets. If a given convex function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is not Lipschitz and one desires to approximate f by smooth convex functions uniformly on the domain U of f, then one has to work harder as, in absence of *strong convexity* of f, partitions of unity cannot be used to path local approximations into a global one without destroying convexity. In a recent paper [1] D. Azagra devised a gluing procedure that permits to show that global approximation of (not necessarily Lipschitz or strongly) convex functions by smooth (or even real analytic) convex functions is indeed feasible. The main result in this direction is the following.

Theorem 1.1 (D. Azagra, [1])**.** *Let U be an open convex subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be a convex function and let $\varepsilon > 0$. There exists a real analytic convex function $g : U \to \mathbb{R}$ such that $f - \varepsilon \leq g \leq f$ on U.*

We also refer to [47] and [62] for information about this problem in the setting of finite-dimensional Riemannian manifolds, and to [3, 5] for the case of infinite-dimensional Riemannian manifolds.

In this chapter we will consider the question whether or not global approximation of continuous convex functions can be performed in Banach spaces. Let us briefly review the main techniques available in this setting for approximating convex functions by smooth convex functions. On the one hand, there are very fine results of Deville, Fonf, Hájek and Talponen [27, 28, 32] showing that if X is the Hilbert space (or more generally a separable Banach space with a C^m equivalent norm) then every bounded closed convex body in X can be approximated by real-analytic (resp. C^m smooth) convex bodies. Via the implicit function theorem this yields that for every convex function $f : X \to \mathbb{R}$ which is bounded on bounded sets, for every $\varepsilon > 0$, and for every bounded set $B \subset X$, there exists a C^m smooth convex function $g : B \to \mathbb{R}$ such that $|f - g| \leq \varepsilon$ on B. Unfortunately, these approximations g are only defined on a bounded subset of X, so they cannot be used along with the techniques of [1] to solve the global approximation problem we are concerned with.

On the other hand, if $f : X \to \mathbb{R}$ is convex and Lipschitz and the dual space X^* is LUR (we refer the reader to [26, 33] for any unexplained terms in Banach space theory), then it is well known that the Moreau-Yosida regularizations of f (also called the Moreau envelope of f, see the book by Rockafellar and Wets [58]), defined by $f_\lambda(x) = \inf_{y \in X}\{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}$ for $x \in X$, $\lambda > 0$, are C^1 smooth and convex, and approximate f uniformly on X as $\lambda \to 0^+$ (see the proof of Theorem 1.4 below). If f is not Lipschitz but it is bounded on bounded subsets of X, then the f_λ approximate f uniformly on bounded subsets of X. And, if f is only continuous, then the convergence of the f_λ to f is uniform only on compact subsets of X. By combining the gluing technique of [1, Theorem 1.2] with these results, one can deduce the following.

Theorem 1.2. [1, Corollary 1.5] *Let U be an open convex subset of a Banach space X such that X^* has an equivalent LUR norm. Then, for every real number $\varepsilon > 0$ and every convex function $f : U \to \mathbb{R}$ which is bounded on bounded subsets B such that* $\operatorname{dist}(B, \partial U) > 0$; *there exists a convex function $g : U \to \mathbb{R}$ of class $C^1(U)$ such that $f - \varepsilon \leq g \leq f$ on U.*

However, as shown in [17, Theorem 2.2] or [19, Theorem 8.2.2], for every infinite-dimensional Banach space X there exist continuous convex functions defined on all of X which are not bounded on bounded sets of X. There are plenty of such examples, and they can be taken to be either smooth or nonsmooth.

Example 1.3. Consider $X = \ell_2$.

(1) The function $f(x) = \sum_{n=1}^{\infty} |x_n|^{2n}$ is real-analytic on X, but is not bounded on the ball $B(0, 2)$ of center 0 and radius 2 in X.

(2) If $\varphi : [0, \infty) \to [0, \infty)$ is a convex function such that $t \leq \varphi(t) \leq 2t$ and φ is not differentiable at any rational number, then it is not difficult to see that the function $g(x) = \sum_{n=1}^{\infty} \varphi(|x_n|)^{2n}$ is continuous and convex on X, is not bounded on $B(0, 2)$, and the set $\{x \in X \,:\, g \text{ is not differentiable at } x\}$ is dense.

In view of these remarks, even in the case when X is the separable Hilbert space, the following result is new.

Theorem 1.4. *Let U be an open convex subset of a Banach space X such that X^* has an equivalent LUR norm. Then, for every real number $\varepsilon > 0$ and every continuous and convex function $f : U \to \mathbb{R}$, there exists a convex function $g : U \to \mathbb{R}$ of class $C^1(U)$ such that $f - \varepsilon \leq g \leq f$ on U.*

This will be proved by combining the above mentioned result on the Moreau-Yosida regularization of a convex function with the following refinement of [1, Theorem 1.2] which tells us that, in general, the problem of global approximation of *continuous* convex functions by C^m smooth convex functions can be reduced to the problem of global approximation of *Lipschitz* convex functions by C^m smooth convex functions.

Theorem 1.5. *Let X be a Banach space with the following property: every* Lipschitz *convex function on X can be approximated by convex functions of class C^m, uniformly on X. Then, for every $U \subseteq X$ open and convex, every continuous convex function on U can be approximated by C^m convex functions, uniformly on U.*

In order to know whether or not similar results are true for higher order of smoothness classes, and in view of Theorem 1.5 above, one would only need to solve the following problem.

Open Problem 1.6. *Let X be a Hilbert space (or in general a Banach space possessing an equivalent norm of class C^m), $f : X \to \mathbb{R}$ a Lipschitz and convex function, and $\varepsilon > 0$. Does there exist $\varphi : X \to \mathbb{R}$ of class C^∞ (resp. C^m) and convex such that $|f - \varphi| \leq \varepsilon$ on X?*

As a matter of fact, by combining Theorem 1.5 and the proof of [1, Theorem 1.2], it would also be enough to solve the following.

Open Problem 1.7. *Let X be a Hilbert space (or in general a Banach space possessing an equivalent norm of class C^m), $f : X \to \mathbb{R}$ a Lipschitz and convex function, B a bounded convex subset of X, and $\varepsilon > 0$. Does there exist $g : X \to \mathbb{R}$ of class C^∞ (resp. C^m) and convex such that $g \leq f$ on X, and $f - \varepsilon \leq g$ on B?*

1.2 Proof of the C^1 approximation theorem

In this section we present the proofs of Theorems 1.5 and 1.4. Let us first recall a couple of tools from [1].

Lemma 1.8 (Smooth maxima)**.** *For every $\varepsilon > 0$ there exists a C^∞ function $M_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ with the following properties:*

1. *M_ε is convex;*
2. $\max\{x, y\} \le M_\varepsilon(x, y) \le \max\{x, y\} + \frac{\varepsilon}{2}$ *for all* $(x, y) \in \mathbb{R}^2$.
3. *$M_\varepsilon(x, y) = \max\{x, y\}$ whenever $|x - y| \ge \varepsilon$.*
4. *$M_\varepsilon(x, y) = M_\varepsilon(y, x)$.*
5. *Lip$(M_\varepsilon) = 1$ with respect to the norm $\|\cdot\|_\infty$ in $\mathbb{R}^2$.*
6. *$y - \varepsilon \le x < x' \implies M_\varepsilon(x, y) < M_\varepsilon(x', y)$.*
7. *$x - \varepsilon \le y < y' \implies M_\varepsilon(x, y) < M_\varepsilon(x, y')$.*
8. *$x \le x', y \le y' \implies M_\varepsilon(x, y) \le M_\varepsilon(x', y')$, with a strict inequality in the case when both $x < x'$ and $y < y'$.*

We call M_ε a smooth maximum. In order to prove this lemma, one first constructs a C^∞ function $\theta : \mathbb{R} \to (0, \infty)$ such that:

1. $\theta(t) = |t|$ if and only if $|t| \ge \varepsilon$;
2. θ is convex and symmetric;
3. $\mathrm{Lip}(\theta) = 1$,

and then one puts

$$M_\varepsilon(x, y) = \frac{x + y + \theta(x - y)}{2}.$$

See [1, Lemma 2.1] for details. Let us also restate Proposition 2.2 from [1].

Proposition 1.9. *Let U be an open convex subset of X, M_ε as in the preceding Lemma, and let $f, g : U \to \mathbb{R}$ be convex functions. For every $\varepsilon > 0$, the function $M_\varepsilon(f, g) : U \to \mathbb{R}$ has the following properties:*

1. *$M_\varepsilon(f, g)$ is convex.*
2. *If f is C^m on $\{x : f(x) \ge g(x) - \varepsilon\}$ and g is C^m on $\{x : g(x) \ge f(x) - \varepsilon\}$ then $M_\varepsilon(f, g)$ is C^m on U. In particular, if f, g are C^m, then so is $M_\varepsilon(f, g)$.*
3. *$M_\varepsilon(f, g) = f$ if $f \ge g + \varepsilon$.*
4. *$M_\varepsilon(f, g) = g$ if $g \ge f + \varepsilon$.*
5. $\max\{f, g\} \le M_\varepsilon(f, g) \le \max\{f, g\} + \varepsilon/2$.
6. *$M_\varepsilon(f, g) = M_\varepsilon(g, f)$.*
7. *Lip$(M_\varepsilon(f, g)_{|B}) \le \max\{Lip(f_{|B})$, Lip$(g_{|B})\}$ for every ball $B \subset U$ (in particular $M_\varepsilon(f, g)$ preserves common local Lipschitz constants of f and g).*
8. *If f, g are strictly convex on a set $B \subseteq U$, then so is $M_\varepsilon(f, g)$.*

9. *If $f_1 \le f_2$ and $g_1 \le g_2$ then $M_\varepsilon(f_1, g_1) \le M_\varepsilon(f_2, g_2)$.*

We are now ready to prove our results. The norm on X will be denoted by $\|\cdot\|$.

Proof of Theorem 1.5. Given a continuous convex function $f : U \to \mathbb{R}$ and $\varepsilon > 0$, we define, for each $n \in \mathbb{N}$,

$$E_n = \{x \in U \mid f \text{ is } n\text{-Lipschitz on an open neighbourhood of } x\}.$$

It is obvious that E_n is an open subset of U and $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$. Since f is continuous and convex, f is locally Lipschitz and then, for every point $x \in U$, there is an open set $x \in U_x \subset U$ and a positive integer n for which f is n-Lipschitz on U_x. This proves that $U = \bigcup_{n=1}^{\infty} E_n$. Now we set

$$\begin{array}{rcc} f_n : X & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \inf_{y\in U}\{f(y) + n\|x-y\|\} \end{array}, \qquad n \in \mathbb{N}.$$

Claim 1.10. *For every $n \in \mathbb{N}$, the function f_n has the following properties:*

(i) $f_n \le f$ *on* U.

(ii) f_n *is* n*-Lipschitz on* X.

(iii) f_n *is convex on* X.

(iv) $f = f_n$ *on* E_n*. In particular,* $f_n = f_{n+1}$ *on* E_n.

Proof of Claim 1.10. Although the first three statements are well-known facts about infimal convolution on Banach spaces (see [64] for a survey paper on these topics) we expose their proofs for the reader's convenience. The statement (i) is obvious by the definition of f_n.

(ii) Given $x, z \in X$ and $\varepsilon > 0$ we can find $y \in U$ such that $f_n(z) \ge f(y) + n\|z-y\| - \varepsilon$. This yields

$$f_n(x) - f_n(z) \le f(y) + n\|x-y\| - f_n(z) \le n\|x-y\| - n\|z-y\| + \varepsilon \le n\|x-z\| + \varepsilon.$$

This shows that $f_n(x) - f_n(z) \le n\|x-z\|$ and reversing x and z gives $|f_n(x) - f_n(z)| \le n\|x-z\|$.

(iii) Given $x, z \in X$ and $\varepsilon > 0$ we can find points $y_x, y_z \in U$ such that

$$f_n(x) \ge f(y_x) + n\|x - y_x\| - \varepsilon \quad \text{and} \quad f_n(z) \ge f(y_z) + n\|z-y_z\| - \varepsilon.$$

Fix $\lambda \in [0,1]$. Since U is convex we have $\lambda y_x + (1-\lambda)y_z \in U$. Using the last inequalities and the convexity of f, we can write

$$\begin{aligned} f_n(\lambda x + (1-\lambda)z) &\le f(\lambda y_x + (1-\lambda)y_z) + n\|\lambda(x-y_x) + (1-\lambda)(z-y_z)\| \\ &\le \lambda f(y_x) + (1-\lambda)f(y_z) + \lambda n\|x-y_x\| + (1-\lambda)n\|z-y_z\| \\ &\le \lambda f_n(x) + (1-\lambda)f_n(z) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the above shows that f_n is convex on X.

(iv) We only need to check that $f \le f_n$ on E_n. Let x be a point of E_n and let U_x be an open subset of U containing x for which f is n-Lipschitz on U_x. Then the function $h_x : U \to \mathbb{R}$ given by

$$h_x(y) = f(y) + n||x-y|| - f(x), \quad \text{for all} \quad y \in U,$$

has a local minimum at the point x, where $h_x(x) = 0$. Because f is convex, h_x is convex as well and then this local minimum is in fact a global one, and therefore $f(x) \le f(y) + n\|x-y\|$ for all $y \in U$. □

Since f_n is Lipschitz, by assumption, for each $n \in \mathbb{N}$ we can find a function h_n of class $C^m(X)$ and convex such that

$$f_n - \sum_{j=0}^{n-1} \frac{\varepsilon}{2^j} \le h_n \le f_n - \sum_{j=0}^{n-2} \frac{\varepsilon}{2^j} - \frac{\varepsilon}{2^n} \quad \text{on} \quad X. \tag{1.2.1}$$

Note that Claim 1.10 together with the inequalities (1.2.1) yield

$$f - \sum_{j=0}^{n-1} \frac{\varepsilon}{2^j} \le h_n \quad \text{on} \quad E_n \quad \text{and} \quad h_n \le f - \sum_{j=0}^{n-2} \frac{\varepsilon}{2^j} - \frac{\varepsilon}{2^n} \quad \text{on} \quad U. \tag{1.2.2}$$

Now, using the smooth maxima of Lemma 1.8, we define a sequence $\{g_n\}_{n\ge 1}$ of functions inductively by setting $g_1 = h_1$ and $g_n = M_{\varepsilon/10^n}(g_{n-1}, h_n)$, for all $n \ge 2$. According to Proposition 1.9, we have that each g_n is convex and of class C^m on X. We also know that

$$\max\{g_{n-1}, h_n\} \le g_n \le \max\{g_{n-1}, h_n\} + \frac{\varepsilon}{10^n} \quad \text{on} \quad X \tag{1.2.3}$$

and

$$g_n(x) = \max\{g_{n-1}(x), h_n(x)\} \quad \text{whenever} \quad |g_{n-1}(x) - h_n(x)| \ge \frac{\varepsilon}{10^n}. \tag{1.2.4}$$

In addition, the sequence $\{g_n\}_{n\ge 1}$ satisfies the following properties.

Claim 1.11. *For every $n \ge 2$, we have*

(i) $f - \varepsilon - \frac{\varepsilon}{2} - \cdots - \frac{\varepsilon}{2^{n-1}} \le g_n$ *on* E_n.

(ii) $g_n = g_{n-1}$ *on* E_{n-1}.

(iii) $g_n \le f - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2} + \cdots + \frac{\varepsilon}{10^n}$ *on* U.

Proof of Claim 1.11. Property (i) is an obvious consequence of inequalities (1.2.2) and (1.2.3). The statement (ii) can be proved as follows. Given $x \in E_{n-1}$, we have that $f_n(x) = f_{n-1}(x)$ by Claim 1.10. It is clear from (1.2.3) that $g_{n-1}(x) \ge h_{n-1}(x)$ and then, using (1.2.1), we obtain

$$g_{n-1}(x) \ge h_{n-1}(x) \ge f_{n-1}(x) - \sum_{j=0}^{n-2} \frac{\varepsilon}{2^j} = f_n(x) - \sum_{j=0}^{n-2} \frac{\varepsilon}{2^j} \ge h_n(x) + \frac{\varepsilon}{2^n} \ge h_n(x) + \frac{\varepsilon}{10^n}.$$

This implies that $g_n(x) = g_{n-1}(x)$ by virtue of (1.2.4). We next show (iii) by induction. In the case $n = 2$, the functions f, h_1, h_2 and g_1 satisfy

$$g_1 = h_1 \le f - \frac{\varepsilon}{2} \quad \text{and} \quad h_2 \le f - \varepsilon - \frac{\varepsilon}{4} \quad \text{on} \quad U$$

thanks to (1.2.2). Since $g_2 = M_{\varepsilon/10^2}(g_1, h_2)$ these inequalities lead us to

$$g_2(x) \le \max\{h_2(x), g_1(x)\} + \frac{\varepsilon}{10^2} \le \max\left\{f(x) - \varepsilon - \frac{\varepsilon}{4}, f(x) - \frac{\varepsilon}{2}\right\} + \frac{\varepsilon}{10^2} = f(x) - \frac{\varepsilon}{2} + \frac{\varepsilon}{10^2}.$$

for every $x \in U$. This proves the statement for $n = 2$. Now we assume that for an integer $n \ge 2$ we have (iii), and we check that the same holds for $n+1$. Let us fix $x \in U$. From (1.2.3) we have

$$g_{n+1}(x) \le \max\{h_{n+1}(x), g_n(x)\} + \frac{\varepsilon}{10^{n+1}}.$$

The induction hypothesis and (1.2.2) with $n+1$ in place of n gives the following two inequalities

$$g_n(x) \le f(x) - \frac{\varepsilon}{2} + \sum_{j=2}^{n} \frac{\varepsilon}{10^j},$$

$$h_{n+1}(x) \leq f(x) - \sum_{j=0}^{n-1} \frac{\varepsilon}{2^j} - \frac{\varepsilon}{2^{n+1}}.$$

Combining the three above inequalities it follows

$$g_{n+1}(x) \leq \max \left\{ f(x) - \sum_{j=0}^{n-1} \frac{\varepsilon}{2^j} - \frac{\varepsilon}{2^{n+1}}, f(x) - \frac{\varepsilon}{2} + \sum_{j=2}^{n} \frac{\varepsilon}{10^j} \right\} + \frac{\varepsilon}{10^{n+1}} = f(x) - \frac{\varepsilon}{2} + \sum_{j=2}^{n+1} \frac{\varepsilon}{10^j}.$$

and this proves (iii) with $n+1$. □

Our approximating function is defined by

$$g(x) = \lim_{n\to\infty} g_n(x) \quad \text{for all} \quad x \in U.$$

Claim 1.11 tells us that $g_{n+k} = g_n$ on each E_n for all $k \geq 1$. It is then clear that g is well defined and $g = g_n$ on E_n for all n. Thus g coincides on E_n with a function of class C^m, where each E_n is an open subset of U and $U = \bigcup_{n=1}^{\infty} E_n$. This shows that g is of class C^m on U. Moreover the function g, being a limit of convex functions, is convex as well. To complete the proof of Theorem 1.5 let us see that $f - 2\varepsilon \leq g \leq f$ on U. Indeed, let $x \in U$ and take an integer $n \geq 2$ for which $x \in E_n$. Using Claim 1.11 and the fact that $g_n(x) = g(x)$ we obtain

$$f(x) - 2\varepsilon \leq f(x) - \sum_{j=0}^{n-1} \frac{\varepsilon}{2^j} \leq g(x) \leq f(x) - \frac{\varepsilon}{2} + \sum_{j=2}^{n} \frac{\varepsilon}{10^j} \leq f(x).$$

Therefore $f - 2\varepsilon \leq g \leq f$ on U. □

Proof of Theorem 1.4. Theorem 1.4 actually is a corollary of Theorem 1.5, because a Banach space X whose dual X^* is LUR has the approximation property mentioned in the hypotheses of Theorem 1.5 for the class C^1. This can be shown by using the infimal convolutions

$$f_\lambda(x) = \inf_{y\in X} \left\{ f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\},$$

where $\|\cdot\|$ is an equivalent norm in X whose dual norm is LUR. It is well known that if f is convex and Lipschitz then f_λ is C^1 smooth and convex, and converges to f uniformly on X, as $\lambda \to 0^+$. For the smoothness part of this assertion, see [34, Proposition 2.3]. On the other hand, we next offer a proof of the fact that if f is Lipschitz then f_λ converges to f uniformly on X as $\lambda \to 0^+$. Observe first that in this case the infimum defining $f_\lambda(x)$ can be restricted to the ball $B\left(x, 2\lambda \mathrm{Lip}(f)\right)$; indeed, if $d(x,y) > 2\lambda\mathrm{Lip}(f)$ then we have

$$f(y) + \frac{1}{2\lambda} d(x,y)^2 \geq f(x) - \mathrm{Lip}(f) d(x,y) + \frac{1}{2\lambda} d(x,y)^2 \geq f(x) \geq f_\lambda(x).$$

Now, one has

$$\begin{aligned}
0 \leq f(x) - f_\lambda(x) &= f(x) - \inf_{y\in B(x, 2\lambda\mathrm{Lip}(f))} \left\{ f(y) + \frac{1}{2\lambda} d(x,y)^2 \right\} \\
&\leq \sup_{y\in B(x, 2\lambda\mathrm{Lip}(f))} \left\{ |f(x) - f(y)| + \frac{1}{2\lambda} d(x,y)^2 \right\} \\
&\leq \mathrm{Lip}(f)\left(2\lambda\mathrm{Lip}(f)\right) + \frac{\left(2\lambda\mathrm{Lip}(f)\right)^2}{2\lambda},
\end{aligned}$$

and the last term converges to 0 as $\lambda \to 0^+$, so the assertion is proved. □

1.3 A characterization of Banach spaces with separable duals

From Theorem 1.4 we will also deduce the following characterization of the class of separable Banach spaces for which the problem of global approximation of continuous convex functions by C^1 convex functions has a positive solution.

Corollary 1.12. *For a separable Banach space X, the following statements are equivalent.*

(i) *X^* is separable.*

(ii) *For every $U \subseteq X$ open and convex, every continuous convex function $f : U \to \mathbb{R}$ and every real number $\varepsilon > 0$, there exists $g : X \to \mathbb{R}$ of class $C^1(U)$ and convex such that $f - \varepsilon \leq g \leq f$ on U.*

Proof. $(i) \implies (ii)$: If X^* is separable, it is well known (see [33, Theorem 8.6] for instance) that there is an equivalent norm in X whose dual norm is LUR on X^*, and therefore by using Theorem 1.4 we obtain (ii).
$(ii) \implies (i)$: Take a convex function $\varphi \in C^1(X)$ such that

$$\|x\| - \frac{1}{4} \leq \varphi(x) \leq \|x\| \quad \text{for all} \quad x \in X.$$

It is easy to construct a function $h \in C^1(\mathbb{R})$ such that $h(x) = 1$ for all $x \leq 0$ and $h(x) = 0$ for all $x \geq 3/4$. Now, if we define the function $\psi := h \circ \varphi$ it is obvious that ψ is of class $C^1(X)$ with $\psi(0) = 1$. We also note that if $\|x\| \geq 1$, then $\varphi(x) \geq 3/4$ and this implies that $\psi(x) = 0$. This shows that ψ is a bump function of class $C^1(X)$. Because X is separable, and according to [33, Theorem 8.6], the dual space X^* is separable too. $\square$

Chapter 2

$C^{1,\omega}$ extensions of convex functions in Hilbert Spaces

Throughout this chapter, by a 1-jet defined on $E \subset X$, where X is a Hilbert space, we will understand a pair of functions $f : E \to \mathbb{R}$ and $G : E \to X$.

2.1 The $C^{1,1}$ and $C^{1,1}_{\text{conv}}$ extension problem for jets

Let us first recall the $C^{1,1}$ version of the classical Whitney extension theorem, see [70, 46, 63] for instance.

Theorem 2.1 ($C^{1,1}$ Whitney-Glaeser Extension Theorem)**.** *If E is a subset of $\mathbb{R}^n$ and we are given functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, then there exists a function $F \in C^{1,1}(\mathbb{R}^n)$ with $F = f$ on E and $\nabla F = G$ on E if and only if the 1-jet (f, G) satisfies the following property: there exists a constant $M > 0$ such that*

$$|f(x) - f(y) - \langle G(y), x-y\rangle| \leq M|x-y|^2, \quad \textit{and} \quad |G(x) - G(y)| \leq M|x-y| \qquad (\widetilde{W^{1,1}})$$

for all $x, y \in E$.

Let us briefly explain how the extension F of Theorem 2.1 is defined. Observe that if f and G are as in Theorem 2.1, we can trivially extend (f, G) to the closure $\overline{E}$ of E so that the inequalities $(\widetilde{W^{1,1}})$ hold on $\overline{E}$ with the same constant M. One of the main ingredients in the construction of the extension F is the *Whitney decomposition* of open sets into a suitable family of cubes, which we call *Whitney cubes*. Let us gather some of the most important properties of the *Whitney decomposition* of the set $\mathbb{R}^n \setminus \overline{E}$.

Proposition 2.2. *There exists a countable family of compact cubes $\{Q_k\}_k$ such that if we consider the corresponding cubes $\{Q_k^*\}_k$ with the same center as Q_k and dilated by the factor $9/8$, the families $\{Q_k\}_k$ and $\{Q_k^*\}_k$ satisfy the following properties.*

1. $\bigcup_k Q_k = \bigcup_k Q_k^* = \mathbb{R}^n \setminus \overline{E}$.
2. *The interiors of Q_k are mutually disjoint.*
3. $\operatorname{diam}(Q_k) \leq d(Q_k, \overline{E}) \leq 4 \operatorname{diam}(Q_k)$ *for all k.*
4. *If two cubes Q_k and Q_j touch each other, that is $\partial Q_k \cap \partial Q_j \neq \emptyset$, then* $\operatorname{diam}(Q_k) \approx \operatorname{diam}(Q_j)$.
5. *If two cubes Q_k^* and Q_j^* are not disjoint, then* $\operatorname{diam}(Q_k) \approx \operatorname{diam}(Q_j)$.
6. *Every point of $\mathbb{R}^n \setminus \overline{E}$ is contained in an open neighbourhood which intersects at most $N = (12)^n$ cubes of the family $\{Q_k^*\}_k$.*

Here the notation $A_j \approx B_l$ means that there exist positive constants γ, Γ, depending only on the dimension n, such that $\gamma A_j \leq B_l \leq \Gamma A_j$ for all j, l satisfying the properties specified in each case. Also, one can associate to the *Whitney cubes* $\{Q_k,\ Q_k^*\}_k$ of $\mathbb{R}^n \setminus \overline{E}$ an smooth partition of unity $\{\varphi_k\}_k$ with the following properties.

Proposition 2.3. *There exists a sequence of functions $\{\varphi_k\}_k$ defined on $\mathbb{R}^n \setminus \overline{E}$ such that*

1. $\varphi_k \in C^\infty(\mathbb{R}^n \setminus \overline{E})$.
2. $0 \leq \varphi_k \leq 1$ *on* $\mathbb{R}^n \setminus \overline{E}$ *and* $\operatorname{supp}(\varphi_k) \subseteq Q_k^*$.
3. $\sum_k \varphi_k = 1$ *on* $\mathbb{R}^n \setminus \overline{E}$.
4. *For every multi-index α there exists a constant $A_\alpha > 0$, depending only on α and on the dimension n, such that*
$$|\partial^\alpha \varphi_k(x)| \leq A_\alpha \operatorname{diam}(Q_k)^{-|\alpha|},$$
for all $x \in \mathbb{R}^n \setminus \overline{E}$ and for all k. Here $\partial^\alpha \varphi_k$ denotes the derivative $\frac{\partial^{|\alpha|}\varphi_k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ for every k and every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$.

A partition of unity $\{\varphi_k\}_k$ with the above properties is called a *Whitney partition of unity*. One can find a detailed exposition of the constructions of Propositions 2.2 and 2.3 in [63, Chapter VI]. The extension F can be explicitly defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{E} \\ \sum_k \left(f(p_k) + \langle G(p_k), x - p_k\rangle\right) \varphi_k(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{E}, \end{cases} \tag{2.1.1}$$

where each p_k is a point of $\overline{E}$ which minimizes the distance of $\overline{E}$ to the cube Q_k. In [46] it was also proved that the function F constructed in this way has the property that $\operatorname{Lip}(\nabla F) \leq k(n)M$, where $k(n)$ is a constant depending only on n (but going to infinity as $n \to \infty$), and $\operatorname{Lip}(\nabla F)$ denotes the Lipschitz constant of the gradient ∇F.

In [69, 53] it was shown, by very different means, that this $C^{1,1}$ version of the Whitney extension theorem holds true if we replace $\mathbb{R}^n$ with any Hilbert space and, moreover, there is an extension operator $(f, G) \mapsto (F, \nabla F)$ which is minimal, in the following sense. Given a Hilbert space X with norm denoted by $\|\cdot\|$, a subset E of X, and functions $f : E \to \mathbb{R}$, $G : E \to X$, a necessary and sufficient condition for the 1-jet (f, G) to have a $C^{1,1}$ extension $(F, \nabla F)$ to the whole space X is that

$$\Gamma(f, G, E) := \sup_{x,y \in E} \left(\sqrt{A_{x,y}^2 + B_{x,y}^2} + |A_{x,y}| \right) < \infty, \tag{2.1.2}$$

where

$$A_{x,y} = \frac{2(f(x) - f(y)) + \langle G(x) + G(y), y - x\rangle}{\|x - y\|^2} \quad \text{and}$$

$$B_{x,y} = \frac{\|G(x) - G(y)\|}{\|x - y\|} \quad \text{for all} \quad x, y \in E, x \neq y.$$

Moreover, the extension $(F, \nabla F)$ can be taken with best Lipschitz constants, in the sense that

$$\Gamma(F, \nabla F, X) = \Gamma(f, G, E) = \|(f, G)\|_E,$$

where

$$\|(f, G)\|_E := \inf\{\operatorname{Lip}(\nabla H) \ : \ H \in C^{1,1}(X) \text{ and } (H, \nabla H) = (f, G) \text{ on } E\}$$

is the trace seminorm of the jet (f, G) on E; see [53] and [54, Lemma 15].

While the operators $(f, G) \mapsto (F, \nabla F)$ given by the constructions in [53, 54, 69] are not linear, they have the useful property that, when we put them to work on $X = \mathbb{R}^n$, they satisfy $\operatorname{Lip}(\nabla F) \leq$

$\eta\|(f,G)\|_E$ for some $\eta > 0$ independent of n (in fact for $\eta = 1$); hence one can say that they are bounded, with norms independent of the dimension n, provided that we endow $C^{1,1}(X)$ with the seminorm given by $C^{1,1}(X) \ni F \mapsto \mathrm{Lip}(\nabla F)$ and we equip the space of jets (f,G) with the trace seminorm $\|(f,G)\|_E$. In contrast, the Whitney extension operator is linear (see (2.1.1)) and also bounded in this sense, but with norm going to ∞ as $n \to \infty$. On the negative side, the formulas in [54] depending on Wells's construction are more complicated than the proof of [53], which uses Zorn's lemma and in particular is not constructive, and the proof of [69] is extremely complicated and not entirely constructive. For more information about Whitney-type extension theorems for functions or jets, about constructing continuous linear extension operators with nearly optimal norms, and about extending these results to other spaces of functions such as Sobolev spaces, see [15, 16, 20, 23, 31, 36, 37, 38, 39, 40, 41, 46, 49, 50, 53, 54, 60, 61, 65, 68] and the references therein.

In this chapter, among other things, we will remedy these drawbacks by providing a very simple, explicit formula for $C^{1,1}$ extension of jets in Hilbert spaces: let us say that a jet (f,G) on $E \subset X$ satisfies condition $(W^{1,1})$ provided that there exists a number $M > 0$ such that

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2 \qquad (W^{1,1})$$

for all $x, y \in E$. In Proposition 2.8 below we will prove some properties of this condition but what we can see at first sight is that condition $(W^{1,1})$ is equal to Wells's necessary and sufficient condition in [69, Theorem 2]. Moreover, $(W^{1,1})$ is also absolutely equivalent to (2.1.2) and, in fact, the number $\Gamma(f,G,E)$ is the smallest $M > 0$ for which (f,G) satisfies $(W^{1,1})$ with constant $M > 0$; see [54, Lemma 15]. And, although condition $(\widetilde{W^{1,1}})$ was originally stated for the finite dimensional setting, it is obvious that it makes sense in any Banach space and then we can compare this condition with $(W^{1,1})$.

Remark 2.4. Condition $(W^{1,1})$ is absolutely equivalent to $(\widetilde{W^{1,1}})$, in the sense that if $(W^{1,1})$ is satisfied with some constant $M > 0$, then $(\widetilde{W^{1,1}})$ is satisfied with constant κM (where κ is an absolute constant independent of the space X; in particular κ does not depend on the dimension of X), and vice versa.

Proof. Let E be a subset of a Hilbert space X and $(f,G) : E \to \mathbb{R} \times X$ a 1-jet. Given $M > 0$, let us momentarily say that (f,G) satisfies the condition $(W^{1,1}_M)$ on E if the inequality defining condition $(W^{1,1})$ is satisfied for (f,G) with constant $M > 0$. Also, given $M_1, M_2 > 0$, we will say that (f,G) satisfies the condition $(\widetilde{W^{1,1}_{M_1,M_2}})$ on E if the inequalities

$$|f(y) - f(x) - \langle G(x), y-x\rangle| \leq M_1\|x-y\|^2, \quad \|G(x)-G(y)\| \leq M_2\|x-y\|,$$

are satisfied for every $x, y \in E$.

We first claim that $(W^{1,1}_M)$ implies $(\widetilde{W^{1,1}_{\frac{M}{2},M}})$. Indeed, for all $x, y \in E$, we have

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$

and

$$f(x) \leq f(y) + \frac{1}{2}\langle G(y)+G(x), x-y\rangle + \frac{M}{4}\|y-x\|^2 - \frac{1}{4M}\|G(y)-G(x)\|^2.$$

By summing both inequalities we get $\|G(x)-G(y)\| \leq M\|x-y\|$. On the other hand, thanks to the inequality $(W^{1,1}_M)$, we can write

$$\begin{aligned}
&f(y)-f(x)-\langle G(x),y-x\rangle \leq \frac{1}{2}\langle G(x)+G(y),y-x\rangle-\langle G(x),y-x\rangle\\
&+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2\\
&=\frac{1}{2}\langle G(y)-G(x),y-x\rangle+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2\\
&=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left(\|x-y\|^2+\frac{1}{M^2}\|G(x)-G(y)\|^2-2\langle\frac{1}{M}(G(y)-G(x)),y-x\rangle\right)\\
&=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(y-x)\right\|^2\leq\frac{M}{2}\|x-y\|^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&f(x)-f(y)-\langle G(x),x-y\rangle \leq \frac{1}{2}\langle G(x)+G(y),x-y\rangle-\langle G(x),x-y\rangle\\
&+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2\\
&=\frac{1}{2}\langle G(y)-G(x),x-y\rangle+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2\\
&=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left(\|x-y\|^2+\frac{1}{M^2}\|G(x)-G(y)\|^2-2\langle\frac{1}{M}(G(y)-G(x)),x-y\rangle\right)\\
&=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(x-y)\right\|^2\leq\frac{M}{2}\|x-y\|^2.
\end{aligned}$$

This leads us to

$$|f(y)-f(x)-\langle G(x),y-x\rangle|\leq\frac{M}{2}\|x-y\|^2,$$

which proves that (f,G) satisfies $(\widetilde{W^{1,1}_{\frac{M}{2},M}})$ on E.

Now let us prove that $(\widetilde{W^{1,1}_{M_1,M_2}})$ implies $(W^{1,1}_M)$, where $M=(3+\sqrt{10})\max\{M_1,M_2\}$. Using that $f(y)-f(x)-\langle G(x),y-x\rangle\leq M_1\|x-y\|^2$, we can write

$$\begin{aligned}
f(y)-f(x)&-\frac{1}{2}\langle G(x)+G(y),y-x\rangle-\frac{M}{4}\|x-y\|^2+\frac{1}{4M}\|G(x)-G(y)\|^2\\
&\leq\langle G(x),y-x\rangle+M_1\|x-y\|^2-\frac{1}{2}\langle G(x)+G(y),y-x\rangle-\frac{M}{4}\|x-y\|^2\\
&=\frac{1}{2}\langle G(x)-G(y),y-x\rangle+\left(M_1-\frac{M}{4}\right)\|x-y\|^2+\frac{1}{4M}\|G(x)-G(y)\|^2\\
&\leq\frac{1}{2}ab+\left(M_1-\frac{M}{4}\right)a^2+\frac{1}{4M}b^2,
\end{aligned}$$

where $a=\|x-y\|$ and $b=\|G(x)-G(y)\|$. Since G is M_2-Lipschitz, we have the inequality $b\leq M_2a$. Hence the last term in the above chain of inequalities is smaller than or equal to

$$(\tfrac{1}{2}M_2+(M_1-\tfrac{M}{4})+\tfrac{1}{4M}M_2^2)a^2\leq(\tfrac{1}{2}K+(K-\tfrac{M}{4})+\tfrac{1}{4M}K^2)a^2,$$

where $K=\max\{M_1,M_2\}$. Then, the last term is smaller than or equal to 0 if and only if $-M^2+6MK+K^2\leq 0$. But, in fact, for $M=(3+\sqrt{10})K$ the term $-M^2+6MK+K^2$ is equal to 0. This proves that (f,G) satisfies $(W^{1,1}_M)$ on E. □

In Theorem 2.27 below we will show that, for every 1-jet (f,G) defined on E and satisfying the property $(W^{1,1})$ with constant M on E, the formula

$$\begin{aligned}
&F=\operatorname{conv}(g)-\tfrac{M}{2}\|\cdot\|^2,\ \text{ where} \qquad (2.1.3)\\
&g(x)=\inf_{y\in E}\{f(y)+\langle G(y),x-y\rangle+\tfrac{M}{2}\|x-y\|^2\}+\tfrac{M}{2}\|x\|^2,\quad x\in X,
\end{aligned}$$

defines a $C^{1,1}(X)$ function with $F = f$ and $\nabla F = G$ on E and $\mathrm{Lip}(\nabla F) \leq M$. Here conv$(g)$ denotes the convex envelope of g, defined by

$$\mathrm{conv}(g)(x) = \sup\{h(x) \,:\, h \text{ is convex, proper and lower semicontinuous}, \, h \leq g\}. \qquad (2.1.4)$$

Another expression for conv(g) is given by

$$\mathrm{conv}(g)(x) = \inf\left\{\sum_{j=1}^{k} \lambda_j g(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{k} \lambda_j = 1, \, x = \sum_{j=1}^{k} \lambda_j x_j, \, k \in \mathbb{N}\right\}. \qquad (2.1.5)$$

In the case that X is finite dimensional, say $X = \mathbb{R}^n$, this expression can be made simpler: by using Carathéodory's Theorem one can show that it is enough to consider convex combinations of at most $n+1$ points. That is to say, if $g : \mathbb{R}^n \to \mathbb{R}$ then

$$\mathrm{conv}(g)(x) = \inf\left\{\sum_{j=1}^{n+1} \lambda_j g(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1, \, x = \sum_{j=1}^{n+1} \lambda_j x_j\right\};$$

see [57, Corollary 17.1.5] for instance.

Let us briefly explain what is the idea behind formula (2.1.3). It is well known that a function $F : X \to \mathbb{R}$ is of class $C^{1,1}$, with $\mathrm{Lip}(\nabla F) = M$, if and only if $F + \frac{M}{2}\|\cdot\|^2$ is convex and $F - \frac{M}{2}\|\cdot\|^2$ is concave. So, if we are given a 1-jet (f, G) defined on $E \subset X$ which can be extended to $(F, \nabla F)$ with $F \in C^{1,1}(X)$ and $\mathrm{Lip}(\nabla F) \leq M$, then the function $H = F + \frac{M}{2}\|\cdot\|^2$ will be convex and of class $C^{1,1}$. Conversely, if we can find a convex and $C^{1,1}$ function H such that $(H, \nabla H)$ is an extension of the jet $E \ni y \mapsto \left(f(y) + \frac{M}{2}\|y\|^2, G(y) + My\right)$, then $X \ni y \mapsto \left(H(y) - \frac{M}{2}\|y\|^2, \nabla H(y) - My\right)$ will be a $C^{1,1}$ extension of (f, G). Thus we can reduce the $C^{1,1}$ extension problem for jets to the $C^{1,1}_{\mathrm{conv}}$ extension problem for jets. Here $C^{1,1}_{\mathrm{conv}}(X)$ stands for the set of all convex functions $\varphi : X \to \mathbb{R}$ of class $C^{1,1}$.

Now, how can we solve the $C^{1,1}_{\mathrm{conv}}$ extension problem for jets? In [10] the following necessary and sufficient condition for $C^{1,1}_{\mathrm{conv}}$ extension of jets was given: for any $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$, $G : E \to X$, we say that (f, G) satisfies condition $(CW^{1,1})$ on E with constant $M > 0$, provided that

$$f(x) \geq f(y) + \langle G(y), x - y\rangle + \frac{1}{2M}\|G(x) - G(y)\|^2 \quad \text{for all} \quad x, y \in E.$$

In [10] it is shown that a 1-jet (f, G) has an extension $(F, \nabla F)$ with $F \in C^{1,1}_{\mathrm{conv}}(X)$ if and only if (f, G) satisfies $(CW^{1,1})$; moreover in this case one can take $F \in C^{1,1}_{\mathrm{conv}}$ such that $\mathrm{Lip}(\nabla F) \leq k(n)M$, where $k(n)$ is a constant only depending on n. The construction in [10] is explicit, but has the same disadvantage as the Whitney extension operator has, namely that $\lim_{n\to\infty} k(n) = \infty$. In [11] this result is extended to the Hilbert space setting, but the proof, inspired by [53], is not constructive as it relies in Zorn's Lemma. However, by following the ideas of the proof of [10], but using a simple formula instead of the Whitney extension theorem, we will show in Theorem 2.11 below that if a 1-jet (f, G) defined on a subset E of a Hilbert space satisfies condition $(CW^{1,1})$ then the function F defined by

$$F = \mathrm{conv}(g), \quad \text{where} \quad g(x) = \inf_{y\in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2\}, \quad x \in X, \qquad (2.1.6)$$

is a $C^{1,1}$ convex function such that $F_{|E} = f$, $(\nabla F)_{|E} = G$, and $\mathrm{Lip}(\nabla F) \leq M$. Moreover, if H is another $C^{1,1}$ convex function with $H = f$ and $\nabla H = G$ on E and $\mathrm{Lip}(\nabla H) \leq M$, then $H \leq F$. This strategy allows us to solve the $C^{1,1}_{\mathrm{conv}}$ extension problem for jets with best constants and, after checking that if (f, G) satisfies $(W^{1,1})$ then $\left(f(y) + \frac{M}{2}\|y\|^2, G(y) + My\right)$ satisfies $(CW^{1,1})$, also allows us to show that the expression

$$F(x) = \mathrm{conv}\left(z \mapsto \inf_{y\in E}\{f(y) + \tfrac{M}{2}\|y\|^2 + \langle G(y) + My, z - y\rangle + M\|z - y\|^2\}\right)(x) - \tfrac{M}{2}\|x\|^2,$$

which is clearly equal to (2.1.3), provides an extension formula that solves the minimal $C^{1,1}$ extension problem for jets, in the sense that $\mathrm{Lip}(\nabla F) \leq M$. Besides we will also prove that if H is another $C^{1,1}$ function with $H = f$ and $\nabla H = G$ on E and $\mathrm{Lip}(\nabla H) \leq M$, then $H \leq F$. Since the extension of (f, G) constructed by Wells in [69] also has this property, it follows that in fact (2.1.3) coincides with Wells's extension. The point is of course that both our formula (2.1.3) and the proof are much simpler than Wells's construction and proof.

2.2 The $C^{1,\omega}_{\text{conv}}$ extension problem for jets

By a modulus of continuity ω we understand a concave and strictly increasing function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$. It is well-known that for every uniformly continuous function $f : X \to Y$ between two metric spaces there exists a modulus of continuity ω such that $d_Y(f(x), f(z)) \leq \omega\,(d_X(x, z))$ for every $x, z \in X$. Slightly abusing terminology, we will say that a mapping $G : X \to Y$ is *uniformly continuous with modulus of continuity* ω if there exists $M \geq 0$ such that

$$d_Y\,(G(x), G(y)) \leq M\omega\,(d_X(x, y))$$

for all $x, y \in X$. A version of the Whitney Extension Theorem for functions of class $C^{1,\omega}(\mathbb{R}^n)$, i.e. functions of class C^1 such the their first derivatives are uniformly continuous with modulus of continuity ω, was proved by G. Glaeser in [46].

Theorem 2.5 ($C^{1,\omega}$ Whitney-Glaeser Extension Theorem)**.** *If E is a subset of $\mathbb{R}^n$, $\omega : [0, +\infty) \to [0, +\infty)$ is a modulus of continuity and we are given functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, then there exists a function $F \in C^{1,\omega}(\mathbb{R}^n)$ with $F = f$ on E and $\nabla F = G$ on E if and only if the 1-jet (f, G) satisfies the following property: there exists a constant $M > 0$ such that*

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M\omega(|x - y|)|x - y|, \quad \textit{and} \quad |G(x) - G(y)| \leq M\omega(|x - y|) \quad (W^{1,\omega})$$

for all $x, y \in E$.

The extension F above is defined by means of the expression (2.1.1) and in [46] it was proved that one can arrange that

$$\sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x - y|)} \leq k(n)M,$$

where $k(n)$ is a constant only depending on n (but with $\lim_{n\to\infty} k(n) = +\infty$). In [10, Theorem 1.4] we proved that a necessary and sufficient condition on a 1-jet (f, G) defined on E for having a $C^{1,\omega}$ convex extension F is that

$$f(x) \geq f(y) + \langle G(y), x - y\rangle + |G(x) - G(y)|\omega^{-1}\left(\frac{1}{2M}|G(x) - G(y)|\right), \quad x, y \in E. \quad (CW^{1,\omega})$$

Moreover, we obtained a good control on the modulus of continuity of F in terms of the constant M, namely,

$$\sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x - y|)} \leq k(n)M, \tag{2.2.1}$$

where, again, $k(n)$ only depends on the dimension n (but tends to ∞ as n goes to ∞).

The latent potential in formula (2.1.6), at least in the convex case, is not confined to $C^{1,1}$ extension problems in Hilbert spaces. Indeed, in Theorem 2.40 below we will prove, by means of a similar formula, that if X is a Hilbert space and ω is a modulus of continuity, with $\omega(\infty) = \infty$, then the condition $(CW^{1,\omega})$ is necessary and sufficient on a 1-jet (f, G) defined on a subset E of a Hilbert space for having an extension $(F, \nabla F)$ such that $F : X \to \mathbb{R}$ is convex and of class $C^{1,\omega}$, with

$$\sup_{x,y\in X,\, x\neq y} \frac{\|\nabla F(x) - \nabla F(y)\|}{\omega(\|x - y\|)} \leq 8M.$$

Not only does this provide a new result for the infinite-dimensional case, but also shows that the constants $k(n)$ of (2.2.1) can be supposed to be independent of the dimension n, at least if $\omega(\infty) = \infty$ and in particular for all of the classes $C^{1,\alpha}_{\text{conv}}(\mathbb{R}^n)$. We will provide a detailed exposition of this results in Section 2.8 below.

Of course, Theorem 2.40 (for the $C^{1,\omega}$ class) is essentially much more general than Theorem 2.11 (for the $C^{1,1}$ class), but we deliberately present these two results in two different sections, for the following two reasons.

1. In Theorem 2.11 we are able to obtain best possible Lipschitz constants of the gradients of the extension, whereas in Theorem 2.40 we only get them up to a factor 8.
2. The proof of Theorem 2.40 is more technical and uses some machinery from Convex Analysis, such as Fenchel conjugates, smoothness and convexity moduli, etc, which could obscure the main ideas and prevent some readers interested only in the proofs of the $C^{1,1}$ results from easily understanding them.

Unfortunately, it seems very unlikely that one could use this kind of formulas to solve $C^{1,\alpha}$ extension problems for general (not necessarily convex) 1-jets in Hilbert spaces. The exponent $\alpha = 1$ is somewhat miraculous in this respect: even for the simplest case that $X = \mathbb{R}$, it is not true in general that, given a function $f \in C^{1,\alpha}(\mathbb{R})$, there exists a constant C such that $f + C|\cdot|^{1+\alpha}$ is convex, as the following example shows.

Example 2.6. If $0 < \alpha < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ -(t-1)^{1+\alpha} & \text{if} \quad t \geq 1, \end{cases}$$

then f is of class $C^{1,\alpha}(\mathbb{R})$ and there is no constant $C > 0$ for which $f + C|\cdot|^{1+\alpha}$ is convex.

Proof. The function f is clearly differentiable on $\mathbb{R}$ and

$$f'(t) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ -(1+\alpha)(t-1)^{\alpha} & \text{if} \quad t \geq 1 \end{cases}$$

It is then obvious that f' is α-Hölder continuous on $\mathbb{R}$. For any $C > 0$, the second derivative of the function $f + C|\cdot|^{1+\alpha}$ at $t > 1$ is

$$-(1+\alpha)\alpha\left((t-1)^{\alpha-1} - Ct^{\alpha-1}\right),$$

which tends to $-\infty$ as $t \to 1^+$. This shows that $f + C|\cdot|^{1+\alpha}$ is not convex on a neighbourhood of 1. $\square$

Finally, let us mention that when the results presented in Sections 2.3, 2.6 and 2.8 were completed, a preprint of A. Daniilidis, M. Haddou, E. Le Gruyer and O. Ley [24] concerning the same problem in Hilbert spaces was made public too. The formula for $C^{1,1}_{\text{conv}}$ extension of 1-jets given in [24] is different from the formula we provide in this thesis. As these authors show, their formula cannot work for the Hölder differentiability classes $C^{1,\alpha}_{\text{conv}}$ with $\alpha \neq 1$. Two advantages of the present approach are the fact that our formula does work for theses classes, and its simplicity.

2.3 Optimal $C^{1,1}$ convex extensions of jets by explicit formulas in Hilbert spaces

Definition 2.7. *Given an arbitrary subset E of X, and a 1-jet $f : E \to \mathbb{R}$, $G : E \to X$, we will say that (f, G) satisfies the condition $(CW^{1,1})$ on E with constant $M > 0$, provided that*

$$f(x) \geq f(y) + \langle G(y), x - y\rangle + \frac{1}{2M}\|G(x) - G(y)\|^2 \quad \textit{for all} \quad x, y \in E.$$

The following proposition shows that this condition is necessary for a 1-jet to have a $C^{1,1}$ convex extension to all of X.

Proposition 2.8. *Let $f \in C^{1,1}(X)$ be convex, and assume that f is not affine. Then*

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq \frac{1}{2M} \|\nabla f(x) - \nabla f(y)\|^2$$

for all $x, y \in X$, where

$$M = \sup_{x,y \in X,\, x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|}.$$

On the other hand, if f is affine, it is obvious that $(f, \nabla f)$ satisfies $(CW^{1,1})$ on every $E \subset X$, for every $M > 0$.

Proof. Suppose that there exist different points $x, y \in X$ such that

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle < \frac{1}{2M} \|\nabla f(x) - \nabla f(y)\|^2,$$

and we will get a contradiction.

Case 1. Assume further that $M = 1$, $f(y) = 0$, and $\nabla f(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) < \frac{1}{2} \|\nabla f(x)\|^2.$$

Call $a = \|\nabla f(x)\| > 0$, $b = f(x)$, set

$$v = -\frac{1}{\|\nabla f(x)\|} \nabla f(x),$$

and define

$$\varphi(t) = f(x + tv)$$

for every $t \in \mathbb{R}$. We have $\varphi(0) = b$, $\varphi'(0) = -a$, and $\varphi'(t) = \langle \nabla f(x + tv), v \rangle$ is 1-Lipschitz because so is ∇f and $\|v\| = 1$. This implies that

$$|\varphi(t) - b + at| \leq \frac{t^2}{2}$$

for every $t \in \mathbb{R}^+$, hence also that

$$\varphi(t) \leq -at + b + \frac{t^2}{2} \text{ for all } t \in \mathbb{R}^+.$$

By assumption we have $b < \frac{1}{2}a^2$, and therefore

$$f\left(x + av\right) = \varphi\left(a\right) \leq -a^2 + b + \frac{a^2}{2} < 0,$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $\nabla f(y) = 0$. This shows that

$$f(x) \geq \frac{1}{2} \|\nabla f(x)\|^2.$$

Case 2. Assume only that $M = 1$. Define

$$g(z) = f(z) - f(y) - \langle \nabla f(y), z - y \rangle$$

for every $z \in X$. Then $g(y) = 0$ and $\nabla g(y) = 0$. By Case 1, we get

$$g(x) \geq \frac{1}{2} \|\nabla g(x)\|^2,$$

and since $\nabla g(x) = \nabla f(x) - \nabla f(y)$ the Proposition is thus proved in the case when $M = 1$.
Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $\psi = \frac{1}{M} f$, which satisfies the assumption of Case 2. Therefore

$$\psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle \geq \frac{1}{2} \|\nabla \psi(x) - \nabla \psi(y)\|^2,$$

which is equivalent to the desired inequality. □

We will need to use the following characterization of $C^{1,1}$ differentiability of convex functions. Of course the result is well known, but we will provide a short proof for completeness, and also in order to remark that the implication $(ii) \implies (i)$ is true for not necessarily convex functions as well, a fact that we will have to use later on.

Proposition 2.9. *For a continuous convex function $f : X \to \mathbb{R}$, the following statements are equivalent.*

(i) *There exists $M > 0$ such that*

$$f(x+h) + f(x-h) - 2f(x) \leq M\|h\|^2 \quad \textit{for all} \quad x, h \in X.$$

(ii) *f is differentiable on X with* $\mathrm{Lip}(\nabla f) \leq M$.

Proof. First we prove that (ii) implies (i), which is also valid for non-convex functions. Using that $\mathrm{Lip}(\nabla f) \leq M$, it follows from Taylor's theorem that

$$f(x+h) - f(x) - \langle \nabla f(x), h \rangle \leq \frac{M}{2} \|h\|^2.$$

Similarly we have

$$f(x-h) - f(x) - \langle \nabla f(x), -h \rangle \leq \frac{M}{2} \|h\|^2,$$

and combining both inequalities we get (i).

Now we do assume that f is a convex function and let us show that (i) implies (ii). Since

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{\|h\|} = 0,$$

for all $x \in X$ and f is convex and continuous, f is differentiable on X. In order to prove that $\mathrm{Lip}(\nabla f) \leq M$ it is enough to see that the function $F : X \to \mathbb{R}$ defined by $F(x) = \frac{M}{2}\|x\|^2 - f(x)$, $x \in X$, is convex. Since f is a continuous function, the convexity of F is equivalent to:

$$F\left(\frac{x+y}{2}\right) \leq \tfrac{1}{2}F(x) + \tfrac{1}{2}F(y) \quad \text{for all} \quad x, y \in X.$$

To see this, given $x, y \in X$, we can write

$$F\left(\frac{x+y}{2}\right) = \tfrac{1}{2}F(x) + \tfrac{1}{2}F(y) + \frac{1}{2}\left(f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) - M\left\|\frac{x-y}{2}\right\|^2 \right)$$

Applying (ii) with $h = \frac{x-y}{2}$ we obtain that

$$f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \leq M\left\|\frac{x-y}{2}\right\|^2,$$

which in turns implies $F\left(\frac{x+y}{2}\right) \leq \frac{1}{2}F(x) + \frac{1}{2}F(y)$. □

Recall that for a function $f : X \to \mathbb{R}$, the convex envelope of f is defined by

$$\mathrm{conv}(f)(x) = \sup\{\phi(x) \,:\, \phi \text{ is convex and lower semicontinuous, } \phi \leq f\}.$$

Another expression for $\mathrm{conv}(f)$ is:

$$\mathrm{conv}(f)(x) = \inf \left\{ \sum_{j=1}^{n} \lambda_j f(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, x = \sum_{j=1}^{n} \lambda_j x_j, n \in \mathbb{N} \right\}.$$

The following result shows that the operator $f \mapsto \mathrm{conv}(f)$ not only preserves $C^{1,1}$ smoothness of functions f and Lipschitz constants of their gradients ∇f, but also that, even for some nondifferentiable functions f, their convex envelopes $\mathrm{conv}(f)$ will be of class $C^{1,1}$, with best possible constants, provided that the functions f satisfy suitable one-sided estimates. This is a slight (but very significant for our purposes) improvement of particular cases of the results in [48], [22, Theorem 7], and [51].

Theorem 2.10. *Let X be a Banach space. Suppose that a function $f : X \to \mathbb{R}$ has a convex, lower semicontinuous minorant, and satisfies*

$$f(x+h) + f(x-h) - 2f(x) \leq M\|h\|^2 \quad \textit{for all} \quad x, h \in X.$$

Then $\psi := \mathrm{conv}(f)$ is a continuous convex function satisfying the same property. In view of Proposition 2.9, we conclude that ψ is of class $C^{1,1}(X)$, with $\mathrm{Lip}(\nabla\psi) \leq M$. In particular, for a function $f \in C^{1,1}(X)$, we have that $\mathrm{conv}(f) \in C^{1,1}(X)$, with $\mathrm{Lip}(\nabla\psi) \leq \mathrm{Lip}(\nabla f)$.

Proof. The function ψ is well defined as $\psi \leq f$ and f has a convex, lower semicontinuous minorant. Now let us check the mentioned inequality. Given $x, h \in X$ and $\varepsilon > 0$, we can pick $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

$$\psi(x) \geq \sum_{i=1}^{n} \lambda_i f(x_i) - \varepsilon, \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i x_i = x.$$

Since $x \pm h = \sum_{i=1}^{n} \lambda_i (x_i \pm h)$, we have $\psi(x \pm h) \leq \sum_{i=1}^{n} \lambda_i f(x_i \pm h)$. This leads us to

$$\psi(x+h) + \psi(x-h) - 2\psi(x) \leq \sum_{i=1}^{n} \lambda_i \left(f(x_i+h) + f(x_i-h) - 2f(x_i) \right) + 2\varepsilon$$

By the assumption on f, we obtain

$$f(x_i+h) + f(x_i-h) - 2f(x_i) \leq M\|h\|^2 \quad i = 1, \ldots, n.$$

Therefore

$$\psi(x+h) + \psi(x-h) - 2\psi(x) \leq M\|h\|^2 + 2\varepsilon. \tag{2.3.1}$$

Since $\varepsilon > 0$ is arbitrary, we get the desired inequality. It is clear that ψ, being a supremum of a family of lower semicontinuous convex functions that are pointwise uniformly bounded (by the function f), is convex, proper and lower semicontinuous. And because all lower semicontinuous, proper and convex functions are continuous at interior points of their domains (see [19, Proposition 4.1.5] for instance), we also have that ψ is continuous. □

Theorem 2.11. *Given a 1-jet (f, G) defined on E satisfying property $(CW^{1,1})$ with constant M on E, the formula*

$$F = \mathit{conv}(g), \quad g(x) = \inf_{y \in E} \{ f(y) + \langle G(y), x - y \rangle + \tfrac{M}{2}\|x-y\|^2 \}, \quad x \in X,$$

defines a $C^{1,1}$ convex function such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $\mathrm{Lip}(\nabla F) \leq M$.

Moreover, if H is another $C^{1,1}$ convex function with $H_{|_E} = f$, $(\nabla H)_{|_E} = G$, and $\mathrm{Lip}(\nabla H) \leq M$, then $H \leq F$.

Proof. We start with a lemma which tells us that the function g lies above every affine function $x \mapsto f(z) + \langle G(z), x - z\rangle,\ z \in E$.

Lemma 2.12. *We have that*

$$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2$$

for every $y, z \in E,\ x \in X$.

Proof. Given $y, z \in E,\ x \in X$, condition $(CW^{1,1})$ implies

$$\begin{aligned}
& f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2 \\
& \geq f(z) + \langle G(z), y - z\rangle + \tfrac{1}{2M}\|G(y) - G(z)\|^2 + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2 \\
& = f(z) + \langle G(z), x - z\rangle + \tfrac{1}{2M}\|G(y) - G(z)\|^2 + \langle G(z) - G(y), y - x\rangle + \tfrac{M}{2}\|x - y\|^2 \\
& = f(z) + \langle G(z), x - z\rangle + \tfrac{1}{2M}\|G(y) - G(z) + 2M(y - x)\|^2 \\
& \geq f(z) + \langle G(z), x - z\rangle.
\end{aligned}$$

□

Observe that Lemma 2.12 shows that $m \leq g$, where g is defined as in Theorem 2.11, and

$$m(x) := \sup_{z \in E}\{f(z) + \langle G(z), x - z\rangle\}, \quad x \in X. \tag{2.3.2}$$

Bearing in mind the definitions of g and m we then deduce that $f \leq m \leq g \leq f$ on E. Thus $g = f$ on E. It is worth noting that the function g is not differentiable in general. Nonetheless the function $F = \operatorname{conv}(g)$ is of class $C^{1,1}$ because, as we next show, g satisfies the one-sided estimate of Theorem 2.10.

Lemma 2.13. *We have*

$$g(x + h) + g(x - h) - 2g(x) \leq M\|h\|^2 \quad \textit{for all} \quad x, h \in X.$$

Proof. Given $x, h \in X$ and $\varepsilon > 0$, by definition of g, we can pick $y \in E$ with

$$g(x) \geq f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2 - \varepsilon.$$

We then have

$$\begin{aligned}
g(x + h) + g(x - h) - 2g(x) &\leq f(y) + \langle G(y), x + h - y\rangle + \tfrac{M}{2}\|x + h - y\|^2 \\
&\quad + f(y) + \langle G(y), x - h - y\rangle + \tfrac{M}{2}\|x - h - y\|^2 \\
&\quad - 2\left(f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2\right) + 2\varepsilon \\
&= \tfrac{M}{2}\left(\|x + h - y\|^2 + \|x - h - y\|^2 - 2\|x - y\|^2\right) + 2\varepsilon \\
&= M\|h\|^2 + 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, the above chain of inequalities proves our lemma. □

By Lemma 2.13 and Theorem 2.10 we then obtain that $F = \operatorname{conv}(g)$ is convex and of class $C^{1,1}$, with $\operatorname{Lip}(\nabla F) \leq M$. We also note that the function m of (2.3.2), being a supremum of continuous convex functions, is convex and lower semicontinuous on X. By definition of F, we thus have $m \leq F \leq g$, where both m and g coincide with f on E. Thus $F = f$ on E.

In order to prove that ∇F coincides with G on E, we use the following well known criterion for differentiability of convex functions in Banach spaces, whose proof is presented in this thesis for the sake of completeness.

Lemma 2.14. *If $\phi : X \to \mathbb{R}$ is convex and lower semicontinuous and $\psi : X \to \mathbb{R}$ is differentiable at $y \in X$ with $\phi \leq \psi$, and $\phi(y) = \psi(y)$, then ϕ is differentiable at y, with $\nabla\phi(y) = \nabla\psi(y)$.*

(This fact can also be phrased as: a convex function ϕ is differentiable at y if and only if ϕ is superdifferentiable at y.)

Proof. Using that $\phi(y) = \psi(y)$ and $\phi \leq \psi$ on X, the differentiability of ψ at y gives

$$\frac{\phi(x) - \phi(y) - \langle \nabla\psi(y), x - y\rangle}{\|x - y\|} \leq \frac{\psi(x) - \psi(y) - \langle \nabla\psi(y), x - y\rangle}{\|x - y\|} \to 0 \tag{2.3.3}$$

as $\|x - y\| \to 0^+$. On the other hand, because ϕ is convex and lower semicontinuous on X, there exists some $\xi \in \partial\phi(y)$, that is, the subdifferential of ϕ at the point y. Assume that $\xi \neq \nabla\psi(y)$. Then we must have for $v = \frac{\xi - \nabla\psi(y)}{\|\xi - \nabla\psi(y)\|}$ and every $t > 0$ that

$$\frac{\phi(y + tv) - \phi(y) - t\langle \nabla\psi(y), v\rangle}{t} \geq \langle \xi - \nabla\psi(y), v\rangle = \|\xi - \nabla\psi(y)\|,$$

where the left sided term tends to 0 as $t \to 0$ by (2.3.3). This yields $\|\xi - \nabla\psi(y)\| \leq 0$, a contradiction. Thus $\xi = \nabla\psi(y)$ and then $\nabla\psi(y) \in \partial\phi(y)$, which leads us to

$$0 \leq \frac{\phi(x) - \phi(y) - \langle \nabla\psi(y), x - y\rangle}{\|x - y\|} \to 0$$

as $\|x - y\| \to 0^+$ by virtue of (2.3.3). Therefore ϕ is differentiable at y with $\nabla\phi(y) = \nabla\psi(y)$. $\square$

Because $m \leq F$ on X and $F = m$ on E, where m is convex and lower semicontinuous and F is differentiable on X, Lemma 2.14 implies that m is differentiable on E with $\nabla m(x) = \nabla F(x)$ for all $x \in E$. It is clear, by definition of m, that $G(x) \in \partial m(x)$ (denoting the subdifferential of m at x) for every $x \in E$, and this observation shows that $\nabla F = G$ on E.

Finally, consider another convex extension $H \in C^{1,1}(X)$ of the jet (f, G) with $\mathrm{Lip}(\nabla H) \leq M$. Using Taylor's theorem and the assumptions on H we have that

$$H(x) \leq f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2, \quad x \in X,\ y \in E.$$

Taking the infimum over $y \in E$ we get $H \leq g$ on X. On the other hand, bearing in mind that H is convex, the definition of the convex envelope of a function implies $H = \mathrm{conv}(H) \leq \mathrm{conv}(g) = F$ on X. This completes the proof of Theorem 2.11. $\square$

2.4 Interpolation of arbitrary subsets by boundaries of $C^{1,1}$ convex bodies

We can use the above results to solve a geometrical problem concerning characterizations of subsets of X which can be interpolated by boundaries of $C^{1,1}$ convex bodies with prescribed unit outer normals. If C is a subset of X and we are given a Lipschitz map $N : C \to X$ such that $\|N(y)\| = 1$ for every $y \in C$, it is natural to ask what conditions on C and N are necessary and sufficient for C to be a subset of the boundary of a $C^{1,1}$ convex body V such that $N(y)$ is outwardly normal to ∂V at y for every $y \in C$. This is equivalent to the following question: given an arbitrary subset C of $\mathbb{R}^n$ and a collection $\mathcal{H}$ of affine hyperplanes of $\mathbb{R}^n$ such that every $H \in \mathcal{H}$ passes through a point $x_H \in C$, what conditions are necessary and sufficient for the existence of a $C^{1,1}$ convex hypersurface S in $\mathbb{R}^n$ such that H is tangent to S at x_H for every $H \in \mathcal{H}$? We will also solve the same problem in the setting of Hilbert spaces for bounded convex bodies.

Throughout this section X will denote a Hilbert space, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will be respectively the norm and the inner product on X, and S_X will be the unit sphere of X. Let us start by clarifying what we mean by a $C^{1,1}$ convex hypersurface. Some authors define them as boundaries of (not necessarily bounded) convex bodies such that the outer normal is locally Lipschitz; however, we will require the existence of a global Lipschitz constant for the outer normal, and therefore we will make a distinction between $C^{1,1}$ convex hypersurfaces and $C^{1,1}_{\text{loc}}$ convex hypersurfaces.

Recall that a convex body of class C^p is a closed convex set V with nonempty interior such that its boundary ∂V is a one-codimensional submanifold of X of class C^p. Assuming, without loss of generality, that $0 \in \operatorname{int} V$, this is equivalent to saying that the Minkowski functional of V, defined by

$$\mu_V(x) = \inf\{t > 0 : x \in tV\},$$

is a continuous sublinear functional which is of class C^p on $X \setminus \mu_V^{-1}(0)$. This implies the existence of convex functions $\psi : X \to \mathbb{R}$ of class $C^p(X)$ such that $V = \psi^{-1}(-\infty, 1]$. Conversely, if V is of this form and $\psi^{-1}(-\infty, 1) \neq \emptyset$ then it is clear that V is a C^p convex body. Thus a C^p convex body is simply a nondegenerate sublevel set of a C^p convex function of class C^p.

Now, a $C^{1,1}$ convex body in a Hilbert space X can be defined as a C^1 convex body V such that the outer unit normal $N : \partial V \to S_X$ is Lipschitz. Again this is equivalent to saying that $V = \psi^{-1}(-\infty, 1]$ for some C^1 convex function $\psi : X \to \mathbb{R}$ such that $\psi^{-1}(-\infty, 1) \neq \emptyset$ and $N = \nabla\psi / \|\nabla\psi\| : \partial V \to S_X$ is Lipschitz. This is the definition we will find most convenient to use.

Definition 2.15. *We will say that a closed convex subset V of X is a convex body of class $C^{1,1}$ if*

1. *V has nonempty interior.*
2. *V can be written as a sublevel set $\psi^{-1}((-\infty, 1])$ of a $C^1(X)$ convex function $\psi : X \to \mathbb{R}$ such that $\nabla\psi(x) \neq 0$ for every $x \in \partial V$.*
3. *The outer unit normal $n_V : \partial V \to S_X$ of V defined as*

$$n_V(x) = \frac{\nabla\psi(x)}{\|\nabla\psi(x)\|}, \quad x \in \partial V,$$

is a Lipschitz mapping.

We will say that a subset S of X is a $C^{1,1}$ convex hypersurface provided that there exists a $C^{1,1}$ convex body V in X such that $S = \partial V$.

The following proposition sums up some elementary properties of smooth convex bodies; in particular we recall the well known fact that the definition of n_V does not depend on the choice of the function ψ. If V is closed and convex and such that ∂V is an hypersurface (i.e., a one-codimensional submanifold) of class C^1, we will say that a vector w is *outwardly normal to ∂V at a point $x \in \partial V$* if w belongs to the orthogonal complement of the (vectorial) tangent space of ∂V at x and $x + \lambda w \in X \setminus V$ for every $\lambda > 0$.

Proposition 2.16. *Suppose that $\psi : X \to \mathbb{R}$ is a convex function of class $C^1(X)$ such that $\psi(x_0) < 1$ for some $x_0 \in X$ and let us denote $V = \{x \in X \, : \, \psi(x) \leq 1\}$. Then the following is true.*

(1) *V is closed and convex.*

(2) *$\partial V = \{x \in X \, : \, \psi(x) = 1\}$ and $\operatorname{int}(V) = \{x \in X \, : \, \psi(x) < 1\}$.*

(3) *V is bounded if and only if ψ is coercive (meaning that $\lim_{\|x\| \to \infty} \psi(x) = \infty$).*

(4) *$\nabla\psi(x) \neq 0$ for every $x \in \partial V$.*

(5) *∂V is a one-codimensional submanifold of X of class C^1 and the (vectorial) tangent space $T_x\partial V$ of ∂V at x is the orthogonal complement of $\nabla\psi(x)$. In fact, $\nabla\psi(x)$ is outwardly normal to ∂V at x for each $x \in \partial V$.*

(6) *If $x \in X \setminus V$ and $x_V \in \partial V$ is such that $\|x - x_V\| = d(x, V)$, then $x - x_V$ is outwardly normal to ∂V at the point x_V. Therefore $x - x_V$ is paralell to $\nabla\psi(x_V)$ and*

$$n_V(x_V) = \frac{\nabla\psi(x_V)}{\|\nabla\psi(x_V)\|} = \frac{x - x_V}{\|x - x_V\|}.$$

In particular, the definition of n_V does not depend on the choice of ψ.

(7) $V \subset \{x \in X \,:\, \langle n_V(z), x - z\rangle \leq 0\}$ *for every $z \in \partial V$.*

Proof.

(1) It follows immediately from the convexity and the continuity of F.

(2) Let us first see that $\mathrm{int}(V) = \{x \in X \,:\, \psi(x) < 1\}$. Indeed, by continuity, $\{x \in X \,:\, \psi(x) < 1\}$ is an open subset contained in V, and then is contained in $\mathrm{int}(V)$. Given any $x \in \mathrm{int}(V)$, we can find a point $y \in \mathrm{int}(V)$ such that the line segment $[x_0, y]$ is contained in $\mathrm{int}(V)$ and $x \in (x_0, y)$. The convexity of ψ implies that, for some $\lambda \in (0, 1)$,

$$\psi(x) \leq \lambda\psi(x_0) + (1-\lambda)\psi(y) \leq \lambda\psi(x_0) + (1-\lambda) < \lambda + (1-\lambda) = 1,$$

that is $\psi(x) < 1$. This proves $\mathrm{int}(V) = \{x \in X \,:\, \psi(x) < 1\}$. It is now clear that

$$\partial V = V \setminus \mathrm{int}(V) = \{x \in X \,:\, \psi(x) = 1\}.$$

(3) If V is unbounded, we can find a sequence $(x_k)_k \in V$ such that $\lim_k \|x_k\| = \infty$ and then $\lim_k \psi(x_k) = \infty$ by coercivity of ψ. This is a contradiction since $\psi(x_k) \leq 1$ for every k. Conversely, if V is bounded an has a nonempty interior then there exist $x_0 \in X$ and $R, r > 0$ such that $B(x_0, r) \subset V \subset B(x_0, R)$. Now, for every $x \in X \setminus V$, let $y_x \in \partial V$ denote the unique point of intersection of ∂V with the ray $\{t(x - x_0) : t > 0\}$. If we write $y_x = \frac{\|y_x - x_0\|}{\|x - x_0\|}x + \left(1 - \frac{\|y_x - x_0\|}{\|x - x_0\|}\right)x_0$, the convexity of ψ leads us to

$$\psi(y_x) \leq \frac{\|y_x - x_0\|}{\|x - x_0\|}\psi(x) + \left(1 - \frac{\|y_x - x_0\|}{\|x - x_0\|}\right)\psi(x_0),$$

which in turn yields

$$\frac{\psi(x) - \psi(x_0)}{\|x - x_0\|} \geq \frac{\psi(y_x) - \psi(x_0)}{\|y_x - x_0\|} \geq \frac{1 - \psi(x_0)}{R}.$$

This implies that $\lim_{\|x\|\to\infty} \psi(x)$ because $\psi(x_0) < 1$.

(4) Because $\mathrm{int}(V) \neq \emptyset$, by (2) there exists a point $x_0 \in \mathrm{int}(V)$ with $F(x_0) < 1$. Assume that $\nabla F(x) = 0$ for some $x \in \partial V$. Using again (2), we have $\psi(x) = 1$ and, since ψ is a convex function, ψ attains a global minimum at x. Therefore $1 = \psi(x) \leq \psi(x_0) < 1$, a contradiction.

(5) We have that 1 is a regular value of ψ by virtue of (4). This shows that ∂V, being the level set $\{x \in X \,:\, \psi(x) = 1\}$ of ψ, is a one-codimensional submanifold of class C^1. It is well known that, in this case, the tangent space $T_x\partial V$ of ∂V at x is the orthogonal complement of $\nabla\psi(x)$. Finally, let us see that for every $x \in \partial V$, $\nabla\psi(x)$ is outwardly normal to V at the point x. Indeed, assume that there is some $\lambda > 0$ with $z + \lambda\nabla\psi(x) \in V$. The convexity of ψ implies that

$$\psi(x + \lambda\nabla\psi(x)) - \psi(x) \geq \langle \nabla\psi(x), x + \lambda\nabla\psi(x) - x\rangle = \lambda\|\nabla\psi(x)\|^2 > 0,$$

that is $\psi(x + \lambda\nabla\psi(x)) > \psi(x) = 1$, which is absurd.

(6) Because X is a Hilbert space, given $x \in X \setminus V$ we can find a unique $x_V \in \partial V$ such that $\|x - x_V\| = d(x, V)$. In order to see that $x - x_V$ is orthogonal to $T_{x_V}\partial V$, let us define the function $f(p) = \|x - p\|^2$ for all $p \in X$. It is obvious that f is of class $C^\infty(X)$ with $f(p) \geq f(x_V)$ for all $p \in C$. Let $u \in T_{x_C}\partial C$

and $\alpha : (-r, r) \to \partial V$ be a curve in ∂V such that $\alpha(0) = x_V$ and $\alpha'(0) = u$. The function $g = f \circ \alpha : (-r, r) \to \mathbb{R}$ is of class C^∞ and $g(0) \leq g(t)$ for every $t \in (-r, r)$. Hence

$$0 = g'(0) = \langle \nabla f(\alpha(0)), \alpha'(0) \rangle = \langle \nabla f(x_V), u \rangle = 2\langle x - x_V, u \rangle.$$

Since u is arbitrary in $T_{x_V}\partial V$, the above proves that $x - x_V \in [T_{x_V}\partial V]^\perp$. Besides, it is clear that $x_V + \lambda(x - x_V)$ is outside V for every $\lambda > 0$. We have thus shown that $x - x_V$ is outwardly normal to ∂V at x. Finally, since $T_{x_V}\partial V$ is a one-codimensional subspace of X, the vectors $x - x_V$ and $\nabla\psi(x_V)$ are necessarily paralell and because both $x - x_V$ and $\nabla\psi(x_V)$ are outwardly normal, we must have

$$n_V(x_V) = \frac{\nabla\psi(x_V)}{\|\nabla\psi(x_V)\|} = \frac{x - x_V}{\|x - x_V\|}.$$

(7) If $z \in \partial V$ and $x \in X$ is such that $\langle n_V(z), x - z \rangle > 0$, using (2) and (6) we obtain

$$\psi(x) \geq \psi(z) + \langle \nabla\psi(z), x - z \rangle = 1 + \|\nabla\psi(z)\| \langle n_V(z), x - z \rangle > 1.$$

This shows that $x \in X \setminus V$. $\square$

2.4.1 The oriented distance function to convex subsets

We will make intensive use of the oriented distance function associated to convex subsets in the proof of our interpolation result.

Definition 2.17. *Given a subset A of X with nonempty boundary, the oriented distance of A is the function $b_A : X \to \mathbb{R}$ defined by*

$$b_A(x) = \begin{cases} d(x, A) & \text{if} \quad x \in X \setminus A \\ 0 & \text{if} \quad x \in \partial A \\ -d(x, \partial A) & \text{if} \quad x \in \operatorname{int}(A). \end{cases}$$

Let us gather some properties concercing the convexity and smoothness of the oriented distance function to convex bodies of class C^1 or $C^{1,1}$ as well as its relation with the outer unit normal. Most of the statements of the following lemma are consequences of the results by M. C. Delfour and J. P. Zolesio [25] in the case when $X = \mathbb{R}^n$. However, we are going to present a detailed proof of these properties because we have not been able to find any reference for the infinite dimensional setting and the proofs in [25] cannot be easily adapted. For the sake of simplicity we will write d_A instead of $d(\cdot, A)$ to refer the distance to any closed subset A of X. For any point $x \in X$ and any closed subset A, we will say that the distances $d_A(x)$ or $b_A(x)$ are attained at x if the infimum/supremum defining d_A or b_A is attained. Finally, for every $x \in X$ and every closed subset A such that the distance $d_A(x)$ is attained at a unique point of A, we will denote by $P_A(x)$ the mentioned point.

Lemma 2.18. *Let V a closed convex subset of X. The following properties are satisfied for b_V.*

(1) *b_V is 1-Lipschitz on X.*

(2) *For every $x \in X$, the distance $d_V(x)$ is attained at a unique point $P_V(x) \in V$, with $P_V(x) = P_{\partial V}(x)$ whenever $x \in X \setminus V$, the mapping $X \ni x \to P_V(x) \in V$ is 1-Lipschitz and b_V is differentiable at every $x \in X \setminus V$, with $\nabla b_V(x) = \nabla d_V(x) = d_V(x)^{-1}(x - P_{\partial V}(x))$. Moreover, for every $r > 0$, the function ∇b_V is Lipschitz on $V_r^+ := \{x \in X \ : \ d(x, V) \geq r\}$ and $\operatorname{Lip}(\nabla b_V, V_r^+) \leq 3r^{-1}$.*

(3) *The mapping $d_V^2 : X \to \mathbb{R}$ is of class $C^{1,1}(X)$, with $\nabla d_V^2(x) = 2(x - P_{\partial V}(x))$ for every $x \in X \setminus V$, $\nabla d_V^2(x) = 0$ for every $x \in V$ and $\operatorname{Lip}(\nabla d_V^2) \leq 4$.*

If we further assume that V is a convex body of class $C^{1,1}$ and n_V is as in Definition 2.15 then we have

(4) *If* $x \in \operatorname{int}(V)$ *is such that* $d_{\partial V}(x) < \operatorname{Lip}(n_V)^{-1}$, *the distance* $d_{\partial V}(x)$ *is attained.*

(5) *For every* $x \in X$ *and every* $y \in \partial V$ *such that* $\|x-y\| = d_{\partial V}(x)$ *we have that* $x-y = b_V(x) n_V(y)$. *Conversely, if* $y \in \partial V$ *is such that* $x-y = b_V(x)n_V(y)$, *then* $\|x-y\| = d_{\partial V}(x)$.

(6) *If* $x \in X$ *is such that* $d_{\partial V}(x) < \operatorname{Lip}(n_V)^{-1}$, *then the distance* $d_{\partial V}(x)$ *is attained at a unique point, which we denote by* $P_{\partial V}(x)$. *Moreover, the mapping*

$$U := \{z \in X \, : \, d_{\partial V}(z) < \operatorname{Lip}(n_V)^{-1}\} \ni x \mapsto P_{\partial V}(x)$$

is 3*-Lipschitz.*

(7) b_V *is differentiable on* U *and* $\nabla b_V = n_V \circ P_{\partial V}$ *on* U. *In particular* $\nabla b_V = n_V$ *on* ∂V. *In addition,* ∇b_V *is* $3\operatorname{Lip}(n_V)$*-Lipschitz on* U.

(8) *On the set* $U^+ := b_V^{-1}\left((-\operatorname{Lip}(n_V)^{-1}, +\infty) \right)$, *the function* b_V *is differentiable, with* $\nabla b_V(x) = \frac{x-P_{\partial V}(x)}{d_V(x)} = n_V(P_{\partial V}(x))$ *for every* $x \in U^+$, *and* ∇b_V *is Lipschitz on* U^+, *with* $\operatorname{Lip}(\nabla b_V, U^+) \leq 3\operatorname{Lip}(n_V)$.

(9) b_V *is convex on* X.

Proof.

(1) The distance function to any closed subset is 1-Lipschitz. Hence b_V is 1-Lipschitz by definition.

(2) By the convexity of V and the strict convexity of the norm $\|\cdot\|$, for every $x \in X \setminus V$, the distance $b_V(x) = d_V(x)$ is attained at a unique point $P_{\partial V}(x) \in \partial V$ and the mapping $X \setminus V \ni x \to P_{\partial V}(x) \in \partial V$ is 1-Lipschitz. Moreover we know that d_V is differentiable at every $x \in X \setminus V$ with $\nabla b_V(x) = \nabla d_V(x) = \frac{x-P_{\partial V}(x)}{d_V(x)}$; see [30, pg. 56-57] for instance. Moreover, given $r>0$ and two points $x, y \in V_r^+$ we can write, using that d_V and $P_{\partial V}$ are 1-Lipschitz on X,

$$\begin{aligned}
&\|\nabla b_V(x) - \nabla b_V(y)\| \\
&\quad = \left\| \frac{x-P_{\partial V}(x)}{d_V(x)} - \frac{y-P_{\partial V}(y)}{d_V(y)} \right\| = \frac{\|(x-P_{\partial V}(x))d_V(y) - (y-P_{\partial V}(y))d_V(x)\|}{d_V(x)d_V(y)} \\
&\quad \leq \frac{\|(x-y) + (P_{\partial V}(y) - P_{\partial V}(x))\|}{d_V(x)} + \frac{\|(d_V(y)-d_V(x))(y-P_{\partial V}(y))\|}{d_V(x)d_V(y)} \\
&\quad \leq \frac{2\|x-y\|}{d_V(x)} + \frac{\|d_V(y)-d_V(x)\|}{d_V(x)} \leq \frac{3\|x-y\|}{d_V(x)} \leq 3r^{-1}\|x-y\|.
\end{aligned}$$

(3) We immediately obtain from (2) that d_V^2 is differentiable on $X \setminus V$ with $\nabla d_V^2(x) = 2(x - P_{\partial V}(x))$ for every $x \in X \setminus V$. In order to see that d_V^2 is differentiable on V, let $x_0 \in V$ and write

$$\lim_{x \to x_0} \frac{d_V^2(x)}{\|x-x_0\|} = \lim_{x \to x_0} \frac{\|x - P_V(x)\|^2}{\|x-x_0\|} \leq \lim_{x \to x_0} \frac{\|x-x_0\|^2}{\|x-x_0\|} = 0,$$

which shows that $\nabla b_V^2(x_0) = 0$. Finally, for every $x, y \in X$,

$$\|\nabla b_V^2(x) - \nabla b_V^2(y)\| = 2\|(x-y) + (P_V(y) - P_V(x))\| \leq 4\|x-y\|,$$

because P_V is 1-Lipschitz. Therefore b_V^2 is of class $C^{1,1}(X)$ and $\operatorname{Lip}(b_V^2) \leq 4$.

(4) Given $x \in \operatorname{int}(V)$ and $r := d_{\partial V}(x) < \operatorname{Lip}(n_V)^{-1}$, we can find a sequence $(z_n)_n$ in ∂V such that $\lim_n \|z_n - x_0\| = r$. If we set $x_n := z_n - r n_V(z_n)$, we claim that $(x_n)_n$ converges to x_0. Indeed, if n is large enough so that $r > \frac{1}{n}$, the point $x_0 + (r - \frac{1}{n}) n_V(z_n)$ belongs to the interior of the ball $B(x_0, r)$ and then $x_0 + (r-\frac{1}{n})n_V(z_n) \in V$. Hence, by Proposition 2.16 (7), we have that

$$\left\langle n_V(z_n), x_0 + (r - \tfrac{1}{n}) n_V(z_n) - z_n \right\rangle \leq 0.$$

This allows us to write

$$\begin{aligned}\|x_n - x_0\|^2 &= \|z_n - x_0 - rn_V(z_n)\|^2 = \|z_n - z_0\|^2 + r^2\|n_V(z_n)\|^2 + 2r\big\langle n_V(z_n), x_0 - z_n\big\rangle \\ &= \|z_n - z_0\|^2 + r^2 + 2r\big\langle n_V(z_n), x_0 + (r - \tfrac{1}{n})n_V(z_n) - z_n\big\rangle - 2r\big\langle n_V(z_n), (r - \tfrac{1}{n})n_V(z_n)\big\rangle \\ &\leq \|z_n - z_0\|^2 + r^2 - 2r\big\langle n_V(z_n), (r - \tfrac{1}{n})n_V(z_n)\big\rangle = \|z_n - z_0\|^2 + r^2 - 2r(r - \tfrac{1}{n}).\end{aligned}$$

The last term tends to $r^2 + r^2 - 2r^2 = 0$ as $n \to \infty$. This shows that $\lim_n \|x_n - x_0\| = 0$. Now, since n_V is Lipschitz we can write, for every $n, m \in \mathbb{N}$,

$$\|z_n - z_m\| = \|x_n + rn_V(z_n) - x_m + rn_V(z_m)\| \leq \|x_n - x_m\| + r\operatorname{Lip}(n_V)\|z_n - z_m\|.$$

This leads us to

$$(1 - r\operatorname{Lip}(n_V))\|z_n - z_m\| \leq \|x_n - x_m\|, \quad n, m \in \mathbb{N},$$

which shows that $(z_n)_n$ is a Cauchy sequence because so is $(x_n)_n$ and $r < \operatorname{Lip}(n_V)$. Thus there exists some $z_0 \in \partial V$ with $d_{\partial V}(x) = \lim_n \|z_n - x\| = \|z_0 - x\|$.

(5) Let ψ be as in Definition 2.15. If $x \in X \setminus V$, we know that the distance is attained at a unique point $P_{\partial V}(x)$ by (2). According to Proposition 2.16, the vectors $\nabla\psi(P_{\partial V}(x))$ and $x - P_{\partial V}(x)$ are paralell and outwardly normal to ∂V at $P_{\partial V}(x)$ and

$$n_V(P_{\partial V}(x)) = \frac{\nabla\psi(P_{\partial V}(x))}{\|\nabla\psi(P_{\partial V}(x))\|} = \frac{x - P_{\partial V}(x)}{\|x - P_{\partial V}(x)\|} = \frac{x - P_{\partial V}(x)}{b_V(x)},$$

which proves the assertion for points $x \in X \setminus V$. Now, consider points $x \in \operatorname{int}(V)$ and $y \in \partial V$ such that $\|x - y\| = d_{\partial V}(x)$. The function $h : X \to \mathbb{R}$ defined by $h(p) = \|p - x\|^2$, for every $p \in X$, is of class C^∞. Given any u in the tangent space $T_y\partial V$ of ∂V at y we consider a curve $\alpha : (-\varepsilon, \varepsilon) \to \partial V$ of class C^1 such that $\alpha(0) = y$ and $\alpha'(0) = u$ and set $g = h \circ \alpha$, which is C^1 on $(-\varepsilon, \varepsilon)$. Since $g(0) = \|x - y\| = d_{\partial V}(x)$, we must have $g(t) \geq g(0)$ for every $t \in (-\varepsilon, \varepsilon)$ and then

$$0 = g'(0) = \langle \nabla h(\alpha(0)), \alpha'(0)\rangle = \langle \nabla h(y), v\rangle = 2\langle y - x, u\rangle.$$

Because u is arbitrary on $T_y\partial V$, this shows that $y - x$ belongs to the orthogonal complement of $T_y\partial V$, which coincides with the line generated by $n_V(y)$ by Proposition 2.16. Hence $y - x$ is paralell to $n_V(y)$ and then $y - x = \beta n_V(y)$ for some β. It is obvious that $y - x$ is inward to ∂V while $n_V(y)$ is outward, and then it is clear that $\beta < 0$ and $\beta = -\|x - y\| = -d_{\partial V}(y) = b_V(y)$. Therefore $x - y = b_V(x)n_V(y)$. Conversely, if $y \in \partial V$ is such that $x - y = b_V(x)n_V(y)$, it is obvious that $\|x - y\| = d_{\partial V}(x)$ because $\|n_V(y)\| = 1$.

(6) Let x be a point with $d_{\partial V}(x) < \operatorname{Lip}(n_V)^{-1}$, or equivalently $|b_V(x)| < \operatorname{Lip}(n_V)^{-1}$. We know from (4) that the distance $d_{\partial V}(x)$ is attained at some $y \in \partial V$. Assume that there are different points $y_1, y_2 \in \partial V$ such that $d_{\partial V}(x) = \|x - y_1\| = \|x - y_2\|$. It then follows from (5) that

$$x - y_1 = b_V(x)n_V(y_1), \quad x - y_2 = b_V(x)n_V(y_2)$$

and substracting both equations we get $y_1 - y_2 = b_V(x)\left(n_V(y_2) - n_V(y_1)\right)$. This implies that

$$\|y_1 - y_2\| \leq |b_V(x)|\|n_V(y_2) - n_V(y_1)\| \leq |b_V(x)|\operatorname{Lip}(n_V)\|y_1 - y_2\| < \|y_1 - y_2\|,$$

a contradiction. Therefore, the point y is the unique y for which we have $\|x - y\| = d_{\partial V}(x)$. In order to see that $P_{\partial V}$ is Lipschitz on $U := \{z \in X \ : \ d_{\partial V}(z) < \operatorname{Lip}(n_V)^{-1}\}$, let z_1 and z_2 be two points in U, and write $P_{\partial V}(z_i) = z_i - b_V(z_i)n_V(P_{\partial V}(z_i))$ for $i = 1, 2$. It follows that

$$\begin{aligned}&\|P_{\partial V}(z_1) - P_{\partial V}(z_2)\| \\ &\qquad\leq \|z_1 - z_2\| + |b_V(z_1) - b_V(z_2)|\|n_V(P_{\partial V}(z_1))\| + |b_V(z_2)|\|n_V(P_{\partial V}(z_1)) - n_V(P_{\partial V}(z_2))\| \\ &\qquad\leq \|z_1 - z_2\| + \|z_1 - z_2\| + \operatorname{Lip}(n_V)^{-1}\operatorname{Lip}(n_V)\|z_1 - z_2\| = 3\|z_1 - z_2\|.\end{aligned}$$

(7) By (6) the distance $d_{\partial V}(x)$ is attained at a unique point and then $d_{\partial V}(x)$ is differentiable at every $x \in U \setminus \partial V$ and $\nabla d_{\partial V}(x) = \frac{x - P_{\partial V}(x)}{d_{\partial V}(x)}$, see [30, pg. 56-57]. That is, b_V is differentiable on $U \setminus \partial V$ and $\nabla b_V(x) = \frac{x-P_{\partial V}(x)}{b_V(x)} = n_V(P_{\partial V}(x))$ for every $x \in U \setminus \partial V$. In addition,

$$\|\nabla b_V(x) - \nabla b_V(y)\| \leq \operatorname{Lip}(n_V) \operatorname{Lip}(P_{\partial V}) \|x-y\| \leq 3 \operatorname{Lip}(n_V)\|x-y\|$$

for every $x, y \in U \setminus \partial V$. Now assume that $x \in \partial V$ and let $\varepsilon > 0$. Because $\|n_V(x)\| = 1$ and the norm $\|\cdot\|$ on X is differentiable at $n_V(x)$ with gradient equal to $n_V(x)$, there exists $\delta > 0$ such that

$$\|y + n_V(x)\| - 1 - \langle n_V(x), y\rangle \leq \frac{\varepsilon}{2}\|y\|, \quad \text{whenever} \quad \|y\| \leq \delta. \tag{2.4.1}$$

Let us assume that

$$0 < \|h\| < \min\{\delta, \delta \operatorname{Lip}(n_V)^{-1}, \varepsilon\delta^2 \operatorname{Lip}(n_V \circ P_{\partial V}, U)^{-1}\}. \tag{2.4.2}$$

We claim that $x - tn_V(x) \in \operatorname{int}(V)$ whenever $0 < t \leq \operatorname{Lip}(n_V)^{-1}$. Indeed, otherwise we would have that $x - tn_V(x) \in \partial V$ for some $t \in (0, \operatorname{Lip}(n_V)^{-1}]$ and then, by convexity and differentiability of ψ, (this function ψ is as in Definition 2.15)

$$\begin{aligned}
0 &= \psi(x) - \psi(x - tn_V(x)) \\
&\geq \big\langle \nabla\psi(x - tn_V(x)), tn_V(x)\big\rangle = t\|\nabla\psi(x - tn_V(x))\| \big\langle n_V(x - tn_V(x)), n_V(x)\big\rangle \\
&= \frac{t}{2}\|\nabla\psi(x - tn_V(x))\| \left(\|n_V(x - tn_V(x))\|^2 + \|n_V(x)\|^2 - \|n_V(x - tn_V(x)) - n_V(x)\|^2\right) \\
&\geq \frac{t}{2}\|\nabla\psi(x - tn_V(x))\|(2 - \operatorname{Lip}^2(n_V)t^2) \geq \frac{t}{2}\|\nabla\psi(x - tn_V(x))\|.
\end{aligned}$$

Since $x - tn_V(x) \in \partial V$, we have that $t\|\nabla\psi(x - tn_V(x))\| > 0$, and the above chain of inequalities yields a contradiction. Hence, if we set $t = \delta^{-1}\|h\|$, it follows that $t < \operatorname{Lip}(n_V)^{-1}$ by (2.4.2) and then $x - tn_V(x) \in \operatorname{int}(V)$ by the above claim. Moreover, b_V is differentiable on the line segment $(x, x - tn_V(x)]$ with derivative equal to $n_V \circ P_{\partial V}$, a Lipschitz function. Thus we can write

$$\begin{aligned}
b_V(x - tn_V(x)) - b_V(x) + t &= \int_0^1 \big\langle \nabla b_V(x - st\, n_V(x)) - n_V(x), -tn_V(x)\big\rangle ds \\
&\leq \int_0^1 \big\langle \left(n_V \circ P_{\partial V}\right)(x - st\, n_V(x)) - n_V(x), -tn_V(x)\big\rangle ds \leq \tfrac{1}{2}\operatorname{Lip}(n_V \circ P_{\partial V}, U)t^2.
\end{aligned}$$

Using (2.4.2), the above shows that

$$b_V(x - tn_V(x)) + t \leq \tfrac{1}{2}\operatorname{Lip}(n_V \circ P_{\partial V})t^2 = \tfrac{1}{2}\operatorname{Lip}(n_V \circ P_{\partial V})\delta^{-2}\|h\|^2 \leq \frac{\varepsilon}{2}\|h\|. \tag{2.4.3}$$

The facts that b_V is 1-Lipschitz and $\|t^{-1}h\| = \delta$, together with (2.4.3) and (2.4.1) allow us to write

$$\begin{aligned}
& b_V(x+h) - b_V(x) - \langle n_V(x), h\rangle \\
&= b_V(x+h) - b_V(x - tn_V(x)) + b_V(x - tn_V(x)) - b_V(x) - \langle n_V(x), h\rangle \\
&\leq \|h + tn_V(x)\| - t - \langle n_V(x), h\rangle + \frac{\varepsilon}{2}\|h\| = t\left(\|t^{-1}h + n_V(x)\| - 1 - \langle n_V(x), t^{-1}h\rangle\right) + \frac{\varepsilon}{2}\|h\| \\
&\leq \frac{\varepsilon t}{2}\|t^{-1}h\| + \frac{\varepsilon}{2}\|h\| = \varepsilon\|h\|.
\end{aligned}$$

On the other hand, because $x + n_V(x)$ is outside V, we have $b_V(x + n_V(x)) = d_V(x + n_V(x)) = 1$. Since $\|h\| \leq \delta$ by virtue of (2.4.2), we obtain, using that b_V is 1-Lipschitz together with (2.4.1), that

$$\begin{aligned}
& b_V(x+h) - b_V(x) - \langle n_V(x), h\rangle \\
&= b_V(x+h) - b_V(x + n_V(x)) + b_V(x + n_V(x)) - b_V(x) - \langle n_V(x), h\rangle \\
&\geq -\|h - n_V(x)\| + 1 - \langle n_V(x), h\rangle = -\left(\|-h + n_V(x)\| - 1 - \langle n_V(x), -h\rangle\right) \geq -\frac{\varepsilon}{2}\|h\|.
\end{aligned}$$

In conclusion

$$\frac{|b_V(x+h)-b_V(x)-\langle n_V(x),h\rangle|}{\|h\|} \leq \varepsilon \quad \text{whenever} \quad \|h\| \quad \text{satisfies} \quad (2.4.2),$$

that is, b_V is differentiable at x with $\nabla b_V(x) = n_V(x)$. We thus have the formula $\nabla b_V = n_V \circ P_{\partial V}$ on U, which proves that ∇b_V is Lipschitz on U.

(8) Combining (2) and (7), we obtain that b_V is differentiable on U^+ with $\nabla b_V = n_V \circ P_{\partial V}$ on U^+ and $\operatorname{Lip}(\nabla b_V, U^+) \leq 3 \operatorname{Lip}(n_V)$.

(9) Outside V we have that $b_V = d_V$, and d_V is convex on X. Indeed, given $x, y \in X \setminus V$, $\lambda \in [0,1]$ and $z_\lambda = \lambda x + (1-\lambda)y$, the point $\lambda P_V(x) + (1-\lambda)P_V(y)$ belongs to V by convexity and then

$$\begin{aligned} d_V(z_\lambda) &\leq \|z - (\lambda P_V(x) + (1-\lambda)P_V(y))\| \\ &\leq \lambda\|x - P_V(x)\| + (1-\lambda)\|y - P_V(y)\| \leq \lambda d_V(x) + (1-\lambda)d_V(y). \end{aligned}$$

Hence b_V is convex on any line segment contained in $X \setminus \operatorname{int} V$. Let us now see that b_V is convex on $\operatorname{int}(V)$. If $[x,y]$ is a line segment contained in $\operatorname{int}(V)$ and

$$z_\lambda := (1-\lambda)x + \lambda y, \quad \lambda \in [0,1],$$

is a point of $[x,y]$, for every $\varepsilon > 0$ we can find a point $p_\lambda \in \partial V$ such that

$$\|z_\lambda - p_\lambda\| \leq d(z_\lambda, \partial V) + \varepsilon = -b_V(z_\lambda) + \varepsilon.$$

Let W_λ denote the tangent hyperplane to ∂V at p_λ; since V is convex we have that $W_\lambda \cap \operatorname{int} V = \emptyset$. Then, if p_x and p_y denote the orthogonal projections of x and y onto W_λ, we have $p_x, p_y \in X \setminus V$, and therefore

$$d(x, \partial V) \leq \|x - p_x\| \quad \text{and} \quad d(y, \partial V) \leq \|y - p_y\|.$$

On the other hand, the function

$$[0,1] \ni t \mapsto d\left((1-t)x + ty, W_\lambda\right),$$

being the orthogonal projection onto an affine hyperplane, is obviously affine, so we have

$$\begin{aligned} -b_V(z_\lambda) + \varepsilon &\geq \|z_\lambda - p_\lambda\| \geq d(z_\lambda, W_\lambda) = (1-\lambda)d(x, W_\lambda) + \lambda d(y, W_\lambda) \\ &= (1-\lambda)\|x - p_x\| + \lambda\|y - p_y\| \geq (1-\lambda)d(x, \partial V) + \lambda d(y, \partial V) = -(1-\lambda)b_V(x) - \lambda b_V(y), \end{aligned}$$

that is to say,

$$b_V\left((1-\lambda)x + \lambda y\right) = b_V(z_\lambda) \leq (1-\lambda)b_V(x) + \lambda b_V(y) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, the above shows that b_V is convex on $\operatorname{int} V$, and by continuity it follows that b_V is convex on V. Finally, if $x \in X \setminus V$ and $y \in \operatorname{int} V$, hence the line segment $[x,y]$ is transversal to ∂V, we may write $[x,y] = [x,z] \cup [z,y]$, where $z \in \partial V$, $[x,z] \subset X \setminus \operatorname{int} V$ and $[z,y] \subset V$. Consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(t) = b_V\left((1-t)x + ty\right)$, and let $t_0 \in (0,1)$ be the number such that $z = (1-t_0)x + t_0 y$. We know that φ is convex on $[0, t_0]$, and φ is convex on $[t_0, 1]$ as well. Besides φ is differentiable at t_0 because b_V is differentiable on a neighbourhood of ∂V by (7). Hence φ is convex on $[0,1]$. It follows that b_V is convex on $[x,y]$. Therefore b_V is convex on X. □

2.4.2 An interpolation theorem for $C^{1,1}$ convex hypersurfaces

We are now ready to formulate and prove the announced characterization of those subsets which can be interpolated by boundaries of $C^{1,1}$ convex bodies, see Definition 2.15.

Definition 2.19. *Given a subset C of X and a mapping $N : C \to S_X$, we will say that N satisfies condition $(\mathcal{KW}^{1,1})$ on C provided that*

$$\langle N(y), y-x\rangle \geq \delta \|N(y)-N(x)\|^2 \quad \text{for all} \quad x, y \in C, \tag{$\mathcal{KW}^{1,1}$}$$

for some $\delta > 0$.

If V is a convex body of class $C^{1,1}$ and $u \in S_X$, we will say that u is outwardly normal to ∂V at $y \in \partial V$ if u coincides with the outer unit normal n_V of V.

Theorem 2.20. *Let C be a subset of X, and let $N : C \to S_X$ be a mapping. Then the following statements are equivalent.*

1. *There exists a $C^{1,1}$ convex body V such that $C \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in C$.*
2. *N satisfies condition $(\mathcal{KW}^{1,1})$ on C for some $\delta > 0$.*

Moreover, if we further assume that C is bounded then V can be taken bounded as well.

Proof.
(2) $\implies$ (1): Let us assume that $N : C \to S_X$ satisfies $(\mathcal{KW}^{1,1})$ on C. Consider the sets

$$E_0 = \{x = z - 2\delta N(z) \, : \, z \in C\}, \quad E = C \cup E_0.$$

We claim that E_0 and C are disjoint. Indeed, if there exists $z \in C$ such that $z' = z - 2\delta N(z) \in C$, then $z' - z = -2\delta N(z)$ and condition $(\mathcal{KW}^{1,1})$ leads us to

$$-2\delta\langle N(z'), N(z)\rangle = \langle N(z'), z'-z\rangle \geq 2\delta\|N(z)-N(z')\|^2 = -2\delta\langle N(z'), N(z)\rangle + 2\delta,$$

a contradiction. Assuming $\delta < 1$ (which we can clearly do) let us define a 1-jet (f, G) on C by

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in C \\ 1-\delta & \text{if} \quad y \in E_0, \end{cases} \quad \text{and} \quad G(y) = \begin{cases} N(y) & \text{if} \quad y \in C \\ 0 & \text{if} \quad y \in E_0. \end{cases}$$

Let us check that (f, G) satisfies condition $(CW^{1,1})$ on E. Given $x, y \in C$ we have

$$f(x) - f(y) - \langle G(y), x-y\rangle - \delta\|G(x)-G(y)\|^2 = \langle N(y), y-x\rangle - \delta\|N(x)-N(y)\|^2,$$

and the above term is nonnegative thanks to condition $(\mathcal{KW}^{1,1})$. If $x, y \in E_0$, then

$$f(x) - f(y) - \langle G(y), x-y\rangle - \delta\|G(x)-G(y)\|^2 = 0.$$

In the case when $x \in C$ and $y \in E_0$ we have

$$f(x) - f(y) - \langle G(y), x-y\rangle - \delta\|G(x)-G(y)\|^2 = \delta - \delta\|N(x)\|^2 = 0.$$

If both x, y belong to E_0, we can write $x = z - 2\delta N(z)$ for some $z \in C$ and, by condition $(\mathcal{KW}^{1,1})$,

$$\begin{aligned} f(x) - f(y) - \langle G(y), x-y\rangle - \delta\|G(x)-G(y)\|^2 &= -2\delta - \langle N(y), x-y\rangle \\ = -2\delta - \langle N(y), z - 2\delta N(z) - y\rangle &\geq -2\delta + 2\delta\langle N(y), N(z)\rangle + \delta\|N(z)-N(y)\|^2 = 0. \end{aligned}$$

Thus (f, G) satisfies condition $(CW^{1,1})$ on $E = C \cup E_0$ and we can use Theorem 2.11 in order to find a convex function $F \in C^{1,1}(X)$ such that $(F, \nabla F) = (f, G)$ on E. Observe that $F \geq 1-\delta$ on X by convexity. For the set $A = \overline{\mathrm{co}}(E)$, the closed convex hull of E, let us define

$$\psi = F + d(\cdot, A)^2 \quad \text{on} \quad X. \tag{2.4.4}$$

Since A is closed and convex, $d(\cdot, A)^2$ is a $C^{1,1}$ convex function on X by virtue of Lemma 2.18 (3). Thus, the function ψ is convex and of class $C^{1,1}$ with $\psi = F$ and $\nabla\psi = \nabla F$ on A. Let us define $V = \psi^{-1}((-\infty, 1])$. Because $\psi = F = 1 - \delta < 1$ on E_0, Proposition 2.16 says that V is closed and convex, with $E_0 \subset \operatorname{int}(V)$ and $\partial V = \psi^{-1}(1)$. In particular, V has nonempty interior. Also, we have $C \subset \partial V$ because $\psi = F = 1$ on C. In order to prove that V is a $C^{1,1}$ convex body it only remains to check that the outer unit normal n_V of V is Lipschitz on ∂V. To this purpose, we need to see first that $\inf_{\partial V} \|\nabla\psi\| > 0$. For every $x \in X$, we can find $z \in \operatorname{co}(E_0 \cup C)$ such that $2d(x, A) \geq \|x - z\|$. We can write $z = \sum_{i=1}^m \lambda_i z_i + \sum_{i=m+1}^n \lambda_i z_i$, where $z_1, \ldots, z_m \in C$, $z_{m+1}, \ldots, z_n \in E_0$, and $\lambda_1, \ldots, \lambda_n \geq 0$, $\sum_{i=1}^n \lambda_i = 1$. And for each z_i, $i = 1, \ldots, m$, we can take $y_i := z_i - 2\delta N(z_i) \in E_0$ satisfying $\|y_i - z_i\| \leq 2\delta$. The point $y := \sum_{i=1}^m \lambda_i y_i + \sum_{i=m+1}^n \lambda_i z_i$ belongs to the convex hull $\operatorname{co}(E_0)$ of E_0 and then $\psi(y) = F(y) \leq 1 - \delta$ by convexity of F (recall that $F = 1 - \delta$ on E_0). We thus have

$$\|z - y\| = \left\| \sum_{i=1}^m \lambda_i (z_i - y_i) \right\| \leq 2\delta \sum_{i=1}^m \lambda_i \leq 2\delta \sum_{i=1}^n \lambda_i = 2\delta.$$

This proves that $\|x - y\| \leq \|x - z\| + \|z - y\| \leq 2d(x, A) + 2\delta$. Now, if $x \in \partial V$, we have that $\psi(x) = F(x) + d(x, A)^2 = 1$ and then $d(x, A) \leq 1$ because $F \geq 0$ on X. If y is as above then $\psi(y) = F(y) \leq 1 - \delta$ and $\|x - y\| \leq 2 + 2\delta$. The convexity of F yields

$$\|\nabla\psi(x)\| \|y - x\| \geq \langle \nabla\psi(x), x - y \rangle \geq \psi(x) - \psi(y) = \delta,$$

which implies $\|\nabla\psi(x)\| \geq \frac{\delta}{2+2\delta}$. The outer unit normal n_V of ∂V is given by

$$n_V(x) = \frac{\nabla\psi(x)}{\|\nabla\psi(x)\|}, \quad x \in \partial V,$$

and the fact that $\inf_{\partial V} \|\nabla\psi\| > 0$ allows us to prove that n_V is Lipschitz on ∂V. Indeed, given $x, y \in \partial V$, we can write

$$\begin{aligned}
\|n_V(x) - n_V(y)\| &= \left\| \frac{\nabla\psi(x)}{\|\nabla\psi(x)\|} - \frac{\nabla\psi(y)}{\|\nabla\psi(y)\|} \right\| = \frac{\Big\| \|\nabla\psi(y)\| \nabla\psi(x) - \|\nabla\psi(x)\| \nabla\psi(y) \Big\|}{\|\nabla\psi(x)\| \|\nabla\psi(y)\|} \\
&= \frac{\Big\| \left(\|\nabla\psi(y)\| - \|\nabla\psi(x)\| \right) \nabla\psi(x) + \|\nabla\psi(x)\| \left(\nabla\psi(x) - \nabla\psi(y) \right) \Big\|}{\|\nabla\psi(x)\| \|\nabla\psi(y)\|} \\
&\leq \frac{\|\nabla\psi(x) - \nabla\psi(y)\|}{\|\nabla\psi(y)\|} + \frac{\|\nabla\psi(x) - \nabla\psi(y)\|}{\|\nabla\psi(y)\|} = \frac{2\|\nabla\psi(x) - \nabla\psi(y)\|}{\|\nabla\psi(y)\|}.
\end{aligned}$$

If m denotes the number $\inf_{\partial V} \|\nabla\psi\|$, the preceding chain of inequalities and the fact that $\nabla\psi$ is Lipschitz lead us to

$$\|n_V(x) - n_V(y)\| \leq \frac{2\|\nabla\psi(x) - \nabla\psi(y)\|}{\|\nabla\psi(y)\|} \leq 2m^{-1} \|\nabla\psi(x) - \nabla\psi(y)\| \leq 2m^{-1} \operatorname{Lip}(\nabla\psi) \|x - y\|$$

for every $x, y \in \partial V$. Finally, if $x \in C$, then $\nabla\psi(x) = \nabla F(x) = N(x)$, that is $N = n_V$ on C.

In addition, let us see that if C is bounded, the convex body V is also bounded. Indeed, the sets E_0 and $A = \overline{\operatorname{co}}(E_0 \cup C)$ are bounded because so is C and because $\|N\| = 1$. Thus the distance $d^2(\cdot, A)$ is coercive, that is, $\lim_{\|x\| \to \infty} d^2(x, A) = +\infty$. Since F is bounded below by $1 - \delta$ on X, the function ψ of (2.4.4) is coercive too and therefore $V = \psi^{-1}((-\infty, 1])$ is a bounded subset by virtue of Proposition 2.16 (3).

(1) $\implies$ (2): If there exists such a convex body V, the outer unit normal n_V of V is Lipschitz and we know from Lemma 2.18 that the oriented distance function b_V to V is convex, 1-Lipschitz on X and of class $C^{1,1}$ on the set

$$U^+ := \{x \in X \, : \, b_V(x) > -\operatorname{Lip}(n_V)^{-1}\}.$$

Let us denote $r = \operatorname{Lip}(n_V)^{-1}$, $\varepsilon = \frac{r}{4}$ and define $F = M_\varepsilon(b_V, -\frac{r}{2})$ on X, the ε-smooth maximum of b_V and $\frac{-r}{2}$, see Lemma 1.8 and Proposition 1.9. Since b_V is convex, F is convex on X as well. Observe that on the set $\{b_V \geq \frac{-r}{4}\}$ we have that $b_V \geq \varepsilon - \frac{r}{2}$, and the properties of smooth maxima give $F = b_V$ on $\{b_V \geq \frac{-r}{4}\}$, and then F is differentiable with $\nabla F = \nabla b_V$ on this set. In particular, the gradient ∇F of F is Lipschitz on this set, $F = b_V = 0$ and $\nabla F = \nabla b_V$ on ∂V. On the other hand, on the set $\{b_V \leq \frac{-3r}{4}\}$ we have $\frac{-r}{2} \geq b_V(x) + \varepsilon$, which implies that $F = \frac{-r}{2}$ on $\{b_V \leq \frac{-3r}{4}\}$; in particular, F is $C^{1,1}$ on this set. Finally, on $\{x \in X \ : \ \frac{-3r}{4} < b_V(x) < \frac{-r}{4}\}$, the definition of F is given by

$$F(x) = \frac{b_V(x) - \frac{r}{2} + \theta(b_V(x) + \frac{r}{2})}{2},$$

where b_V is differentiable with Lipschitz derivative and $\theta : \mathbb{R} \to (0, +\infty)$ is a C^∞ function, see the comments after Lemma 1.8. Using that b_V is 1-Lipschitz on X and that ∇b_V is Lipschitz on U^+ (which contains the region we are working on), we can write

$$\begin{aligned}
2\|\nabla F(x) - \nabla F(y)\| &= \|\nabla b_V(x) - \nabla b_V(y) + \theta'(b_V(x) + \tfrac{r}{2})\nabla b_V(x) - \theta'(b_V(y) + \tfrac{r}{2})\nabla b_V(y)\| \\
&\leq (1 + \theta'(b_V(y) + \tfrac{r}{2}))\operatorname{Lip}(\nabla b_V, U^+)\|x - y\| + |\theta'(b_V(x) + \tfrac{r}{2}) - \theta'(b_V(y) + \tfrac{r}{2})|\|\nabla b_V(x)\| \\
&\leq \left(1 + \sup_{[\frac{-r}{4}, \frac{r}{4}]} |\theta'|\right)\operatorname{Lip}(\nabla b_V, U^+)\|x - y\| + \left(\sup_{[\frac{-r}{4}, \frac{r}{4}]} |\theta''|\right)\|x - y\|.
\end{aligned}$$

This proves that F is $C^{1,1}$ on $\{x \in X \ : \ \frac{-3r}{4} < b_V(x) < \frac{-r}{4}\}$ too. In conclusion F is a convex function of class $C^{1,1}(X)$ such that $F = 0$ and $\nabla F = n_V$ on ∂V. According to Proposition 2.8, F satisfies the condition $(CW^{1,1})$ on ∂V for some δ, i.e.,

$$F(x) - F(y) - \langle \nabla F(y), x - y\rangle \geq \delta\|\nabla F(x) - \nabla F(y)\|^2 \quad \text{for all} \quad x, y \in \partial V.$$

and therefore

$$\langle n_V(y), y - x\rangle \geq \delta\|n_V(x) - n_V(y)\|^2 \quad \text{for all} \quad x, y \in \partial V.$$

In particular, since our given function $N : C \to S_X$ coincides with n_V on C we have that N satisfies $(\mathcal{KW}^{1,1})$ on C.

$\square$

2.5 Sup-inf explicit formulas of $C^{1,1}$ convex extensions on $\mathbb{R}^n$

In this section, we present an alternative sup-inf formula for the extension F given in Theorem 2.11 for $C^{1,1}_{\text{conv}}$ functions on $\mathbb{R}^n$. This formula is inspired by that of [54, Theorem 26] for $C^{1,1}$ (not necessarily convex) functions but, in our case, the formula will be simpler.

Throughout this section, we will denote the euclidean norm on $\mathbb{R}^n$ by $|\cdot|$. Let us assume that E is a nonempty subset of $\mathbb{R}^n$ and (f, G) is a 1-jet on E satisfying the condition $(CW^{1,1})$ (see Definition 2.7) with constant $M > 0$. For every $a, b \in E, \ x \in \mathbb{R}^n$, we define

$$\begin{aligned}
\alpha_{a,b} &:= M\Big(f(b) - f(a) - \langle G(a), b - a\rangle\Big) - \tfrac{1}{2}|G(a) - G(b)|^2, \\
\beta^x_{a,b} &:= \Big|\tfrac{1}{2}\Big(G(b) - G(a) + M(x - b)\Big)\Big|^2, \\
Z^x_{a,b} &:= \tfrac{1}{2}\Big(G(a) + G(b) + M(x - b)\Big), \\
r^x_{a,b} &:= \sqrt{\alpha_{a,b} + \beta^x_{a,b}}
\end{aligned}$$

and the set

$$\Lambda_x := \bigcap_{a,b \in E} B(Z^x_{a,b}, r^x_{a,b}),$$

where each $B(Z^x_{a,b}, r^x_{a,b})$ denotes the closed ball centered at $Z^x_{a,b}$ with radius $r^x_{a,b}$. Finally let us define the functions

$$\psi^+(x,a,v) = f(a) + \langle v, x-a\rangle - \tfrac{1}{2M}|G(a)-v|^2$$
$$\psi^-(x,a,v) = f(a) + \langle G(a), x-a\rangle + \tfrac{1}{2M}|G(a)-v|^2,$$

for all $x \in \mathbb{R}^n,\ a \in E,\ v \in \mathbb{R}^n$.

Remark 2.21.
(1) *For every* $x \in \mathbb{R}^n$, *we have* $v \in \Lambda_x$ *if and only if*

$$\sup_{a\in E} \psi^-(x,a,v) \leq \inf_{a\in E} \psi^+(x,a,v).$$

(2) *Given* $x_0 \in \mathbb{R}^n$, *the* 1*-jet* $(\widetilde{f}, \widetilde{G})$ *extends* (f,G) *from* E *to* $E\cup\{x_0\}$ *satisfying inequality* $(CW^{1,1})$ *on* $E\cup\{x_0\}$ *with constant* $M>0$ *if and only if*

$$\sup_{a\in E} \psi^-(x,a,\widetilde{G}(x)) \leq \widetilde{f}(x) \leq \inf_{a\in E} \psi^+(x,a,\widetilde{G}(x)), \quad x \in E\cup\{x_0\}.$$

Proof.
(1) We have that $\sup_{a\in E}\psi^-(x,a,v) \leq \inf_{a\in E}\psi^+(x,a,v)$ if and only if, for all $a,b\in E$,

$$f(a) + \langle G(a), x-a\rangle + \frac{1}{2M}|v-G(a)|^2 \leq f(b) - \langle v, b-x\rangle - \frac{1}{2M}|v-G(b)|^2.$$

Multiplying by M we have that

$$\frac{1}{2}\left(|v-G(a)|^2 + |v-G(b)|^2\right) + M\langle v, b-x\rangle \leq M(f(b)-f(a)) + M\langle G(a), a-x\rangle.$$

Applying the Paralelogram Law to the left-side term we obtain

$$\frac{1}{4}\left(|2v-G(a)-G(b)|^2 + |G(b)-G(a)|^2\right) + M\langle v, b-x\rangle \leq M(f(b)-f(a)) + M\langle G(a), a-x\rangle,$$

or equivalently

$$\left|v - \frac{G(a)+G(b)}{2}\right|^2 + M\langle v, b-x\rangle \leq M(f(b)-f(a)) + M\langle G(a), a-x\rangle - \frac{1}{4}|G(b)-G(a)|^2.$$

This can be written as

$$\begin{aligned}\left|v - \frac{G(a)+G(b)}{2}\right|^2 &- 2\Big\langle v - \frac{G(a)+G(b)}{2}, \frac{M}{2}(x-b)\Big\rangle + \frac{M^2}{4}|x-b|^2 \\ &\leq M(f(b)-f(a)) + M\langle G(a), a-x\rangle - \frac{1}{4}|G(b)-G(a)|^2 \\ &\quad + 2\Big\langle \frac{G(a)+G(b)}{2}, \frac{M}{2}(x-b)\Big\rangle + \frac{M^2}{4}|x-b|^2,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left|\left(v - \frac{G(a)+G(b)}{2}\right) - \frac{M}{2}(x-b)\right|^2 \\ &\quad\leq M\Big(f(b)-f(a)-\langle G(a), b-a\rangle\Big) - \frac{1}{2}|G(a)-G(b)|^2 + M\langle G(a), b-a\rangle \\ &\quad\quad + \Big\langle G(a)+G(b), \frac{M}{2}(x-b)\Big\rangle + \frac{M^2}{4}|x-b|^2 + M\langle G(a), a-x\rangle + \frac{1}{4}|G(a)-G(b)|^2.\end{aligned}$$

By the definition of $Z_{a,b}$ and $\alpha_{a,b}$ we obtain

$$
\begin{aligned}
&|v - Z_{a,b}|^2 \leq \alpha_{a,b} + M\langle G(a), b-a\rangle + \left\langle G(a)+G(b), \frac{M}{2}(x-b)\right\rangle \\
&+ \frac{M^2}{4}|x-b|^2 + M\langle G(a), a-x\rangle + \frac{1}{4}|G(a)-G(b)|^2 \\
&= \alpha_{a,b} + \left\langle G(b), \frac{M}{2}(x-b)\right\rangle - \left\langle G(a), \frac{M}{2}(x-b)\right\rangle + \frac{1}{4}|G(a)-G(b)|^2 + \frac{M^2}{4}|x-b|^2 \\
&= \alpha_{a,b} + \frac{1}{4}|G(a)-G(b)|^2 + 2\left\langle \frac{G(b)-G(a)}{2}, \frac{M}{2}(x-b)\right\rangle + \frac{M^2}{4}|x-b|^2 \\
&= \alpha_{a,b} + \left|\frac{1}{2}(G(b)-G(a)) + \frac{M}{2}(x-b)\right|^2 = \alpha_{a,b} + \beta_{a,b}.
\end{aligned}
$$

Obviously, the above is equivalent to $v \in B(Z^x_{a,b}, r^x_{a,b})$ for every $a, b \in E$, that is $v \in \Lambda_x$.

(2) The 1-jet $(\widetilde{f}, \widetilde{G})$ extends (f, G) from E to $E \cup \{x_0\}$ and satisfies inequality $(CW^{1,1})$ on $E \cup \{x_0\}$ with constant $M > 0$ if and only if

$$
\begin{aligned}
&\widetilde{f}(x) - f(a) - \langle G(a), x-a\rangle \geq \frac{1}{2M}|\widetilde{G}(x) - G(a)|^2 \quad \text{and} \\
&f(a) - \widetilde{f}(x) - \langle \widetilde{G}(x), a-x\rangle \geq \frac{1}{2M}|\widetilde{G}(x) - G(a)|^2 \quad \text{for all} \quad x \in E \cup \{x_0\},\ a \in E.
\end{aligned}
$$

Note that these inequalities are equivalent to

$$
\begin{aligned}
&\widetilde{f}(x) \geq f(a) + \langle G(a), x-a\rangle + \frac{1}{2M}|\widetilde{G}(x) - G(a)|^2 \quad \text{and} \\
&\widetilde{f}(x) \leq f(b) - \langle \widetilde{G}(x), b-x\rangle - \frac{1}{2M}|\widetilde{G}(x) - G(b)|^2 \quad \text{for all} \quad x \in E \cup \{x_0\},\ a, b \in E,
\end{aligned}
$$

which in turn is equivalent to

$$
\sup_{a\in E} \psi^-(x, a, \widetilde{G}(x)) \leq \widetilde{f}(x) \leq \inf_{a \in E} \psi^+(x, a, \widetilde{G}(x)), \quad x \in E \cup \{x_0\}.
$$

□

If F denotes the extension of the 1-jet (f, G) defined in Theorem 2.11, we know that $(F, \nabla F)$ satisfies condition $(CW^{1,1})$ on $\mathbb{R}^n$ with constant $M > 0$ (See Proposition 2.8). We obtain, by virtue of Remark 2.21, that

$$
\nabla F(x) \in \Lambda_x \quad \text{and} \quad \sup_{a\in E} \psi^-(x, a, \nabla F(x)) \leq F(x) \leq \inf_{a\in E} \psi^+(x, a, \nabla F(x)), \quad x \in \mathbb{R}^n. \tag{2.5.1}
$$

In particular, the set Λ_x is nonempty for every $x \in \mathbb{R}^n$. Now, since ψ^+ is continuous, the mapping

$$
\mathbb{R}^n \ni v \longmapsto \inf_{a\in E} \psi^+(x, a, v)
$$

is upper semicontinuous and therefore the function

$$
h(x) := \sup_{v \in \Lambda_x} \inf_{a\in E} \psi^+(x, a, v), \quad x \in \mathbb{R}^n
$$

is well defined.

Lemma 2.22. *For every $x \in \mathbb{R}^n$, there exists a unique $v_x \in \Lambda_x$ such that*

$$
h(x) = \inf_{a\in E} \psi^+(x, a, v_x).
$$

Proof. Fix $x \in \mathbb{R}^n$. It is clear that Λ_x is a compact and convex subset of $\mathbb{R}^n$. Then the supremum defining $h(x)$ must be attained at some $v_x \in \Lambda_x$. In order to prove the uniqueness, we use the following. For all $v_1, v_2 \in \Lambda_x,\ \lambda \in [0,1]$ and $a \in E$, we have $\lambda v_1 + (1-\lambda)v_2 \in \Lambda_x$ and

$$\psi^+(x,a,\lambda v_1 + (1-\lambda)v_2) = f(a) + \langle \lambda v_1 + (1-\lambda)v_2, x-a\rangle - \tfrac{1}{2M}|G(a) - \lambda v_1 - (1-\lambda)v_2|^2, \quad (2.5.2)$$

where $|G(a) - \lambda v_1 - (1-\lambda)v_2|^2$ can be written as

$$\begin{aligned}
&\lambda^2|G(a)-v_1|^2 + (1-\lambda)^2|G(a)-v_2|^2 + 2\lambda(1-\lambda)\langle G(a)-v_1, G(a)-v_2\rangle \\
&\quad = \lambda|G(a)-v_1|^2 + (1-\lambda)|G(a)-v_2|^2 \\
&\qquad - \lambda(1-\lambda)\left(|G(a)-v_1|^2 + |G(a)-v_2|^2 - 2\langle G(a)-v_1, G(a)-v_2\rangle\right) \\
&\quad = \lambda|G(a)-v_1|^2 + (1-\lambda)|G(a)-v_2|^2 - \lambda(1-\lambda)|v_1-v_2|^2.
\end{aligned}$$

By plugging this on (2.5.2) we obtain

$$\begin{aligned}
\psi^+(x,a,\lambda v_1 + (1-\lambda)v_2) &= \lambda\left(f(a) + \langle G(a), v_1\rangle\right) + (1-\lambda)\left(f(a) + \langle G(a), v_2\rangle\right) \\
&\quad - \tfrac{1}{2M}\left(\lambda|G(a)-v_1|^2 + (1-\lambda)|G(a)-v_2|^2 - \lambda(1-\lambda)|v_1-v_2|^2\right).
\end{aligned}$$

Therefore

$$\psi^+(x,a,\lambda v_1 + (1-\lambda)v_2) = \lambda\psi^+(x,a,v_1) + (1-\lambda)\psi^+(x,a,v_2) + \tfrac{\lambda(1-\lambda)}{2M}|v_1-v_2|^2,$$

and this leads us to

$$\inf_{a\in E}\psi^+(x,a,\lambda v_1 + (1-\lambda)v_2) \geq \lambda \inf_{a\in E}\psi^+(x,a,v_1) + (1-\lambda)\inf_{a\in E}\psi^+(x,a,v_2) + \tfrac{\lambda(1-\lambda)}{2M}|v_1-v_2|^2.$$

We then have proved that the function

$$\Lambda_x \ni v \longmapsto \inf_{a\in E}\psi^+(x,a,v)$$

is strictly concave. Therefore, the supremum defining $h(x)$ is attained at a unique $v_x \in \Lambda_x$. □

Theorem 2.23. *We have, for every $x \in \mathbb{R}^n$,*

$$F(x) = \sup_{v\in\Lambda_x}\inf_{a\in E}\psi^+(x,a,v) = \inf_{a\in E}\psi^+(x,a,v_x), \quad \nabla F(x) = v_x,$$

where F is that of Theorem 2.11 and $v_x \in \Lambda_x$ is as in Lemma 2.22.

Proof. We define the 1-jet $(h, \nabla h)$ on $\mathbb{R}^n$ by setting

$$h(x) := \sup_{v\in\Lambda_x}\inf_{a\in E}\psi^+(x,a,v), \quad \nabla h(x) := v_x, \quad x \in \mathbb{R}^n.$$

If $x \in E$, we see from the definition of $\alpha_{x,x}, \beta^x_{x,x}$ and $r^x_{x,x}$ that $\Lambda_x = \{G(x)\}$. Since $\nabla h(x) \in \Lambda_x$, we get $\nabla h(x) = G(x) = \nabla F(x)$. By the definition of $\nabla h(x) = v_x$ (see Lemma 2.22) we also have

$$h(x) = \inf_{a\in E}\psi^+(x,a,G(x)) \leq \psi^+(x,x,G(x)) = f(x) = F(x),$$

and then (2.5.1) shows that

$$F(x) \leq \inf_{a\in E}\psi^+(x,a,G(x)) = h(x)$$

This shows that $h = F$ on E. Now, let us consider a point $x \in \mathbb{R}^n \setminus E$. Using (2.5.1), it follows that $\nabla F(x) \in \Lambda_x$ and

$$F(x) \leq \inf_{a\in E}\psi^+\left(x,a,\nabla F(x)\right) \leq \sup_{v\in\Lambda_x}\inf_{a\in E}\psi^+\left(x,a,\nabla F(x)\right) = h(x).$$

On the other hand, Remark 2.21 (1) tells us that

$$\sup_{a\in E} \psi^-(x,a,v_x) \leq h(x) = \inf_{a\in E} \psi^+(x,a,v_x),$$

which shows that 1-jet $(h, \nabla h(x))$ satisfies condition $(CW^{1,1})$ on $E\cup\{x\}$ with constant $M>0$. Then, Theorem 2.11 provides a $C^{1,1}$ convex extension $(h^*, \nabla h^*)$ of $(h, \nabla h(x))$ to all of $\mathbb{R}^n$ satisfying that $\mathrm{Lip}(\nabla h^*) \leq M$. Since $(h^*, \nabla h^*)$ coincides with $(h, \nabla h(x)) = (f, G)$ on E, the last part of Theorem 2.11 says that we must have $h^* \leq F$ on $\mathbb{R}^n$, which implies in particular that $h(x) \leq F(x)$. We have thus shown that $F = h$ on $\mathbb{R}^n$. Finally, given $x \in \mathbb{R}^n \setminus E$,

$$h(x) = F(x) \leq \inf_{a\in E} \psi^+(x,a,\nabla F(x)),$$

thanks to (2.5.1). The definition of h together with Lemma 2.22 allows us to conclude that $\nabla F(x) = v_x$. □

2.6 Optimal $C^{1,1}$ extensions of jets by explicit formulas in Hilbert spaces

In this section we will prove that formula (2.1.3) defines a $C^{1,1}$ extension of the jet (f,G) on E, provided that this jet satisfies a necessary and sufficient condition found by Wells in [69], which is equivalent to the classical Whitney condition for $C^{1,1}$ extension $(\widetilde{W^{1,1}})$ by virtue of Remark 2.4.

Definition 2.24. *We will say that a* 1*-jet* (f,G) *defined on a subset* E *of a Hilbert space satisfies condition* $(W^{1,1})$ *with constant* $M>0$ *on* E *provided that*

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$

for all $x,y\in E$.

Let us first see why this condition is necessary for $C^{1,1}$ extension.

Proposition 2.25.

(i) *If* (f,G) *satisfies* $(W^{1,1})$ *on* E *with constant* M*, then* G *is* M*-Lipschitz on* E*.*

(ii) *If* F *is a function of class* $C^{1,1}(X)$ *with* $\mathrm{Lip}(\nabla F)\leq M$*, then* $(F,\nabla F)$ *satisfies* $(W^{1,1})$ *on* $E=X$ *with constant* M*.*

Proof.

(i) Given $x,y\in E$, we have

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$f(x) \leq f(y) + \frac{1}{2}\langle G(y)+G(x), x-y\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2.$$

By combining both inequalities we immediately obtain $\|G(x)-G(y)\| \leq M\|x-y\|$.

(ii) Fix $x,y\in X$ and $z=\frac{1}{2}(x+y)+\frac{1}{2M}(\nabla F(y)-\nabla F(x))$. Using Taylor's theorem we obtain

$$F(z) \leq F(x) + \langle \nabla F(x), \tfrac{1}{2}(y-x)+\tfrac{1}{2M}(\nabla F(y)-\nabla F(x))\rangle + \tfrac{M}{2}\left\|\tfrac{1}{2}(y-x)+\tfrac{1}{2M}(\nabla F(y)-\nabla F(x))\right\|^2$$

and

$$F(z) \geq F(y) + \langle \nabla F(y), \tfrac{1}{2}(x-y)+\tfrac{1}{2M}(\nabla F(y)-\nabla F(x))\rangle - \tfrac{M}{2}\left\|\tfrac{1}{2}(x-y)+\tfrac{1}{2M}(\nabla F(y)-\nabla F(x))\right\|^2.$$

We thus have

$$\begin{aligned}F(y) &\leq F(x) + \langle \nabla F(x), \tfrac{1}{2}(y-x) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\rangle \\ &\quad + \tfrac{M}{2}\left\|\tfrac{1}{2}(y-x) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\right\|^2 \\ &\quad - \langle \nabla F(y), \tfrac{1}{2}(x-y) - \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\rangle \\ &\quad + \tfrac{M}{2}\left\|\tfrac{1}{2}(x-y) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\right\|^2 \\ &= F(x) + \frac{1}{2}\langle \nabla F(x) + \nabla F(y), y-x\rangle + \tfrac{M}{4}\|x-y\|^2 - \tfrac{1}{4M}\|\nabla F(x) - \nabla F(y)\|^2\end{aligned}$$

□

The following lemma will allow us to deal with the $C^{1,1}$ extension problem for 1-jets by relying on our previous solution of the $C^{1,1}$ convex extension problem for 1-jets.

Lemma 2.26. *Given an arbitrary subset E of a Hilbert space X and a 1-jet (f, G) defined on E, we have the following: (f, G) satisfies $(W^{1,1})$ on E, with constant $M > 0$, if and only if the 1-jet $(\tilde{f}, \tilde{G})$ defined by $\tilde{f}(x) = f(x) + \frac{M}{2}\|x\|^2$, $\tilde{G}(x) = G(x) + Mx$, $x \in E$, satisfies property $(CW^{1,1})$ on E, with constant $2M$.*

Proof. Suppose first that (f, G) satisfies $(W^{1,1})$ on E with constant $M > 0$. We have, for all $x, y \in E$,

$$\begin{aligned}&\tilde{f}(x) - \tilde{f}(y) - \langle \tilde{G}(y), x-y\rangle - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y)\|^2 \\ &\quad = f(x) - f(y) + \frac{M}{2}\|x\|^2 - \frac{M}{2}\|y\|^2 - \langle G(y) + My, x-y\rangle \\ &\qquad - \frac{1}{4M}\|G(x) - G(y) + M(x-y)\|^2 \\ &\quad \geq \frac{1}{2}\langle G(x) + G(y), x-y\rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M}\|G(x) - G(y)\|^2 \\ &\qquad + f(x) - f(y) + \frac{M}{2}\|x\|^2 - \frac{M}{2}\|y\|^2 - \langle G(y) + My, x-y\rangle \\ &\qquad - \frac{1}{4M}\|G(x) - G(y) + M(x-y)\|^2 \\ &\quad = \frac{M}{2}\|x\|^2 + \frac{M}{2}\|y\|^2 - M\langle x, y\rangle - \frac{M}{2}\|x-y\|^2 = 0.\end{aligned}$$

Conversely, if $(\tilde{f}, \tilde{G})$ satisfies $(CW^{1,1})$ on E with constant $2M$, we have

$$\begin{aligned}&f(x) + \frac{1}{2}\langle G(x) + G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2 - f(y) \\ &\quad = \tilde{f}(x) - \frac{M}{2}\|x\|^2 + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G}(y) - M(x+y), y-x\rangle + \frac{M}{4}\|x-y\|^2 \\ &\qquad - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 - \tilde{f}(y) + \frac{M}{2}\|y\|^2 \\ &\quad = \tilde{f}(x) - \tilde{f}(y) + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G}(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 \\ &\qquad - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 \\ &\quad \geq \langle \tilde{G}(y), x-y\rangle + \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y)\|^2 + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G(y)}, y-x\rangle \\ &\qquad + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 = 0.\end{aligned}$$

□

We are now ready to stablish our formula for $C^{1,1}$ extension of general 1-jets.

Theorem 2.27. *Let E be a subset of a Hilbert space X. Given a 1-jet (f,G) satisfying property $(W^{1,1})$ with constant M on E, the formula*

$$F = conv(g) - \tfrac{M}{2}\|\cdot\|^2,$$
$$g(x) = \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}\|x-y\|^2\} + \tfrac{M}{2}\|x\|^2, \quad x\in X,$$

defines a $C^{1,1}(X)$ function with $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $\mathrm{Lip}(\nabla F) \leq M$.

Moreover, if H is another $C^{1,1}$ function with $H = f$ and $\nabla H = G$ on E and $\mathrm{Lip}(\nabla H) \leq M$, then $H \leq F$.

Proof. From Lemma 2.26, we know that the 1-jet $(\tilde{f}, \tilde{G})$ defined by

$$\tilde{f}(x) = f(x) + \frac{M}{2}\|x\|^2, \quad \tilde{G}(x) = G(x) + Mx, \quad x\in E,$$

satisfies property $(CW^{1,1})$ on E with constant $2M$. Then, by Theorem 2.11, the function

$$\tilde{F} = \mathrm{conv}(g), \quad \tilde{g}(x) = \inf_{y\in E}\{\tilde{f}(y) + \langle\tilde{G}(y), x-y\rangle + M\|x-y\|^2\}, \quad x\in X,$$

is convex and of class $C^{1,1}$ with $(\tilde{F}, \nabla\tilde{F}) = (\tilde{f}, \tilde{G})$ on E and $\mathrm{Lip}(\nabla\tilde{F}) \leq 2M$. The definitions of $\tilde{f}$ and $\tilde{G}$ imply that

$$\tilde{g}(x) = \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}\|x-y\|^2\} + \tfrac{M}{2}\|x\|^2, \quad x\in X.$$

Now, according to Proposition 2.8, the jet $(\tilde{F}, \nabla\tilde{F})$ satisfies condition $(CW^{1,1})$ with constant $2M$ on the whole X. Thus, if F is the function defined by

$$F(x) = \tilde{F}(x) - \frac{M}{2}\|x\|^2, \quad x\in X,$$

we get, thanks to Lemma 2.26, that the jet $(F, \nabla F)$ satisfies condition $(W^{1,1})$ with constant M on X. Hence, by Proposition 2.25, F is of class $C^{1,1}(X)$, with $\mathrm{Lip}(\nabla F) \leq M$. From the definition of $\tilde{f}, \tilde{G}, \tilde{F}$ and F it is immediate that $F = f$ and $\nabla F = G$ on E.

Finally, suppose that H is another $C^{1,1}(X)$ function with $H = f$ and $\nabla H = G$ on E and $\mathrm{Lip}(\nabla H) \leq M$. Using all of these assumptions together with Taylor's Theorem we have that

$$H(x) + \frac{M}{2}\|x\|^2 \leq f(y) + \langle G(y), x-y\rangle + \frac{M}{2}\|x-y\|^2 + \frac{M}{2}\|x\|^2,$$

for all $x\in X, y\in E$. Taking the infimum over E we get that

$$H(x) + \frac{M}{2}\|x\|^2 \leq g(x), \quad x\in X.$$

Since H is $C^{1,1}(X)$ with $\mathrm{Lip}(\nabla H) \leq M$, the jet $(H, \nabla H)$ satisfies the condition $(W^{1,1})$ on E with constant M. Using Lemma 2.26, we obtain that $(\tilde{H}, \nabla\tilde{H})$ (defined as in that Lemma) satisfies $(CW^{1,1})$ on E with constant $2M$. In particular the function $X \ni x \mapsto \tilde{H}(x) = H(x) + \frac{M}{2}\|x\|^2$ is convex, which implies that

$$\tilde{H} = \mathrm{conv}(\tilde{H}) \leq g.$$

Therefore, $\tilde{H} \leq \tilde{F}$ on X, from which we obtain that $H \leq F$ on X. $\square$

2.7 Kirszbraun Extension Theorem

As a consequence of Theorem 2.27, we can give a short proof of Kirszbraun-Valentine's extension theorem for Lipschitz functions (see [52, 65]), providing an explicit formula for the extension.

Corollary 2.28 (Kirszbraun-Valentine's Theorem). *Let X, Y be two Hilbert spaces, E a subset of X and $G : E \to Y$ a Lipschitz mapping. There exists $\widetilde{G} : X \to Y$ with $\widetilde{G} = G$ on E and* $\mathrm{Lip}(\widetilde{G}) = \mathrm{Lip}(G)$.

Proof. Consider on $X \times Y$ the scalar product given by $\langle (x,y),(x',y')\rangle = \langle x,x'\rangle_X + \langle y,y'\rangle_Y$. We will denote by $\|\cdot\|$, $\|\cdot\|_X$ and $\|\cdot\|_Y$ the norms on $X \times Y$, X and Y respectively. Also, we denote by $\langle \cdot,\cdot\rangle_X$ and $\langle \cdot,\cdot\rangle_Y$ the scalar products on X and Y respectively. It is then clear that $X \times Y$ is a Hilbert space. Now we consider the 1-jet (f^*, G^*) defined on $E \times \{0\}$ by $f^*(x,0) = 0$ and $G^*(x,0) = (0, G(x))$, for all $x \in E$. If we denote $M = \mathrm{Lip}(G)$, we have that

$$
\begin{aligned}
f^*(x,0) &- f^*(y,0) + \frac{1}{2}\langle G^*(x,0) + G^*(y,0), (y,0)-(x,0)\rangle \\
&\quad + \frac{M}{4}\|(x,0)-(y,0)\|^2 - \frac{1}{4M}\|G^*(x,0) - G^*(y,0)\|^2 \\
&= \frac{1}{2}\langle (0,G(x)) - (0,G(y)), (y,0)-(x,0)\rangle + \frac{M}{4}\|x-y\|_X^2 - \frac{1}{4M}\|G(x)-G(y)\|_Y^2 \\
&= \frac{M}{4}\|x-y\|_X^2 - \frac{1}{4M}\|G(x)-G(y)\|_Y^2 \geq 0.
\end{aligned}
$$

Thus, (f^*, G^*) satisfies condition $(W^{1,1})$ on $E \times \{0\}$ with constant M. By Theorem 2.27, the function

$$
\begin{aligned}
&F = \mathrm{conv}(g) - \tfrac{M}{2}\|\cdot\|^2, \\
&g(x,y) = \inf_{z\in E}\{f^*(z,0) + \langle G^*(z,0),(x-z,y)\rangle + \tfrac{M}{2}\|(x-z,y)\|^2\} + \tfrac{M}{2}\|(x,y)\|^2, \quad (x,y) \in X \times Y,
\end{aligned}
$$

is of class $C^{1,1}(X \times Y)$ with $(F, \nabla F) = (f^*, G^*)$ on $E \times \{0\}$ and $\mathrm{Lip}(\nabla F) \leq M$. We see that the expression defining g can be simplified as

$$
g(x,y) = \inf_{z\in E}\{\langle G(z), y\rangle_Y + \tfrac{M}{2}\|(x-z,y)\|^2\} + \tfrac{M}{2}\|(x,y)\|^2, \quad (x,y) \in X \times Y.
$$

If we denote by $P : X \times Y \to Y$ the canonical projection, then the function $\widetilde{G} := P(\nabla F)$ coincides with G on E and $\mathrm{Lip}(\widetilde{G}) \leq \mathrm{Lip}(\nabla F) \leq M$. □

2.8 $C^{1,\omega}$ convex extensions of jets by explicit formulas in Hilbert spaces

In this section we will present the solution to the problem of finding $C^{1,\omega}$ convex extensions of 1-jets in the Hilbert space by means of explicit formulas. Unless otherwise stated, we will assume that X is a Hilbert space and $\omega : [0,+\infty) \to [0,+\infty)$ is a concave and increasing function such that $\omega(0) = 0$ and $\lim_{t\to+\infty}\omega(t) = +\infty$. Also, we will denote

$$
\varphi(t) = \int_0^t \omega(s)ds \tag{2.8.1}
$$

for every $t \geq 0$. It is obvious that φ is differentiable with $\varphi' = \omega$ on $[0,+\infty)$ and, because ω is strictly increasing, φ is strictly convex. The function ω has an inverse $\omega^{-1} : [0,+\infty) \to [0,+\infty)$ which is convex and strictly increasing, with $\omega^{-1}(0) = 0$. We also note that

$$
\begin{aligned}
\omega(ct) \leq c\omega(t) \quad &\text{and} \quad \omega^{-1}(ct) \geq c\omega^{-1}(t) \quad \text{for} \quad c \geq 1,\ t \geq 0 \\
\omega(ct) \geq c\omega(t) \quad &\text{and} \quad \omega^{-1}(ct) \leq c\omega^{-1}(t) \quad \text{for} \quad c \leq 1,\ t \geq 0.
\end{aligned}
$$

In the sequel we will make intensive use of the Fenchel conjugate of a function on the Hilbert space. Recall that, given a function $g : X \to \mathbb{R}$, the Fenchel conjugate of g is defined by

$$g^*(x) = \sup_{z\in X}\{\langle x, z\rangle - g(z)\}, \quad x \in X,$$

where g^* may take the value $+\infty$ at some x. We next gather a couple of elementary properties of this operator which we will need later on. A detailed exposition can be found in [19, Chapter 2, Section 3] or [74, Chapter 2, Section 3] for instance.

Proposition 2.29. *We have:*

(i) $(ag)^* = ag^*(\frac{\cdot}{a})$ *and* $\left(ag(\frac{\cdot}{a})\right)^* = ag^*$ *for* $a > 0$.

(ii) *If* $\rho : \mathbb{R} \to \mathbb{R}$ *is even, then* $(\rho \circ \|\cdot\|)^* = \rho^*(\|\cdot\|)$.

Abusing terminology, we will consider the Fenchel conjugate of nonnegative functions only defined on $[0, +\infty)$, say $\delta : [0, +\infty) \to [0, +\infty)$. In order to avoid problems, we will assume that all the functions involved are extended to all of $\mathbb{R}$ by setting $\delta(t) = \delta(-t)$ for $t < 0$. Hence δ will be an even function on $\mathbb{R}$ and therefore

$$\delta^*(t) = \sup_{s\in\mathbb{R}}\{ts - \delta(s)\} = \sup_{s\geq 0}\{ts - \delta(s)\}, \quad \text{for} \quad t \geq 0.$$

For our modulus of continuity ω, and the corresponding function φ, (see 2.8.1) the following proposition provides a formula for φ^* in terms of ω.

Proposition 2.30. [See [74, Lemma 3.7.1, pg. 227].] *We have that* $\varphi^*(t) = \int_0^t \omega^{-1}(s)ds$ *for all* $t \geq 0$ *and* $\varphi(t) + \varphi^*(s) = ts$ *if and only if* $s = \omega(t)$.

Let us now recall the definition of uniformly convex functions and the modulus of convexity. See [74] for a detailed exposition of this topic.

Definition 2.31. *A function* $f : X \to \mathbb{R}$ *is said to be uniformly convex, with modulus of convexity* δ *(being* $\delta : [0, +\infty) \to [0, +\infty)$ *a nondecreasing function with* $\delta(0) = 0$*) provided that*

$$\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y) + \lambda(1-\lambda)\delta(\|x - y\|)$$

for all $\lambda \in [0, 1]$ *and* $x, y \in X$.

Theorem 2.32. [67, Theorem 3]. *Let* X *be a Hilbert space. If* $\rho : [0, +\infty) \to [0, +\infty)$ *is an increasing function with* $\rho(ct) \geq c\rho(t)$ *for all* $c \geq 1$ *and* $t \geq 0$, *then the function* $\Phi : X \to \mathbb{R}$ *defined by* $\Phi(x) = \int_0^{\|x\|} \rho(t)dt$, $x \in X$, *is uniformly convex, with modulus of convexity* $\delta(t) = \int_0^t \rho(s/2)ds$, $t \geq 0$. *In addition,*

(i) *If the function* $(0, +\infty) \ni t \longmapsto \dfrac{\rho(t)}{t}$ *is convex we can take* $\delta(t) = 2\int_0^t \rho\left(\frac{s}{2}\right) ds$.

(ii) *If the function* $(0, +\infty) \ni t \longmapsto \dfrac{\rho(t)}{t}$ *is concave we can take* $\delta(t) = \int_0^t \rho(s)ds$.

In particular, the Theorem applies for increasing convex functions $\rho : [0, +\infty) \to [0, +\infty)$ with $\rho(0) = 0$.

For the sake of completeness and because we have not been able to find an English version of [67], we are going to provide the proof of Theorem 2.32. We first need to prove the following lemma.

Lemma 2.33. *Let* $\Phi : X \to \mathbb{R}$ *be a differentiable function such that*

$$\langle \nabla\Phi(x) - \nabla\Phi(y), x - y\rangle \geq \phi(\|x - y\|) \quad \text{for all} \quad x, y \in X, \tag{2.8.2}$$

where $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ *is a nondecreasing function with* $\phi(0) = 0$ *and* $\phi(t) > 0$ *for* $t > 0$. *Then* Φ *is uniformly convex on* X *with modulus of convexity* $\delta(t) = \int_0^t \frac{\phi(s)}{s}ds$, *provided that* $\delta(t)$ *is finite for every* $t > 0$.

Proof. Fix $\alpha \in (0,1)$ and $x, y \in X$ with $x \neq y$. We define

$$\gamma(t) = y + \frac{t}{\|x-y\|}(x-y), \quad h(t) = \Phi(\gamma(t)), \quad \text{for all} \quad t \in \mathbb{R}.$$

Because ψ is differentiable on X, the function h is differentiable on $\mathbb{R}$ with

$$h'(t) = \left\langle \nabla\Phi(\gamma(t)), \frac{x-y}{\|x-y\|} \right\rangle, \quad t \in \mathbb{R}.$$

By (2.8.2) we obtain for $t > s$ that

$$\begin{aligned} h'(t) - h'(s) &= \left\langle \nabla\Phi(\gamma(t)) - \nabla\Phi(\gamma(s)), \frac{x-y}{\|x-y\|} \right\rangle \\ &= \left\langle \nabla\Phi(\gamma(t)) - \nabla\Phi(\gamma(s)), \frac{\gamma(t)-\gamma(s)}{t-s} \right\rangle \geq \frac{\phi\left(\|\gamma(t)-\gamma(s)\|\right)}{t-s} = \frac{\phi(t-s)}{t-s}. \end{aligned}$$

We thus get

$$h'(t) - h'(s) \geq \frac{\phi(t-s)}{t-s} \quad \text{for} \quad t > s. \tag{2.8.3}$$

Now we write

$$\begin{aligned} \alpha\psi(x) + &(1-\alpha)\psi(y) - \psi(\alpha x + (1-\alpha)y) \\ &= \alpha h(\|x-y\|) + (1-\alpha)h(0) - h(\alpha\|x-y\|) \\ &= \alpha\left(h(\|x-y\| - h(\alpha\|x-y\|)\right) + (1-\alpha)\left(h(0) - h(\alpha\|x-y\|)\right) \\ &= \alpha \int_{\alpha\|x-y\|}^{\|x-y\|} h'(t)ds - (1-\alpha)\int_0^{\alpha\|x-y\|} h'(t)ds. \end{aligned}$$

By changing the variable $t = \alpha\|x-y\| + (1-\alpha)s$ in the first integral and $t = \alpha\|x-y\| - \alpha s$ in the second one, we obtain that

$$\begin{aligned} \alpha \int_{\alpha\|x-y\|}^{\|x-y\|} & h'(t)ds - (1-\alpha)\int_0^{\alpha\|x-y\|} h'(t)ds \\ &= \alpha(1-\alpha)\int_0^{\|x-y\|} \Big[h'\left(\alpha\|x-y\| + (1-\alpha)s\right) - h'\left(\alpha\|x-y\| - \alpha s\right)\Big] ds. \end{aligned}$$

From (2.8.3), the last term is greater than or equal to

$$\alpha(1-\alpha)\int_0^{\|x-y\|} \frac{\phi(s)}{s} ds = \alpha(1-\alpha)\delta(\|x-y\|),$$

and this proves the Lemma. □

Proof of Theorem 2.32. By the assumptions on ρ, the functions $t \mapsto \rho(t),\ t \mapsto \frac{\rho(t)}{t}$ are nondecreasing. Also, it is clear that Φ is differentiable on X with $\nabla\Phi(0) = 0$ and $\nabla\Phi(x) = \frac{\rho(\|x\|)}{\|x\|}x$ for $x \neq 0$. We now fix $x, y \in X$ with $x \neq y$ and $x, y \neq 0$. We can write

$$\begin{aligned} \langle \nabla\Phi(x) - \nabla\Phi(y), x-y \rangle &= \frac{1}{2}\left[2\|x\|^2\frac{\rho(\|x\|)}{\|x\|} + 2\|y\|^2\frac{\rho(\|y\|)}{\|y\|} - 2\langle x, y\rangle\left(\frac{\rho(\|x\|)}{\|x\|} + \frac{\rho(\|y\|)}{\|y\|}\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{\rho(\|x\|)}{\|x\|} - \frac{\rho(\|y\|)}{\|y\|}\right)(\|x\|^2 - \|y\|^2) + \|x-y\|^2\left(\frac{\rho(\|x\|)}{\|x\|} + \frac{\rho(\|y\|)}{\|y\|}\right)\right]. \end{aligned}$$

Because $t \longmapsto \dfrac{\rho(t)}{t}$ is nondecreasing, the identity above gives

$$\langle \nabla\Phi(x) - \nabla\Phi(y), x-y \rangle \geq \frac{\|x-y\|^2}{2}\left(\frac{\rho(\|x\|)}{\|x\|} + \frac{\rho(\|y\|)}{\|y\|}\right). \tag{2.8.4}$$

It is obvious that either $\|x\|$ or $\|y\|$ is bigger than $\frac{1}{2}\|x\|+\frac{1}{2}\|y\|$, so

$$\frac{\rho(\|x\|)}{\|x\|}+\frac{\rho(\|y\|)}{\|y\|}\geq\frac{\rho\left(\frac{1}{2}\|x\|+\frac{1}{2}\|y\|\right)}{\frac{1}{2}\|x\|+\frac{1}{2}\|y\|}. \tag{2.8.5}$$

We also have that

$$\frac{\rho\left(\frac{1}{2}\|x\|+\frac{1}{2}\|y\|\right)}{\frac{1}{2}\|x\|+\frac{1}{2}\|y\|}\geq\frac{2}{\|x-y\|}\rho\left(\frac{\|x-y\|}{2}\right), \tag{2.8.6}$$

as $\|x-y\|\leq\|x\|+\|y\|$. Combining inequalities (2.8.4), (2.8.5) and (2.8.6) we obtain

$$\langle\nabla\Phi(x)-\nabla\Phi(y),x-y\rangle\geq\|x-y\|\rho\left(\frac{\|x-y\|}{2}\right).$$

Note that the last inequality also holds for $x=0$ or $y=0$. Thus the first statement of the theorem is proved by virtue of Lemma 2.33.

Now suppose that $t\mapsto\frac{\rho(t)}{t}$ is convex. Then, we have

$$\frac{\rho\left(\frac{1}{2}\|x\|+\frac{1}{2}\|y\|\right)}{\frac{1}{2}\|x\|+\frac{1}{2}\|y\|}\leq\frac{1}{2}\frac{\rho(\|x\|)}{\|x\|}+\frac{1}{2}\frac{\rho(\|y\|)}{\|y\|}$$

for $x,y\neq 0$. Using (2.8.4) and a similar argument as in the proof (2.8.6) we get that

$$\langle\nabla\Phi(x)-\nabla\Phi(y),x-y\rangle\geq 2\|x-y\|\rho\left(\frac{\|x-y\|}{2}\right),$$

which is also true for $x=0$ or $y=0$. Bearing in mind Lemma 2.33, the statement (i) of the theorem is proved.

Finally, let us suppose that $t\mapsto\frac{\rho(t)}{t}$ is concave. We have, for all $x,y\neq 0$, that

$$\frac{\rho\left(\|x\|+\|y\|\right)}{\|x\|+\|y\|}\frac{\|x\|}{\|x\|+\|y\|}\leq\frac{\rho(\|x\|)}{\|x\|},\quad\frac{\rho\left(\|x\|+\|y\|\right)}{\|x\|+\|y\|}\frac{\|y\|}{\|x\|+\|y\|}\leq\frac{\rho(\|y\|)}{\|y\|},$$

which implies that

$$\frac{\rho(\|x\|)}{\|x\|}+\frac{\rho(\|y\|)}{\|y\|}\geq\frac{\rho\left(\|x\|+\|y\|\right)}{\|x\|+\|y\|}.$$

The last inequality together with (2.8.4) and a similar argument as in the proof of (2.8.6) lead us to

$$\langle\nabla\Phi(x)-\nabla\Phi(y),x-y\rangle\geq\frac{1}{2}\|x-y\|\rho\left(\|x-y\|\right),$$

which, again, is also true for $x=0$ or $y=0$. Thanks to Lemma 2.33, the statement (ii) is proved. □

In the same spirit as in Proposition 2.9, we show that, in order to prove that a continuous convex funcion f defined on a Banach space is differentiable with an ω-continuous derivative, it is enough to check a simple inequality which only involves the values of f and the function φ of (2.8.1).

Proposition 2.34. *Let X be a Banach space. If $f:X\to\mathbb{R}$ is a continuous convex function and*

$$f(x+h)+f(x-h)-2f(x)\leq C\varphi(2\|h\|),\quad \textit{for all}\quad x,h\in X,$$

then f is of class $C^{1,\omega}(X)$ and $\|Df(x)-Df(y)\|\leq 4C\omega\left(2\|x-y\|\right)$ for all $x,y\in X$.

Proof. The inequality of the assumption together with (2.8.1) tells us that, for every $x, h \in X$,

$$\frac{f(x+h)+f(x-h)-2f(x)}{\|h\|} \leq C\frac{\varphi(2\|h\|)}{\|h\|} \leq 2C\omega(2\|h\|),$$

where the last term tends to 0 as $\|h\| \to 0^+$. Because f is continuous and convex, this implies that f is differentiable on X. together with the continuity of f proves the existence of Df. Consider $x, y, h \in X$ with $\|h\| = \|x-y\|$. Using repeatedly the convexity of f and then the assumption, we get

$$\begin{aligned}
(Df(x)-Df(y))(h) &\leq f(x+h)-f(x)-Df(y)(h) \\
&\leq f(x+h)-f(x)+f(x)-f(y)-Df(y)(x-y)-Df(y)(h) \\
&\leq f(x+h)-f(y)-Df(y)(x+h-y) \\
&\leq f(x+h)-f(y)-f(2y-x-h)-f(y) \\
&\leq f(y+(x+h-y))+f(y-(x+h-y))-2f(y) \\
&\leq C\varphi\left(2\|x+h-y\|\right) \leq C\varphi\left(4\|x-y\|\right).
\end{aligned}$$

Thus

$$\|Df(x)-Df(y)\| \leq 4C\,\frac{\varphi(4\|x-y\|)}{4\|x-y\|}.$$

Note that, by concavity of ω, it follows that

$$\frac{\varphi(t)}{t} = \int_0^1 \omega(tu)du \leq \omega\left(\frac{t}{2}\right) \quad t \geq 0.$$

Therefore $\|Df(x)-Df(y)\| \leq 4C\omega\left(2\|x-y\|\right).$ □

Now we prove an inequality involving the function φ and the norm $\|\cdot\|$ which will be very useful in the proof of the main theorem.

Lemma 2.35. *Let $(X, \|\cdot\|)$ be a Hilbert space, and φ be defined by* (2.8.1). *Then the function $\psi(x) = \varphi(\|x\|),\ x \in X$, satisfies the following inequality*

$$\psi(x+h)+\psi(x-h)-2\psi(x) \leq \psi(2h) \quad \textit{for all} \quad x, h \in X.$$

Also, ψ is of class $C^{1,\omega}(X)$ with $\|\nabla\psi(x)-\nabla\psi(y)\| \leq 4\omega(2\|x-y\|)$ for all $x, y \in X$.

Furthermore, we can arrange that:

(i) $\psi(x+h)+\psi(x-h)-2\psi(x) \leq 2\psi(h)$ for all $x, h \in X$ if the function $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is convex, and

(ii) $\psi(x+h)+\psi(x-h)-2\psi(x) \leq \frac{1}{2}\psi(2h)$ for all $x, h \in X$ if the function $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is concave.

Proof. By combining the fact that $(\rho \circ \|\cdot\|)^* = \rho^*(\|\cdot\|)$ for any even $\rho : [0,+\infty) \to [0,+\infty)$ (see Proposition 2.29 and the subsequent comment) with Proposition 2.30, we obtain that $\psi^*(x) = \int_0^{\|x\|} \omega^{-1}(s)ds,\ x \in X$, where ω^{-1} is a convex function. Thus, we can apply Theorem 2.32 with $\rho = \omega^{-1}$ and $\Phi = \psi^*$ to deduce that

$$\lambda\psi^*(x)+(1-\lambda)\psi^*(y) \geq \psi^*(\lambda x+(1-\lambda)y)+\lambda(1-\lambda)\delta(\|x-y\|),$$

for all $x, y \in X,\ \lambda \in [0,1]$, where $\delta(t) = \int_0^t \omega^{-1}\left(\frac{s}{2}\right)ds,\ t \geq 0$. Then it is clear that

$$\begin{aligned}
\delta_{\psi^*}(\varepsilon) &:= \inf\left\{\frac{1}{2}\psi^*(x)+\frac{1}{2}\psi^*(y)-\psi^*\left(\frac{x+y}{2}\right) \ : \|x-y\| \geq \varepsilon,\ x, y \in X\right\} \\
&\geq \inf\left\{\frac{1}{4}\delta(\|x-y\|) \ : \|x-y\| \geq \varepsilon,\ x, y \in X\right\} \geq \frac{1}{4}\delta(\varepsilon)
\end{aligned}$$

for all $\varepsilon \geq 0$. Let us denote

$$\rho_\psi(t) := \sup\left\{\frac{1}{2}\psi(x+ty) + \frac{1}{2}\psi(x-ty) - \psi(x) \ : x, y \in X, \ \|y\| = 1\right\}$$

for all $t \geq 0$. Since ψ is continuous and convex on X, we can use [19, Theorem 5.4.1(a), pg. 252] to deduce

$$\rho_\psi(t) = \sup\left\{t\tfrac{\varepsilon}{2} - \delta_{\psi^*}(\varepsilon) \ : \varepsilon \geq 0\right\}, \quad t \geq 0.$$

Applying the preceding estimation to δ_{ψ^*} we see that

$$\rho_\psi(t) \leq \tfrac{1}{2}\sup\left\{t\varepsilon - \tfrac{1}{2}\delta(\varepsilon) \ : \varepsilon \geq 0\right\} = \tfrac{1}{2}\left(\tfrac{1}{2}\delta\right)^*(t), \quad t \geq 0.$$

By definition of δ it is clear that $\frac{1}{2}\delta(t) = \int_0^{t/2}\omega^{-1}(s)ds$. Using Proposition 2.29 together with Proposition 2.30 we have that $\left(\frac{1}{2}\delta\right)^*(t) = \int_0^{2t}\omega(s)ds, \ t \geq 0$. Then it follows

$$\rho_\psi(t) \leq \frac{1}{2}\int_0^{2t}\omega(s)ds, \quad t \geq 0$$

and therefore

$$\psi(x+ty) + \psi(x-ty) - 2\psi(x) \leq \int_0^{2t}\omega(s)ds, \quad \text{for all } t \geq 0, \ x, y \in X, \ \text{with } \|y\| = 1,$$

which is equivalent to the desired inequality. For the second part, according to Theorem 2.32, we have $\delta(t) = 2\int_0^t \omega^{-1}\left(\frac{s}{2}\right)ds$, $t \geq 0$ in the case (i) and $\delta(t) = \int_0^t \omega^{-1}(s)ds$, $t \geq 0$ in the case (ii). Then, using Proposition 2.29 together with Proposition 2.30 we obtain $\left(\frac{1}{2}\delta\right)^*(t) = 2\int_0^t\omega(s)ds, \ t \geq 0$ in the case (i) and $\left(\frac{1}{2}\delta\right)^*(t) = \frac{1}{2}\int_0^{2t}\omega(s)ds, \ t \geq 0$ in the case (ii). By repeating the same calculations as above, we immediately obtain the inequalities (i) and (ii).

Finally, for every ω, it follows from Proposition 2.34 that $\psi \in C^{1,\omega}(X)$ and $\|\nabla\psi(x) - \nabla\psi(y)\| \leq 4\omega(2\|x-y\|)$ for all $x, y \in X$.

□

A suitable condition for our extension problem is as follows.

Definition 2.36. *Given an arbitrary subset E of a Hilbert space X, and a 1-jet $f : E \to \mathbb{R}$, $G : E \to X$, we will say that (f, G) satisfies condition $(CW^{1,\omega})$ on E with constant $M > 0$, provided that*

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + M\varphi^*\left(\frac{1}{M}\|G(x) - G(y)\|\right) \quad \textit{for all} \quad x, y \in E. \qquad (CW^{1,\omega})$$

We can compare this with Definition 2.7, that is, if $\omega(t) = t$, then $\varphi(t) = \frac{1}{2}t^2$ and $\varphi^* = \varphi$. Thus, condition $(CW^{1,\omega})$ coincides with $(CW^{1,1})$ for $\omega(t) = t$. Throughout the rest of the Section, for a mapping $G : E \to X$, where E is a subset of X, we will denote

$$M_\omega(G) = \sup_{x \neq y, \, x, y \in E} \frac{\|G(x) - G(y)\|}{\omega(\|x-y\|)}.$$

We next make some remarks on this new condition $(CW^{1,\omega})$.

Remark 2.37. *The following is true.*

(i) If (f, G) satisfies $(CW^{1,\omega})$ on E with constant M, then

$$\|G(x) - G(y)\| \leq 2M\omega\left(\frac{\|x-y\|}{2}\right) \quad x, y \in E.$$

In particular $M_\omega(G) \leq 2M$.

(ii) *The inequality defining condition* $(CW^{1,\omega})$ *can be rewritten as*

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + (M\varphi)^* \left(\|G(x)-G(y)\|\right) \quad \textit{for all} \quad x, y \in E,$$

Proof.
(i) We fix $x, y \in E$ and set $t = \frac{1}{M}\|G(x)-G(y)\|$. We have that

$$M\varphi^*(t) = \|G(x)-G(y)\| \frac{\varphi^*(t)}{t}.$$

Using first Proposition 2.30 and then Jensen's inequality (recall that ω^{-1} is a convex function) we obtain

$$\frac{\varphi^*(t)}{t} = \int_0^1 \omega^{-1}(tu)du \geq \omega^{-1}\left(\frac{t}{2}\right) = \omega^{-1}\left(\frac{1}{2M}\|G(x)-G(y)\|\right)$$

and then

$$M\varphi^*\left(\frac{1}{M}\|G(x)-G(y)\|\right) \geq \|G(x)-G(y)\|\omega^{-1}\left(\frac{1}{2M}\|G(x)-G(y)\|\right).$$

Now, using the inequality defining the condition $(CW^{1,\omega})$ we have

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + M\varphi^*\left(\frac{1}{M}\|G(x)-G(y)\|\right)$$
$$f(y) \geq f(x) + \langle G(x), y-x\rangle + M\varphi^*\left(\frac{1}{M}\|G(x)-G(y)\|\right)$$

hence

$$\langle G(x)-G(y), x-y\rangle \geq 2M\varphi^*\left(\frac{1}{M}\|G(x)-G(y)\|\right) \geq \|G(x)-G(y)\|\omega^{-1}\left(\frac{1}{2M}\|G(x)-G(y)\|\right).$$

We conclude that

$$\|G(x)-G(y)\| \leq 2M\omega\left(\frac{\|x-y\|}{2}\right).$$

(ii) This follows from elementary properties of the conjugate of a function; see Proposition 2.29. □

Remark 2.38. An alternative formulation of the condition $(CW^{1,\omega})$ for a 1-jet (f, G) on E is that

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \|G(x)-G(y)\|\omega^{-1}\left(\frac{1}{2M}\|G(x)-G(y)\|\right) \tag{2.8.7}$$

for all $x, y \in E$. This is the condition for $C^{1,\omega}_{\text{conv}}$ extension of 1-jets that we introduced in [12]. If we denote the above condition by $(\widetilde{CW^{1,\omega}})$, we have that $(\widetilde{CW^{1,\omega}})$ and $(CW^{1,\omega})$ are actually equivalent.

Proof. Since ω^{-1} is convex, we have that

$$\varphi^*(t) = \int_0^t \omega^{-1}(s)ds \geq t\omega^{-1}(t/2) \quad \text{for all} \quad t \geq 0. \tag{2.8.8}$$

On the other hand, because ω^{-1} is increasing, it follows

$$\varphi^*(t) = \int_0^t \omega^{-1}(s)ds \leq t\omega^{-1}(t) \quad \text{for all} \quad t \geq 0. \tag{2.8.9}$$

Taking first $t = \frac{1}{M}\|G(x)-G(y)\|$ in (2.8.8) and then $t = \frac{1}{2M}\|G(x)-G(y)\|$ in (2.8.9) and also bearing in mind Proposition 2.29 (i) we obtain

$$(M\varphi)^*\left(\|G(x)-G(y)\|\right) \geq \|G(x)-G(y)\|\omega^{-1}\left(\frac{1}{2M}\|G(x)-G(y)\|\right) \geq (2M\varphi)^*(\|G(x)-G(y)\|).$$

By comparing condition $(CW^{1,\omega})$ (Definition 2.36) with $(\widetilde{CW^{1,\omega}})$ (inequality (2.8.7)) we then see that both conditions are equivalent. □

Let us now see that $(CW^{1,\omega})$ is a necessary condition for $C^{1,\omega}$ convex extension of 1-jets.

Proposition 2.39. *Let $f \in C^{1,\omega}(X)$ be convex, and assume that f is not affine. Then the 1-jet $(f, \nabla f)$ satisfies the condition $(CW^{1,\omega})$ with constant $M > 0$ on $E = X$, where*

$$M = \sup_{x,y \in X,\, x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\omega(\|x-y\|)}.$$

On the other hand, if f is affine, it is obvious that $(f, \nabla f)$ satisfies $(CW^{1,\omega})$ on every $E \subset X$, for every $M > 0$.

Proof. Suppose that there exist different points $x, y \in X$ such that

$$f(x) - f(y) - \langle \nabla f(y), x-y \rangle < M\varphi^* \left(\frac{1}{M} \|\nabla f(x) - \nabla f(y)\| \right),$$

and we will get a contradiction.
Case 1. Assume further that $M = 1$, $f(y) = 0$, and $\nabla f(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) < \varphi^* \left(\|\nabla f(x)\| \right).$$

Set

$$v = -\frac{1}{\|\nabla f(x)\|} \nabla f(x),$$

and define

$$h(t) = f(x + tv)$$

for every $t \in \mathbb{R}$. We have $h(0) = f(x)$, $h'(0) = -\|\nabla f(x)\|$, and $h'(t) = \langle \nabla f(x+tv), v \rangle$. This implies that

$$\left| h(t) - f(x) + \|\nabla f(x)\| t \right| \leq \int_0^t \omega(s) ds = \varphi(t)$$

for every $t \in \mathbb{R}^+$, hence also that

$$h(t) \leq -\|\nabla f(x)\| t + f(x) + \varphi(t) \text{ for all } t \in \mathbb{R}^+.$$

By using the assumption on $f(x)$ and Proposition 2.30 we have

$$f\left(x + \omega^{-1}(\|\nabla f(x)\|) v \right) < \varphi^*(\|\nabla f(x)\|) - \|\nabla f(x)\| \omega^{-1}(\|\nabla f(x)\|) + \varphi\left(\omega^{-1}(\|\nabla f(x)\|) \right) = 0,$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $\nabla f(y) = 0$. This shows that

$$f(x) \geq \varphi^* \left(\|\nabla f(x)\| \right).$$

Case 2. Assume only that $M = 1$. Define

$$g(z) = f(z) - f(y) - \langle \nabla f(y), z - y \rangle$$

for every $z \in X$. Then $g(y) = 0$ and $\nabla g(y) = 0$. By Case 1, we get

$$g(x) \geq \varphi^* \left(\|\nabla g(x)\| \right),$$

and since $\nabla g(x) = \nabla f(x) - \nabla f(y)$ the Proposition is thus proved in the case when $M = 1$.
Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $g = \frac{1}{M} f$, which satisfies the assumption of Case 2. Therefore

$$g(x) - g(y) - \langle \nabla g(y), x - y \rangle \geq \varphi^* \left(\|\nabla g(x) - \nabla g(y)\| \right),$$

which is equivalent to the desired inequality. □

Let us now present the main result of this section.

Theorem 2.40. *Given a* 1*-jet* (f, G) *defined on* E *satisfying the property* $(CW^{1,\omega})$ *with constant* M *on* E*, the formula*

$$F = conv(g), \quad g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x-y\rangle + M\varphi\left(\|x-y\|\right)\}, \quad x \in X,$$

defines a $C^{1,\omega}$ *convex function with* $F_{|_E} = f$ *and* $(\nabla F)_{|_E} = G$*, and*

$$\|\nabla F(x) - \nabla F(y)\| \leq 4M\omega\left(2\|x-y\|\right) \quad \textit{for all} \quad x, y \in X.$$

In particular, $M_\omega(\nabla F) \leq 8M$.

For the proof of Theorem 2.40 we will need to use the following well-known inequality, whose proof is immediate from the definition of the Fenchel conjugate.

Proposition 2.41 (Generalized Young's inequality for the Fenchel conjugate)**.** *Let* $\rho : \mathbb{R} \to \mathbb{R}$ *be a convex function. Then*

$$ab \leq \rho(a) + \rho^*(b) \quad \textit{for all} \quad a, b > 0.$$

As in the proof of Theorem 2.11, we will see that the function g of Theorem 2.40 lies above the affine functions $x \mapsto f(z) + \langle G(z), x-z\rangle$ for every $z \in E$.

Lemma 2.42. *We have*

$$f(z) + \langle G(z), x-z\rangle \leq f(y) + \langle G(y), x-y\rangle + M\varphi(\|x-y\|)$$

for every $y, z \in E,\ x \in X$.

Proof. Given $y, z \in E,\ x \in X$, condition $(CW^{1,\omega})$ with constant M (together with Remark 2.37 (ii)) leads us to

$$\begin{aligned} & f(y) + \langle G(y), x-y\rangle + M\varphi(\|x-y\|) \\ & \geq f(z) + \langle G(z), x-z\rangle + (M\varphi)^*(\|G(y)-G(z)\|) \\ & \quad + \langle G(z)-G(y), y-x\rangle + M\varphi(\|x-y\|) \\ & \geq f(z) + \langle G(z), x-z\rangle - ab + M\varphi(a) + (M\varphi)^*(b), \end{aligned}$$

where $a = \|y-x\|$ and $b = \|G(z)-G(y)\|$. Applying Proposition 2.41 we obtain that the last term is greater than or equal to $f(z) + \langle G(z), x-z\rangle$. □

The previous Lemma shows that $m \leq g$, where g is that of Theorem 2.40, and

$$m(x) := \sup_{z \in E}\{f(z) + \langle G(z), x-z\rangle\}, \quad x \in X.$$

By definition of g and m it is then obvious that $f \leq m \leq g \leq f$ on E. Thus $g = f$ on E. We next show that g satisfies the one-sided estimate of Proposition 2.34.

Lemma 2.43. *We have*

$$g(x+h) + g(x-h) - 2g(x) \leq M\varphi(\|2h\|) \quad \textit{for all} \quad x, h \in X.$$

Proof. Given $x, h \in X$ and $\varepsilon > 0$, by definition of g, we can pick $y \in E$ with

$$g(x) \geq f(y) + \langle G(y), x-y\rangle + M\varphi(\|x-y\|) - \varepsilon.$$

We then have

$$\begin{aligned} g(x+h)+g(x-h)-2g(x) &\leq f(y)+\langle G(y), x+h-y\rangle + M\varphi(\|x+h-y\|) \\ &\quad + f(y)+\langle G(y), x-h-y\rangle + M\varphi(\|x-h-y\|) \\ &\quad - 2\left(f(y)+\langle G(y), x-y\rangle + M\varphi(\|x-y\|)\right) + 2\varepsilon \\ &= M\left(\varphi(\|x+h-y\|)+\varphi(\|x-h-y\|)-2\varphi(\|x-y\|)\right)+2\varepsilon \\ &\leq M\varphi(2\|h\|)+2\varepsilon, \end{aligned}$$

where the last inequality follows from Lemma 2.35. □

Now, if we define $F = \operatorname{conv}(g)$, with the same proof as that of Theorem 2.10, we get that

$$F(x+h)+F(x-h)-2F(x) \leq M\varphi(\|2h\|) \quad \text{for all} \quad x, h \in X.$$

Because F is convex, by virtue of Proposition 2.34, we have that $F \in C^{1,\omega}(X)$ with

$$\|\nabla F(x)-\nabla F(y)\| \leq 4M\omega(2\|x-y\|) \quad \text{for all} \quad x, y \in X.$$

Finally, the same argument involving the function m as that at the end of Section 2.3 shows that $F = f$ and $\nabla F = G$ on E. The proof of Theorem 2.40 is complete. □

In addition, for some particular modulus of continuity, we can improve the estimation on the modulus of continuity of the gradient of the extension F provided by Theorem 2.40.

Theorem 2.44. *Considering ω, (f, G) and F as in Theorem 2.40, the following holds.*

(i) $\|\nabla F(x)-\nabla F(y)\| \leq 4M\omega\left(\|x-y\|\right)$ for all $x, y \in X$ if the function $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is convex.

(ii) $\|\nabla F(x)-\nabla F(y)\| \leq 2M\omega\left(2\|x-y\|\right)$ for all $x, y \in X$ if the function $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is concave.

In both cases, we obtain $M_\omega(\nabla F) \leq 4M$.

Furthermore, if $\omega(t)=t^\alpha$, $0<\alpha\leq 1$, we can arrange $M_\omega(\nabla F) \leq \frac{2^{2+\alpha}}{1+\alpha}M$ whenever $\alpha < \frac{1}{2}$ and $M_\omega(\nabla F) \leq \frac{2^{1+2\alpha}}{1+\alpha}M$ whenever $\alpha \geq \frac{1}{2}$.

Proof. By repeating the proof of Theorem 2.40 and using the second part of Lemma 2.35 we immediate see that

$$F(x+h)+F(x-h)-2F(x) \leq 2M\varphi(\|h\|) \quad \text{for all} \quad x, h \in X,$$

whenever $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is convex and

$$F(x+h)+F(x-h)-2F(x) \leq \frac{M}{2}\varphi(\|2h\|) \quad \text{for all} \quad x, h \in X$$

whenever $(0,+\infty) \ni t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is concave. Hence, using the same calculations as in Proposition 2.34, we get that

$$\|\nabla F(x)-\nabla F(y)\| \leq 4M\,\frac{\varphi\left(2\|x-y\|\right)}{2\|x-y\|} \quad \text{for all} \quad x, y \in X, \quad \text{in the case } (i)$$

and

$$\|\nabla F(x)-\nabla F(y)\| \leq 2M\,\frac{\varphi\left(4\|x-y\|\right)}{4\|x-y\|} \quad \text{for all} \quad x, y \in X, \quad \text{in the case } (ii).$$

Thus, the inequality $\varphi(t)/t \leq \omega\left(t/2\right)$ leads us to

$$\|\nabla F(x)-\nabla F(y)\| \leq 4M\omega\left(\|x-y\|\right) \quad \text{for all} \quad x, y \in X, \quad \text{in the case } (i)$$

and

$$\|\nabla F(x) - \nabla F(y)\| \leq 2M\omega\left(2\|x-y\|\right) \quad \text{for all} \quad x, y \in X, \quad \text{in the case } (ii).$$

As for the Hölder case, we see that $t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is convex for $\alpha < 1/2$ and $t \mapsto \dfrac{\omega^{-1}(t)}{t}$ is concave for $\alpha \geq 1/2$. Since we can compute $\frac{\varphi(t)}{t} = \frac{t^{1+\alpha}}{1+\alpha}$, it follows that

$$\|\nabla F(x) - \nabla F(y)\| \leq \frac{2^{2+\alpha}}{1+\alpha} M \|x-y\|^{\alpha} \quad \text{for all} \quad x, y \in X, \quad \text{whenever} \quad \alpha < \frac{1}{2}$$

and

$$\|\nabla F(x) - \nabla F(y)\| \leq \frac{2^{1+2\alpha}}{1+\alpha} M \|x-y\|^{\alpha} \quad \text{for all} \quad x, y \in X, \quad \text{whenever} \quad \alpha \geq \frac{1}{2}.$$

This completes the proof of the Theorem. □

Remark 2.45. Observe that if we are given a concave and strictly increasing modulus of continuity $\omega : [0, +\infty) \to [0, +\infty)$ such that $\lim_{t \to +\infty} \omega(t)$ is finite, we can define a new concave and strictly increasing modulus of continuity $\widetilde{\omega} : [0, +\infty) \to [0, +\infty)$ with $\omega \leq \widetilde{\omega}$ and $\lim_{t \to +\infty} \widetilde{\omega}(t) = +\infty$ by setting $\widetilde{\omega} = \omega$ on $[0, t_0]$ and $\widetilde{\omega}(t) = \omega(t_0) + \omega'(t_0)(t - t_0)$ for every $t \geq t_0$, where t_0 is a differentiability point of ω. Therefore, for any uniformly continuous function $G : E \to X$ there exists a modulus of continuity ω with the property that $\lim_{t \to +\infty} \omega(t) = +\infty$ and a constant $M > 0$ such that

$$\|G(x) - G(y)\| \leq M\omega(\|x-y\|), \quad x, y \in E.$$

This shows that Theorem 2.40 solves the Whitney extension problems for C^1 convex functions with uniformly continuous derivatives in full generality.

Chapter 3

$C^{1,\alpha}$ extensions of convex functions in superreflexive spaces

3.1 Moduli of smoothness and Pisier's renorming theorem

In this chapter we show that we can even go beyond the Hilbertian case by proving that a result similar to Theorem 2.40 holds for the class $C^{1,\alpha}_{\text{conv}}(X)$ whenever $(X, \|\cdot\|)$ is a superreflexive Banach space whose norm $\|\cdot\|$ has modulus of smoothness of power type $1+\alpha$, with $\alpha \in (0,1]$. Let us first recall some elementary definitions.

Given two Banach spaces X and Y, let us denote

$$d(X,Y) = \inf\{\|T\|\|T\|^{-1} \,:\, T : E \to F \text{ is an isomorphism }\},$$

with the convention $\inf(\emptyset) = +\infty$. We will say that Y is finitely representable in X if for every subspace M of Y of finite dimension and every $\varepsilon > 0$, there exists a subspace N of X such that $d(M,N) \leq 1+\varepsilon$.

Definition 3.1. *A Banach space X is said to be superreflexive if every Banach space Y which is finitely representable in X is reflexive.*

A very useful characterization of superreflexive Banach spaces is given by Pisier's renorming Theorem.

Theorem 3.2. [56, Theorem 3.1]. *Let X be a Banach space. X is superreflexive if and only if there exists an equivalent norm $\|\cdot\|$ on X such that $\|\cdot\|$ is uniformly smooth with modulus of smoothness of power type $p = 1+\alpha$ for some $0 < \alpha \leq 1$.*

For general reference about renorming properties of superreflexive spaces see, for instance [26, 33].

Throughout this chapter, and unless otherwise stated, X will denote a superreflexive Banach space, $\|\cdot\|$ an equivalent norm on X and $\|\cdot\|_*$ the dual norm of $\|\cdot\|$ on X^*. By Theorem 3.2, we may assume that the norm $\|\cdot\|$ is uniformly smooth with modulus of smoothness of power type $p = 1+\alpha$ for some $0 < \alpha \leq 1$. Hence, there exists a constant $C \geq 2$, depending only on this norm, such that

$$\|x+h\|^{1+\alpha} + \|x-h\|^{1+\alpha} - 2\|x\|^{1+\alpha} \leq C\|h\|^{1+\alpha} \quad \text{for all} \quad x, h \in X. \tag{3.1.1}$$

For a mapping $G : E \to X^*$, where E is a subset of X, we will denote

$$M_\alpha(G) = \sup_{x \neq y,\, x,y \in E} \frac{\|G(x) - G(y)\|_*}{\|x-y\|^\alpha}.$$

By a 1-jet defined on E we mean a pair of functions (f, G), where $f : E \to \mathbb{R}$ and $G : E \to X^*$.

3.2 $C^{1,\alpha}$ convex extensions of jets by explicit formulas in superreflexive spaces

We now give the definition of the suitable condition for $C^{1,\alpha}$ convex extension, which we will see later on is both necessary and sufficient.

Definition 3.3. *Given an arbitrary subset E of X, and a 1-jet $f : E \to \mathbb{R}$, $G : E \to X^*$, we will say that (f, G) satisfies the condition $(CW^{1,\alpha})$ on E with constant $M > 0$, provided that*

$$f(x) \geq f(y) + G(y)(x-y) + \frac{\alpha}{(1+\alpha)M^{1/\alpha}} \|G(x) - G(y)\|_*^{1+\frac{1}{\alpha}} \quad \textit{for all} \quad x, y \in E.$$

Note that this condition coincides with $(CW^{1,\omega})$ of Definition 2.36 in Hilbert spaces in the case that $\omega(t) = t^\alpha$, because, in this case, the function φ of (2.8.1) is $\varphi(t) = \frac{1}{1+\alpha} t^{1+\alpha}$ and then $\varphi^*(t) = \frac{\alpha}{1+\alpha} t^{1+\frac{1}{\alpha}}$.

Also, let us see that if a 1-jet (f, G) satisfies condition $(CW^{1,\alpha})$, then G is α-Holder continuous.

Remark 3.4. *If (f, G) satisfies $(CW^{1,\alpha})$ on E with constant $M > 0$, then $M_\alpha(G) \leq \left(\frac{1+\alpha}{2\alpha}\right)^\alpha M$.*

Proof. Using inequality $(CW^{1,\alpha})$ we obtain for all $x, y \in E$

$$f(x) \geq f(y) + G(y)(x-y) + \frac{\alpha}{(1+\alpha)M^{1/\alpha}} \|G(x) - G(y)\|_*^{1+\frac{1}{\alpha}},$$
$$f(y) \geq f(x) + G(x)(y-x) + \frac{\alpha}{(1+\alpha)M^{1/\alpha}} \|G(x) - G(y)\|_*^{1+\frac{1}{\alpha}}.$$

By summing both inequalities we obtain

$$\|G(x) - G(y)\|_* \|x-y\| \geq (G(x) - G(y))(x-y) \geq \frac{2\alpha}{(1+\alpha)M^{1/\alpha}} \|G(x) - G(y)\|_*^{1+\frac{1}{\alpha}}$$

which immediately implies the desired estimate. □

We next prove that condition $(CW^{1,\alpha})$ is indeed necessary for the existence of a $C^{1,\alpha}$ convex extension in Banach spaces (not necessarily superreflexive).

Proposition 3.5. *Let X be a Banach space, let $f \in C^{1,\alpha}(X)$ be convex with $M_\alpha(Df) \leq M$, and assume that f is not affine. Then (f, Df) satisfies the condition $(CW^{1,\alpha})$ on X with constant M.*

On the other hand, if f is affine and continuous, it is obvious that (f, Df) satisfies $(CW^{1,\alpha})$ on every $E \subset X$, for every $M > 0$.

Proof. Suppose that there exist different points $x, y \in X$ such that

$$f(x) - f(y) - Df(y)(x-y) < \frac{\alpha}{(1+\alpha)M^{\frac{1}{\alpha}}} \|Df(x) - Df(y)\|_*^{1+\frac{1}{\alpha}},$$

and we will get a contradiction.

Case 1. Assume further that $M = 1$, $f(y) = 0$, and $Df(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) \leq \frac{\alpha}{1+\alpha} \|Df(x)\|_*^{1+\frac{1}{\alpha}} - r \quad \text{for some} \quad r > 0. \tag{3.2.1}$$

Let us fix $0 < \varepsilon \leq \frac{r}{2} \big\|Df(x)\big\|_*^{-(1+\frac{1}{\alpha})}$ and pick $v_\varepsilon \in X$ with $\|v_\varepsilon\| = 1$ and

$$Df(x)(v_\varepsilon) \leq (\varepsilon - 1)\|Df(x)\|_*. \tag{3.2.2}$$

We define $\varphi(t) = f(x + tv_\varepsilon)$ for every $t \in \mathbb{R}$. We have $\varphi(0) = f(x)$, $\varphi'(0) = Df(x)(v_\varepsilon)$, and $\varphi'(t) = Df(x + tv_\varepsilon)(v_\varepsilon)$. This implies that

$$|\varphi(t) - \varphi(0) - \varphi'(0)t| \leq \int_0^t s^\alpha ds = \frac{t^{1+\alpha}}{1+\alpha}$$

for every $t \in \mathbb{R}^+$, hence also that

$$\varphi(t) \leq f(x) + tDf(x)(v_\varepsilon) + \frac{t^{1+\alpha}}{1+\alpha} \text{ for all } t \in \mathbb{R}^+.$$

Using first (3.2.1) and then (3.2.2) we have

$$\begin{aligned} & f\left(x + \|Df(x)\|_*^{1/\alpha} v_\varepsilon\right) = \varphi\left(\|Df(x)\|_*^{1/\alpha}\right) \\ & \leq \frac{\alpha}{1+\alpha}\|Df(x)\|_*^{1+\frac{1}{\alpha}} - r + \|Df(x)\|_*^{1/\alpha} Df(x)(v_\varepsilon) + \frac{1}{1+\alpha}\|Df(x)\|_*^{1+\frac{1}{\alpha}} \\ & \leq -r + \varepsilon\|Df(x)\|_*^{1+\frac{1}{\alpha}} \leq -\frac{r}{2} < 0, \end{aligned}$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $Df(y) = 0$. This shows that

$$f(x) \geq \frac{\alpha}{1+\alpha}\|Df(x)\|_*^{1+\frac{1}{\alpha}}.$$

Case 2. Assume only that $M = 1$. Define $g(z) = f(z) - f(y) - Df(y)(z - y)$ for every $z \in X$. Then $g(y) = 0$ and $Dg(y) = 0$. By Case 1, we get

$$g(x) \geq \frac{\alpha}{1+\alpha}\|Dg(x)\|_*^{1+\frac{1}{\alpha}},$$

and since $Dg(x) = Df(x) - Df(y)$ the Proposition is thus proved in the case when $M = 1$.
Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $\psi = \frac{1}{M}f$, which satisfies the assumption of Case 2. Therefore

$$\psi(x) - \psi(y) - D\psi(y)(x - y) \geq \frac{\alpha}{1+\alpha}\|D\psi(x) - D\psi(y)\|_*^{1+\frac{1}{\alpha}},$$

which is equivalent to the desired inequality. □

We will make use of the following differentiability criterium for continuous convex functions.

Proposition 3.6. *If f is a continuous convex function on X and*

$$f(x + h) + f(x - h) - 2f(x) \leq C\|h\|^{1+\alpha}, \quad \textit{for all} \quad x, h \in X,$$

then f is of class $C^{1,\alpha}(X)$ and $M_\alpha(Df) \leq 2^{1+\alpha}C$.

Proof. Similar to the proof of Proposition 2.34. □

The main result of this chapter is the following.

Theorem 3.7. *Given a 1-jet (f, G) defined on E satisfying the property $(CW^{1,\alpha})$ with constant M on E, the formula*

$$F = \textit{conv}(g), \quad g(x) = \inf_{y \in E}\{f(y) + G(y)(x - y) + \tfrac{M}{1+\alpha}\|x - y\|^{1+\alpha}\}, \quad x \in X,$$

defines a $C^{1,\alpha}$ convex function with $F_{|E} = f$, $(DF)_{|E} = G$, and

$$M_\alpha(DF) \leq \frac{2^{1+\alpha}C}{1+\alpha}M,$$

where C is the constant of (3.1.1).

Proof of Theorem 3.7. The general scheme of the proof is similar to that of Theorem 2.11. Let us first recall Young's inequality.

Proposition 3.8 (Young's inequality)**.** *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \varepsilon a^p + \frac{b^q}{q(\varepsilon p)^{q/p}} \quad \textit{for all} \quad a, b, \varepsilon > 0.$$

One of the main steps of the proof of Theorem 3.7 is proving the following inequality, which essentially tells us that the function g lies above every affine function $x \mapsto f(z) + G(z)(x-z),\ z \in E$.

Lemma 3.9. *We have*

$$f(z) + G(z)(x-z) \leq f(y) + G(y)(x-y) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha}$$

for every $y, z \in E,\ x \in X$.

Proof. Given $y, z \in E,\ x \in X$, condition $(CW^{1,\alpha})$ with constant M implies

$$\begin{aligned} & f(y) + G(y)(x-y) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha} \\ & \quad \geq f(z) + G(z)(y-z) + \tfrac{\alpha}{(1+\alpha)M^{1/\alpha}}\|G(y)-G(z)\|_*^{1+\frac{1}{\alpha}} \\ & \qquad + G(y)(x-y) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha} \\ & \quad = f(z) + G(z)(x-z) + \tfrac{\alpha}{(1+\alpha)M^{1/\alpha}}\|G(y)-G(z)\|_*^{1+\frac{1}{\alpha}} \\ & \qquad + (G(z)-G(y))(y-x) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha} \\ & \quad \geq f(z) + G(z)(x-z) - ab + \tfrac{M}{1+\alpha}a^{1+\alpha} + \tfrac{\alpha}{(1+\alpha)M^{1/\alpha}}b^{1+\frac{1}{\alpha}}, \end{aligned}$$

where $a = \|y-x\|$ and $b = \|G(z)-G(y)\|_*$. By applying Proposition 3.8 with

$$p = 1+\alpha, \quad q = 1+\tfrac{1}{\alpha} \quad \varepsilon = \tfrac{M}{1+\alpha},$$

we obtain that

$$-ab + \tfrac{M}{1+\alpha}a^{1+\alpha} + \tfrac{\alpha}{(1+\alpha)M^{1/\alpha}}b^{1+\frac{1}{\alpha}} \geq 0.$$

This proves the Lemma. □

Observe that Lemma 3.9 shows that $m \leq g$, where g is defined as in Theorem 3.7, and

$$m(x) := \sup_{z\in E}\{f(z) + G(z)(x-z)\}, \quad x \in X.$$

Then, using the definition of g and m, we also have that $f \leq m \leq g \leq f$ on E. Thus $g = f$ on E.

Lemma 3.10. *We have*

$$g(x+h) + g(x-h) - 2g(x) \leq \frac{CM}{1+\alpha}\|h\|^{1+\alpha} \quad \textit{for all} \quad x, h \in X,$$

where C is as in (3.1.1).

Proof. Given $x, h \in X$ and $\varepsilon > 0$, by definition of g, we can pick $y \in E$ with

$$g(x) \geq f(y) + G(y)(x-y) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha} - \varepsilon.$$

We then have

$$\begin{aligned} g(x+h)+g(x-h)-2g(x) &\leq f(y)+G(y)(x+h-y)+\tfrac{M}{1+\alpha}\|x+h-y\|^{1+\alpha} \\ &\quad + f(y)+G(y)(x-h-y)+\tfrac{M}{1+\alpha}\|x-h-y\|^{1+\alpha} \\ &\quad - 2\left(f(y)+G(y)(x-y)+\tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha}\right)+2\varepsilon \\ &= \tfrac{M}{1+\alpha}\left(\|x+h-y\|^{1+\alpha}+\|x-h-y\|^{1+\alpha}-2\|x-y\|^{1+\alpha}\right)+2\varepsilon \\ &\leq \frac{CM}{1+\alpha}\|h\|^{1+\alpha}+2\varepsilon, \end{aligned}$$

where the last inequality follows from inequality (3.1.1). □

Then, by defining $F = \text{conv}(g)$ (see definition (2.1.5)) and using Lemma 3.10, the same proof as that of Theorem 2.10 lead us to the inequality.

$$F(x+h)+F(x-h)-2F(x) \leq \frac{CM}{1+\alpha}\|h\|^{1+\alpha} \quad \text{for all} \quad x, h \in X.$$

Because F is convex and continuous, by virtue of Proposition 3.6, we have that $F \in C^{1,\alpha}(X)$ with

$$M_\alpha(DF) \leq \frac{2^{1+\alpha}C}{1+\alpha}M.$$

Finally, the same argument involving the function m as that at the end of Chapter 2, Section 2.3 shows that $F = f$ and $DF = G$ on E. □

3.3 Example in general Banach spaces

Let us finish this chapter with some comments and an example which show that we cannot expect the above results to hold true for a general Banach space X, unless X is superreflexive.

On the one hand, we claim that a necessary condition for the validity of a Whitney extension theorem of class $C^{1,\omega}(X)$ for a Banach space X is that there is a smooth bump function whose derivative is ω-continuous on X. Indeed, let $C = \{x \in X : \|x\| \geq 1\} \cup \{0\}$, and define $f : C \to \mathbb{R}$ and $G : C \to X^*$ by

$$f(x) = 0 \quad \text{if} \quad \|x\| \geq 1, \quad f(0) = 1, \quad \text{and} \quad G(x) = 0 \quad \text{for all} \quad x \in C.$$

Observe that for all $x, y \in C$,

$$|f(x)-f(y)-G(y)(x-y)| = |f(x)-f(y)| = \begin{cases} 0 & \text{if } x = y = 0 \\ 0 & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 1 & \text{if } x = 0 \text{ and } y \neq 0 \\ 1 & \text{if } x \neq 0 \text{ and } y = 0 \end{cases} \leq \|x-y\|^2$$

and $\|G(x)-G(y)\|_* = 0 \leq \|x-y\|$. Thus, the jet (f, G) satisfies the assumptions $(\widetilde{W^{1,1}})$ (or, equivalently, $(W^{1,1})$, by Remark 2.4) of the Whitney extension theorem. If a Whitney-type extension theorem were true for X, then there would exist a $C^{1,\omega}$ function $F : X \to \mathbb{R}$ such that $F(x) = 0$ for $\|x\| \geq 1$ and $F(0) = 1$. Then according to [26, Theorem V.3.2] the space X would be superreflexive.

It is unknown whether for every superreflexive Banach space X (other than a Hilbert space) a Whitney-type extension theorem for the class $C^{1,\omega}$ holds true at least for some modulus ω. It is also unknown whether a Whitney-type extension theorem holds true for every class $C^{1,\omega}(X)$ if X is a Hilbert space and ω is not linear. However the results of Chapter 2 provide some answers to analogous questions for the classes $C^{1,\omega}_{\text{conv}}(X)$.

On the other hand, one could ask whether superreflexivity of X is necessary in order to obtain Whitney-type extension theorems for the classes $C^{1,\omega}_{\text{conv}}(X)$, and wonder whether Banach spaces like c_0, with sufficiently many differentiable functions (and even with real-analytic equivalent norms), could admit such Whitney-type theorems. The following example answers the second question in the negative.

Example 3.11. *Let $X = c_0$ (the Banach space of all sequences of real numbers that converge to 0, endowed with the supremum norm). Then for every modulus of continuity ω, there are discrete sets $C \subset X$ and 1-jets (f, G) with $f : C \to \mathbb{R}$, $G : C \to X^*$ satisfying condition $(CW^{1,\omega})$ on C, and such that for no $F \in C^{1,\omega}_{conv}(X)$ do we have $F_{|_C} = f$ and $(DF)_{|_C} = G$.*

Proof. Fix a modulus of continuity ω. Let $\{e_j\}_{j=1}^\infty$ be the canonical basis of X (that is to say $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, etc.), and let $\{e_j^*\}_{j=1}^\infty$ be the associated coordinate functionals; thus we have that $\|e_j\| = 1$, $e_i^*(e_j) = \delta_{ij}$, and $\|e_j^*\|_* = 1$. Let

$$C = \{\pm e_j : j \in \mathbb{N}\} \cup \{0\},$$

and define $f : C \to \mathbb{R}$ and $G : C \to X^*$ by

$$f(0) = 0, \; f(\pm e_j) = \frac{1}{2} \quad \text{for all} \quad j \in \mathbb{N}, \quad \text{and} \quad G(0) = 0, \; G(\pm e_j) = \pm e_j^* \quad \text{for all} \quad j \in \mathbb{N}.$$

Let us first check that

$$f(x) - f(y) - G(y)(x - y) \geq \frac{1}{2} \quad \text{for all} \quad x, y \in C, \; x \neq y. \tag{3.3.1}$$

Indeed, if $y = 0$ and $x = \pm e_j$, then $f(y) = 0$, $G(y) = 0$ and $f(x) = \frac{1}{2}$; and the inequality follows immediately. If $x = 0$ and $y = \pm e_j$, then

$$f(x) - f(y) - G(y)(x - y) = f(y) + G(y)(y) = -\frac{1}{2} + e_j^*(e_j) = \frac{1}{2}.$$

If $x = \varepsilon e_j$ and $y = \theta e_i$, where $\varepsilon, \theta \in \{-1, 1\}$ and $i \neq j$, then

$$f(x) - f(y) - G(y)(x - y) = \frac{1}{2} - \frac{1}{2} + \theta e_i^*(\theta e_i^* - \varepsilon e_j^*) = \theta^2 = 1 \geq \frac{1}{2}.$$

We have thus shown inequality (3.3.1). On the other hand, taking $M = \frac{2}{\omega(1/4)}$, we have that

$$M\varphi^*\left(\frac{1}{M}\|G(x) - G(y)\|_*\right) \leq M\varphi\left(\frac{2}{M}\right) = M\int_0^{2/M} \omega^{-1}(t)dt \leq 2\omega^{-1}\left(\frac{2}{M}\right) = \frac{1}{2}.$$

Combining the above inequality with (3.3.1), we get that (f, G) satisfies condition $(CW^{1,\omega})$ on C. Assume now that there exists $F \in C^{1,\omega}_{\text{conv}}(X)$ such that (F, DF) extends the jet (f, G). If $\|x\| = 1$ then, by taking $j \in \mathbb{N}$ such that $|x_j| = 1$, we have, with $y = \text{sign}(x_j)e_j$, that

$$F(x) \geq F(y_j) + DF(y_j)(x - y_j) = \frac{1}{2} + |x_j| - 1 = \frac{1}{2},$$

and by convexity it follows that $F(x) \geq 1/2$ for all $\|x\| \geq 1$, while $F(0) = 0$. Moreover F has the following properties.

Claim 3.12. *There exists $M > 0$ for which:*

(i) $\|DF(x) - DF(y)\|_* \leq M\omega(\|x - y\|)$ for all $x, y \in X$.

(ii) $\|DF(x)\|_* \leq M\omega(1)$ for $\|x\| \leq 1$.

(iii) $|F(x) - F(y)| \leq 2M\omega(\|x - y\|)$ for all $\|x\|, \|y\| \leq 1$.

Proof of Claim 3.12. The first one is obvious from the fact that F is of class $C^{1,\omega}(X)$. The second one follows from (i) together with the fact that $DF(0) = 0$. In order to prove (iii), we note that the concavity of ω gives us that $\omega(1)\|x - y\| \leq 2\omega\left(\frac{1}{2}\|x - y\|\right)$ whenever $\|x\|, \|y\| \leq 1$. Thus, using first the Mean Value Theorem and then property (ii), it follows

$$|F(x) - F(y)| \leq M\omega(1)\|x - y\| \leq 2\omega\left(\frac{1}{2}\|x - y\|\right) \leq 2M\omega(\|x - y\|), \quad \text{for all} \quad \|x\|, \|y\| \leq 1.$$

$\square$

We now take a function $h : \mathbb{R} \to [0,1]$ of class $C^\infty(\mathbb{R})$ such that $h = 1$ on $(-\infty, 0]$ and $h = 0$ on $[\frac{1}{2}, +\infty)$. We have that $h^{(j)} = 0$ on $(-\infty, 0] \cup [\frac{1}{2}, +\infty)$ for every $j \geq 0$ and, in particular, h' and h'' are bounded on $\mathbb{R}$. Finally, let us define $\varphi = h \circ F$ on X. From the properties of F and h, it is clear that φ is of class $C^1(X)$ with $0 \leq \varphi \leq 1,\ \varphi(0) = 1$ and $\varphi(x) = 0,\ D\varphi(x) = 0$ whenever $\|x\| \geq 1$. Let us check that, in fact, $\varphi \in C^{1,\omega}(X)$. Let us denote by B_X and S_X the closed unit ball and the unit sphere of X respectively. We fix $x, y \in X$ and study three cases separately.

Case 1. $x, y \in B_X$. We can write

$$\begin{aligned} D\varphi(x) - D\varphi(y) &= h'(F(x))DF(x) - h'(F(y))DF(y) \\ &= h'(F(y))\,(DF(x) - DF(y)) + \big(h'(F(x)) - h'(F(y))\big)\,DF(x), \end{aligned}$$

which leads us to

$$\|D\varphi(x) - D\varphi(y)\|_* \leq |h'(F(y))|\|DF(x) - DF(y)\|_* + |h'(F(x)) - h'(F(y))|\|DF(x)\|_*. \tag{3.3.2}$$

The first term is bounded above by $M\,(\sup_{\mathbb{R}} |h'|)\,\omega(\|x-y\|)$ by virtue of Claim 3.12 (i). For the second one, we use the Mean Value Theorem for h' and Claim 3.12 (ii) and (iii) to obtain

$$\begin{aligned} &|h'(F(x)) - h'(F(y))|\|DF(x)\|_* \\ &\qquad \leq M\omega(1)\left(\sup_{\mathbb{R}} |h''|\right)|F(x) - F(y)| \leq 2\,\omega(1)M^2\left(\sup_{\mathbb{R}} |h''|\right)\omega(\|x-y\|). \end{aligned}$$

By plugging the last inequalities in (3.3.2), it follows

$$\|D\varphi(x) - D\varphi(y)\|_* \leq M'\omega(\|x-y\|), \quad \text{where} \quad M' = M\left(\sup_{\mathbb{R}} |h'|\right) + 2\,\omega(1)M^2\left(\sup_{\mathbb{R}} |h''|\right).$$

Case 2. $x, y \notin B_X$. In this case it is obvious that $\|D\varphi(x) - D\varphi(y)\|_* \leq M\omega(\|x-y\|)$ as $D\varphi(x) = D\varphi(y) = 0$.

Case 3. $x \in B_X$ but $y \notin B_X$. We consider a point $z \in [x,y] \cap S_X$, where $[x,y]$ denotes the segment from x to y. It is clear that $D\varphi(z) = D\varphi(y) = 0$. Then applying Case 1 to x and z, we obtain

$$\|D\varphi(x) - D\varphi(y)\|_* = \|D\varphi(x) - D\varphi(z)\|_* \leq M'\omega(\|x-z\|) \leq M'\omega(\|x-y\|).$$

We then have shown that φ is a bump function on X of class $C^{1,\omega}(X)$. Again, using for instance [26, Theorem V.3.2], $X = c_0$ would be a superreflexive space, which is absurd.

□

Observe also that Proposition 3.5 shows that $(CW^{1,\alpha})$ is a necessary condition for $C^{1,\alpha}_{\text{conv}}(X)$ extension. The above example shows that in the case that $X = c_0$ this condition is no longer sufficient, and therefore any characterization of the class of 1-jets defined on subsets of c_0 which admit $C^{1,\alpha}_{\text{conv}}$ extensions to c_0 would have to involve some new conditions.

Chapter 4

C^1 extensions of convex functions on $\mathbb{R}^n$

4.1 C^1 convex extensions from compact subsets

Let us first recall Whitney's Extension Theorem for C^1. See [70, 63] for an explicit reference.

Theorem 4.1. *Let E be a closed subset of $\mathbb{R}^n$ and let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be two functions with G continuous. A necessary and sufficient condition for the existence of a function $F \in C^1(\mathbb{R}^n)$ with $F = f$ and $\nabla F = G$ on E is that*

$$\lim_{|z-y|\to 0^+} \frac{f(z) - f(y) - \langle G(y), z-y\rangle}{|z-y|} = 0 \quad \textit{uniformly on} \quad y, z \in K \tag{W^1}$$

for every compact subset K of E.

The extension F is constructed in the same way as (2.1.1), Chapter 2 for the classes $C^{1,1}$ and $C^{1,\omega}$.

In this chapter we deal with a similar extension problem for the class of C^1 convex functions: given a closed subset E of $\mathbb{R}^n$, a continuous mapping $G : E \to \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$, how can we decide whether there is a convex function $F \in C^1(\mathbb{R}^n)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$?

Besides the very basic character of this problem, there are other reasons for wanting to solve this kind of problems, as extension techniques for convex functions have natural applications in Analysis, Differential Geometry, PDE theory (in particular Monge-Ampère equations), Economics, and Quantum Computing. See the introductions of [43, 72] for background about convex extensions problems.

We will first focus our attention on the case when our domain E is compact, and we will not study the problem for an arbitrary E until Section 4.6. Since for a function $\varphi \in C^1(\mathbb{R}^n)$ and a compact set $E \subset \mathbb{R}^n$ there always exists a modulus of continuity for the restriction $(\nabla\varphi)_{|_E}$, Theorem 2.40 provides a solution to our C^1 convex extension problem when E is compact. However, given such a 1-jet (f, G) on a compact set E, unless $\omega(t) = t$ or one has a clue about what ω might do the job, in practice it may be difficult to find a modulus of continuity ω such that (f, G) satisfies $(CW^{1,\omega})$, and for this reason it is also desirable to have a criterion for C^1 convex extendibility which does not involve dealing with moduli of continuity.

On the other hand, there is evidence suggesting that, if E is not assumed to be compact or G is not uniformly continuous, this problem has a more complicated solution; see [59, Example 4], [66, Example 3.2], and Example 5.34 in Chapter 5. These examples show in particular that there exists a closed convex set $V \subset \mathbb{R}^2$ with nonempty interior and a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$ so that f is convex on an open convex neighborhood of V and yet there is no convex function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F = f$ on V. Such V and f may be defined for instance by

$$V = \{(x, y) \in \mathbb{R}^2 \,:\, x > 0,\ xy \geq 1\},$$

and

$$f(x,y) = -2\sqrt{xy} + \frac{1}{x+1} + \frac{1}{y+1}$$

for every $(x,y) \in V$. Nevertheless, we will show that there cannot be any such examples with V compact (see Theorem 4.2 below).

If we are given a 1-jet (f,G) on E (that is, $f : E :\to \mathbb{R}$ and $G : E \to \mathbb{R}^n$) such that G is continuous and (f,G) satisfies condition (W^1) of Theorem 4.1, there always exists function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on E. In the special case that E has nonempty interior, if $f : E \to \mathbb{R}$ is convex and (f,G) satisfies (W^1), we will see that, without any further assumptions on (f,G), f *always* has a convex C^1 extension to all of $\mathbb{R}^n$ with $(\nabla F)_{|_E} = G$.

Theorem 4.2. *Let E be a compact convex subset of $\mathbb{R}^n$ with non-empty interior. Let $f : E \to \mathbb{R}$ be a convex function, and $G : E \to \mathbb{R}^n$ be a continuous mapping satisfying Whitney's extension condition (W^1) on E. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$.*

However, if E and f are convex but int(E) is empty then, in order to obtain differentiable convex extensions of f to all of $\mathbb{R}^n$ Whitney condition (W^1) is not enough and we will need to complement it with a global geometrical condition.

Definition 4.3. *Given a* 1*-jet (f,G) defined on E, we will say that (f,G) satisfies condition (CW^1) on E provided that*

$$f(x) = f(y) + \langle G(y), x-y\rangle \implies G(x) = G(y), \quad \textit{for all} \quad x,y \in E. \qquad (CW^1)$$

Observe that condition (CW^1) for a 1-jet (f,G) defined on E says that if there exist $x,y \in E$ such that the point $(x, f(x)) \in \mathbb{R}^{n+1}$ belongs to the hyperplane $H_y = \{(p,z) \in \mathbb{R}^n \times \mathbb{R} \ : \ z = f(y) + \langle G(y), p-y\rangle\}$, then the hyperplane $H_x = \{(p,z) \in \mathbb{R}^n \times \mathbb{R} \ : \ z = f(x) + \langle G(x), p-x\rangle\}$ has the same linear part as H_y and therefore $H_x = H_y$ because both contain the point $(x, f(x))$.

This condition (CW^1) together with (W^1) are necessary and sufficient for C^1 convex extension from *convex compact* subsets.

Theorem 4.4. *Let E be a compact convex subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be a convex function, and $G : E \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on E, if and only if f and G satisfy (W^1) and (CW^1) on E.*

Furthermore, in the general case of a non-convex compact set E, we will just have to add another global geometrical condition to (CW^1).

Definition 4.5. *Given a* 1*-jet (f,G) defined on E, we will say that (f,G) satisfies condition (C) on E provided that*

$$f(x) \geq f(y) + \langle G(y), x-y\rangle \quad \textit{for all} \quad x,y \in E. \qquad (C)$$

Observe that the above condition says that the function f must lie above every putative tangent $f(y) + \langle G(y), \cdot - y\rangle$. Let us make a remark on condition (C) which will simplify the statement of our main theorem.

Remark 4.6. *If (f,G) satisfies condition (C) and G is continuous, then (f,G) satisfies Whitney's condition (W^1).*

Proof. Thanks to condition (C), we can write

$$f(y) \geq f(x) + \langle G(x), y-x\rangle \quad x,y \in E.$$

We thus have

$$0 \leq \frac{f(x) - f(y) - \langle G(y), x-y\rangle}{|x-y|} \leq \frac{\langle G(x) - G(y), x-y\rangle}{|x-y|} \leq |G(x) - G(y)|, \quad x,y \in E.$$

Since G is continuous and E is compact, the term $|G(x) - G(y)|$ tends to 0 as $|x-y| \to 0^+$ uniformly on $x,y \in E$. This proves that Whitney's condition (W^1) is satisfied. □

Theorem 4.7. *Let E be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be an arbitrary function, and $G : E \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on E, if and only if (f, G) satisfies the conditions (C) and (CW^1) on E.*

Moreover, in the case that conditions (C) and (CW^1) are satisfied, the extension F can be taken to be Lipschitz on $\mathbb{R}^n$ with

$$\mathrm{Lip}(F) = \sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq \kappa \sup_{y\in E} |G(y)|,$$

where κ is an absolute constant.

In particular, if F is the function of Theorem 4.7, assuming $0 \in E$ and we define

$$\|F\|_1 := |F(0)| + \sup_{x\in\mathbb{R}^n} |\nabla F(x)|, \tag{4.1.1}$$

we obtain an extension F of f such that

$$\|F\|_1 \leq \kappa \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|_E} = f, (\nabla\varphi)_{|_E} = G\}. \tag{4.1.2}$$

Therefore the norm of our extension is nearly optimal in this case too.

It is worth noting that this kind of control of $\mathrm{Lip}(F)$ in terms of $\sup_{y\in E} |G(y)|$ solely cannot be obtained in general, as the following example shows.

Example 4.8. For every $m \in \mathbb{N}$ we define the 1-jet (f_m, G_m) on the set $E = \{0, 1\} \subset \mathbb{R}$ by $f_m(0) = 0, f_m(1) = m$ and $G_m(0) = G_m(1) = 1$. Then, for any $C^1(\mathbb{R})$ extension F of (f_m, G_m) there exists a point $z \in (0, 1)$ for which $F'(z) = m$ by the Mean Value Theorem. Thus $\mathrm{Lip}(F) = \sup_{\mathbb{R}} |F'| \geq m$ while $\sup_E |G| = 1$.

For 1-jets (f, G) not satisfying (C), the proof of Whitney's extension theorem only permits to obtain extensions $(F, \nabla F)$ (of jets (f, G) on E) which satisfy estimations of the type

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\mathrm{Lip}(f) + \sup_{y\in E} |G(y)| \right)$$

or of the type

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\sup_{y\in E} |f(y)| + \sup_{y\in E} |G(y)| \right).$$

On the other hand, in [50] M. Jiménez-Sevilla and L. Sánchez-González proved a generalization of the Whitney extension Theorem for C^1 to the setting of Banach separable spaces satisfying a condition involving approximation of Lipschitz function by C^1-smooth functions. It turns out that every separable Hilbert space satisfies this property and then, in particular, it follows that if $f : E \to \mathbb{R}$ and $G : E \to X$ are functions defined on a closed subset E of a separable Hilbert space X such that G is continuous and (f, G) satisfying Whitney's condition (W^1) on E, then there exists an extension F of class $C^1(X)$ of the 1-jet (f, G). Moreover, it is also shown in [50] that this extension F can be taken so as to satisfy

$$\mathrm{Lip}(F) \leq \kappa^* \left(\mathrm{Lip}(f) + \sup_{y\in E} \|G(y)\|_* \right),$$

where $\kappa^* > 1$ is constant which only depends on the space X and $\|\cdot\|_*$ denotes the norm on X^*. Therefore, if we consider the Hilbert space $X = \ell_2(\mathbb{N})$, every euclidean space $\mathbb{R}^n$ can be seen as a subspace of X and then, as a corollary of the results of [50], we can state the following theorem.

Theorem 4.9 (Jiménez-Sevilla–Sánchez-González, [50]). *There exists an absolute constant $\kappa^* > 1$ for which the following holds. Let $n \geq 1$. Let E be a closed subset of $\mathbb{R}^n$ and let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be two functions with G continuous. A necessary and sufficient condition for the existence of a function $F \in C^1(\mathbb{R}^n)$ with $F = f$ and $\nabla F = G$ on E is that*

$$\lim_{|z-y|\to 0^+} \frac{f(z) - f(y) - \langle G(y), z - y\rangle}{|z-y|} = 0 \quad \textit{uniformly on} \quad y, z \in K \qquad (W^1)$$

for every compact subset K of E. Moreover the function F can be taken so as to satisfy

$$\mathrm{Lip}(F) \leq \kappa^* \left(\mathrm{Lip}(f) + \sup_{y\in E} |G(y)| \right).$$

Observe that in Theorem 4.9, although the constant k^* does not depend on the dimension n, the estimation of $\mathrm{Lip}(F)$ still depends on $\mathrm{Lip}(f)$, which, a priori, has nothing to do with $\|G\|_\infty := \sup_{y\in E} |G(y)|$. Nevertheless, let us see that, in the convex extension problem we are dealing with, condition (C) allows us to estimate $\mathrm{Lip}(f)$ in terms of $\|G\|_\infty$.

Remark 4.10. *E be a compact subset of $\mathbb{R}^n$ and let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ a be two functions with G continuous and such that (f, G) satisfies condition (C) on E. Then* $\mathrm{Lip}(f) \leq \|G\|_\infty$.

Proof. Since (f, G) satisifies (C) we can write the inequalities

$$-\|G\|_\infty |x-y| \leq \langle G(y), x - y\rangle \leq f(x) - f(y) \leq \langle G(x), y - x\rangle \leq \|G\|_\infty |x-y|$$

for every $x, y \in E$. This proves the desired estimation. □

We will see in Section 4.2 below how Remark 4.10 allows us to obtain the estimation on $\mathrm{Lip}(F)$ of Theorem 4.7.

In the particular case when E is finite, Theorem 4.7 provides necessary and sufficient conditions for interpolation of finite sets of data by C^1 convex functions.

Corollary 4.11. *Let S be a finite subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ be a function. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ on S if and only if there exists a mapping $G : S \to \mathbb{R}^n$ such that f and G satisfy conditions (C) and (CW^1) on S.*

In [55, Theorem 14] it is proved that, for every finite set of *strictly convex data* in $\mathbb{R}^n$ there always exists a C^∞ convex function (or even a convex polynomial) that interpolates the given data. However, in the case that the data are convex but not strictly convex, the above corollary seems to be new.

Theorem 4.2 is a consequence of Theorem 4.4 and of the following result.

Lemma 4.12. *Let $f \in C^1(\mathbb{R}^n)$, $C \subset \mathbb{R}^n$ be a compact convex set with nonempty interior, $x_0, y_0 \in C$. Assume that f is convex on C and*

$$f(x_0) = f(y_0) + \langle \nabla f(y_0), x_0 - y_0\rangle.$$

Then $\nabla f(x_0) = \nabla f(y_0)$.

Proof.

Case 1. Suppose first that $f(x_0) = f(y_0) = 0$. We may of course assume that $x_0 \neq y_0$ as well. Then we also have $\langle \nabla f(y_0), x_0 - y_0\rangle = 0$. If we consider the C^1 function $\varphi(t) = f(y_0 + t(x_0 - y_0))$, we have that φ is convex on the interval $[0, 1]$ and $\varphi'(0) = 0$, hence $0 = \varphi(0) = \min_{t\in[0,1]} \varphi(t)$, and because $\varphi(0) = \varphi(1)$ and the set of minima of a convex function on a convex set is convex, we deduce that $\varphi(t) = 0$ for all $t \in [0, 1]$. This shows that f is constant on the segment $[x_0, y_0]$ and in particular we have

$$\langle \nabla f(z), z_0 - z_0'\rangle = 0 \quad \text{for all} \quad z, z_0, z_0' \in [x_0, y_0].$$

Now pick a point a_0 in the interior of C and a number $r_0 > 0$ so that $B(a_0, r_0) \subset \mathrm{int}(C)$. Since C is a compact convex body, every ray emanating from a point $a \in B(a_0, r_0)$ intersects the boundary of C at exactly one point. This implies that (even though the segment $[x_0, y_0]$ might entirely lie on the boundary ∂C), for every $a \in B(a_0, r_0)$, the interior of the triangle Δ_a with vertices x_0, a, y_0, relative to the affine plane spanned by these points, is contained in the interior of C; we will denote $\mathrm{relint}\,(\Delta_a) \subset \mathrm{int}(C)$.

Let p_0 be the unique point in $[x_0, y_0]$ such that $|a_0 - p_0| = d\,(a_0, [x_0, y_0])$ (the distance to the segment $[x_0, y_0]$), set $w_0 = a_0 - p_0$, and denote $v_a := a - p_0$ for each $a \in B(a_0, r_0)$. Thus for every $a \in B(a_0, r_0)$ we can write $v_a = u_a + w_0$, where $u_a := a - a_0 \in B(0, r_0)$, and in particular we have $\{v_a : a \in B(a_0, r_0)\} = B(w_0, r_0)$.

Claim 4.13. *For every z_0, z_0' in the relative interior of the segment $[x_0, y_0]$, we have $\nabla f(z_0) = \nabla f(z_0')$.*

Let us prove our claim. It is enough to show that $\langle \nabla f(z_0) - \nabla f(z_0'), v_a\rangle = 0$ for every $a \in B(a_0, r_0)$ (because if a linear form vanishes on a set with nonempty interior, such as $B(w_0, r_0)$, then it vanishes everywhere). So take $a \in B(a_0, r_0)$. Since z_0 and z_0' are in the relative interior of the segment $[x_0, y_0]$ and $\mathrm{relint}\,(\Delta_a) \subset \mathrm{int}(C)$, there exists $t_0 > 0$ such that $z_0 + tv_a, z_0' + tv_a \in \mathrm{int}(C)$ for every $t \in (0, t_0]$.

If we had $\langle \nabla f(z_0') - \nabla f(z_0), v_a\rangle > 0$ then, because f is convex on C and $f(z_0) = f(z_0') = 0$, $\langle \nabla f(z_0'), z_0 - z_0'\rangle = 0$, we would get

$$f(z_0 + tv_a) = f(z_0' + z_0 - z_0' + tv_a) \geq \langle \nabla f(z_0'), z_0 - z_0' + tv_a\rangle = \langle \nabla f(z_0'), tv_a\rangle,$$

hence

$$\lim_{t \to 0^+} \frac{f(z_0 + tv_a)}{t} \geq \langle \nabla f(z_0'), v_a\rangle > \langle \nabla f(z_0), v_a\rangle = \lim_{t \to 0^+} \frac{f(z_0 + tv_a)}{t},$$

a contradiction. By interchanging the roles of z_0, z_0', we see that the inequality $\langle \nabla f(z_0') - \nabla f(z_0), v_a\rangle < 0$ also leads to a contradiction. Therefore $\langle \nabla f(z_0') - \nabla f(z_0), v_a\rangle = 0$ and Claim 4.13 is proved.

Now, by using the continuity of ∇f, we conclude that $\nabla f(x_0) = \nabla f(y_0)$.

Case 2. In the general situation, let us consider the function h defined by

$$h(x) = f(x) - f(y_0) - \langle \nabla f(y_0), x - y_0\rangle, \;\; x \in \mathbb{R}^n.$$

It is clear that h is convex on C, and $h \in C^1(\mathbb{R}^n)$. We also have

$$\nabla h(x) = \nabla f(x) - \nabla f(y_0),$$

and in particular $\nabla h(y_0) = 0$. Besides, using the assumption that $f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0\rangle$, we have $h(x_0) = 0 = h(y_0)$, and $h(x_0) - h(y_0) = \langle \nabla h(y_0), x_0 - y_0\rangle$. Therefore we can apply Case 1 with h instead of f and we get that $\nabla h(x_0) = \nabla h(y_0) = 0$, which implies that $\nabla f(x_0) = \nabla f(y_0)$. $\square$

From the above Lemma it is clear that (CW^1) is a necessary condition for a convex function $f : E \to \mathbb{R}$ (and a mapping $G : E \to \mathbb{R}^n$) to have a convex, C^1 extension F to all of $\mathbb{R}^n$ with $\nabla F = G$ on E, and also that if the jet (f, G) satisfies (W^1) and $\mathrm{int}(E) \neq \emptyset$ then (f, G) automatically satisfies (CW^1) on E as well. It is also obvious that Theorem 4.4 is an immediate consequence of Theorem 4.7, and that the condition (C) is also necessary in Theorem 4.7. Thus, in order to prove Theorems 4.2, 4.4, and 4.7 it will be sufficient to establish the *if* part of Theorem 4.7.

4.2 Proof of the results on compact subsets

In this section, we are going to prove the *if* part of Theorem 4.7. Consider a 1-jet $(f, G) : E \to \mathbb{R} \times \mathbb{R}^n$ defined on a compact subset E of $\mathbb{R}^n$ such that G is continuous and (f, G) satisfies conditions (C) and (CW^1) on E. By Remark 4.6, (f, G) also satisfies condition Whitney's condition (W^1) and then we can apply Theorem 4.9 to obtain a function $\tilde{f} \in C^1(\mathbb{R}^n)$ such that $(\tilde{f}, \nabla \tilde{f}) = (f, G)$ on E and

$$\mathrm{Lip}(\tilde{f}) \leq \kappa^* \left(\mathrm{Lip}(f) + \|G\|_\infty\right),$$

where $\kappa^* > 1$ is an absolute constant. Moreover, we see from Remark 4.10 that $\text{Lip}(f) \leq \|G\|_\infty$ because (f, G) satisfies (C) on E. Thus we further have

$$\text{Lip}(\tilde{f}) \leq 2\kappa^* \|G\|_\infty. \tag{4.2.1}$$

Let us consider the function $m := m(f, G) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m(x) = \sup_{y \in E} \{f(y) + \langle G(y), x - y \rangle\}, \quad x \in \mathbb{R}^n. \tag{4.2.2}$$

Since E is compact and the function $y \mapsto f(y) + \langle G(y), x - y \rangle$ is continuous, it is obvious that $m(x)$ is well defined, and in fact the supremum is attained, for every $x \in \mathbb{R}^n$. Furthermore, if we set

$$K := \max_{y \in E} |G(y)| \tag{4.2.3}$$

then each affine function $x \mapsto f(y) + \langle G(y), x - y \rangle$ is K-Lipschitz, and therefore m, being the supremum of a family of convex and K-Lipschitz functions, is convex and K-Lipschitz on $\mathbb{R}^n$. Moreover, we have

$$m = f \text{ on } E. \tag{4.2.4}$$

Indeed, if $x \in E$ then, because f satisfies (C) on E, we have $f(x) \geq f(y) + \langle G(y), x - y \rangle$ for every $y \in E$, hence $m(x) \leq f(x)$. On the other hand, we also have $f(x) \leq m(x)$ because of the definition of $m(x)$ and the fact that $x \in E$.

In the case when E is convex and has nonempty interior, it is clear that if $h : \mathbb{R}^n \to \mathbb{R}$ is convex and $h = f$ on E, then $m \leq h$. Thus, in this case, m is the minimal convex extension of f to all of $\mathbb{R}^n$, which accounts for our choice of notation. However, if E is convex but has empty interior then there is no minimal convex extension operator. We refer the interested reader to [59] for necessary and sufficient conditions for m to be finite everywhere, in the situation when $f : E \to \mathbb{R}$ is convex but not necessarily everywhere differentiable.

If the function $m(f, G)$ were differentiable on $\mathbb{R}^n$, there would be nothing else to say. Unfortunately, it is not difficult to construct examples showing that $m(f, G)$ need not be differentiable outside E (even when E is convex and (f, G) satisfies (CW^1), see Examples 4.21 and 4.22 of Section 4.3). Nevertheless, a crucial step in our proof is the following fact: m is differentiable on E, provided that (f, G) satisfies conditions (C) and (CW^1) on E.

Lemma 4.14. *For each $x_0 \in E$, the function m is differentiable at x_0, with $\nabla m(x_0) = G(x_0)$.*

Proof. Notice that, by definition of m we have, for every $x \in \mathbb{R}^n$,

$$\langle G(x_0), x - x_0 \rangle + m(x_0) = \langle G(x_0), x - x_0 \rangle + f(x_0) \leq m(x).$$

Since m is convex, this means that $G(x_0)$ belongs to $\partial m(x_0)$ (the subdifferential of m at x_0). If m were not differentiable at x_0 then there would exist a number $\varepsilon > 0$ and a sequence (h_k) converging to 0 in $\mathbb{R}^n$ such that

$$\frac{m(x_0 + h_k) - m(x_0) - \langle G(x_0), h_k \rangle}{|h_k|} \geq \varepsilon \quad \text{for every } k \in \mathbb{N}. \tag{4.2.5}$$

Because the sup defining $m(x_0 + h_k)$ is attained, we obtain a sequence $(y_k) \subset E$ such that

$$m(x_0 + h_k) = f(y_k) + \langle G(y_k), x_0 + h_k - y_k \rangle,$$

and by compactness of E we may assume, up to passing to a subsequence, that (y_k) converges to some point $y_0 \in E$. Because $f = m$ on E, and by continuity of f, G and m we then have

$$\begin{aligned} f(x_0) = m(x_0) &= \lim_{k \to \infty} m(x_0 + h_k) \\ &= \lim_{k \to \infty} (f(y_k) + \langle G(y_k), x_0 + h_k - y_k \rangle) = f(y_0) + \langle G(y_0), x_0 - y_0 \rangle, \end{aligned}$$

that is, $f(x_0) = f(y_0) + \langle G(y_0), x_0 - y_0\rangle$. Since $x_0, y_0 \in E$ and (f, G) satisfies (CW^1) on E, this implies that $G(x_0) = G(y_0)$. And because $m(x_0) \geq f(y_k) + \langle G(y_k), x_0 - y_k\rangle$ by definition of m, we then have

$$\begin{aligned} &\frac{m(x_0 + h_k) - m(x_0) - \langle G(x_0), h_k\rangle}{|h_k|} \\ &\leq \frac{f(y_k) + \langle G(y_k), x_0 + h_k - y_k\rangle - f(y_k) - \langle G(y_k), x_0 - y_k\rangle - \langle G(x_0), h_k\rangle}{|h_k|} \\ &= \frac{\langle G(y_k) - G(x_0), h_k\rangle}{|h_k|} \leq |G(y_k) - G(x_0)| = |G(y_k) - G(y_0)|, \end{aligned}$$

from which we deduce, using the continuity of G, that

$$\limsup_{k\to\infty} \frac{m(x_0 + h_k) - m(x_0) - \langle G(x_0), h_k\rangle}{|h_k|} \leq 0,$$

in contradiction with (4.2.5). □

Now we proceed with the rest of the proof of Theorem 4.7. Our strategy will be to use the differentiability of m on ∂E in order to construct a (not necessarily convex) differentiable function g such that $g = f$ and $\nabla g = G$ on E, $g \geq m$ on $\mathbb{R}^n$, and $\lim_{|x|\to\infty} g(x) = \infty$. Then we will define F as the convex envelope of g, which will be of class $C^1(\mathbb{R}^n)$ and $(F, \nabla F)$ will coincide with (f, G) on E.

For each $\varepsilon > 0$, let $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined by

$$\theta_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ t^2 & \text{if} \quad t \leq \frac{K+\varepsilon}{2} \\ (K+\varepsilon)\left(t - \frac{K+\varepsilon}{2}\right) + \left(\frac{K+\varepsilon}{2}\right)^2 & \text{if} \quad t > \frac{K+\varepsilon}{2} \end{cases} \tag{4.2.6}$$

(recall that $K = \|G\|_\infty \leq \mathrm{Lip}(\tilde{f})$). Observe that $\theta_\varepsilon \in C^1(\mathbb{R})$, $\mathrm{Lip}(\theta_\varepsilon) = K + \varepsilon$. Now set

$$\Phi_\varepsilon(x) = \theta_\varepsilon\left(d(x, E)\right), \tag{4.2.7}$$

where $d(x, E)$ stands for the distance from x to E, notice that $\Phi_\varepsilon(x) = d(x, E)^2$ on an open neighborhood of E, and define

$$H_\varepsilon(x) = |\tilde{f}(x) - m(x)| + 2\Phi_\varepsilon(x), \quad x \in \mathbb{R}^n.$$

Note that $\mathrm{Lip}(\Phi_\varepsilon) = \mathrm{Lip}(\theta_\varepsilon)$ because $d(\cdot, E)$ is 1-Lipschitz, and therefore

$$\mathrm{Lip}(H_\varepsilon) \leq \mathrm{Lip}(\widetilde{f}) + K + 2(K + \varepsilon) \leq 4\,\mathrm{Lip}(\widetilde{f}) + 2\varepsilon. \tag{4.2.8}$$

Claim 4.15. *H_ε is differentiable on E, with $\nabla H_\varepsilon(x_0) = 0$ for every $x_0 \in E$.*

Proof. The function $d(\cdot, E)^2$ is obviously differentiable, with a null gradient, at x_0, hence we only have to see that $|\widetilde{f} - m|$ is differentiable, with a null gradient, at x_0. We have that $m(x_0) = f(x_0) = \widetilde{f}(x_0)$ by (4.2.4) and also $\nabla m(x_0) = G(x_0) = \nabla\widetilde{f}(x_0)$ by Lemma 4.14. This implies that

$$\frac{|\widetilde{f}(x) - m(x)|}{|x - x_0|} = \frac{|\widetilde{f}(x) - \widetilde{f}(x_0) - \langle\nabla\widetilde{f}(x_0), x - x_0\rangle|}{|x - x_0|} + \frac{|m(x) - m(x_0) - \langle\nabla m(x_0), x - x_0\rangle|}{|x - x_0|}$$

tends to 0 as $|x - x_0| \to 0^+$, which proves our Claim. □

Now, because Φ_ε is continuous and positive on $\mathbb{R}^n \setminus E$, using mollifiers and a partition of unity, one can construct a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n \setminus E)$ such that

$$|\varphi_\varepsilon(x) - H_\varepsilon(x)| \leq \Phi_\varepsilon(x) \quad \text{for every } x \in \mathbb{R}^n \setminus E, \tag{4.2.9}$$

and
$$\mathrm{Lip}(\varphi_\varepsilon) \leq \mathrm{Lip}(H_\varepsilon) + \varepsilon \tag{4.2.10}$$
(see for instance [47, Proposition 2.1] for a proof in the more general setting of Riemannian manifolds, or [4] even for possibly infinite-dimensional Riemannian manifolds). Let us define $\widetilde{\varphi} = \widetilde{\varphi_\varepsilon} : \mathbb{R}^n \to \mathbb{R}$ by
$$\widetilde{\varphi} = \begin{cases} \varphi_\varepsilon(x) & \text{if} \quad x \in \mathbb{R}^n \setminus E \\ 0 & \text{if} \quad x \in E. \end{cases}$$

Claim 4.16. *The function $\widetilde{\varphi}$ is differentiable on $\mathbb{R}^n$, and it satisfies $\nabla\widetilde{\varphi}(x_0) = 0$ for every $x_0 \in E$.*

Proof. It is obvious that $\widetilde{\varphi}$ is differentiable on $\mathrm{int}(E) \cup (\mathbb{R}^n \setminus E)$. We also have $\nabla\widetilde{\varphi} = 0$ on $\mathrm{int}(E)$, trivially. We only have to check that $\widetilde{\varphi}$ is differentiable, with a null gradient, on ∂E. If $x_0 \in \partial E$ we have (recalling that $\Phi_\varepsilon(x) = d(x,E)^2$ on a neighborhood of E) that
$$\frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)|}{|x - x_0|} = \frac{|\widetilde{\varphi}(x)|}{|x - x_0|} \leq \frac{|H_\varepsilon(x)| + d(x,E)^2}{|x - x_0|} \to 0$$
as $|x - x_0| \to 0^+$, because both H_ε and $d(\cdot, E)^2$ vanish at x_0 and are differentiable, with null gradients, at x_0. Therefore $\widetilde{\varphi}$ is differentiable at x_0, with $\nabla\widetilde{\varphi}(x_0) = 0$. □

Note also that
$$\mathrm{Lip}(\widetilde{\varphi}) = \mathrm{Lip}(\varphi_\varepsilon) \leq \mathrm{Lip}(H_\varepsilon) + \varepsilon \leq 4\,\mathrm{Lip}(\widetilde{f}) + 3\varepsilon. \tag{4.2.11}$$

Next we define
$$g = g_\varepsilon := \tilde{f} + \widetilde{\varphi}. \tag{4.2.12}$$
The function g is differentiable on $\mathbb{R}^n$, and coincides with $\tilde{f} = f$ on E. Moreover, we also have $\nabla g = \nabla \tilde{f} = G$ on E (because $\nabla\widetilde{\varphi} = 0$ on E). And, for $x \in \mathbb{R}^n \setminus E$, we have
$$g(x) \geq \tilde{f}(x) + H(x) - \Phi_\varepsilon(x) = \tilde{f}(x) + |\tilde{f}(x) - m(x)| + \Phi_\varepsilon(x) \geq m(x) + \Phi_\varepsilon(x).$$
This shows that $g \geq m$. On the other hand, because m is K-Lipschitz, we have
$$m(x) \geq m(0) - K|x|,$$
and because E is bounded, say $E \subset B(0,R)$ for some $R > 0$, also
$$\begin{aligned}\Phi_\varepsilon(x) &= (K+\varepsilon)d(x,E) - \frac{(K+\varepsilon)^2}{4} \\ &\geq (K+\varepsilon)d(x,B(0,R)) - \frac{(K+\varepsilon)^2}{4} = (K+\varepsilon)\left(|x| - R - \frac{K+\varepsilon}{4}\right)\end{aligned}$$
for $|x| \geq R + \frac{K+\varepsilon}{2}$. Hence
$$g(x) \geq m(x) + \Phi_\varepsilon(x) \geq m(0) - K|x| + (K+\varepsilon)\left(|x| - R - \frac{K+\varepsilon}{4}\right),$$
for $|x|$ large enough, which implies
$$\lim_{|x|\to\infty} g(x) = \infty.$$
Also, notice that according to (4.2.11) and the definition of g, we have
$$\mathrm{Lip}(g) \leq \mathrm{Lip}(\widetilde{f}) + \mathrm{Lip}(\widetilde{\varphi}) \leq 5\,\mathrm{Lip}(\widetilde{f}) + 3\varepsilon.$$
Now we will use a differentiability property of the convex envelope of a function $\psi : \mathbb{R}^n \to \mathbb{R}$, defined by
$$\mathrm{conv}(\psi)(x) = \sup\{h(x) \,:\, h \text{ is convex}, h \leq \psi\}.$$

Another expression for conv(ψ), which follows from Carathéodory's Theorem, is

$$\operatorname{conv}(\psi)(x) = \inf\left\{\sum_{j=1}^{n+1}\lambda_j\psi(x_j) \,:\, \lambda_j \geq 0,\, \sum_{j=1}^{n+1}\lambda_j = 1,\, x = \sum_{j=1}^{n+1}\lambda_j x_j\right\},$$

see [57, Corollary 17.1.5] for instance. The following result is a restatement of a particular case of the main theorem in [51]; see also [48].

Theorem 4.17 (Kirchheim-Kristensen)**.** *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\lim_{|x|\to\infty}\psi(x) = \infty$, then conv($\psi$) $\in C^1(\mathbb{R}^n)$.*

Although not explicitly stated in that paper, the proof of [51] also shows that

$$\operatorname{Lip}\left(\operatorname{conv}(\psi)\right) \leq \operatorname{Lip}(\psi).$$

If we define $F = \operatorname{conv}(g)$ we thus get that F is convex on $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n)$. Moreover, thanks to (4.2.1),

$$\operatorname{Lip}(F) \leq \operatorname{Lip}(g) \leq 5\operatorname{Lip}(\widetilde{f}) + 3\varepsilon \leq 10\,\kappa^* \sup_{y\in E}|G(y)| + 3\varepsilon. \tag{4.2.13}$$

Let us now check that $F = f$ on E. Since m is convex on $\mathbb{R}^n$ and $m \leq g$, we have that $m \leq F$ on $\mathbb{R}^n$ by definition of conv(g). On the other hand, since $g = f$ on E we have, for every convex function h with $h \leq g$, that $h \leq f$ on E, and therefore, for every $y \in E$,

$$F(y) = \sup\{h(y) \,:\, h \text{ is convex }, h \leq g\} \leq f(y) = m(y).$$

This shows that $F(y) = f(y)$ for every $y \in E$. In order to see that we also have $\nabla F(y) = G(y)$ for every $y \in E$, we use the following differentiability criterium, whose proof was given in Chapter 2, Lemma 2.14.

Lemma 4.18. *If $\phi : \mathbb{R}^n \to \mathbb{R}$ is convex and $\psi : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $y \in \mathbb{R}^n$ with $\phi \leq \psi$, and $\phi(y) = \psi(y)$, then ϕ is differentiable at y, with $\nabla\phi(y) = \nabla\psi(y)$.*

Since we know that $m \leq F$, $m(y) = f(y) = F(y)$ for all $y \in E$, and $F \in C^1(\mathbb{R}^n)$, it follows from Lemma 4.18 (by taking $\phi = m$ and $\psi = F$), and from Lemma 4.14, that

$$G(y) = \nabla m(y) = \nabla F(y) \text{ for all } y \in E.$$

Finally, note that, inequality (4.2.13) implies (by assuming that $\varepsilon \leq \kappa^*\|G\|_\infty/3$, which we may do) that

$$\operatorname{Lip}(F) \leq 11\,\kappa^* \sup_{y\in E}|G(y)| \tag{4.2.14}$$

and also, assuming $0 \in E$, that

$$\|F\|_1 \leq 11\,\kappa^* \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|E} = f, (\nabla\varphi)_{|E} = G\}. \tag{4.2.15}$$

The proof of Theorem 4.7 is complete.

Remark 4.19. *The function F provided by Theorem 4.7 can be taken so as to satisfy $\lim_{|x|\to\infty}F(x) = \infty$.*

Proof. We have seen in Section 4.2 that the extension F of Theorem 4.7 is defined as $F = \operatorname{conv}(g)$, where g is defined in (4.2.12) and has the property that

$$g(x) \geq m(x) + \theta_\varepsilon(d(x, E)), \quad x \in \mathbb{R}^n,$$

where θ_ε is defined in (4.2.6). If $R > 0$ is a number such that $E \subset B(0,R)$, then, because θ_ε is nondecreasing,

$$g(x) \geq m(x) + \theta_\varepsilon(d(x, B(0,R))), \quad x \in \mathbb{R}^n.$$

Notice that, since θ_ε is convex and nondecreasing, $c := m + \theta_\varepsilon(d(\cdot, B(0,R))$ is a convex function. Moreover, if $|x| \geq R + \frac{K+\varepsilon}{2}$, (here K denotes $\sup_{y\in E} |G(y)|$) then

$$m(x) + \theta_\varepsilon(d(x, B(0,R))) \geq m(0) - K|x| + (K+\varepsilon)\left(|x| - R - \frac{K+\varepsilon}{2}\right) + \left(\frac{K+\varepsilon}{2}\right)^2,$$

and the last term tends to ∞ as $|x| \to \infty$. Since $F = \operatorname{conv}(g)$ and c is a convex function with $c \leq g$, we must have $F \geq c$, which implies that $\lim_{|x|\to\infty} F(x) = \infty$. $\square$

We finish this section by proving that a natural variation of condition (CW^1) allows us to stablish an extension Theorem for C^1 convex function from bounded (not necessarily closed) subsets of $\mathbb{R}^n$.

Theorem 4.20. *Let E be a bounded (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be an arbitrary function, and $G : E \to \mathbb{R}^n$ be a bounded continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on E, if and only if (f, G) satisfies the following two conditions.*

(i) *$f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *For every pair of sequences $(x_k)_k, (y_k)_k \subset E$, then*

$$\lim_k (f(x_k) - f(y_k) - \langle G(y_k), x_k - y_k\rangle) = 0 \implies \lim_k |G(x_k) - G(y_k)| = 0.$$

Moreover, in the case that both conditions are satisfied, the extension F can be taken so that

$$\operatorname{Lip}(F) = \sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq \kappa \sup_{y\in E} |G(y)|,$$

where κ is an absolute constant. Furthermore, F can be taken with the property that $\lim_{|x|\to\infty} F(x) = +\infty$.

Proof.

Only if part: Assume that F is a convex function of class $C^1(\mathbb{R}^n)$ such that $(F, \nabla F)$ agrees with (f, G) on E. By convexity and differentiability of F, we have $F(x) \geq F(y) + \langle \nabla F(y), x - y\rangle$ for every $x, y \in E$, which clearly implies condition (i). Now, given two sequences $(x_k)_k$ and $(y_k)_k$ of E with

$$\lim_k (F(x_k) - F(y_k) - \langle \nabla F(y_k), x_k - y_k\rangle) = 0,$$

suppose that $(G(x_k) - G(y_k))_k$ does not converge to 0. Then, after passing to subsequences, we may and do assume that $(x_k)_k$ converges to x and $(y_k)_k$ converges to y for some $x, y \in \overline{E}$ and for some $\varepsilon > 0$ we have

$$|\nabla F(x_k) - \nabla F(y_k)| \geq \varepsilon \quad \text{for every} \quad k \in \mathbb{N}. \tag{4.2.16}$$

The continuity of F and ∇F yields

$$F(x) = F(y) + \langle \nabla F(y), x - y\rangle.$$

Using the necessity of Theorem 4.7, that is, Lemma 4.12, we obtain that

$$\lim_k |\nabla F(x_k) - \nabla F(y_k)| = |\nabla F(x) - \nabla F(y)| = 0,$$

which contradicts (4.2.16). We have thus shown the necessity of condition (ii).

If part: Let us first prove that f is K-Lipschitz on E with $K = \|G\|_\infty = \sup_{y\in E}|G(y)|$. Using condition (i), we can write the inequalities

$$\langle G(y), x-y\rangle \le f(x) - f(y) \le \langle G(x), x-y\rangle \quad x, y \in E.$$

It is then obvious that $|f(x) - f(y)| \le K|x-y|$ for every $x, y \in E$. Let us now prove that G is uniformly continuous on E. Indeed, let $(x_k)_k$ and $(y_k)_k$ two sequences on E with $\lim_k |x_k - y_k| = 0$. Using condition (i), we get

$$0 \le f(x_k) - f(y_k) - \langle G(y_k), x_k - y_k\rangle \le \langle G(x_k) - G(y_k), x_y - y_k\rangle \le 2K|x_k - y_k|,$$

where the last term tends to 0 as $k \to \infty$. By condition (ii), we must have that $\lim_k |G(x_k)-G(y_k)| = 0$, which proves that G is uniformly continuous on E. Thus, both f and G can be uniquely extended with continuity to the closure $\overline{E}$ of E. Notice that the continuity of f and G on $\overline{E}$ shows that

$$f(x) \ge f(y) + \langle G(y), x-y\rangle \quad \text{for all} \quad x, y \in \overline{E}.$$

Also, given two points $x, y \in \overline{E}$ such that $f(x) = f(y) + \langle G(y), x-y\rangle$, we can find two sequences $(x_k)_k$ and $(y_k)_k$ on E with $\lim_k x_k = x$ and $\lim_k y_k = y$. Again, the continuity of f and G on $\overline{E}$ leads us to

$$\lim_k (f(x_k) - f(y_k) - \langle G(y_k), x_k - y_k\rangle) = 0,$$

which in turn implies, by condition (ii), that

$$|G(x) - G(y)| = \lim_k |G(x_k) - G(y_k)| = 0.$$

That is $G(x) = G(y)$. We have thus shown that the pair (f, G) satisfies conditions (C) and (CW^1) on the compact set $\overline{E}$ of $\mathbb{R}^n$. By Theorem 4.7, there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $(F, \nabla F) = (f, G)$ on E. Moreover, F can be taken so that

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \le \kappa \sup_{y\in\overline{E}} |G(y)| = \kappa \sup_{y\in E} |G(y)|,$$

where κ is an absolute constant. Furthermore, in view of Remark 4.19, this function F can be taken with the property that $\lim_{|x|\to\infty} F(x) = +\infty$. $\square$

4.3 Some relevant examples

In this section we will consider some examples relevant to the preceding results and proofs. We first observe that the functions $m(f) = m(f, \nabla f)$ in (4.2.2) need not be differentiable outside E, even in the case when E is a convex body and f is C^∞ on E. To see this, we provide two different examples.

Example 4.21. Let g be the function $g(x, y) = \max\{x+y-1, -x+y-1, \frac{1}{3}y\}$. Using for instance the smooth maxima mentioned in Lemma 1.8, one can smooth away the edges of the graph of g produced by the intersection of the plane $z = \frac{1}{3}y$ with the planes $z = y \pm x - 1$, thus obtaining a smooth convex function f defined on $E := g^{-1}(-\infty, 0] \cap \{(x, y) : y \ge -1\}$. However, $m(f)$ will not be everywhere differentiable, because for $y \ge 2$ we have $m(f)(x, y) = \max\{x+y-1, -x+y-1\}$, and this max function is not smooth on the line $x = 0$.

Example 4.22. [71, Example 9] Let $E = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \le 1, \ y \le 0\}$ and

$$f(x,y) = \begin{cases} 0 & \text{if } y + |x| \le 0, \\ x^2 + y^2 & \text{if } y - |x| \ge 0, \\ \frac{1}{2}(|x| + y)^2 & \text{otherwise} \end{cases}$$

Then f is a C^1 convex function on $\mathbb{R}^2$ but $m(f)$ is not differentiable at any point $(0, y)$ with $y > 0$.

The following example shows that when E has empty interior there are convex functions $f : E \to \mathbb{R}$ and continuous mappings $G : E \to \mathbb{R}^n$ which satisfy (W^1) but do not satisfy (CW^1).

Example 4.23. Let E be the segment $\{0\} \times [0, 1]$ in $\mathbb{R}^2$, and f, G be defined by $f(0, y) = 0$ and $G(0, y) = (y, 0)$. If we define $\widetilde{f}(x, y) = xy$ then it is clear that $\widetilde{f}$ is a C^1 extension of f to $\mathbb{R}^2$ which satisfies $\nabla \widetilde{f}(0, y) = G(0, y)$ for $(0, y) \in E$. Therefore the 1-jet (f, G) satisfies Whitney's extension condition (W^1). However, for every $(0, y), (0, y') \in E$, we have that

$$f(0, y) - f(0, y') - \langle G(0, y'), (0, y) - (0, y')\rangle = \langle G(0, y'), (0, y' - y)\rangle = \langle (y', 0), (0, y' - y)\rangle = 0$$

but $G(0, y) = (y, 0) \neq (y', 0) = G(0, y')$ for every $y, y' \in [0, 1]$ with $y \neq y'$. Thus (f, G) does not satisfy (CW^1) on E. In particular f does not have any convex C^1 extension F to $\mathbb{R}^n$ with $\nabla F = G$ on E.

Finally, let us mention that in Theorem 4.17 the condition of coercivity for ψ (that is, $\lim_{|x|\to\infty} \psi(x) = +\infty$) cannot be removed in general as shown by the next example due to J. Benoist and J-B. Hiriart-Urruty in [14].

Example 4.24. The function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\psi(x, y) = \sqrt{x^2 + e^{-y^2}}$ for all $(x, y) \in \mathbb{R}^2$ is of class C^∞. Hoewever, the convex envelope conv(ψ) of ψ satisfies conv$(\psi)(x, y) = |x|$ for every $(x, y) \in \mathbb{R}^2$. That is, conv(ψ) is not differentiable.

4.4 Interpolation of compact subsets by boundaries of C^1 convex bodies

In this section we present a geometrical application of Theorem 4.7 concerning characterizations of compact subsets K of $\mathbb{R}^n$ which can be interpolated by boundaries of C^1 convex bodies (with prescribed unit outer normals on K). This result may be compared to [43], where M. Ghomi showed how to construct C^m smooth strongly convex bodies V with prescribed strongly convex submanifolds and tangent planes. Theorem 4.29 below allows us to deal with arbitrary compacta instead of manifolds, and to drop the strong convexity assumption. Unlike the $C^{1,1}$ case, the oriented distance function to a convex body V of class C^1 is not necessarily of class C^1 on a neighbourhood of ∂V, as M. C. Delfour and J. P. Zolesio noted in [25, Remark 5.6]. For this reason we cannot make use of the tools in Subsection 2.4.1, Lemma 2.18 to construct C^1 convex functions whose derivatives are equal (or, at least, are proportional) to the outer unit normal n_V on the boundary ∂V of a given convex body V. Instead we will make an intensive use of the differentiability properties of the Minkowski functional associated to convex bodies containing the origin as an interior point. In fact, we will define the outer unit normal of C^1 convex bodies in terms of the Minkowski functional.

4.4.1 The Minkowski functional. Elementary properties and differentiability

Definition 4.25. *Given a nonempty subset V of a Hilbert space $(X, \|\cdot\|)$ we define the Minkowski functional of V by*

$$\mu_V(x) = \inf\{t \geq 0 \, : \, x \in tV\}, \quad x \in X.$$

The following proposition sums up some well-known properties of the Minkowski functional associated to convex subsets with nonempty interior. See [33, Chapter (II)] for details.

Proposition 4.26. *If $V \subseteq X$ is convex with $0 \in \text{int}(V)$ we have:*

(1) $0 \leq \mu_V(x) < +\infty$ *for all $x \in X$, and $\mu_V(0) = 0$.*

(2) $\mu_V = \mu_{\text{int}(V)} = \mu_{\overline{V}}$.

(3) *If $0 < t < \infty$, then $\mu_V(x) < t$ if and only if $x \in t\,\text{int}(V) = \text{int}(tV)$.*

(4) *μ_V is a positively homogeneous subadditive functional. In particular, μ_V is convex.*

(5) $\{x \in X \,:\, \mu_V(x) < 1\} = \operatorname{int}(V) \subset V \subset \overline{V} = \{x \in X \,:\, \mu_V(x) \leq 1\}$,

(6) *If $r > 0$ is such that $B(0,r) \subset V$, then $\mu_V(x) \leq r^{-1}\|x\|$ for all $x \in X$. Also,*

(7) *μ_V is r^{-1}-Lipschitz, and*

(8) *$\mu_V(x) - 1 \leq r^{-1} d(x,V)$ for all $x \in X$.*

Suppose in addition that $V \subset X$ is bounded and $R > 0$ is such that $V \subset B(0,R)$.

(9) *There exists $R > 0$ such that $\mu_V(x) \geq R^{-1}\|x\|$ for all $x \in X$.*

(10) *For all $x \in X$, we have $d(x, \partial V) \leq R|\mu_V(x) - 1|$. In particular $d(x,V) \leq R(\mu_V(x) - 1)$ if $x \in X \setminus V$.*

Now we study the differentiability of the Minskowski functional associated to convex bodies.

Proposition 4.27. *Let F be a $C^1(X)$ convex function such that $F(0) < 1$. If $V = \{x \in X \,:\, F(x) \leq 1\}$, then the Minkowski functional μ_V of V is of class $C^1(X \setminus \mu_V^{-1}(0))$ and*

$$\nabla \mu_V(x) = \frac{\mu_V(x)}{\left\langle \nabla F\left(\frac{x}{\mu_V(x)}\right), x \right\rangle} \nabla F\left(\frac{x}{\mu_V(x)}\right) \quad \textit{for all} \quad x \in X \setminus \mu_V^{-1}(0).$$

Proof. We know from Proposition 2.16 that V is a closed and convex with $0 \in \operatorname{int}(V)$ and that $\partial V = \{x \in X \,:\, F(x) = 1\}$. The convexity of F gives

$$\langle \nabla F(x), x \rangle \geq F(x) - F(0) > 0 \quad \text{for all} \quad x \in \partial V. \tag{4.4.1}$$

Because μ_V is convex, in order to show that $\mu_V \in C^1(X \setminus \mu_V^{-1}(0))$ it is enough to check that μ_V is differentiable at every $x_0 \in X \setminus \mu_V^{-1}(0)$. Let us distinguish two cases.

Case 1. Suppose first that $x_0 \in \partial V$. Consider the function $H(x,y) = F(y \cdot x) - 1$ for every $(x,y) \in X \times \mathbb{R}$. Since F is of class $C^1(X)$, the function H is of class $C^1(X \times \mathbb{R})$ and it is clear that $H(x_0, 1) = 0$ and $D_y H(x_0, 1) = DF(x_0)(x_0) > 0$ by (4.4.1). The Implicit Function Theorem provides an open neighbourhood $U \subset X$ of x_0 and a unique function $\varphi : U \to \mathbb{R}$ of class $C^1(U)$ such that $\varphi(x_0) = 1$ and $F(\varphi(x) \cdot x) - 1 = H(x, \varphi(x)) = 0$ for every $x \in U$. Moreover, the derivative of φ at each $x \in U$ is

$$D\varphi(x) = \frac{-1}{D_y H(x, \varphi(x))} D_x H(x, \varphi(x)) = \frac{-\varphi(x)}{DF(\varphi(x) \cdot x)(x)} DF(\varphi(x) \cdot x).$$

In particular

$$D\varphi(x_0) = \frac{-1}{DF(x_0)(x_0)} DF(x_0).$$

Obviously U can be chosen small enough so that φ and μ_V are strictly positive on U. Because $F(\varphi(x) \cdot x) = 1$ on U, then $\varphi(x) \cdot x \in \partial V$ and $1 = \mu_V(\varphi(x) \cdot x) = \varphi(x)\mu_V(x)$ for every $x \in U$. This shows that $\mu_V = \frac{1}{\varphi}$ on U and then μ_V is differentiable at x_0 with

$$\nabla \mu_V(x_0) = \frac{-1}{\varphi^2(x_0)} \nabla \varphi(x_0) = -\nabla \varphi(x_0) = \frac{1}{\langle \nabla F(x_0), x_0 \rangle} \nabla F(x_0).$$

Case 2. If $x_0 \in X \setminus \mu_V^{-1}(0)$ is arbitrary, using that $\frac{x_0}{\mu_V(x_0)} \in \partial V$ and that μ_V is positively homogeneous we can write

$$\begin{aligned}
&\frac{\left| \mu_V(x_0 + h) - \mu_V(x_0) - \left\langle \nabla \mu_V\left(\frac{x_0}{\mu_V(x_0)}\right), h \right\rangle \right|}{\|h\|} \\
&= \frac{\left| \mu_V\left(\frac{x_0}{\mu_V(x_0)} + \frac{h}{\mu_V(x_0)}\right) - \mu_V\left(\frac{x_0}{\mu_V(x_0)}\right) - \left\langle \nabla \mu_V\left(\frac{x_0}{\mu_V(x_0)}\right), \frac{h}{\mu_V(x_0)} \right\rangle \right|}{\left\| \frac{h}{\mu_V(x_0)} \right\|},
\end{aligned}$$

and the last term tends to 0 as $\|h\| \to 0^+$ by Case 1. This shows that μ_V is differentiable at x_0 and

$$\nabla\mu_V(x_0) = \nabla\mu_V\left(\frac{x_0}{\mu_V(x_0)}\right) = \frac{\mu_V(x_0)}{\left\langle \nabla F\left(\frac{x_0}{\mu_V(x_0)}\right), x_0\right\rangle}\nabla F\left(\frac{x_0}{\mu_V(x_0)}\right).$$

□

4.4.2 An interpolation theorem for C^1 compact convex bodies

Definition 4.28. *We will say that a subset $V \subset \mathbb{R}^n$ is a compact convex body if V is compact and convex with nonempty interior. If we further assume that $0 \in \operatorname{int}(V)$, we will say that V is of class C^1 if its Minkowski functional μ_V is of class C^1 on $\mathbb{R}^n \setminus \{0\}$. In this case, we define the outer unit normal to ∂V by*

$$n_V(x) = \frac{\nabla\mu_V(x)}{|\nabla\mu_V(x)|}, \quad x \in \partial V.$$

Finally, we will say that a vector $u \in \mathbb{S}^{n-1}$ is outwardly normal to ∂V at a point $y \in \partial V$ if $u = n_V(y)$.

Here and below $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$. Now we have all the ingredients we need to state and prove our interpolation theorem for C^1 compact convex bodies containing the origin as an interior point. The pertinent conditions are:

$$\begin{array}{llll} (\mathcal{O}) & \langle N(y), y\rangle > 0 & \text{for all} & y \in K; \\ (\mathcal{K}) & \langle N(y), x-y\rangle \leq 0 & \text{for all} & x, y \in K; \\ (\mathcal{KW}^1) & \langle N(y), x-y\rangle = 0 \implies N(x) = N(y) & \text{for all} & x, y \in K, \end{array}$$

and our result for the class C^1 then reads as follows.

Theorem 4.29. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{S}^{n-1}$ be a continuous mapping. Then the following statements are equivalent.*

1. *There exists a C^1 compact convex body V with $0 \in int(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
2. *K and N satisfy conditions $(\mathcal{O})$, $(\mathcal{K})$, and $(\mathcal{KW}^1)$.*

Proof. (2) $\implies$ (1): We set $E = K \cup \{0\}$. Thanks to condition $(\mathcal{O})$, continuity of N and compactness of K, we can find a number $\alpha > 0$ sufficiently close to 1 so that

$$0 < 1 - \alpha < \min_{y\in K}\langle N(y), y\rangle. \tag{4.4.2}$$

Notice in particular that $0 \notin K$. We now define a 1-jet (f, G) on E by

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in K \\ \alpha & \text{if} \quad y = 0, \end{cases} \quad \text{and} \quad G(y) = \begin{cases} N(y) & \text{if} \quad y \in K \\ 0 & \text{if} \quad y = 0. \end{cases}$$

Because N is continuous, G is continuous as well and let us check that f and G satisfy conditions (C) and (CW^1) of Theorem 4.7 on the set E. Given $x, y \in E$, we see from the definition of f and G that

$$f(x) - f(y) - \langle G(y), x-y\rangle = \begin{cases} -\langle N(y), x-y\rangle & \text{if} \quad x, y \in K \\ 1-\alpha & \text{if} \quad x \in K,\ y = 0 \\ \alpha - 1 + \langle N(y), y\rangle & \text{if} \quad x = 0,\ y \in K. \end{cases}$$

Using condition $(\mathcal{K})$ in the case $x, y \in K$ and the choice of α (see (4.4.2)) if $x = 0$ or $y = 0$, we obtain that

$$f(x) - f(y) - \langle G(y), x-y\rangle \geq 0, \quad x, y \in E.$$

This shows that condition (C) is satisfied. Also, note that if $x = 0$ or $y = 0$, the above inequality is in fact a strict inequality, which implies that condition (CW^1) is trivially satisfied in these cases. In the case when $x, y \in K$, if we assume

$$-\langle N(y), x-y\rangle = f(x) - f(y) - \langle G(y), x-y\rangle = 0,$$

then condition $(\mathcal{KW}^1)$ implies that $G(x) = N(x) = N(y) = G(y)$. We have proved that (f, G) satisfies condition (CW^1). Therefore, according to Theorem 4.7, there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on E. Moreover, from Remark 4.19, the function F can be taken so as to satisfy $\lim_{|x|\to\infty} F(x) = \infty$. If we define $V = \{x \in \mathbb{R}^n \, : \, F(x) \leq 1\}$, Proposition 2.16 tells us that V is a compact convex body with $0 \in \operatorname{int}(V)$ (because $F(0) = \alpha < 1$) and $\{x \in \mathbb{R}^n \, : \, F(x) = 1\} = \partial V$. Since $F = f = 1$ on K, we obtain that $K \subseteq \partial V$. Because F is of class $C^1(\mathbb{R}^n)$ and, according to Proposition 4.27, the Minkowski functional μ_V of V is of class $C^1(\mathbb{R}^n \setminus \mu_V^{-1}(0))$. Note, since V is bounded, Proposition 4.26 (9) gives that $\mu_V^{-1}(0) = \{0\}$. We have thus shown that V is of class C^1. In fact, Proposition 4.27 tells us that the gradients $\nabla F(x)$ and $\nabla \mu_V(x)$ are a positive multiple of each other. This implies that, for each $x \in K$,

$$N(x) = \frac{\nabla F(x)}{|\nabla F(x)|} = \frac{\nabla \mu_V(x)}{|\nabla \mu_V(x)|} = n_V(x),$$

which shows that $N(x)$ is outwardly normal to ∂V at x.

(1) $\implies$ (2): Let μ_V be the Minkowski functional of V. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \theta\left(\mu_V(x)\right), \quad x \in \mathbb{R}^n,$$

where $\theta : \mathbb{R} \to [0, +\infty)$ is a C^1 Lipschitz and increasing convex function with $\theta(t) = t^2$ whenever $|t| \leq 2$ and $\theta(t) = at$ whenever $|t| \geq 2$, for a suitable $a > 0$. The Minkowski functional is Lipschitz and convex because V is a convex body and this implies that F is Lipschitz. Also, because θ is convex and increasing, F is convex as well. Since V is of class C^1, the function μ_V is of $C^1(\mathbb{R}^n \setminus \{0\})$ and then $F \in C^1(\mathbb{R}^n \setminus \{0\})$. Let us see that in fact F is differentiable at 0 with $\nabla F(0) = 0$. Indeed, because $0 \in \operatorname{int}(V)$, we can find $r > 0$ with $B(0, r) \subset V$, which implies that $\mu_V(x) \leq r^{-1}|x|$ for every $x \in \mathbb{R}^n$. This yields

$$\lim_{|x|\to 0} \frac{|F(x) - F(0)|}{|x|} = \lim_{|x|\to 0} \frac{\mu_V^2(x)}{|x|} \leq \lim_{|x|\to 0} \frac{r^{-1}|x|\mu_V(x)}{|x|} = \lim_{|x|\to 0} \mu_V(x) = 0.$$

Hence F is differentiable at 0 with $\nabla F(0) = 0$. By Theorem 4.7 we then have that F satisfies conditions (C) and (CW^1) on ∂V. In addition, note that $\partial V = \mu_V^{-1}(1) = F^{-1}(1)$ and, in particular, $F = 1$ on E. Besides,

$$\nabla F(x) = 2\mu_V(x)\nabla\mu_V(x) = 2\nabla\mu_V(x) \quad \text{whenever} \quad x \in \partial V.$$

Using this together with the fact that $K \subseteq \partial V$ and $N(x)$ is outwardly normal to ∂V at x for every $x \in K$, we have that

$$N(x) = \frac{\nabla \mu_V(x)}{|\nabla \mu_V(x)|} = \frac{\nabla F(x)}{|\nabla F(x)|}, \quad x \in E.$$

Let us check that N and K satisfy conditions $(\mathcal{O})$, $(\mathcal{K})$ and $(\mathcal{KW}^1)$. For every $y \in K$, the convexity of F together with $\nabla F(0) = 0$ give

$$0 \leq F(y) - F(0) - \langle \nabla F(0), y\rangle \leq \langle \nabla F(y) - \nabla F(0), y\rangle = |\nabla F(y)|\langle N(y), y\rangle$$

which clearly implies, that $\langle N(y), y\rangle \geq 0$ and if $\langle N(y), y\rangle = 0$, then the above inequality yields $F(y) = 0$, which is a contradiction because $y \in K \subseteq \partial V = F^{-1}(1)$. Hence, condition $(\mathcal{O})$ is satisfied. In order to check condition $(\mathcal{K})$, we consider two points $x, y \in K$ and use condition (C) of F on K (that is, the convexity of F) to obtain

$$\langle N(y), y-x\rangle = |\nabla F(y)|^{-1}\left(-\langle \nabla F(y), x-y\rangle\right) = |\nabla F(y)|^{-1}\left(F(x) - F(y) - \langle \nabla F(y), x-y\rangle\right) \leq 0,$$

which proves $(\mathcal{K})$. Finally, if $x, y \in K$ are such that $\langle N(y), y - x\rangle = 0$, using the above identity we have that

$$F(x) - F(y) - \langle \nabla F(y), x - y\rangle = 0,$$

and then condition (CW^1) tells us

$$N(y) = \frac{\nabla F(y)}{|\nabla F(y)|} = \frac{\nabla F(x)}{|\nabla F(x)|} = N(x),$$

which shows the necessity of $(\mathcal{KW}^1)$. □

4.5 Convex functions and self-contracted curves

Very recently, in [29], E. Durand-Cartagena and A. Lemenant have applied Theorem 4.7 to find a characterization of strongly self-contracted curves in $\mathbb{R}^n$. If $T \in (0, +\infty]$, a differentiable curve $\gamma : [0, T] \to \mathbb{R}^n$ is said to be *strongly self-contracted* if for every $t, s \in [0, T]$ with $t < s$ and $\gamma'(t) \neq 0$ we have that

$$\langle \gamma'(t), \gamma(s) - \gamma(t)\rangle > 0.$$

It turns out that, assuming $C^{1,\alpha}$ regularity, these curves are characterized to be solutions of the gradient flow equation

$$\gamma'(t) = -\nabla f(\gamma(t)).$$

The exact statement of this characterization is as follows.

Theorem 4.30. [29, Theorem 1.2]. *Let $\gamma : [0, L] \to \mathbb{R}^n$ be an arc-length parameterized curve of class $C^{1,\alpha}([0, L])$ for some $\alpha \in (\frac{1}{2}, 1]$. Then γ is strongly self-contracted if and only if there exist a C^1 convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a parametrization $\tilde{\gamma} : [0, T] \to \mathbb{R}^n$ of γ with $T < +\infty$ such that*

$$\tilde{\gamma}'(t) = -\nabla f(\tilde{\gamma}(t)), \quad t \in [0, T].$$

4.6 C^1 convex extensions from arbitrary subsets

We are now going to give the solution to the following problem in full generality.

Problem 4.31. *Given E a subset of $\mathbb{R}^n$, and functions $f : E \to \mathbb{R}$ and $G : E \to \mathbb{R}^n$, how can we decide whether there is a* convex *function $F \in C^1(\mathbb{R}^n)$ such that $F(x) = f(x)$ and $\nabla F(x) = G(x)$ for all $x \in E$?*

In Section 4.1 we gave the solution to Problem 4.31 in the particular case that E is a compact (or bounded) subset, see Theorem 4.7 (and Theorem 4.20). We have seen that in this especial situation the two necessary and sufficient conditions on a 1-jet (f, G) with G continuous that we obtained for C^1_{conv} extendibility are:

$$f(x) \geq f(y) + \langle G(y), x - y\rangle \quad \text{for all} \quad x, y \in E \tag{C}$$

(which ensures convexity), and

$$f(x) = f(y) + \langle G(y), x - y\rangle \implies G(x) = G(y) \quad \text{for all} \quad x, y \in E \tag{CW^1}$$

(which tells us that if two points of the graph of f lie on a line segment contained in a hyperplane which we want to be tangent to the graph of an extension at one of the points, then our putative tangent hyperplanes at both points must be the same). In fact, it is shown in Remark 4.6 that the continuity of G plus conditions (C) imply Whitney's condition (W^1).

In Section 4.1 we also gave an example showing that the above conditions are no longer sufficient when E is not compact (even if E is an unbounded convex body). The reasons for this insufficiency can be mainly classified into two kinds of difficulties that only arise if the set E is unbounded and G is not uniformly continuous on E:

1. There may be no convex extension of f to the whole of $\mathbb{R}^n$.
2. Even when there are convex extensions of f defined on all of $\mathbb{R}^n$, and even when some of these extensions are differentiable in some neighborhood of E, there may be no $C^1(\mathbb{R}^n)$ convex extension of f.

Let us show how one can overcome these difficulties by adding new necessary conditions to (C), (CW^1) in order to obtain a complete solution to Problem 4.31.

As is perhaps inevitable, our solution to Problem 4.31 contains several technical conditions which may be quite difficult to grasp at a first reading. For this reason we will reverse the logical order of the exposition: we will start by providing some corollaries and examples. Only at last will the main theorem be stated.

The first kind of complication we have mentioned is well understood thanks to [59], and is not difficult to deal with: the requirement that

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty \text{ for every sequence } (x_k)_k \subset E \text{ with } \lim_{k\to\infty} |G(x_k)| = +\infty \quad (EX)$$

guarantees that there exist convex functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi_{|_E} = f$.

The second kind of difficulty, however, is of a subtler geometrical character, and is related, on the one hand, to the rigid global behavior of convex functions (see Theorem 4.41 below) and, on the other hand, to the fact that a differentiable (or even real-analytic) convex function $f : \mathbb{R}^n \to \mathbb{R}$ may have what one can call *corners at infinity*. In a short while we will be giving a precise meaning to this vague expression, but let us first ask ourselves this question: what would appear to be a natural generalization of condition (CW^1) of Definition 4.3 to the noncompact setting? As a first guess it may be natural to consider a replacement of (CW^1) with the following condition: if $(x_k)_k, (z_k)_k$ are sequences in E then

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0. \tag{4.6.1}$$

A natural variant of this condition is:

$$\lim_{k\to\infty} \frac{f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle}{|x_k - z_k|} = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0,$$

and it is clear that both conditions are the same as (CW^1) if E is compact. However, if E is unbounded none of these conditions is necessary for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ such that $(F, \nabla F) = (f, G)$ on E, as the following example shows.

Example 4.32. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{x^2 + e^{-2y}}$. This is a real analytic strictly convex function on $\mathbb{R}^2$ and one can check that the Hessian $D^2 f$ is strictly positive everywhere, see Example 4.39 for details. We have

$$\nabla f(x, y) = \left(\frac{x}{\sqrt{x^2 + e^{-2y}}}, -\frac{e^{-2y}}{\sqrt{x^2 + e^{-2y}}} \right),$$

and by considering the sequences

$$z_k = \left(\frac{1}{k}, k\right), \quad x_k = (0, k), \quad k \in \mathbb{N},$$

we see that

$$\begin{aligned}
\lim_{k\to\infty} & \frac{f(x_k) - f(z_k) - \langle \nabla F(z_k), x_k - z_k\rangle}{|x_k - z_k|} \\
&= \lim_{k\to\infty} \frac{\sqrt{e^{-2k}} - \sqrt{k^{-2} + e^{-2k}} + k^{-2}\left(k^{-2} + e^{-2k}\right)^{-1/2}}{k^{-1}} \\
&= \lim_{k\to\infty} \sqrt{k^2 e^{-2k}} - \sqrt{1 + k^2 e^{-2k}} + \frac{1}{\sqrt{1 + k^2 e^{-2k}}} \\
&= \lim_{k\to\infty} \frac{-1}{\sqrt{k^2 e^{-2k}} + \sqrt{1 + k^2 e^{-2k}}} + \frac{1}{\sqrt{1 + k^2 e^{-2k}}} = 0;
\end{aligned}$$

which in our case implies

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle \nabla F(z_k), x_k - z_k\rangle) = 0,$$

and yet we have that

$$\liminf_{k\to\infty} |\nabla f(x_k) - \nabla f(z_k)| \geq \lim_{k\to\infty} \left| \frac{k^{-1}}{\sqrt{k^{-2}+e^{-2k}}} \right| = \lim_{k\to\infty} \left| \frac{1}{\sqrt{1+k^2e^{-2k}}} \right| = 1 \neq 0.$$

So our first guess turned out to be wrong, and we have to be more careful. In view of the above example, and at least if we are looking for extensions $(F, \nabla F)$ with $F \in C^1(\mathbb{R}^n)$ convex and essentially coercive (that is, C^1 convex extensions $F(x)$ which, up to a linear perturbation, tend to ∞ as $|x|$ goes to infinity), it could make sense to restrict condition (4.6.1) to sequences $(x_k)_k$ which are bounded. On the other hand, if $(G(z_k))_k$ is not bounded as well, then by using condition (EX), up to extracting a subsequence, we would have

$$\lim_{k\to\infty} \frac{\langle G(z_k), z_k\rangle - f(z_k)}{|G(z_k)|} = \infty,$$

hence

$$\langle G(z_k), z_k\rangle - f(z_k) = M_k|G(z_k)|, \quad \text{with } \lim_{k\to\infty} M_k = \infty,$$

and it follows that

$$\begin{aligned} f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle &= f(x_k) - f(z_k) + \langle G(z_k), z_k\rangle - \langle G(z_k), x_k\rangle \\ &\geq f(x_k) + (M_k - |x_k|)\,|G(z_k)| \to \infty \end{aligned}$$

because $(f(x_k))_k$ and $(x_k)_k$ are bounded and $M_k \to \infty$. Thus we have learned that we cannot have

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0$$

unless $(G(x_k))_k$ is bounded. An educated guess for a good substitute of (CW^1) could then be to require that

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0 \tag{4.6.2}$$

for all sequences $(x_k)_k$ and $(z_k)_k$ in E such that $(x_k)_k$ and $(G(z_k))_k$ are bounded.

We will prove in Section 4.8 below that this new condition is necessary for the existence of a function F which solves our problem. Now, if we add (4.6.2) to (EX) and (C), will this new set of conditions be sufficient as well? The answer to this question depends on how large the set span$\{G(x) - G(y) : x, y \in E\}$ is. If this set coincides with $\mathbb{R}^n$ then those conditions are sufficient, and otherwise they are not; this is the content of the following easy (but especially useful) consequence of the main result of this section, Theorem 4.43.

Corollary 4.33. *Given an arbitrary subset E of $\mathbb{R}^n$ and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, assume that the following conditions are satisfied.*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(x_k)_k$ and $(G(z_k))_k$ are bounded and*

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

(iv) *span* $(\{G(x) - G(y) : x, y \in E\}) = \mathbb{R}^n$.

Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and F is essentially coercive.

Here, by saying that F is *essentially coercive* we mean that there exists a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{|x|\to\infty} (F(x) - \ell(x)) = \infty.$$

Although Corollary 4.33 is a consequence of Theorem 4.43 below, the proof of Theorem 4.20 can be easily adapted to produce a simpler proof of Corollary 4.33.

By comparing Example 4.32 with Corollary 4.33 we may arrive at a remarkable conclusion: our given jet (f, G) may well have some *corners at infinity* and, for C^1 convex extension purposes, that will not matter at all as long as the jet (f, G) forces all possible convex extensions to be essentially coercive (equivalently, as long as span$\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$). Let us now explain what we mean by a jet having a corner at infinity.

Definition 4.34. *Let X be a proper linear subspace of $\mathbb{R}^n$ and let us denote by $X^\perp$ its orthogonal complement. We say that a jet $(f, G) : E \subset \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ has a corner at infinity in a direction of $X^\perp$ provided that there exist two sequences $(x_k)_k, (z_k)_k$ in E such that, if $P_X : \mathbb{R}^n \to X$ is the orthogonal projection, then we have that $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded,* $\lim_{k\to\infty} |x_k| = \infty$,

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0,$$

and yet

$$\limsup_{k\to\infty} |G(x_k) - G(z_k)| > 0.$$

We will also say that jet (f, G) has a corner at infinity in the direction of the line $\{tv : t \in \mathbb{R}\}$ (where $v \in \mathbb{R}^n \setminus \{0\}$) provided that there exist sequences $(x_k)_k, (z_k)_k$ satisfying the above properties with P_X being the orthogonal projection onto the hyperplane X perpendicular to v.

For instance, the function f of Example 4.32, when restricted to the sequences $(x_k)_k$, $(z_k)_k$ defined there, gives an instance of a jet that has a corner at infinity directed by the line $x = 0$, see Figure 4.1 below. Of course, the pair $(f, \nabla f)$, unrestricted, provides another instance. In this case it is natural to say that the function f itself has a corner at infinity. More pathological examples can be given in higher dimensions.

Example 4.35. Consider the following two examples.

(1) Let $1 \leq k \leq n$ be an integer and define the function

$$f(x_1, x_2, \ldots, x_n) = \sqrt{\sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{n} e^{-2x_j}}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.6.3}$$

Then f is a convex function of class C^∞ with strictly positive Hessian at every point, which has a corner at infinity in the direction of e_j for every $j = k+1, \ldots, n$ and f is essentially coercive.

(2) Let $n \geq 3$ and $2 \leq k < n$ be integers and define the function

$$f(x_1, \ldots, x_n) = \sqrt{x_1^2 + \sum_{j=2}^{k} e^{-2x_j}}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.6.4}$$

Then f is convex and of class C^∞, f has a corner at infinity in the direction of e_j, for every $j = 2, \ldots, k$, and is not essentially coercive. Nevertheless f is *essentially k-coercive* (meaning that f can be written as $f = c \circ P$, where P is the orthogonal projection onto a k-dimensional subspace of X of $\mathbb{R}^n$ and $c : X \to \mathbb{R}$ is essentially coercive).

Proof.

(1) Let us first check that f is strictly convex on $\mathbb{R}^n$. Given $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, the gradient of f at x is

$$\nabla f(x)=\frac{1}{f(x)}\left(x_1,\dots,x_k,-e^{-2x_{k+1}},\dots,-e^{-2x_n}\right). \tag{4.6.5}$$

The second derivatives of f at x are

$$\frac{\partial^2 f}{\partial x_j^2}(x)=\frac{1}{f(x)^3}\left(\sum_{\ell=1,\,\ell\neq j}^{k}x_\ell^2+\sum_{\ell=k+1}^{n}e^{-2x_\ell}\right),\quad \frac{\partial^2 f}{\partial x_i\partial x_j}(x)=-\frac{x_ix_j}{f(x)^3},\quad 1\le i,j\le k,\ i\neq j,$$

$$\frac{\partial^2 f}{\partial x_j^2}(x)=\frac{e^{-2x_j}}{f(x)^3}\left(2\sum_{\ell=1}^{k}x_\ell^2+2\sum_{\ell=k+1}^{n}e^{-2x_\ell}-e^{-2x_j}\right),\quad k+1\le j\le n,$$

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(x)=-\frac{e^{-2x_i}e^{-2x_j}}{f(x)^3},\quad k+1\le i,j\le n,\ i\neq j,$$

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(x)=\frac{x_ie^{-2x_j}}{f(x)^3},\quad k+1\le j\le n,\ 1\le i\le k.$$

We thus have, for every $v=(v_1,\dots,v_n)\in\mathbb{R}^n$ with $|v|=1$, that

$$\begin{aligned}
D^2f(x)(v^2)&=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)v_iv_j=\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_j^2}(x)v_j^2+2\sum_{1\le i<j\le n}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)v_iv_j\\
&=\sum_{j=1}^{k}\frac{\partial^2 f}{\partial x_j^2}(x)v_j^2+2\sum_{1\le i<j\le k}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)v_iv_j\\
&\quad+\sum_{j=k+1}^{n}\frac{\partial^2 f}{\partial x_j^2}(x)v_j^2+2\sum_{j=k+1}^{n}\sum_{i=1}^{j-1}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)v_iv_j.
\end{aligned}$$

On the one hand

$$\begin{aligned}
&\sum_{j=1}^{k}\frac{\partial^2 f}{\partial x_j^2}(x)v_j^2+2\sum_{1\le i<j\le k}\frac{\partial^2 f}{\partial x_i\partial x_j}(x)v_iv_j\\
&=\frac{1}{f(x)^3}\left(\sum_{\ell=k+1}^{n}e^{-2x_\ell}\sum_{j=1}^{k}v_j^2+\sum_{j=1}^{k}\sum_{\ell=1,\,\ell\neq j}^{k}v_j^2x_\ell^2-2\sum_{1\le i<j\le k}x_ix_jv_iv_j\right)\\
&=\frac{1}{f(x)^3}\left(\sum_{\ell=k+1}^{n}e^{-2x_\ell}\sum_{j=1}^{k}v_j^2+\sum_{1\le i<j\le k}(v_j^2x_i^2+v_i^2x_j^2)-2\sum_{1\le i<j\le k}x_ix_jv_iv_j\right)\\
&=\frac{1}{f(x)^3}\left(\sum_{\ell=k+1}^{n}e^{-2x_\ell}\sum_{j=1}^{k}v_j^2+\sum_{1\le i<j\le k}(v_jx_i+v_ix_j)^2\right)\ge\frac{\sum_{\ell=k+1}^{n}e^{-2x_\ell}}{f(x)^3}\sum_{j=1}^{k}v_j^2.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\sum_{j=k+1}^{n} \frac{\partial^2 f}{\partial x_j^2}(x) v_j^2 + 2 \sum_{j=k+1}^{n} \sum_{i=1}^{j-1} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) v_i v_j \\
&= \sum_{j=k+1}^{n} \frac{\partial^2 f}{\partial x_j^2}(x) v_j^2 + 2 \sum_{j=k+1}^{n} \sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) v_i v_j + 2 \sum_{j=k+1}^{n} \sum_{i=k+1}^{j-1} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) v_i v_j \\
&= \frac{1}{f(x)^3} \Bigg[\sum_{j=k+1}^{n} e^{-2x_j} \left(2 \sum_{\ell=1}^{k} x_\ell^2 + 2 \sum_{\ell=k+1}^{n} e^{-2x_\ell} - e^{-2x_j} \right) v_j^2 \\
&\qquad + 2 \sum_{j=k+1}^{n} \sum_{i=1}^{k} e^{-2x_j} x_i v_i v_j - 2 \sum_{j=k+1}^{n} \sum_{i=k+1}^{j-1} e^{-2x_i} e^{-2x_j} v_i v_j \Bigg] \\
&= \frac{1}{f(x)^3} \Bigg[\sum_{j=k+1}^{n} e^{-2x_j} \left(2 \sum_{\ell=1}^{k} x_\ell^2 v_j^2 + 2 \sum_{i=1}^{k} x_i v_i v_j \right) \\
&\qquad + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} v_j^2 + 2 \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} v_j^2 \right) - 2 \sum_{j=k+1}^{n} \sum_{i=k+1}^{j-1} e^{-2x_i} e^{-2x_j} v_i v_j \Bigg].
\end{aligned}
$$

It is clear that the last term is greater than or equal to

$$
\begin{aligned}
&\frac{1}{f(x)^3} \Bigg[\sum_{j=k+1}^{n} e^{-2x_j} \left(2 \sum_{\ell=1}^{k} x_\ell^2 v_j^2 - \sum_{i=1}^{k} x_i^2 v_j^2 - \sum_{i=1}^{k} v_i^2 \right) \\
&\qquad + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} v_j^2 + 2 \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} v_j^2 - \sum_{i=k+1}^{j-1} e^{-2x_i} v_i^2 - \sum_{i=k+1}^{j-1} e^{-2x_i} v_j^2 \right) \Bigg] \\
&= \frac{1}{f(x)^3} \Bigg[\sum_{j=k+1}^{n} e^{-2x_j} \left(2 \sum_{\ell=1}^{k} x_\ell^2 v_j^2 - \sum_{i=1}^{k} v_i^2 \right) \\
&\qquad + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} v_j^2 + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} v_j^2 + \sum_{\ell=j+1}^{n} e^{-x_\ell} v_j^2 - \sum_{i=k+1}^{j-1} e^{-2x_i} v_i^2 \right) \Bigg] \\
&\geq \frac{1}{f(x)^3} \Bigg[- \sum_{j=k+1}^{n} e^{-2x_j} \sum_{i=1}^{k} v_i^2 + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} v_j^2 + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} v_j^2 \right) \\
&\qquad + \sum_{j=k+1}^{n} \sum_{\ell=j+1}^{n} e^{-2x_j} e^{-2x_\ell} v_j^2 - \sum_{j=k+1}^{n} \sum_{\ell=k+1}^{j-1} e^{-2x_\ell} e^{-2x_j} v_\ell^2 \Bigg] \\
&= \frac{1}{f(x)^3} \Bigg[- \sum_{j=k+1}^{n} e^{-2x_j} \sum_{i=1}^{k} v_i^2 + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} v_j^2 + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} v_j^2 \right) \\
&\qquad + \sum_{j=k+1}^{n} \sum_{\ell=j+1}^{n} e^{-2x_j} e^{-2x_\ell} v_j^2 - \sum_{\ell=k+1}^{n} \sum_{j=\ell+1}^{n} e^{-2x_\ell} e^{-2x_j} v_\ell^2 \Bigg] \\
&= \frac{1}{f(x)^3} \left[- \sum_{j=k+1}^{n} e^{-2x_j} \sum_{i=1}^{k} v_i^2 + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} \right) v_j^2 \right]
\end{aligned}
$$

If we have that $v_{k+1} = \cdots = v_n = 0$, then

$$D^2 f(x)(v^2) = \sum_{j=1}^{k} \frac{\partial^2 f}{\partial x_j^2}(x) v_j^2 + 2 \sum_{1 \leq i < j \leq k} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) v_i v_j = \frac{\sum_{\ell=k+1}^{n} e^{-2x_\ell}}{f(x)^3} \sum_{j=1}^{k} v_j^2 > 0.$$

Let us now assume $(v_{k+1}, \ldots, v_n) \neq 0$. Then, then preceding inequalities yield

$$\begin{aligned} D^2 f(x)(v^2) &\geq \frac{\sum_{\ell=k+1}^{n} e^{-2x_\ell}}{f(x)^3} \sum_{j=1}^{k} v_j^2 \\ &\quad + \frac{1}{f(x)^3} \left[- \sum_{j=k+1}^{n} e^{-2x_j} \sum_{i=1}^{k} v_i^2 + \sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} \right) v_j^2 \right] \\ &= \frac{1}{f(x)^3} \left[\sum_{j=k+1}^{n} e^{-2x_j} \left(e^{-2x_j} + \sum_{\ell=k+1,\, \ell \neq j}^{n} e^{-2x_\ell} \right) v_j^2 \right] > 0. \end{aligned}$$

We have thus shown that f has strictly positive Hessian at each point of $\mathbb{R}^n$ and, in particular, f is strictly convex on $\mathbb{R}^n$. Let us now see that f is essentially coercive. We define the linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ by $\ell(x) = x_1 + \cdots + x_k$, for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The convavity of the function $(0, +\infty) \ni t \mapsto t^{1/2}$ yields

$$\begin{aligned} f(x) + \ell(x) &= \sqrt{\sum_{j=1}^{k} \frac{n x_j^2}{n} + \sum_{j=k+1}^{n} \frac{n e^{-2x_j}}{n}} + \ell(x) \geq \frac{1}{n} \left(\sum_{j=1}^{k} \sqrt{n} |x_j| + \sum_{j=k+1}^{n} \sqrt{n} e^{-x_j} \right) + \ell(x) \\ &= \sum_{j=1}^{k} \frac{|x_j|}{\sqrt{n}} + \sum_{j=k+1}^{n} \left(\frac{e^{-x_j}}{\sqrt{n}} + x_j \right), \quad \text{for every} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n; \end{aligned}$$

and it is clear that the last term defines a coercive function on $\mathbb{R}^n$. Finally, let us prove that f has a corner at infinity in every direction e_j, $j = k+1, \ldots, n$. In order to do so, we fix $j \in \{k+1, \ldots, n\}$, denote X the orthogonal complement of the line directed by e_j and define sequences

$$x_j^\ell = \ell e_j, \quad z_j^\ell = \frac{1}{\ell} e_1 + \ell e_j, \quad \ell \in \mathbb{N}.$$

If P_X denotes the orthogonal projection onto X, it is obvious that the sequence $(P_X(x_j^\ell))_\ell$ is identically zero and, in particular, is bounded. Using (4.6.5), it follows that

$$f(x_j^\ell) = e^{-\ell}, \quad f(z_j^\ell) = \sqrt{\frac{1}{\ell^2} + e^{-2\ell}},$$

$$\nabla f(x_j^\ell) = -2e^{-\ell} e_j, \quad \nabla f(z_j^\ell) = \frac{1}{\sqrt{\frac{1}{\ell^2} + e^{-2\ell}}} \left(\frac{1}{\ell} e_1 - 2e^{-2\ell} e_j \right), \quad j \in \mathbb{N}.$$

We observe that

$$\nabla f(z_j^\ell) = \frac{1}{\sqrt{\frac{1}{\ell^2} + e^{-2\ell}}} \left(\frac{1}{\ell} e_1 - 2e^{-\ell} e_j \right) = \frac{1}{\sqrt{1 + \ell^2 e^{-2\ell}}} e_1 - \frac{2e^{-2\ell} \ell}{\sqrt{1 + \ell^2 e^{-2\ell}}} e_j, \tag{4.6.6}$$

and then the sequence $\left(\nabla f(z_j^\ell) \right)_\ell$ is clearly bounded. Also, we can write

$$\lim_\ell \left(f(x_j^\ell) - f(z_j^\ell) - \langle \nabla f(z_j^\ell), x_j^\ell - z_j^\ell \rangle \right) = \lim_\ell \left(e^{-\ell} - \sqrt{\frac{1}{\ell^2} + e^{-2\ell}} + \frac{1}{\ell \sqrt{1 + \ell^2 e^{-2\ell}}} \right) = 0.$$

On the other hand, (4.6.6) yields

$$|\nabla f(x_j^\ell) - \nabla f(z_j^\ell)| \geq \frac{1}{\sqrt{1+\ell^2 e^{-2\ell}}},$$

which tends to 1 as ℓ goes to infinity. According to Definition 4.34, we have thus shown that f has a corner at infinity in the direction e_j.

(2) Let X be the subspace spanned by the set $\{e_1, \ldots, e_k\}$ and let $P : \mathbb{R}^n \to X$ be the orthogonal projection onto X. It is obvious that f can be written as $f = c \circ P$, where c is the function $c(y) = f(y, 0, \ldots, 0)$ for every $y \in X$. The function c defined on X is one of the examples of (1) and then c is convex and essentially coercive on X. This implies that f is convex on $\mathbb{R}^n$ and k-essentially coercive. Finally, given $j \in \{2, \ldots, k\}$, let us denote by Y_j the orthogonal complement of the line directed by e_j and define the sequences

$$x_j^\ell = \ell e_j, \quad z_j^\ell = \frac{1}{\ell} e_1 + \ell e_j, \quad \ell \in \mathbb{N}.$$

It is clear that $\left(P_{Y_j}(x_j^\ell)\right)_\ell$ is identically zero. Since both sequences $(x_j^\ell)_\ell$ and $(z_j^\ell)_\ell$ are contained in X, the calculations in (1) show that $\left(\nabla c(z_j^\ell)\right)_\ell$ is bounded,

$$\lim_\ell \left(c(x_j^\ell) - c(z_j^\ell) - \langle \nabla c(z_j^\ell), x_j^\ell - z_j^\ell \rangle \right) = 0 \quad \text{and} \quad \liminf_\ell |\nabla f(x_j^\ell) - \nabla f(z_j^\ell)| \geq 1.$$

Because $f = c \circ P$ and

$$\nabla f(x_1, \ldots, x_n) = \left(\nabla c(x_1, \ldots, x_k), 0, \ldots, 0\right), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

we obtain that f has a corner at infinity in the direction e_j. $\square$

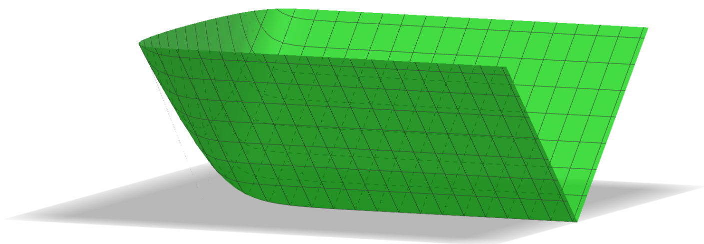

Figure 4.1: $f(x,y) = \sqrt{x^2 + e^{-2y}}$, $(x,y) \in \mathbb{R}^2$. This function has a corner at infinity directed by the line $x = 0$ and it is essentially coercive.

In general it can be shown that the presence of a corner at infinity in the graph of a differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$ forces essential k-coercivity of f, for some $k \geq 2$, in a subspace of directions containing the directions of the corner. This is a consequence of the main result of this section, Theorem 4.43.

We will not explicitly use the notion of corner at infinity in our proofs. Our reasons for introducing these objects are the facts that: 1) one way or another, corners at infinity will be to blame for most of the predicaments and technicalities involved in any attempt to solve Problem 4.31; and 2) we firmly believe that the reader will be more able to understand the statements and proofs of the following results once he has been acquainted with this notion. As a matter of fact, the most technical conditions of Theorems 4.38 and 4.43 below can be rephrased more intuitively in terms of corners at infinity and essential coercivity of data in the directions of those corners.

Unfortunately Corollary 4.33 does not provide a characterization of the 1-jets which admit essentially coercive C^1 convex extensions. This is due to the fact that a jet (f, G) defined on a set E may admit such an extension and yet span$\{G(x)-G(y) : x, y \in E\} \neq \mathbb{R}^n$; that is to say, condition (iv) is not necessary, as shown by the trivial example of the jet (f_0, G_0) with $E_0 = \{0\} \subset \mathbb{R}^2$, $f_0(0) = 0$, $G_0(0) = 0$, which admits a C^1 convex and coercive extension given by $(F_0, \nabla F_0)$, where $F_0(x, y) = x^2 + y^2$.

Of course, a C^1 convex extension problem for a given 1-jet (f, G) may have solutions which are not essentially coercive; in fact it may happen that none of its solutions are essentially coercive. A sister of Corollary 4.33 which provides a more general, but still partial solution to Problem 4.31, is the following.

Corollary 4.36. *Given an arbitrary subset E of $\mathbb{R}^n$ and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, assume that the following conditions are satisfied:*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x-y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *Let $P = P_Y : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $Y :=$ span$\{G(x)-G(y) : x, y \in E\}$. If $(x_k)_k, (z_k)_k$ are sequences in E such that $(P(x_k))_k$ and $(G(z_k))_k$ are bounded and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$.

Condition (iii) of the above corollary can be intuitively rephrased by saying that: 1) our jet satisfies a natural generalization of condition (CW^1); and 2) (f, G) cannot have corners at infinity in any direction contained in the orthogonal complement of the subspace $Y =$ span$\{G(x) - G(y) : x, y \in E\}$.

It could be natural to hope for the conditions of Corollary 4.36 to be necessary as well, thus providing a nice characterization of those 1-jets which admit C^1 convex extensions. Unfortunately the solution to Problem 4.31 is necessarily more complicated, as the following example shows.

Example 4.37. Let $E_1 = \{(x, y) \in \mathbb{R}^2 : y = \log|x|, |x| \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_1(x, y) = |x|$, $G_1(x, y) = (-1, 0)$ if $x < 0$, $G_1(x, y) = (1, 0)$ if $x > 0$. In this case we have $Y :=$ span$\{G_1(x, y) - G_1(x', y') : (x, y), (x', y') \in E_1\} = \mathbb{R} \times \{0\}$, and we will see in Example 4.39 that condition (iii) of Theorem 4.36 is not satisfied. However, we will see that, for $\varepsilon > 0$ small enough, if we set $E_1^* = E_1 \cup \{(0, 1)\}$, $f_1^* = f_1$ on E_1, $f_1^*(0, 1) = \varepsilon$, $G_1^* = G_1$ on E_1, and $G_1^*(0, 1) = (0, \varepsilon)$, then the conditions of Corollary 4.33 are satisfied for (f_1, G_1). This implies that the problem of finding a C^1 convex extension of the jet (f_1^*, G_1^*) does have a solution, and therefore the same is true for the jet (f_1, G_1).

This example shows that in some cases the C^1 convex extension problem for a 1-jet (f, G) may be geometrically underdetermined in the sense that we may not have been given enough differential data so as to have condition (iii) of the above corollary satisfied with $Y =$ span$\{G(x) - G(y) : x, y \in E\}$, and yet it may be possible to find a few more jets (β_j, w_j) associated to finitely many points $p_j \in \mathbb{R}^n \setminus \overline{E}$, $j = 1, \ldots, m$, so that, if we define $E^* = E \cup \{p_1, \ldots, p_m\}$ and extend the functions f and G from E to E^* by setting

$$f(x_j) := \beta_j, \quad G(p_j) := w_j \quad \text{for} \quad j = 1, \ldots, m, \tag{4.6.7}$$

then the new extension problem for (f, G) defined on E^* does satisfy condition (iii) of Corollary 4.36. Notice that, the larger Y grows, the weaker condition (iii) of Corollary 4.36 becomes.

We are now prepared to state a first version of our main result.

Theorem 4.38. *Given an arbitrary subset E of $\mathbb{R}^n$ and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, the following is true. There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, and $(\nabla F)_{|_E} = G$, if and only if the following conditions are satisfied.*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x-y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *Let $Y :=$ span$\{G(x) - G(y) \ : \ x, y \in E\}$. There exists a linear subspace $X \supseteq Y$ such that, either $Y = X$, or else, if we denote $k = \dim Y$, $d = \dim X$ and $P_X : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection from $\mathbb{R}^n$ onto X, there exist points $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, numbers $\beta_1, \ldots, \beta_{d-k} \in \mathbb{R}$, and vectors $w_1, \ldots, w_{d-k} \in \mathbb{R}^n$ such that:*

(a) $X =$ *span* $(\{u - v \ : \ u, v \in G(E) \cup \{w_1, \ldots, w_{d-k}\}\})$.

(b) $\beta_j > \max_{1 \leq i \neq j \leq d-k}\{\beta_i + \langle w_i, p_j - p_i\rangle\}$ *for all* $1 \leq j \leq d-k$.

(c) $\beta_j > \sup_{z \in E, \, |G(z)| \leq N}\{f(z) + \langle G(z), p_j - z\rangle\}$ *for all* $1 \leq j \leq d-k$ *and* $N \in \mathbb{N}$.

(d) $\inf_{x \in E, \, |P_X(x)| \leq N}\{f(x) - \max_{1\leq j\leq d-k}\{\beta_j + \langle w_j, x - p_j\rangle\}\} > 0$ *for all* $N \in \mathbb{N}$.

(iv) *If X and P_X are as in (iii), and $(x_k)_k, (z_k)_k$ are sequences in E such that $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle \right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

As we see, the difference between Theorem 4.38 and Corollary 4.36 is in the technical condition (iii), which can be informally summed up by saying that, whenever the jets $(f(x), G(x))$, $x \in E$, do not provide us with enough differential data so that condition (iii) of Corollary 4.36 holds, there is enough room in $\mathbb{R}^n \setminus \overline{E}$ to add finitely many new jets (β_j, w_j), associated to new points p_j, $j = 1, \ldots, d-k$, in such a way that the new extension problem does satisfy the conditions of Corollary 4.36. Hence the new extension problem will be one for which, even though there may be corners at infinity, those corners at infinity will necessarily be directed by subspaces which are contained in the span of the putative derivatives, and the new data will force essential coercivity of all possible extensions in the directions of those corners.

In Section 4.10 below we will show that, in the particular case that G is bounded (and so we may expect to find an F with a bounded gradient), these complicated conditions about compatibility of the old and new data admit a much nicer geometrical reformulation, see Theorem 4.57 below.

Let us consider some examples that will hopefully offer further clarification of these comments.

Example 4.39. Consider the following 1-jets (f_j, G_j) defined on subsets E_j of $\mathbb{R}^n$:

1. $E_1 = \{(x, y) \in \mathbb{R}^2 : y = \log|x|, |x| \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_1(x, y) = |x|$, $G_1(x, y) = (-1, 0)$ if $x < 0$, $G_1(x, y) = (1, 0)$ if $x > 0$.

2. $E_2 = \{(x, y) \in \mathbb{R}^2 : y = \log|x|, |x| \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_2 = \varphi$, $G_2 = \nabla\varphi$, where $\varphi(x, y) = \sqrt{x^2 + e^{-2y}}$.

3. $E_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, y = \log|x|, |x| \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_3 = \varphi$, $G_3 = \nabla\varphi$, where $\varphi(x, y, z) = \sqrt{x^2 + e^{-2y}}$.

4. $E_4 = E_1 \cup \{(x, y) \in \mathbb{R}^2 : |x| \geq 1\}$, $f_4(x, y) = |x|$, $G_4(x, y) = (-1, 0)$ if $x < 0$, $G_4(x, y) = (1, 0)$ if $x > 0$.

We claim that:

(1) For the jet (f_1, G_1), and with the notation of Theorem 4.43, we have $Y = \mathbb{R}\times\{0\}$, but the smallest possible X we can take is $X = \mathbb{R}^2$ (and all possible extensions F must be essentially coercive on $\mathbb{R}^2$).

(2) For the jet (f_2, G_2) we have $Y = \mathbb{R}^2$, and all possible extensions F must be essentially coercive on $\mathbb{R}^2$.

(3) For the jet (f_3, G_3) we have $Y = \mathbb{R}^2 \times \{0\}$, and we can take either $X = Y$ or $X = \mathbb{R}^3$.

(4) For the jet (f_4, G_4) we have $Y = \mathbb{R}\times\{0\}$, but one cannot apply Theorem 4.43 with any X. There exists no $F \in C^1_{\text{conv}}(\mathbb{R}^2)$ such that $(F, \nabla F)$ extends (f_4, G_4).

Proof.

(1) It is clear that $Y = \text{span}\{G(x,y) - G(x',y') \,:\, (x,y),(x',y') \in E_1\} = Y\times\{0\}$. Let us see that condition (v) of Theorem 4.43 is not satisfied for $X = Y$. Indeed, the sequences $z_k = (1/k, -\log k),\ \tilde{z_k} = (-1/k, -\log k),\ k\in\mathbb{N}$, belong to E_1 and

$$f(z_k) - f(\tilde{z_k}) - \langle G(\tilde{z_k}), z_k - \tilde{z_k}\rangle = \tfrac{1}{k} - \tfrac{1}{k} - \langle(-1,0),(\tfrac{2}{k},0)\rangle = \tfrac{2}{k}$$

tends to 0 as k goes to infinity; but $\lim_k |G(z_k) - G(\tilde{z_k})| = 2$. This shows that we cannot apply Theorem 4.43 with $X = Y$. However, let us see that conditions $(i) - (v)$ of Theorem 4.43 are fulfilled with $X = \mathbb{R}^2$. The first condition (i) follows from the fact that the function $\varphi(x,y) = |x|$ is convex on $\mathbb{R}^2$ and for every $x\neq 0$, the function φ is differentiable at x with $\nabla\varphi(x,y) = (\text{sign}(x), 0)$. Conditions (ii) and (iii) trivially hold. Let us now check condition (iv). We define $\beta = e^{-1}$, $p = (0,1)$ and $w = (0,\beta)$. It is then clear that $X = \text{span}\{u - v \;:\; u,v\in G(E_1)\cup\{w\}\}$. We immediately see that conditions $(iv)(a)$ and $(iv)(b)$ are satisfied. To check condition $(iv)(c)$, take a point $(x,y)\in E_1$ and write

$$\beta - f(x,y) - \langle G(x,y), p - (x,y)\rangle = \beta - |x| - \langle(\text{sign}(x),0),(0,1)-(x,y)\rangle = \beta - |x| + x\,\text{sign}(x) = \beta.$$

In order to show that condition $(iv)(d)$ is satisfied, we take a point $(x,y)\in E_1$ and we write

$$f(x,y) - \beta - \langle w, (x,y) - p\rangle = |x| - \beta - \langle(0,\beta),(x,y)-(0,1)\rangle = |x| - \beta y = |x| - e^{-1}\log|x|.$$

The function $(0,+\infty)\ni t\to t - e^{-1}\log t$ attains a global minimum at $t = e^{-1}$. This shows that $|t| - e^{-1}\log|t| \geq 2e^{-1} = 2\beta$ for every $t\neq 0$, which in turn implies that

$$f(x,y) - \beta - \langle w,(x,y)-p\rangle \geq 2\beta, \quad \text{for all} \quad (x,y)\in E_1.$$

Finally, let us check that condition (v) is true for f, G and $X = \mathbb{R}^2$. Given two bounded sequences $(z_k)_k$ and $(\tilde{z_k})_k$ in E_1, we write $z_k = (x_k, \log(x_k))$ and $\tilde{z_k} = (\tilde{x_k}, \log(\tilde{z_k}))$ for every k. This yields

$$\begin{aligned} f(z_k) - &f(\tilde{z_k}) - \langle G(\tilde{z_k}), z_k - \tilde{z_k}\rangle \\ &= |x_k| - |\tilde{x_k}| - \langle(\text{sign}(\tilde{x_k}),0), z_k - \tilde{z_k}\rangle = |x_k| - \text{sign}(\tilde{x_k})x_k = x_k\left(\text{sign}(x_k) - \text{sign}(\tilde{x_k})\right), \end{aligned}$$

for every k. If we assume that

$$\lim_k \left(f(z_k) - f(\tilde{z_k}) - \langle G(\tilde{z_k}), z_k - \tilde{z_k}\rangle\right) = 0,$$

the preceding equations lead us to that either $(x_k)_k$ tends to 0 or else $\text{sign}(x_k) = \text{sign}(\tilde{x_k})$ whenever $k\geq k_0$ for some k_0. But $\lim_k x_k = 0$ implies that $\lim_k \log|x_k| = -\infty$ and the sequence $(z_k)_k$ would be unbounded, which is a contradiction. Then, we must have

$$\lim_k |G(z_k) - G(\tilde{z_k})| = \lim_k |\,\text{sign}(x_k) - \text{sign}(\tilde{x_k})| = 0,$$

which proves condition (v).

(2) Let us see that the function $\varphi(x,y)=\sqrt{x^2+e^{-2y}}$, for $(x,y)\in\mathbb{R}^2$ is a C^∞ convex function on $\mathbb{R}^2$. The function φ is clearly C^∞ on $\mathbb{R}^2$ and, after elementary calculations, we obtain

$$\nabla\varphi(x,y)=\left(\frac{x}{\sqrt{x^2+e^{-2y}}},\frac{-e^{-2y}}{\sqrt{x^2+e^{-2y}}}\right),\quad H_\varphi(x,y)=\frac{e^{-2y}}{(x^2+e^{-2y})^{3/2}}\begin{pmatrix}1 & x\\ x & 2x^2+e^{-2y}\end{pmatrix}$$

for all $(x,y)\in\mathbb{R}^2$, where $H_\varphi(x,y)$ denotes the Hessian matrix of φ at the point (x,y). If follows immediately that H_φ is positive definite at every point, which implies that φ is convex (in fact, strictly convex) on $\mathbb{R}^2$. On the other hand, the points $(x,y)=(1,0)$, $(x',y')=(-1,0)$ and $(x'',y'')=(2,\log 2)$ belong to E_2 and

$$\nabla\varphi(x,y)-\nabla\varphi(x',y')=(\sqrt{2},0),$$
$$\nabla\varphi(x,y)-\nabla\varphi(x'',y'')=\left(2^{-1/2}-4(17)^{-1/2},-2^{-1/2}+2^{-1}(17)^{-1/2}\right).$$

This clearly implies that $Y=\operatorname{span}\{G(x,y)-G(x',y')\ :\ (x,y),\,(x',y')\in E_2\}=\mathbb{R}^2$. In particular, we have that $\operatorname{span}\{\nabla\varphi(x,y)-\nabla\varphi(x',y')\ :\ (x,y),\,(x',y')\in\mathbb{R}^2\}=\mathbb{R}^2$, that is $X_\varphi=\mathbb{R}^2$ (see Theorem 4.41 for notation). Thus we can use the *only if* part of Theorem 4.43 (which will be proved in Section 4.8) to obtain that (f_2,G_2) satisfies conditions $(i)-(v)$ with $X=\mathbb{R}^2$.

(3) With identical calculations as for (f_2,g_2,E_2) we obtain that $Y=\mathbb{R}^2\times\{0\}$. The function φ is a C^1 convex function on $\mathbb{R}^2$ with $X_\varphi=\mathbb{R}^2\times\{0\}$ and the *only if* part of Theorem 4.43 implies that (f_3,G_3) satisfies conditions $(i)-(v)$ with $X=\mathbb{R}^2\times\{0\}$. On the other hand, if we consider the function $\psi(x,y,z)=\varphi(x,y,z)+z^2$ for every $(x,y,z)\in\mathbb{R}^3$, we see that ψ is C^1 and convex on $\mathbb{R}^3$, with $X_\psi=\mathbb{R}^3$ and $(\psi,\nabla\psi)=(f_3,G_3)$ on E_3 too. Again, the *only if* part of Theorem 4.43 shows that the 1-jet (f_3,G_3) satisfies conditions $(i)-(v)$ with $X=\mathbb{R}^3$.

(4) It is clear that $Y=\mathbb{R}\times\{0\}$. Using the same calculations as in the first example (note that E_4 contains E_1), we see that condition (v) of Theorem 4.43 is not satisfied with $X=Y$. On the other hand, if $X=\mathbb{R}^2$, let us see that condition (iv) is not satisfied. Indeed, assume there exist a point $p=(p_1,p_2)\in\mathbb{R}^2\setminus\overline{E_4}$, a number $\beta\in\mathbb{R}$ and a vector $w=(w_1,w_2)\in\mathbb{R}^n$ such that conditions $(iv)(a)-(d)$ are satisfied with p,β and w. From condition $(iv)(d)$, we must have

$$|x|-\beta-w_1(x-p_1)-w_2(y-p_2)=f(x,y)-\beta-\langle w,(x,y)-p\rangle>0\quad\text{for every}\quad (x,y)\in E_4.$$

Then, if we consider points in E_4 of the form $(1,y)$ the above inequality tells us that

$$1-\beta-w_1(1-p_1)-w_2(y-p_2)>0\quad\text{for every}\quad y\in\mathbb{R}.$$

If we first let $y\to+\infty$ and then $y\to-\infty$, we obtain both $w_2\le 0$ and $w_2\ge 0$, and then $w_2=0$. This implies that $w\in\mathbb{R}\times\{0\}$, which contradicts condition $(iv)(a)$. Therefore, condition (iv) does not hold with $X=\mathbb{R}^2$.

□

Even though Theorem 4.38 fully solves Problem 4.31, an important question (as coercivity of a convex function may be relevant or even essential to a number of possible applications, e.g. in PDE theory) remains open: how can we characterize those 1-jets (f,G) such that there exists an essentially coercive convex function $F\in C^1(\mathbb{R}^n)$ so that $(F,\nabla F)$ extends (f,G)? The answer is: those jets are the jets which satisfy the conditions of Theorem 4.38 with $X=\mathbb{R}^n$. More generally, one could ask for C^1 convex extensions with prescribed global behavior (meaning extensions which are essentially coercive only in some directions, and affine in others). This ties in with a question which will be extremely important in our proofs: what is the global geometrical shape of the C^1 convex extension we are trying to build?

In this regard, it will be convenient for us to state a refinement of Theorem 4.38 which characterizes the set of 1-jets admitting C^1 convex extensions with a prescribed global behavior, and which requires our introducing some definitions and notation.

Definition 4.40. Let Z be a real vector space, and $P : Z \to X$ be the orthogonal projection onto a subspace $X \subseteq Z$. We will say that a function f defined on a subset E of Z is *essentially P-coercive* provided that there exists a linear function $\ell : Z \to \mathbb{R}$ such that for every sequence $(x_k)_k \subset E$ with $\lim_{k\to\infty} |P(x_k)| = \infty$ one has

$$\lim_{k\to\infty} (f - \ell)(x_k) = \infty.$$

We will say that f is *essentially coercive* whenever f is essentially I-coercive, where $I : Z \to Z$ is the identity mapping.

If X is a linear subspace of $\mathbb{R}^n$, we will denote by $P_X : \mathbb{R}^n \to X$ the orthogonal projection, and we will say that $f : E \to \mathbb{R}$ is *essentially coercive in the direction of X* whenever f is essentially P_X-coercive.

We will also denote by $X^\perp$ the orthogonal complement of X in $\mathbb{R}^n$. For a subset V of $\mathbb{R}^n$, $\operatorname{span}(V)$ will stand for the linear subspace spanned by the vectors of V. Finally, we define $C^1_{\text{conv}}(\mathbb{R}^n)$ as the set of all functions $f : \mathbb{R}^n \to \mathbb{R}$ which are convex and of class C^1.

In [1] essentially coercive convex functions were called *properly convex*, and some approximation results, which fail for general convex functions, were shown to be true for this class of functions. The following result was also implicitly proved in [1, Lemma 4.2]. Since this will be a very important tool in the statements and proofs of all the results of this section, and because we have introduced new terminology and added conclusions, we will provide a self-contained proof in Section 4.7 for the readers' convenience.

Theorem 4.41. *For every convex function $f : \mathbb{R}^n \to \mathbb{R}$ there exist a unique linear subspace X_f of $\mathbb{R}^n$, a unique vector $v_f \in X_f^\perp$, and a unique essentially coercive convex function $c_f : X_f \to \mathbb{R}$ such that f can be written in the form*

$$f(x) = c_f(P_{X_f}(x)) + \langle v_f, x\rangle \text{ for all } x \in \mathbb{R}^n.$$

Moreover, if Y is a linear subspace of $\mathbb{R}^n$ such that f is essentially coercive in the direction of Y, then $Y \subseteq X_f$.

The following Proposition shows that the directions X_f given by these decompositions are stable by approximation.

Proposition 4.42. *With the notation of the preceding theorem, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and A is a positive number such that $f(x) \leq g(x) + A$ for all $x \in \mathbb{R}^n$, then $X_f \subseteq X_g$.*

In particular, if $|f - g| \leq A$ then $X_f = X_g$.

Proof. The inequality $f(x) \leq g(x) + A$ and the essential coercivity of f in the direction X_f implies that g is essentially coercive in the direction X_f. Then $X_f \subseteq X_g$ by the last part of Theorem 4.41. □

We are finally ready to state the announced refinement of Theorem 4.38 which characterizes precisely what 1-jets (f, G) admit extensions $(F, \nabla F)$ such that $F \in C^1_{\text{conv}}(\mathbb{R}^n)$ and X_F coincides with a prescribed linear subspace X of $\mathbb{R}^n$.

Theorem 4.43. *Given an arbitrary subset E of $\mathbb{R}^n$, a linear subspace $X \subset \mathbb{R}^n$, the orthogonal projection $P := P_X : \mathbb{R}^n \to X$, and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, the following is true. There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|E} = f$, $(\nabla F)_{|E} = G$, and $X_F = X$, if and only if the following conditions are satisfied.*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) $Y :=$ *span* $(\{G(x) - G(y) : x, y \in E\}) \subseteq X$.

(iv) *If* $Y \neq X$ *and we denote* $k = \dim Y$ *and* $d = \dim X$, *there exist points* $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, *numbers* $\beta_1, \ldots, \beta_{d-k} \in \mathbb{R}$, *and vectors* $w_1, \ldots, w_{d-k} \in \mathbb{R}^n$ *such that:*

(a) $X =$ *span* $(\{u - v \,:\, u, v \in G(E) \cup \{w_1, \ldots, w_{d-k}\}\})$.

(b) $\beta_j > \max_{1 \le i \neq j \le d-k}\{\beta_i + \langle w_i, p_j - p_i\rangle\}$ *for all* $1 \le j \le d-k$.

(c) $\beta_j > \sup_{z \in E, \, |G(z)| \le N}\{f(z) + \langle G(z), p_j - z\rangle\}$ *for all* $1 \le j \le d-k$ *and* $N \in \mathbb{N}$.

(d) $\inf_{x \in E, \, |P(x)| \le N}\{f(x) - \max_{1 \le j \le d-k}\{\beta_j + \langle w_j, x - p_j\rangle\}\} > 0$ *for all* $N \in \mathbb{N}$.

(v) *If* $(x_k)_k, (z_k)_k$ *are sequences in* E *such that* $(P(x_k))_k$ *and* $(G(z_k))_k$ *are bounded and*

$$\lim_{k \to \infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0,$$

then $\lim_{k \to \infty} |G(x_k) - G(z_k)| = 0$.

In particular, by considering the case that $X = \mathbb{R}^n$, we obtain a characterization of the 1-jets which admit C^1 convex extensions F such that $X_F = \mathbb{R}^n$, that is, C^1 convex extensions which are essentially coercive in $\mathbb{R}^n$.

It is clear that Theorem 4.38 and Corollaries 4.33 and 4.36 are immediate consequences of the above theorem. The proof of Theorem 4.43 will be given in Sections 4.8 and 4.9.

Remark 4.44. In practice, if $Y \neq X$ and we are able to calculate (or at least appropriately estimate) the minimal convex extension of the jet (f, G), defined by

$$m(x) = m(f, G)(x) = \sup_{y \in E}\{f(y) + \langle G(y), x - y\rangle\},$$

then a natural way to check condition (iv) is as follows. Define, for each $u \in X$, $p \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, the sets

$$S(m, u, p, \beta) = \{x \in \mathbb{R}^n : m(x) < \beta + \langle u, x - p\rangle\},$$

and consider vectors $\{u_1, \ldots, u_{d-k}\}$ such that $X = \text{span}\,(\{u - v \,:\, u, v \in G(E) \cup \{u_1, \ldots, u_{d-k}\}\})$. Find $p_1 \in \mathbb{R}^n \setminus \overline{E}$, $\beta_1 \in \mathbb{R}$ such that $S(m, u_1, p_1, \beta_1) \neq \emptyset$ and

$$m(x) \ge \beta_1 + \langle u_1, x - p_1\rangle + r \quad \text{for all} \quad x \in E,$$

for some $r > 0$. Also, find $q_1 \in S(m, u_1, p_1, \beta_1)$ sufficiently close to p_1 such that

$$m(q_1) \le \beta_1 + \langle u_1, q_1 - p_1\rangle - r' \quad \text{and} \quad |\langle u_1, p_1 - q_1\rangle| \le \frac{r'}{2},$$

for some $r' > 0$ with $r' \le r$. Then set $E_1^* = E \cup \{q_1\}$, and define $f_1^* := E_1^* \to \mathbb{R}$, $G_1^* : E_1^* \to \mathbb{R}^n$ by

$$f_1^*(q_1) = \beta_1, \; f_1^*(x) = f(x) \text{ if } x \in E; \quad G_1^*(q_1) = u_1, \; G_1^*(x) = G(x) \text{ if } x \in E.$$

Notice that the new putative tangent hyperplane $h(x) = \beta_1 + \langle G_1^*(q_1), x - q_1\rangle$ that we have added to our problem lies strictly below the graph of the old function f. Indeed, because $f = m$ on E, we have for all $x \in E$:

$$\begin{aligned} f(x) - f_1^*(q_1) - \langle G_1^*(q_1), x - q_1\rangle &= m(x) - \beta_1 - \langle u_1, x - p_1\rangle + \langle u_1, q_1 - p_1\rangle \\ &\ge r + \langle u_1, q_1 - p_1\rangle \ge \frac{r}{2}. \end{aligned}$$

On the other hand the old hyperplanes $x \mapsto f(y) + \langle G(y), x - y\rangle$, $y \in E$, lie strictly below the point $(q_1, f_1^*(q_1))$, as for all $y \in E$ we have

$$f_1^*(q_1) - f(y) - \langle G(y), q_1 - y\rangle \ge \beta_1 - m(q_1) \ge r' + \langle u_1, p_1 - q_1\rangle \ge \frac{r'}{2}.$$

Next, for the jet (f_1^*, G_1^*) defined on E_1^* we consider the analogous C^1_{conv} extension problem. Now we have that

$$Y_1 := \text{span}\{G_1^*(x) - G_1^*(y) : x, y \in E_1^*\}$$

has dimension $k+1$ and contains Y. Proceeding as before we consider the minimal function

$$m_1(x) = m(f_1^*, G_1^*)(x)$$

and find $p_2, q_2 \in \mathbb{R}^n$, $\beta_2 \in \mathbb{R}$ with the same properties as p_1, q_1, β_1 with respect to E_1^* instead of E. Then we set $E_2^* = E_1^* \cup \{q_2\}$ and define $f_2^* := E_2^* \to \mathbb{R}$, $G_2^* : E_2^* \to \mathbb{R}^n$ by

$$f_2^*(q_2) = \beta_2, \; f_1^*(x) = f_1^*(x) \; \text{ if } x \in E_1^*; \quad G_2^*(q_2) = u_2, \; G_2^*(x) = G_1^*(x) \; \text{ if } x \in E_1^*.$$

By continuing the process in this manner we will obtain, in $d-k$ steps, points q_j, vectors $w_j = u_j$ and numbers β_j, $j = 1, \ldots, d-k$, satisfying condition (iv) of Theorem 4.43.

4.7 Global behaviour of convex functions

In this section, we shall prove Theorem 4.41 as well as some properties related to the subspace X_f mentioned in that theorem which will be crucial in the proof of Theorem 4.43. Let us first recall some terminology from [1]. We say that a function $C : \mathbb{R}^n \to \mathbb{R}$ is a k-dimensional *corner function* on $\mathbb{R}^n$ if it is of the form

$$C = \max\{\, \ell_1 + b_1,\, \ell_2 + b_2,\, \ldots,\, \ell_k + b_k \,\},$$

where the $\ell_j : \mathbb{R}^n \to \mathbb{R}$ are linear functions such that the functions $L_j : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$, $1 \le j \le k$, are linearly independent in $(\mathbb{R}^{n+1})^*$, and the $b_j \in \mathbb{R}$. This is equivalent to saying that the functions $\{\ell_2 - \ell_1, \ldots, \ell_k - \ell_1\}$ are linearly independent in $(\mathbb{R}^n)^*$.

We also say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is supported by C at a point $x \in \mathbb{R}^n$ provided we have $C \le f$ on $\mathbb{R}^n$ and $C(x) = f(x)$.

Let us first study the essential coercivity of corner functions, which will be helpful in the proof of Theorem 4.41.

Lemma 4.45. *Every $(k+1)$-dimensional corner function on $\mathbb{R}^n$ is essentially P_X-coercive, where $P_X : \mathbb{R}^n \to X$ denotes the orthogonal projection onto a k-dimensional subspace X of $\mathbb{R}^n$. In fact, if $\ell_1, \ldots, \ell_{k+1} \in (\mathbb{R}^n)^*$ and $b_1, \ldots, b_{k+1} \in \mathbb{R}$ are such that $\{\ell_j - \ell_1\}_{j=2}^{k+1}$ are linearly independent in $(\mathbb{R}^n)^*$ and $C = \max_{1 \le j \le k+1}\{\ell_j + b_j\}$, then C is essentially P_X-coercive, where*

$$X = \left(\bigcap_{j=2}^{k+1} \ker(\ell_j - \ell_1) \right)^{\perp}.$$

Proof. Let C be a $(k+1)$-dimensional corner function and let $\ell_1, \ldots, \ell_{k+1} \in (\mathbb{R}^n)^*$, $b_1, \ldots, b_{k+1} \in \mathbb{R}$ and X as in the statement. For every $j = 1, \ldots, k+1$ we denote by v_j the unique vector of $\mathbb{R}^n$ such that $\ell_j(x) = \langle v_j, x \rangle$ for every $x \in \mathbb{R}^n$. Hence, the function C can be written as

$$C(x) = \max\{\langle v_1, x \rangle + b_1, \ldots, \langle v_{k+1}, x \rangle + b_{k+1}\}, \quad x \in \mathbb{R}^n,$$

where the vectors $v_1, \ldots, v_{k+1} \in \mathbb{R}^n$ satisfy that $\{v_2 - v_1, \ldots, v_{k+1} - v_1\}$ is linearly independent in $\mathbb{R}^n$ and $X = \text{span}\{v_2 - v_1, \ldots, v_{k+1} - v_1\}$. Let us denote

$$u_j = v_1 - v_j, \quad j = 2, \ldots, k+1, \quad \text{and} \quad v = \lambda \sum_{i=1}^{k+1} v_i, \quad \lambda = \frac{1}{k+1}.$$

Note that $X = \operatorname{span}\{u_2, \dots, u_{k+1}\}$. We now write, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} C(x) - \langle v, x\rangle &= \max\left\{ \langle v_j, x\rangle - \lambda \sum_{i=1}^{k+1} \langle v_i, x\rangle + b_j \ : \ j = 1, \dots, k+1 \right\} \\ &= \max\left\{ \langle v_j - v_1, x\rangle + \lambda \sum_{i=1}^{k+1} \langle v_1 - v_i, x\rangle + b_j \ : \ j = 1, \dots, k+1 \right\} \\ &= \max\left\{ \langle v_j - v_1, x\rangle + \lambda \sum_{i=2}^{k+1} \langle v_1 - v_i, x\rangle + b_j \ : \ j = 1, \dots, k+1 \right\} \\ &= \max\left\{ \lambda \sum_{i=2}^{k+1} \langle u_i, x\rangle + b_1, -\langle u_j, x\rangle + \lambda \sum_{i=2}^{k+1} \langle u_i, x\rangle + b_j \ : \ j = 2, \dots, k+1 \right\}. \end{aligned}$$

For every $i \in \{2, \dots, k+1\}$ we have that $\langle u_i, x\rangle = \langle u_i, P_X(x)\rangle$ because $u_i \in X$. This allows us to write

$$C(x) = c(P_X(x)) + \langle v, x\rangle, \quad x \in \mathbb{R}^n, \quad \text{where,}$$

$$c(y) = \max\left\{ \lambda \sum_{i=2}^{k+1} \langle u_i, y\rangle + b_1, -\langle u_j, y\rangle + \lambda \sum_{i=2}^{k+1} \langle u_i, y\rangle + b_j \ : \ j = 2, \dots, k+1 \right\}, \quad y \in X. \tag{4.7.1}$$

The function c is obviously a convex function on X (in fact, a corner function on X) and let us see that c is coercive on X. For the sake of contradiction, let $(y_\ell)_\ell$ be a sequence on X with $\lim_\ell |y_\ell| = +\infty$ such that $(c(y_\ell))_\ell$ is bounded above. Here $|\cdot|$ denotes the Euclidean norm on X (the restriction of the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ to X). If we define

$$\|y\| := \max\{|\langle u_2, y\rangle|, \dots, |\langle u_{k+1}, y\rangle|\}, \quad y \in X,$$

the fact that $X = \operatorname{span}\{u_2, \dots, u_{k+1}\}$ tells us that $\|\cdot\|$ is a norm on X, which is necessarily equivalent to the Euclidean norm $|\cdot|$ on X. This implies that $\lim_\ell \|y_\ell\| = +\infty$ and then we can find a subsequence of $(y_\ell)_\ell$, which we will keep denoting by $(y_\ell)_\ell$, such that, for every $j = 2, \dots, k+1$, the sequence $(\langle u_j, y_\ell\rangle)_\ell$ either tends to $+\infty$ or tends to $-\infty$ or else is bounded. Also, we have that $\lim_\ell |\langle u_j, y_\ell\rangle| = +\infty$ for at least one $j \in \{2, \dots, k+1\}$. We set

$$J^+ = \{j \in \{2, \dots, k+1\} \ : \ \lim_\ell \langle u_j, y_\ell\rangle = +\infty\}, \quad J^- = \{j \in \{2, \dots, k+1\} \ : \ \lim_\ell \langle u_j, y_\ell\rangle = -\infty\}.$$

The preceding observations show that $J^+ \cup J^-$ is nonempty. Assume first that $J^- = \emptyset$. Then each sequence $(\langle u_i, y_\ell\rangle)_\ell$, $i = 2, \dots k+1$, is bounded below and there is some $j_* \in J^+$. We thus have from (4.7.1) that

$$c(y_\ell) \geq \lambda \sum_{i=2}^{k+1} \langle u_i, y_\ell\rangle + b_1 \geq \frac{1}{2} \langle u_{j_*}, y_\ell\rangle \quad \text{for} \quad \ell \quad \text{large enough.}$$

This implies that $\lim_\ell c(y_\ell) = +\infty$, which contradicts that $(c(y_\ell))_\ell$ is bounded above. Assume now that $J^- \neq \emptyset$ and set $m = \operatorname{card}(J^-) \leq k$. Let $\alpha \in \mathbb{R}$ be a constant such that

$$\lambda \sum_{i \notin J^-} \langle u_i, y_\ell\rangle + \min_{j=2,\dots,k+1} b_j \geq \alpha \quad \text{for every} \quad \ell.$$

Using again (4.7.1) we can write

$$c(y_\ell) \geq \max_{j \in J^-} \left\{ -\langle u_j, y_\ell\rangle + \lambda \sum_{i \in J^-} \langle u_i, y_\ell\rangle + \lambda \sum_{i \notin J^-} \langle u_i, y_\ell\rangle + b_j \right\} \geq \alpha + \max_{j \in J^-} \left\{ -\langle u_j, y_\ell\rangle + \lambda \sum_{i \in J^-} \langle u_i, y_\ell\rangle \right\};$$

and thanks to the trivial inequality $\max(a_1,\ldots,a_m) \geq \frac{1}{m}(a_1+\cdots+a_m)$ (which holds for all real numbers $a_1,\ldots,a_m$), the above term is greater than or equal to

$$\frac{1}{m}\left(\sum_{j\in J^-} -\langle u_j, y_\ell\rangle + m\lambda \sum_{i\in J^-}\langle u_i, y_\ell\rangle\right) = \frac{1}{m}\left(\sum_{j\in J^-} -(1-m\lambda)\langle u_j, y_\ell\rangle\right).$$

Because $m\lambda = \frac{m}{k+1}$ is strictly smaller than 1, the last term tends to $+\infty$ as $\ell \to +\infty$, which implies that $\lim_\ell c(y_\ell) = +\infty$, a contradiction. Therefore c must be coercive and this completes the proof of our Lemma. $\square$

Let us now prove that, assuming that a convex function f admits a decomposition such as that of Theorem 4.41, then the subspace X_f is uniquely determined in terms of the subdifferentials of f. In order to do so, we use the following auxiliar lemma, which is important by itself.

Lemma 4.46. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function such that f admits a decomposition $f = c\circ P_X + \langle v,\cdot\rangle$, where $P_X:\mathbb{R}^n\to X$ is the orthogonal projection onto X, $c: X\to\mathbb{R}^n$ is convex and essentially coercive and $v\in\mathbb{R}^n$. Given $x\in\mathbb{R}^n$ and $\eta\in\partial f(x)$, we have $\eta - v\in X$ and $\eta - v\in\partial c(P_X(x))$.*

Proof. Suppose that $x\in\mathbb{R}^n$ and $\eta\in\partial f(x)$ but $\eta - v\notin X$. Then we can find $w\in X^\perp$ with $\langle \eta - v, w\rangle = 1$. It follows from $f = c\circ P_X + \langle v,\cdot\rangle$ that

$$\langle \eta, w\rangle \leq f(x+w) - f(x) = c(P_X(x+w)) + \langle v, x+w\rangle - c(P_X(x)) - \langle v, x\rangle = \langle v, w\rangle.$$

This implies that $\langle \eta - v, w\rangle \leq 0$, a contradiction. This shows that $\eta - v\in X$. Now, let $z\in X$ and $x\in\mathbb{R}^n$. We have

$$c(z) - c(P_X(x)) = f(z) - \langle v, z\rangle - f(x) + \langle v, x\rangle \geq \langle \eta - v, z - x\rangle = \langle \eta - v, z - P_X(x)\rangle.$$

Therefore, $\eta - v\in\partial c(P_X(x))$. $\square$

Lemma 4.47. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function and assume that f can be written as $f = c\circ P_X + \langle v,\cdot\rangle$, where $P_X:\mathbb{R}^n\to X$ is the orthogonal projection onto the subspace X of $\mathbb{R}^n$, $c: X\to\mathbb{R}$ is convex and essentially coercive and $v\in\mathbb{R}^n$. Then*

$$X = \mathit{span}\{\xi_x - \xi_y \,:\, \xi_x\in\partial f(x),\ \xi_y\in\partial f(y),\ x,y\in\mathbb{R}^n\}.$$

In particular, if f is differentiable on $\mathbb{R}^n$, then

$$X = \mathit{span}\{\nabla f(x) - \nabla f(y) \,:\, x,y\in\mathbb{R}^n\}.$$

Proof. Let us denote by Z the subspace of the right side term. Given two points $x,y\in\mathbb{R}^n$ and $\xi_x\in\partial f(x)$, $\xi_y\in\partial f(y)$, we know by Lemma 4.46 that $\xi_x - v$ and $\xi_y - v$ belong to X, which implies that $\xi_x - \xi_y\in X$. This shows that $Z\subseteq X$. In order to prove that $X\subseteq Z$, assume that there exists some $w\in X\setminus\{0\}$ with $w\perp Z$. We take $x_0\in\mathbb{R}^n$ and $\xi_t\in\partial f(x_0+tw)$ for every $t\in\mathbb{R}$, that is, $\xi_0\in\partial f(x_0)$. Thus $\xi_0 - \xi_t\in Z$ for every $t\in\mathbb{R}$ and then

$$0\leq f(x_0+tw) - f(x_0) - \langle \xi_0, tw\rangle \leq \langle \xi_t - \xi_0, tw\rangle = 0,$$

that is, $f(x_0+tw) = f(x_0) + \langle \xi_0, tw\rangle$ for every $t\in\mathbb{R}$. The decomposition $f = c\circ P_X + \langle v,\cdot\rangle$ of f yields

$$c\,(P_X(x_0)+tw) = f(x_0+tw) - \langle v, x_0+tw\rangle = f(x_0) + \langle \xi_0, w\rangle - \langle v, x_0+tw\rangle,\quad t\in\mathbb{R},$$

which is an affine function on $\mathbb{R}$ and contradicts the fact that c is essentially coercive on X. Therefore we must have $X = Z$.

$\square$

Finally, let us prove that if our convex function to be decomposed as in Theorem 4.41 is given by a supremum of a (possible infinite) family of affine functions, then the subspace of the decomposition can be written in terms of the linear parts of these affine functions. These will be very useful in the proof of the *if* part of Theorem 4.43.

Lemma 4.48. *Let $f : E \to \mathbb{R}$ and $G : E \to \mathbb{R}^n$ two mappings defined on a subset E of $\mathbb{R}^n$ satisfying that $f(x) \geq f(y) + \langle G(y), x-y\rangle$ for all $x, y \in E$. Consider the function*

$$m(x) = \sup_{y\in E}\{f(y) + \langle G(y), x-y\rangle\}, \quad x \in \mathbb{R}^n,$$

and assume that $m(x)$ is finite for every $x \in \mathbb{R}^n$ and that m admits the decomposition $m = c\circ P_X + \langle v, \cdot\rangle$, where $P_X : \mathbb{R}^n \to X$ is the orthogonal projection onto the subspace X of $\mathbb{R}^n$, $c : X \to \mathbb{R}^n$ is convex and essentially coercive, and $v \in \mathbb{R}^n$. Then $X = span\{G(x) - G(y) \ : \ x, y \in E\}$.

Proof. Let us denote $Z = \operatorname{span}\{G(x) - G(y) \ : \ x, y \in E\}$. By definition of m, it is clear that $m(x) = f(x)$ for every $x \in E$. Moreover, $G(x) \in \partial m(x)$ for every $x \in E$. By Lemma 4.47, $G(x) - G(y) \in X$ for every $x, y \in E$, which proves that $Z \subseteq X$. Now, if $X \neq Z$, we can take a vector $w \in X \setminus \{0\}$ such that $w \perp Z$. If we consider a point $x_0 \in E$ we obtain, for all $t \in \mathbb{R}$, that

$$\begin{aligned} m(x_0 + tw) - m(x_0) - \langle G(x_0), tw\rangle &= m(x_0 + tw) - f(x_0) - \langle G(x_0), tw\rangle \\ &= \sup_{z\in E}\{f(z) - f(x_0) + \langle G(z) - G(x_0), tw\rangle + \langle G(z), x_0 - z\rangle\} \\ &= \sup_{z\in E}\{f(z) - f(x_0) + \langle G(z), x_0 - z\rangle\} \leq 0. \end{aligned}$$

Because $G(x_0) \in \partial m(x_0)$, we also have $m(x_0 + tw) - m(x_0) - \langle G(x_0), tw\rangle \geq 0$, which implies

$$c(P_X(x_0) + tw) = m(x_0 + tw) - \langle v, x_0 + tw\rangle = m(x_0) - \langle v, x_0\rangle + \langle G(x_0) - v, tw\rangle$$

for all $t \in \mathbb{R}$, and in particular the function $\mathbb{R} \ni t \mapsto c(P_X(x_0) + tw)$ cannot be essentially coercive, contradicting the essentiall coercivity of c. Therefore we must have $X = Z$.

□

Once we have established all these properties, let us prove Theorem 4.41.

Proof of Theorem 4.41. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let us show that f admits a decomposition $f = c\circ P_X + \langle v, \cdot\rangle$, where P_X is the orthogonal projection onto the subspace X of $\mathbb{R}^n$, $c : X \to \mathbb{R}$ is convex and essentially coercive and $v \in X^{\perp}$. Let us study two cases separately.

Case 1. We will first assume that f is differentiable (and therefore of class C^1, since f is convex). If f is affine, say $f(x) = a\langle u, x\rangle + b$, then the result is trivially true with $X = \{0\}$, $c(0) = b$, and $v = au$. On the other hand, if f is essentially coercive then the result also holds obviously with $X = \mathbb{R}^n$, $v = 0$, and $c = f$. So we may assume that f is neither affine nor essentially coercive. In particular there exist $x_0, y_0 \in \mathbb{R}^n$ with $Df(x_0) \neq Df(y_0)$. It is then clear that $L_1(x, x_{n+1}) = x_{n+1} - Df(x_0)(x)$ and $L_2(x, x_{n+1}) = x_{n+1} - Df(y_0)(x)$ are two linearly independent linear functions on $(\mathbb{R}^{n+1})^*$, hence f is supported at x_0 by the two-dimensional corner

$$\mathbb{R}^n \ni x \mapsto \max\{f(x_0) + Df(x_0)(x - x_0), f(y_0) + Df(y_0)(x - y_0)\}.$$

Let us then define k as the greatest integer number so that f is supported at x_0 by a $(k+1)$-dimensional corner. The assumption together with Lemma 4.45 give $1 \leq k < n$. Then we also have that there exist $\ell_1, \ldots, \ell_{k+1} \in (\mathbb{R}^n)^*$ and $b_1, \ldots, b_{k+1} \in \mathbb{R}$ so that $\{\ell_j - \ell_1\}_{j=2}^{k+1}$ are linearly independent in $(\mathbb{R}^n)^*$ and the corner function $C = \max_{1\leq j\leq k+1}\{\ell_j + b_j\}$ supports f at x_0. The subspace $\bigcap_{j=2}^{k+1} \ker(\ell_j - \ell_1)$ has dimension $n - k$ and then we can find linearly independent vectors $w_1, \ldots, w_{n-k}$ such that $\bigcap_{j=2}^{k+1} \ker(\ell_j - \ell_1) = \operatorname{span}\{w_1, \ldots, w_{n-k}\}$. We now claim

$$\frac{d}{dt}(f - \ell_1)(y + tw_q) = 0 \quad \text{for all} \quad y \in \mathbb{R}^n,\ t \in \mathbb{R},\ q = 1, \ldots, n-k. \tag{4.7.2}$$

Indeed, assume that there exist $y \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $q \in \{1, \ldots, n-k\}$ such that $\frac{d}{dt}(f - \ell_1)(y + tw_q)|_{t=t_0} \neq 0$. It follows that $(Df(y+t_0w_q)-\ell_1)(w_q) \neq 0$, which in turn implies that $Df(y+t_0w_q)-\ell_1$ is linearly independent with $\{\ell_j - \ell_1\}_{j=2}^{k+1}$ because $w_q \in \bigcap_{j=2}^{k+1} \ker(\ell_j - \ell_1)$. Therefore, by convexity of f, the function

$$x \mapsto \max\{\ell_1(x) + b_1, \ldots, \ell_{k+1}(x) + b_{k+1}, Df(y + t_0w_q)(x - y - t_0w_q) + f(y + t_0w_q)\}$$

is a $(k + 2)$-dimensional corner supporting f at x_0, which contradicts the choice of k. This proves that (4.7.2) is true and then the Mean Value Theorem yields

$$(f-\ell_1)\left(y+\sum_{j=1}^{n-k} t_jw_j\right) = (f-\ell_1)\left(y+\sum_{j=2}^{n-k} t_jw_j\right) = \cdots = (f-\ell_1)(y+t_{n-k}w_{n-k}) = (f-\ell_1)(y) \tag{4.7.3}$$

for every $y \in \mathbb{R}^n$ and $t_1, \ldots, t_{n-k} \in \mathbb{R}$. Let P_X be the orthogonal projection of $\mathbb{R}^n$ onto the subspace $X := \operatorname{span}\{w_1, \ldots, w_{n-k}\}^\perp$. We may define

$$\widetilde{c}(z) = (f - \ell_1)(z) \quad \text{for all} \quad z \in X,$$

which obviously is a convex function. For every $x \in \mathbb{R}^n$, we can write

$$x = P_X(x) + P_{X^\perp}(x) = P_X(x) + \sum_{j=1}^{n-k} t_jw_j \quad \text{for some} \quad t_1, \ldots, t_{n-k} \in \mathbb{R}$$

and then (4.7.3) gives $(f - \ell_1)(x) = \widetilde{c}(P_X(x))$. Now let us write

$$\ell_1(x) = \langle u, x\rangle + \langle v, x\rangle \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $u \in X$ and $v \in X^\perp$. We then have

$$f(x) = c(P_X(x)) + \langle v, x\rangle \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $c : X \to \mathbb{R}$ is defined by

$$c(x) = \widetilde{c}(x) + \langle u, x\rangle.$$

Finally, let us see that c is essentially coercive on X. Since $C = \max_{1\leq j\leq k+1}\{\ell_j + b_j\}$ is a $(k + 1)$-dimensional corner function and $X^\perp = \bigcap_{j=2}^{k+1} \operatorname{Ker}(\ell_j - \ell_1)$, Lemma 4.45 tells us that C is essentially P_X-coercive, that is, the restriction of C to X is essentially coercive on X. Besides,

$$c(x) = f(x) - \langle v, x\rangle = f(x) \geq C(x) \quad \text{for every} \quad x \in X,$$

and therefore c is essentially coercive on X.

Case 2. In the case that $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not everywhere differentiable, we can use [1, Corollary 1.3] in order to find a C^1 (or even C^∞) convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f - 1 \leq g \leq f$ on $\mathbb{R}^n$. Then we may apply Case 1 in order to find a subspace $X \subseteq \mathbb{R}^n$, an essentially coercive convex function $c : X \to \mathbb{R}$ and a vector $v \in X^\perp$ such that

$$g(z) = c(P_X(z)) + \langle v, z\rangle$$

for all $z \in \mathbb{R}^n$, where $P_X : \mathbb{R}^n \to X$ is the orthogonal projection. Given $x \in X$ and $\xi \in X^\perp$, the function

$$\mathbb{R} \ni t \mapsto g(x + t\xi) = c(P_X(x)) + \langle v, \xi\rangle t + \langle v, x\rangle$$

is affine. Since $f \leq g + 1$ and f is convex, the subdifferentials of the function $\mathbb{R} \ni t \mapsto f(x + t\xi)$ must coincide with $\langle v, \xi\rangle$. In other words, the function $\mathbb{R} \ni t \mapsto f(x + t\xi)$ is affine and with linear part equal to $\langle v, \xi\rangle$. This shows that

$$f(x + t\xi) = f(x) + t\langle v, \xi\rangle$$

for every $x \in X$, $\xi \in X^\perp$ and $t \in \mathbb{R}$. Hence, by writing $z \in \mathbb{R}^n$ as $z = P_X(z) + P_{X^\perp}(z)$ and bearing in mind that $v \in X^\perp$ we obtain

$$f(z) = f(P_X(z)) + \langle v, P_{X^\perp}(z)\rangle = \varphi(P_X(z)) + \langle v, z\rangle \quad \text{for all} \quad z \in \mathbb{R}^n,$$

where $\varphi : X \to \mathbb{R}$ is defined by $\varphi(x) = f(x)$ for all $x \in X$. Moreover, because $g \le \varphi$ on X and g is essentially P_X-coercive it is clear that φ is essentially coercive on X. This shows the existence of the decomposition in the statement.

Now let us see that this decomposition is unique. In order to do so, let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and assume that we have two subspaces Z_1, Z_2, two convex and essentially coercive functions $\varphi_1 : Z_1 \to \mathbb{R}$, $\varphi_2 : Z_2 \to \mathbb{R}$ and two vectors $\xi_1 \in Z_1^\perp$, $\xi_2 \in Z_2^\perp$ for which

$$f(x) = \varphi_1(P_{Z_1}(x)) + \langle \xi_1, x\rangle, \tag{4.7.4}$$

and

$$f(x) = \varphi_2(P_{Z_2}(x)) + \langle \xi_2, x\rangle, \tag{4.7.5}$$

for every $x \in \mathbb{R}^n$. Thanks to Lemma 4.47 we know that both Z_1 and Z_2 must coincide with

$$\operatorname{span}\{\xi_x - \xi_y \ : \ \xi_x \in \partial f(x),\ \xi_y \in \partial f(y),\ x, y \in \mathbb{R}^n\},$$

in particular $Z_1 = Z_2$. Next, let us see that $\xi_1 = \xi_2$. If we set $x = 0$, then (4.7.4) and (4.7.5) yield $\varphi_1(0) = f(0) = \varphi_2(0)$ and then, for every $v \in Z_1^\perp$, we have

$$\varphi_1(0) + \langle \xi_1, v\rangle = f(v) = \varphi_2(0) + \langle \xi_2, v\rangle = \varphi_1(0) + \langle \xi_2, v\rangle,$$

which implies that $\langle \xi_1, v\rangle = \langle \xi_2, v\rangle$ for all $v \in Z_1^\perp$. Because $\xi_1, \xi_2 \in Z_1^\perp$ this shows that $\xi_1 = \xi_2$. Once we know that $Z_1 = Z_2$ and $\xi_1 = \xi_2$, it immediately follows from (4.7.4) and (4.7.5) that $\varphi_1 = \varphi_2$. This shows that the decomposition is unique.

Finally let us prove that if f is essentially coercive in the direction of a subspace Y then $Y \subseteq X_f$. So, let us assume that there exists a linear form ℓ on $\mathbb{R}^n$ such that $f(x) - \ell(x) \to \infty$ as $|P_Y(x)| \to \infty$ but $Y \nsubseteq X_f$. Then there exists a vector $\xi \in X_f^\perp \setminus Y^\perp$ and this implies that the function

$$\mathbb{R} \ni t \mapsto f(t\xi) - \ell(t\xi) = c(P_{X_f}(t\xi)) + t\langle v, \xi\rangle - \ell(\xi)t = c(0) + t\langle v, \xi\rangle - \ell(\xi)t \tag{4.7.6}$$

is affine. Since $\xi \notin Y^\perp$, then $|P_Y(\xi)| > 0$ and $|P_Y(t\xi)| \to \infty$ as $|t| \to \infty$, which implies that $f(t\xi) - \ell(t\xi) \to \infty$ as $|t| \to \infty$ by the assumption on f. This contradicts the fact that the function of (4.7.6) is affine. Therefore we must have $Y = X_f$. The proof of Theorem 4.41 is thus complete. □

4.8 Proving the necessity of the conditions

We start with the proof of Theorem 4.43. In this section we are going to prove the *only if* part. So, let F be a convex function of class $C^1(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends (f, G) from E, and $X_F = X$. Let us check that conditions $(i) - (v)$ are satisfied for (f, G) and X.

4.8.1 Condition (i)

The inequality $f(x) - f(y) - \langle G(y), x - y\rangle \ge 0$ for all $x, y \in E$ follows from the fact that F is convex and differentiable with $(F, \nabla F) = (f, G)$ on E.

4.8.2 Condition (ii)

Assume that $(|\nabla F(x_k)|)_k$ tends to $+\infty$ for a sequence $(x_k)_k \subset \mathbb{R}^n$ but

$$\frac{\langle \nabla F(x_k), x_k \rangle - F(x_k)}{|\nabla F(x_k)|}$$

does not go to $+\infty$. Then, passing to a subsequence, we may assume that there exists $M > 0$ such that $\langle \nabla F(x_k), x_k \rangle - F(x_k) \leq M|\nabla F(x_k)|$ for all k. We denote $z_k = 2M \frac{\nabla F(x_k)}{|\nabla F(x_k)|}$. By convexity, we have, for all k, that

$$0 \leq F(z_k) - F(x_k) - \langle \nabla F(x_k), z_k - x_k \rangle \leq F(z_k) - M|\nabla F(x_k)|,$$

which contradicts the assumption that $|\nabla F(x_k)| \to \infty$.

4.8.3 Condition (iii)

Making use of Theorem 4.41 and bearing in mind that $X_F = X$, we can write $F = c \circ P_X + \langle v, \cdot \rangle$, where $P_X : \mathbb{R}^n \to X$ is the orthogonal projection onto the subspace X, the function $c : X \to \mathbb{R}$ is convex and essentially coercive on X, and $v \perp X$. Because F is differentiable on $\mathbb{R}^n$ and $c = F - \langle v, \cdot \rangle$ on X, the function c is differentiable on X and $\nabla F(x) = \nabla c(P_X(x)) + v$ for all $x \in \mathbb{R}^n$. Since $F = G$ on E, we get that

$$G(x) - G(y) = \nabla F(x) - \nabla F(y) = \nabla c(P_X(x)) - \nabla c(P_X(y)) \in X, \quad \text{for all} \quad x, y \in E.$$

4.8.4 Condition (v)

Let us consider sequences $(x_k)_k, (z_k)_k$ on E such that $(P_X(x_k))_k$ and $(\nabla F(z_k))_k$ are bounded and

$$\lim_{k\to\infty} (F(x_k) - F(z_k) - \langle \nabla F(z_k), x_k - z_k \rangle) = 0. \tag{4.8.1}$$

Suppose that $|\nabla F(x_k) - \nabla F(z_k)|$ does not converge to 0. Then, using that $(P_X(x_k))_k$ is bounded, there exist some $x_0 \in X$ and $\varepsilon > 0$ for which, possibly after passing to a subsequence, $P_X(x_k)$ converges to x_0 and $|\nabla F(x_k) - \nabla F(z_k)| \geq \varepsilon$ for every k. By using the decomposition $F = c \circ P_X + \langle v, \cdot \rangle$ and some elementary properties of orthogonal projections together with (4.8.1) we obtain

$$\lim_{k\to\infty} (c(P_X(x_k)) - c(P_X(z_k)) - \langle \nabla c(P_X(z_k)), P_X(x_k) - P_X(z_k) \rangle) = 0.$$

Since $\nabla F(y) - v = \nabla c(P_X(y))$ for all $y \in \mathbb{R}^n$ we have that $(\nabla c(P_X(z_k)))_k$ is bounded and

$$|\nabla c(P_X(x_k)) - \nabla c(P_X(z_k))| \geq \varepsilon$$

for every k. Besides

$$\lim_{k\to\infty} (c(x_0) - c(P_X(z_k)) - \langle \nabla c(P_X(z_k)), x_0 - P_X(z_k) \rangle) = 0.$$

The contradiction follows from the following lemma.

Lemma 4.49. *Let $h : X \to \mathbb{R}$ be a differentiable convex function, $x_0 \in X$, and $(y_k)_k$ be a sequence in X such that $(\nabla h(y_k))_k$ is bounded and*

$$\lim_{k\to\infty} (h(x_0) - h(y_k) - \langle \nabla h(y_k), x_0 - y_k \rangle) = 0.$$

Then $\lim_{k\to\infty} |\nabla h(x_0) - \nabla h(y_k)| = 0.$

Proof. Suppose not. Then, up to extracting a subsequence, we would have $|\nabla h(x_0) - \nabla h(y_k)| \geq \varepsilon$, for some positive ε and for every k. Now, for every k, we set

$$\alpha_k := h(x_0) - h(y_k) - \langle \nabla h(y_k), x_0 - y_k \rangle, \quad v_k := \frac{\nabla h(y_k) - \nabla h(x_0)}{|\nabla h(y_k) - \nabla h(x_0)|}.$$

In Lemma 4.12 it is proved that $\alpha_k = 0$ implies $|\nabla h(x_0) - \nabla h(y_k)| = 0$, which is absurd. Thus we must have $\alpha_k > 0$ for every k. By convexity we have

$$\begin{aligned}\sqrt{\alpha_k}\langle \nabla h(x_0 + \sqrt{\alpha_k} v_k), v_k\rangle &\geq h(x_0 + \sqrt{\alpha_k} v_k) - h(x_0)\\ &\geq h(y_k) + \langle \nabla h(y_k), x_0 + \sqrt{\alpha_k} v_k - y_k\rangle - h(x_0)\\ &= -\alpha_k + \sqrt{\alpha_k}\langle \nabla h(y_k), v_k\rangle\end{aligned}$$

for all k. Hence, we obtain

$$\langle \nabla h(x_0 + \sqrt{\alpha_k} v_k) - \nabla h(x_0), v_k\rangle \geq -\sqrt{\alpha_k} + |\nabla h(y_k) - \nabla h(x_0)| \geq -\sqrt{\alpha_k} + \varepsilon.$$

But the above inequality is impossible, as ∇h is continuous and $\alpha_k \to 0$. □

4.8.5 Condition (iv)

By applying Theorem 4.41 we may write

$$F(x) = c(P_X(x)) + \langle v, x\rangle,$$

with $c : X \to \mathbb{R}$ convex and essentially coercive, and $v \perp X$. And from Lemma 4.47

$$X = \text{span}\{\nabla F(x) - \nabla F(y) : x, y \in \mathbb{R}^n\}.$$

Let us denote $Y := \text{span}\{\nabla F(x) - \nabla F(y) : x, y \in E\} \subset X$ and assume that $Y \neq X$. Let k and d denote the dimensions of Y and X respectively. We can find points $x_0, x_1, \ldots, x_k \in E$ such that $Y = \text{span}\{\nabla F(x_j) - \nabla F(x_0) : j = 1, \ldots, k\}$. We claim that there exists $p_1 \in \mathbb{R}^n$ such that $\nabla F(p_1) - \nabla F(x_0) \notin Y$. Indeed, otherwise we would have that $\nabla F(p) - \nabla F(x_0) \in Y$ for all $p \in \mathbb{R}^n$, which implies that

$$\nabla F(p) - \nabla F(q) = (\nabla F(p) - \nabla F(x_0)) - (\nabla F(q) - \nabla F(x_0)) \in Y, \quad \text{for all} \quad p, q \in \mathbb{R}^n.$$

This is a contradiction since $X \neq Y$. Then the subspace Y_1 spanned by Y and the vector $\nabla F(p_1) - \nabla F(x_0)$ has dimension $k+1$. If $d = k+1$, we are done. If $d > k+1$, using the same argument as above, we can find a point $p_2 \in \mathbb{R}^n$ such that $\nabla F(p_2) - \nabla F(x_0) \notin Y_1$. By induction, we obtain points $p_1, \ldots, p_{d-k} \in \mathbb{R}^n$ such that the set $\{\nabla F(p_j) - \nabla F(x_0)\}_{j=1}^{d-k}$ is linearly independent and $X = Y \oplus \text{span}\{\nabla F(p_j) - \nabla F(x_0) : j = 1, \ldots, d-k\}$, which shows that

$$X = \text{span}\left\{u - w : u, w \in \nabla F(E) \cup \{\nabla F(p_1), \ldots, \nabla F(p_{d-k})\}\right\}.$$

This shows the necessity of $(iv)(a)$. Obviously we have $\nabla F(p_j) - \nabla F(x_0) \in X \setminus Y$ for all $j = 1, \ldots, d-k$, and we claim that

$$p_j \in \mathbb{R}^n \setminus \overline{E} \quad \text{for all} \quad j = 1, \ldots, d-k.$$

Indeed, if there exists a sequence $(q_\ell)_\ell \subset E$ with $(q_\ell)_\ell \to p_j$ for some $j = 1, \ldots, d-k$, then, because Y is closed and ∇F is continuous, $\nabla F(p_j) - \nabla F(x_0) = \lim_\ell (\nabla F(q_\ell) - \nabla F(x_0)) \in Y$, which is a contradiction. By the (already shown) necessity of condition (v), applied with $E^* = E \cup \{p_1, \ldots, p_{d-k}\}$ in place of E, we have that

$$\lim_{\ell\to\infty} |\nabla F(x_\ell) - \nabla F(z_\ell)| = 0 \tag{4.8.2}$$

whenever $(x_\ell)_\ell, (z_\ell)_\ell$ are sequences in E^* such that $(P_X(x_\ell))_\ell$ and $(\nabla F(z_\ell))_\ell$ are bounded and

$$\lim_{\ell\to\infty} \left(F(x_\ell) - F(z_\ell) - \langle \nabla F(z_\ell), x_\ell - z_\ell\rangle\right) = 0.$$

But the fact that $\text{dist}(\nabla F(p_j) - \nabla F(x_0), Y) > 0$ for each $j = 1, \ldots, d-k$ prevents the limiting condition (4.8.2) from holding true with $(z_\ell)_\ell \subset \{p_1, \ldots, p_{d-k}\}$ and $(x_\ell)_\ell \subset E$. This implies that the inequalities

$$\begin{aligned} &F(p_j) \geq F(p_i) + \langle \nabla F(p_i), p_j - p_i\rangle, \quad 1 \leq i, j \leq d-k,\ i \neq j,\\ &F(p_j) \geq \sup_{z\in E, |\nabla F(z)|\leq N} \{F(z) + \langle \nabla F(z), p_j - z\rangle\}, \quad 1 \leq j \leq d-k,\ N \in \mathbb{N}, \text{ and}\\ &F(x) \geq F(p_j) + \langle \nabla F(p_j), x - p_j\rangle, \quad 1 \leq j \leq d-k,\ x \in \mathbb{R}^n, \end{aligned}$$

which generally hold by convexity of F, must all be strict. Moreover, the last of these inequalities, together with (4.8.2), also implies that

$$\inf_{x\in E,\, |P_X(x)|\leq N} \{F(x) - \max_{1\leq j\leq d-k} \{F(p_j) + \langle \nabla F(p_j), x - p_j\rangle\}\} > 0$$

for all $N \in \mathbb{N}$. Setting $w_j = \nabla F(p_j)$ and $\beta_j = F(p_j)$, $j = 1, \ldots, d-k$, this shows the necessity of $(iv)(b) - (d)$.

4.9 Proving the sufficiency of the conditions

We are now going to prove the *if* part of Theorem 4.43. So, let us assume that E is an arbitrary subset and that (f, G) and X satisfy conditions $(i) - (v)$ of Theorem 4.43.

First of all, with the notation of condition (iv), if $Y \neq X$, we define

$$E^* = E \cup \{p_1, \ldots, p_{d-k}\}$$

and extend the functions f and G to E^* by setting

$$f(p_j) := \beta_j, \quad G(p_j) := w_j \quad \text{for} \quad j = 1, \ldots, d-k. \tag{4.9.1}$$

If $Y = X$, we just set $E^* = E$ and ignore any reference to the points p_j and their companions w_j and β_j in what follows.

Lemma 4.50. *We have:*

(1) $X = span\left(\{G(x) - G(y) \,:\, x, y \in E^*\}\right)$.

(2) *There exists* $r > 0$ *such that* $f(p_i) - f(p_j) - \langle G(p_j), p_i - p_j\rangle \geq r$ *for all* $1 \leq i \neq j \leq d-k$.

(3) *For every* $N \in \mathbb{N}$, *there exists* $r_N > 0$ *with* $f(p_i) - f(z) - \langle G(z), p_i - z\rangle \geq r_N$ *for all* $z \in E$ *with* $|G(z)| \leq N$ *and all* $1 \leq i \leq d-k$.

(4) *For every* $N \in \mathbb{N}$, *there exists* $r_N > 0$ *with* $f(x) - f(p_i) - \langle G(p_i), x - p_i\rangle \geq r_N$ *for all* $x \in E$ *with* $|P_X(x)| \leq N$ *and all* $1 \leq i \leq d-k$.

Proof. This follows immediately from condition (iv) and the definitions of (4.9.1). □

Lemma 4.51. *The jet* (f, G) *defined on* E^* *satisfies the inequalities of the assumption* (i) *on* E^*, *that is,*

$$f(x) \geq f(y) + \langle G(y), x - y\rangle, \quad x, y \in E^*.$$

Moreover, if $(x_k)_k, (z_k)_k$ *are sequences in* E^* *such that* $(P_X(x_k))_k$ *and* $(G(z_k))_k$ *are bounded, then*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0.$$

Proof. Suppose that$(x_k)_k, (z_k)_k$ are sequences in E^* such that $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded and $\lim_{k\to\infty}(f(x_k)-f(z_k)-\langle G(z_k), x_k-z_k\rangle)=0$. In view of Lemma 4.50 $(2), (3)$ and (4), it is immediate that there exists k_0 such that either there is some $1\le i\le d-k$ with $x_k=z_k=p_i$ for all $k\ge k_0$ or else $x_k, z_k\in E$ for all $k\ge k_0$. In the first case, the conclusion is trivial. In the second case, $\lim_{k\to\infty}|G(x_k)-G(z_k)|=0$ follows from condition (v) of Theorem 4.43. □

We now consider the *minimal convex extension* of the jet (f,G) from E^*, defined by

$$m(x)=m(f,G,E^*)(x):=\sup_{y\in E^*}\{f(y)+\langle G(y), x-y\rangle\},\quad x\in\mathbb{R}^n.$$

It is clear that m, being the supremum of a family of affine functions, is a convex function on $\mathbb{R}^n$. In fact, we have the following.

Lemma 4.52. *$m(x)$ is finite for every $x\in\mathbb{R}^n$. In addition, $m=f$ on E^* and $G(x)\in\partial m(x)$ for all $x\in E^*$.*

Here $\partial m(x):=\{\xi\in\mathbb{R}^n : m(y)\ge m(x)+\langle\xi, y-x\rangle$ for all $y\in\mathbb{R}^n\}$ is the subdifferential of m at x.

Proof. Fix a point $z_0\in E^*$. For any given point $x\in\mathbb{R}^n$ it is clear that there exists a sequence $(y_k)_k$ (possibly stationary) in E^* such that

$$f(z_0)+\langle G(z_0), x-z_0\rangle\le f(y_k)+\langle G(y_k), x-y_k\rangle\quad\text{for all}\quad k,$$

and $f(y_k)+\langle G(y_k), x-y_k\rangle\to m(x)$ as $k\to\infty$. On the other hand, by the first statement of Lemma 4.51, we have

$$f(y_k)+\langle G(y_k), x-y_k\rangle\le f(z_0)+\langle G(y_k), y_k-z_0\rangle+\langle G(y_k), x-y_k\rangle=f(z_0)+\langle G(y_k), x-z_0\rangle.$$

for every k. Then it is clear that $m(x)<+\infty$ when $(G(y_k))_k$ is a bounded sequence. We next show that this sequence can never be unbounded. Indeed, in such case, by the condition (ii) in Theorem 4.43 (which obviously holds with E^* in place of E), we would have a subsequence for which $\lim_{k\to\infty}|G(y_k)|=+\infty$ which in turn implies

$$\lim_{k\to\infty}\frac{\langle G(y_k), y_k\rangle-f(y_k)}{|G(y_k)|}=+\infty.$$

Hence, by the assumption on $(y_k)_k$ we would have

$$\frac{f(y_k)-\langle G(y_k), y_k\rangle}{|G(y_k)|}\ge\frac{f(z_0)+\langle G(z_0), x-z_0\rangle}{|G(y_k)|}-\Big\langle\frac{G(y_k)}{|G(y_k)|}, x\Big\rangle.$$

Since $\lim_{k\to\infty}|G(y_k)|=+\infty$, the right-hand term is bounded below, and this leads to a contradiction. Therefore $m(x)<+\infty$ for all $x\in\mathbb{R}^n$. In addition, by using the definition of m and the first statement of Lemma 4.51 for the jet (f,G), we obtain that $m=f$ on E^* and that $G(x)$ belongs to $\partial m(x)$ for all $x\in E^*$. □

Making use of Theorem 4.41, we can write

$$m=c\circ P_{X_m}+\langle v,\cdot\rangle\quad\text{on}\quad\mathbb{R}^n,$$

where $c: X_m\to\mathbb{R}$ is convex and essentially coercive on X_m and $v\perp X_m$. Thanks to the first part of Lemma 4.51, we can apply Lemma 4.48 to obtain that $X_m=\operatorname{span}\{G(x)-G(y) : x,y\in E^*\}$, which in turn coincides with X by virtue of Lemma 4.50 (1). We thus have $X_m=X$ and

$$m=c\circ P_X+\langle v,\cdot\rangle\quad\text{on}\quad\mathbb{R}^n. \tag{4.9.2}$$

By combining Lemma 4.46 with the second part of Lemma 4.52 we obtain that

$$G(x)-v\in\partial c(P_X(x))\subset X\quad\text{for all}\quad x\in E^*. \tag{4.9.3}$$

We are now going to study the differentiability of the function c.

Lemma 4.53. *The function c is differentiable on $\overline{P_X(E^*)}$, and, if $y \in P_X(E^*)$, then $\nabla c(y) = G(x) - v$, where $x \in E^*$ is such that $P_X(x) = y$.*

Proof. Let us suppose that c is not differentiable at some $y_0 \in \overline{P_X(E^*)}$. Then, by the convexity of c on X, we may assume that there exist a sequence $(h_k)_k \subset X$ with $|h_k| \searrow 0$ and a number $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{c(y_0 + h_k) + c(y_0 - h_k) - 2c(y_0)}{|h_k|} \quad \text{for all} \quad k.$$

We now consider sequences $(y_k)_k \subset P_X(E^*)$ and $(x_k)_k \subset E^*$ with

$$P_X(x_k) = y_k \quad \text{and} \quad y_k \to y_0.$$

In particular, the sequence $(P_X(x_k))_k$ is bounded. Since each h_k belongs to X, we can use (4.9.2) to rewrite the last inequality as

$$\varepsilon \leq \frac{m(y_0 + h_k) + m(y_0 - h_k) - 2m(y_0)}{|h_k|} \quad \text{for all} \quad k. \tag{4.9.4}$$

By the definition of m we can pick two sequences $(z_k)_k, (\widetilde{z_k})_k \subset E^*$ with the following properties:

$$\begin{aligned} m(y_0 + h_k) &\geq f(z_k) + \langle G(z_k), y_0 + h_k - z_k\rangle \geq m(y_0 + h_k) - \frac{|h_k|}{2^k}, \\ m(y_0 - h_k) &\geq f(\widetilde{z_k}) + \langle G(\widetilde{z_k}), y_0 - h_k - \widetilde{z_k}\rangle \geq m(y_0 - h_k) - \frac{|h_k|}{2^k} \end{aligned}$$

for every k. We claim that $(G(z_k))_k$ must be bounded. Indeed, otherwise, possibly after passing to a subsequence and using the condition (ii) of Theorem 4.43, we would obtain that

$$\lim_{k\to\infty} G(z_k) = \lim_{k\to\infty} \frac{\langle G(z_k), z_k\rangle - f(z_k)}{|G(z_k)|} = +\infty.$$

Due to the choice of $(z_k)_k$ we must have

$$\begin{aligned} m(y_0) &= \lim_{k\to\infty} \left(f(z_k) + \langle G(z_k), x_0 + h_k - z_k\rangle\right) \\ &= \lim_{k\to\infty} |G(z_k)| \left(\frac{f(z_k) - \langle G(z_k), z_k\rangle}{|G(z_k)|} + \left\langle \frac{G(z_k)}{|G(z_k)|}, x_0 + h_k \right\rangle \right) = -\infty, \end{aligned}$$

which is absurd. Similarly one can show that $(G(\widetilde{z_k}))_k$ is bounded. Now we write

$$\begin{aligned} f(x_k) - &f(z_k) - \langle G(z_k), x_k - z_k\rangle \\ &= f(x_k) - \langle v, x_k\rangle - (m(y_0 + k_k) - \langle v, y_0 + h_k\rangle) \\ &\quad + m(y_0 + h_k) - f(z_k) - \langle G(z_k), y_0 + h_k - z_k\rangle \\ &\quad + \langle G(z_k) - v, y_0 + h_k - x_k\rangle. \end{aligned}$$

By (4.9.2), the first term in the sum equals $c(P_X(x_k)) - c(y_0 + h_k)$, which converges to 0 because $P_X(x_k) \to y_0$ and c is continuous. Thanks to the choice of the sequence $(z_k)_k$, the second term also converges to 0. From (4.9.3), we have $G(z_k) - v \in X$ for all k, and then the third term in the sum is actually $\langle G(z_k) - v, y_0 - P_X(x_k) + h_k\rangle$, which converges to 0, as $(G(z_k))_k$ is bounded and $P_X(x_k) \to y_0$. We then have

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

where $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded sequences. We obtain from the second part of Lemma 4.51 that $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$, and similarly one can show that $\lim_{k\to\infty} |G(x_k) - G(\widetilde{z_k})| = 0$. This obviously implies

$$\lim_{k\to\infty} |G(z_k) - G(\widetilde{z_k})| = 0. \tag{4.9.5}$$

By the choice of the sequence $(z_k)_k$, $(\widetilde{z}_k)_k$ and by inequality (4.9.4) we have, for every k,

$$\begin{aligned}\varepsilon &\leq \frac{f(z_k) + \langle G(z_k), y_0 + h_k - z_k\rangle}{|h_k|} + \frac{f(\widetilde{z}_k) + \langle G(\widetilde{z}_k), y_0 - h_k - \widetilde{z}_k\rangle}{|h_k|} \\ &\quad - \frac{f(z_k) + \langle G(z_k), y_0 - z_k\rangle + f(\widetilde{z}_k) + \langle G(\widetilde{z}_k), y_0 - \widetilde{z}_k\rangle}{|h_k|} \\ &= \left\langle G(z_k) - G(\widetilde{z}_k), \frac{h_k}{|h_k|}\right\rangle + \frac{1}{2^{k-1}} \leq |G(z_k) - G(\widetilde{z}_k)| + \frac{1}{2^{k-1}}.\end{aligned}$$

Then (4.9.5) leads us to a contradiction. We conclude that c is differentiable on $\overline{P_X(E^*)}$.

We now prove the second part of the Lemma. Consider $y \in P_X(E^*)$ and $x \in E^*$ with $P_X(x) = y$. Using (4.9.3), we have $G(x) - v \in \partial c(y)$. Because c is differentiable at y, we further obtain that $G(x) - v = \nabla c(y)$. □

In order to complete the proof of Theorem 4.43, we will need the following lemma.

Lemma 4.54. *Let $h : X \to \mathbb{R}$ be a convex and coercive function such that h is differentiable on a closed subset A of X. There exists $H \in C^1(X)$ convex and coercive such that $H = h$ and $\nabla H = \nabla h$ on A.*

Proof. Since h is convex, its gradient ∇h is continuous on A (see [57, Corollary 24.5.1] for instance). Then, for all $x, y \in A$, we have

$$0 \leq \frac{h(x) - h(y) - \langle \nabla h(y), x - y\rangle}{|x - y|} \leq \left\langle \nabla h(x) - \nabla h(y), \frac{x - y}{|x - y|}\right\rangle \leq |\nabla h(x) - \nabla h(y)|,$$

where the last term tends to 0 as $|x - y| \to 0$ uniformly on $x, y \in K$ for every compact subset K of A. This shows that the pair $(h, \nabla h)$ defined on A satisfies the conditions of the classical Whitney Extension Theorem for C^1 functions, see Theorem 4.1. Therefore, there exists a function $\widetilde{h} \in C^1(X)$ such that $\widetilde{h} = h$ and $\nabla \widetilde{h} = \nabla h$ on A. We now define

$$\phi(x) := |h(x) - \widetilde{h}(x)| + 2d(x, A)^2, \quad x \in X. \tag{4.9.6}$$

Claim 4.55. *ϕ is differentiable on A, with $\nabla\phi(x_0) = 0$ for every $x_0 \in A$.*

Proof. The function $d(\cdot, A)^2$ is obviously differentiable, with a null gradient, at every $x_0 \in A$. Since $\widetilde{h} = h$ and $\nabla\widetilde{h} = \nabla h$ on A, the same argument as in the proof of Claim 4.15 shows that $|h - \widetilde{h}|$ is differentiable, with a null gradient, at every $x_0 \in A$. □

Now, because $d(\cdot, A)^2$ is continuous and positive on $X \setminus A$, according to Whitney's approximation theorem [70] we can find a function $\varphi \in C^\infty(X \setminus A)$ such that

$$|\varphi(x) - \phi(x)| \leq d(x, A)^2 \quad \text{for every } x \in X \setminus A. \tag{4.9.7}$$

Let us define $\widetilde{\varphi} : X \to \mathbb{R}$ by $\widetilde{\varphi} = \varphi$ on $X \setminus A$ and $\widetilde{\varphi} = 0$ on A.

Claim 4.56. *The function $\widetilde{\varphi}$ is differentiable on X and $\nabla\widetilde{\varphi} = 0$ on A.*

Proof. It is obvious that $\widetilde{\varphi}$ is differentiable on $\operatorname{int}(A) \cup (X \setminus A)$ and $\nabla\widetilde{\varphi} = 0$ on $\operatorname{int}(A)$. We only have to check that $\widetilde{\varphi}$ is differentiable on ∂A. If $x_0 \in \partial A$ we have

$$\frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)|}{|x - x_0|} = \frac{|\widetilde{\varphi}(x)|}{|x - x_0|} \leq \frac{|\phi(x)| + d(x, A)^2}{|x - x_0|} \to 0$$

as $|x - x_0| \to 0^+$, because both ϕ and $d(\cdot, A)^2$ vanish at x_0 and are differentiable, with null gradients, at x_0. Therefore $\widetilde{\varphi}$ is differentiable at x_0, with $\nabla\widetilde{\varphi}(x_0) = 0$. □

Now we set

$$g := \tilde{h} + \tilde{\varphi}$$

on X. It is clear that $g = h$ on A. Also, by Claim 4.56, g is differentiable on X with $\nabla g = \nabla h$ on A. By combining (4.9.6) and (4.9.7) we obtain that

$$g(x) \geq \tilde{h}(x) + \phi(x) - d(x, A)^2 \geq h(x) \quad \text{for all} \quad x \in X \setminus A.$$

Therefore $g \geq h$ on X and in particular g is coercive on X, because so is h, by assumption. We next consider the *convex envelope* of g. If we define

$$H = \operatorname{conv}(g)$$

we immediately get by Theorem 4.17, that H is convex on X and $H \in C^1(X)$. By definition of H we have that $h \leq H \leq g$ on X, which implies that H is coercive. Also, because $g = h$ on A, we have that $H = h$ on A. Since h is convex and H is differentiable on X with $h = H$ on A and $h \leq H$ on X, Lemma 2.14 shows that $\nabla H = \nabla h$ on A. This completes the proof of Lemma 4.54. $\square$

Now we are able to finish the proof of Theorem 4.43. Setting $A := \overline{P_X(E^*)}$, we see from Lemma 4.53 that c is differentiable on A. Moreover, since $c : X \to \mathbb{R}$ is convex and essentially coercive on X, there exists $\eta \in X$ such that $h := c - \langle \eta, \cdot \rangle$ is convex, differentiable on A and coercive on X. Applying Lemma 4.54 to h, we obtain $H \in C^1(X)$ convex and coercive on X with $(H, \nabla H) = (h, \nabla h)$ on A. Thus, the function $\varphi := H + \langle \eta, \cdot \rangle$ is convex, essentially coercive on X and of class $C^1(X)$ with $(\varphi, \nabla \varphi) = (c, \nabla c)$ on A. We next show that $F := \varphi \circ P_X + \langle v, \cdot \rangle$ is the desired extension of (f, G). Since φ is $C^1(X)$ and convex, it is clear that F is $C^1(\mathbb{R}^n)$ and convex as well. Bearing in mind Theorem 4.41 and the fact that φ is essentially coercive, it follows that $X_F = X$. Also, since $\varphi(y) = c(y)$ for $y \in P_X(E)$, we obtain from (4.9.2) and Lemma 4.52 that

$$F(x) = \varphi(P_X(x)) + \langle v, x \rangle = c(P_X(x)) + \langle v, x \rangle = m(x) = f(x).$$

Finally, from the second part of Lemma 4.53, we have, for all $x \in E$, that

$$\nabla F(x) = \nabla \varphi(P_X(x)) + v = G(x) - v + v = G(x).$$

The proof of Theorem 4.43 is complete. $\square$

4.10 A C^1 extension theorem for Lipschitz convex functions

In the special case that the function G of Theorem 4.43 is bounded, one should expect to find Lipschitz convex functions $F \in C^1(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends (f, G) and $\operatorname{Lip}(F) \lesssim \|G\|_\infty$. As we have observed in Section 4.1, this kind of control of $\operatorname{Lip}(F)$ in terms of $\sup_{y \in E} |G(y)|$ solely cannot be obtained, in general, for nonconvex jets, but it is possible in the convex case, at least when E is bounded; see Theorem 4.20. The next result tells us that this is indeed feasible, and moreover shows that the technical conditions of (iv) in Theorem 4.43 can be replaced (just in this Lipschitz case) by a nicer geometric condition which tells us that the complement of the closure of E in $\mathbb{R}^n$ contains the union of a certain finite collection of cones.

Theorem 4.57. *Given an arbitrary subset E of $\mathbb{R}^n$, a linear subspace $X \subset \mathbb{R}^n$, the orthogonal projection $P := P_X : \mathbb{R}^n \to X$, and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, the following is true. There exists a Lipschitz convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if the following conditions are satisfied.*

(i) *G is continuous and bounded and $f(x) \geq f(y) + \langle G(y), x - y \rangle$ for all $x, y \in E$.*

(ii) *$Y := \operatorname{span}(\{G(x) - G(y) : x, y \in E\}) \subseteq X$.*

(iii) *If $Y \neq X$ and we denote $k = \dim Y$ and $d = \dim X$, there exist points $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, a number $\varepsilon \in (0,1)$, and linearly independent normalized vectors $w_1, \ldots, w_{d-k} \in X \cap Y^{\perp}$ such that, for every $j = 1, \ldots, d-k$, the cone $V_j := \{x \in \mathbb{R}^n \,:\, \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x - p_j)|\}$ does not contain any point of $\overline{E}$. Here $P_Y : \mathbb{R}^n \to Y$ denotes the orthogonal projection onto Y.*

(iv) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(P_X(x_k))_k$ is bounded and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0.$

Moreover, there exists an absolute constant $\kappa > 0$ such that, whenever these conditions are satisfied, the extension F can be taken so that

$$\mathrm{Lip}(F) = \sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq \kappa \sup_{y\in E} |G(y)|.$$

Remark 4.58. If E is a bounded subset, and we have a 1-jet (f, G) on E satisfying the conditions of Theorem 4.20, then conditions $(ii), (iii)$ and (iv) of Theorem 4.57 are trivially satisfied with any subspace X containing Y. This shows that, in this case, for any X containing $Y := \mathrm{span}\{G(x) - G(y) \,:\, x, y \in E\}$, the extension F can be taken so that $X_F = X$.

A consequence of Theorem 4.57 is the following corollary.

Corollary 4.59. *Given an arbitrary subset E of $\mathbb{R}^n$ and two functions $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, let us denote by $P_X : \mathbb{R}^n \to X$ the orthogonal projection onto the subspace $X = \mathrm{span}\{G(x) - G(y) \,:\, x, y \in E\}$. There exists a Lipschitz convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if the following conditions are satisfied.*

(i) *G is continuous and bounded and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(P_X(x_k))_k$ is bounded and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0.$

4.11 Necessity of the conditions for C^1 convex Lipschitz extensions

In this section we are going to prove the *only if* part of Theorem 4.57. So, let F be a Lipschitz convex function of class $C^1(\mathbb{R}^n)$ with $(F, \nabla F) = (f, G)$ on E and $X_F = X$ and let us see that (f, G) and X satisfy conditions $(i) - (iv)$ on the set E. Also, we learn from Lemma 4.47 that

$$X_F = \mathrm{span}\{\nabla F(x) - \nabla F(y) \,:\, x, y \in \mathbb{R}^n\}.$$

We already know from Section 4.8 that conditions $(i), (ii)$ and (iv) are necessary for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ with $(f, G) = (F, \nabla F)$ on E and $X_F = X$. Let us prove that condition (iii) is satisfied as well. If $\mathrm{Lip}(F) = 0$ then F is constant, so we have $X = X_F = \{0\} = Y$, and condition (iii) is trivially satisfied. Otherwise we have $X = X_F \neq \{0\}$, and assuming that $Y \neq X$ we may find points $x_0, x_1, \ldots, x_k \in E$ and $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$ such that

$$Y = \mathrm{span}\{G(x_j) - G(x_0) \,:\, j = 1, \ldots, k\},$$

$$\nabla F(p_j) - G(x_0) \in X \setminus Y \quad \text{for every} \quad j = 1, \ldots, d-k$$

and the set $\{\nabla F(p_j) - G(x_0) \,:\, j = 1, \ldots, d-k\}$ is linearly independent. Now we define, for each $j = 1, \ldots, d-k$, the subspace Y_j spanned by Y and the vector $\nabla F(p_j) - G(x_0)$. Obviously we can find $w_j \in Y_j \cap Y^\perp$ with $|w_j| = 1$ and $Y_j = Y \oplus [w_j]$, for every $j = 1, \ldots, d-k$. Moreover, w_j can be taken so that

$$\mu_j := \langle \nabla F(p_j) - G(x_0), w_j \rangle > 0, \quad \text{for all} \quad j = 1, \ldots, d-k.$$

Let us take $\varepsilon > 0$ small enough so that

$$\varepsilon < \frac{\mu_j}{2\operatorname{Lip}(F) + 2\|G\|_\infty} \quad \text{for all} \quad j = 1, \ldots, d-k.$$

Note that, because $\mu_j \leq 2\operatorname{Lip}(F)$ for each j, we have that $\varepsilon < 1$. Now, assume that there exists some $x \in \overline{E}$ with $x \in V_j := \{x \in \mathbb{R}^n \, : \, \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x - p_j)|\}$ for some $j = 1, \ldots, d-k$. Using the convexity of F we can write

$$\begin{aligned} & F(x) - F(p_j) - \langle \nabla F(p_j), x - p_j \rangle \\ & \quad \leq \langle \nabla F(x) - \nabla F(p_j), x - p_j \rangle = \langle \nabla F(x) - G(x_0), x - p_j \rangle + \langle G(x_0) - \nabla F(p_j), x - p_j \rangle \\ & \quad = \langle \nabla F(x) - G(x_0), x - p_j \rangle - \mu_j \langle w_j, x - p_j \rangle + \langle P_Y(G(x_0) - \nabla F(p_j)), x - p_j \rangle. \end{aligned}$$

Since we are assuming that $x \in \overline{E}$, the continuity of ∇F yields $\nabla F(x) - G(x_0) \in Y$. Then, the last term coincides with

$$\begin{aligned} & \langle \nabla F(x) - G(x_0), P_Y(x - p_j) \rangle - \mu_j \langle w_j, x - p_j \rangle + \langle P_Y(G(x_0) - \nabla F(p_j)), P_Y(x - p_j) \rangle \\ & \quad \leq (2\|G\|_\infty + 2\operatorname{Lip}(F)) \, |P_Y(x - p_j)| - \mu_j \langle w_j, x - p_j \rangle \leq 0, \end{aligned}$$

where the last inequality follows from the definition of ε and the fact that $x \in V_j$. We have thus shown that

$$F(x) - F(p_j) - \langle \nabla F(p_j), x - p_j \rangle = 0.$$

By condition (CW^1) (see Definition 4.3 and Lemma 4.12) we must have $\nabla F(p_j) = \nabla F(x)$, where $x \in \overline{E}$. It then follows that $\nabla F(p_j) - G(x_0) = \nabla F(x) - G(x_0) \in Y$, which contradicts the choice of p_j. Therefore $\overline{E}$ and $\bigcup_{j=1}^{d-k} V_j$ are disjoint.

4.12 Sufficiency of the conditions for C^1 convex Lipschitz extensions

Let us denote by m the *minimal convex extension* of the jet 1-jet (f, G) from E, that is

$$m(x) = \sup_{y \in E} \{ f(y) + \langle G(y), x - y \rangle \}, \quad x \in \mathbb{R}^n.$$

Because G is bounded, the function m is a supremum of $\|G\|_\infty$-Lipschitz convex functions on $\mathbb{R}^n$ and therefore m is convex and $\|G\|_\infty$-Lipschitz as well. In particular, the supremum defining $m(x)$ is finite for every $x \in \mathbb{R}^n$. By Theorem 4.41, we can write

$$m = c \circ P_{X_m} + \langle v, \cdot \rangle,$$

where $v \in \mathbb{R}^n$ and $c : X_m \to \mathbb{R}$ is a coercive convex function. Moreover, we know from Lemma 4.48 that $X_m = Y = \operatorname{span}\{G(x) - G(y) \, : \, x, y \in E\}$ and therefore

$$m = c \circ P_Y + \langle v, \cdot \rangle. \tag{4.12.1}$$

Let us prove some properties of m, c and v.

Lemma 4.60. *Let us denote $K = \|G\|_\infty = \sup_{y \in E} |G(y)|$. We have that:*

(1) *The function m is K-Lipschitz on $\mathbb{R}^n$.*

(2) *The vector v belongs to the subdifferential of m at some point $y_0 \in Y$, and $|v| \leq K$.*

(3) *There exists points $x_1, \ldots, x_k \in E$ such that $\{G(x_j) - v\}_{j=1}^k$ is a basis of Y.*

(4) *The function c is $2K$-Lipschitz on Y.*

(5) *There exists numbers $0 < \alpha \leq 2K$ and $\beta \in \mathbb{R}$ such that $c(y) \geq \alpha|y| + \beta$ for every $y \in Y$.*

Proof.

(1) As we have said before, the function m is a supremum of K-Lipschitz affine functions on $\mathbb{R}^n$ and therefore m is K-Lipschitz as well.

(2) Since c is coercive on Y, the function c attains a global minimum. Thus there exists a point $y_0 \in Y$ with $c(y) \geq c(y_0)$ for every $y \in Y$. We then have, for every $x \in \mathbb{R}^n$, that

$$m(x) = c(P_Y(x)) + \langle v, x\rangle \geq c(y_0) + \langle v, x\rangle = c(y_0) + \langle v, y_0\rangle + \langle v, x - y_0\rangle = m(y_0) + \langle v, x - y_0\rangle,$$

which implies that $v \in \partial m(y_0)$. Since m is K-Lipschitz, we obtain, for every $x \in \mathbb{R}^n$,

$$K|x - y_0| + m(y_0) \geq m(x) \geq m(y_0) + \langle v, x - y_0\rangle,$$

which implies that $\langle v, \frac{x-y_0}{|x-y_0|}\rangle \leq K$, for every $x \in \mathbb{R}^n \setminus \{y_0\}$. This shows that $|v| \leq K$.

(3) Recall that $\eta - v \in Y$ for every $\eta \in \partial m(x)$ by virtue of Lemma 4.46. In particular we have $G(x) - v \in Y$ for every $x \in E$. Let us take some $x_1 \in Y$ with $G(x_1) - v \neq 0$. If $\dim(Y) = 1$, there is nothing to say. If $\dim(Y) > 1$, we claim that there exists some $x_2 \in E$ such that $G(x_2) - v$ and $G(x_1) - v$ are linearly independent. Indeed, assume that $G(x) - v$ and $G(x_1) - v$ are proportional for every $x \in E$. Then we would have for every $x, y \in E$ that

$$G(x) - G(y) = (G(x) - v) + (v - G(y))$$

is proportional to $G(x_1) - v$, hence $\dim(Y) = 1$, a contradiction. Using repeteadly this argument we obtain (3).

(4) follows at once from (1), (2), and the fact that $c = m - \langle v, \cdot\rangle$ on Y.

(5) We first claim that there exist two positive numbers α and r such that $c(x) \geq \alpha|y|$ whenever $y \in Y$ is such that $|y| \geq r$. Indeed, otherwise we can find a sequence $(y_\ell)_\ell \in Y$ with

$$|y_\ell| \geq \ell \quad \text{and} \quad c(y_\ell) \leq \frac{1}{\ell}|y_\ell| \quad \text{for every} \quad \ell \in \mathbb{N}.$$

By convexity of c we obtain

$$c\left(\frac{\ell}{|y_\ell|}y_\ell\right) \leq \frac{\ell}{|y_\ell|}c(y_\ell) + \left(1 - \frac{\ell}{|y_\ell|}\right)c(0) \leq 1 + \left(1 - \frac{\ell}{|y_\ell|}\right)c(0) \leq 1 + c(0).$$

But the sequence $\left(\frac{\ell}{|y_\ell|}y_\ell\right)_\ell$ clearly converges to ∞, and the coercivity of c implies that $\lim_\ell c\left(\frac{\ell}{|y_\ell|}y_\ell\right) = +\infty$, which contradicts the above inequality. Thus there exists $\alpha, r > 0$ with

$$c(y) \geq \alpha|y| \quad \text{whenever} \quad |y| \geq r.$$

Now, if $|y| \leq r$, we can write

$$c(y) \geq \inf_{B(0,r)} g = \alpha r + \inf_{B(0,r)} c - \alpha r \geq \alpha|y| + \left(\inf_{B(0,r)} c - \alpha r\right),$$

where $B(0, r)$ denotes the closed ball centered at 0 and with radius r on Y. If we set

$$\beta = \min\left\{\inf_{B(0,r)} c - \alpha r, 0\right\},$$

we obviously have $c(y) \geq \alpha|y| + \beta$ for every $y \in Y$. Now, because c is $2K$-Lipschitz, we have that

$$c(0) + 2K|y| \geq c(y) \geq \alpha|y| + \beta, \quad y \in Y.$$

This clearly implies that $\alpha \leq 2K$. □

4.12.1 Defining new data

Let us consider $w_1, \ldots, w_{d-k} \in Y^\perp \cap X,\ \varepsilon \in (0,1),\ p_1, \ldots, p_{d-k}$ and $V_1, \ldots, V_{d-k}$ as in condition (iii) of Theorem 4.57. Using the constants α and β of Lemma 4.60 (5), we consider a positive $T > 0$ large enough so that

$$(\varepsilon\,\alpha)T \geq 2 - \beta - \max_{j=1,\ldots,d-k} \{\alpha |P_Y(p_j)| + m(p_j) - \langle v, p_j\rangle\}$$

and

$$(\varepsilon\,\alpha)T \min_{1\leq i\neq j\leq d-k} \{1 - \langle w_i, w_j\rangle\} \geq 1 + \max_{1\leq i,j\leq d-k} \{c(P_Y(p_j)) - c(P_Y(p_i)) + \varepsilon\alpha\langle w_j, p_i - p_j\rangle\}.$$

Note that, since the vectors $\{w_i\}_{i=1}^{d-k}$ have norm equal to 1, then $\langle w_i, w_j\rangle = 1$ if and only if $w_i = w_j$, which is equivalent (as the vectors $\{w_1, \ldots, w_{d-k}\}$ are linearly independent) to $i = j$. So, it is clear that we can find a positive $T > 0$ satisfying both inequalities. We define the following new data:

$$q_j = p_j + T w_j, \quad f(q_j) = m(q_j) + 1, \quad G(q_j) = v + \varepsilon\alpha w_j, \quad j = 1, \ldots, d-k. \tag{4.12.2}$$

Note that $q_i = q_j$ if and only if $p_i - p_j = T(w_j - w_i)$. Since $w_i \neq w_j$ whenever $i \neq j$, it is clear that we can take T large enough so that the points q_i and q_j are distinct if $i \neq j$. On the other hand, because each w_j is orthogonal to Y, we immediately see that $q_j \in V_j$ and, in particular, $q_j \notin \overline{E}$ for every $j = 1, \ldots, d-k$.

Lemma 4.61. *The following inequalities are satisfied.*

(1) $f(q_j) - f(x) - \langle G(x), q_j - x\rangle \geq 1$ *for every* $x \in E,\ j = 1, \ldots, d-k$.

(2) $f(x) - f(q_j) - \langle G(q_j), x - q_j\rangle \geq 1$ *for every* $x \in E,\ j = 1, \ldots, d-k$.

(3) $f(q_i) - f(q_j) - \langle G(q_j), q_i - q_j\rangle \geq 1$ *for every* $1 \leq i \neq j \leq d-k$.

Proof.

(1) Since $f(q_j) = m(q_j) + 1$, the definition of m leads us to

$$f(q_j) - f(x) - \langle G(x), q_j - x\rangle = m(q_j) - f(x) - \langle G(x), q_j - x\rangle + 1 \geq 1,$$

for $x \in E,\ j = 1, \ldots, d-k$.

(2) We fix $x \in E$ and $j = 1, \ldots, d-k$. The decomposition of m yields

$$m(q_j) = c(P_Y(p_j) + P_Y(Tw_j)) + \langle v, q_j\rangle = c(P_Y(p_j)) + \langle v, q_j\rangle = m(p_j) + \langle v, q_j - p_j\rangle.$$

We obtain from this the following:

$$\begin{aligned}
f(x) - f(q_j) - \langle G(q_j), x - q_j\rangle &= m(x) - m(p_j) + \langle v, p_j - q_j\rangle - \langle G(q_j), x - q_j\rangle - 1 \\
&= c\circ(P_Y(x)) + \langle v, x\rangle - m(p_j) + \langle v, p_j - q_j\rangle - \langle v + \varepsilon\alpha w_j, x - q_j\rangle - 1 \\
&= c\circ(P_Y(x)) - m(p_j) + \langle v, p_j\rangle - \varepsilon\alpha\langle w_j, x - q_j\rangle - 1 \\
&= c\circ(P_Y(x)) - m(p_j) + \langle v, p_j\rangle - \varepsilon\alpha\langle w_j, x - p_j\rangle - \varepsilon\alpha\langle w_j, p_j - q_j\rangle - 1 \\
&= c\circ(P_Y(x)) - m(p_j) + \langle v, p_j\rangle - \varepsilon\alpha\langle w_j, x - p_j\rangle + \varepsilon\alpha T - 1.
\end{aligned}$$

Now, using Lemma 4.60 (5), the last term is bigger than or equal to

$$\begin{aligned}
&\alpha|P_Y(x)| + \beta - m(p_j) + \langle v, p_j\rangle - \varepsilon\alpha\langle w_j, x - p_j\rangle + \varepsilon\alpha T - 1 \\
&\qquad \geq \alpha|P_Y(x - p_j)| - \alpha|P_Y(p_j)| + \beta - m(p_j) + \langle v, p_j\rangle - \varepsilon\alpha\langle w_j, x - p_j\rangle + \varepsilon\alpha T - 1 \\
&\qquad \geq \alpha|P_Y(x - p_j)| - \varepsilon\alpha\langle w_j, x - p_j\rangle + 1,
\end{aligned}$$

where the last inequality follows from the choice of T. Now, since $x \in E$, the condition (iii) tells us that x does not belong to the cone V_j, which implies that the last term is greater than or equal to

$$\varepsilon\alpha\langle w_j, x - p_j\rangle - \varepsilon\alpha\langle w_j, x - p_j\rangle + 1 = 1.$$

This establishes the inequalities of (2).

(3) Consider $1 \leq i \neq j \leq d-k$. Notice that

$$f(q_i)-f(q_j) = c(P_Y(p_i+Tw_i))-c(P_Y(p_j+Tw_j))+\langle v, q_i-q_j\rangle = c(P_Y(p_i))-c(P_Y(p_j))+\langle v, q_i-q_j\rangle.$$

This implies

$$\begin{aligned} f(q_i) - f(q_j) - \langle G(q_j), q_i - q_j\rangle &= c(P_Y(p_i)) - c(P_Y(p_j)) + \langle v, q_i - q_j\rangle - \langle v + \varepsilon\,\alpha\, w_j, q_i - q_j\rangle \\ &= c(P_Y(p_i)) - c(P_Y(p_j)) - \varepsilon\,\alpha\langle w_j, p_i - p_j + T(w_i - w_j)\rangle \\ &= c(P_Y(p_i)) - c(P_Y(p_j)) - \varepsilon\,\alpha\langle w_j, p_i - p_j\rangle + \varepsilon\,\alpha\,T\,(1 - \langle w_i, w_j\rangle) \geq 1, \end{aligned}$$

where the last inequality follows from the choice of T. □

4.12.2 Properties of the new jet

We now define the set $E^* = E \cup \{q_1, \ldots, q_{d-k}\}$. Note that we have already extended the definition of (f, G) to E^*.

Lemma 4.62. *We have that:*

(1) $X = span\{G(x) - G(y) \ : \ x, y \in E^*\}$.

(2) *G is continuous on E^* and $f(x) \geq f(y) + \langle G(y), x-y\rangle$ for all $x, y \in E^*$.*

(3) *$|G(x)| \leq 3K$ for every $x \in E^*$.*

(4) *If $(x_\ell)_\ell, (z_\ell)_\ell$ are sequences in E^* such that $(P_X(x_\ell))_\ell$ is bounded and*

$$\lim_{\ell\to\infty} (f(x_\ell) - f(z_\ell) - \langle G(z_\ell), x_\ell - z_\ell\rangle) = 0,$$

then $\lim_{\ell\to\infty} |G(x_\ell) - G(z_\ell)| = 0$.

Proof.

(1) By Lemma 4.60 (3), there are points $x_1, \ldots, x_k \in E$ with $Y = \text{span}\{G(x_j) - v \ : \ j = 1, \ldots, k\}$, where v is that of (4.12.1). Since the vectors $w_1, \ldots, w_{d-k}$ are linearly independent, the definitions of (4.12.2) show that

$$\text{span}\{G(q_j) - v \ : \ j = 1, \ldots, d-k\} = \text{span}\{(\varepsilon\,\alpha)w_j \ : \ j = 1, \ldots, d-k\} = X \cap Y^\perp.$$

We thus have that

$$X = \text{span}\{G(x_1) - v, \ldots, G(x_k) - v, G(q_1) - v, \ldots, G(q_{d-k}) - v\}.$$

For every two points $x, y \in E^*$, we can write

$$G(x) - G(y) = (G(x) - v) - (G(y) - v),$$

but notice that $G(z) - v \in Y = \text{span}\{G(x_i) - v\}_{i=1}^k$ for every $z \in E$ and obviously $G(z) - v \in \text{span}\{G(q_j) - v\}_{j=1}^{d-k}$ if $z \in E^* \setminus E$. This implies that $G(x) - G(y) \in X$ for every $x, y \in E^*$. Conversely, if $z \in E^*$, we can write

$$G(z) - v = (G(z) - G(x_1)) + (G(x_1) - v),$$

where the first term belongs to $\text{span}\{G(x) - G(y) \ : \ x, y \in E^*\}$ and the second one belongs to $Y = \text{span}\{G(x) - G(y) \ : \ x, y \in E\}$. We conclude that $X = \text{span}\{G(x) - G(y) \ : \ x, y \in E^*\}$.

(2) The points $q_1, \ldots, q_{d-k}$ are distinct and none of them belong to $\overline{E}$. Because G is continuous on E, G is in fact continuous on E^*. Condition (i) of Theorem 4.57 together with Lemma 4.61 tell us that

$$f(x) \geq f(y) + \langle G(y), x-y \rangle \quad \text{for all} \quad x, y \in E^*.$$

(3) From (4.12.2), $G(q_j) = v + (\varepsilon\, \alpha) w_j$, for $j = 1, \ldots, d-k$. Now Lemma 4.60 tells us that $|v| \leq K$ and $\alpha \leq 2K$, where K denotes $\sup_{y \in E} |G(y)|$. Since $\varepsilon \in (0,1)$ and the vectors w_j's have norm equal to 1, we can write $|G(p_j)| \leq |v| + \alpha \leq 3K$.

(4) Suppose that$(x_\ell)_\ell, (z_\ell)_\ell$ are sequences in E^* such that $(P_X(x_\ell))_k$ is bounded and

$$\lim_{\ell \to \infty} \left(f(x_\ell) - f(z_\ell) - \langle G(z_\ell), x_\ell - z_\ell \rangle \right) = 0.$$

In view of Lemma 4.61, it is immediate that there exists ℓ_0 such that either there is some $1 \leq j \leq d-k$ with $x_\ell = z_\ell = q_j$ for all $\ell \geq \ell_0$ or else $x_\ell, z_\ell \in E$ for all $\ell \geq \ell_0$. In the first case, the conclusion is trivial. In the second case, $\lim_{\ell \to \infty} |G(x_\ell) - G(z_\ell)| = 0$ follows from condition (iv) of Theorem 4.57. □

We now define $m^*(x) = \sup_{y \in E^*} \{ f(y) + \langle G(y), x-y \rangle \}$ for every $x \in \mathbb{R}^n$. We learnt from Lemma 4.48 that $X_{m^*} = \text{span}\{G(x) - G(y) \,:\, x, y \in E^*\}$. And from Lemma 4.62 (1), $X_{m^*} = X$. The function m^* is convex and using Lemma 4.62 (2) it is clear that $m^* = f$ on E^*. Also, for every $x \in E^*$, we have that $G(x) \in \partial m^*(x)$ and, by virtue of Lemma 4.62 (3), m^* is $3K$-Lipschitz on $\mathbb{R}^n$. The function m^* has the decomposition

$$m^* = c^* \circ P_X + \langle v^*, \cdot \rangle \quad \text{on} \quad \mathbb{R}^n, \tag{4.12.3}$$

where $c^* : X \to \mathbb{R}$ is convex and coercive on X, and $v^* \in \mathbb{R}^n$. With the same proof as that of Lemma 4.60 (2), we see that $v^* \in \partial m^*(z_0)$ for some $z_0 \in X$, the function c^* is $6K$-Lipschitz and $|v^*| \leq 3K$. We study the differentiability of c^* in the following Lemma.

Lemma 4.63. *The function c^* is differentiable on $\overline{P_X(E^*)}$, and, if $y \in P_X(E^*)$, then $\nabla c^*(y) = G(x) - v^*$, where $x \in E^*$ is such that $P_X(x) = y$.*

Proof. Thanks to Lemma 4.62 (4) we can repeat the proof of Lemma 4.53, which proved the result for the general (not necessarily Lipschitz) case. □

4.12.3 Construction of the extension

Lemma 4.64. *Let $h : X \to \mathbb{R}$ be a convex, Lipschitz and coercive function such that h is differentiable on a closed subset A of X. There exists $H \in C^1(X)$ convex, Lipschitz and coercive such that $H = h$ and $\nabla H = \nabla h$ on A. Moreover, H can be taken so that* $\text{Lip}(H) \leq \kappa_0 \, \text{Lip}(h)$, *where $\kappa_0 > 1$ is an absolute constant.*

Proof. We are going to use the ideas of the proof of Theorem 4.7 (see Section 4.2) and of the proof of Lemma 4.54. Since h is convex, its gradient ∇h is continuous on A. Then, for all $x, y \in A$, we have

$$0 \leq \frac{h(x) - h(y) - \langle \nabla h(y), x-y \rangle}{|x-y|} \leq \left\langle \nabla h(x) - \nabla h(y), \frac{x-y}{|x-y|} \right\rangle \leq |\nabla h(x) - \nabla h(y)|,$$

where the last term tends to 0 as $|x-y| \to 0^+$ uniformly on $x, y \in K$ for every compact subset K of A. This shows that the pair $(h, \nabla h)$ defined on A satisfies the conditions of the classical Whitney Extension Theorem for C^1 functions. If we use Theorem 4.9 instead of Theorem 4.1, we obtain a function $\widetilde{h} \in C^1(X)$ such that $\widetilde{h} = h$, $\nabla \widetilde{h} = \nabla h$ on A and we can arrange $\text{Lip}(\widetilde{h}) \leq \kappa^* \, \text{Lip}(h)$, where $\kappa^* > 1$ is an absolute constant.

Let us denote $L = \text{Lip}(h)$. For each $\varepsilon > 0$, let $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined by

$$\theta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2 & \text{if } t \leq \frac{L+\varepsilon}{2} \\ (L+\varepsilon)\left(t - \frac{L+\varepsilon}{2}\right) + \left(\frac{L+\varepsilon}{2}\right)^2 & \text{if } t > \frac{L+\varepsilon}{2}. \end{cases}$$

Observe that $\theta_\varepsilon \in C^1(\mathbb{R})$, $\text{Lip}(\theta_\varepsilon) = L + \varepsilon$. Now set

$$\Phi_\varepsilon(x) = \theta_\varepsilon\left(d(x, A)\right),$$

where $d(x, A)$ stands for the distance from x to A, notice that $\Phi_\varepsilon(x) = d(x, A)^2$ on an open neighborhood of A, and define

$$H_\varepsilon(x) = |\widetilde{h}(x) - h(x)| + 2\Phi_\varepsilon(x).$$

Note that $\text{Lip}(\Phi_\varepsilon) = \text{Lip}(\theta_\varepsilon)$ because $d(\cdot, A)$ is 1-Lipschitz, and therefore

$$\text{Lip}(H_\varepsilon) \leq \text{Lip}(\widetilde{h}) + L + 2(L + \varepsilon) \leq (3 + \kappa^*)L + 2\varepsilon. \tag{4.12.4}$$

Claim 4.65. *H_ε is differentiable on A, with $\nabla H_\varepsilon(x) = 0$ for every $x \in A$.*

Proof. Same as that of Claim 4.55. □

Now, because Φ_ε is continuous and positive on $X \setminus A$, by using mollifiers and a partition of unity, one can construct a function $\varphi_\varepsilon \in C^\infty(X \setminus A)$ such that

$$|\varphi_\varepsilon(x) - H_\varepsilon(x)| \leq \Phi_\varepsilon(x) \quad \text{for every } x \in X \setminus A, \tag{4.12.5}$$

and

$$\text{Lip}(\varphi_\varepsilon) \leq \text{Lip}(H_\varepsilon) + \varepsilon. \tag{4.12.6}$$

Let us define $\widetilde{\varphi} = \widetilde{\varphi_\varepsilon} : X \to \mathbb{R}$ by

$$\widetilde{\varphi}(x) = \begin{cases} \varphi_\varepsilon(x) & \text{if } x \in X \setminus A \\ 0 & \text{if } x \in A. \end{cases}$$

Claim 4.66. *The function $\widetilde{\varphi}$ is differentiable on X and it satisfies $\nabla\widetilde{\varphi}(x_0) = 0$ for every $x_0 \in A$.*

Proof. Same as that of Claim 4.56. □

Note also that

$$\text{Lip}(\widetilde{\varphi}) = \text{Lip}(\varphi_\varepsilon) \leq \text{Lip}(H_\varepsilon) + \varepsilon \leq (3 + \kappa)L + 3\varepsilon. \tag{4.12.7}$$

Next we define

$$g = g_\varepsilon := \widetilde{h} + \widetilde{\varphi}. \tag{4.12.8}$$

The function g is differentiable on X, and coincides with h on A. Moreover, we also have $\nabla g = \nabla h$ on A (because $\nabla\widetilde{\varphi} = 0$ on A). And, for $x \in X \setminus A$, we have

$$g(x) \geq \widetilde{h}(x) + H_\varepsilon(x) - \Phi_\varepsilon(x) = \widetilde{h}(x) + |h(x) - \widetilde{h}(x)| + \Phi_\varepsilon(x) \geq h(x) + \Phi_\varepsilon(x).$$

This shows that $g \geq h$, which in turn implies that g is coercive. Also, notice that according to (4.12.7) and the definition of g, we have

$$\text{Lip}(g) \leq \text{Lip}(\widetilde{h}) + \text{Lip}(\widetilde{\varphi}) \leq \kappa^* L + (3 + \kappa^*)L + 3\varepsilon = (3 + 2\kappa^*)L + 3\varepsilon. \tag{4.12.9}$$

If we define $H = \text{conv}(g)$ we thus get, thanks to Theorem 4.17, that H is convex on X and $H \in C^1(X)$, with

$$\text{Lip}(H) \leq \text{Lip}(g) \leq (3 + 2\kappa^*)L + 3\varepsilon. \tag{4.12.10}$$

Thus, we can take ε small enough so that $\text{Lip}(H) \leq 2\kappa_0 L$, where $\kappa_0 = 3 + 2\kappa^*$. Finally, we know (using Lemma 2.14) that $H = h$ and $\nabla H = \nabla h$ on A. Also, because h is a coercive convex function, we have that $H \geq h$ is coercive as well. This completes the proof of Lemma 4.64. □

Now we are able to finish the proof of Theorem 4.57. Setting $A := \overline{P_X(E^*)}$, we see from Lemma 4.63 that c^* is differentiable on A. Moreover, since $c^* : X \to \mathbb{R}$ is convex and coercive on X, Lemma 4.64 provides us with a Lipschitz, convex and coercive function H of class $C^1(X)$ such that $(H, \nabla H) = (c^*, \nabla c^*)$ on A and

$$\mathrm{Lip}(H) \leq \kappa_0 \,\mathrm{Lip}(c^*) \leq 6MK,$$

where κ_0 is the absolute constant of Lemma 4.64. Recall that K denotes $\sup_{y\in E} |G(y)|$. We next show that $F := H \circ P_X + \langle v^*, \cdot\rangle$ is the desired extension of (f, G). Since H is $C^1(X)$ and convex, it is clear that F is $C^1(\mathbb{R}^n)$ and convex as well. Because H is coercive on X, it follows (using Theorem 4.41) that $X_F = X$. Also, since $H(y) = c^*(y)$ for $y \in P_X(E)$, we obtain from (4.12.3) that

$$F(x) = H(P_X(x)) + \langle v^*, x\rangle = c^*(P_X(x)) + \langle v^*, x\rangle = m^*(x) = f(x).$$

Besides, from the second part of Lemma 4.63, we have, for all $x \in E$, that

$$\nabla F(x) = \nabla H(P_X(x)) + v^* = G(x) - v^* + v^* = G(x).$$

Finally, note that

$$\mathrm{Lip}(F) \leq \mathrm{Lip}(H) + |v^*| \leq 6\kappa_0 K + 3K = (6\kappa_0 + 3)K = (6\kappa_0 + 3) \sup_{y\in E} |G(y)|.$$

The proof of Theorem 4.57 is complete □

4.13 Interpolation of arbitrary subsets by boundaries of C^1 convex bodies

Finally, let us turn our attention to a geometrical problem which is closely related to our results.

Problem 4.67. *Given an arbitrary subset E of $\mathbb{R}^n$ and a unitary vector field $N : E \to \mathbb{R}^n$, what conditions will be necessary and sufficient in order to guarantee the existence of a convex hypersurface M of class C^1 with the properties that $E \subset M$ and $N(x)$ is normal to M at each $x \in E$?.*

As we have already mentioned, this question is equivalent to ask: given an arbitrary subset E of a Hilbert space X and a collection $\mathcal{H}$ of affine hyperplanes of X such that every $H \in \mathcal{H}$ passes through a point $x_H \in E$, what conditions are necessary and sufficient for the existence of a C^1 convex hypersurface S in X such that H is tangent to S at x_H for every $H \in \mathcal{H}$?

This is a problem which we solved in Section 4.4 in the case that E is a compact subset. Now, with the help of Theorem 4.57, we are able to give the solution to Problem 4.67 in full generality.

Definition 4.68. *We say that a subset V of $\mathbb{R}^n$ is a (possibly unbounded) convex body provided that V is closed and convex, with nonempty interior. Assuming that $0 \in int(V)$, we will say that V is of class C^1 provided that its Minkowski functional*

$$\mu_V(x) = \inf\{\lambda > 0 : \tfrac{1}{\lambda}x \in V\}$$

is of class C^1 on the open set $\mathbb{R}^n \setminus \mu_V^{-1}(0)$. The outer unit normal n_V of V is defined by

$$n_V(x) = \frac{1}{|\nabla \mu_V(x)|} \nabla \mu_V(x), \quad x \in \partial V.$$

Finally, we will say that a vector $u \in \mathbb{S}^{n-1}$ is outwardly normal to ∂V at a point $y \in \partial V$ if $u = n_V(y)$.

Recall that $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$.

Theorem 4.69. *Let E be an arbitrary subset of $\mathbb{R}^n$, $N : E \to \mathbb{S}^{n-1}$ a continuous mapping, X a linear subspace of $\mathbb{R}^n$, and $P : \mathbb{R}^n \to X$ the orthogonal projection. Then there exists a (possibly unbounded) convex body V of class C^1 such that $E \subset \partial V$, $0 \in int(V)$, $N(x) = n_V(x)$ for all $x \in E$, and $X = span\,(n_V(\partial V))$, if and only if the following conditions are satisfied.*

1. *$\langle N(y), x-y\rangle \le 0$ for all $x, y \in E$.*

2. *For all sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded, we have that*
$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k\rangle = 0 \implies \lim_{k\to\infty} |N(z_k) - N(x_k)| = 0.$$

3. *$0 < \inf_{y\in E}\langle N(y), y\rangle$.*

4. *Denoting $d = dim(X)$, $Y = span(N(E))$, $\ell = dim(Y)$, we have that $Y \subseteq X$, and if $Y \neq X$ and $P_Y : \mathbb{R}^n \to Y$ is the orthogonal projection then there exist linearly independent normalized vectors $w_1, \dots, w_{d-\ell} \in X \cap Y^{\perp}$, points $p_1, \dots, p_{d-\ell} \in \mathbb{R}^n$, and a number $\varepsilon \in (0,1)$ such that*
$$\left(\overline{E} \cup \{0\}\right) \cap \left(\bigcup_{j=1}^{d-\ell} V_j\right) = \emptyset,$$
where $V_j := \{x \in \mathbb{R}^n \ : \ \varepsilon\langle w_j, x-p_j\rangle \ge |P_Y(x-p_j)|\}$ for every $j = 1, \dots, d-\ell$.

Proof. Let us assume first that there exists such a convex body V, and let us check that N and $P = P_X : \mathbb{R}^n \to X$ satisfy conditions $(1)-(4)$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by
$$F(x) = \theta\left(\mu_V(x)\right), \quad x \in \mathbb{R}^n,$$
where $\theta : \mathbb{R} \to [0, +\infty)$ is a C^1 Lipschitz and increasing convex function with $\theta(t) = t^2$ whenever $|t| \le 2$ and $\theta(t) = at$ whenever $|t| \ge 2$, for a suitable $a > 0$. Because V is a convex body with $0 \in \operatorname{int}(V)$, the Minkowski functional μ_V of V Lipschitz and convex thanks to Proposition 4.26; and this implies that F is Lipschitz. Also, because θ is convex and increasing, F is convex as well. In addition, note that $\partial V = \mu_V^{-1}(1) = F^{-1}(1)$, and in particular $F = 1$ on E. Since V is of class C^1, the function μ_V is of class $C^1\left(\mathbb{R}^n \setminus \mu_V^{-1}(0)\right)$ and then F is $C^1(\mathbb{R}^n \setminus \mu_V^{-1}(0))$. Let us see that in fact F is differentiable with null gradient at all points of $\mu_V^{-1}(0)$. Indeed, because $0 \in \operatorname{int}(V)$, we can find $r > 0$ with
$$B(0,r) \subset \operatorname{int}(V), \tag{4.13.1}$$
where $B(0,r)$ denotes the closed ball centered at 0 and with radius r. Then μ_V is r^{-1}-Lipschitz on $\mathbb{R}^n$ by Proposition 4.26 and for every $x_0 \in \mu_V^{-1}(0)$ we have $F(x_0) = 0$ and
$$\lim_{x\to x_0} \frac{|F(x)-F(x_0)|}{|x-x_0|} = \lim_{x\to x_0} \frac{\mu_V^2(x)}{|x-x_0|} \le \lim_{x\to x_0} \frac{r^{-2}|x-x_0|^2}{|x-x_0|} = 0.$$
We have thus shown that F is differentiable on $\mathbb{R}^n$. Moreover, the gradient of F is given by
$$\nabla F(x) = \begin{cases} 0 & \text{if} \quad \mu_V(x) = 0, \\ 2\mu_V(x)\nabla\mu_V(x) & \text{if} \quad 0 < \mu_V(x) \le 2, \\ a\nabla\mu_V(x) & \text{if} \quad \mu_V(x) \ge 2. \end{cases}$$
This shows that $\nabla F(x)$ is a positive multiple of $\nabla\mu_V(x)$ for every $x \in \mathbb{R}^n \setminus \mu_V^{-1}(0)$ and then
$$n_V(x) = \frac{\nabla\mu_V(x)}{|\nabla\mu_V(x)|} = \frac{\nabla F(x)}{|\nabla F(x)|}, \quad x \in \partial V,$$
$$N(x) = \frac{\nabla F(x)}{|\nabla F(x)|}, \quad x \in E.$$
Using $\nabla F(0) = 0$ together with Lemma 4.47 it follows
$$\begin{aligned} X_F &= \operatorname{span}\{\nabla F(x) - \nabla F(y) \ : \ x, y \in \mathbb{R}^n\} = \operatorname{span}\{\nabla F(x) \ : \ x \in \mathbb{R}^n\} = \operatorname{span}\{\nabla\mu_V(x) \ : \ x \in \mathbb{R}^n\} \\ &= \operatorname{span}\{\nabla\mu_V(x) \ : \ x \in \partial V\} = \operatorname{span}\{n_V(x) \ : \ x \in \partial V\} = X. \end{aligned}$$

Therefore $(F, \nabla F)$ satisfies conditions $(i) - (iv)$ of Theorem 4.57 on the set $E^* := E \cup \{0\}$ with projection $P = P_X : \mathbb{R}^n \to X$. Then condition (1) follows from (i) and the fact that $F = 1$ on E. In order to check (2), take two sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded. Now suppose that

$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k\rangle = 0.$$

Then we also have, using that $F(x_k) = 1 = F(z_k)$, that

$$\lim_{k\to\infty} \left(F(x_k) - F(z_k) - \langle \nabla F(z_k), x_k - z_k\rangle\right) = 0,$$

and according to (iv) of Theorem 4.57 we obtain

$$\lim_{k\to\infty} |\nabla F(x_k) - \nabla F(z_k)| = 0. \tag{4.13.2}$$

Suppose, seeking a contradiction that we do not have $\lim_{k\to\infty} |N(x_k) - N(z_k)| = 0$. Then, after possibly passing to subsequences, we may assume that there exists some $\varepsilon > 0$ such that

$$|N(x_k) - N(z_k)| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Since $F(x_k) = 1,\ F(0) = 0$ and $\nabla F(x_k) \in X$, the convexity of F yields

$$0 \leq F(0) - F(x_k) - \langle \nabla F(x_k), -x_k\rangle = -1 + \langle \nabla F(x_k), x_k\rangle = -1 + \langle \nabla F(x_k), P(x_k)\rangle$$

and this shows that $\inf_k |\nabla F(x_k)| > 0$. Thanks to (4.13.2), we have $\inf_k |\nabla F(z_k)| > 0$ too and both $(\nabla F(x_k))_k$ and $(\nabla F(z_k))_k$ are bounded above because F is Lipschitz. So we may assume, possibly after extracting subsequences again, that $(\nabla F(x_k))_k$ and $(\nabla F(z_k))_k$ converge, respectively, to vectors $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$. By (4.13.2) we then get $\xi = \eta$, hence also

$$\varepsilon \leq |N(x_k) - N(z_k)| = \left| \frac{\nabla F(x_k)}{|\nabla F(x_k)|} - \frac{\nabla F(z_k)}{|\nabla F(z_k)|} \right| \to \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| = 0,$$

a contradiction.

Let us now check (3). Consider $r > 0$ as in (4.13.1). If $y \in E \subseteq \partial V$ is parallel to $N(y)$, then $\langle N(y), y\rangle = |y| \geq r$. Otherwise, by convexity of V, the triangle of vertices 0, $rN(y)$ and y, with angles α, β, γ at those vertices, is contained in V. So is the triangle of vertices 0, $rN(y)$, p, where p is the intersection of the line segment $[0, y]$ with the line $L = \{rN(y) + tv : t \in \mathbb{R}\}$, where v is perpendicular to $N(y)$ in the plane span$\{y, N(y)\}$ (see Figure 4.2 below). Then we have that $|p| < |y|$, and $|p| \cos\alpha = r$, hence

$$\langle N(y), y\rangle = |y| \cos\alpha > |p| \cos\alpha = r > 0.$$

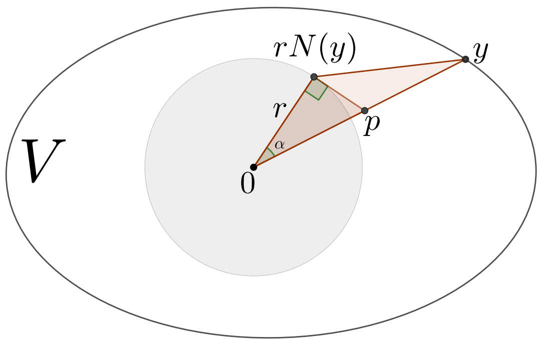

Figure 4.2

Finally condition (4) follows immediately from (iii) of Theorem 4.57 applied with $E^* = E \cup \{0\}$ (and from the fact that $\nabla F(0) = 0$).

Conversely, assume that $N : E \to \mathbb{S}^{n-1}$ and $P = P_X : \mathbb{R}^n \to X$ satisfy conditions $(1)-(4)$, and let us construct a suitable V with the help of Theorem 4.57. Thanks to condition (3), we can choose $r > 0$ with

$$0 < r < \inf_{y \in E} \langle N(y), y \rangle, \tag{4.13.3}$$

and then we define $E^* = E \cup \{0\}$, $f : E^* \to \mathbb{R}$, $G : E^* \to \mathbb{R}^n$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x = 0, \end{cases} \qquad \text{and} \qquad G(x) = \begin{cases} \frac{2}{r} N(x) & \text{if} \quad x \in E \\ 0 & \text{if} \quad x = 0. \end{cases}$$

It is clear that condition (3) implies that $\text{dist}(0, \overline{E}) > 0$, hence the continuity of G on E^* follows immediately from the continuity of N on E. We need to prove that the 1-jet (f, G) satisfies on E^* conditions $(i) - (iv)$ of Theorem 4.57 with P_X. Let us first check that

$$f(x) - f(y) - \langle G(y), x - y \rangle \geq 0 \quad \text{for all} \quad x, y \in E^*.$$

Indeed, if $x, y \in E$, then

$$f(x) - f(y) - \langle G(y), x - y \rangle = \langle G(y), y - x \rangle = \frac{2}{r} \langle N(y), y - x \rangle,$$

which is nonnegative by condition (1). The situation when $x \in E$ and $y = 0$ is trivial because

$$f(x) - f(y) - \langle G(y), x - y \rangle = 1.$$

Finally, if $x = 0$ and $y \in E$ we have

$$f(x) - f(y) - \langle G(y), x - y \rangle = f(0) - f(y) - \langle G(y), 0 - y \rangle = -1 + \frac{2}{r} \langle N(y), y \rangle \geq 1, \tag{4.13.4}$$

by virtue of (4.13.3). Therefore condition (i) of Theorem 4.57 is fulfilled. Condition (ii) of Theorem 4.57 follows from (4) and from the fact that $G(0) = 0$ because

$$Y = \text{span}\{N(x) \, : \, x \in E\} = \text{span}\{G(x) \, : \, x \in E\} = \text{span}\{G(x) - G(y) \, : \, x, y \in E^*\}.$$

On the other hand it is clear that (iii) follows immediately from (4). It only remains for us to check (iv). Consider two sequences $(x_k)_k$ and $(z_k)_k$ in E^* such that $(P_X(x_k))_k$ is bounded. In the case that both $(x_k)_k$ and $(z_k)_k$ belong to E, we have that

$$f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle = \langle G(z_k), z_k - x_k \rangle = \frac{2}{r} \langle N(z_k), z_k - x_k \rangle$$

for every k. If we have that

$$\lim_k \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \right) = 0,$$

then $\lim_k \langle N(z_k), z_k - x_k \rangle = 0$ and condition (2) yields

$$\lim_k |G(x_k) - G(z_k)| = \frac{2}{r} \lim_k |N(x_k) - N(z_k)| = 0,$$

which proves condition (iv) in this case. In the case that $x_k = 0$ for every k and $(z_k)_k$ belongs to E, the same calculations as in (4.13.4) lead us to

$$f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \geq 1,$$

and then condition (iv) is trivially satisfied in this case. We have thus shown that condition (iv) is satisfied. Thus we may apply Theorem 4.57 in order to find a Lipschitz convex function $F \in C^1(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends the jet (f, G), and, with the notation of Theorem 4.41, $X_F = X$. In fact, from the proof of Theorem 4.57 (more precisely, from Lemma 4.60 and Subsection 4.12.1), we know that there are points $q_1, \ldots, q_{d-\ell} \in \mathbb{R}^n$, a vector $v \in \partial m(y_0)$ with $y_0 \in Y$ and a positive constant λ such that $\nabla F(q_i) = v + \lambda w_i$ for every $i = 1, \ldots, d-\ell$. Here m denotes the function

$$m(x) = \sup_{y \in E^*} \{ f(y) + \langle G(y), x - y \rangle \}, \quad x \in \mathbb{R}^n.$$

We also learnt from Section 4.12 that $Y = \text{span}\{G(x) - G(y) \, : \, x, y \in E^*\}$ coincides with X_m, the subspace associated to the decomposition of m provided by Theorem 4.41. This subspace X_m can be also written as

$$Y = X_m = \text{span}\{\xi_x - \xi_y \, : \, \xi_x \in \partial m(x), \, \xi_y \in \partial m(y), \, x, y \in \mathbb{R}^n\}$$

by virtue of Lemma 4.47. Since $G(0) = 0$, we have that $0 \in \partial m(0)$ and then, because $v \in \partial m(y_0)$, the vector v belongs to $X_m = Y$. Recall that Y denotes the subspace $\text{span}(N(E))$ and coincides with $\text{span}(G(E))$. Because $\dim(X) = d$, $\dim(Y) = \ell$ and $w_1, \ldots, w_{d-\ell} \in X \cap Y^\perp$ are linearly independent, all these observations lead us to

$$\begin{aligned} X &= \text{span}\left(G(E) \cup \{w_1, \ldots, w_{d-\ell}\}\right) \\ &= \text{span}\left(G(E) \cup \{v + \lambda w_1, \ldots, v + \lambda w_{d-\ell}\}\right) = \text{span}\left(\nabla F(E) \cup \{\nabla F(q_1), \ldots, \nabla F(q_{d-\ell})\}\right). \end{aligned} \tag{4.13.5}$$

Moreover, from Lemma 4.61, we have that

$$F(q_i) \geq f(0) + \langle G(0), q_i \rangle + 1 = 1, \quad \text{for all} \quad i = 1, \ldots, d - \ell. \tag{4.13.6}$$

We then define $V = F^{-1}((-\infty, 1])$. According to Proposition 2.16, V is closed and convex and, because $F(0) = 0 < 1$, $0 \in \text{int}(V)$. Hence V is a convex body. In addition, it is clear that $E \subset \partial V$ as $F = f = 1$ on E. Also, because F is of class $C^1(\mathbb{R}^n)$ and, according to Proposition 4.27, the Minkowski functional μ_V of V is of class $C^1(\mathbb{R}^n \setminus \mu_V^{-1}(0))$. This shows that V is of class C^1. In fact, Proposition 4.27 tells us that the gradient $\nabla F(x)$ is a positive multiple of $\nabla \mu_V(x)$ for every $x \in \mathbb{R}^n \setminus \mu_V^{-1}(0)$. This implies that

$$n_V(x) = \frac{\nabla \mu_V(x)}{|\nabla \mu_V(x)|} = \frac{\nabla F(x)}{|\nabla F(x)|}, \quad x \in \partial V,$$

and then

$$N(x) = \frac{G(x)}{|G(x)|} = \frac{\nabla F(x)}{|\nabla F(x)|} = n_V(x), \quad x \in E.$$

Thus $N(x)$ is outwardly normal to ∂V at x. Finally, let us show that $X = \text{span}(n_V(\partial V))$. Proposition 4.27 says that $\nabla F(x)$ and $\nabla F\left(\frac{x}{\mu_V(x)}\right)$ are proportional to $\nabla \mu_V(x)$ and $\nabla \mu_V\left(\frac{x}{\mu_V(x)}\right)$ respectively and that $\nabla \mu_V(x) = \nabla \mu_V\left(\frac{x}{\mu_V(x)}\right)$. That is, $\nabla F(x)$ is proportional to $\nabla F\left(\frac{x}{\mu_V(x)}\right)$ for every $x \in \mathbb{R}^n \setminus \mu_V^{-1}(0)$. This shows that

$$\text{span}(n_V(\partial V)) = \text{span}\{\nabla F(x) \, : \, x \in \partial V\} = \text{span}\{\nabla F(x) \, : \, x \in \mathbb{R}^n \setminus \mu_V^{-1}(0)\} \tag{4.13.7}$$

Thanks to (4.13.6), the points $q_1, \ldots, q_{d-\ell}$ do not belong to $\text{int}(V)$ and then $\mu_V(q_i) \geq 1$ for every $i = 1, \ldots, d-\ell$. The fact that $\mu_V = 1$ on E together with (4.13.5) yields

$$X = \text{span}\left(\nabla F(E) \cup \{\nabla F(q_1), \ldots, \nabla F(q_{d-\ell})\}\right) \subseteq \text{span}\{\nabla F(x) \, : \, x \in \mathbb{R}^n \setminus \mu_V^{-1}(0)\}.$$

Finally, Lemma 4.47 and the fact that $\nabla F(0) = 0$ imply that $X = X_F = \text{span}\{\nabla F(x) \, : \, x \in \mathbb{R}^n\}$, which shows that $\text{span}(n_V(\partial V)) = X$ by virtue of (4.13.7). $\square$

As before, in the case that $X = \text{span}(N(E))$, the above result is much easier to use.

Corollary 4.70. *Let E be an arbitrary subset of $\mathbb{R}^n$, $N : E \to \mathbb{S}^{n-1}$ a continuous mapping, X a linear subspace of $\mathbb{R}^n$ such that $X = span(N(E))$, and $P : \mathbb{R}^n \to X$ the orthogonal projection. Then there exists a (possibly unbounded) convex body V of class C^1 such that $E \subset \partial V$, $0 \in int(V)$, $N(x) = n_V(x)$ for all $x \in E$, and $X = span\,(n_V(\partial V))$, if and only if the following conditions are satisfied:*

1. *$\langle N(y), x - y\rangle \leq 0$ for all $x, y \in E$.*
2. *For all sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded, we have that*
$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k\rangle = 0 \implies \lim_{k\to\infty} |N(z_k) - N(x_k)| = 0.$$
3. *$0 < \inf_{y\in E}\langle N(y), y\rangle$.*

Proof. It follows immediately from Theorem 4.69. □

4.14 The problem in the setting of Hilbert spaces

It is natural to wonder whether it is possible to establish a Whitney Extension Theorem for C^1 convex functions similar to Theorems 4.7, 4.20 or 4.43 in the setting of infinite dimensional Hilbert Spaces. In Chapter 2 we solved this question for the class of C^1 convex functions with uniformly continuous derivatives, see the comments in Remark 2.45. In this section we show that there exist bounded, smooth convex functions defined on an open neighborhood of a closed ball in $X := \ell_2^{(\mathbb{R})}$, the space of square summable real valued sequences, which have no continuous convex extensions to all of X. This indicates that even for the best possible convex domain and the best class of differentiability of the jet, we should look for new conditions (stronger than (CW^1)) in order to solve the C^1 convex extension problem for 1-jets in separable Hilbert spaces.

Let us denote by C the closed unit ball of X. The natural complexification of the space is $X_{\mathbb{C}} = \ell_2$. Also let $U = \{x \in X : \|x\| < 2\}$, $U_{\mathbb{C}} = \{x \in X_{\mathbb{C}} : \|x\| < 2\}$, and $S_X = \{x \in X : \|x\| = 1\}$.

Example 4.71. *There exists a function $F : U \to \mathbb{R}$ such that*

(i) *F is real analytic on U;*

(ii) *F is convex on U with $D^2F(x)(v^2) \geq 1$ for every $x \in U$, $v \in S_X$;*

(iii) *F is bounded on C, and*

(iv) *$F_{|C}$ has no continuous convex extension to the whole space X.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of X, and consider the sequence of vectors $\{\widetilde{e_n}\}_{n\geq 2} \subset C$ defined as follows:
$$\widetilde{e_n} = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_n, \quad n \geq 2.$$
For every $n \geq 2$, we define the linear functional $h_n \in X^*$ by $h_n(x) = \langle x, \widetilde{e_n}\rangle$ for all $x \in X$. Equivalently, for every $x = (x_n)_{n\geq 1} \in X$, we have $h_n(x) = \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_n$ for every $n \geq 2$. Now let us define
$$\begin{array}{rccl} f: & U & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sum_{n=2}^{\infty}(h_n(x))^{2n}, \end{array}$$
or equivalently $f(x) = \sum_{n\geq 2}\left(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_n\right)^{2n}$ for all $x = (x_n)_n \in U$. Let us first check that f is well defined. Given $x \in U$, take $r = 2 - |x_1| > 0$. Because $x \in \ell_2^{(\mathbb{R})}$, there is some $n_0 \in \mathbb{N}$ such that

$|x_n| \leq \frac{r}{2\sqrt{3}}$ whenever $n \geq n_0$. Therefore, if $n \geq n_0$, we have

$$\begin{aligned}\left|\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_n\right| &\leq \frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}|x_n| = \frac{1}{2}(2-r) + \frac{\sqrt{3}}{2}|x_n| \\ &\leq \frac{1}{2}(2-r) + \frac{r}{4} = 1 - \frac{r}{4} =: \lambda.\end{aligned}$$

Since $\lambda < 1$,

$$\sum_{n \geq n_0} \left|\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_n\right|^{2n} \leq \sum_{n \geq n_0} \lambda^{2n}$$

converges and this shows that $f(x)$ is finite.

Claim 4.72. *f is bounded above by $M := \frac{49}{24}$ on C.*

Proof. Given $x \in C$, and $x = (x_n)_{n \geq 1}$, since $\sum_{n \geq 1} x_n^2 \leq 1$, we have that $\sum_{n \geq 2} x_n^2 \leq 1 - x_1^2$; and this implies that there is at most one coordinate $N \geq 2$ such that $x_N^2 > \frac{1-x_1^2}{2}$. Hence, the rest of the coordinates satisfy

$$|x_n| \leq \sqrt{\frac{1-x_1^2}{2}} \quad \text{for every} \quad n \geq 2 \quad \text{with} \quad n \neq N.$$

And, of course, $|x_N| \leq \sqrt{1-x_1^2}$. This yields

$$\begin{aligned}f(x) &\leq \sum_{n \geq 2} \left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}|x_n|\right)^{2n} = \left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}|x_N|\right)^{2N} + \sum_{n \geq 2,\, n \neq N} \left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}|x_n|\right)^{2n} \\ &\leq \left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}\sqrt{1-x_1^2}\right)^{2N} + \sum_{n \geq 2,\, n \neq N} \left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}\sqrt{\frac{1-x_1^2}{2}}\right)^{2n}.\end{aligned}$$

In order to get a bound for the first sum in the last term, we consider the function $g(t) = \frac{t}{2} + \frac{\sqrt{3}}{2}\sqrt{1-t^2}$ for $t \in [0,1]$. A simple calculation shows that g has a maximum at $t = \frac{1}{2}$ and then $g(t) \leq g(1/2) = 1$ for all $t \in [0,1]$. Therefore

$$\left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}\sqrt{1-x_1^2}\right)^{2N} \leq 1.$$

The second sum can be bounded as follows. Take $h(t) = \frac{t}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{1-t^2}}{\sqrt{2}}$, $t \in [0,1]$. This function h attains a maximum at $t = \sqrt{\frac{2}{5}}$. Hence $h(t) \leq h\left(\sqrt{\frac{2}{5}}\right) = \sqrt{\frac{5}{8}}$, for every $t \in [0,1]$, which in turn implies

$$\left(\frac{1}{2}|x_1| + \frac{\sqrt{3}}{2}\sqrt{\frac{1-x_1^2}{2}}\right)^{2n} \leq \left(\sqrt{\frac{5}{8}}\right)^{2n} = \left(\frac{5}{8}\right)^{n} \quad \text{for all} \quad n \geq 2, \quad n \neq N.$$

Therefore, $f(x) \leq 1 + \sum_{n \geq 2,\, n \neq N} \left(\frac{5}{8}\right)^n \leq 1 + \sum_{n \geq 2} \left(\frac{5}{8}\right)^n = \frac{49}{24}$. □

Claim 4.73. *f is real analytic on U.*

Proof. Consider the complex function

$$\begin{aligned}\widetilde{f}: \; U_{\mathbb{C}} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n=2}^{\infty} \left(\tfrac{1}{2}z_1 + \tfrac{\sqrt{3}}{2}z_n\right)^{2n}.\end{aligned}$$

Obviously the restriction of $\widetilde{f}$ to U is the function f, and we can see that $\tilde{f}$ is well defined with the same calculations as we made above for f. Of course it is enough to prove that $\widetilde{f}$ is holomorphic on $U_{\mathbb{C}}$, for which in turn it is enough to check that, given $z \in U_{\mathbb{C}}$ there are $r > 0$ and a sequence $\{M_n\}_{n\geq 2}$ of positive numbers such that

$$\sum_{n\geq 2} M_n < +\infty \quad \text{and} \quad \left|\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_n\right|^{2n} \leq M_n \quad \text{for all} \quad y \in \overline{B}_{\mathbb{C}}(z,r) \subseteq U_{\mathbb{C}}, \quad n \geq 2,$$

where $\overline{B}_{\mathbb{C}}(z,r) = \{y \in X_{\mathbb{C}} : \|z-y\| \leq r\}$. Indeed, fix $z \in U_{\mathbb{C}}$. We take $r > 0$ such that $\overline{B}_{\mathbb{C}}(z,r) \subset U_{\mathbb{C}}$ with $\|z\| + r < 2$ and $r \leq \frac{2-|z_1|}{4(1+\sqrt{3})}$. Find $n_0 \in \mathbb{N}$ such that $|z_n| \leq \frac{2-|z_1|}{2\sqrt{3}}$ whenever $n \geq n_0$. Of course these $r > 0$ and $n_0 \in \mathbb{N}$ only depend on z. Define the numbers

$$\lambda_n = \begin{cases} 1+\sqrt{3} & \text{if} \quad 2 \leq n \leq n_0 - 1, \\ \dfrac{6+|z_1|}{8} & \text{if} \quad n \geq n_0, \end{cases}$$

and $M_n = \lambda_n^{2n}$ for all $n \geq 2$. Since $|z_1| < 2$, the sum $\sum_{n\geq 2} M_n$ converges. If $y \in \overline{B}_{\mathbb{C}}(z,r)$, with $y = (y_n)_{n\geq 1}$, then $|y_n| \leq r + |z_n|$ for every $n \geq 1$. Therefore, if $n \geq n_0$, because $|z_n| \leq \frac{2-|z_1|}{2\sqrt{3}}$ and $r \leq \frac{2-|z_1|}{4(1+\sqrt{3})}$ we have

$$\begin{aligned} \left|\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_n\right| &\leq \frac{1}{2}|y_1| + \frac{\sqrt{3}}{2}|y_n| \leq \frac{1}{2}(|z_1|+r) + \frac{\sqrt{3}}{2}(|z_n|+r) \\ &\leq \frac{1+\sqrt{3}}{2}\frac{2-|z_1|}{4(1+\sqrt{3})} + \frac{|z_1| + \frac{1}{2}(2-|z_1|)}{2} = \lambda_n. \end{aligned}$$

And for integers $2 \leq n \leq n_0 - 1$, we have the obvious inequality $\left|\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_n\right| \leq 1+\sqrt{3} = \lambda_n$. Hence

$$\left|\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_n\right|^{2n} \leq M_n \quad \text{for every} \quad n \geq 2$$

and this proves our statement. □

Now, the convexity of f can be checked as follows. The function $f_n = g_n \circ h_n$, being h_n a linear functional and $\mathbb{R} \ni t \to g_n(t) = t^{2n}$ a convex function for all $n \geq 2$, is convex on U, and f, being the sum of convex functions, is convex on U as well. Now define $F := f + N$, where $N : X \to \mathbb{R}$ is the function defined by $N(x) = \frac{\|x\|^2}{2}$ for all $x \in X$. Since X is a Hilbert space, the function N is analytic on X. Of course N is bounded on C and $D^2N(x)(v^2) = \|v\|^2 = 1$ for all $v \in S_X$ and all $x \in X$. Hence F is real analytic, is bounded on C and, since f is convex and differentiable, $D^2F(x)(v^2) = D^2f(x)(v^2) + D^2N(x)(v^2) \geq 1$ for all $x \in U$ and all $v \in S_X$. We then have proved (i), (ii) and (iii) of our Theorem.

In order to prove (iv), consider the minimal convex extension of F,

$$m_C(F)(x) = \sup_{y\in C}\{F(y) + \langle \nabla F(y), x-y\rangle\}, \quad x \in X.$$

Observe that (iv) will be proved as soon as we find points $x \in X$ with $m_C(F)(x) = +\infty$. We next prove that in fact $m_C(F) = +\infty$ for all x of the form $x = re_1$, with $r > 2$. So fix $r > 2$ and $x = re_1$. For any $k \geq 2$ and $n \geq 2$ we inmediately see that $\langle \widetilde{e_n}, \widetilde{e_k}\rangle = 1/4$ for $n \neq k$ and $\langle \widetilde{e_k}, \widetilde{e_k}\rangle = 1$. Then

$$f(\widetilde{e_k}) = 1 + \sum_{n\geq 2,\, n\neq k} \left(\frac{1}{4}\right)^{2n} \quad \text{and} \quad N(\widetilde{e_k}) = \frac{1}{2}, \quad k \geq 2.$$

Since f is analytic, we can calculate its derivatives by differentiating the series term by term, and then

$$\langle \nabla f(\widetilde{e_k}), v\rangle = \sum_{n\geq 2} 2n\langle \widetilde{e_k}, \widetilde{e_n}\rangle^{2n-1}\langle v, \widetilde{e_n}\rangle = \sum_{n\geq 2,\, n\neq k} 2n\left(\frac{1}{4}\right)^{2n-1}\langle v, \widetilde{e_n}\rangle + 2k\,\langle v, \widetilde{e_k}\rangle$$

for every $v \in X$ and $k \geq 2$. On the other hand, $\langle \nabla N(\widetilde{e_k}), v\rangle = \langle \widetilde{e_k}, v\rangle$ for all $v \in X$. For $v = x - \widetilde{e_k}$, we have

$$\langle v, \widetilde{e_n}\rangle = \left\langle re_1, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_n \right\rangle - \langle \widetilde{e_k}, \widetilde{e_n}\rangle = \frac{r}{2} - \begin{cases} 1 & \text{if} \quad n = k, \\ \frac{1}{4} & \text{if} \quad n \neq k. \end{cases}$$

Gathering the above inequalities we obtain, for $k \geq 2$,

$$\begin{aligned} &F(\widetilde{e_k}) + \langle \nabla F(\widetilde{e_k}), x - \widetilde{e_k}\rangle = f(\widetilde{e_k}) + N(\widetilde{e_k}) + \langle \nabla f(\widetilde{e_k}), x - \widetilde{e_k}\rangle + \langle \nabla N(\widetilde{e_k}), x - \widetilde{e_k}\rangle \\ &= 1 + \sum_{n\geq 2,\, n\neq k} \left(\frac{1}{4}\right)^{2n} + \frac{1}{2} + \sum_{n\geq 2,\, n\neq k} 2n \left(\frac{1}{4}\right)^{2n-1} \left(\frac{r}{2} - \frac{1}{4}\right) + 2k\left(\frac{r}{2} - 1\right) + \left(\frac{r}{2} - 1\right) \\ &\geq k(r-2); \end{aligned}$$

and the last term tends to $+\infty$ as k goes to $+\infty$. We thus have proved that $m_C(F)(x) = +\infty$ for those points $x \in X$ of the form $x = re_1,\ r > 2$. $\square$

4.15 Convex functions and Lusin properties

Very recently, in [7], D. Azagra and P. Hajłasz have found an interesting application of Corollary 4.33 concerning the Lusin property for convex functions.

If $\mathcal{L}^n$ denotes Lebesgue's measure on $\mathbb{R}^n$, we will say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has a Lusin property of type $C^1_{\text{conv}}(\mathbb{R}^n)$ if for every $\varepsilon > 0$ there exists a convex function $g \in C^1(\mathbb{R}^n)$ such that $\mathcal{L}^n\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) < \varepsilon$. In [7] the following characterization of convex functions satisfying this property is proved.

Theorem 4.74 (Azagra-Hajlasz). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and assume that f is not of class C^1. Then f is essentially coercive if and only if for every $\varepsilon > 0$ there exists a convex function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $\mathcal{L}^n\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) \leq \varepsilon$.*

Chapter 5

C^m extensions of convex functions on $\mathbb{R}^n$

5.1 Whitney's Extension Theorem for C^m

The fundamental starting point for our extension results is the classical Whitney Extension Theorem for C^m, see [70].

Theorem 5.1 (Whitney's Extension Theorem for C^m). *Let m be a positive integer, $C \subset \mathbb{R}^n$ be a closed subset of $\mathbb{R}^n$ and $\{f_\alpha\}_{|\alpha|\leq m}$ a family of real valued functions defined on C. Let us write*

$$f_\alpha(x) = \sum_{|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y)$$

for all $x, y \in C$ and every multi-index α with $|\alpha| \leq m$. Then there exists a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $C^m(\mathbb{R}^n)$ such that $\partial^\alpha F = f_\alpha$ on C for every $|\alpha| \leq m$ if and only if

$$\lim_{|x-y|\to 0} \frac{|R_\alpha(x,y)|}{|x-y|^{m-|\alpha|}} = 0 \quad \textit{uniformly on} \quad x, y \in K \tag{W^m}$$

for every compact subset K of C and all $|\alpha| \leq m$.

Note that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n$ of $\mathbb{R}^n$, we denote $|\alpha| = \alpha_1+\cdots+\alpha_n$, the order of α, and $\alpha! = \alpha_1!\cdots\alpha_n!$. Also, given a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Finally, by $\partial^\alpha F$ we mean

$$\partial^\alpha F = \frac{\partial^{|\alpha|} f}{(\partial x_1)^{\alpha_1}\cdots(\partial x_n)^{\alpha_n}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n).$$

The extension F of Theorem 5.1 can be explicitly defined by means of the formula

$$F(x) = \begin{cases} f(x) & \text{if } x \in C \\ \sum_k \left(\sum_{|\alpha|\leq m} \frac{f_\alpha(p_k)}{\alpha!}(x-p_k)^\alpha \right) \varphi_k(x) & \text{if } x \in \mathbb{R}^n \setminus C, \end{cases} \tag{5.1.1}$$

where each p_k is a point of C which minimizes the distance of C to the cube Q_k, $\{Q_k\}_k$ are the *Whitney cubes* of the complement of C and $\{\varphi_k\}_k$ is the *Whitney partition of unity* associated to $\{Q_k\}_k$. See Propositions 2.2 and 2.3 for notation and terminology.

Equivalently Whitney's Extension Theorem for C^m can be formulated in terms of families of polynomials of degree up to m.

Theorem 5.2. *Let C be a closed subset of $\mathbb{R}^n$, and $m \in \mathbb{N}$. If $\{P_y^m\}_{y\in C}$ is a family of polynomials of degree less than or equal to m, a necessary and sufficient condition on the family $\{P_y^m\}_{y\in C}$ for the existence of a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $C^m(\mathbb{R}^n)$ with $J_y^m F = P_y^m$ is that*

$$\lim_{\delta\to 0^+} \rho_m(K,\delta) = 0 \, \textit{for each compact subset } K \textit{ of } C, \tag{W^m}$$

where

$$\rho_m(K,\delta) = \sup\left\{ \frac{\|D^j P_y(z) - D^j P_z(z)\|}{|y-z|^{m-j}} \ : \ j=0,\ldots,m,\ y,z\in K,\ 0<|y-z|\le\delta \right\}.$$

Here $J_y^m F$, denotes the Taylor polynomial of order m of F at y. Also, for every $j\in\{0,\ldots,m\}$ and every $y,z\in A$ we denote

$$\|D^j P_y(z)\| = \sup_{u_1,\ldots,u_j\,\in\,\mathbb{S}^{n-1}} |D^j P_y(z)(u_1,\ldots,u_j)|,$$

where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$. Let us briefly explain why Theorems 5.1 and 5.2 are equivalent. If we are given a family of functions $\{f_\alpha\}_{|\alpha|\le m}$ defined on a closed subset C of $\mathbb{R}^n$, then we can define for each $y\in C$ a polynomial P_y^m of degree less than equal to m by setting

$$P_y^m(x) = \sum_{|\beta|\le m} \frac{f_\beta(y)}{\beta!}(x-y)^\beta, \quad x\in\mathbb{R}^n.$$

We then have that $\partial^\alpha P_x^m(x) = f_\alpha(x)$ for every $x\in C$. Conversely if we are provided with a family of polynomials $\{P_y^m\}_{y\in C}$ of degree less than or equal to m, then we can define, for each multi-index α with $|\alpha|\le m$, the function f_α on C via $f_\alpha(x)=\partial^\alpha P_x^m(x)$ for every $x\in C$; and we have the identity.

$$P_y^m(x) = \sum_{|\beta|\le m}\frac{\partial^\beta P_y^m(y)}{\beta!}(x-y)^\beta = \sum_{|\beta|\le m}\frac{f_\beta(y)}{\beta!}(x-y)^\beta, x\in\mathbb{R}^n,\ y\in C.$$

Therefore if we are given either a family of functions $\{f_\alpha\}_{|\alpha|\le m}$ on C or a family of polynomials $\{P_y^m\}_{y\in C}$ of degree up to m, with the above remarks we have that

$$\left| f_\alpha(x) - \sum_{|\beta|\le m-|\alpha|}\frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \right| = |\partial^\alpha P_x^m(x) - \partial^\alpha P_y^m(x)|$$

for every multi-index α with $|\alpha|\le m$ and every $x,y\in C$. This clearly shows the equivalence between both (W^m) conditions of Theorems 5.1 and 5.2.

Furthermore, we can give an equivalent version of the Whitney's Extensión Theorem in terms of symmetric k-linear forms. Given an integer $k\ge 0$, by a symmetric k-linear form A on $\mathbb{R}^n$, we understand a mapping $A:(\mathbb{R}^n)^k\to\mathbb{R}$ such that A is linear on each coordinate and $A(v_1,\ldots,v_k) = A(v_{\sigma(1)},\ldots,v_{\sigma(k)})$ for every permutation $\sigma:\{1,\ldots,k\}\to\{1,\ldots,k\}$ and all $v_1,\ldots,v_k\in\mathbb{R}^n$. In the trivial case $k=0$, by a k-linear form we merely understand a real number. The vector space of all the symmetric k-linear forms on $\mathbb{R}^n$ will be denoted by $\mathcal{L}^k(\mathbb{R}^n,\mathbb{R})$.

Theorem 5.3. *Let C be a closed subset of $\mathbb{R}^n$, and $m\in\mathbb{N}$. Assume that we are given a family of functions $\{A_k\}_{k=0}^m$, where $A_k : C\to\mathcal{L}^k(\mathbb{R}^n,\mathbb{R})$ for every $k=0,\ldots,m$. Then, a necessary and sufficient condition on the family $\{A_k\}_{k=0}^m$ for the existence of a function $F:\mathbb{R}^n\to\mathbb{R}$ of class $C^m(\mathbb{R}^n)$ with $D^kF=A_k$ on C is that for every compact subset K of C*

$$\lim_{|y-z|\to 0^+}\frac{\left\| A_k(z) - \sum_{\ell=0}^{m-k}\frac{1}{\ell!}A_{k+\ell}(y)(z-y)^\ell \right\|}{|y-z|^{m-k}} = 0 \text{ uniformly on } y,z\in K, \qquad (W^m)$$

where we denote $\|L\| = \sup_{v_1,\ldots,v_k\in\mathbb{S}^{n-1}}|L(v_1,\ldots,v_k)|$ for every $L\in\mathcal{L}^k(\mathbb{R}^n,\mathbb{R})$.

If we are given a family of polynomials $\{P_y^m\}_{y\in C}$ of degree up to m, then we can construct a family $\{A_k\}_{k=0}^m$, $A_k : C \to \mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$, by setting $A_k(y) = D^k P_y^m(y)$ and Taylor's theorem gives us

$$D^k P_y(x) = \sum_{\ell=0}^{m-k} \frac{1}{\ell!} D^{k+\ell} P_y(y)(x-y)^\ell = \sum_{\ell=0}^{m-k} \frac{1}{\ell!} A_{k+\ell}(y)(x-y)^\ell, \quad \in \mathbb{R}^n,\ y \in C$$

Conversely, if we have a family $\{A_k\}_{k=0}^m$, where each A_k is a $\mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$-valued function defined on C, then we can define for every $y \in C$ a polynomial P_y^m of degree less than or equal to m via the formula

$$P_y(x) = \sum_{\ell=0}^{m} \frac{1}{\ell!} A_\ell(y)(x-y)^\ell, \quad x \in \mathbb{R}^n,$$

and it is clear that $D^k P_y^m(y) = A_k(y)$ for every $k = 0, \ldots, m$ and every $y \in C$. In view of this remarks, it is clear that Theorems 5.2 and 5.3 are equivalent.

From now on, by a m-jet on a closed subset C of $\mathbb{R}^n$ we will understand either a family of real-valued functions $\{f_\alpha\}_{|\alpha|\leq m}$ defined on C satisfying condition (W^m) of Theorem 5.1, or a family of polynomials $\{P_y^m\}_{y\in C}$ of degree less than or equal to m satisfying condition (W^m) of Theorem 5.2, or a family $\{A_k\}_{k=0}^m$ of functions defined on C such that each A_k is $\mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$-valued and satisfies condition (W^m) of Theorem 5.3.

5.2 The C^m_{conv} extension problem for jets

It is natural to wonder what further conditions (if any) on a m-jet $\{f_\alpha\}_{|\alpha|\leq m}$, $m \geq 2$, defined on a closed subset C would be necessary and sufficient to ensure that the extension F of Theorem 5.1 can be taken to be convex. This is a problem that we were able to solve for the $C^{1,\omega}$ class on Hilbert or superreflexive spaces in Chapters 2 and 3 and for the C^1 class on $\mathbb{R}^n$ in Chapter 4. As in those cases, we cannot expect that a construction similar to Whitney's (5.1.1) provides a solution because partitions of unity destroy the possible convexity of our jet on C. Here, by *convexity of a jet* we understand that the jet can be extended to a C^m function f on $\mathbb{R}^n$ whose second derivative D^2F is semidefinite positive at every point of C. Of course, if C is convex, this implies that f is convex on C.

Unlike the C^1 and $C^{1,\omega}$ cases, if the domain C is not assumed to be convex, the problem seems to have a much more complicated solution. For instance, if our domain C is finite, a natural approach to construct a C^m convex extension of a m-jet which has a semidefinite positive Hessian on C, would be to imitate some of the ideas of the proof Theorem 4.7 for the C^1 class. That is, if we are able to find a $C^m(\mathbb{R}^n)$ (not necessarily convex) function g which extends our jet from C and lies above the *minimal convex extension*

$$m(x) := \max_{y\in C} \left\{ f_0(y) + \sum_{|\alpha|=1} f_\alpha(y)(x-y)^\alpha \right\},$$

then we consider $f = \text{conv}(g)$, that is, the convex envelope of g (see (2.1.5)). We saw in Theorems 2.10 and 4.17 that the convex envelope of a differentiable function is of class $C^{1,1}$ whenever the derivative of the function is uniformly continuous or of class C^1 whenever the function is coercive. Unfortunately these results concerning the differentiabilty of convex envelopes are optimal and, no matter what order of smoothness the function f could have, the best possible class of differentiability we can obtain with this tool is C^1 or $C^{1,1}$. To see this, let us consider the following example.

Example 5.4. Let $C = \{-1, 1\}$ and define $f_0(x) = f_1(x) = 0$ and $f_2(x) = 1$ for every $x \in C$. The function $g(x) = \frac{1}{8}(x^2-1)^2$, $x \in \mathbb{R}$, is a C^∞ extension of the jet (f_0, f_1, f_2) from C and g lies above the function identically zero, the minimal convex extension of the jet. However, the convex envelope $f = \text{conv}(g)$ of g is the function $f(x) = \frac{1}{8}(x^2-1)^2$ whenever $|x| \geq 1$ and $f = 0$ on $[-1, 1]$; which is of class $C^{1,1}$ but it is not of class C^2. In fact, there is no convex function F of class

$C^2(\mathbb{R})$ such that $F = f_0,\ F' = f_1$ and $F'' = f_2$ on C. Indeed, any convex extension F of class $C^2(\mathbb{R})$ of the 2-jet (f_0, f_1, f_2) is necessarily equal to 0 on the interval $[-1, 1]$ because F' must satisfy $0 = F'(-1) \leq F'(t) \leq F'(1) = 0$ for every $t \in [-1, 1]$. Therefore $F''_+(-1) = F''_-(1) = 0$ and F'' does not extend f_2.

In view of the existence of this kind of examples even in the one dimensional case and for finite domains, we will restrict our attention to the problem when the domain is closed and convex. On the other hand, if C is not assumed to be compact, we need to deal with the problem that our m-jet may have *corners at infinity* (see Definition 4.34) because these geometrical phenomena persist no matter what order of differentiability our jet could have, as we saw in Example 4.35. For this reason, it is reasonable not to study the problem for unbounded closed domains until we have a full solution to the problem when our domain is compact, as we did in Chapter 4 for the class C^1.

Therefore, at least in a first approach to the problem, it seems reasonable to assume that C is convex and compact, which we will do in the rest of this chapter, and ask ourselves if our extension problem can always be solved in this relatively simple case.

The problem of extending convex functions f which are the restriction of a C^m (not necessarily convex function) to a convex function of class C^m was considered by M. Ghomi [44] and M. Yan [72]. A consequence of their results is that, under the assumption that f has a strictly positive Hessian on the boundary ∂C, there always exists a function $F \in C^m(\mathbb{R}^n)$ such that F is convex and $F = f$ on C. See also [43, 45, 21] for related problems. Observe that for a nonconvex domain C, Example 5.4 above shows that the requirement that our jet has strictly positive hessian on C (meaning that the jet admits a C^m extension with positive definite second derivative at every point of C) does not ensure the existence of C^m convex extensions. The situation does not improve if C is convex but unbounded. Indeed, a modification of [59, Example 4], which will be presented in Section 5.9 below, shows that there exist strongly convex functions f which have smooth convex extensions to small open neighborhoods of C, but no convex extensions to $\mathbb{R}^n$. Therefore the results by M. Ghomi [44] and M. Yan [72] for functions with strictly positive Hessian on the boundary of the domain are no longer true if the domain is not assumed to be a convex compactum.

Of course, strict positiveness of the Hessian is a very strong condition which is far from being necessary, and it would be desirable to get rid of this requirement altogether, if possible. However, some other assumptions must be made in its place, at least when $m \geq 3$, as already in one dimension there are examples of C^3 convex functions g defined on compact intervals I which cannot be extended to $C^3(J)$ convex functions for any open interval J containing I. To see this, let us consider a couple of examples.

Example 5.5. We claim the following.

(1) The function $g(x) = x^2 - x^3$ defined for $x \in I := [0, \frac{1}{3}]$ satisfies Whitney's condition (W^m) for every $m \in \mathbb{N}$ and $g'' \geq 0$ on I but there is no C^3 convex extensions of g to any open interval containing I.

(2) The above example generalizes to arbitrary dimension n by considering for instance

$$f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 - x_1^3, \ \ (x_1, \ldots, x_n) \in B(0, 1/3). \tag{5.2.1}$$

The function f has semidefinite positive Hessian at every point of the ball $B(0, 1/3)$ and satisfies Whitney conditions (W^m) of every order on $B(0, \frac{1}{3})$, but there is no C^3 convex extension of f to any open neighbourhood of $B(0, 1/3)$.

Proof. The function g of (1) admits a C^∞ extension to all of $\mathbb{R}$ and then g satisfies Whitney's conditions of every order. Also, we have that $g''(x) = 2 - 6x$ for every $x \in I$ and $g''' = -6$ on I. This shows that $g'' \geq 0$ on I and any convex C^3 extension $h : J \to \mathbb{R}$ of g to an open interval J containing I must satisfy that $h''' < 0$ on a neighbourhood of the point $x = 1/3$. Since $h''(1/3) = g''(1/3) = 0$, we obtain, by virtue of the Mean Value Theorem, that

$$h''(x) = h'''(z_x)(x - \tfrac{1}{3}), \quad \text{for some} \quad z_x \in (1/3, x), \quad \text{and all} \quad x > 1/3,\ x \in J.$$

In view of the above remarks about h, the values $h'''(z_x)$ are negative if x is close enough to $1/3$. This implies that $h'' < 0$ on some open interval contained in J, which contradicts the convexity of h on J. $\square$

In particular these examples show that the condition $D^2 f \geq 0$ on C is not sufficient to ensure the existence of a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C for any $m \geq 3$. Therefore, we should look for new conditions on the derivatives of f on C (beyond $D^2 f \geq 0$ on C) that are necessary and sufficient to guarantee that f has a C^m convex extension F to all of $\mathbb{R}^n$.

5.3 New conditions for the C^m convex extension problem

Let us introduce new conditions for the C^m convex extension problem for m-jets.

Definition 5.6 ((CW^m) condition for m-jets)**.** *Let $m \in \mathbb{N}$, $m \geq 2$ and C a compact subset of $\mathbb{R}^n$. We will say that a family of polynomials $\{P_y^m\}_{y\in C}$ of degree up to m satisfies the condition (CW^m) provided that*

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(D^2 P_y^m(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^2 P_y^m(y)(w^{m-2}, v^2) \right) \geq 0 \qquad (CW^m)$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$. This of course means that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$D^2 P_y^m(y)(v^2) + t\, D^3 P_y^m(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P_y^m(y)(w^{m-2}, v^2) \geq -\varepsilon t^{m-2}$$

for all $y \in C$, $v, w \in \mathbb{S}^{n-1}$, $0 < t \leq t_\varepsilon$.

We will also say that $\{P_y^m\}_{y\in C}$ satisfies (CW^m) with a strict inequality if there are some $\eta > 0$ and $t_0 > 0$ such that

$$D^2 P_y^m(y)(v^2) + t\, D^3 P_y^m(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P_y^m(y)(w^{m-2}, v^2) \geq \eta t^{m-2}$$

for all $y \in C$, $v, w \in \mathbb{S}^{n-1}$, $0 < t \leq t_0$.

Observe that for $m = 2$, the above condition (CW^2) merely says that $D^2 P_y^2(y)(v^2) \geq 0$ for every $y \in C$ and every $v \in \mathbb{S}^{n-1}$.

Definition 5.7 ((CW^m) condition for functions)**.** *We will say that a function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^m satisfies condition (CW^m) on a compact subset C provided*

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(D^2 F(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m F(y)(w^{m-2}, v^2) \right) \geq 0 \qquad (CW^m)$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$

We will also say that F satisfies (CW^m) with a strict inequality on C if there are some $\eta > 0$ and $t_0 > 0$ such that

$$D^2 F(y)(v^2) + t\, D^3 F(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m F(y)(w^{m-2}, v^2) \geq \eta t^{m-2}$$

for all $y \in C$, $v, w \in \mathbb{S}^{n-1}$, $0 < t \leq t_0$.

Since for a function F of class $C^m(\mathbb{R}^n)$ and a subset C of $\mathbb{R}^n$, the family of polynomials $\{J_y^m F\}_{y\in C}$ (where each $J_y^m F$ denotes the Taylor polynomial of F of order m at the point y) satisfies that $D^m(J_y^m F)(y) = D^m F(y)$ for every $y \in C$; the condition (CW^m) on C for F given in Definition 5.7 is equivalent to condition (CW^m) on C for the family $\{J_y^m F\}_{y\in C}$ given in Definition 5.6. An equivalent definition of condition (CW^m) for k-linear forms instead of polynomials or functions is the following.

Definition 5.8 ((CW^m) condition for linears forms)**.** *Given a family of functions $\{A_k\}_{k=0}^m$ defined on a compact subset C such that each A_k is $\mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$-valued, we will say that $\{A_k\}_{k=0}^m$ satisfies condition (CW^m) on C provided that*

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(A_2(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} A_m(y)(w^{m-2}, v^2) \right) \geq 0 \qquad (CW^m)$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$

We will also say that $\{A_k\}_{k=0}^m$ satisfies (CW^m) with a strict inequality on C if there are some $\eta > 0$ and $t_0 > 0$ such that

$$A_2(y)(v^2) + t\, A_3(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} A_m(y)(w^{m-2}, v^2) \geq \eta t^{m-2}$$

for all $y \in C$, $v, w \in \mathbb{S}^{n-1}$, $0 < t \leq t_0$.

Note that, for $m = 2$, the condition (CW^m) above merely says that $A_2(y)$ is semidefinite positive for every $y \in C$.

Let us now prove that condition (CW^m) is a necessary condition on a family of polynomials $\{P_y^m\}_{y\in C}$ of degree up to m for the existence of a convex function F of class C^m with $J_y^m F = P_y^m$ for every $y \in C$. In view of the comment subsequent to Definition 5.7 , this is equivalent to proving that any convex C^m function F satisfies condition (CW^m) (in the sense of Definition 5.7) on every compact convex subset of $\mathbb{R}^n$.

Lemma 5.9. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^m, $m \geq 2$, such that F is convex on a neigbourhood of a convex compact subset C. Then F satisfies condition (CW^m) on C.*

Proof. Let U be a convex open subset of $\mathbb{R}^n$ with $C \subset U$ and such that F is convex on U. By continuity of F, we further have that F is convex on the closure $\overline{U}$ on U. Since F is of class C^m, we must have $D^2F(x)(v^2) \geq 0$ for every $x \in U$, $v \in \mathbb{S}^{n-1}$. By the compactness of C, there are points $z_1, \ldots, z_N \in C$ and positive numbers $r_1, \ldots, r_N$ such that

$$C \subseteq \bigcup_{j=1}^{N} \overline{B}(z_j, r_j) \subset \bigcup_{j=1}^{N} \overline{B}(z_j, 2r_j) \subset U, \qquad (5.3.1)$$

where each $\overline{B}(z_j, r_j)$ is the closed ball centered at z_j and radius r_j. Let us consider $0 < t \leq \min\{r_j \ : j = 1, \ldots, N\}$. If $y \in C$ and $w \in \mathbb{S}^{n-1}$, then y belongs to some ball $\overline{B}(z_j, r_j)$ by virtue of (5.3.1), and hence $y + tw \in \overline{B}(r_j, 2r_j) \subset U$. This implies that $D^2F(y + tw)(v^2) \geq 0$ for every $v \in \mathbb{S}^{n-1}$. Making use of Taylor's Theorem for the second derivative D^2F of F at the point y, we obtain that

$$\begin{aligned} 0 &\leq D^2F(y + tw)(v^2) \\ &= D^2F(y)(v^2) + t\, D^3F(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^mF(y)(w^{m-2}, v^2) + R_m(t, y, v, w), \end{aligned}$$

where, by compactness of C,

$$\lim_{t\to 0^+} \frac{R_m(t, y, v, w)}{t^{m-2}} = 0 \quad \text{uniformly on } y \in C, w, v \in \mathbb{S}^{n-1}.$$

We have thus shown that

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(D^2F(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^mF(y)(w^{m-2}, v^2) \right) \geq 0$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$, that is, F satisfies condition (CW^m) on C. $\square$

The relation between the conditions (CW^m) and (CW^{m+1}) is given in the following remark.

Remark 5.10. *Let C be a compact subset of $\mathbb{R}^n$. Assume that $\{P_y^{m+1}\}_{y\in C}$ satisfies (W^{m+1}) and (CW^{m+1}) on C for some $m \geq 2$. Then $\{P_y^m\}_{y\in C}$ satisfies (CW^m) on C too, where each P_y^m is obtained from P_y^{m+1} by discarding its $(m+1)$-homogeneous terms.*

Proof. Since the family of polynomials $\{P_y^{m+1}\}_{y\in C}$ satisfies condition (CW^{m+1}) on C, given $\varepsilon > 0$, we can find a positive $t_\varepsilon > 0$ such that

$$\begin{aligned} &Q(P_y^{m+1}, t, v, w) \\ &:= \frac{D^2P_y^{m+1}(y)(v^2) + tD^3P_y^{m+1}(y)(w, v^2) + \cdots + \frac{t^{m-1}}{(m-1)!}D^{m+1}P_y^{m+1}(y)(w^{m-1}, v^2)}{t^{m-1}} \geq -\frac{\varepsilon}{2}. \end{aligned}$$

for every $0 < t \leq t_\varepsilon$, $y \in C$, $v, w \in \mathbb{R}^n$ with $|v| = |w| = 1$. On the other hand, Whitney's condition (W^{m+1}) for the family $\{P_y^{m+1}\}_{y\in C}$ tells us that there exists $r > 0$ such that

$$\|D^{m+1}P_z^{m+1}(z) - D^{m+1}P_y^{m+1}(z)\| \leq 1 \quad \text{whenever} \quad |y - z| \leq r, \; y, z \in C.$$

Thus, because C is compact, $\sup_{z\in C} \|D^{m+1}P_z^{m+1}(z)\|$ is finite and then we can choose t_ε so that

$$0 < t_\varepsilon \leq \frac{\varepsilon}{2\left(1 + \sup_{z\in C} \|D^{m+1}P_z^{m+1}(z)\|\right)}.$$

Since we have

$$P_y^{m+1}(x) = \sum_{k=0}^{m+1} \frac{1}{k!} D^k P_y^{m+1}(y)(x - y)^k, \quad x \in \mathbb{R}^n, \; y \in C,$$

each polynomial P_y^m can be written as

$$P_y^m(x) = \sum_{k=0}^{m} \frac{1}{k!} D^k P_y^{m+1}(y)(x - y)^k, \quad x \in \mathbb{R}^n, \; y \in C$$

and then $D^kP_y^m(y) = D^kP_y^{m+1}(y)$ for every $k = 0, \ldots, m$ and every $y \in C$. Using the preceding observations it follows that for every $0 < t \leq t_\varepsilon$, $y \in C$, $v, w \in \mathbb{R}^n$ with $|v| = |w| = 1$,

$$\begin{aligned} Q(P_y^m, t, v, w) :&= \frac{D^2P_y^m(y)(v^2) + tD^3P_y^m(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!}D^mP_y^m(y)(w^{m-2}, v^2)}{t^{m-2}} \\ &= \frac{D^2P_y^{m+1}(y)(v^2) + tD^3P_y^{m+1}(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!}D^mP_y^{m+1}(y)(w^{m-2}, v^2)}{t^{m-2}} \\ &= tQ(P_y^{m+1}, t, v, w) - \frac{t}{(m-1)!}D^{m+1}P_y^{m+1}(y)(w^{m-2}, v^2) \\ &\geq -\frac{\varepsilon}{2}t - t\sup_{z\in C}\|D^{m+1}P_z^{m+1}(z)\| \geq -\frac{\varepsilon}{2}t - \frac{\varepsilon}{2}t = -\varepsilon t \geq -\varepsilon. \end{aligned}$$

Therefore the family of polynomials $\{P_y^m\}_{y\in C}$ satisfies condition (CW^m) on C. □

We are now going to make some observations on the formulation of the condition (CW^m) when the convex compact set C has nonempty interior.

Remark 5.11. If $C \subset \mathbb{R}^n$ is compact and convex with nonempty interior and we have a function $f : C \to \mathbb{R}$, then, because the interior of C is dense on C, the derivatives D^jF of any extension $F \in C^m(\mathbb{R}^n)$ of f are uniquely determined by the values of f on C. Thus, if there exists a family $\{P_y^m\}_{y\in C}$ of polynomials of degree up to m with $P_y^m(y) = f(y)$ for every $y \in C$ and satisfying Whitney's condition (W^m) on C, this family is unique. Therefore, in the case that C has nonempty interior, the condition (CW^m) on C for may be reformulated as follows

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(D^2 f(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \right) \geq 0 \qquad (CW^m)$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$, understanding that $D^j f$ denotes the derivative of order j at y of any C^m extension of f to $\mathbb{R}^n$.

In view of the above remark, one might then think that for our convex extension problem, by considering the relative interior of the convex compact set C, there would be no loss of generality in assuming that C has nonempty interior (and therefore considering that (CW^m) holds only for v, w in the linear span of the directions $y - y'$ with $y, y' \in C$). However, since we are looking for convex analogues of the classical Whitney's extension theorem (which deals with prescribing differential data as well as extending functions) such an approach would make us lose some valuable insight about the question as to what extent one can prescribe values and derivatives of convex functions on a given compact convex set with empty interior. Indeed, for a convex compact set C with empty interior and a convex function $f : C \to \mathbb{R}$, there are infinitely many convex functions $F : C \to \mathbb{R}$ with very different derivatives on C and such that $F = f$ on C. Let us look, for instance, at the extreme situation in which C is a singleton, say $C = \{0\}$. One of our results in this chapter (see Theorem 5.24 below) implies that, for any $m \geq 2$ and any polynomial P of degree up to m such that

$$\liminf_{t\to 0^+} \frac{D^2 P(0)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P(0)(w^{m-2}, v^2)}{t^{m-2}} \geq 0$$

uniformly on $v, w \in \mathbb{S}^{n-1}$, there exists a convex function F of class C^m such that the Taylor polynomial of F of order m at 0 is P. Consequently, there are many degrees of freedom in prescribing derivatives of convex functions of class C^m at a given point.

On the other hand, if C is a convex compact set with nonempty interior (what is usually called a convex body) and $f : C \to \mathbb{R}$ is a convex function which has a (not necessarily convex) C^m extension to an open neighbourhood of C, then it is clear that f automatically satisfies $D^2 f(x) \geq 0$ on the interior of C, that is f satisfies (CW^2) on the interior of C. Conversely, if f satisfies (CW^2) on the interior of C then it immediately follows, using Taylor's theorem, that $D^2 f(x)$ is positive semidefinite for all x in the open convex set $\text{int}(C)$, hence f is convex on $\text{int}(C)$, and by continuity we infer that f is also convex on C.

Remark 5.12. The above observation together with Remark 5.9 show that if C is a convex compact subset of $\mathbb{R}^n$ with nonempty interior, $m \in \mathbb{N}, m \geq 2$, and $\{P_y^m\}_{y \in C}$ is a family of polynomials of degree up to m, then a necessary condition for the existence of a convex function F of class $C^m(\mathbb{R}^n)$ with $J_y^m F = P_y^m$ for every $y \in C$ is that

1. $\{P_y^m\}_{y \in C}$ satisfies (W^m) on C and (CW^m) on ∂C and $D^2 P_y^m(y)(v^2) \geq 0$ for every $y \in \text{int}(C)$ and every $v \in \mathbb{S}^{n-1}$; or equivalently:

2. $\{P_y^m\}_{y \in C}$ satisfies (W^m) on C and (CW^m) on ∂C and the function $C \ni y \mapsto P_y^m(y)$ is convex.

Remark 5.13. One might wonder whether the conditions (CW^m) could be deduced from the condition $D^2 f \geq 0$ on C, at least in the case that C has nonempty interior. The answer is negative: the function f defined in (5.2.1) satisfies $D^2 f \geq 0$ on the ball $C = \overline{B}(0, 1/3)$ but f does not satisfy condition (CW^3).

Proof. The function f is defined on $C = \overline{B}(0, 1/3)$ by $f(x) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_1^3$. The second derivative of f is

$$D^2 f(x) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(x) e_i^* \otimes e_i^* = (2 - 6x_1) e_1^* \otimes e_1^* + \sum_{i=2}^n 2 e_i^* \otimes e_i^*,$$

where each e_i^* denotes the linear function $(x_1, \ldots, x_n) \mapsto x_i$. We have, for every $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ with $|v| = 1$, that

$$D^2 f(x)(v^2) = (2 - 6x_1)v_1^2 + \sum_{i=2}^{n} 2v_i^2 \geq 0 \quad \text{for every} \quad x \in C.$$

The third derivative of f is given by the expression

$$D^3 f(x) = -6\, e_1^* \otimes e_1^* \otimes e_1^*,$$

and then, for every $v = (v_1, \ldots, v_n),\ w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ with $|v| = |w| = 1$,

$$D^3 f(x)(w, v^2) = -6v_1^2 w_1, \quad x \in C.$$

We thus have, for every $t > 0, y = (y_1, \ldots, y_n) \in \partial C, v, w \in \mathbb{R}^n$ with $|v| = |w| = 1$, that

$$\frac{D^2 f(y)(v^3) + tD^3 f(y)(w, v^2)}{t} = \frac{(2 - 6y_1)v_1^2 + \sum_{i=2}^{n} 2v_i^2 - 6tv_1^2 w_1}{t}.$$

If we take $y = \frac{1}{3}e_1$ and $v = w = e_1$, the above expression is equal to -6 for every $t > 0$. In particular, f does not satisfy condition (CW^3) on ∂C. $\square$

5.4 C^m extensions from compact convex subsets

The best we have been able to obtain for general compact convex subsets if the following.

Theorem 5.14. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}$ with $m \geq n + 3$, and let $\{P_y^m\}_{y \in C}$ be a family of polynomials of degree less than or equal to m. Assume that $\{P_y^m\}_{y \in C}$ satisfies conditions (W^m) and (CW^m) on C. Then there exists a convex function $F \in C^{m-n-1}(\mathbb{R}^n)$ such that $J_y^{m-n-1}F = P_y^{m-n-1}$ for every $y \in C$, where each P_y^{m-n-1} denotes the polynomial obtained from P_y^m by discarding its homogeneous terms of degree greater than $m - n - 1$.*

On the other hand, if C has nonempty interior and $\{P_y^m\}_{y \in C}$ satisfies conditions (W^m) on C and (CW^m) on ∂C, and the function $C \ni y \mapsto P_y^m(y)$ is convex, then there exists a $C^{m-n-1}(\mathbb{R}^n)$ convex function F, with $J_y^{m-n-1}F = P_y^{m-n-1}$ for every $y \in C$.

The above result is probably not optimal, at least in the case when C has nonempty interior. If C has empty interior then conditions (CW^m) and (W^m) are not sufficient for a family $\{P_y^m\}_{y \in C}$ of polynomials of degree less than or equal to m to have a C^m convex extension to $\mathbb{R}^n$, as we will see in Example 5.35 in Section 5.9 below. In fact, this example shows that one cannot expect to find smooth convex extensions of jets satisfying (W^m) and (CW^m) on C without losing at least two orders of smoothness. However, it is conceivable that these conditions (CW^m) might be sufficient in the case when C has nonempty interior. In Chapter 6, we will prove an optimal result for functions of class C^∞.

Throughout the rest of this section we will give the proof of Theorem 5.14. Since the family $\{P_y^m\}_{y \in C}$ satisfies Whitney's condition (W^m) on C, we may assume that there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ of class C^m such that $J_y^m f = P_y^m$ for every $y \in C$ and f satisfies condition (CW^m) on C in the sense of Definition 5.7. Observe that, in view of Remark 5.10, we have that $D^2 f(x)(v^2) \geq 0$ for every $x \in C$ and every $v \in \mathbb{S}^{n-1}$. On the other hand, if C has nonempty interior and we assume that the family $\{P_y^m\}_{y \in C}$ satisfies condition (CW^m) only on ∂C and also that the function $C \ni y \mapsto P_y^m(y)$ is convex, then the function f is convex on C, which implies that $D^2 f(x)(v^2) \geq 0$ for every $x \in C$ and every $v \in \mathbb{S}^{n-1}$ because C has nonempy interior. This indicates that with either of the two conditions of Theorem 5.14 (for arbitrary compact convex sets or for compact convex bodies), the function f satisfies

$$f \quad \text{satisfies} \quad (CW^m) \text{ on } \partial C \quad \text{and} \quad D^2 f(x)(v^2) \geq 0, \quad x \in C,\ v \in \mathbb{S}^{n-1}. \tag{5.4.1}$$

Moreover, multiplying f by a suitable bump function we can also assume that f has support contained on $C + \overline{B}(0, 2)$. We will split the proof into several subsections.

5.4.1 Idea of the proof.

Let us give a rough sketch of the proof of Theorem 5.14 so as to guide the reader through the inevitable technicalities. This proof has two main parts. In the first part we will estimate the possible lack of convexity of f outside C by using conditions (CW^m), Lemma 5.15. In fact, we will construct a nondecreasing continuous function $\omega : \mathbb{R} \to [0, +\infty)$ such that $\omega \geq 0$, $\omega^{-1}(0) = (-\infty, 0]$, and $\min_{|v|=1} D^2 f(x)(v^2) \geq -\omega(d(x, C))d(x, C)^{m-2}$ for every $x \in \mathbb{R}^n$. In the second part of the proof we will compensate the lack of convexity of f outside C with the construction of a C^2 function $\psi(\mathbb{R}^n)$ such that $\psi \geq 0$, $\psi^{-1}(0) = C$, and $\min_{|v|=1} D^2\psi(x)(v^2) \geq 2\omega(d(x, C))d(x, C)^{m-2}$ on $\mathbb{R}^n$. Then, by setting $F := f + \psi$ we will conclude the proof of Theorem 5.14. We will see that the highest order of differentiability we can obtain for the function ψ is $m - n - 1$. However, we will use a similar plan for the proof of Theorem 5.27 (and for the proof of Theorem 6.11 of Chapter 6), and we will see that, in these cases, we can obtain smoothness of order $m - 1$ for the corresponding function ψ.

There are many ways to construct such a function ψ. For an arbitrary compact convex subset, the essential point is to write C as an intersection of a family of half-spaces, and then to make a weighted sum, or an integral, of suitable convex functions composed with the linear forms that provide those half-spaces. If the sequence of linear forms is appropiately distributed, in the weighted sum approach, or if one uses a measure equivalent to the standard measure on $\mathbb{S}^{n-1}$, in the integral approach, then the different functions ψ produced by these methods will have equivalent convexity properties. See [2] for an instance of the weighted sum approach, and [45, Proposition 2.1] for the integral approach. Of course our situation is more complicated than that of these references, as we need to find quantitative estimations of the convexity of ψ outside C which are good enough to outweigh our previous estimations of the lack of convexity of f outside C. It turns out that, in the present C^m case, this goal can be achieved with either method of construction of ψ. Here we will follow the integral approach of Ghomi's in [45, Proposition 2.1], as it will lead us to easier calculations.

5.4.2 The function ω

Lemma 5.15. *There exists a non decreasing continuous function $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ such that*

$$D^2 f(x)(v^2) \geq -\omega(d(x, C))d(x, C)^{m-2} \quad \textit{for all} \quad x \in \mathbb{R}^n,\ v \in \mathbb{S}^{n-1}.$$

Proof. Let us denote

$$Q_m(t, y, v, w) = \frac{D^2 f(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2)}{t^{m-2}}$$

for all $t > 0, y \in \partial C,\ v, w \in \mathbb{S}^{n-1}$ and

$$\varepsilon_m(t) = \sup_{\{z \in \mathbb{R}^n,\, z' \in \partial C,\, |z - z'| \leq t\}} \|D^m f(z) - D^m f(z')\|.$$

By using condition (CW^m) and uniform continuity of $D^m f$, given a positive integer p, there exists $r_p > 0$ such that

$$Q_m(t, y, v, w) \geq -\frac{1}{2p} \quad \text{and} \quad \varepsilon_m(t) \leq \frac{1}{2p} \tag{5.4.2}$$

for every $y \in \partial C,\ v, w \in \mathbb{S}^{n-1}$ and $0 < t \leq r_p$. We may suppose that this sequence $\{r_p\}_{p \geq 1}$ is strictly decreasing to 0. Since the derivatives of f up to order m are bounded on $\mathbb{R}^n$ we can find a constant $M > 1$ such that

$$\varepsilon_m(t) - Q_m(t, y, v, w) \leq M \quad \text{for all} \quad y \in \partial C, v, w \in \mathbb{S}^{n-1}, t \geq r_1. \tag{5.4.3}$$

Now, given $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$, we denote by $y \in \partial C$ the metric projection of x onto C, $w = (x - y)/|x - y|$ and $t = d(x, C)$. By Taylor's theorem and the definition of Q_m and ε_m, we have

$$D^2 f(x)(v^2) \geq t^{m-2} Q_m(t, y, v, w) - t^{m-2}\varepsilon_m(t) = -t^{m-2}\left(\varepsilon_m(t) - Q_m(t, y, v, w)\right).$$

We define $\omega : [0, +\infty) \to [0, +\infty)$ by setting

$$\omega(0) = 0,\ \omega(r_p) = \frac{1}{p-1} \quad p \geq 2, \quad \omega(r_1) = M,$$
$$\omega \quad \text{affine on each} \quad [r_{p+1}, r_p] \quad p \geq 1, \quad \omega(t) = M \quad t \geq r_1.$$

Since the sequence $\{r_p\}_{p\geq 1}$ is strictly decreasing to 0, it is clear that ω is a non decreasing continuous function such that $\omega(t) \geq \frac{1}{p}$ for every $t \geq r_{p+1}$ and every $p \geq 2$, and that $\omega(t) \geq 1$ for every $t \geq r_2$. Using inequalities (5.4.2) and (5.4.3) we deduce that

$$D^2 f(x)(v^2) \geq -M t^{m-2} \quad \text{for} \quad t \geq r_1$$
$$D^2 f(x)(v^2) \geq -\frac{1}{p} t^{m-2} \quad \text{for} \quad t \leq r_p, \quad p \in \mathbb{N}$$

and by the properties of ω we conclude

$$D^2 f(x)(v^2) \geq -\omega(d(x, C)) d(x, C)^{m-2} \quad \text{for every} \quad x \in \mathbb{R}^n \setminus C,\ v \in \mathbb{S}^{n-1},$$

where the above inequality trivially extends to $x \in C$ thanks to (5.4.1). $\square$

5.4.3 The function φ

Using the function ω defined in Lemma 5.15, we introduce two new functions

$$g(t) = \begin{cases} \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-n-1}} \omega(2^{m-n-2}s) ds\, dt_{m-n-1} \cdots dt_2 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases} \tag{5.4.4}$$

Since ω is continuous, the function g is of class $C^{m-n-1}(\mathbb{R})$ with $g^{(k)}(0) = 0$ for every $1 \leq k \leq m-n-1$. In addition, $g^{-1}(0) = (-\infty, 0]$ and $g''(t) = \tilde{\varepsilon}(t) > 0$ for all $t > 0$. In particular, g is convex on $\mathbb{R}$ and positive, with a strictly positive second derivative, on $(0, +\infty)$.

Now, for every vector $w \in \mathbb{S}^{n-1}$, define $h(w) = \max_{z\in C}\langle z, w\rangle$, the support function of C (for information about support functions of convex sets, see [57] for instance). Since C is compact, it is clear that the function h is Lipschitz on $\mathbb{S}^{n-1}$ with Lipschitz constant equal to $\operatorname{diam}(C)$ and then, in particular, h is continuous. We also define the function

$$\begin{array}{rccl} \phi: & \mathbb{S}^{n-1} \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (w, x) & \longmapsto & \phi(w, x) = g(\langle x, w\rangle - h(w)). \end{array}$$

For every $w = (w_1, \ldots, w_n) \in \mathbb{S}^{n-1}$, the function $\phi(w, \cdot)$ is of class $C^{m-n-1}(\mathbb{R}^n)$ because so is g, and for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq m-n-1$, we have

$$\frac{\partial^\alpha}{\partial x^\alpha}\phi(w, x) = g^{(|\alpha|)}(\langle x, w\rangle - h(w)) w^\alpha,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. In addition, we note that when $x \in C$, we have $\langle x, w\rangle \leq h(w)$ for every $w \in \mathbb{S}^{n-1}$. Therefore, the properties of g and its derivatives imply that $\phi(w, \cdot)$ is a function of class $C^{m-n-1}(\mathbb{R}^n)$ whose derivatives of order less than or equal to $m-n-1$ and $\phi(w, \cdot)$ itself vanish on C for every $w \in \mathbb{S}^{n-1}$. Moreover, the function $\phi(w, \cdot)$, being a composition of a convex function with a non-decreasing convex function, is convex as well.
Finally, we define the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} \phi(w, x)\, dw \quad \text{for every} \quad x \in \mathbb{R}^n.$$

Here the integral is taken with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on the sphere $\mathbb{S}^{n-1}$. For information about integration with respect to Hausdorff measures, see [35] for instance. In the case $n = 1$, recall that $\mathcal{H}^{n-1} = \mathcal{H}^0$ is the counting measure on $\mathbb{S}^0 = \{-1, 1\}$ and then $\varphi = \phi(-1, \cdot) + \phi(1, \cdot)$. Anyhow, in Section 5.6 below we will see that, in the one dimensional case, the statement and the proof of Theorem 5.14 can be very much improved and simplified and we will not need to deal with these functions ϕ and φ, see Proposition 5.25. By the properties of ϕ, we see that $\varphi = 0$ on C and φ is convex on $\mathbb{R}^n$. Because $\phi(w, \cdot)$ is of class $C^{m-n-1}(\mathbb{R}^n)$, the derivatives $(w, x) \mapsto \frac{\partial^\alpha}{\partial x^\alpha}\phi(w, x)$ are continuous for every multi-index α with $|\alpha| \leq m - n - 1$, and $\mathbb{S}^{n-1}$ is compact, it follows from standard results on differentiation under the integral sign that the function φ is of class $C^{m-n-1}(\mathbb{R}^n)$ as well and that $\partial^\alpha \varphi(x) = 0$ for every $x \in C$ and every multi-index α with $|\alpha| \leq m - n - 1$. In other words, $J_x^{m-n-1}\varphi = 0$ for all $x \in C$. The second derivative of φ is

$$D^2\varphi(x)(v^2) = \int_{\mathbb{S}^{n-1}} g''(\langle x, w\rangle - h(w))\langle w, v\rangle^2 \, dw, \quad x \in \mathbb{R}^n, \ v \in \mathbb{S}^{n-1}. \tag{5.4.5}$$

5.4.4 Selection of angles and directions

For given $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$ we will now find a region $W = W(x, v)$ of $\mathbb{S}^{n-1}$ of sufficient volume (depending only, and conveniently, on $d(x, C)$) on which we have good lower estimates for $g''(\langle x, w\rangle - h(w))\langle w, v\rangle^2$. This will involve a careful selection of angles and directions.

Fix a point $x \in \mathbb{R}^n \setminus C$, let x_C be the metric projection of x onto the compact convex C, and set

$$u_x = \frac{1}{|x - x_C|}(x - x_C),$$

and

$$\alpha_x = \frac{d(x, C)}{d(x, C) + \operatorname{diam}(C)}.$$

Lemma 5.16. *With the above notation we have $\langle x, u_x\rangle - h(u_x) = d(x, C)$ and*

$$d(x, C) \geq \langle x, w\rangle - h(w) \geq \frac{1}{2} d(x, C)$$

for all $w \in \mathbb{S}^{n-1}$ such that $\widehat{w\, u_x} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$.

Here $\widehat{w\, u_x}$ denotes the length of the shortest geodesic (or angle) between w and u_x in $\mathbb{S}^{n-1}$.

Proof. Let us check that $\langle x, u_x\rangle - h(u_x) = d(x, C)$. Suppose that there exist $z \in C$ and $\eta \in \mathbb{R}$ with $\langle x - z, u_x\rangle \leq |x - x_C| - \eta$ and denote $z_t = x_C + t(z - x_C)$ for all $t \in \mathbb{R}$. We immediately see that $z_t \in C$ whenever $t \in [0, 1]$. Also, if we define the function $f(t) = |z_t - x|^2$ for $t \in \mathbb{R}$ it is obvious that $f \in C^\infty(\mathbb{R})$. Moreover, we can write

$$f(t) = \langle x_C - x + t(z - x_C), x_C - x + t(z - x_C)\rangle = t^2|z - x_C|^2 + 2t\langle x_C - x, z - x_C\rangle + |x_C - x|^2.$$

We see from this that $f'(t) = 2t|z - x_C|^2 + 2\langle x_C - x, z - x_C\rangle$ and, in particular,

$$\begin{aligned} f'(0) &= 2\langle x_C - x, z - x_C\rangle = 2(\langle x_C - x, z - x\rangle + \langle x_C - x, x - x_C\rangle) \\ &= -2|x_C - x| \left(\langle u_x, z - x\rangle + |x_C - x|\right) \leq -2\eta|x - x_C|^2 < 0. \end{aligned}$$

Thus there exists $\varepsilon \in (0, 1)$ such that

$$|z_t - x|^2 = f(t) < f(0) = |x - x_C|^2 \quad \text{for all} \quad t \in (0, \varepsilon),$$

and this contradicts the fact that x_C is a point of C which minimizes the distance of C to the point z. For the second part, given $w \in \mathbb{S}^{n-1}$ with $\widehat{w\, u_x} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$, let us denote $\theta = \widehat{w\, u_x}$. Since C is compact, we can find $\xi \in C$ such that $h(w) = \langle \xi, w\rangle$. Using that $\langle x, u_x\rangle - h(u_x) = |x - x_C|$ and $|w - u_x| \le \theta$, we have

$$\begin{aligned}
\langle x, w\rangle - h(w) &= \langle x, w - u_x\rangle + |x - x_C| + h(u_x) - h(w)\\
&\ge \langle x, w - u_x\rangle + |x - x_C| + \langle \xi, u_x - w\rangle\\
&= \langle x - \xi, w - u_x\rangle + |x - x_C|\\
&\ge -\left(\operatorname{diam}(C) + |x - x_C|\right)\theta + |x - x_C|\\
&\ge -\left(\operatorname{diam}(C) + |x - x_C|\right)\frac{\alpha_x}{2} + |x - x_C|\\
&= \frac{1}{2}|x - x_C|.
\end{aligned}$$

The other inequality, $d(x, C) \ge \langle x, w\rangle - h(w)$ follows from

$$\langle x, w\rangle - h(w) = \langle x, w\rangle - \sup_{z\in C}\langle z, w\rangle \le \langle x - x_C, w\rangle \le |x - x_C| = d(x, C).$$

□

The region $W \subset \mathbb{S}^{n-1}$ that we need will be a hyperspherical cap on the sphere $\mathbb{S}^{n-1}$, that is, the portion of the sphere between two paralell hyperplanes. An hyperspherical can be also seen as the set of points w in the sphere $\mathbb{S}^{n-1}$ such that w forms an angle between β_1 and β_2 with a given point $w_0 \in \mathbb{S}^{n-1}$, where $0 \le \beta_1 < \beta_2 \le \pi$. The following proposition gives us an explicit value for the Hausdorff measure $\mathcal{H}^{n-1}$ of hyperspherical caps on $\mathbb{S}^{n-1}$.

Proposition 5.17. *Let $0 < \beta < \frac{\pi}{2}$ and $w_0 \in \mathbb{S}^{n-1}$, where $n \ge 2$. The Hausdorff measure $\mathcal{H}^{n-1}$ on the sphere $\mathbb{S}^{n-1}$ of the hyperspherical cap $A = \left\{w \in \mathbb{S}^{n-1} \,:\, \widehat{w\,w_0} \in [0, \beta]\right\}$, is*

$$\mathcal{H}^{n-1}(A) = \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \int_0^\beta \sin^{n-2}(t)dt,$$

where $\mathcal{H}^{n-2}$ denotes the $(n-2)$-dimensional Hausdorff measure on the sphere $\mathbb{S}^{n-2}$ of $\mathbb{R}^{n-1}$.

Proof. If $d \ge 2$ is an integer, from [35, Chapter 3, pg. 250] we deduce that, for every $\mathcal{H}^{d-1}$-measurable subset B of $\mathbb{S}^{d-1}$,

$$\mathcal{L}^d(B^*) = \int_0^{+\infty} r^{d-1}\int_{\mathbb{S}^{d-1}} \chi_{B^*}(ru)\, d\mathcal{H}^{d-1}(u)dr = \int_0^{+\infty} r^{d-1}dr \int_{\mathbb{S}^{d-1}} \chi_B(w)\, d\mathcal{H}^{d-1}(w) = \frac{1}{d}\mathcal{H}^{d-1}(B), \tag{5.4.6}$$

where $B^* = \{tu \,:\, t \in [0, 1],\, u \in B\}$, χ_B and χ_{B^*} denote the characteristic functions of B and B^* respectively and $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. If we set $B = \mathbb{S}^{d-1}$, then B^* is the closed unit ball $B_d(0, 1)$ of $\mathbb{R}^d$ and (5.4.6) gives $\mathcal{H}^{d-1}(\mathbb{S}^{d-1}) = d\mathcal{L}^d(B_d(0, 1))$. We consider the standard hyperspherical coordinates on $\mathbb{R}^d$. That is, we set $U_d = (0, +\infty) \times Q_{d-1}$, where $Q_{d-1} = (0, \pi)^{d-2} \times (-\pi, \pi)$ and $\Psi_d = (\Psi_d^1, \ldots, \Psi_d^d) : U_d \to \mathbb{R}$, where

$$\left\{\begin{array}{lcl}
\Psi_d^1(r, \phi) & = & r\cos\phi_1\\
\Psi_d^2(r, \phi) & = & r\sin\phi_1\cos\phi_2\\
\quad\vdots & & \\
\Psi_d^{d-1}(r, \phi) & = & r\sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2}\cos\phi_{d-1}\\
\Psi_d^d(r, \phi) & = & r\sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2}\sin\phi_{d-1},
\end{array}\right. \tag{5.4.7}$$

for every $(r, \phi) = (r, \phi_1, \ldots, \phi_{d-1}) \in U_d$. The image of U_d is $\Psi_d(U_d) = \mathbb{R}^d \setminus \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \,:\, x_d = 0,\, x_{d-1} \le 0\}$ and $\Psi_d : U_d \to \Psi(U_d)$ is a dipheomorphism of class C^∞ with jacobian J_{Ψ_d} equal to

$$J_{\Psi_d}(r, \phi) = r^{d-1}J_{d-1}^*(\phi), \quad J_{d-1}^*(\phi) = \prod_{j=1}^{d-2}\sin^{d-j-1}(\phi_j), \quad \text{for every} \quad (r, \phi) \in U_d. \tag{5.4.8}$$

Since the set $\{x=(x_1,\ldots,x_d)\in\mathbb{R}^d \ : \ x_d=0, \ x_{d-1}\leq 0\}$ has $\mathcal{L}^d$ measure equal to zero, we can write

$$\mathcal{L}^d(B_d(0,1))=\int_{\Psi_d^{-1}(B_d(0,1))}|J_{\Psi_d}(r,\phi)|drd\phi=\int_0^1 r^{d-1}dr\int_{Q_{d-1}}J^*_{d-1}(\phi)d\phi=\frac{1}{d}\int_{Q_{d-1}}J^*_{d-1}(\phi)d\phi,$$

where the last integral is taken with respect to the Lebesgue measure $\mathcal{L}^{d-1}$ on the cube Q_{d-1}. We thus have

$$\mathcal{H}^{d-1}(\mathbb{S}^{d-1})=d\mathcal{L}^d(B_d(0,1))=\int_{Q_{d-1}}J^*_{d-1}(\phi)d\phi, \quad \text{for every} \quad d\geq 2. \tag{5.4.9}$$

Now we use (5.4.6) for $d=n$, B equal to our hyperspherical cap A and $B^*=A^*=\{tw : t\in[0,1], w\in A\}$ to obtain $\mathcal{H}^{n-1}(A)=n\mathcal{L}^n(A^*)$. Since the Lebesgue measure is invariant under isometries, we may and do assume that $A^*=\{tw \ : \ t\in[0,1], \ \widehat{w\,e_1}\in[0,\beta]\}$, where $e_1=(1,0,0,\ldots)$. Bearing in mind the parametrization of (5.4.7), observe that a point (r,ϕ) of U_n belongs to $\Psi_n^{-1}(A^*)$ if and only if $(r,\phi_2,\ldots,\phi_{n-1})\in[0,1]\times Q_{d-2}$ and $\phi_1\in[0,\beta]$. Then (5.4.8) gives

$$\begin{aligned}
\mathcal{H}^{n-1}(A)=n\mathcal{L}^n(A^*)&=n\int_{\Psi_n^{-1}(A^*)}|J_{\Psi_n}(r,\phi)|drd\phi=n\int_{\Psi_n^{-1}(A^*)}r^{n-1}J^*_{d-1}(\phi)drd\phi\\
&=n\int_{\Psi_n^{-1}(A^*)}r^{n-1}J^*_{d-2}(\phi_2,\ldots,\phi_{d-1})\sin(\phi_1)^{n-2}drd\phi\\
&=\int_0^1 nr^{n-1}dr\int_0^\beta\sin^{n-2}(\phi_1)d\phi_1\int_{Q_{d-2}}J^*_{d-2}(\phi_2,\ldots,\phi_{d-1})d\phi_2\ldots d\phi_{n-1}\\
&=\int_0^\beta\sin^{n-2}(t)dt\int_{Q_{d-2}}J^*_{d-2}=\mathcal{H}^{n-2}(\mathbb{S}^{n-2})\int_0^\beta\sin^{n-2}(t)dt,
\end{aligned}$$

where the last equation follows from (5.4.9). This proves the assertion. $\square$

Finally we construct the desired region W. See Figure 5.1 below for a picture on $\mathbb{R}^2$.

Lemma 5.18. *Given any $v\in\mathbb{S}^{n-1}$ with $\langle u_x,v\rangle\geq 0$, there exists a vector $w_0=w_0(x,v)\in\mathbb{S}^{n-1}$ such that if we define*

$$W=W_{x,v}:=\left\{w\in\mathbb{S}^{n-1} \ : \ \widehat{w\,w_0}\in\left[0,\frac{\alpha_x}{12}\right]\right\},$$

then:

(1) *For every $w\in W$, we have $\widehat{u_x\,w}\in\left[\frac{\alpha_x}{3},\frac{\alpha_x}{2}\right]$.*

(2) *For every $w\in W$, we have $\langle w,v\rangle\geq\sin(\frac{\alpha_x}{3})$.*

(3) *$\mathcal{H}^{n-1}(W)\geq V(n)\alpha_x^{n-1}$, where $V(n)>0$ is a constant depending only on the dimension n.*

Proof. We prove (1) and (2) at the same time by studying two cases separately.
Case 1. $u_x\neq v$. Take an w_0 in the unit circle of the plane spanned by the vectors u_x and v, in such a way that $\widehat{w_0\,u_x}=\frac{5\alpha_x}{12}$, and that the arc in that circle joining u_x with w_0 has the same orientation as the arc joining u_x with v. Set $W=\{w\in\mathbb{S}^{n-1} \ : \ \widehat{w\,w_0}\in[0,\frac{\alpha_x}{12}]\}$ and let $w\in W$.
First, recalling that the angles shorter than π give the usual distance between points of $\mathbb{S}^{n-1}$, we may use the triangle inequality to estimate

$$\widehat{u_x\,w}\leq\widehat{u_x\,w_0}+\widehat{w_0\,w}\leq\frac{5\alpha_x}{12}+\frac{\alpha_x}{12}=\frac{\alpha_x}{2}$$

and

$$\widehat{u_x\,w}\geq\widehat{u_x\,w_0}-\widehat{w_0\,w}\geq\frac{5\alpha_x}{12}-\frac{\alpha_x}{12}=\frac{\alpha_x}{3},$$

that is $\widehat{u_x\, w} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$. It only remains to see that $\langle w, v\rangle \geq \sin(\alpha_x/3)$ for all $w \in W$. First, it is clear that $\widehat{v\, w_0} \leq \frac{\pi}{2} - \frac{5\alpha_x}{12}$. Now, for an arbitrary $w \in W$, we have

$$\widehat{v\, w} \leq \widehat{v\, w_0} + \widehat{w_0\, w} \leq \frac{\pi}{2} - \frac{5\alpha_x}{12} + \frac{\alpha_x}{12} = \frac{\pi}{2} - \frac{\alpha_x}{3}.$$

Therefore $\langle v, w\rangle = \cos(\widehat{v\, w}) \geq \cos(\frac{\pi}{2} - \frac{\alpha_x}{3}) = \sin \frac{\alpha_x}{3}$.
Case 2. $u_x = v$. Take w_0 in the sphere $\mathbb{S}^{n-1}$ such that $\widehat{w_0\, u_x} = \frac{5\alpha_x}{12}$. If we define

$$W = \left\{ w \in \mathbb{S}^{n-1} \ : \ \widehat{w\, w_0} \in \left[0, \frac{\alpha_x}{12}\right] \right\},$$

following the same estimations as in Case 1 we obtain $\widehat{u_x\, w} \in [\frac{\alpha_x}{3}, \frac{\alpha_x}{2}]$ for every $w \in W$. And we conclude that $\langle w, v\rangle = \langle w, u_x\rangle \geq \sin \frac{\alpha_x}{3}$.
Let us now prove (3). Note that for those angles β such that $0 \leq \beta \leq \frac{\alpha_x}{12} \leq \frac{\pi}{3}$, it is clear that $\sin \beta \geq \frac{1}{2}\beta$. Thanks to Proposition 5.17 we obtain

$$\begin{aligned}\mathcal{H}^{n-1}(W) &= \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \int_0^{\alpha_x/12} \sin^{n-2}(\beta) d\beta \geq \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \int_0^{\alpha_x/12} \left(\tfrac{1}{2}\beta\right)^{n-2} d\beta \\ &= \frac{\alpha_x^{n-1}\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{(12)^{n-1}(n-1)2^{n-2}} = \frac{\alpha_x^{n-1}\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{12(n-1)(24)^{n-2}} = V(n)\alpha_x^{n-1},\end{aligned}$$

where

$$V(n) = \frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{12(n-1)(24)^{n-2}}$$

for every $n \geq 2$. This proves (3). $\square$

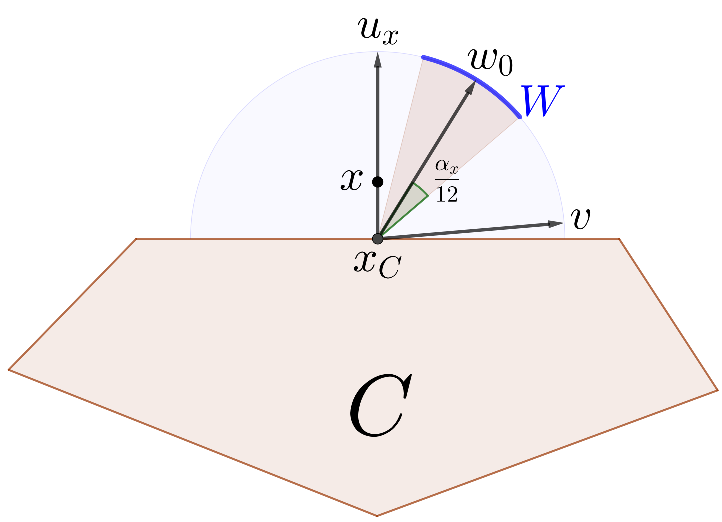

Figure 5.1: The spherical cap $W = W(x, v)$ of Lemma 5.18 in two dimensions.

5.4.5 A convex extension on a neighbourhood of the domain

Let us denote

$$k(n, m, C) = \frac{V(n)}{36 \cdot 2^{2+3+\cdots+(m-n-2)}(1 + \mathrm{diam}(C))^{n+1}},$$

where $V(n)$ is the constant of Lemma 5.18.

Lemma 5.19. *Consider the function $H = f + \frac{2}{k(n,m,C)}\varphi$ defined on $\mathbb{R}^n$. Then, for every $x \in \mathbb{R}^n$ such that $d(x, C) \leq 1$, and for every $v \in \mathbb{S}^{n-1}$, we have*

$$D^2H(x)(v^2) \geq 0$$

with strict inequality whenever $0 < d(x, C) \leq 1$. Also, the function H is of class $C^{m-n-1}(\mathbb{R}^n)$ with $H = f$ on C and $J_y^{m-n-1}H = P_y^{m-n-1}$ for all $y \in C$.

Proof. If $x \in C$ there is nothing to prove because φ is convex on $\mathbb{R}^n$ and $D^2f(x) \geq 0$ for every $v \in \mathbb{S}^{n-1}$ by (5.4.1). We now claim that

$$D^2\varphi(x)(v^2) > 0 \quad \text{for every} \quad x \in \mathbb{R}^n \setminus C,\ v \in \mathbb{S}^{n-1}. \tag{5.4.10}$$

Indeed, if $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$ is a direction then Lemma 5.16 provides an open subset U_x of $\mathbb{S}^{n-1}$ such that $\langle x, w\rangle - h(w) > 0$ for every $w \in U_x$. Obviously, there is an open subset V_x of U_x for which $\langle v, w\rangle \neq 0$ for every $w \in U_x$. It follows by the properties of the function g and equation (5.4.5) that

$$D^2\varphi(x)(v^2) \geq \int_{V_x} g''\left(\langle x, w\rangle - h(w)\right)\langle v, w\rangle^2 dw > 0,$$

which proves our Claim. Once we have checked this, suppose that $x \in \mathbb{R}^n \setminus C$ with $d(x, C) \leq 1$ and denote

$$t := d(x, C).$$

Fix also a direction $v \in \mathbb{S}^{n-1}$. Since $D^2H(x)(v^2) = D^2H(x)((-v)^2)$, we may suppose that $\langle v, u_x\rangle \geq 0$, where $u_x = (x - x_C)/|x - x_C|$ and x_C is the metric projection of x onto C. Let us consider the angle $\alpha = \alpha_x$ and the subset $W = W_{x,v}$ of $\mathbb{S}^{n-1}$ as in Lemmas 5.16 and 5.18 respectively. By Lemma 5.18 (2) we know that $\langle v, w\rangle \geq \sin\left(\frac{\alpha}{3}\right)$ whenever $w \in W$. It then follows from the identity (5.4.5) that

$$D^2\varphi(x)(v^2) \geq \int_W g''(\langle x, w\rangle - h(w))\sin^2\left(\frac{\alpha}{3}\right) dw.$$

Since $t \leq 1$, the angle α satisfies

$$\alpha = \frac{t}{t + \operatorname{diam}(C)} \geq \frac{t}{1 + \operatorname{diam}(C)}.$$

For any $w \in W$, Lemma 5.18 (1) gives that $\widehat{u_x\, w} \in \left[\frac{\alpha}{3}, \frac{\alpha}{2}\right]$ and; on the other hand, Lemma 5.16 says that, in this case,

$$\frac{t}{2} \leq \langle x, w\rangle - h(w) \leq t$$

Because g'' is non decreasing, we have that

$$g''(\langle x, w\rangle - h(w)) \geq g''\left(\frac{t}{2}\right) \quad \text{for all} \quad w \in W.$$

These estimations lead us to

$$D^2\varphi(x)(v^2) \geq \mathcal{H}^{n-1}(W)g''\left(\frac{t}{2}\right)\sin^2\left(\frac{\alpha}{3}\right).$$

Note that Lemma 5.18 (3) also shows that there exists a positive constant $V(n)$ only depending on n such that $\mathcal{H}^{n-1}(W) \geq V(n)\alpha^{n-1}$. Because $\alpha \leq 1$, we must have $\sin^2\left(\frac{\alpha}{3}\right) \geq \frac{\alpha^2}{36}$ and then the Hessian of φ at x on the direction v satisfies

$$D^2\varphi(x)(v^2) \geq \frac{V(n)}{36}\left(\frac{t}{1 + \operatorname{diam}(C)}\right)^{n+1} g''\left(\frac{t}{2}\right). \tag{5.4.11}$$

We next give a lower bound for $g''(t/2)$. By the construction of g we have

$$g''(t/2) = \int_0^{t/2}\int_0^{t_2}\cdots\int_0^{t_{m-n-3}} \omega(2^{m-n-2}s)ds\, dt_{m-n-3}\cdots dt_2,$$

where, in the special case $m = n+3$, the above expression means $g''(t/2) = \omega(t)$. Using that ω is nonnegative and nondecreasing we may estimate:

$$\begin{aligned} g''(t/2) &\geq \int_{t/4}^{t/2}\int_0^{t_2}\cdots\int_0^{t_{m-n-3}} \omega(2^{m-n-2}s)ds\, dt_{m-n-3}\cdots dt_2 \\ &\geq \frac{t}{4}\int_0^{t/4}\int_0^{t_2}\cdots\int_0^{t_{m-n-4}} \omega(2^{m-n-2}s)ds\, dt_{m-n-4}\cdots dt_2 \\ &\geq \frac{t}{4}\cdot\frac{t}{8}\int_0^{t/8}\int_0^{t_2}\cdots\int_0^{t_{m-n-5}} \omega(2^{m-n-2}s)ds\, dt_{m-n-5}\cdots dt_2 \\ &\geq \frac{t}{4}\cdot\frac{t}{8}\cdots\frac{t}{2^{m-n-3}}\cdot\frac{t}{2^{m-n-2}}\omega(t) = \frac{t^{m-n-3}}{2^{2+3+\cdots+(m-n-2)}}\omega(t). \end{aligned}$$

By plugging this estimation in (5.4.11), we obtain that

$$D^2\varphi(x)(v^2) \geq k(n,m,C)t^{m-2}\omega(t) > 0.$$

On the other hand, Lemma 5.15 implies that

$$D^2 f(x)(v^2) \geq -\omega(t)t^{m-2}.$$

Therefore, the function $H = f + \frac{2}{k(n,m,C)}\varphi$ satisfies $D^2H(x)(v^2) \geq 0$ on the neighbourhood $\{x \in \mathbb{R}^n : d(x,C) \leq 1\}$ of C, with strict inequality whenever $0 < d(x,C) \leq 1$. Finally, since $\varphi \in C^{m-n-1}(\mathbb{R}^n)$ with $J_y^{m-n-1}\varphi = 0$ for every $y \in C$, we also have that the function H is of class $C^{m-n-1}(\mathbb{R}^n)$ with $H = f$ on C and $J_y^{m-n-1}H = J_y^{m-n-1}f = P_y^{m-n-1}$ for all $y \in C$. □

5.4.6 Conclusion of the proof: convexity of the extension on $\mathbb{R}^n$.

To complete the proof of Theorem 5.14 we only have to change the funcion H of Lemma 5.19 slightly.

Lemma 5.20. *There exists a number $a > 0$ such that the function $F := f + a\varphi$ is of class $C^{m-n-1}(\mathbb{R}^n)$, concides with f on C, satisfies $J_y^{m-n-1}F = P_y^{m-n-1}$ for every $y \in C$, is convex on $\mathbb{R}^n$, and has a strictly positive Hessian on $\mathbb{R}^n \setminus C$.*

Proof. Let us denote $\psi = \frac{2}{k(n,m,C)}\varphi$, where $k(n,m,C)$ is that of Subsection 5.4.5. We recall that $f = 0$ outside $C + B(0,2)$. Since $C_1 := \{x \in \mathbb{R}^n : 1 \leq d(x,C) \leq 2\}$ is a compact subset where ψ has a strictly positive Hessian (see inequality (5.4.10)), and using again that f has compact support, we can find $M \geq 1$ such that

$$\sup_{x\in\mathbb{R}^n,\, v\in\mathbb{S}^{n-1}} |D^2 f(x)(v^2)| \leq M \quad \text{and} \quad \inf_{x\in C_1,\, v\in\mathbb{S}^{n-1}} D^2\psi(x)(v^2) \geq \frac{1}{M}. \tag{5.4.12}$$

Let us take $A = 2M^2$ and $F = f + A\psi$. If $d(x,C) \leq 1$ (this includes the situation $x \in C$) and $v \in \mathbb{S}^{n-1}$ we have, by Lemma 5.19, that

$$D^2F(x)(v^2) = 2M^2D^2\psi(x)(v^2) + D^2f(x)(v^2) \geq D^2\psi(x)(v^2) + D^2f(x)(v^2) \geq 0.$$

In the case when $d(x,C) \in [1,2]$, given any $|v| = 1$, the inequalities of (5.4.12) lead us to

$$D^2F(x)(v^2) = 2M^2D^2\psi(x)(v^2) + D^2f(x)(v^2) \geq 2M - M = M > 0.$$

Finally, in the region $\{x \in \mathbb{R}^n : d(x,C) > 2\}$, we have that $f = 0$. Hence

$$D^2F(x)(v^2) = 2M^2D^2\psi(x)(v^2) > 0$$

thanks to (5.4.10). Therefore, by setting $a = 2A/k(n,m,C)$, we get that $F = f + A\psi = f + a\varphi$ is of class $C^{m-n-1}(\mathbb{R}^n)$, satisfies $F(y) = f(y)$ and $J_y^{m-n-1}F = J_y^{m-n-1}f = P_y^{m-n-1}$ for every $y \in C$, and $D^2F(x) \geq 0$ on $\mathbb{R}^n$ with strict inequality on $\mathbb{R}^n \setminus C$. In particular, F is convex on $\mathbb{R}^n$. □

5.5 Assuming a strict inequality on the boundary

We are going to show that, in the special case when condition (CW^k) is satisfied with a strict inequality for some k, the problem becomes much easier to solve, because in this situation f must be convex on a neighbourhood of C, and then we may use the following proposition.

Proposition 5.21. *Let $m \in \mathbb{N}$. If $C \subset \mathbb{R}^n$ is compact, and if there exists an open convex neighbourhood U of C such that $f : U \to \mathbb{R}$ is C^m and convex, then there exists a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C.*

Proof. Because f is of class $C^m(U)$ and U is an open neighbourhood of C, the derivatives of f up to order m satisfy Whitney condition (W^m) on the closure of an intermediate open convex neighbourhood V of C, that is, $C \subset V \subset \overline{V} \subset U$. Thus Whitney's Extension Theorem provides an extension of class $C^m(\mathbb{R}^n)$, which we keep denoting by f, and f is convex on this open neighbourhood V of C. Furthermore, by compactness of ∂C, it is clear that we can find points $z_1, \ldots, z_N \in \partial C$ and positive numbers $r_1, \ldots, r_N$ such that

$$\partial C \subset \bigcup_{j=1}^{N} \overline{B}(z_j, r_j), \quad \bigcup_{j=1}^{N} \overline{B}(z_j, 2r_j) \subset V.$$

If we take $r = \min\{r_1, \ldots, r_N\}$, then the set $C + \overline{B}(0, r)$ is contained in V because for every $x = z + w$, where $z \in \partial C$ and $|w| \leq r$, we can find some $j \in \{1, \ldots, N\}$ with $|z - z_j| \leq r_j$, which shows that

$$|x - z_j| \leq |x - z| + |z - z_j| \leq r + r_j \leq 2r_j.$$

We thus have obtained that $C + \overline{B}(0, r)$ is a neighbourhood of C where f is convex. Also, observe that multiplying by a suitable bump function, we may and do assume that f has compact support contained in $C + \overline{B}(0, 2r)$. In a similar way to the proof of Theorem 5.14, we are going to construct a function $\varphi : \mathbb{R}^n \to [0, +\infty)$ of class C^m such that $\varphi^{-1}(0) = C$ and $D^2\varphi$ is strictly positive on $\mathbb{R}^n \setminus C$. Consider a function $\delta : \mathbb{R} \to [0, +\infty)$ of class C^∞ such that $\delta = 0$ on $(-\infty, 0]$ and $\delta > 0$ on $(0, +\infty)$. Then the function

$$g(t) = \int_0^t \int_0^s \delta(u)\, du ds, \quad t \in \mathbb{R},$$

is a C^∞ nonnegative function with $g = 0$ on $(-\infty, 0]$, $g > 0$ on $(0, +\infty)$ and $g'' = \delta > 0$ on $(0, +\infty)$. We next consider the function $h(w) = \sup_{z \in C} \langle z, w \rangle$ and define the function

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} g\left(\langle x, w \rangle - h(w)\right) dw, \quad x \in \mathbb{R}^n.$$

With the same arguments as in Subsection 5.4.3, we obtain that φ is a nonnegative function of class $C^m(\mathbb{R}^n)$ (in fact, of class $C^\infty(\mathbb{R}^n)$) with $\varphi^{-1}(0) = C$. Then second derivative of φ is

$$D^2\varphi(x)(v^2) = \int_{\mathbb{S}^{n-1}} g''\left(\langle x, w \rangle - h(w)\right) \langle v, w \rangle^2 dw, \quad x \in \mathbb{R}^n,\ v \in \mathbb{S}^{n-1}.$$

It is then clear that $D^2\varphi$ is semidefinite positive on $\mathbb{R}^n$ and, using the same calculations as in the proof of Lemma 5.19 (see inequality (5.4.10)) we obtain that $D^2\varphi$ is definite positive on $\mathbb{R}^n \setminus C$. Because f is convex on the set $C + \overline{B}(0, r)$, it is clear that the function $f + \varphi$ is convex on $C + \overline{B}(0, r)$ and has strictly positive Hessian on the set $\{x \in \mathbb{R}^n \,:\, 0 < d(x, C) \leq r\}$. With the same proof as that of Lemma 5.20 it follows that multypling φ by a positive constant $a > 0$ big enough, the function $f + a\varphi$ is convex and of class C^m on $\mathbb{R}^n$, and coincides with f on C. $\square$

It is worth noting that, in the above Proposition, the assumption that C is compact cannot be removed in general. In Example 5.34 below we will present an smooth convex function defined in a closed subset of $\mathbb{R}^n$ which has a strictly positive Hessian and admits an smooth convex extension to a neighbourhood of its domain and yet it does not admit a convex extension to all of $\mathbb{R}^n$.

As a straightforward consequence of Proposition 5.21 we obtain the following.

Corollary 5.22. *Let $m \in \mathbb{N}$, $m \geq 2$. Let C be a convex compact subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a convex function having a (not necessarily convex) C^m extension to an open neighbourhood of C. If f satisfies (CW^k) with a strict inequality on C for some $2 \leq k \leq m$, then there exists a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C.*

Proof. Since f satisfies (CW^k) with a strict inequality on C (see Definition 5.7), there exists some $t_0 > 0$ such that

$$D^2 f(y)(v^2) + t\, D^3 f(y)(w, v^2) + \cdots + \frac{t^{k-2}}{(k-2)!} D^k f(y)(w^{k-2}, v^2) \geq \eta t^{k-2}$$

for all $y \in C$, $w, v \in \mathbb{S}^{n-1}$, $0 < t \leq t_0$ and, on the other hand, by Taylor's theorem and uniform continuity of $D^m f$,

$$\begin{aligned} &D^2 f(y + tw)(v^2) \\ &\quad = D^2 f(y)(v^2) + t\, D^3 f(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) + R_m(t, y, v, w), \end{aligned}$$

where

$$\lim_{t \to 0^+} \frac{R_m(t, y, v, w)}{t^{m-2}} = 0 \quad \text{uniformly on} \quad y \in C, w, v \in \mathbb{S}^{n-1}.$$

We may assume $t_0 \leq 1$. Then we may also find $t_0' \in (0, t_0)$ such that $R_m(t, y, v, w) \geq -\frac{\eta}{2} t^{m-2}$ for all $y \in C$, $w, v \in \mathbb{S}^{n-1}$, $0 < t \leq t_0'$, and it follows that

$$D^2 f(y + tw)(v^2) \geq \frac{\eta}{2} t^{m-2}$$

for all $y \in C$, $w, v \in \mathbb{S}^{n-1}$, $0 < t \leq t_0'$. This implies that $D^2 f(x)(v^2) \geq 0$ for all $v \in \mathbb{S}^{n-1}$ whenever $d(x, C) \leq t_0'$, and therefore that f is convex on $U := \{x \in \mathbb{R}^n : d(x, C) < t_0'\}$. Our corollary then follows from Proposition 5.21. □

The easiest instance of application of this corollary is of course when f has a strictly positive Hessian on ∂C, in which case we recover the aforementioned consequence of the results of M. Ghomi's [44] and M. Yan's [72].

Let us also note that in this case f automatically satisfies (CW^p) for all the rest of p's.

Proposition 5.23. *Let $m \in \mathbb{N}$, $m \geq 2$. If $f \in C^m(\mathbb{R}^n)$ satisfies (CW^k) with a strict inequality on ∂C for some $k \geq 2$, then f satisfies (CW^p) with a strict inequality on ∂C for every $p \in \{2, \ldots, m\}$.*

Proof. Obviously we can assume that $k < m$. There exists $\delta > 0$ and $0 < t_0 \leq 1$ such that

$$Q_k(f, y, t, v, w) := \frac{D^2 f(y)(v^2) + t D^3 f(y)(w, v^2) + \cdots + \frac{t^{k-2}}{(k-2)!} D^k f(y)(w^{k-2}, v^2)}{t^{k-2}} \geq \eta$$

for all $0 < t \leq t_0$, $y \in \partial C$ and $v, w \in \mathbb{R}^n$ with $|v| = |w| = 1$. On the other hand, since the derivatives $D^k f$, $k = 0, \ldots, m$, are continuous and C is compact, $\sup_{z \in C} \|D^j f(z)\|$ is finite for every $j = 0, \ldots, m$ and we can choose t_0 small enough so that

$$t_0 \leq \frac{\eta}{2\left(1 + \sup_{z \in C,\, j=0,\ldots,m} \|D^j f(z)\|\right)}.$$

Then we can write

$$
\begin{aligned}
&D^2 f(y)(v^2) + tD^3 f(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \\
&= t^{k-2} Q_k(f, y, t, v, w) + \frac{t^{k-1}}{(k-1)!} D^{k+1} f(y)(w^{k-1}, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \\
&\geq \eta t^{k-2} + \frac{t^{k-1}}{(k-1)!} D^{k+1} f(y)(w^{k-1}, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \\
&\geq t^{k-2} \left(\eta - t \sup_{z \in C,\, j=0,\ldots,m} \| D^j f(z) \| \right) \geq t^{k-2} \frac{\eta}{2},
\end{aligned}
$$

for $0 < t \leq t_0,\ y \in \partial C,\ v, w \in \mathbb{R}^n$ with $|v| = |w| = 1$. Therefore the function f satisfies condition (CW^m) with strict inequality on ∂C. □

5.6 The two easiest situations

5.6.1 The case when the domain is a singleton

Let us prove that, when the domain is a singleton, condition (CW^m) is necessary and sufficient for the C^m convex extension problem. Note that por a point $x_0 \in \mathbb{R}$ and a polynomial P of degree up to m on $\mathbb{R}^n$, the fact that P satisfies (CW^m) on $C = \{x_0\}$ means that

$$
\liminf_{t \to 0^+} \frac{1}{t^{m-2}} \left(D^2 P(x_0)(v^2) + tD^3 P(x_0)(w, v^2) + \cdots \frac{t^{m-2}}{(m-2)!} D^m P(x_0)(w^{m-2}, v^2) \right) \geq 0
$$

uniformly on $v, w \in \mathbb{S}^{n-1}$.

Theorem 5.24. *Let $C = \{x_0\}$, where x_0 is a point of $\mathbb{R}^n$. Let $m \geq 2$ an integer and let P be a polynomial of degree less than or equal to m. There exists a convex function $F \in C^m(\mathbb{R}^n)$ such that $J^m_{x_0} F = P$ if and only if P satisfies the condition (CW^m) at the point x_0.*

Proof. We will essentially repeat the strategy of the proof of Theorem 5.14. Because $C = \{x_0\}$, the polynomial P trivially satifies Whitney's condition (W^m) at the point x_0 and then we may and do assume that there exists a function f (not necessarily convex) of class $C^m(\mathbb{R}^n)$ such that $J^m_{x_0} f = P$. Also, because f satisfies condition (CW^m) on $C = \{x_0\}$, Lemma 5.15 provides us with a continuous non decreasing function $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ and

$$
D^2 f(x)(v^2) \geq -\omega(|x - x_0|)|x - x_0|^{m-2} \quad x \in \mathbb{R}^n, \quad v \in \mathbb{S}^{n-1}.
$$

As in the proof of Theorem 5.14, we consider the functions

$$
g(t) = \begin{cases} \int_0^t \int_0^{t_2} \cdots \int_0^{t_m} \omega\left(2^{m-1} s\right) ds\, dt_m \cdots dt_2 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}
$$

$$
\varphi(x) = \int_{\mathbb{S}^{n-1}} g(\langle x, w \rangle - h(w))\, dw, \quad x \in \mathbb{R}^n.
$$

Since ω is continuous, the function g is of class $C^m(\mathbb{R})$ with $g^{(k)}(0) = 0$ for every $1 \leq k \leq m$. The same arguments and calculations of Subsection 5.4.3 allow us to deduce that φ is of class $C^m(\mathbb{R}^n)$ with $\varphi^{-1}(0) = C$ and $J^m_{x_0} \varphi = 0$. Given $x \in \mathbb{R}^n \setminus \{x_0\}$ with $t := |x - x_0| \leq 1$ and a direction $v \in \mathbb{S}^{n-1}$, we learn from Subsection 5.4.6 that

$$
D^2 \varphi(x)(v^2) \geq \mathcal{H}^{n-1}(W_{x,v}) g''\left(\frac{t}{2}\right) \sin^2\left(\frac{\alpha_x}{3}\right).
$$

Here

$$\alpha_x := \frac{t}{t + \mathrm{diam}(C)}$$

and $W_{x,v}$ is defined in Lemma 5.18 with $\mathcal{H}^{n-1}(W_{x,\alpha}) \geq V(n)\alpha^{n-1}$, where $V(n)$ is positive and only depends on n. Since C is a singleton, we obviously have that $\alpha_x = 1$ for every point $x \in \mathbb{R}^n \setminus \{x_0\}$. We thus have the estimation

$$D^2\varphi(x)(v^2) \geq V(n) \sin^2\left(\frac{1}{3}\right) g''\left(\frac{t}{2}\right).$$

In order to estimate the term $g''\left(\frac{t}{2}\right)$, we make similar calculations as that of Subsection 5.4.6 to obtain

$$g''\left(\frac{t}{2}\right) \geq \frac{t}{4} \cdot \frac{t}{8} \cdots \frac{t}{2^{m-2}} \cdot \frac{t}{2^{m-1}}\omega(t) = \frac{t^{m-2}}{2^{2+3+\cdots+(m-1)}}\omega(t).$$

This implies that

$$D^2\varphi(x)(v^2) \geq \frac{V(n)\sin^2\left(\frac{1}{3}\right)}{2^{2+3+\cdots+(m-1)}}\omega(|x - x_0|)|x - x_0|^{m-2},$$

for every $x \in \mathbb{R}^n$ and every direction $v \in \mathbb{S}^{n-1}$. Using the same argument as at the end of Subsection 5.4.6, we construct a convex function $F \in C^m(\mathbb{R}^n)$ with $J^m_{x_0}F = P$. $\square$

5.6.2 The one dimensional case

In dimension $n = 1$ the boundary of every compact interval I has only two points and there are only two directions in which to differentiate. Hence Definition 5.7 of condition (CW^m) can be very much simplified and this allows us to stablish an *if and only if* theorem for C^m convex extensions of convex functions.

Proposition 5.25. *Let I be a closed interval in $\mathbb{R}$, and $m \in \mathbb{N}$ with $m \geq 2$. Let $f : I \to \mathbb{R}$ be a convex function of class C^m in the interior of I, and assume that f has one-sided derivatives of order up to m, denoted by $f^{(k)}(a^+)$ and $f^{(k)}(b^-)$, at the extreme points of I. Then f has a convex extension F of class $C^m(\mathbb{R})$ with $F^{(k)}(a) = f^{(k)}(a^+)$ and $F^{(k)}(b) = f^{(k)}(b^-)$ if and only if the first (if any) non-zero derivative which occurs in the finite sequence $\{f^{(2)}(b^-), f^{(3)}(b^-), \ldots, f^{(m)}(b^-)\}$ is positive and of even order, and similarly for $\{f^{(2)}(a^+), f^{(3)}(a^+), \ldots, f^{(m)}(a^+)\}$.*

Proof. Let F be a convex function of class $C^m(\mathbb{R}^n)$. Given any point $x \in \mathbb{R}$, we claim that either $F^{(2)}(x) = F^{(3)}(x) = \cdots = F^{(m)}(x) = 0$ or else the first non-zero derivative of the finite sequence $\{F^{(2)}(x), F^{(3)}(x), \ldots, F^{(m)}(x)\}$ is positive and of even order. Let $2 \leq k \leq m$ the order of the first non-zero derivative. Indeed, if $k = 2$, then $F^{(k)}(x) > 0$ by convexity and we are done. Assume that $k > 2$. We have that

$$F^{(2)}(x) = \cdots = F^{(k-1)}(x) = 0 \quad \text{but} \quad F^{(k)}(x) \neq 0. \tag{5.6.1}$$

By continuity of $F^{(k)}$, there exists some δ such that if $|x - z| \leq \delta$ then $F^{(k)}(z) \neq 0$ and $\mathrm{sign}(F^{(k)}(z)) = \mathrm{sign}(F^{(k)}(x))$. Combining (5.6.1) with Taylor's theorem, we obtain, for any $y \in \mathbb{R}$ with $0 \leq |y - x| \leq \delta$, a point $z \in (x, y)$ such that

$$F(y) = F(x) + F'(x)(y - x) + \frac{F^{(k)}(z)}{k!}(y - x)^k.$$

Since F is convex, it follows that $F^{(k)}(z)(y - x)^k \geq 0$. And because $0 < |x - z| < |x - y| \leq \delta$ we have that $F^{(k)}(z)(y - x)^k > 0$ and

$$\mathrm{sign}(F^{(k)}(x)) = \mathrm{sign}(F^{(k)}(z)) = \mathrm{sign}((y - x)^k) \quad \text{whenever} \quad 0 < |x - y| \leq \delta.$$

This implies that k is even and $F^{(k)}(x) > 0$. We then have proved our claim. Now, assume also that $F = f$ on $I = [a, b]$ and $F^{(k)}(a) = f^{(k)}(a^+)$ and $F^{(k)}(b) = f^{(k)}(b^-)$ for every $k = 0, \ldots, m$. If follows immediately that the first (if any) non-zero derivative in the finite sequence $\{f^{(k)}(b^-)\}_{k=2}^m$ is positive and of even order, and similarly for $\{f^{(k)}(a^+)\}_{k=2}^m$.

Conversely, assume that the first (if any) non-zero derivative in the finite sequences $\{f^{(k)}(b^-)\}_{k=2}^m$ and $\{f^{(k)}(a^+)\}_{k=2}^m$ are positive and of even order. The function defined by

$$F(x)=\begin{cases} f(x) & \text{if } x\in[a,b] \\ f(a)+f'(a^+)(x-a)+\frac{1}{2}f^{(2)}(a^+)(x-a)^2+\cdots+\frac{1}{m!}f^{(m)}(a^+)(x-a)^m & \text{if } x\le a \\ f(b)+f'(b^-)(x-b)+\frac{1}{2}f^{(2)}(b^-)(x-b)^2+\cdots+\frac{1}{m!}f^{(m)}(b^-)(x-b)^m & \text{if } x\ge b \end{cases}$$

is of class $C^m(\mathbb{R})$ with $F=f$ on (a,b) and $F^{(k)}(a)=f^{(k)}(a^+)$, $F^{(k)}(b)=f^{(k)}(b^-)$ for all $k=0,\dots,m$. Since f is convex on $[a,b]$, the second derivative F'' of F is nonnegative on $[a,b]$. We are now going to prove that there exists a function $\overline{F}$ of class $C^m(\mathbb{R})$ with $\overline{F}^{(j)}=F^{(j)}$ on $(-\infty,b]$ for all $j=0,\dots,m$ and $\overline{F}$ has nonnegative second derivative on $[a,+\infty)$. If $f^{(2)}(b^-)=\cdots=f^{(m)}(b^-)=0$, then $F(x)=f(b)+f'(b^-)(x-b)$ for all $x\ge b$ and, in particular, $F''(x)=0$ on the interval $[a,+\infty)$. Thus, in this case, it is enough to take $\overline{F}=F$. Now assume that $2\le k\le m$ is the order of the first non-zero term in $\{f^{(j)}(b^-)\}_{j=2}^m$. By assumption k is even and $f^{(k)}(b-)>0$. It follows from Taylor's theorem that

$$\begin{aligned} F''(x) &= \frac{1}{(k-2)!}f^{(k)}(b^-)(x-b)^{k-2}+\cdots+\frac{1}{(m-2)!}f^{(m)}(b^-)(x-b)^{m-2} \\ &= (x-b)^{k-2}\left(\frac{1}{(k-2)!}f^{(k)}(b^-)+\cdots+\frac{1}{(m-2)!}f^{(m)}(b^-)(x-b)^{m-k-2}\right), \end{aligned}$$

for $x\ge b$. Thus there exists some $\delta>0$ such that $F''(x)>0$ whenever $b\le x\le b+\delta$. Now we pick a nonnegative function $g\in C^\infty(\mathbb{R})$ such that $g=0$ on $(-\infty,b+\delta/2]$ and $g>0$ on $(b+\delta/2,+\infty)$. Then the function $h:\mathbb{R}\to\mathbb{R}$ defined by

$$h(x)=\begin{cases} \int_0^x\int_0^t g(s)ds\,dt & \text{if } x>b+\frac{\delta}{2} \\ 0 & \text{if } x\le b+\frac{\delta}{2} \end{cases}$$

is also nonnegative and of class $C^\infty(\mathbb{R})$ with $h=0$ on $(-\infty,b+\frac{\delta}{2}]$ and $h''=g$ on $\mathbb{R}$. By the properties of g we see that $h''\ge 0$ on $\mathbb{R}$, which implies that h is convex, and $h''>0$ on $[b+\delta/2,+\infty)$. We also consider a function $\theta:\mathbb{R}\to[0,1]$ of class C^∞ with $\theta=1$ on $(-\infty,b+\delta]$ and $\theta=0$ on $[b+2\delta,+\infty)$. The function $\tilde{F}=\theta F$ is of class $C^m(\mathbb{R})$ with $\tilde{F^{(j)}}=F^{(j)}$ on $(-\infty,b+\delta]$ for all $j=0,\dots,m$ and $\tilde{F}=0$ on $[b+2\delta,+\infty)$. Since h'' is strictly positive on $[b+\delta,b+2\delta]$, we can define

$$A=\frac{1+\sup\{1+|\tilde{F}''|\ :\ x\in[b+\delta,b+2\delta]\}}{\inf\{h''\ :\ x\in[b+\delta,b+2\delta]\}}>0. \tag{5.6.2}$$

We consider the function $\overline{F}=\tilde{F}+Ah$ on $\mathbb{R}$. It is clear that $\overline{F}$ is of class $C^m(\mathbb{R})$ with $\overline{F}^{(j)}=\tilde{F}^{(j)}=F^{(j)}$ on $(-\infty,b+\delta/2]$ for all $j=0,\dots,m$. This shows that $\overline{F}''\ge 0$ on $[a,b+\delta/2]$. On the interval $[b+\delta/2,b+\delta]$ we have that $\overline{F}''=F''+Ag''\ge F''>0$. By virtue of (5.6.2) we can write

$$\overline{F}''=\tilde{F}''+Ah''\ge\tilde{F}''+1+|\tilde{F}''|\ge 1 \quad \text{on} \quad [b+\delta,b+2\delta].$$

Finally, $\overline{F}''=Ah''>0$ on $[b+2\delta,+\infty)$. In conclusion, the function $\overline{F}$ is of class $C^m(\mathbb{R})$ with $\overline{F}^{(j)}=F^{(j)}$ on $(-\infty,b]$ for all $j=0,\dots,m$ and $\overline{F}$ has nonnegative second derivative on $[a,+\infty)$. By repeating the same arguments with the function $\overline{F}$ instead of F at the extreme point a, we obtain a function $\overline{\overline{F}}\in C^m(\mathbb{R})$ with $\overline{\overline{F}}^{(j)}=F^{(j)}$ on $[a,b]$ for all $j=0,\dots,m$ and $\overline{\overline{F}}''\ge 0$ on $\mathbb{R}$. In particular, $\overline{\overline{F}}$ is convex, $\overline{\overline{F}}=f$ on (a,b) and $\overline{\overline{F}}^{(j)}(a)=f^{(j)}(a^+)$, $\overline{\overline{F}}^{(j)}(b)=f^{(j)}(b^-)$ for all $j=0,\dots,m$. $\square$

5.7 Assuming further conditions on the domain: almost optimal results

In this section we find a class of relatively nice convex bodies for which Theorem 5.14 can be very much improved.

5.7.1 Definition of (FIO) body of class C^m

Definition 5.26 (FIO^m bodies)**.** *Given an integer $m \geq 2$, we will say that a subset C of $\mathbb{R}^n$ is an ovaloid of class C^m if there exist $M > 0$ and a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *ψ is of class $C^m(\mathbb{R}^n)$.*

(ii) *$D^2\psi(x)(v^2) \geq M$ for all $x \in \mathbb{R}^n$ and for all $v \in \mathbb{S}^{n-1}$.*

(iii) *$C = \psi^{-1}\left((-\infty, 1]\right)$.*

We will also say that a set C is (FIO^m), or an FIO body of class C^m, if C is the intersection of a finite family of ovaloids of class C^m.

By restricting our attention to the class of (FIO) bodies, we can find convex extensions of functions satisfying (W^m) and (CW^m) with a loss of just one order of smoothness.

Theorem 5.27. *Let C be a convex subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}$ with $m \geq 3$, and let $\{P_y^m\}_{y \in C}$ be a family of polynomials of degree less than or equal to m. Assume that C is (FIO^{m-1}) and that the family $\{P_y^m\}_{y\in C}$ satisfies (W^m) on C, (CW^m) on ∂C and the function $C \ni y \mapsto P_y^m(y)$ is convex. Then there exists a convex function $F \in C^{m-1}(\mathbb{R}^n)$ such that $J_y^{m-1}F = P_y^{m-1}$ for every $y \in C$, where each P_y^{m-1} is obtained from P_y^m by discarding its homogeneous terms of order m.*

We will give the proof of Theorem 5.27 into several subsections. The idea of the proof is similar to that of Theorem 5.14 (see Subsection 5.4.1); but, in this case, the function ψ which compensates the lack of convexity of any C^m (not necessarily convex) extension of the family $\{P_y^m\}_{y\in C}$ is essentially given by the functions ψ_j's defining the ovaloids C_j's (see definition 5.26), where $C = \bigcap_{j=1}^N C_j$. Nevertheless we will need to prove several properties of these functions ψ_j's and their derivatives.

5.7.2 Sublevel sets of strongly convex functions

Here we gather some elementary properties of ovaloids that we will need in the proof of Theorem 5.27.

Proposition 5.28. *Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^m(\mathbb{R}^n)$, with $m \geq 2$, such that there exists a constant $M > 0$ with $D^2\psi(x)(v^2) \geq M$ for all $x \in \mathbb{R}^n$ and for all $v \in \mathbb{S}^{n-1}$. If we denote $C = \{x \in \mathbb{R}^n \, : \, \psi(x) \leq 1\}$, then the following is true.*

(1) *C is a convex compact set, $\partial C = \{x \in \mathbb{R}^n \, : \, \psi(x) = 1\}$ and $\operatorname{int}(C) = \{x \in \mathbb{R}^n \, : \, \psi(x) < 1\}$.*

(2) *If $\operatorname{int}(C) = \emptyset$, then C is a singleton.*

If we further assume that $\operatorname{int}(C) \neq \emptyset$ then we also have:

(3) *$\nabla\psi$ does not vanish on ∂C and ∂C is a one-codimensional manifold of class C^m.*

(4) *ψ attains a unique minimum in $\operatorname{int}(C)$.*

(5) *There is a constant $\beta > 0$ such that*

$$\psi(x) - 1 \geq \beta d(x, C) \quad \textit{for every} \quad x \in \mathbb{R}^n \setminus C.$$

Proof.

(1) It is clear that C, being a sublevel set of the continuous function ψ, is a closed subset. In order to see that C is also bounded, let us check that ψ is a coercive function, that is, $\lim_{|x|\to\infty}\psi(x) = +\infty$. Consider a sequence $(x_k)_k \subset \mathbb{R}^n$ with $\lim_k |x_k| = +\infty$ and fix a point $x_0 \in \mathbb{R}^n$. By Taylor's theorem, there exists, for every $k \geq 1$, a point $z_k \in [x_k, x_0]$ such that

$$\psi(x_k) = \psi(x_0) + \langle \nabla\psi(x_0), x_k - x_0\rangle + \frac{1}{2}D^2\psi(z_k)(x_k - x_0)^2.$$

This leads us to

$$\psi(x_k) \geq -|\psi(x_0)| - |\nabla\psi(x_0)||x_k - x_0| + \frac{1}{2}M|x_k - x_0|^2, \quad k \geq 1,$$

which in turn implies, because $\lim_k |y_k - x| = +\infty$, that $\lim_k \psi(y_k) = +\infty$. This proves the coercivity of ψ. Now it is clear that C is a bounded subset because otherwise we would have a sequence $(x_k)_k$ with $\lim_k |x_k| = +\infty$ but $\psi(x_k) \leq 1$ for all k, contradicting the coercivity of ψ. The convexity of C follows from the convexity (in fact, strong convexity) of ψ on $\mathbb{R}^n$. In order to prove that $\mathrm{int}(C) = \{x \in \mathbb{R}^n \ : \ \psi(x) < 1\}$, observe that for every $x \in \mathrm{int}(C)$, we can find a line segment $[y, z]$ contained in $\mathrm{int}(C)$ such that $x \in (y, z)$, that is, $x = \lambda z + (1-\lambda)y$ for some $\lambda \in (0, 1)$. The strict convexity of ψ allows us to write

$$\psi(x) < \lambda\psi(z) + (1-\lambda)\psi(y) \leq \lambda + (1-\lambda) = 1.$$

We thus have that $\mathrm{int}(C) \subset \{x \in \mathbb{R}^n \ : \ \psi(x) < 1\}$ and the converse inclusion is a consequence of the continuity of ψ. The fact that $\partial C = \{x \in \mathbb{R}^n \ : \ \psi(x) = 1\}$ follows immediately.

(2) Assume that $\mathrm{int}(C) = \emptyset$ and that there are two points $x, y \in C$ with $x \neq y$. Using (1), it is clear that $C = \{x \in \mathbb{R}^n \ : \ \psi(x) = 1\}$. By convexity of C, the point $z = \frac{x+y}{2}$ belongs to C. Because ψ is strictly (in fact, strongly) convex on $\mathbb{R}^n$, we obtain

$$1 = \psi(z) < \frac{1}{2}\psi(x) + \frac{1}{2}\psi(y) = 1,$$

which is absurd.

(3) By replacing the smoothness C^1 with C^m, it follows immediately from Proposition 2.16.

(4) Because ψ is continuous and C is a compact subset, the function ψ attains a local minimum in C. Since ψ is a convex function on $\mathbb{R}^n$, this local minimum is in fact a global one. Moreover, because ψ is strictly convex, this minimum is attained at a unique point, say $x_0 \in C$. But $x_0 \notin \partial C$ by (4), which implies that $x_0 \in \mathrm{int}(C)$.

(5) We learn from (3) that $\nabla\psi \neq 0$ on ∂C. Then, by the compactness of ∂C and the continuity of $\nabla\psi$, we can find $\beta > 0$ such that $|\nabla\psi(x)| \geq \beta$ for all $x \in \partial C$. If $x \notin C$, by taking $x_C \in \partial C$ with $|x - x_C| = d(x, C)$, the convexity of ψ leads us to

$$\psi(x) - 1 = \psi(x) - \psi(x_C) \geq \langle \nabla\psi(x_C), x - x_C\rangle.$$

By Proposition 2.16, $\nabla\psi(x_C)$ is a positive multiple of $x - x_C$ and then the last product coincides with $|\nabla\psi(x_C)||x - x_C| \geq \beta d(x, C)$. □

5.7.3 The distance to the intersection of convex sets

For the proof of Theorem 5.27 we will need to estimate the distance to a finite intersection of convex subsets in terms of the supremum of the distances to each subset. In order to do this, we will momentarily make use of some properties of the Minkowski functional asociated to convex subsets.

Proposition 5.29. *If $C \subseteq X$ is convex with $0 \in \mathrm{int}(C)$ we have:*

(1) *If $C = \bigcap_{k=1}^N C_k$, where each C_k is a convex subset with $0 \in \mathrm{int}\, C$ then $\mu_C = \max_{1\leq k\leq N}\mu_{C_k}$.*

Suppose in addition that $C \subset X$ is bounded.

(2) *If $C = \bigcap_{k=1}^N C_k$, where each C_k is convex and bounded with $0 \in \operatorname{int}(C)$, we have*

$$\max_{1 \le k \le N} d(x, C_k) \le d(x, C) \le \frac{R}{r} \max_{1 \le k \le N} d(x, C_k) \quad \textit{for all} \quad x \in X,$$

whenever $r, R > 0$ are such that $\overline{B}(0, r) \subseteq C \subseteq \overline{B}(0, R)$.

(3) *If $C = \bigcap_{k=1}^N C_k$, where each C_k is convex and bounded with $\operatorname{int}(C) \neq \emptyset$, but not necessarily $0 \in \operatorname{int}(C)$, we have*

$$\max_{1 \le k \le N} d(x, C_k) \le d(x, C) \le \frac{R}{r} \max_{1 \le k \le N} d(x, C_k) \quad \textit{for all} \quad x \in X,$$

where $r, R > 0$ are such that $\overline{B}(x_0, r) \subseteq C \subseteq \overline{B}(x_0, R)$ and $x_0 \in \operatorname{int}(C)$.

Proof.

(1) By Proposition 4.26 (3) we have that $\mu_C(x) < t$ if and only if $x \in t \operatorname{int}(C) = t \bigcap_{j=1}^N \operatorname{int}(C_j)$. This is equivalent to $\mu_{C_j}(x) < t$ for every $j = 1, \ldots, N$, that is, $\max\{\mu_{C_j}(x) \ : \ j = 1, \ldots, N\} < t$. This proves that $\mu_C = \max_{1 \le k \le N} \mu_{C_k}$.

(2) When $x \in C$ there is nothing to prove. If $x \notin C$, using Proposition 4.26 (8) and (10) we obtain

$$d(x, C) \le R\left(\mu_C(x) - 1\right) = R\left(\max_{1 \le k \le N} \mu_{C_k}(x) - 1\right) = R\left(\max_{1 \le k \le N} (\mu_{C_k}(x) - 1)\right).$$

By (2), the last term is less than or equal to $\frac{R}{r} \max_{1 \le k \le N} d(x, C_k)$.

(3) After a translation, the same proof as in (2) holds. $\square$

5.7.4 Proof of the extension result for (FIO) bodies

First of all, let us make a small remark.

Remark 5.30. *If a set C is (FIO^m), (see Definition 5.26), then either C has nonempty interior or C is a single point.*

Proof. Let $C = \bigcap_{j=1}^N C_j$, where each C_j is an ovaloid of class C^m. Suppose that $\operatorname{int}(C) = \bigcap_{j=1}^N \operatorname{int}(C_j) = \emptyset$, and let us show that C is a single point. Indeed, assuming that there exist $x, y \in C$ with $x \neq y$, the point $z = \frac{x+y}{2}$ belongs to C by convexity. On the other hand, we have that $z \notin C_j \setminus \operatorname{int}(C_j)$ for some $j \in \{1, \ldots, N\}$, which implies that $\psi_j(z) = 1$. Also, because $x, y \in C_j$, we obviously have $\psi_j(x), \psi_j(y) \le 1$. Bearing in mind that ψ_j is a strictly convex function on $\mathbb{R}^n$, we obtain

$$1 = \psi_j(z) < \frac{1}{2}\psi_j(x) + \frac{1}{2}\psi_j(y) \le 1,$$

which is absurd. $\square$

In view of the above Remark, we may thus suppose that C has nonempty interior, as the result follows immediately from Theorem 5.24 in the case that C is a singleton. We are now ready to complete the proof of Theorem 5.27.

Fix $m \in \mathbb{N}$ with $m \ge 3$ and suppose that C is (FIO^{m-1}) with nonempty interior. Since the family of polynomials $\{P_y^m\}_{y \in C}$ satisfies Whitney's condition (W^m) or order m, we may assume that there exists some $f \in C^m(\mathbb{R}^n)$ satisfying the property (CW^m) on ∂C, $D^2 f(x)$ is semidefinite positive for every $x \in C$ and $J_y^m f = P_y^m$ for every $y \in C$. According to Definition 5.26, we can write $C = \bigcap_{j=1}^N C_j$, where for each $1 \le j \le N$ there are $M_j > 0$ and a function $\psi_j : \mathbb{R}^n \to \mathbb{R}$ of class $C^{m-1}(\mathbb{R}^n)$

such that $C_j = \psi_j^{-1}(-\infty, 1]$ and $D^2\psi_j(x)(v^2) \geq M_j$ for all $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$. Let us denote $M = \min\{M_j \, : \, 1 \leq j \leq N\}$. By Proposition 5.28 (5), for each $j \in \{1, \ldots, N\}$, the set C_j is a convex compactum and there is a constant $\beta_j > 0$ with $\psi_j(x) - 1 \geq \beta_j d(x, C_j)$ whenever $x \notin C_j$. Set $\beta = \min\{\beta_j \, : \, j = 1, \ldots, N\}$. Using Proposition 5.29 (3), we obtain $L > 0$ with $d(x, C) \leq L \max_{1 \leq j \leq N} d(x, C_j)$ for all $x \in \mathbb{R}^n$. To sum up, we have found positive constants L, β, M satisfying

$$d(x, C) \leq L \max_{1 \leq j \leq N} d(x, C_j) \quad \text{for all} \quad x \in \mathbb{R}^n; \tag{5.7.1}$$

$$\psi_j(x) - 1 \geq \beta d(x, C_j) \quad \text{for all} \quad x \notin C_j, \quad 1 \leq j \leq N; \tag{5.7.2}$$

$$D^2\psi_j(x)(v^2) \geq M \quad \text{for all} \quad x \in \mathbb{R}^n, \, v \in \mathbb{S}^{n-1}, \quad 1 \leq j \leq N. \tag{5.7.3}$$

Since $f : \mathbb{R}^n \to \mathbb{R}$ satisfies (CW^m) on C, the estimation given in Lemma 5.15 involving the function ω holds for f. For these positive constants $L, \beta > 0$, we define the following functions

$$g(t) = \begin{cases} \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-1}} \omega\left(2^{m-2}s\right) ds \, dt_{m-1} \cdots dt_2 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

$$h(t) = g(L\beta^{-1}t), \quad t \in \mathbb{R},$$

and

$$\varphi(x) = \sum_{j=1}^N h(\psi_j(x) - 1), \quad x \in \mathbb{R}^n.$$

It is clear that $g \in C^{m-1}(\mathbb{R})$ with $g^{(k)}(0) = 0$ for all $0 \leq k \leq m-1$. By the definition of the ψ's and h, we have that $\varphi^{-1}(0) = C$ and $\varphi \in C^{m-1}(\mathbb{R}^n)$. It is clear that every partial derivative $\partial^\alpha \varphi$ is a sum of functions all of them multiplied by derivatives of the form $h^{(\ell)}(\psi_j - 1)$ and then, because $h^{(k)} = 0$ for every $0 \leq k \leq m-1$ and $\psi_j = 1$ on C, we have that $\partial^\alpha \varphi(x) = 0$ for all $x \in C$ and all $|\alpha| \leq m-1$, that is, $J_x^{m-1}\varphi = 0$ for all $x \in C$. A simple calculation and the fact that $g'' \geq 0$ lead us to

$$\begin{aligned} D^2\varphi(x)(v^2) &= \sum_{j=1}^N h''(\psi_j(x) - 1)\langle \nabla \psi_j(x), v\rangle^2 + \sum_{j=1}^N h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2) \\ &\geq \sum_{j=1}^N h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2), \end{aligned}$$

for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$. Now, we study the convexity of φ outside of C. Fix $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$. From (5.7.3) we deduce

$$D^2\varphi(x)(v^2) \geq \sum_{j=1}^N h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2) \geq M \sum_{j=1}^N h'(\psi_j(x) - 1).$$

But the above sum is greater than or equal to $Mh'(\psi_j(x) - 1)$, where we consider an index $j := j_x$ with $d(x, C_j) = \max_{1 \leq i \leq N} d(x, C_i)$. Of course, for this index j, we have that $x \notin C_j$. This implies $\psi_j(x) > 1$ and therefore

$$D^2\varphi(x)(v^2) \geq Mh'(\psi_j(x) - 1) = ML\beta^{-1}g'(L\beta^{-1}(\psi_j(x) - 1)).$$

Using inequalities (5.7.2) and (5.7.1) and the choice of j, we obtain

$$\psi_j(x) - 1 \geq \beta d(x, C_j) \geq \beta L^{-1} d(x, C).$$

The above inequality and the fact that g' is non decreasing imply that

$$g'(L\beta^{-1}(\psi_j(x) - 1)) \geq g'(d(x, C)) = g'(t),$$

where $t := d(x, C)$. Recalling that ω is nonnegative and nondecreasing, we obtain

$$\begin{aligned}
g'(t) &= \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-2}} \omega\left(2^{m-2}s\right) ds\, dt_{m-2} \cdots dt_2 \\
&\geq \int_{t/2}^t \int_0^{t_2} \cdots \int_0^{t_{m-2}} \omega(2^{m-2}s) ds\, dt_{m-2} \cdots dt_2 \\
&\geq \frac{t}{2} \int_0^{t/2} \int_0^{t_2} \cdots \int_0^{t_{m-3}} \omega(2^{m-2}s) ds\, dt_{m-3} \cdots dt_2 \\
&\geq \frac{t}{2} \cdot \frac{t}{4} \int_0^{t/4} \int_0^{t_2} \cdots \int_0^{t_{m-4}} \omega(2^{m-2}s) ds\, dt_{m-4} \cdots dt_2 \\
&\geq \frac{t}{2} \cdot \frac{t}{4} \cdots \frac{t}{2^{m-3}} \int_0^{t/2^{m-3}} \omega(2^{m-2}s) ds \\
&\geq \frac{t}{2} \cdot \frac{t}{4} \cdots \frac{t}{2^{m-3}} \cdot \frac{t}{2^{m-2}} \omega(t) = \frac{t^{m-2}}{2^{1+2+3+\cdots+(m-2)}} \omega(t).
\end{aligned}$$

Therefore

$$D^2\varphi(x)(v^2) \geq ML\beta^{-1} g'(t) = k(n, m, C) t^{m-2} \omega(t),$$

where

$$k(n, m, C) := \frac{ML\beta^{-1}}{2^{1+2+3+\cdots+(m-2)}}.$$

On the other hand, Lemma 5.15 gives us the following inequality:

$$D^2 f(x)(v^2) \geq -\omega(t) t^{m-2}.$$

Hence $F := f + \frac{2}{k(n,m,C)} \varphi$ has a strictly positive Hessian on $\mathbb{R}^n \setminus C$, is of class $C^{m-1}(\mathbb{R}^n)$, and coincides with f on C. Since $J_x^{m-1}\varphi = 0$ for all $x \in C$, we have that $J_x^{m-1} F = J_x^{m-1} f = P_x^{m-1}$ for all $x \in C$. Because f is convex on C and the extension F is differentiable, we have that F is convex in $\mathbb{R}^n$. The proof of Theorem 5.27 is complete.

5.8 Relation between (CW^2) and (CW^1)

It is natural to ask whether conditions (CW^m) defined on this chapter for compact convex domains imply the condition (CW^1) defined in Chapter 4 for C^1 convex extensions of functions. As we saw in Theorem 4.4, condition (CW^1) together with condition (C) (which is automatically fulfilled if we assume that our function f is convex on the domain) are necessary and sufficient conditions on a 1-jet for having a C^1 convex extension from a compact convex domain. We are now going to prove that (CW^2) implies (CW^1) on compact convex subsets.

Proposition 5.31. *Let E be a closed convex subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ a function of class $C^2(\mathbb{R}^n)$ such that f satisfies the condition (CW^2) on E and*

$$M := \sup\{D^2 f(x)(v^2) \,:\, |v| = 1,\, x \in E\} < +\infty. \tag{5.8.1}$$

Then $M\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq |\nabla f(x) - \nabla f(y)|^2$ for all $x, y \in E$.

Proof. Given two symmetric linear operators $A, B : \mathbb{R}^n \to \mathbb{R}^n$, by $A \leq B$ we mean that

$$A(v^2) := v^t A v \leq v^t B v =: B(v^2) \quad \text{for all} \quad v \in \mathbb{R}^n,\ |v| = 1.$$

Let us fix $x \in E$. The fact that f satisfies (CW^2) on E together with (5.8.1) lead us to $0 \leq D^2 f(x) \leq MI$, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It immediately follows that

$$-MI \leq 2D^2 f(x) - MI \leq MI,$$

which in turn implies

$$|(2D^2f(x) - MI)(v^2)| \leq M \quad \text{for all} \quad |v| = 1.$$

Recall that, for a selfadjoint linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$, we have the identity

$$\|A\| := \sup\{|A(v)| \,:\, |v| = 1\} = \sup\{\left|A(v^2)\right| \,:\, |v| = 1\}.$$

We then have shown that $\|2D^2f(x) - MI\| \leq M$. Now, we fix $x, y \in E$ and $v \in \mathbb{R}^n$ with $|v| = 1$. Since E is convex and ∇f is differentiable on $\mathbb{R}^n$, we can use the mean value theorem to obtain some $z \in E$ for which

$$\big\langle (2\nabla f(x) - Mx) - (2\nabla f(y) - My), v\big\rangle = \big\langle (2D^2f(z) - MI)(v), x - y\big\rangle \leq M|x - y|.$$

Beacuse $|v| = 1$ is arbitrary, the above inequality shows that $2\nabla f - MI : \mathbb{R}^n \to \mathbb{R}^n$ is M-Lipschitz on E. Thus, for every $x, y \in E$, we have

$$\begin{aligned} M^2|x - y|^2 &\geq \left|(2\nabla f(x) - Mx) - (2\nabla f(y) - My)\right|^2 = \left|2(\nabla f(x) - \nabla f(y)) - M(x - y)\right|^2 \\ &= 4|\nabla f(x) - \nabla f(y)|^2 + M^2|x - y|^2 - 4M\langle \nabla f(x) - \nabla f(y), x - y\rangle. \end{aligned}$$

This immediately implies the desired inequality. □

Lemma 5.32. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^2(\mathbb{R}^n)$ such that f satisfies (CW^2) on a compact convex subset E of $\mathbb{R}^n$. Then $(f, \nabla f)$ satisfies condition (CW^1) on E, that is,*

$$f(x) = f(y) + \langle \nabla f(y), x - y\rangle \implies \nabla f(x) = \nabla f(y), \quad \textit{for every} \quad x, y \in E.$$

Proof. Given $x, y \in E$ with $f(x) = f(y) + \langle \nabla f(y), x - y\rangle$, we need to prove that $\nabla f(x) = \nabla f(y)$.
Case 1. We first suppose that $f(y) = 0$ and $\nabla f(y) = 0$. We thus have $f(x) = 0$. The fact that $D^2 f$ is positive semidefinite implies in particular that f is convex on E. Therefore, if we define $h(t) = f(tx + (1 - t)y)$ for all $t \in \mathbb{R}$, we see that h is of class $C^2(\mathbb{R})$ and h is convex on $[0, 1]$ with $h(0) = 0 = h(1) = h'(0) = 0$. By convexity and differentiability we must have $h = 0$ on $[0, 1]$, which implies that $\langle \nabla f(x), x - y\rangle = h'(1) = 0$. Hence we get that $\langle \nabla f(x) - \nabla f(y), x - y\rangle = 0$. Since E is compact and D^2f is continuous, inequality (5.8.1) holds for some $M \geq 0$. According to Proposition 5.31 we have $\nabla f(x) = \nabla f(y)$.
Case 2. Let $f(y)$ and $\nabla f(y)$ be arbitrary. Defining $g(z) = f(z) - f(y) - \langle \nabla f(y), z - y\rangle$ for all $z \in \mathbb{R}^n$, we have that $g(x) = g(y) = 0$ and $\nabla g(y) = 0$. Since f satisfies (CW^2) on E, it is clear that g satisfies (CW^2) on E as well. According to Case 1, we have $\nabla g(x) = 0$, which is equivalent to $\nabla f(x) = \nabla f(y)$. This proves the Lemma. □

Thanks to Lemma 5.32, we obtain the following corollary which tells us that if we assume condition (CW^2) on a compact convex subset, then at least we always have a C^1 convex extension to all of $\mathbb{R}^n$. We will use the formulation of Whitney's extension theorem for linear forms rather than the formulation for polynomials, see Theorem 5.3.

Corollary 5.33. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$, $G : C \to (\mathbb{R}^n)^*$ and $H : C \to \mathcal{L}^2(\mathbb{R}^n, \mathbb{R})$ be three functions such that (f, G, H) satisfies Whitney's condition (W^2). If $H(y)$ is semidefinite positive for every $y \in C$, then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ and $DF = G$ on E.*

Proof. Recall that condition (CW^2) on the set C for a 2-jet (f, G, H) merely says that $H(y)(v^2) \geq 0$ for every $v \in \mathbb{S}^{n-1}$ and every $y \in C$. By Whitney's Extension Theorem (see Theorem 5.3), we may and do assume that f is extended to a (not necessarily convex) function of class $C^2(\mathbb{R}^n)$ with $Df = G$ and $D^2f = H$ on C. On the other hand, we have that $D^2f(x)(v^2) = H(x)(v^2) \geq 0$ for every $x \in C$ and every $v \in \mathbb{S}^{n-1}$. Hence, according to Lemma 5.32, $(f, \nabla f)$ satisfies condition (CW^1) on C. Also, since f is convex on C and f is differentiable, then

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle \quad x, y \in C;$$

which shows that $(f, \nabla f)$ satisfies condition (C) on C. Therefore, Theorem 4.4 provides us with a convex function F of class C^1 such that $F = f$ and $\nabla F = \nabla f$ on E. □

5.9 Remarks and Counterexamples

The following example is a variation of [59, Example 4] and shows that Theorem 5.14 fails if we drop the assumption that C be compact, even in the presence of strictly positive Hessians. Moreover, this proves that, in contrast to Proposition 5.21, the fact that f has a smooth convex extension to an open convex neighbourhood of C does not imply that f can be extended to a convex function on $\mathbb{R}^n$.

Example 5.34. Let $C = \{(x, y) \in \mathbb{R}^2 \, : \, x > 0, \, xy \geq 1\}$, and define

$$f(x, y) = -2\sqrt{xy} + \frac{1}{x+1} + \frac{1}{y+1}$$

for every $(x, y) \in C$. The set C is convex and closed, with a nonempty interior. Let us extend the definition of f to the set $B = \{(x, y) \in \mathbb{R}^2 \, : \, x, y \geq 0\}$ by setting $f(x, y) = -2\sqrt{xy} + \frac{1}{x+1} + \frac{1}{y+1}$ for all $(x, y) \in B$. It is clear tha f is of class C^∞ on the interior $\operatorname{int}(B)$ of B and the first and second derivatives of f are

$$\nabla f(x, y) = \left(-x^{-\frac{1}{2}} y^{\frac{1}{2}} - \frac{1}{(x+1)^2}, \; -x^{\frac{1}{2}} y^{-\frac{1}{2}} - \frac{1}{(y+1)^2} \right).$$

and

$$H_f(x, y) = \begin{pmatrix} \frac{1}{2} x^{-\frac{3}{2}} y^{\frac{1}{2}} + \frac{2}{(x+1)^2} & -\frac{1}{2} x^{-\frac{1}{2}} y^{-\frac{1}{2}} \\ -\frac{1}{2} x^{-\frac{1}{2}} y^{-\frac{1}{2}} & \frac{1}{2} x^{\frac{1}{2}} y^{-\frac{3}{2}} + \frac{2}{(y+1)^2} \end{pmatrix}$$

for every $(x, y) \in \operatorname{int}(B)$. It is then clear that f has a strictly positive Hessian on $\operatorname{int}(B)$. In particular, f is convex on $\operatorname{int}(B)$ and then, by continuity of f on B, f is convex on B. We claim that f does not have any convex extension to all of $\mathbb{R}^2$. In order to prove this it is sufficient to see that, for instance, $m_C(f)(-1, -1) = \infty$, where $m_C(f)$ is the minimal convex extension of f from C, defined by

$$m_C(f)(x, y) = \sup_{(u,v) \in C} \{ f(u, v) + \langle \nabla f(u, v), (x, y) - (u, v) \rangle \}, \quad \text{for all} \quad (x, y) \in C.$$

As a matter of fact, we are going to see that $m_C(f)(x, y) = \infty$ for every $(x, y) \in \mathbb{R}^2$ such that $x < 0$ or $y < 0$. Considering the curve $\gamma(t) = (t, \frac{1}{t})$, $t > 0$, which parameterizes the boundary of C, we have

$$\begin{aligned} m_C(f)(x, y) &\geq f\left(t, \tfrac{1}{t}\right) + \langle \nabla f\left(t, \tfrac{1}{t}\right), (x - t, y - \tfrac{1}{t}) \rangle \\ &= -2 + \frac{1}{t} + \frac{1}{1 + t^{-1}} - (x - t)\left(\frac{1}{t} + \frac{1}{(1+t)^2} \right) - (y - t^{-1})\left(t + \frac{1}{(1 + t^{-1})^2} \right) \\ &= 1 - \frac{x}{t} - yt - \frac{yt^2 - 2t + x}{(1+t)^2} \end{aligned}$$

for every $t > 0$. By letting $t \to 0^+$ when $x < 0$ and $t \to +\infty$ if $y < 0$, we obtain that the above term tends to $+\infty$, which shows that $m_C(f)(x, y) = +\infty$.

The following example shows that if C has empty interior then one cannot expect to find smooth convex extensions (of functions satisfying (W^m) and (CW^m) on C) without experiencing a certain loss of differentiability. The example also shows that in $\mathbb{R}^2$ this loss amounts to at least two orders of smoothness, and that the situation does not improve as m grows large (unless $m = \infty$, as we will see in the next chapter, Chapter 6).

Example 5.35. Consider the function $\theta(y) = \frac{1 - \cos(2\pi y)}{2\pi}$, $y \in \mathbb{R}$. Clearly, $\theta \in C^\infty(\mathbb{R})$, with $\theta(0) = \theta(1) = 0$, $\theta(1/2) = \frac{1}{\pi}$ and $\theta'(y) = \sin(2\pi y)$. Let $m \geq 2$ be an even integer and and define $h(x, y) = \theta(y) x^m$, $(x, y) \in \mathbb{R}^2$. Let $C := \{0\} \times [0, 1]$. We have $D^k h = 0$ on C for all $k \in \{0, \ldots, m-1\}$, and

$$D^m h(x, y) = m! \theta(y) e_1^* \overbrace{\otimes \cdots \otimes}^{m} e_1^* \quad \text{for} \quad (x, y) \in C$$

(here e_1^* denotes the linear function $(x_1, x_2) \mapsto x_1$). Therefore $D^m h(0, 0) = D^m h(0, 1) = 0$, and $D^m h(0, \frac{1}{2}) = \frac{m!}{\pi} e_1^* \otimes \cdots \otimes e_1^*$. We claim the following.

(1) There is no convex function $F \in C^m(\mathbb{R}^2)$ such that $D^k F = D^k h$ on C for $k \in \{0, \ldots, m\}$.

(2) h satisfies conditions (W^k) for every order k and (CW^{m+1}) (and in particular (CW^m) too) on C.

(3) h does not satisfy condition (CW^{m+2}) on C.

Proof. The statement (1) follows immediately from the following remark.

Remark 5.36. *If $m \geq 2$, there exists no convex function $f \in C^m(\mathbb{R}^2)$ such that $D^k f(0,y) = 0$ for all $k \in \{0, \ldots, m-1\}$, $y \in [0,1]$, and such that $D^m f(0,0) = D^m f(0,1) = 0$ and $D^m f(0, \frac{1}{2}) = Ae_1^* \otimes \cdots \otimes e_1^*$, where $A > 0$ is a constant.*

Proof of Remark 5.36. For the sake of contradiction, suppose there is such an f. Using Taylor's theorem we have

$$f(x,y) = \frac{1}{m!} D^m f(0, y_0)(x, y - y_0)^m + R(x, y, y_0) \quad (x,y) \in \mathbb{R}^2, \ y_0 \in [0,1],$$

where $\dfrac{R(x,y)}{|(x, y-y_0)|^m} \to 0$ as $(x,y) \to (0, y_0)$, uniformly on $y_0 \in [0,1]$. Fix $0 < \varepsilon < \frac{A}{2m!}$, and take $\delta = \delta(\varepsilon) > 0$ such that if $y_0 \in [0,1]$ and $(x,y) \in \mathbb{R}^2$ satisfy $(x^2 + (y-y_0)^2)^{1/2} \leq \delta$ then

$$\left| f(x,y) - \frac{1}{m!} D^m f(0, y_0)(x, y-y_0)^m \right| = |R(x,y)| \leq \varepsilon (x^2 + (y-y_0)^2)^{\frac{m}{2}}.$$

Evaluating for $y = y_0 = 1/2$ we obtain

$$\left| f(x, \tfrac{1}{2}) - A \frac{x^m}{m!} \right| \leq \varepsilon |x|^m, \text{ if } |x| \leq \delta.$$

For $y = y_0 \in \{0, 1\}$ and $|x| \leq \delta$ we get

$$\max\{|f(x,0)|, |f(x,1)|\} \leq \varepsilon |x|^m.$$

Fix $x_0 > 0$ with $x_0 \leq \delta$. We then have

$$f(x_0, \tfrac{1}{2}) \geq A \frac{x_0^m}{m!} - \varepsilon x_0^m > 2\varepsilon x_0^m - \varepsilon x_0^m = \varepsilon x_0^m \geq \max\{f(x_0, 0), f(x_0, 1)\}.$$

This implies that $[0,1] \ni t \mapsto \varphi(t) = f(x_0, t)$ satisfies $\varphi(\frac{1}{2}) > \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(1)$, and in particular f cannot be convex. $\square$

Let us now prove (2). For any function $g : \mathbb{R}^n \to \mathbb{R}$, which is k times differentiable at some point $x \in \mathbb{R}^n$, we will compute the derivative $D^k g(x)$ of g at x via the formula

$$D^k g(x) = \sum_{i_1, \ldots, i_k = 1}^{n} \frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) \, e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*, \tag{5.9.1}$$

where each $e_{i_j}^*$ denotes the linear function $(x_1, \ldots, x_n) \mapsto x_{i_j}$. Of course, because h is of class $C^\infty(\mathbb{R}^2)$, h satisfies condition (W^k) for every k on the set C. Let us check that h satisfies (CW^{m+1}) on C. We must see that, given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$Q_{m+1}(y,t,v,w) = \frac{\frac{1}{(m-2)!} D^m h(0,y)(v^2, w^{m-2}) + \frac{t}{(m-1)!} D^{m+1} h(0,y)(v^2, w^{m-1})}{t} \geq -\varepsilon,$$

for every $y \in [0,1], \ v, w \in \mathbb{S}^1, \ 0 < t \leq t_\varepsilon$. With the help of (5.9.1) we obtain

$$D^m h(0,y)(v^2, w^{m-2}) = \sum_{i_1, \ldots, i_m = 1}^{2} \frac{\partial^m h}{\partial x_{i_1} \cdots \partial x_{i_m}}(0,y) \, v_{i_1} v_{i_2} w_{i_3} \cdots w_{i_m}$$

where x_1 and x_2 stand for the variables x and y respectively. Because $\frac{\partial^k h}{\partial x^k}(0,y) = 0$ whenever $k \neq m$, it is clear that the only possible nonzero term in the above sum is that in which the m-tuple $(i_1,\ldots,i_m)$ satisfies $i_1 = \cdots = i_m = 1$. This shows that

$$D^m h(0,y)(v^2, w^{m-2}) = \frac{\partial^m h}{\partial x^m}(0,y) v_1^2 w_1^{m-2} = m!\, \theta(y) v_1^2 w_1^{m-2} \tag{5.9.2}$$

For the derivatives of order $m+1$, we use again (5.9.1) to obtain

$$D^{m+1} h(0,y)(v^2, w^{m-1}) = \sum_{i_1,\ldots,i_{m+1}=1}^{2} \frac{\partial^{m+1} h}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}}(0,y)\, v_{i_1} v_{i_2} w_{i_3} \cdots w_{i_{m+1}}$$

Since $\frac{\partial^k h}{\partial x^k}(0,y) = 0$ whenever $k \neq m$, it is clear that the only possible nonzero terms in the above sum are those whose $(m+1)$-tuple $(i_1,\ldots,i_{m+1})$ contains m 1's and one 2's. Among these admissible tuples, let us study two cases separately. In the case when $i_1 = i_2 = 1$, the $(m-1)$-tuple $(i_3,\ldots,i_{m+1})$ must contain $(m-2)$ 1's and one 2's and the product $v_{i_1} v_{i_2} w_{i_3} \cdots w_{i_{m+1}}$ coincides with $v_1^2 w_1^{m-2} w_2$. Observe that there are $(m-1)$ possible $(m+1)$-tuples $(i_1,\ldots,i_{m+1})$ satisfying this. In the case when $(i_1,i_2) \neq (1,1)$, we must have $i_3 = \cdots = i_{m+1} = 1$ and the product $v_{i_1} v_{i_2} w_{i_3} \cdots w_{i_{m+1}}$ coincides with $v_1 v_2 w_1^{m-1}$. Note that there only 2 possible $(m+1)$-tuples $(i_1,\ldots,i_{m+1})$ satisfying this. All these observations lead us to

$$\begin{aligned} D^{m+1} h(0,y)(v^2, w^{m-1}) &= \frac{\partial^{m+1} h}{\partial x^m \partial y}(0,y) \left[(m-1) v_1^2 w_1^{m-2} w_2 + 2 v_1 v_2 w_1^{m-1}\right] \\ &= m!\, \theta'(y) \left[(m-1) v_1^2 w_1^{m-2} w_2 + 2 v_1 v_2 w_1^{m-1}\right]. \end{aligned} \tag{5.9.3}$$

For our given $\varepsilon > 0$, let us fix t_ε such that

$$0 < t_\varepsilon \leq \min\left(1, \frac{\varepsilon}{4\pi(2m+3)(m+1)m(m-1)}\right).$$

Take $y \in [0,1],\ v, w \in \mathbb{S}^1$ and $0 < t \leq t_\varepsilon$. We have

$$Q_{m+1}(y,t,v,w) := \frac{1}{t}\left[\frac{m!}{(m-2)!}\theta(y) v_1^2 w_1^{m-2} + \frac{m!}{(m-1)!} t\theta'(y)\left((m-1) v_1^2 w_1^{m-2} w_2 + 2 v_1 v_2 w_1^{m-1}\right)\right].$$

Since m is even, we have $w_1^{m-2} \geq 0$, and we obtain from the preceding equation that

$$Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(y)|v_1| - (m+1)t|\theta'(y)|\right). \tag{5.9.4}$$

Let us now distinguish the following cases.

Case 1. Assume $y \in [\frac{1}{4}, \frac{3}{4}]$. Then $2\pi y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Therefore $\cos(2\pi y) \leq 0$, which implies $\theta(y) \geq \frac{1}{2\pi}$. Since we always have $|\theta'(y)| = |\sin(2\pi y)| \leq 1$, it follows from (5.9.4) that

$$Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\frac{|v_1|}{2\pi} - (m+1)t\right).$$

Subcase 1.1. Assume $|v_1| \geq 2\pi(m+1)t$. Then it is clear that $Q_{m+1}(y,t,v,w) \geq 0 \geq -\varepsilon$.

Subcase 1.2. Assume $2\pi(m+1)t^2 \leq |v_1| \leq 2\pi(m+1)t$. Then, since $|w_1|, t, 1-t \leq 1$, we obtain

$$\begin{aligned} &Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left((m+1)t^2 - (m+1)t\right) \\ &= (m+1)m(m-1)|v_1||w_1|^{m-2}(t-1) \geq -2\pi(m+1)^2 m(m-1)t|w_1|^{m-2}(1-t) \\ &\geq -2\pi t(m+1)^2 m(m-1) \geq -2\pi t_\varepsilon (m+1)^2 m(m-1) \geq -\varepsilon. \end{aligned}$$

Subcase 1.3. Assume $|v_1| \leq 2\pi(m+1)t^2$. We have

$$\begin{aligned}
&Q_{m+1}(y,t,v,w) \geq -\frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left((m+1)t - \frac{|v_1|}{2\pi}\right)\\
&\geq -\frac{2\pi m(m-1)(m+1)t^2|w_1|^{m-2}}{t}\left(m+1+\frac{1}{2\pi}\right) \geq -2\pi(m+1)m(m-1)(m+2)\,t\\
&\geq -2\pi(m+1)m(m-1)(m+2)\,t_\varepsilon \geq -\varepsilon.
\end{aligned}$$

Case 2. Assume $y \in [0, \frac{1}{4})$. Then $\pi y \in [0, \frac{\pi}{4})$ and $2\pi y \in [0, \frac{\pi}{2})$. We have

$$\theta(y) = \frac{1-\cos(2\pi y)}{2\pi} = \frac{\sin^2(\pi y)}{\pi}.$$

On the other hand,

$$|\theta'(y)| = |\sin(2\pi y)| = \sin(2\pi y) = 2\sin(\pi y)\cos(\pi y).$$

By substituting in (5.9.4), we get

$$\begin{aligned}
Q_{m+1}(y,t,v,w) &\geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\frac{\sin^2(\pi y)|v_1|}{\pi} - (m+1)\sin(2\pi y)t\right)\\
&= \frac{m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)}{t}\left(\frac{\sin(\pi y)|v_1|}{\pi} - 2(m+1)\cos(\pi y)t\right).
\end{aligned}$$

Subcase 2.1. Assume $\sin(\pi y)|v_1| \geq 2\pi(m+1)\cos(\pi y)t$. Then obviously $Q_{m+1}(y,t,v,w) \geq 0 \geq -\varepsilon$.
Subcase 2.2. Assume $2\pi(m+1)\cos(\pi y)t^2 \leq \sin(\pi y)|v_1| \leq 2\pi(m+1)\cos(\pi y)t$. We have

$$Q_{m+1}(y,t,v,w) \geq m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)2(m+1)\cos(\pi y)(t-1),$$

whose absolute value is less than or equal to

$$\begin{aligned}
&2(m+1)m(m-1)|v_1|\sin(\pi y) \leq 4\pi(m+1)^2 m(m-1)\cos(\pi y)t\\
&\leq 4\pi(m+1)^2 m(m-1)t \leq 4\pi(m+1)^2 m(m-1)t_\varepsilon \leq \varepsilon.
\end{aligned}$$

This shows that $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.
Subcase 2.3. Assume $\sin(\pi y)|v_1| \leq 2\pi(m+1)\cos(\pi y)t^2$. Recall that

$$Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)}{t}\left(\frac{\sin(\pi y)|v_1|}{\pi} - 2(m+1)\cos(\pi y)t\right).$$

The absolute value of the last term is less than or equal to

$$\begin{aligned}
&\frac{m(m-1)|v_1|\sin(\pi y)\left(\frac{1}{\pi}+2(m+1)\right)}{t} \leq \frac{m(m-1)2\pi(m+1)\cos(\pi y)t^2(1+2(m+1))}{t}\\
&\leq 2\pi(m+1)m(m-1)(2m+3)t \leq 2\pi(2m+3)(m+1)m(m-1)t_\varepsilon \leq \varepsilon.
\end{aligned}$$

Hence $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.
Case 3. Assume finally that $y \in (\frac{3}{4}, 1]$. Take $z = 1-y$. Clearly $\cos(2\pi z) = \cos(2\pi y)$, and $\sin(2\pi z) = -\sin(2\pi y)$. Therefore $\theta(z) = \theta(y)$ and $|\theta'(y)| = |\theta'(z)|$, hence

$$\begin{aligned}
Q_{m+1}(y,t,v,w) &\geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(y)|v_1| - (m+1)t|\theta'(y)|\right)\\
&= \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(z)|v_1| - (m+1)t|\theta'(z)|\right),
\end{aligned}$$

and since $z \in [0, \frac{1}{4})$, we can apply Case 2 with z instead of y to obtain $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.

We have thus shown the statement (2).

Finally, let us prove statement (3). For every $y \in [0,1],\ v, w \in \mathbb{S}^1$, we obtain from (5.9.1) that

$$D^{m+2}h(0,y)(v^2, w^m) = \sum_{i_1,\dots,i_{m+2}=1}^{2} \frac{\partial^{m+2}h}{\partial x_{i_1}\cdots\partial x_{i_{m+2}}}(0,y)v_{i_1}v_{i_2}w_{i_3}\dots w_{i_{m+2}},$$

where x_1 and x_2 stands for the variables x and y respectively. If we set $v = (0,1)$ and $w = (1,0)$, the only possible nonzero term is that for which the $(m+2)$-tuple $(i_1, \dots, i_{m+2})$ satisfies $i_1 = i_2 = 2$ and $i_3 = \cdots = i_{m+2} = 1$. This shows that

$$D^{m+2}h(0,y)(v^2, w^m) = \frac{\partial^{m+2}h}{\partial x^m \partial y^2}(0,y)\, v_2^2 w_1^m = m!\, \theta''(y) \tag{5.9.5}$$

Combining (5.9.2), (5.9.3) and (5.9.5) we get

$$\begin{aligned}
&Q_{m+2}(y,t,v,w)\\
&:= \frac{1}{t^m}\left(D^2h(0,y)(v^2) + \cdots + \frac{t^{m-1}}{(m-1)!}D^{m+1}h(0,y)(v^2,w^{m-1}) + \frac{t^m}{m!}D^{m+2}h(0,y)(v^2,w^m)\right)\\
&= \frac{m(m-1)\theta(y)v_1^2w_1^{m-2}}{t^2} + \frac{(m-1)\theta'(y)\left((m-1)v_1^2w_1^{m-2}w_2 + 2v_1v_2w_1^{m-1}\right)}{t} + \theta''(y) = \theta''(y).
\end{aligned}$$

But the function $\theta''(y) = 2\pi\cos(2\pi y)$ is strictly negative on the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$ and so is $Q_{m+2}(y,t,v,w)$ on that interval. Therefore, condition (CW^{m+2}) is not satisfied for h on C. $\square$

Chapter 6

C^∞ extensions of convex functions on $\mathbb{R}^n$.

6.1 Whitney's Extension Theorem for C^∞

The Whitney's Extension Theorem for C^∞ reads as follows.

Theorem 6.1 (Whitney's Extension Theorem for C^∞)**.** *Let $C \subset \mathbb{R}^n$ be a closed subset of $\mathbb{R}^n$ and $\{f_\alpha\}_{\alpha \in (\mathbb{N}\cup\{0\})^n}$ an infinite family of real valued functions defined on C. Let us write, for every positive integer $m \geq 0$,*

$$f_\alpha(x) = \sum_{|\beta| \leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha^m(x,y)$$

for all $x, y \in C$ and every multi-index α with $|\alpha| \leq m$. Then there exists a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha F = f_\alpha$ on C for every multi-index α if and only for every $m \geq 0$,

$$\lim_{|x-y| \to 0} \frac{|R_\alpha^m(x,y)|}{|x-y|^{m-|\alpha|}} = 0 \quad \textit{uniformly on} \quad x, y \in K$$

for every compact subset K of C and all $|\alpha| \leq m$.

An infinite family of real valued functions $\{f_\alpha\}_{\alpha \in (\mathbb{N}\cup\{0\})^n}$ defined on C is called a ∞-jet on C. Observe that Theorem 6.1 essentially says that a ∞-jet $\{f_\alpha\}_{\alpha \in (\mathbb{N}\cup\{0\})^n}$ defined on C has a extension $F \in C^\infty(\mathbb{R}^n)$, that is, $\partial^\alpha F = f_\alpha$ on C for every multi-index α, if and only if the m-jet $\{f_\alpha\}_{|\alpha| \leq m}$ satisfies condition (W^m) of Theorem 5.1 on C for every $m \geq 0$. If $\{Q_k\}_k$ are the *Whitney cubes* of $\mathbb{R}^n \setminus C$ (see Proposition 2.2), p_k denotes a point of C which minimizes the distance of C to the cube Q_k and $\{\varphi_k\}_k$ is the *Whitney partition of unity* associated to $\{Q_k\}_k$ (see Proposition 2.3), the function F of Theorem 6.1 can be defined via the formula

$$F(x) = \begin{cases} f(x) & \text{if } x \in C \\ \sum_k \left(\sum_{|\alpha| \leq \nu_k} \frac{f_\alpha(p_k)}{\alpha!}(x-p_k)^\alpha \right) \varphi_k(x) & \text{if } x \in \mathbb{R}^n \setminus C, \end{cases} \tag{6.1.1}$$

where $(\nu_k)_k$ is an strictly increasing sequence of integers carefully chosen and depending on the funcions $\{f_\alpha\}_{\alpha \in \mathbb{N}\cup\{0\}}$, on the set C and on the dimension n. An explicit exposition of this construction is given in [70].

Let us also restate Theorem 6.1 in terms of families of polynomials.

Definition 6.2. *Given a closed subset C of $\mathbb{R}^n$, we will say that family of polynomials $\{P_y^m\}_{y \in C, m \in \mathbb{N}\cup\{0\}}$ with $\deg(P_y^m) \leq m$ for every $y \in C$ and every $m \in \mathbb{N}\cup\{0\}$, is a compatible family of polynomials for C^∞ extension if for every $k \geq j$ the polynomial P_y^j is the Taylor polynomial of order j at y of the polynomial P_y^k for every $y \in C$.*

Theorem 6.3. *Let C be a closed subset of $\mathbb{R}^n$ and let $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ be a compatible family of polynomials for C^∞ extension with $\deg(P_y^m)\le m$ for every $y\in C$ and every $m\in\mathbb{N}\cup\{0\}$. There exists a function $F\in C^\infty(\mathbb{R}^n)$ with $J_y^m F = P_y^m$ for every $m\in\mathbb{N}\cup\{0\}$ and every $y\in C$ if and only if for each $m\in\mathbb{N}$ the subfamily $\{P_y^m\}_{y\in C}$ satisfies Whitney's condition (W^m) of order m of Theorem 5.2. We will abreviate this by saying that the compatible family $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies condition (W^∞) on C.*

Similarly, Whitney's Extension Theorem for C^∞ can be reformulated in terms of k-linear forms.

Theorem 6.4. *Let C be a closed subset of $\mathbb{R}^n$ and let $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ be a family of functions defined on C such that each A_m is $\mathcal{L}^m(\mathbb{R}^n,\mathbb{R})$-valued. There exists a function $F\in C^\infty(\mathbb{R}^n)$ with $D^mF = A_m$ on C for every $m\in\mathbb{N}\cup\{0\}$ if and only if for each $m\in\mathbb{N}$ the subfamily $\{A_k\}_{k=0}^m$ satisfies Whitney's condition (W^m) of Theorem 5.3. We will abreviate this by saying that the family $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ satisfies condition (W^∞) on C.*

With the same arguments as in Chapter 5, Section 5.1, one can see that Theorems 6.1, 6.3 and 6.4 are equivalent.

6.2 C^∞ convex extension theorem

We are now going to present our main result for smooth convex extension of jets. We saw in Chapter 5 that condition (CW^m) of Definitions 5.6, 5.7 and 5.8 is necessary for C^m convex extension but is not sufficient in general, as we learnt from Example 5.35. Nonetheless, for $m=\infty$ we are able to prove an *if and only if* theorem from compact convex subset (possibly of empty interior).

Let us now define the suitable condition for C^∞ convex extension. Similar to the C^m case (see Chapter 5, Section 5.3) we are going to give several (equivalent) definitions in terms of functions, polynomials and linear forms.

Definition 6.5 (Condition (CW^∞))**.** *Let C be a compact subset of $\mathbb{R}^n$ and let $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ be a compatible family of polynomials for C^∞ extension with $\deg(P_y^m)\le m$ for every $y\in C$ and every $m\in\mathbb{N}\cup\{0\}$. We will say that the family $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies condition (CW^∞) on C provided that, for each $m\ge 2$, every subfamily $\{P_y^m\}_{y\in C}$ satisfies condition (CW^m) of Definition 5.6, that is, for each $m\ge 2$,*

$$\liminf_{t\to 0^+}\frac{1}{t^{m-2}}\left(D^2P_y^m(y)(v^2)+tD^3P_y^m(y)(w,v^2)+\cdots+\frac{t^{m-2}}{(m-2)!}D^mP_y^m(y)(w^{m-2},v^2)\right)\ge 0$$

uniformly on $y\in C$, $v,w\in\mathbb{S}^{n-1}$.

Definition 6.6 ((CW^∞) condition for functions)**.** *We will say that a function $F:\mathbb{R}^n\to\mathbb{R}$ of class C^∞ satisfies condition (CW^∞) on a compact subset C provided that F satisfies, for every $m\ge 2$, condition (CW^m) on C in the sense of Definition 5.7*

Since for a function F of class $C^\infty(\mathbb{R}^n)$ and a subset C of $\mathbb{R}^n$, the family $\{J_y^mF\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ (where each J_y^mF denotes the Taylor polynomial of F of order m at the point y) satisfies that $D^m(J_y^mF)(y) = D^mF(y)$ for every $y\in C$ and every $m\in\mathbb{N}\cup\{0\}$, the condition (CW^∞) on C for F given in Definition 6.6 is equivalent to condition (CW^∞) on C for the family $\{J_y^mF\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ given in Definition 6.5. Finally, let us give the corresponding definition in terms on k-linear forms.

Definition 6.7 ((CW^∞) condition for linears forms)**.** *Given an infinite family of functions $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ defined on a compact subset C such that each A_m is $\mathcal{L}^m(\mathbb{R}^n,\mathbb{R})$-valued, we will say that $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ satisfies condition (CW^∞) on C provided that each subfamily $\{A_k\}_{k=0}^m$, $m\ge 2$, satisfies condition (CW^m) on C in the sense of Definition 5.8.*

It is clear that the three above definitions are equivalent. Let us first see that this condition is necessary for C^∞ convex extension.

Lemma 6.8. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^\infty(\mathbb{R}^n)$ such that F is convex on an open neighbourhood of a compact convex subset C of $\mathbb{R}^n$. Then F satisfies condition (CW^∞) on C.*

Proof. Using Lemma 5.9 we obtain that F satisfies condition (CW^m) on C for every $m \geq 2$, that is F satisfies (CW^∞) on C. □

From Remark 5.12 and Definition 6.5 we immediately observe the following.

Remark 6.9. If C is a convex compact subset of $\mathbb{R}^n$ with nonempty interior and $\{P_y^m\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ is a compatible family of polynomials for C^∞ extension, then a necessary condition for the existence of a convex function F of class $C^\infty(\mathbb{R}^n)$ with $J_y^m F = P_y^m$ for every $y \in C$ and every $m \in \mathbb{N}\cup\{0\}$ is that:

1. $\{P_y^m\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) on C and (CW^∞) on ∂C and $D^2P_y^m(y)(v^2) \geq 0$ for every $y \in \operatorname{int}(C)$ and every $v \in \mathbb{S}^{n-1}$ and every $m \geq 2$,

   or equivalently:

2. $\{P_y^m\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) on C and (CW^∞) on ∂C and each function $C \ni y \mapsto P_y^m(y),\ m \geq 2$, is convex.

Let us also see that condition (CW^k) with a strict inequality for some k (see Definition 5.6) automatically implies condition (CW^∞).

Proposition 6.10. *If C is a compact subset of $\mathbb{R}^n$ and the family $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) on C and there exists some integer $k > 2$ for which the subfamily $\{P_y^k\}y \in C$ satisfies condition (CW^k) on C with a strict inequality, then $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies condition (CW^∞).*

Proof. It immediately follows from Proposition 5.23 that each subfamily $\{P_y^m\}_{y\in C},\ m \geq 2$, satisfies condition (CW^m) (in fact, with a strict inequality) on C, that is, $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies (CW^∞) on C. □

We are now ready to present the main result of this chapter.

Theorem 6.11. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ be a compatible family of polynomials for C^∞ extension. Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with $J_y^m F = P_y^m$ for every $y \in C$ and $m \in \mathbb{N}\cup\{0\}$, if and only if the family $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) and (CW^∞) on C.*

Moreover, if C has nonempty interior, then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with $J_y^m F = P_y^m$ for every $y \in C$ and $m \in \mathbb{N}\cup\{0\}$, if and only if $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) on C, (CW^∞) on ∂C and the function $C \ni y \mapsto P_y^m(y)$ is convex for every $m \geq 2$.

An equivalent reformulation of this result is the following.

Theorem 6.12. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ be a family of functions defined on C such that each A_m is $\mathcal{L}^m(\mathbb{R}^n, \mathbb{R})$-valued. Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with $D^m F = A_m$ on C for every $m \in \mathbb{N}\cup\{0\}$ if and only if $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) and (CW^∞) on C.*

Moreover, if C has nonempty interior, then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with $D^m F = A_m$ on C for every $m \in \mathbb{N}\cup\{0\}$ if and only if $\{A_m\}_{m\in\mathbb{N}\cup\{0\}}$ satisfies (W^∞) on C, (CW^∞) on ∂C and $A_2(y)$ is semidefinite positive for every $y \in C$.

6.3 Proof of the main theorem

In this section we will prove Theorem 6.11. The *only if* part of the Theorem follows immediately from Lemma 6.8. On the other hand, if C has nonempty interior, Remark 6.9 tells us that the conditions (W^∞) on C, (CW^∞) on ∂C and that $C \ni C \mapsto P_y^m(y)$ is convex for each $m \geq 2$ are necessary for the existence of such a function F. Obviously, the major effort goes in proving the *if* part of the Theorem.

First of all, by using Whitney's extension theorem (see Theorem 6.3) we may and do assume that there exists a function $f \in C^\infty(\mathbb{R}^n)$, with $J_y^m f = P_y^m$ for all $m \in \mathbb{N}$ and all $y \in C$, and that f satisfies condition (CW^m) on C for every $m \geq 2$ in the sense of Definition 6.6. On the other hand, if C has nonempty interior and we assume that the family $\{P_y^m\}_{y\in C,\, m\in\mathbb{N}\cup\{0\}}$ satisfies conditions (CW^m) for every $m \geq 2$ only on ∂C and also that each function $C \ni y \mapsto P_y^m(y)$, $m \geq 2$ is convex, then the function f is convex on C, which in turn implies that $D^2 f(x)(v^2) \geq 0$ for every $x \in C$ and every $v \in \mathbb{S}^{n-1}$ because C has nonempty interior. This indicates that with either of the two sets of conditions of Theorem 6.11 (for arbitrary compact convex sets or for compact convex bodies), the function f satisfies

$$f \quad \text{satisfies} \quad (CW^\infty) \text{ on } \partial C \quad \text{and} \quad D^2 f(x)(v^2) \geq 0, \quad x \in C,\, v \in \mathbb{S}^{n-1}. \tag{6.3.1}$$

Since C is compact, multiplying f by a suitable bump function of class C^∞, we may also assume that f has a compact support contained in $C + \overline{B}(0,2)$.

6.3.1 Sketch of the proof.

We will follow a plan of proof similar to that of Theorem 5.14, see Subsection 5.4.1. However, we warn the reader that what we now say we are going to do is not exactly what we will actually do. Our proof could be rewritten to match this sketch exactly, but at the cost of adding further technicalities, which we do not feel would be pertinent. This proof has two main parts. In the first part we will estimate the possible lack of convexity of f outside C by using condition (CW^∞), that is Lemma 6.13. In fact, using a Whitney partition of unity, and some ideas from the proof of the Whitney extension theorem in the C^∞ case, we will construct a function $\eta \in C^\infty(\mathbb{R})$ such that $\eta \geq 0$, $\eta^{-1}(0) = (-\infty, 0]$, and $\min_{|v|=1} D^2 f(x)(v^2) \geq -\eta\,(d(x,C))$ for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$. In the second part of the proof we will compensate the lack of convexity of f outside C with the construction of a function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \geq 0$, $\psi^{-1}(0) = C$, and $\min_{|v|=1} D^2\psi(x)(v^2) \geq 2\eta\,(d(x,C))$. Then, by setting $F := f + \psi$ we will conclude the proof of Theorem 6.11.

As in the proof of Theorem 5.14, in order to construct such a function ψ, we will write C as an intersection of a family of half-spaces, and then we make an integral of suitable convex functions composed with the linear forms that provide those half-spaces.

6.3.2 Lower estimates for the Hessian of f.

We next show how the assumption of conditions (CW^m) for every $m \geq 2$ implies a lower bound for the Hessian of f in terms of the distance to C.

Lemma 6.13. *Given $m \in \mathbb{N}$ if $f \in C^m(\mathbb{R}^n)$ and f satisfies (CW^m) then there is a number $r_m > 0$ such that, whenever $d(x,C) \leq r_m$, we have*

$$D^2 f(x)(v^2) \geq -d(x,C)^{m-2}, \quad \textit{for all} \quad v \in \mathbb{S}^{n-1}.$$

Proof. If $x \in C$, the desired inequality follows immediately thanks to (6.3.1). Now, given $x \in \mathbb{R}^n \setminus C$, $|v| = 1$, $t := d(x,C)$, let y be the unique point of ∂C with the property that $d(x,C) = |x-y|$. Take $w = (x-y)/|x-y|$. We have $y = x + tw$. By Taylor's Theorem, we can write

$$\begin{aligned} D^2 f(x)(v^2) =& D^2 f(y)(v^2) + t\, D^3 f(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \\ &+ \frac{t^{m-2}}{(m-2)!}\left[D^m f(y+sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2)\right], \end{aligned}$$

for some $s \in [0, t]$. Since f satisfies condition (CW^m), there exists a positive number r_m, independent of y, v and w, for which

$$\inf_{0<r\le r_m} \left\{ \frac{D^2 f(y)(v^2) + rD^3 f(y)(w, v^2) + \cdots + \frac{r^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2)}{r^{m-2}} \right\} \ge -\frac{1}{2}.$$

Thus, for $0 < t \le r_m$,

$$D^2 f(x)(v^2) \ge -\frac{t^{m-2}}{2} + \frac{t^{m-2}}{(m-2)!} \left[D^m f(y + sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2) \right].$$

On the other hand, if $s \in [0, t]$, we can write

$$D^m f(y + sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2) \le \|D^m f(x + sw) - D^m f(y)\|,$$

where we denote $\|A\| := \sup_{u_1,\ldots,u_m \in \mathbb{S}^{n-1}} |A(u_1, \ldots, u_m)|$, for every symmetric m-linear form A on $\mathbb{R}^n$. Moreover, the above expression is smaller than or equal to

$$\varepsilon_m(t) := \sup_{\{z\in\mathbb{R}^n,\, z'\in\partial C,\, |z-z'|\le t\}} \|D^m f(z) - D^m f(z')\|.$$

Since $D^m f$ is uniformly continuous, there is $r'_m > 0$ such that if $0 < r \le r'_m$, then $\varepsilon_m(r) \le \frac{1}{2}$ (in fact we have $\lim_{r\to 0^+} \varepsilon_m(r) = 0$). Therefore, if we suppose $0 < t \le \min\{r_m, r'_m\}$, we obtain

$$D^2 f(x)(v)^2 \ge -\frac{t^{m-2}}{2} - \frac{t^{m-2}}{(m-2)!}\varepsilon_m(t) \ge -t^{m-2}.$$

□

6.3.3 A Whitney partition of unity on $(0, +\infty)$

For all $k \in \mathbb{Z}$, we define the closed intervals

$$I_k = [2^k, 2^{k+1}], \quad I_k^* = \left[\frac{3}{4}2^k, \frac{9}{8}2^{k+1}\right].$$

Obviously $(0, +\infty) = \bigcup_{k\in\mathbb{Z}} I_k$. We note that I_k and I_k^* have the same midpoint and $\ell(I_k^*) = \frac{3}{2}\ell(I_k)$, where $\ell(I_k) = 2^k$ denotes the length of I_k. In other words, the interval I_k^* is I_k expanded by the factor $3/2$.

Proposition 6.14. *The intervals I_k, I_k^* satisfy:*

1. *If $t \in I_k^*$, then*
$$\frac{3}{4}\ell(I_k) \le t \le \frac{9}{4}\ell(I_k).$$
2. *If I_k^* and I_j^* are not disjoint, then*
$$\frac{1}{2}\ell(I_k) \le \ell(I_j) \le 2\ell(I_k).$$
3. *Given any $t > 0$, there exists an open neighbourhood $U_t \subset (0, +\infty)$ of t such that U_t intersects at most 2 intervals of the collection $\{I_k^*\}_{k\in\mathbb{Z}}$.*

Proof.

1. If $t \in I_k^* = [\frac{3}{4}2^k, \frac{9}{8}2^{k+1}]$, we have $\ell(I_k) = 2^k$ and

$$\frac{3}{4}\ell(I_k) \leq t \leq \frac{9}{8}2^{k+1} = \frac{9}{4}\ell(I_k).$$

2. First of all, we note that $I_k^* \cap I_{k+2}^* = \emptyset$ for all $k \in \mathbb{Z}$. Indeed,

$$\sup(I_k^*) = \frac{9}{8}2^{k+1} = \frac{9}{16}2^{k+2} < \frac{3}{4}2^{k+2} = \inf(I_{k+2}^*).$$

This shows that if two intervals of the family $\{I_k^*\}_{k\in\mathbb{Z}}$ are not disjoint, then their indices must be two consecutive integers. Hence, if I_k^* and I_j^* are not disjoint, because $\ell(I_k) = 2^k$ and $\ell(I_j) = 2^j$, we immediately obtain

$$\frac{1}{2}\ell(I_k) \leq \ell(I_j) \leq 2\ell(I_k).$$

3. Fix $t > 0$ and $k \in \mathbb{Z}$ with $t \in I_k$. Then $\operatorname{int}(I_k^*)$ is an open neighborhood of t and, thanks to the preceding remark, I_k^* cannot intersect I_{k-2}^* or I_{k+2}^*. Thus I_k^* can only intersect with some of the intervals I_{k-1}^*, I_k^* or I_{k+1}^*. Now we claim that either I_k^* contains an open neighborhood of t which can only intersect with I_k^* or I_{k-1}^* or else I_k^* contains an open neighborhood of t which can only intersect with I_k^* or I_{k+1}^*. To check this, let us study two cases.
Suppose first that $2^k \leq t \leq \frac{9}{8}2^k$. Set $U_t = (t - \frac{2^k}{8}, t + \frac{2^k}{8})$. If $t' \in U_t$, then

$$\inf(I_k^*) = \frac{3}{4}2^k < 2^k - \frac{2^k}{8} \leq t - \frac{2^k}{8} < t' < t + \frac{2^k}{8} \leq \frac{10}{8}2^k < \frac{3}{4}2^{k+1} = \inf(I_{k+1}^*) < \sup(I_k^*).$$

This shows that $U_t \subset \operatorname{int}(I_k^*)$ and that U_t and I_{k+1}^* are disjoint. Thus U_t can only intersect with I_{k-1}^* or I_k^*.
Now, we suppose that $\frac{9}{8}2^k < t \leq 2^{k+1}$. If we take $\delta = \min\{t - \frac{9}{8}2^k, \frac{2^{k+1}}{8}\}$ and set $U_t = (t-\delta, t+\delta)$, we have for all $t' \in U_t$:

$$\inf(I_k^*) < \sup(I_{k-1}^*) = \frac{9}{8}2^k \leq t - \delta < t' < t + \delta \leq \frac{9}{8}2^{k+1} = \sup(I_k^*).$$

Hence $U_t \subset \operatorname{int}(I_k^*)$ and that U_t and I_{k-1}^* are disjoint. Thus U_t can only intersect I_{k+1}^* or I_k^*. $\square$

This is a special case of the decomposition of an open set in Whitney's cubes, see Proposition 2.2 for instance. In the one dimensional case things are much simpler and, for instance, one may replace the number $N = 12$ in Proposition 2.2 with the number 2. Anyhow, dealing with the number 12 instead of 2 would have no harmful effect in our proof.

We now relabel the families $\{I_k\}_k$ and $\{I_k^*\}_k$, $k \in \mathbb{Z}$, as sequences indexed by $k \in \mathbb{N}$, so we will write $\{I_k\}_{k\geq 1}$ and $\{I_k^*\}_{k\geq 1}$. For every $k \geq 1$, we will denote by t_k and ℓ_k the midpoint and the length of I_k, respectively.

Next we recall how to define a Whitney partition of unity subordinated to the intervals I_k^*. Let us take a bump function $\theta_0 \in C^\infty(\mathbb{R})$ with $0 \leq \theta_0 \leq 1$, $\theta_0 = 1$ on $[-1/2, 1/2]$; and $\theta_0 = 0$ on $\mathbb{R} \setminus (-\frac{3}{4}, \frac{3}{4})$. For every k, we define the function θ_k by

$$\theta_k(t) = \theta_0\left(\frac{t - t_k}{\ell_k}\right), \quad t \in \mathbb{R}.$$

It is clear that $\theta_k \in C^\infty(\mathbb{R})$, that $0 \leq \theta_k \leq 1$, that $\theta_k = 1$ on I_k, and that $\theta_k = 0$ outside $\operatorname{int}(I_k^*)$.
Now we consider the function $\Phi = \sum_{k\geq 1}\theta_k$ defined on $(0, +\infty)$. Using Proposition 6.14, every point $t > 0$ has an open neighbourhood which is contained in $(0, +\infty)$ and intersects at most two of the intervals $\{I_k^*\}_k$. Since $\operatorname{supp}(\theta_k) \subset I_k^*$, the sum defining Φ has only two terms and therefore Φ is of class C^∞. For the same reason, $\Phi(t) = \sum_{I_k^* \ni t}\theta_k(t) \leq 2$, for $t > 0$. On the other hand, every $t > 0$ must be contained in some I_k, where the function θ_k takes the constant value 1, so we have $1 \leq \Phi \leq 2$. These properties allow us to define, on $(0, \infty)$, the functions $\theta_k^* = \frac{\theta_k}{\Phi}$. These are C^∞ functions satisfying $\sum_k \theta_k^* = 1$, $0 \leq \theta_k^* \leq 1$, and $\operatorname{supp}(\theta_k^*) \subseteq I_k^*$. Less elementary, but crucial, is the following property. See [70, 63] for a proof in the more general setting of $\mathbb{R}^n$.

Proposition 6.15. *For every $j \in \mathbb{N} \cup \{0\}$, there exist positive constants A_j, A'_j, A''_j such that, for all $t > 0$ and $k \geq 1$,*

(1) $|\theta_k^{(j)}(t)| \leq A'_j \ell_k^{-j}$.

(2) *If $t \in I_k^*$, then* $|\Phi^{(j)}(t)| \leq A''_j \ell_k^{-j}$.

(3) $|(\theta_k^*)^{(j)}(t)| \leq A_j \ell_k^{-j}$.

Proof. For the case $j = 0$ it is enough to take $A_0 = A'_0 = 1$ and $A''_0 = 2$. For $j \geq 1$, we let us check the three inequalities separately.
(1) Since $\operatorname{supp}(\theta_0)$ is compact, there exists $A'_j > 0$ with $|\theta_0^{(j)}(t)| \leq A'_j$ for all $t > 0$. Given $k \geq 1$, we have

$$\theta_k^{(j)}(t) = \frac{1}{\ell_k^j} \theta_0^{(j)} \left(\frac{t - t_k}{\ell_k} \right),$$

and therefore $|\theta_k^{(j)}(t)| \leq A'_j \ell_k^{-j}$.
(2) If $t \in I_k^*$ and $i \in \mathbb{N}$ are such that $t \in I_i^*$, then we have $\frac{1}{2}\ell_k \leq \ell_i \leq 2\ell_k$ by virtue of Proposition 6.14. Then inequality (1) yields

$$|\Phi^{(j)}(t)| = \left| \sum_{I_i^* \ni t} \theta_i^{(j)}(t) \right| \leq \sum_{I_i^* \ni t} A'_j \ell_i^{-j} \leq \sum_{I_i^* \ni t} A'_j \left(\frac{1}{2} \ell_k \right)^{-j}.$$

Because in the above sum there are at most 2 nonzero terms, we have $|\Phi^{(j)}(t)| \leq 2A'_j \frac{\ell_k^{-j}}{2^{-j}} = A''_j \ell_k^{-j}$, where $A''_j := 2^{j+1}$.
(3) Since $\operatorname{supp}(\theta_k^*) \subseteq I_k^*$, we may and do assume that $t \in I_k^*$. Let us prove this statement by induction on j. For $j = 1$, note that $(\Phi\theta_k^*)' = \theta'_k$. Thus we have

$$(\theta_k^*)' = \frac{\theta'_k - \theta_k^* \Phi'}{\Phi}.$$

The facts that $0 \leq \theta_k^* \leq 1$ and $\Phi \geq 1$ together with statements (1) and (2) lead us to

$$|(\theta_k^*)'(t)| \leq \frac{|\theta'_k(t)| + |\theta_k^*(t)||\Phi'(t)|}{|\Phi(t)|} \leq A'_1 \ell_k^{-1} + A''_1 \ell_k^{-1} = A_1 \ell_k^{-1},$$

where $A_1 := A'_1 + A''_1$. Now, let us suppose that, for every $1 \leq l \leq j$, there exist $A_l > 0$ such as in inequality (3). We next use Leibniz's rule in order to compute the $(j+1)$-th derivative of $\theta_k^* \Phi = \theta_k$ and then we write separately the $(j+1)$-th derivative of θ_k^* and the rest of the sum to obtain

$$\Phi(\theta_k^*)^{(j+1)} + \sum_{l=0}^{j} \binom{j+1}{l} (\theta_k^*)^{(l)} \Phi^{(j+1-l)} = \theta_k^{j+1}.$$

Using first the fact that $\Phi \geq 1$ and then inequalities (1) and (2) together with the induction hypothesis, we can estimate $|(\theta_k^*)^{(j+1)}(t)|$ in the following way:

$$\begin{aligned} |(\theta_k^*)^{(j+1)}(t)| &\leq |\theta_k^{(j+1)}(t)| + \sum_{l=0}^{j} \binom{j+1}{l} |(\theta_k^*)^{(l)}(t)||\Phi^{(j+1-l)}(t)| \\ &\leq A'_{j+1} \ell_k^{-(j+1)} + \sum_{l=0}^{j} \binom{j+1}{l} A_l \ell_k^{-l} A''_{j+1-l} \ell_k^{-(j+1-l)} = A_{j+1} \ell_k^{-(j+1)}, \end{aligned}$$

where $A_{j+1} := A'_{j+1} + \sum_{l=0}^{j} \binom{j+1}{l} A_l A''_{j+1-l}$. This concludes the proof of (3). $\square$

6.3.4 The sequence $\{\delta_p\}_p$ and the function ε

Let us consider the numbers r_m of Lemma 6.13. It is clear that we can construct a sequence $\{\delta_p\}_p$ of positive numbers satisfying

$$\delta_p \leq \min\left\{ r_{p+2}, \frac{1}{(p+2)!} \right\}, \text{ for } p \geq 1$$
$$\delta_p < \frac{\delta_{p-1}}{2}, \text{ for } p \geq 2.$$

Of course the sequence $\{\delta_p\}_p$ is strictly decreasing to 0. Now, for every k we define a positive integer γ_k as follows. In the case that $\ell_k \geq \delta_1$, we set $\gamma_k = 1$. In the opposite case, $\ell_k < \delta_1$, we take γ_k as the unique positive integer for which

$$\delta_{\gamma_k+1} \leq \ell_k < \delta_{\gamma_k}.$$

Finally let us define:

$$\varepsilon(t) = \begin{cases} \sum_{k\geq 1} t^{\gamma_k} \theta_k^*(t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

In the following lemma we show that ε is of class C^∞ on $\mathbb{R}$ and satisfies an additional property which will be important in Subsection 6.3.6.

Lemma 6.16. *The function ε satisfies the following properties.*

(1) *ε is of class $C^\infty(\mathbb{R})$ and satisfies $\varepsilon^{(j)}(0) = 0$ for every $j \in \mathbb{N} \cup \{0\}$.*

(2) *If $0 < t \leq \delta_4$ and $q \in \mathbb{N}$ are such that $\delta_{q+1} \leq t < \delta_q$ and $\frac{t}{2} \leq s \leq t$, then $\varepsilon(2s) \geq t^{q+2}$.*

Proof. For the first statement, we immediately see that $\varepsilon^{-1}(0) = (-\infty, 0]$, that $\varepsilon > 0$ on $(0, +\infty)$ and that $\varepsilon \in C^\infty(\mathbb{R} \setminus \{0\})$. In order to prove the differentiability of ε at $t = 0$ and that all the derivatives of ε at $t = 0$ are 0, it is sufficient to prove that for all $j \in \mathbb{N} \cup \{0\}$,

$$\lim_{t \to 0^+} \frac{|\varepsilon^{(j)}(t)|}{t} = 0.$$

To check this, fix $j \in \mathbb{N} \cup \{0\}$ and $\eta > 0$ and take

$$\widetilde{t_j} := \min\left\{ \frac{\eta}{2B_j 4^j (j+1)!}, \delta_{j+5} \right\}, \quad \text{where} \quad B_j = \max\{A_l \,:\, 0 \leq l \leq j\}.$$

Recall that the numbers A_l are those given by Proposition 6.15. Let $0 < t \leq \widetilde{t_j}$. Due to the fact that $\{\delta_p\}_p$ is strictly decreasing, we can find a unique positive integer q such that $\delta_{q+1} \leq t < \delta_q$, and because $t \leq \delta_{j+5} < \delta_1$, we must have $q \geq j + 4$. Now, if k is such that $t \in I_k^*$, Proposition 6.14 tells us that

$$\ell_k \leq \frac{4}{3} t < 2t \leq 2\delta_{j+5} < \delta_1,$$

and using the definition of γ_k, we have

$$\delta_{\gamma_k+1} \leq \ell_k \leq \frac{4}{3} t < 2t < 2\delta_q < \delta_{q-1}.$$

The above inequalities imply that $\gamma_k + 1 > q - 1$, that is $\gamma_k \geq q - 1$. In particular $\gamma_k \geq j + 3$. On the other hand, using Proposition 6.14 again, we obtain:

$$\delta_{\gamma_k} > \ell_k \geq \frac{4t}{9} \geq \frac{t}{4} \geq \frac{\delta_{q+1}}{4} > \delta_{q+3},$$

so $\gamma_k \leq q+2$.
If we use Leibnitz's Rule, we obtain

$$\varepsilon^{(j)}(t) = \sum_{k\geq 1}\sum_{l=0}^{j}\binom{j}{l}\frac{d^l}{dt^l}(t^{\gamma_k})(\theta_k^*)^{(j-l)}(t),$$

and since $\gamma_k \geq j+3$ for those k such that $t \in I_k^*$, we can write

$$\frac{|\varepsilon^{(j)}(t)|}{t} = \Big|\sum_{I_k^*\ni t}\sum_{l=0}^{j}\binom{j}{l}\frac{\gamma_k!}{(\gamma_k-l)!}t^{\gamma_k-l-1}(\theta_k^*)^{(j-l)}(t)\Big| \leq \sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\,\gamma_k!\,t^{\gamma_k-l-1}A_{j-l}\ell_k^{l-j}.$$

Now, by Proposition 6.14 we know that $\ell_k \geq \frac{4}{9}t \geq \frac{1}{4}t$. Moreover, because $\gamma_k \leq q+2$, we have $\gamma_k! \leq (q+2)!$ and the last sum is smaller than or equal to

$$\sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\,(q+2)!\,t^{\gamma_k-l-1}\,A_{j-l}\,\frac{t^{l-j}}{4^{l-j}}.$$

Writing $t^{\gamma_k-l-1} = t^2 t^{\gamma_k-l-3} \leq t\,\delta_q\,t^{\gamma_k-l-3}$, this sum is smaller than or equal to

$$\sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\,(q+2)!\,t\,\delta_q\,t^{\gamma_k-l-3}\,A_{j-l}\,\frac{t^{l-j}}{4^{l-j}} \leq \left(4^j j! B_j \sum_{I_k^*\ni t}\sum_{l=0}^{j}(q+2)!\delta_q\,t^{\gamma_k-j-3}\right)t.$$

Noting that $t \leq \delta_{j+5} < 1$ and $\gamma_k \geq j+3$, we must have $t^{\gamma_k-j-3} \leq 1$. By construction of the sequence $\{\delta_p\}_p$ we have that $(q+2)!\,\delta_q \leq 1$, and using that the sum $\sum_{I_k^*\ni t}$ has at most 2 terms, we obtain

$$\frac{|\varepsilon^{(j)}(t)|}{t} \leq 4^j(j+1)j!\,2B_j t \leq 4^j(j+1)!\,2B_j\widetilde{t_j} \leq \eta.$$

This completes the proof of statement (1).
Now we prove the second statement. First of all, we note that $\delta_{q+1} \leq t \leq 2s \leq 2t < 2\delta_q < \delta_{q-1}$, and in particular $q \geq 3$. Let us suppose that $2s \in I_k^*$. Using Proposition 6.14,

$$\delta_{\gamma_k+1} \leq \ell_k \leq \frac{4}{3}(2s) < 2(2s) < 2\delta_{q-1} < \delta_{q-2},$$

that is $\gamma_k \geq q-2$. If we use Proposition 6.14 again,

$$\delta_{\gamma_k} > \ell_k \geq \frac{4(2s)}{9} \geq \frac{(2s)}{4} \geq \frac{\delta_{q+1}}{4} > \delta_{q+3},$$

and then $\gamma_k \leq q+2$.
Finally, note that $2s \leq 2t < \delta_{q-1} < \delta_1 < 1$, and due to the fact that $\gamma_k \leq q+2$ for those k such that $2s \in I_k^*$, we have that $(2s)^{q+2} \leq (2s)^{\gamma_k}$. This allows us to obtain the desired inequality:

$$t^{q+2} \leq (2s)^{q+2} = \sum_{I_k^*\ni 2s}(2s)^{q+2}\theta_k^*(2s) \leq \sum_{I_k^*\ni 2s}(2s)^{\gamma_k}\theta_k^*(2s) = \varepsilon(2s).$$

□

6.3.5 The function φ

As we said in Subsection 6.3.1 we need to construct a function φ which compensates the *lack of convexity* of f given by Lemma 6.13. We begin by defining

$$\tilde{\varepsilon}(t) = \begin{cases} \dfrac{\varepsilon(2t)}{t^{n+3}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Since $\varepsilon \in C^\infty(\mathbb{R})$, with $\varepsilon^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$, we have that $\tilde{\varepsilon} \in C^\infty(\mathbb{R})$ and $\tilde{\varepsilon}^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$ as well. Now, let us consider the function

$$g(t) = \begin{cases} \int_0^t \int_0^s \tilde{\varepsilon}(r) dr\, ds & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

It is clear that $g \in C^\infty(\mathbb{R})$ and $g^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$. In addition, $g^{-1}(0) = (-\infty, 0]$ and $g''(t) = \tilde{\varepsilon}(t) > 0$ for all $t > 0$. In particular, g is convex on $\mathbb{R}$ and positive, with a strictly positive second derivative on $(0, +\infty)$.
Now, for every vector $w \in \mathbb{S}^{n-1}$, define $h(w) = \max_{z \in C} \langle z, w \rangle$, the support function of C. We also define the function

$$\begin{array}{rccl} \phi: & \mathbb{S}^{n-1} \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (w, x) & \longmapsto & \phi(w, x) = g(\langle x, w \rangle - h(w)). \end{array}$$

Using similar arguments as in Chapter 5, Subsection 5.4.3 it follows that $\phi(w, \cdot)$ is a function of class $C^\infty(\mathbb{R}^n)$ whose derivatives of all order and $\phi(w, \cdot)$ itself vanish on C for every $w \in \mathbb{S}^{n-1}$. Moreover, the function $\phi(w, \cdot)$, being a composition of a convex function with a non-decreasing convex function, is convex as well. Finally, we define the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} \phi(w, x)\, dw \quad \text{for every} \quad x \in \mathbb{R}^n.$$

We have that $\varphi^{-1}(0) = C$ and that φ is convex on $\mathbb{R}^n$. Because $\phi(w, \cdot)$ is in $C^\infty(\mathbb{R}^n)$, the derivatives $(w, x) \mapsto \frac{\partial^\alpha}{\partial x^\alpha} \phi(w, x)$ are continuous for every multi-index α, and $\mathbb{S}^{n-1}$ is compact, it follows from standard results on differentiation under the integral sign that the function φ is of class $C^\infty(\mathbb{R}^n)$ as well, and that $\partial^\alpha \varphi(x) = 0$ for every $x \in C$ and every multi-index α. In other words, $J_x^m \varphi = 0$ for all $m \in \mathbb{N} \cup \{0\}$ and all $x \in C$. Besides, the second derivative of φ is

$$D^2 \varphi(x)(v^2) = \int_{\mathbb{S}^{n-1}} g''(\langle x, w \rangle - h(w)) \langle w, v \rangle^2\, dw, \quad x \in \mathbb{R}^n,\ v \in \mathbb{S}^{n-1}.$$

6.3.6 A smooth convex extension on a neighbourhood of the domain

In the same spirit as in the proof of Theorem 5.14, Subsection 5.4.5, we are going to compensate the lack of convexity of our function f with the function φ that we have just constructed. Using the constant $V(n)$, obtained in Lemma 5.18, define

$$C(n) = \frac{V(n)}{36(1 + \operatorname{diam}(C))^{n+1}}.$$

Lemma 6.17. *With the notation of Subsection 6.3.4, consider the function $H = f + \frac{2}{C(n)} \varphi$ defined on $\mathbb{R}^n$, and take $r = \delta_4$. Then, for every $x \in \mathbb{R}^n$ such that $t := d(x, C) \leq r$, and for every $v \in \mathbb{S}^{n-1}$, we have*

$$D^2 H(x)(v^2) \geq t^q,$$

where q is the unique positive integer such that $\delta_{q+1} \leq t < \delta_q$.

Proof. As we saw in the proof of Lemma 5.19, we have that

$$D^2\varphi(x)(v^2) > 0 \quad \text{for every} \quad x \in \mathbb{R}^n \setminus C,\ v \in \mathbb{S}^{n-1}. \tag{6.3.2}$$

Now, fix x, t, v, q as in the statement. If $x \in C$ there is nothing to prove because φ is convex on $\mathbb{R}^n$ and $D^2 f(x) \geq 0$ by (6.3.1). Let us now suppose that $x \in \mathbb{R}^n \setminus C$. Since $D^2H(x)(v^2) = D^2H(x)((-v)^2)$, we may suppose that $\langle v, u_x \rangle \geq 0$, where

$$u_x = \frac{1}{|x - x_C|}(x - x_C)$$

and $x_C \in \partial C$ is the metric projection of x onto C. Take the angle α_x and the set $W = W(x, v)$ as in Lemmas 5.16 and 5.18 respectively. By the construction of φ, we have

$$D^2\varphi(x)(v^2) = \int_{\mathbb{S}^{n-1}} \tilde{\varepsilon}(\langle x, w\rangle - h(w))\langle w, v\rangle^2 \, dw \geq \int_W \tilde{\varepsilon}(\langle x, w\rangle - h(w))\langle w, v\rangle^2 \, dw > 0, \tag{6.3.3}$$

and for $w \in W$, Lemma 5.18 gives us that $\widehat{w\, u_x} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$; on the other hand Lemma 5.16 says that, in this case,

$$\frac{t}{2} \leq \langle x, w\rangle - h(w) \leq t \leq \delta_4.$$

Using the second statement of Lemma 6.16 we obtain

$$\tilde{\varepsilon}(\langle x, w\rangle - h(w)) = \frac{\varepsilon(2(\langle x, w\rangle - h(w)))}{(\langle x, w\rangle - h(w))^{n+3}} \geq \frac{t^{q+2}}{(\langle x, w\rangle - h(w))^{n+3}} \geq \frac{t^{q+2}}{t^{n+3}} = \frac{t^q}{t^{n+1}}.$$

On the other hand, due to Lemma 5.18 the product $\langle v, w\rangle$ is greater than or equal to $\sin(\frac{\alpha_x}{3})$ for all $w \in W$. By combining the preceding inequalities, we get

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}} \sin^2\left(\frac{\alpha_x}{3}\right) \mathcal{H}^{n-1}(W).$$

By the third part of Lemma 5.18, the last term is greater or equal than

$$\frac{t^q}{t^{n+1}} \sin^2\left(\frac{\alpha_x}{3}\right) V(n)\alpha_x^{n-1},$$

where $V(n)$ is a positive constant only depending on n. Since $\alpha_x \leq 1$, we have that $\sin(\frac{\alpha_x}{3}) \geq \frac{1}{2}\frac{\alpha_x}{3}$, so we obtain

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}} \frac{\alpha_x^2}{36} V(n)\alpha_x^{n-1} = \frac{t^q}{t^{n+1}} \frac{\alpha_x^{n+1}}{36} V(n).$$

Moreover, we have

$$\alpha_x = \frac{t}{t + \operatorname{diam}(C)} \geq \frac{t}{1 + \operatorname{diam}(C)},$$

because $t \leq r = \delta_4 < 1$. Gathering these inequalities, we get

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}} \frac{t^{n+1}}{36(1 + \operatorname{diam}(C))^{n+1}} V(n) = C(n)t^q.$$

Finally, due to the construction of the sequence $\{\delta_p\}_p$, (see Subsection 6.3.4) we have $d(x, C) = t < \delta_q \leq r_{q+2}$, hence Lemma 6.13 ensures that

$$D^2 f(x)(v^2) \geq -t^q.$$

Therefore

$$D^2H(x)(v^2) = D^2 f(x)(v^2) + \frac{2}{C(n)} D^2\varphi(x)(v^2) \geq -t^q + 2t^q = t^q.$$

$\square$

Since $J_y^m \varphi = 0$ for $y \in C$ and each $m \in \mathbb{N} \cup \{0\}$, we have proved that H is of class $C^\infty(\mathbb{R}^n)$, $H = f$ on C, $J_y^m H = J_y^m f = P_y^m$ for every $y \in C$ and every $m \in \mathbb{N}$, and H has a strictly positive Hessian on the set $\{x \in \mathbb{R}^n : 0 < d(x, C) \leq r\}$.

6.3.7 Conclusion of the proof: convexity of the extension on $\mathbb{R}^n$

To complete the proof of Theorem 6.11 we only have to change the funcion H slightly.

Lemma 6.18. *There exists a number $a > 0$ such that the function $F := f + a\varphi$ is of class $C^\infty(\mathbb{R}^n)$, concides with f on C, satisfies $J_y^m F = P_y^m$ for every $y \in C$, $m \in \mathbb{N}$, is convex on $\mathbb{R}^n$, and has a strictly positive Hessian on $\mathbb{R}^n \setminus C$.*

Proof. Let us denote $\psi = \frac{2}{C(n)}\varphi$. We recall that $f = 0$ outside $C + B(0,2)$. Take $r > 0$ as in Lemma 6.17. Since $C_r := \{x \in \mathbb{R}^n \ : \ r \leq d(x,C) \leq 2\}$ is a compact subset where ψ has a strictly positive Hessian (see inequality (6.3.2)), and using again that f has compact support, we can find $M \geq 1$ such that

$$\sup_{x\in\mathbb{R}^n,\, v\in\mathbb{S}^{n-1}} |D^2 f(x)(v^2)| \leq M \quad \text{and} \quad \inf_{x\in C_r,\, v\in\mathbb{S}^{n-1}} D^2\psi(x)(v^2) \geq \frac{1}{M}. \tag{6.3.4}$$

Let us take $A = 2M^2$ and $F = f + A\psi$. If $d(x,C) \leq r$ (this includes the situation $x \in C$) and $v \in \mathbb{S}^{n-1}$ we have, by Lemma 6.17, that

$$D^2F(x)(v^2) = 2M^2 D^2\psi(x)(v^2) + D^2 f(x)(v^2) \geq D^2\psi(x)(v^2) + D^2 f(x)(v^2) \geq 0.$$

In the case when $d(x,C) \in [r,2]$, given any $|v| = 1$, the inequalities of (6.3.4) yield

$$D^2F(x)(v^2) = 2M^2 D^2\psi(x)(v^2) + D^2 f(x)(v^2) \geq 2M - M = M > 0.$$

Finally, in the region $\{x \in \mathbb{R}^n \ : \ d(x,C) > 2\}$, we have that $f = 0$. Hence

$$D^2F(x)(v^2) = 2M^2 D^2\psi(x)(v^2) > 0$$

by virtue of (6.3.2). Therefore, by setting $a = 2A/C(n)$, we get that the function $F = f + A\psi = f + a\varphi$ is of class $C^\infty(\mathbb{R}^n)$, satisfies $F(y) = f(y)$ and $J_y^m F = P_y^m$ for every $y \in C$, $m \in \mathbb{N}$, and has a nonnegative Hessian on $\mathbb{R}^n$. This proves that F is convex on $\mathbb{R}^n$. In fact, the Hessian of f is strictly positive on $\mathbb{R}^n \setminus C$. $\square$

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