On Some Paracompactness-type Properties of Fuzzy Topological Spaces

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Abstract

The aim of this paper is to study some paracompactness-type properties for fuzzy topological spaces. we prove that these properties are good extensions of others defined by A.V.Arkhangel'skii (and studied by S.A. Peregudov) and obtain several results about them.

Keywords: Fuzzy sets; Topology; paracompactness; fuzzy perfect maps.

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1 Introduction

On 1971-1972, A.V. Arhangel'kii introduced some new paracompactness-type properties: a-paracompactness, a^*-paracompactness and b-paracompactness. Later, S.A. Peregudov did some research on these concepts [9,10].

In this paper, we define fuzzy extensions of these notions, prove that these properties are good extensions of those defined by A.V. Arhangel'skiii, and obtain several results about them.

First, we give some definitions:

Definition 1 *Let μ be a fuzzy set in a fuzzy topological space (X,τ). We will say that μ is fuzzy a^*-paracompact (respectively, a-paracompact) if for each open cover (in Lowen's sense) $\mathcal{U}$ of μ and for each $\zeta \in (0,1]$, there exits an open refinement $\mathcal{V}$ of $\mathcal{U}$ which covers $\mu - \xi$ (in Lowen's sense) and every infinite subfamily of $\mathcal{V}$ contains an infinite pairwise disjoint subfamily (respectively, contains a disjoint pair of elements). We will say that (X,τ) is fuzzy a^*-paracompact (respectively, a-paracompact) if each constant fuzzy set in X is fuzzy a^*-paracompact (respectively, a-paracompact).*

Definition 2 *Let μ be a fuzzy set in a fuzzy topological space (X,τ). We will say that μ is fuzzy b-paracompact if for each open cover (in*

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Lowen's sense) $\mathcal{U}$ *of* μ *and for each* $\zeta \in (0,1]$, *there exits an open refinement* $\mathcal{V}$ *of* $\mathcal{U}$ *which covers* $\mu - \xi$ *(in Lowen's sense) and every subfamily of* $\mathcal{V}$ *with the finite intersection property, is finite. We will say that* (X,τ) *is fuzzy b-paracompact if each constant fuzzy set in* X *is fuzzy b-paracompact.*

Definition 3 *[4] Let* (X,T) *be a topological space,* $\omega(T) = \{\mu : (X,T) \to I \mid \mu$ *is a lower semi-continuos map*$\}$.*The fuzzy topological space* $(X,\omega(T))$ *is said induced by* (X,T).

Definition 4 *[4] Let* $\mathcal{P}$ *be a property of topological spaces and* $\mathcal{P}^*$ *be a fuzzy extension of* $\mathcal{P}$. *The concept* $\mathcal{P}^*$ *is a good extension of* $\mathcal{P}$ *if a topological space* (X,T) *verifies* $\mathcal{P}$ *if and only if* $(X,\omega(T))$ *verifies* $\mathcal{P}^*$.

Definition 5 *[7] A fuzzy topological space* (X,τ) *is a weakly inducement of an ordinary topological space* (X,T) *if:*

(a) $T = [\tau]$, *where* $[\tau] = \{A \subset X \ / \ \chi_A \in \tau\}$ *is the original topology of* τ.

(b) every $\mu \in \tau$ *is a lower semi-continuos map from* (X,T) *into* $I = [0,1]$.

2 Results

Theorem 6 *A topological space* $(X,[\tau])$ *is b-paracompact if and only if* (X,τ) *is fuzzy b-paracompact.*

Proof. $(\Longrightarrow)$ *For each* $r \in I$, *let a family* $\mathcal{B} \subset \tau$ *such that* $\bigvee_{G\in\mathcal{B}} G \geq r$. *Let* $\xi \in (0,r]$, *then the family* $\mathcal{U} = \{G^{-1}(r-\xi,1] \ / \ G \in \mathcal{B}\} \subset [\tau]$ *is an open cover of* $(X,[\tau])$, *and from the hypothesis, it has an open refinement* $\mathcal{U}^* \subset [\tau]$ *which is a cover of* $\mu - \xi$ *and every subfamily of* $\mathcal{U}^*$ *with the finite intersection property is finite.*

For each $V \in \mathcal{U}^*$, *there exits* $G_V^{-1}(r-\xi,1] \in \mathcal{U}$ *such that* $V \subset G_V^{-1}(r-\xi,1]$. *Then* $\mathcal{B}^* = \{\chi_V \wedge G_V \ / \ V \in \mathcal{U}^*\} \subset \tau$ *and refines* $\mathcal{B}$.

Now, for each $x \in X$, *there exits* $V \in \mathcal{U}^*$ *such that* $\chi_V(x) = 1$ *and* $G_V(x) > r - \xi$. *Thus,* $(\chi_V \wedge G_V)(x) \geq r - \xi$ *and* $\bigvee_{V\in\mathcal{U}^*} (\chi_V \wedge G_V)(x) \geq r - \xi$.

For each $\mathcal{B}_1^* \subset \mathcal{B}^*$ *such that* $\mathcal{B}_1^*$ *has the finite intersection property, we have that also,* $\{V_j \ / \ \chi_{V_j} \wedge G_{Vj} \in \mathcal{B}_1^*\}$ *has the finite intersection property, because, if for each* $\{\chi_{V_1} \wedge G_{V_1}, ...\chi_{V_n} \wedge G_{V_n}\} \subset \mathcal{B}_1^*$ *is* $(\chi_{V_1} \wedge G_{V_1}) \wedge ... \wedge (\chi_{V_n} \wedge G_{V_n}) \neq 0$, *there exists* $x \in X$ *such that, for each* $i \in \{1,...n\}$ *is* $(\chi_{Vi} \wedge G_{V_i})(x) \neq 0$ *(then,* $x \in V_i$ *and* $G_{V_i}(x) \neq 0$*). So,* $V_1 \cap ... \cap V_n \neq \varnothing$.

From the hypothesis, $\{V_j \ / \ \chi_{V_j} \wedge G_{Vj} \in \mathcal{B}_1^*\}$ *is finite, then* $\mathcal{B}_1^*$ *is finite.*

Thus, (X,τ) *is fuzzy b-paracompact.*

$(\Longleftarrow)$ *Let* $\mathcal{U} \subset [\tau]$ *be an open cover of* $(X,[\tau])$, *then* $\mathcal{U}' = \{\chi_U \ / \ U \in \mathcal{U}\}$ *is an open fuzzy cover of* (X,τ),*and for each* $r \in I$, $\bigvee_{U\in\mathcal{U}} \chi_U \geq r$. *For each* $\zeta \in (0,r]$, *there exits an open refinement* $\mathcal{G}$ *of* $\mathcal{U}'$ *such that every subfamily with the finite intersection property is finite, and* $\bigvee_{G\in\mathcal{G}} G \geq r - \xi$, *which implies that* $\bigvee_{G\in\mathcal{G}} G \geq r - \xi_1$ *for all* $\xi_1 > \xi$. *Let* $\mathcal{U}^* = \{G^{-1}(r-\xi,1] \ / \ G \in \mathcal{G}\}$, *then* $\mathcal{U}^* \subset [\tau]$ *is an open refinement of* $\mathcal{U}$ *(because, for each* $G^{-1}(r-\xi,1] \in \mathcal{U}^*$ *there exits* $V_G \in \mathcal{U}$ *such that* $G \leq \xi_{V_G}$, *and* $G^{-1}(r-\xi,1] \subset V_G$*), and* $\mathcal{U}^*$ *ia an open cover of* X, *because* $\bigvee_{G\in\mathcal{G}} G \geq r - \xi_1$.

For each $\mathcal{U}_1^* \subset \mathcal{U}^*$ *such that* $\mathcal{U}_1^*$ *has the finite intersection property, we have that* $\{G_j \ / \ G_j^{-1}(r-\xi,1] \in \mathcal{U}_1^*\}$ *has also the finite intersection property (because, if for each* $\{G_1^{-1}(r-\xi,1], ..., G_n^{-1}(r-\xi,1]\} \subset \mathcal{U}_1^*$ *is* $G_1^{-1}(r-\xi,1] \cap ... \cap$

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$G_n^{-1}(r-\xi,1] \neq \varnothing$, *then for each* $i = 1,..,n$ *,there is* $V_{G_i} \in \mathcal{U}$ *such that* $G_i \leq \chi_{V_{G_i}}$ *and there is* $x \in X$ *such that* $G_i(x) > r-\xi$ *for all* $i = 1,...,n$. *So,* $(G_1 \wedge ... \wedge G_n)(x) \neq 0$*). By the hypothesis,* $\{G_j \ / \ G_j^{-1}(r-\xi,1] \in \mathcal{U}_1^*\}$ *is finite, then* $\mathcal{U}_1^*$ *is finite.*

Thus, $(X,[\tau])$ *is b-paracompact.*

Remark 7 *If* (X,T) *is a topological space, then* (X,T) *is b-paracompact if and only if* $(X,\omega(T))$ *is fuzzy b-paracompact. (i.e. fuzzy b-paracompactnesss is a good extension of b-paracompactness).*

Proof. *By the above proof (now it is also valid).* ■

Theorem 8 *A topological space* $(X,[\tau])$ *is* a^**-paracompact (resp. a-paracompact) if and only if* (X,τ) *is fuzzy* a^**-paracompact (resp .a-paracompact).*

Proof. ($\Longrightarrow$) *For each* $r \in I$, *let a family* $\mathcal{B} \subset \tau$ *such that* $\bigvee_{G \in \mathcal{B}} G \geq r$. *Let* $\xi \in (0,r]$, *then the family* $\mathcal{U} = \{G^{-1}(r-\xi,1] \ / \ G \in \mathcal{B}\} \subset [\tau]$ *is an open cover of* $(X,[\tau])$, *and by the hypothesis, it has an open refinement* $\mathcal{U}^* \subset [\tau]$ *which is a cover of* $\mu-\xi$ *and every infinite subfamily of* $\mathcal{U}^*$ *contains an infinite pairwise disjoint subfamily (resp. contains a disjoint pair of elements).*

Analogously to the previous theorem, for each $V \in \mathcal{U}^*$, *there is* $G_V^{-1}(r-\xi,1] \in \mathcal{U}$ *such that* $V \subset G_V^{-1}(r-\xi,1]$.*Then* $\mathcal{B}^* = \{\chi_V \wedge G_V \ / \ V \in \mathcal{U}^*\} \subset \tau$, *refines* $\mathcal{B}$, *and*

$$\bigvee_{V \in \mathcal{U}^*} (\chi_V \wedge G_V)\ (x) \geq r-\xi \ .$$

For each infinite subfamily $\mathcal{B}_1^* \subset \mathcal{B}^*$, $\mathcal{B}_1^* = \left\{\chi_{V_j} \wedge G_{Vj} \ / \ j \in J\right\}$ *for some infinite set* J, *then* $\{V_j \ / \ j \in J\} \subset \mathcal{U}^*$ *is infinite and contains an infinite pairwise disjoint subfamily (resp. contains a disjoint pair of elements). Thus,* $\mathcal{B}_1^*$ *has an infinite pairwise disjoint subfamily (resp. has a disjoint pair of elements), because* $V_{j_1} \cap V_{j_2} = \varnothing$ *implies that* $(\chi_{V_{j_1}} \wedge G_{V_{j_1}}) \wedge (\chi_{V_{j_2}} \wedge G_{V_{j2}}) = \chi_{V_{j_1} \cap V_{j_2}} \wedge (G_{V_{j_1}} \wedge G_{V_{j2}}) = 0$.

So, (X,τ) *is fuzzy* a^**-paracompact (resp. a-paracompact).*

($\Longleftarrow$) *Let* $\mathcal{U} \subset [\tau]$ *be an open cover of* $(X,[\tau])$, *then* $\mathcal{U}' = \{\chi_U \ / \ U \in \mathcal{U}\}$ *is an open fuzzy cover of* (X,τ),*and for each* $r \in I$, $\bigvee_{U \in \mathcal{U}} \chi_U \geq r$. *For each* $\zeta \in (0,r]$, *there exists an open refinement* $\mathcal{G}$ *of* $\mathcal{U}'$ *such that every infinite subfamily of* $\mathcal{G}$ *contains an infinite pairwise disjoint subfamily (resp. contains a disjoint pair of elements), and* $\bigvee_{G \in \mathcal{G}} G \geq r-\xi$.

Analogously to the previous theorem, $\mathcal{U}^* = \{G^{-1}(r-\xi,1] \ / \ G \in \mathcal{G}\}$ *refines* $\mathcal{U}$ *and is an open cover of* X.

For each infinite subfamily $\mathcal{U}_1^*$ *of* $\mathcal{U}^*$ *we have that* $\mathcal{U}_1^* = \{G_j^{-1}(r-\xi,1] \ / \ j \in J\}$ *for some infinite set* J. *Then* $\{G_j \ /j \in J\}$ *is infinite and contained in* $\mathcal{G}$. *Thus, there exits an infinite pairwise disjoint subfamily (resp. exists a disjoint pair of elements) of it, and also* $\mathcal{U}_1^*$ *has an infinite pairwise disjoint subfamily (resp. has a disjoint pair of elements), because* $G_{V_{j_1}} \wedge G_{V_{j2}} = 0$ *implies that* $G_{j_1}^{-1}(r-\xi,1] \cap G_{j_2}^{-1}(r-\xi,1] = \varnothing$ *(if* $G_{j_1}(x_0), G_{j_2}(x_0) \in (r-\xi,1]$ *for some* $x_0 \in X$, *then* $(G_{j_1} \wedge G_{j_2})(x_0) \neq 0$, *and this is contradictory).*

So, $(X,[\tau])$ *is* a^**-paracompact (resp. a-paracompact).*

Remark 9 *If* (X,T) *is a topological space, then* (X,T) *is* a^**-paracompact (resp. a-paracompact) if and only if* $(X,\omega(T))$ *is fuzzy* a^**-*

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paracompact (resp.a-paracompact) (i.e. fuzzy a^*-paracompactness and fuzzy a-paracompactness are good extensions of a^*-paracompactness and a-paracompactness).

Proof. *By the above proof (now it is also valid).* ■

Proposition 10 *All fuzzy paracompact and Hausdorff fuzzy topological space is fuzzy a^*-paracompact.*

Proof. *Firstly, fuzzy paracompactness and fuzzy Hausdorffness are good extensions ([1,5]). Also, every Hausdorff paracompact space is a^*-paracompact ([9]), and finally by last Remark, fuzzy a^*-paracompactness is a good extension.*

■

Proposition 11 *The properties fuzzy b-paracompactness, a-paracompactness and a^*-paracompactness are preserved by preimages of fuzzy perfect maps in the sense of Ghosh (rep. Srivastava and Lal, Christoph).*

Proof. It follows from above Remarks and [6,9]. ■

Proposition 12 *Let $f : (X,\tau) \to (Y,\varsigma)$ be a fuzzy perfect map in the Ghosh's sense (resp. Christoph or Srivastava and Lal) of a fuzzy locally compact b-paracompact space (X,τ) onto the fuzzy Hausdorff space (Y,ς). Then, (Y,ς) is fuzzy b-paracompact.*

Proof. We have that fuzzy locally compactness and fuzzy Hausdorffness are good extensions ([3,5]), and also fuzzy b-paracompactness (by the first Remark). Then, the result follows from [9, Th.5]. ■

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