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# Hyperinvariant subspaces for trace class perturbations of normal operators and decomposability <sup>☆</sup>



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## ABSTRACT

We prove that a large class of trace-class perturbations of diagonalizable normal operators on a separable, infinite dimensional complex Hilbert space have non-trivial closed hyperinvariant subspaces. Moreover, a large subclass consists of decomposable operators in the sense of Colojoară and Foaiaş [3].

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## 1. Introduction

A long-standing open problem is whether every rank-one perturbation of a diagonalizable normal operator on a separable infinite-dimensional complex Hilbert space has a non-trivial closed invariant subspace (see, for example, Problem 8K in the 1978 Percy's monograph [18]). About fifteen years ago, remarkable results in this setting were proved by Foiaş, Jung, Ko and Percy in [7–9], and subsequently by Fang and J. Xia for the case of finite-rank perturbations (see [10]). Recently, the authors in [11–13] have undertaken a thorough study based on characterising spectral subspaces which has allowed them to prove that a large class of finite-rank perturbations of diagonalizable normal operators are *decomposable operators* in the sense of Colojoară and Foiaş [3]. As a consequence, every operator  $T$  in such a class has a rich spectral structure and plenty of non-trivial closed hyperinvariant subspaces, which extends all the previously known results on such question claimed as *a stubbornly intractable problem* in [7].

Nevertheless, the character of the finiteness rank of the compact operator plays a fundamental role in the aforementioned study. Indeed, studying the existence of invariant subspaces or the decomposability of perturbations of normal operators, even of hermitian operators, by *general* compact operators is an old problem. Livsič solved the existence problem for nuclear perturbations of self-adjoint operators, Sahnovič for Hilbert-Schmidt perturbations, and Gohberg and Krein, Macaev, and Schwartz for perturbations with compacts belonging to the Schatten von Neumann class  $\mathcal{C}_p$ ,  $1 \leq p < \infty$  (see [5] for more on the subject). In the setting of compact perturbations of normal operators  $N$ , if the spectrum of  $N$  lies on a  $C^2$  Jordan curve  $\gamma$ , Radjabalipour and H. Radjavi [20] showed that the linear bounded operator  $T = N + K$  where  $K$  is a compact operator belonging to  $\mathcal{C}_p$ ,  $1 \leq p < \infty$ , is decomposable if and only if the spectrum  $\sigma(T)$  does not fill the interior of  $\gamma$  (see also [19]).

But, in 1977 Herrero [14] proved that not every compact perturbation of a unitary operator is decomposable. So not every compact perturbation of a normal operator with a spectrum lying on a  $C^2$  Jordan curve is decomposable, and the situation becomes even more hopeless if no assumption on the spectrum is required.

In this framework, the aim of this work is studying trace-class perturbations of diagonalizable normal operators  $(\mathcal{N} + \mathcal{C}_1)$  acting on a separable, infinite dimensional complex Hilbert space  $H$  from the standpoint of view of the existence of invariant subspaces. Indeed, we will prove that there exists a large subclass  $(\mathcal{N} + \mathcal{C}_1)_{h,s}$  having non-trivial closed *hyperinvariant subspaces*, namely, each  $T \in (\mathcal{N} + \mathcal{C}_1)_{h,s}$  has non-trivial closed invariant subspaces which are also invariant for every operator in its commutant  $\{T\}'$ . Moreover, for those  $T \in (\mathcal{N} + \mathcal{C}_1)_{h,s}$  such that the point spectrum of both  $T$  and its adjoint  $T^*$  is at most countable, we will show that they do have more structure regarding invariant

subspaces, namely, they are decomposable operators. Recall that a bounded linear operator  $T$  on  $H$  is decomposable if splitting the spectrum  $\sigma(T)$  provides a decomposition of  $H$  in terms of invariant subspaces. More precisely, if for every open cover  $\{U, V\}$  of  $\mathbb{C}$  there exist two closed invariant subspaces  $H_1, H_2 \subseteq H$  such that

$$\sigma(T|_{H_1}) \subseteq U \text{ and } \sigma(T|_{H_2}) \subseteq V,$$

and  $H = H_1 + H_2$ . Here  $\sigma(T|_{H_i})$  denotes the spectrum of the restriction of  $T$  to the invariant subspace  $H_i$ ,  $i = 1, 2$ . It is worthy to point out that the sum decomposition is, in general, not direct, nor are the spectra of the restrictions necessarily disjoint. Decomposable operators were introduced by Foias [6] in the sixties as a generalization of spectral operators in the sense of Dunford [5] and Foias' original definition was somewhat more technical, but equivalent to the one set down here (see [16] for more on the subject).

In order to state our main result, let  $H$  be a separable, infinite dimensional complex Hilbert space and  $\mathcal{E} = \{e_n\}_{n \geq 1}$  an (ordered) orthonormal basis of  $H$  fixed.

If  $\Lambda = \{\lambda_n\}_{n \geq 1}$  is any bounded sequence in the complex plane  $\mathbb{C}$ , let  $D_\Lambda$  denote the diagonal operator with respect to  $\mathcal{E}$  associated to  $\Lambda$ , namely,

$$D_\Lambda e_n = \lambda_n e_n, \quad (n \geq 1).$$

For  $k \geq 1$ , let  $u_k$  and  $v_k$  non zero vectors in  $H$  and let us denote their Fourier coefficients with respect to  $\mathcal{E}$  as

$$u_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} e_n, \quad v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} e_n \quad (k \geq 1).$$

Our main theorem in this work reads as follows:

**Main Theorem.** *Let  $H$  be a separable, infinite dimensional complex Hilbert space,  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  a bounded sequence and  $\{u_k\}_{k \geq 1}, \{v_k\}_{k \geq 1}$  non zero vectors in  $H$ . Assume*

$$\sum_{(n,k) \in \mathcal{N}_u} |\alpha_n^{(k)}|^2 \log \left( 1 + \frac{1}{|\alpha_n^{(k)}|} \right) + \sum_{(n,k) \in \mathcal{N}_v} |\beta_n^{(k)}|^2 \log \left( 1 + \frac{1}{|\beta_n^{(k)}|} \right) < \infty, \quad (1.1)$$

where  $\mathcal{N}_u := \{(n, k) \in \mathbb{N} \times \mathbb{N} : \alpha_n^{(k)} \neq 0\}$  and  $\mathcal{N}_v := \{(n, k) \in \mathbb{N} \times \mathbb{N} : \beta_n^{(k)} \neq 0\}$ . Then, the trace-class perturbation of  $D_\Lambda$ ,  $T = D_\Lambda + \sum_{k=1}^{\infty} u_k \otimes v_k$ , acting on  $H$  by

$$Tx = \left( D_\Lambda + \sum_{k=1}^{\infty} u_k \otimes v_k \right) x = D_\Lambda x + \sum_{k=1}^{\infty} \langle x, v_k \rangle u_k, \quad (x \in H), \quad (1.2)$$

has non trivial closed hyperinvariant subspaces provided that it is not a scalar multiple of the identity operator. Moreover, if both point spectrum  $\sigma_p(T)$  and  $\sigma_p(T^*)$  are at most countable,  $T$  is decomposable.

Before proceeding further, a couple of comments are in order. Firstly, condition (1.1) implies that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (|\alpha_n^{(k)}|^2 + |\beta_n^{(k)}|^2) < \infty \tag{1.3}$$

and

$$\sum_{(n,k) \in \mathcal{N}_u} |\alpha_n^{(k)}|^2 \log \left( \frac{1}{|\alpha_n^{(k)}|} \right) + \sum_{(n,k) \in \mathcal{N}_v} |\beta_n^{(k)}|^2 \log \left( \frac{1}{|\beta_n^{(k)}|} \right) < \infty. \tag{1.4}$$

In particular, (1.3) yields that  $\sum_{k=1}^{\infty} (\|u_k\|^2 + \|v_k\|^2) < \infty$ , so the compact operator  $K = \sum_{k=1}^{\infty} u_k \otimes v_k$  is trace-class. Secondly, trace-class perturbations of normal operators whose eigenvectors span  $H$  are unitarily equivalent to those expressed by (1.2).

**Remark 1.1.** It should be noted that the condition (1.1) is slightly different to the one imposed in [12,13]. This modification is necessary to ensure the validity of both (1.3) and (1.4). An equivalent condition for obtaining these summability properties is to assume (1.4) and furthermore to suppose that there exist finite subsets  $\mathfrak{N}_u, \mathfrak{N}_v \subset \mathbb{N} \times \mathbb{N}$  such that

$$\sup_{(n,k) \in (\mathbb{N} \times \mathbb{N}) \setminus \mathfrak{N}_u} |\alpha_n^{(k)}| < 1, \quad \sup_{(n,k) \in (\mathbb{N} \times \mathbb{N}) \setminus \mathfrak{N}_v} |\beta_n^{(k)}| < 1.$$

As it was aforementioned, the main theorem of this work extends the results in [7–10, 12,13]. In particular, in the rank-one perturbation case, if  $T = D_{\Lambda} + u_1 \otimes v_1$  is non-scalar, the summability condition (1.1) along with authors’ contributions yields, in particular, that  $T$  has non trivial closed hyperinvariant subspaces as far as

$$\sum_{n=1}^{\infty} (|\alpha_n^{(1)}|^p + |\beta_n^{(1)}|^q) < \infty$$

for every  $(p, q) \in (0, 2] \times (0, 2] \setminus \{(2, r), (r, 2) : r \in (1, 2]\}$  being, indeed, decomposable whenever the point spectrum of  $T$  and  $T^*$  are, at most, denumerable (see Fig. 1). Moreover, in the infinite-rank perturbation case, it comprises the results by Klaja [15] and Theorem 1.2 of Albretch and Chevreau [2].

Before closing this introductory section, we outline the strategy in order to prove our main theorem, pointing out the main differences with the previously aforementioned results. As in the finite-rank case, the proof relies on the construction of a family of spectral idempotents associated to the operator  $T = D_{\Lambda} + \sum_{k=1}^{\infty} u_k \otimes v_k$  that is, idempotents  $J$  belonging to the bicommutant of  $T$  such that their ranges are spectral subspaces of  $T$ . These ranges will be the non-trivial closed hyperinvariant subspaces for  $T$ . Moreover, we

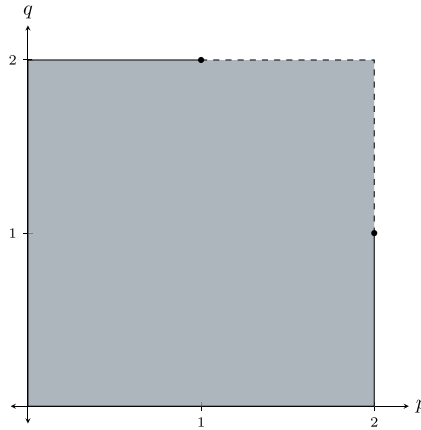


Fig. 1. Decomposability in the rank-one perturbation case.

will be able to construct a sufficiently rich Boolean algebra of such spectral idempotents, which will provide the decomposability as in [9].

Accordingly, our main efforts are addressed to the construction of such operators and for such a task, we will make use of the so-called *unconventional Dunford functional calculus*, but with substantial modifications.

The unconventional functional calculus was explicitly stated in [10] and used in [15,2] to provide non-trivial closed hyperinvariant subspaces and decomposability for certain compact perturbations of diagonalizable normal operators, as we already pointed out. The main idea consists of considering piecewise-differentiable closed curves  $\gamma \subset \mathbb{C}$  intersecting the spectrum  $\sigma(T)$  and defining a right-inverse  $R(z)$  of  $(T - zI)$  for every  $z \in \gamma$ . This inverse turns out to be of the form

$$R(z) = (D_\Lambda - zI)^{-1} + K(z),$$

where  $K(z)$  is a compact operator for each  $z \in \gamma$ . In general,  $(D_\Lambda - zI)^{-1}$  may be unbounded, so  $R(z)$  would be unbounded as well. Nevertheless, the operator  $R(z)$  can be integrated along the curve  $\gamma$ , due to the Borel functional calculus for normal operators (see [15, Proposition 2.6]). Hence, the spectral idempotents (or other kind of operators with equivalent properties) are obtained as the operators defined via

$$\frac{1}{2\pi i} \int_\gamma R(z) dz.$$

Our approach will also rely on integrating expressions of the kind  $(D_\Lambda - zI)^{-1} + K(z)$  throughout a curve (see (4.1)), but under the assumptions of summability (1.1) the operator  $K(z)$  may not even be well defined for those  $z \in \gamma \cap \sigma(T)$ . Thus, we will consider a *formal* right inverse of  $(T - zI)$  and integrate it along a curve  $\gamma$  which will allow us to obtain bounded operators  $J$  (see Theorem 4.1) and which will turn out to

be spectral idempotents (Theorem 4.5). The proof of this latter fact will be carried out through Section 4, for which a characterization of the spectral subspaces associated to  $T$  will be needed (Theorem 2.2).

Integrating formal right inverses of  $(T - zI)$  already appeared implicitly in the authors' work [13] (tracing back to [11, Theorem 6.7]) in connection with the existence of spectral idempotents for finite rank perturbations of diagonalizable normal operators. Nevertheless, in such a context, the definition of the spectral idempotents relies heavily on finite-dimensional linear algebra since it deals with the determinant and cofactors of a certain matrix  $M_T$  associated to the operator (see [13, Lemma 4.5 and Definition 4.8]). Moreover, the expression of the determinant via the associated cofactors plays a decisive role in the construction.

For trace class perturbations of normal operators, the finite-dimensional linear algebra does not apply anymore and the tools coming from Fredholm theory of determinants seem not to be suitable to face the problem. Accordingly, the construction of the idempotents in this setting will depend on the invertibility of an auxiliary operator  $I + Y(z)X(z)$ , which plays a role similar to that of the matrix  $M_T$  along the curve  $\gamma$ . In addition, the properties of continuity of the map  $z \in \gamma \mapsto (I + Y(z)X(z))^{-1}$  will allow us to obtain bounds of conditionally-convergent series and an inversion formula (Theorem 3.6) that will be fundamental to obtain the desired properties for the operators  $J$ . The construction of the operator  $I + Y(z)X(z)$  and the study of its properties will be carried out in Section 3.

## 2. Preliminaries

In this section, we recall some preliminaries and prove a characterization of the spectral subspaces associated to closed sets of the complex plane for trace-class perturbations of diagonalizable normal along the lines of [11, Theorem 2.1].

Throughout this work,  $H$  denotes an infinite dimensional separable complex Hilbert space,  $\mathcal{L}(H)$  the Banach algebra of all bounded linear operators on  $H$  and  $\mathcal{E} = \{e_n\}_{n \geq 1}$  an (ordered) orthonormal basis of  $H$  fixed.

Recall that compact operator  $K \in \mathcal{L}(H)$  belongs to the Schatten class  $\mathcal{C}_p$  with  $1 \leq p < \infty$  if

$$\sum_{n=1}^{\infty} a_n(K)^p < \infty,$$

where  $a_n(K)$  denotes the singular values of  $K$ , namely, eigenvalues of the hermitian operator  $|K| := \sqrt{K^*K}$ . Indeed, for every  $1 \leq p < \infty$ , the norm

$$\|K\|_{\mathcal{C}_p} := \left( \sum_{n=1}^{\infty} a_n(K)^p \right)^{1/p} < \infty$$

makes each Schatten class  $\mathcal{C}_p$  a closed ideal of operators of  $\mathcal{L}(H)$ .

For  $p = 1$ ,  $\|\cdot\|_{\mathcal{C}_1}$  is the trace-class norm and  $K$  is trace-class if there exist two orthogonal sequences  $\{x_i\}$  and  $\{y_i\}$  in  $H$  and positive real numbers  $\{\lambda_i\}_{i \geq 1}$  in  $\ell^1$  such that

$$x \mapsto T(x) = \sum_{i=1}^{\infty} \lambda_i \langle x, x_i \rangle y_i, \quad \forall x \in H,$$

where  $\{\lambda_i\}_{i \geq 1}$  are the singular values of  $K$ , with each value repeated as often as its multiplicity.

As we mentioned in the introduction, the proof of the Main Theorem will be carried out by constructing a sufficiently rich family of spectral idempotents of the operator  $T$ , that is, idempotents with *spectral subspaces* ranges. Spectral subspaces play an important role in order to produce non-trivial closed hyperinvariant subspaces and to introduce them, let us recall a few relevant notions from Local Spectral Theory (we refer to the monograph [16] for more on the subject).

A linear bounded operator  $T$  in  $H$  has the *single-valued extension property* (SVEP) if for every connected open set  $G \subset \mathbb{C}$  and every analytic function  $f : G \rightarrow X$  such that  $(T - \lambda I)f(\lambda) \equiv 0$  on  $G$ , one has  $f \equiv 0$  on  $G$ . Note that every operator  $T$  such that the point spectrum  $\sigma_p(T)$  has empty interior has the SVEP.

When  $T$  has the SVEP and  $x \in H$ , is possible to define the *local resolvent*  $\rho_T(x)$  of  $T$  at  $x$  as the union of all open sets  $U \subset \mathbb{C}$  such that there exists a unique vector-valued holomorphic function  $f_x : U \rightarrow H$  satisfying

$$(T - zI)f_x(z) = x \quad (z \in U). \tag{2.1}$$

The analytic extension of  $f_x$  to  $\rho_T(x)$  is called *local resolvent function* of  $T$  at  $x$ . Likewise, the complement of  $\rho_T(x)$  is called the *local spectrum* of  $T$  at  $x$ :

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Among many other properties, the local spectrum always satisfies that  $\sigma_T(x) \subseteq \sigma(T)$  for every  $x \in X$  and  $\sigma_T(0) = \emptyset$ .

The concept of local spectrum allows to define *the local spectral subspaces* of an operator. Given a subset  $\Omega \subset \mathbb{C}$ , the local spectral subspace of  $T$  associated to  $\Omega$  is

$$H_T(\Omega) = \{x \in H : \sigma_T(x) \subseteq \Omega\}.$$

If  $\Omega_1 \subset \Omega_2$  then  $H_T(\Omega_1) \subset H_T(\Omega_2)$  and  $H_T(\Omega) = H_T(\Omega \cap \sigma(T))$ . It is worthy to note that  $H_T(\Omega)$  is a linear manifold that is hyperinvariant for  $T$  but not necessarily closed even for closed subsets (see [1, Chapter 2] for instance).

We are in position to define the spectral idempotents of an operator, introduced in [9] in connection with the study of decomposability.

**Definition 2.1.** Given  $T \in \mathcal{L}(H)$ , an idempotent operator  $J \in \mathcal{L}(H)$  is said to be a **spectral idempotent** for  $T$  if:

- (1)  $J$  belongs to the bicommutant of  $T$ , that is  $J \in \{T\}''$ .
- (2) The range of  $J$  is the spectral subspace associated to  $\sigma(T|_{\text{ran}(J)})$ , that is,

$$\text{ran}(J) = H_T(\sigma(T|_{\text{ran}(J)})).$$

In such a case,  $J$  is denoted by  $J_T(\sigma)$ , where  $\sigma = \sigma(T|_{\text{ran}(J)})$ .

With this definition at hands, we proceed to prove a characterization of certain spectral subspaces associated to  $T$  which will be useful to define particular spectral idempotents. In order to state it, we introduce the following notation. Given  $\Lambda = (\lambda_n) \subset \mathbb{C}$  any sequence and  $A \subset \mathbb{C}$ ,  $N_A$  will stand for the set of positive integers:

$$N_A = \{n \in \mathbb{N} : \lambda_n \in \Lambda \cap A\}.$$

In addition, given an open set  $U$ , a holomorphic map  $g$  on  $U$  and  $w \in U$ , we define

$$\Gamma(g)(z, w) = \begin{cases} \frac{g(z)-g(w)}{z-w} & z \neq w \\ g'(w) & z = w \end{cases}$$

Observe that  $\Gamma(g)(z, w)$  is continuous in  $U \times U$  and for every  $w \in U$ , the map  $z \mapsto \Gamma(g)(z, w)$  is holomorphic in  $U$ .

**Theorem 2.2.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  be a bounded sequence and  $u_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} e_n$ ,  $v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} e_n$ ,  $k \geq 1$ , non-zero vectors in  $H$  such that  $\sum_{k=1}^{\infty} (\|u_k\|^2 + \|v_k\|^2) < \infty$ . Suppose that  $T = D_{\Lambda} + \sum_{k=1}^{\infty} u_k \otimes v_k \in \mathcal{L}(H)$  is a non-scalar operator having the SVEP and  $F \subset \mathbb{C}$  a closed set such that  $\sigma(T) \cap F \neq \emptyset$ . A vector  $x \in H$  belongs to the spectral subspace  $H_T(F)$  if and only if there exist a sequence  $\{g_k\}_{k \geq 1}$  of unique analytic functions defined on  $F^c = \mathbb{C} \setminus F$  satisfying:*

- (i) If  $x = \sum_{n=1}^{\infty} x_n e_n$ , then

$$x_n = \sum_{k=1}^{\infty} g_k(\lambda_n) \alpha_n^{(k)}$$

for every  $n \in N_{F^c}$ .

- (ii) The function

$$z \in F^c \mapsto \sum_{n \in N_{F^c}} \left( \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n) \alpha_n^{(k)} \right) e_n$$

is a vector valued analytic function on  $F^c$ .

(iii) For each  $k \in \mathbb{N}$ , the identity

$$\begin{aligned} \sum_{n \in N_F} \frac{x_n \overline{\beta_n^{(k)}}}{\lambda_n - z} &= g_k(z) \left( \sum_{n \in N_F} \frac{\alpha_n^{(k)} \overline{\beta_n^{(k)}}}{\lambda_n - z} + 1 \right) \\ &- \sum_{n \in N_{F^c}} \left( \sum_{m=1}^{\infty} \Gamma(g_m)(z, \lambda_n) \alpha_n^{(m)} \right) \overline{\beta_n^{(k)}} \\ &+ \sum_{n \in N_F} \left( \sum_{m \neq k} \frac{g_m(z) \alpha_n^{(m)} \overline{\beta_n^{(k)}}}{\lambda_n - z} \right) \end{aligned}$$

holds for every  $z \in F^c$ .

(iv) The series

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} \right|^2$$

is finite for every  $z \in F^c$ .

In such a case, the unique local resolvent function of  $T$  at  $x$  is given by

$$f_x(z) = \sum_{n \in N_F} \frac{x_n - \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)}}{\lambda_n - z} e_n + \sum_{n \in N_{F^c}} \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n) \alpha_n^{(k)} e_n, \quad (z \in F^c).$$

The proof of Theorem 2.2 is based on that of Theorem 2.1 in [11] with suitable modifications. We detail it for the sake of completeness.

**Proof.** Let us start by proving that the conditions (i)–(iv) are necessary. Let  $x \in H_T(F)$  be fixed. Since  $T$  has the SVEP, there exists a unique vector-valued analytic function  $f_x : F^c \rightarrow H$  such that

$$(T - zI)f_x(z) = x \tag{2.2}$$

for every  $z \in F^c$ . Now,

$$(T - zI)f_x(z) = (D_{\Lambda} - zI)f_x(z) + \sum_{k=1}^{\infty} \langle f_x(z), v_k \rangle u_k.$$

We define

$$g_k(z) := \langle f_x(z), v_k \rangle, \quad (z \in F^c)$$

for each  $k \in \mathbb{N}$ . Obviously,  $g_k$  is an analytic function in  $F^c$ . Moreover, since  $f_x$  unique, it follows that the functions  $g_k$  are unique for each  $k \in \mathbb{N}$ . Now, write

$$f_{x,n}(z) := \langle f_x(z), e_n \rangle.$$

From (2.2) it follows

$$(\lambda_n - z)f_{x,n}(z) + \sum_{k=1}^{\infty} g_k(z)\alpha_n^{(k)} = x_n$$

for every  $n \in \mathbb{N}$  and  $z \in F^c$ . Then, for every  $n \in N_{F^c}$ ,

$$\sum_{k=1}^{\infty} g_k(\lambda_n)\alpha_n^{(k)} = x_n,$$

what proves (i).

To show (ii), equation (2.2) along with (i) yields that

$$f_{x,n}(z) = \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n)\alpha_n^{(k)}$$

for every  $n \in N_{F^c}$  and  $z \in F^c$ . On the other hand, for every  $n \in N_F$

$$f_{x,n}(z) = \frac{x_n - \sum_{k=1}^{\infty} g_k(z)\alpha_n^{(k)}}{\lambda_n - z}$$

for every  $z \in F^c$ . Accordingly,

$$f_x(z) = \sum_{n \in N_F} \frac{x_n - \sum_{k=1}^{\infty} g_k(z)\alpha_n^{(k)}}{\lambda_n - z} e_n + \sum_{n \in N_{F^c}} \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n)\alpha_n^{(k)} e_n. \tag{2.3}$$

Observe that the map

$$z \in F^c \mapsto \sum_{n \in N_{F^c}} \left( \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n)\alpha_n^{(k)} \right) e_n$$

is just the orthogonal projection of  $f_x$  onto  $\overline{\text{span} \{e_n : n \in N_{F^c}\}}$ , so the analyticity of  $f_x$  yields that condition (ii) holds.

To prove (iii), it suffices to recall that  $g_k(z) = \langle f_x(z), v_k \rangle$  for each  $k \in \mathbb{N}$  and use the expression (2.3) to deduce the equations.

Finally, in order to show (iv), let  $z \in F^c$ . By recalling that  $g_k(z) = \langle f_x(z), v_k \rangle$  and upon applying twice the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} \right|^2 &\leq \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} |g_k(z)|^2 \right) \left( \sum_{k=1}^{\infty} |\alpha_n^{(k)}|^2 \right) \\ &= \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_n^{(k)}|^2 \right) \left( \sum_{k=1}^{\infty} |\langle f_x(z), v_k \rangle|^2 \right) \\ &\leq \left( \sum_{k=1}^{\infty} \|u_k\|^2 \right) \left( \|f_x(z)\|^2 \sum_{k=1}^{\infty} \|v_k\|^2 \right) \\ &< \infty, \end{aligned}$$

which shows (iv).

In order to prove the sufficiency, let us assume that there exists a sequence  $\{g_k\}_{k \geq 1}$  of analytic functions on  $F^c$  satisfying (i), (ii), (iii) and (iv) and show that  $x \in H_T(F)$ . Let us define

$$f_x(z) = \sum_{n \in N_F} \frac{x_n - \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)}}{\lambda_n - z} e_n + \sum_{n \in N_{F^c}} \sum_{k=1}^{\infty} \Gamma(g_k)(z, \lambda_n) \alpha_n^{(k)} e_n \quad (z \in F^c).$$

Observe that, by (ii) and (iv),  $f_x$  is analytic on  $F^c$ . Moreover, by (iii), it follows that  $\langle f_x(z), v_k \rangle = g_k(z)$  for every  $z \in F^c$ .

Let us prove that

$$(T - zI)f_x(z) = x$$

for every  $z \in F^c$ . First, we observe that for every  $z \in F^c \setminus \{\lambda_n : n \in F^c\}$

$$\begin{aligned} (T - zI)f_x(z) &= (D_{\Lambda} - z)f_x(z) + \sum_{k=1}^{\infty} g_k(z)u_k \\ &= \sum_{n \in N_F} (x_n - \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)}) e_n + \sum_{n \in N_{F^c}} \sum_{k=1}^{\infty} (g_k(\lambda_n) - g_k(z)) \alpha_n^{(k)} e_n \\ &\quad + \sum_{k=1}^{\infty} g_k(z)u_k \\ &= x - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} e_n + \sum_{k=1}^{\infty} g_k(z)u_k. \end{aligned} \tag{2.4}$$

Now, condition (iv) implies that

$$G(z) := \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} \right) e_n$$

converges in norm for every  $z \in F^c \setminus \{\lambda_n : n \in F^c\}$ . In addition,  $\sum_{k=1}^\infty g_k(z)u_k$  converges for every  $z \in F^c \setminus \{\lambda_n : n \in F^c\}$  due to the boundedness of the operator  $\sum_{k=1}^\infty u_k \otimes v_k$  and the equality  $\langle f_x(z), v_k \rangle = g_k(z)$ . The equality  $\langle G(z), e_n \rangle = \langle \sum_{k=1}^\infty g_k(z)u_k, e_n \rangle$  for every  $z \in F^c \setminus \{\lambda_n : n \in F^c\}$  and  $n \in \mathbb{N}$  shows that

$$G(z) = \sum_{k=1}^\infty g_k(z)u_k$$

for every  $z \in F^c \setminus \{\lambda_n : n \in F^c\}$ . This along with (2.4), yields that  $(T - zI)f_x(z) = x$  for every  $z \in F^c \setminus \{\lambda_n : n \in N_{F^c}\}$ . Since  $\{\lambda_n : n \in N_{F^c}\}$  is countable, the Identity Theorem yields that  $(T - zI)f_x(z) = x$  for every  $z \in F^c$ , as we wish to show.  $\square$

**Remark 2.3.** It is important to note that the hypothesis  $\sum_{k=1}^\infty (\|u_k\|^2 + \|v_k\|^2) < \infty$  in Theorem 2.2 is only required to demonstrate the necessity of the condition (iv). Clearly, this summability hypothesis implies, in particular, that the operator  $K = \sum_{k=1}^\infty u_k \otimes v_k$  is trace class. Indeed, in the absence of this hypothesis, conditions (i) – (iv) suffice to establish that  $x \in H_T(F)$ , independent of any further convergence assumptions on the norms of  $u_k$  and  $v_k$ , apart from the boundedness of the operator  $K$ .

We close this section with a generalization of [12, Lemma 2.2] in the setting of doubly indexed series:

**Lemma 2.4.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1}$  be a bounded sequence of complex numbers and let  $\{\alpha_n^{(k)}\}_{n \geq 1, k \geq 1}$  be a doubly indexed sequence of complex numbers satisfying*

$$\sum_{(n,k) \in \mathcal{N}_u} |\alpha_n^{(k)}|^2 \log \left( 1 + \frac{1}{|\alpha_n^{(k)}|} \right) < \infty \tag{2.5}$$

where  $\mathcal{N}_u = \{(n, k) \in \mathbb{N} \times \mathbb{N} : \alpha_n^{(k)} \neq 0\}$ . Then, for almost every  $\mathbf{x} \in \mathbb{R}$ ,

$$\sum_{n=1}^\infty \sum_{k=1}^\infty \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} < \infty.$$

The proof is similar to that of [15, Lemma 3.1] in which [10, Lemma 2.1] is generalized. The same ideas work to extend [12, Lemma 2.2] to this setting, taking into account that (2.5) implies that both series  $\sum_{n=1}^\infty \sum_{k=1}^\infty |\alpha_n^{(k)}|^2$  and  $\sum_{(n,k) \in \mathcal{N}_u} |\alpha_n^{(k)}|^2 \log \left( \frac{1}{|\alpha_n^{(k)}|} \right)$  are finite, so we omit the proof for the sake of brevity.

### 3. Decomposability set and auxiliary operators

In this section, we prove those key results which will allow us to define the spectral idempotents associated to the trace-class perturbations of diagonalizable normal opera-

tors covered by the Main Theorem. We start introducing the *decomposability set* of such perturbations.

**Definition 3.1.** Let  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  be a bounded sequence not lying in any vertical line and such that the set of accumulation points  $\Lambda'$  is not a singleton and denote by

$$a = \min_{z \in \Lambda'} \operatorname{Re}(z) \quad b = \max_{z \in \Lambda'} \operatorname{Re}(z).$$

Let  $\{u_k\}_{k \geq 1}$  and  $\{v_k\}_{k \geq 1}$  be non zero vectors in  $H$  satisfying (1.1) and assume that the trace-class perturbation  $T = D_\Lambda + \sum_{k=1}^\infty u_k \otimes v_k$ , acting on  $H$  satisfies that  $\sigma_p(T) \cup \sigma_p(T^*)$  is at most countable. The decomposability set of  $T$  consists of those real numbers satisfying:

$$\Delta(T) = \left\{ \mathbf{x} \in (a, b) \setminus \operatorname{Re}(\sigma_p(T) \cup \sigma_p(T^*)) : \sum_{n=1}^\infty \sum_{k=1}^\infty \left( \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} + \frac{|\beta_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} \right) < \infty \right\}.$$

**Remark 3.2.** Note that the assumption that the sequence  $\Lambda$  does not lie in a vertical line is not a restriction in the class of operators considered, since it can be achieved by a rotation. It is included in order that  $a < b$ .

Before going further, observe that  $\Delta(T)$  is non-empty, and even more, it contains almost every point of  $(a, b)$  as a consequence of Lemma 2.4. Likewise, it is clear that the decomposability set of  $T$  does not contain the real part of any eigenvalue of  $D_\Lambda$ ,  $T$  or  $T^*$ .

To avoid some technicalities, we will make the following assumption in the rest of the manuscript, using the notation introduced in the Main Theorem.

(\*) **Assumption:** We suppose that both  $\sigma(D_\Lambda) = \bar{\Lambda}$  and  $\sigma(T)$  are contained in the unit disc  $\mathbb{D}$ .

Note that multiplying by some appropriate complex number the assumption is achievable and harmless regarding the existence of non-trivial closed hyperinvariant subspaces. Likewise, it does not affect to the existence of non trivial spectral idempotents in the bicommutant of  $T$ .

Now, as in [13], given  $\mathbf{x}$  in the real decomposability set we consider the positively oriented curves

$$\gamma_{\mathbf{x}}^+ := \ell_{\mathbf{x}} \cup A_{\mathbf{x}}^+, \quad \gamma_{\mathbf{x}}^- := \ell_{\mathbf{x}} \cup A_{\mathbf{x}}^-,$$

where

$$\ell_{\mathbf{x}} := \{z \in \bar{\mathbb{D}} : \operatorname{Re}(z) = \mathbf{x}\},$$

and

$$A_{\mathbf{x}}^+ := \{e^{i\theta} \in \mathbb{T} : \operatorname{Re}(e^{i\theta}) \geq \mathbf{x}\}, \quad A_{\mathbf{x}}^- := \{e^{i\theta} \in \mathbb{T} : \operatorname{Re}(e^{i\theta}) \leq \mathbf{x}\}.$$

Here  $\mathbb{T}$  denotes the unit circle. Observe that, by definition,  $\gamma_{\mathbf{x}}^+$  and  $\gamma_{\mathbf{x}}^-$  are piecewise differentiable curves that do not intersect the sequence  $\Lambda$  and the point spectra  $\sigma_p(T)$  and  $\sigma_p(T^*)$ .

Let us denote by

$$F_{\mathbf{x}}^+ := \overline{\operatorname{int}(\gamma_{\mathbf{x}}^+)}, \quad F_{\mathbf{x}}^- := \overline{\operatorname{int}(\gamma_{\mathbf{x}}^-)},$$

where  $\operatorname{int}(\gamma_{\mathbf{x}}^+)$  and  $\operatorname{int}(\gamma_{\mathbf{x}}^-)$  stand for the set of points in  $\mathbb{C}$  with index 1 with respect to the closed Jordan curve  $\gamma_{\mathbf{x}}^+$  or  $\gamma_{\mathbf{x}}^-$ , respectively.

In order to define particular square roots of bounded diagonal operators, we fix the square root in  $\mathbb{C} \setminus \{0\}$ :

$$z \in \mathbb{C} \setminus \{0\} \mapsto z^{1/2} := |z|^{1/2} e^{i\arg(z)/2}$$

where  $\arg(z) \in [-\pi, \pi)$ . It is important to remark that this is not a continuous branch of the square root, as there are no branches of the square root that are continuous in such a domain. Nevertheless, the chosen square root is *well defined* as a function in  $\mathbb{C} \setminus \{0\}$ .

Having in mind that  $D_{\Lambda} = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n \in \mathcal{L}(H)$ , we denote

$$(D_{\Lambda} - zI)^{1/2} := \sum_{n=1}^{\infty} (\lambda_n - z)^{1/2} e_n \otimes e_n \quad \text{for } z \in \mathbb{C} \setminus \Lambda, \tag{3.1}$$

$$(D_{\Lambda} - zI)^{-1/2} := \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - z)^{1/2}} e_n \otimes e_n \quad \text{for } z \in \mathbb{C} \setminus \Lambda. \tag{3.2}$$

Observe that  $(D_{\Lambda} - zI)^{1/2}$  is a well defined bounded operator for every  $z \in \mathbb{C} \setminus \Lambda$ , while  $(D_{\Lambda} - zI)^{-1/2}$  is bounded if  $z \notin \overline{\Lambda}$ , and it is well-defined but unbounded if  $z \in \overline{\Lambda} \setminus \Lambda$ .

We are in position to introduce the infinite-dimensional counterpart of those operators considered by Fang and Xia in the proof of [10, Lemma 3.1].

**Proposition 3.3.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  be a bounded sequence not lying in any vertical line such that  $\Lambda'$  is not a singleton and  $\{u_k\}_{k \geq 1}$  and  $\{v_k\}_{k \geq 1}$  non zero vectors in  $H$  satisfying (1.1). Assume that the trace-class perturbation  $T = D_{\Lambda} + \sum_{k=1}^{\infty} u_k \otimes v_k$  satisfies both  $\sigma(D_{\Lambda}) = \overline{\Lambda}$  and  $\sigma(T)$  are contained in  $\mathbb{D}$ . For each  $\mathbf{x} \in \Delta(T)$  the operators*

$$X^+(z) = \sum_{k=1}^{\infty} (D_{\Lambda} - zI)^{-1/2} u_k \otimes e_k, \quad \text{for } z \in \gamma_{\mathbf{x}}^+; \tag{3.3}$$

$$Y^+(z) = \sum_{k=1}^{\infty} e_k \otimes (D_{\Lambda}^* - \bar{z}I)^{-1/2} v_k, \quad \text{for } z \in \gamma_{\mathbf{x}}^+,$$

are well-defined bounded operators in  $H$ . The analogous statement holds for  $\gamma_{\mathbf{x}}^-$ .

**Proof.** Let us fix  $\mathbf{x} \in \Delta(T)$  and  $z \in \gamma_{\mathbf{x}}^+ = \ell_{\mathbf{x}} \cup A_{\mathbf{x}}^+$  and prove the result for  $X^+(z)$ . The proof for  $Y^+(z)$  is similar arguing with the adjoint operator  $Y^+(z)^*$ .

First, let us show that  $(D_{\Lambda} - zI)^{-1/2}u_k$  belong to  $H$  for every  $k \geq 1$ . Since  $\bar{\Lambda} \subseteq \mathbb{D}$ , the operator  $(D_{\Lambda} - zI)^{-1/2}$  is bounded whenever  $z \in A_{\mathbf{x}}^+$ , so  $(D_{\Lambda} - zI)^{-1/2}u_k \in H$ . Accordingly, the real task is when  $z \in \ell_{\mathbf{x}}$  and in such a case, having in mind that  $\ell_{\mathbf{x}} \cap \Lambda = \emptyset$ , we note that

$$\left\| (D_{\Lambda} - zI)^{-1/2}u_k \right\|^2 = \sum_{n=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\lambda_n - z|} \leq \sum_{n=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} < \infty, \tag{3.4}$$

so  $(D_{\Lambda} - zI)^{-1/2}u_k \in H$  as well for every  $k \geq 1$ .

Now, we deal with the operator  $X^+(z)$ . As before, if  $z \in A_{\mathbf{x}}^+$  the proof is obvious, so let us assume  $z \in \ell_{\mathbf{x}}$ . If  $x = \sum_{n=1}^{\infty} x_n e_n \in H$ , applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|X^+(z)x\|^2 &= \left\| \sum_{k=1}^{\infty} x_k (D_{\Lambda} - zI)^{-1/2}u_k \right\|^2 = \left\| \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} x_k \frac{\alpha_n^{(k)}}{(\lambda_n - z)^{1/2}} \right) e_n \right\|^2 \\ &= \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} x_k \frac{\alpha_n^{(k)}}{(\lambda_n - z)^{1/2}} \right|^2 \leq \|x\|^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|}, \end{aligned}$$

which is finite since  $\mathbf{x}$  belongs to the decomposability set  $\Delta(T)$ . As a consequence,  $X^+(z)$  is a well-defined bounded operator in  $H$ , which concludes the proof  $\square$

*A word about notation.* For the sake of simplicity, throughout the rest of text we will state most of the results only for the curves  $\gamma_{\mathbf{x}}^+$  and denote the operators in (3.3) by  $X(z)$  and  $Y(z)$ , respectively. Clearly all the results will also hold when the curves  $\gamma_{\mathbf{x}}^-$  are considered.

Our next goal is proving that  $I + X(z)Y(z)$  and  $I + Y(z)X(z)$  are invertible for every  $z \in \gamma_{\mathbf{x}}^+$ . For such a purpose, we recall that two operators  $T, S \in \mathcal{L}(H)$  are said to be *quasiasimilar* if there exist operators  $A, B \in \mathcal{L}(H)$  such that  $TA = AS$  and  $BT = SB$  and their kernels satisfy

$$\ker A = \ker A^* = \ker B = \ker B^* = \{0\}.$$

In this case, the operators  $A, B$  are called *quasiaffinities*. It is worthy to note that if  $T$  and  $S$  are quasiasimilar,  $\sigma_p(T) = \sigma_p(S)$ .

Next lemma is a generalization of [7, Theorem 2.5] and [12, Theorem 4.8].

**Lemma 3.4.** *Under the hypotheses of Proposition 3.3, let  $\mathbf{x} \in \Delta(T)$  and  $\xi \in \gamma_{\mathbf{x}}^+$ . Then, the linear bounded operators  $T - \xi I$  and*

$$\tilde{T} = (D_\Lambda - \xi I) + \sum_{k=1}^\infty ((D_\Lambda - \xi I)^{-1/2} u_k) \otimes ((D_\Lambda - \xi I)^{1/2})^* v_k$$

are quasisimilar.

The proof follows the same lines of that of [12, Theorem 4.8], and we include it for the sake of completeness.

**Proof.** Note that both operators  $(D_\Lambda - \xi I)^{1/2}$  and  $T - \xi I = (D_\Lambda - \xi I) + \sum_{k=1}^\infty u_k \otimes v_k$  are quasiaffinities, since the curves  $\gamma_{\mathbf{x}}^+$  and  $\gamma_{\mathbf{x}}^-$  do not contain any eigenvalue of  $D_\Lambda$ ,  $T$  or  $T^*$ . Now, define

$$U = (D_\Lambda - \xi I)^{-1/2} (T - \xi I)$$

and observe that it is also a quasiaffinity. Since,  $(T - \xi I) = (D_\Lambda - \xi I)^{1/2} U$  and  $\tilde{T} = U(D_\Lambda - \xi I)^{1/2}$ , we deduce

$$(T - \xi I)(D_\Lambda - \xi I)^{1/2} = (D_\Lambda - \xi I)^{1/2} \tilde{T}.$$

Finally,  $U(T - \xi I) = U(D_\Lambda - \xi I)^{1/2} U = \tilde{T} U$ , and the statement follows.  $\square$

We are in position to state a first step to approach the proof of the Main Theorem.

**Theorem 3.5.** *Under the hypotheses of Proposition 3.3, if  $\mathbf{x}$  is in the decomposability set  $\Delta(T)$  and  $z \in \gamma_{\mathbf{x}}^+$ , the operators*

$$I + X(z)Y(z), \quad I + Y(z)X(z)$$

are invertible in  $H$ .

**Proof.** Let  $\mathbf{x} \in \Delta(T)$  and  $z \in \gamma_{\mathbf{x}}^+ = \ell_{\mathbf{x}} \cup A_{\mathbf{x}}^+$  be fixed. First, note that

$$X(z)Y(z) = \sum_{k=1}^\infty (D_\Lambda - zI)^{-1/2} u_k \otimes (D_\Lambda - zI)^{-1/2})^* v_k.$$

Observe that if  $z$  lies in the segment  $\ell_{\mathbf{x}}$

$$\begin{aligned} & \sum_{k=1}^\infty \left\| (D_\Lambda - zI)^{-1/2} u_k \right\| \left\| (D_\Lambda - zI)^{-1/2})^* v_k \right\| \\ & \leq \left( \sum_{k=1}^\infty \left\| (D_\Lambda - zI)^{-1/2} u_k \right\|^2 \right)^{1/2} \left( \sum_{k=1}^\infty \left\| (D_\Lambda - zI)^{-1/2})^* v_k \right\|^2 \right)^{1/2} \end{aligned}$$

$$\leq \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} \right)^{1/2} \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\beta_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} \right)^{1/2} < \infty,$$

and, accordingly,  $X(z)Y(z)$  is a trace-class operator.

On the other hand, having in mind that  $(D_{\Lambda} - zI)^{-1/2}$  is uniformly bounded when  $z \in A_{\mathbf{x}}^+$  and that the operator  $K = \sum_{k=1}^{\infty} u_k \otimes v_k$  is trace-class, it follows that  $X(z)Y(z)$  is as well trace class whenever  $z \in A_{\mathbf{x}}^+$ . Consequently  $X(z)Y(z)$  is a trace class operator, and hence compact.

In order to prove that  $I + X(z)Y(z)$  is invertible, let us argue by contradiction. Assume that  $I + X(z)Y(z)$  is not invertible. Hence, by compactness, it follows that  $-1 \in \sigma_p(X(z)Y(z))$  and  $-1 \in \sigma_p((X(z)Y(z))^*)$ . That is, there exists  $h \in H \setminus \{0\}$  such that

$$h + (X(z)Y(z))^*h = 0.$$

Applying  $(D_{\Lambda} - zI)^*$ , we deduce that

$$(D_{\Lambda} - zI)^*h + \left( \sum_{k=1}^{\infty} ((D_{\Lambda} - zI)^{1/2})^*v_k \otimes (D_{\Lambda} - zI)^{-1/2}u_k \right)h = 0.$$

Then, 0 is an eigenvalue of the operator

$$S = (D_{\Lambda} - zI)^* + \sum_{k=1}^{\infty} ((D_{\Lambda} - zI)^{1/2})^*v_k \otimes (D_{\Lambda} - zI)^{-1/2}u_k.$$

By Lemma 3.4, it follows that  $T - zI$  is quasisimilar to  $S^*$ , so  $T^* - \bar{z}I$  is quasisimilar to  $S$ . Finally,  $\bar{z}$  is an eigenvalue of  $T^*$ , which is a contradiction.

Hence,  $I + X(z)Y(z)$  is invertible, which deals with the first statement of the Theorem 3.5.

Nevertheless, from here the invertibility of  $I + Y(z)X(z)$  is just a standard fact (see [4, p. 199]) since the operator  $I - Y(z)(I + X(z)Y(z))^{-1}X(z)$  is an inverse for  $I + Y(z)X(z)$ . This concludes the proof.  $\square$

As a consequence of Theorem 3.5, we observe that if  $\mathbf{x} \in \Delta(T)$  and  $z \in \gamma_{\mathbf{x}}^+$  we may write

$$(I + Y(z)X(z))^{-1} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}(z)e_i \otimes e_j \tag{3.5}$$

where

$$a_{i,j}(z) = \langle (I + Y(z)X(z))^{-1}e_j, e_i \rangle. \tag{3.6}$$

Our next step consists of relating the coefficients  $a_{i,j}(z)$ , by means of a functional equation, to the operator  $T = D_\Lambda + \sum_{k=1}^\infty u_k \otimes v_k$  which allows us to construct the idempotents announced in the outline of the proof in introductory section. For such a purpose, we recall in this setting the *Borel series* associated to  $T$ , that is, the series defined by

$$f_T^{(i,j)}(z) = \sum_{n=1}^\infty \frac{\alpha_n^{(i)} \overline{\beta_n^{(j)}}}{\lambda_n - z}, \quad ((i, j) \in \mathbb{N} \times \mathbb{N}) \tag{3.7}$$

for those  $z \in \mathbb{C}$  such that the series converges. It is worthy to remark that  $f_T^{(i,j)}$  defines an analytic function in  $\mathbb{C} \setminus \overline{\Lambda}$ , and played a fundamental role in the existence of the invariant subspaces in the finite-rank perturbation case (see the works [11–13]).

Next theorem is the main result of this section, and the key of the construction of the idempotents to prove the Main Theorem.

**Theorem 3.6.** *Under the hypotheses of Proposition 3.3, let  $\mathbf{x} \in \Delta(T)$  and  $z \in \gamma_{\mathbf{x}}^+$ . There exists a positive constant  $C_{\mathbf{x}}^+ > 0$ , independent of  $z$ , such that for every  $x = \sum_{n=1}^\infty x_n e_n \in H$*

$$\sum_{i=1}^\infty \left| \sum_{j=1}^\infty x_j a_{i,j}(z) \right|^2 \leq C_{\mathbf{x}}^+ \|x\|^2. \tag{3.8}$$

Moreover, for each  $n \in \mathbb{N}$

$$\sum_{k=1}^\infty \sum_{j=1}^\infty x_j a_{k,j}(z) (\delta_{k,n} + f_T^{(k,n)}(z)) = x_n, \tag{3.9}$$

where  $\delta_{k,n} = 1$  if  $n = k$  and 0 otherwise.

In order to prove Theorem 3.6, the following fact is required:

**Proposition 3.7.** *Under the hypotheses of Proposition 3.3, for each  $\mathbf{x} \in \Delta(T)$  the map*

$$z \in \gamma_{\mathbf{x}}^+ \mapsto (I + Y(z)X(z))^{-1} \in \mathcal{L}(H)$$

is continuous in the norm topology of  $\mathcal{L}(H)$ .

**Proof.** Let  $\mathbf{x} \in \Delta(T)$  be fixed. Since  $I + Y(z)X(z)$  is invertible for every  $z \in \gamma_{\mathbf{x}}^+$  (Theorem 3.5), and taking inverses is continuous in the topology of  $\mathcal{L}(H)$ , it suffices to prove that the map

$$z \in \gamma_{\mathbf{x}}^+ \mapsto Y(z)X(z) \in \mathcal{L}(H)$$

is continuous.

For every  $z \in \gamma_{\mathbf{x}}^+ = \ell_{\mathbf{x}} \cup A_{\mathbf{x}}^+$ , it is easy to check that

$$Y(z)X(z) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_T^{(k,n)}(z) e_n \otimes e_k.$$

If  $z \in \ell_{\mathbf{x}}$  and  $x = \sum_{n=1}^{\infty} x_n e_n \in H$ , upon applying Cauchy-Schwarz inequality twice we deduce that

$$\begin{aligned} \|Y(z)X(z)x\|^2 &= \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} x_k f_T^{(k,n)}(z) \right|^2 \\ &\leq \|x\|^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |f_T^{(k,n)}(z)|^2 \\ &= \|x\|^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} \frac{\alpha_j^{(k)} \overline{\beta_j^{(n)}}}{\lambda_j - z} \right|^2 \\ &\leq \|x\|^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} \frac{|\alpha_j^{(k)}|^2}{|\lambda_j - z|} \right) \left( \sum_{j=1}^{\infty} \frac{|\beta_j^{(n)}|^2}{|\lambda_j - z|} \right) \\ &= \|x\|^2 \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{|\alpha_j^{(k)}|^2}{|\lambda_j - z|} \right) \left( \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{|\beta_j^{(n)}|^2}{|\lambda_j - z|} \right) \\ &\leq \|x\|^2 \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{|\alpha_j^{(k)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|} \right) \left( \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{|\beta_j^{(n)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|} \right), \end{aligned}$$

which is finite since  $\mathbf{x}$  belongs to the decomposability set  $\Delta(T)$ .

On the other hand, if  $z \in A_{\mathbf{x}}^+$ , denoting  $d := \operatorname{dist}(\mathbb{T}, \overline{\Lambda}) > 0$  and applying the same arguments, we deduce that

$$\|Y(z)X(z)\|^2 \leq \frac{1}{d^2} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^{(k)}|^2 \right) \left( \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\beta_j^{(n)}|^2 \right),$$

which is also finite.

Note that, as a consequence of the previous estimates, the finite-rank operators

$$U_N(\cdot) := \sum_{n=1}^N \sum_{k=1}^N f_T^{(k,n)}(\cdot) e_n \otimes e_k$$

converge uniformly on  $\gamma_{\mathbf{x}}^+$  to  $Y(\cdot)X(\cdot)$ . Accordingly, it suffices to show that for each  $N \in \mathbb{N}$  the maps

$$z \in \gamma_{\mathbf{x}}^+ \mapsto U_N(z)$$

are continuous, or equivalently, that the maps

$$z \in \gamma_{\mathbf{x}}^+ \mapsto f_T^{(k,n)}(z) \in \mathbb{C}$$

are continuous for every  $k, n \in \mathbb{N}$ . But taking into account that

$$\sum_{j=1}^{\infty} \left| \frac{\alpha_j^{(k)} \overline{\beta_j^{(n)}}}{\lambda_j - z} \right| \leq \left( \sum_{j=1}^{\infty} \frac{|\alpha_j^{(k)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|} \right)^{1/2} \left( \sum_{j=1}^{\infty} \frac{|\beta_j^{(n)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|} \right)^{1/2}$$

for every  $z \in \ell_{\mathbf{x}}$  and

$$\sum_{j=1}^{\infty} \left| \frac{\alpha_j^{(k)} \overline{\beta_j^{(n)}}}{\lambda_j - z} \right| \leq \frac{1}{d} \|u_k\| \|v_n\|$$

for every  $z \in A_{\mathbf{x}}^+$ , it follows that each Borel series  $f_T^{(k,n)}$  converges uniformly on  $\gamma_{\mathbf{x}}^+$ , so they define continuous functions on  $\gamma_{\mathbf{x}}^+$ . This concludes the proof.  $\square$

Finally, we are in position of proving Theorem 3.6.

**Proof of Theorem 3.6.** Let  $\mathbf{x} \in \Delta(T)$  fixed and  $z \in \gamma_{\mathbf{x}}^+$ . To prove (3.8), having in mind the expression of  $a_{i,j}(z)$  given by (3.5), we note that for each  $x = \sum_{n=1}^{\infty} x_n e_n \in H$ ,

$$\|(I + Y(z)X(z))^{-1}x\|^2 = \left\| \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} x_j a_{i,j}(z) \right) e_i \right\|^2 = \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} x_j a_{i,j}(z) \right|^2 \quad (3.10)$$

Now the map  $z \in \gamma_{\mathbf{x}}^+ \mapsto (I + Y(z)X(z))^{-1} \in \mathcal{L}(H)$  is continuous by Proposition 3.7, so the compactness of  $\gamma_{\mathbf{x}}^+$  yields that there exists  $C_{\mathbf{x}}^+ > 0$  such that

$$\|(I + Y(z)X(z))^{-1}\|^2 \leq C_{\mathbf{x}}^+$$

for every  $z \in \gamma_{\mathbf{x}}^+$ . This along with (3.10) yields

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} x_j a_{i,j}(z) \right|^2 \leq C_{\mathbf{x}}^+ \|x\|^2$$

which is (3.8).

Finally, let us prove that (3.9) holds. Fix  $z \in \gamma_{\mathbf{x}}^+$  and note that

$$I + Y(z)X(z) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\delta_{k,n} + f_T^{(k,n)}(z)) e_n \otimes e_k.$$

If  $x = \sum_{n=1}^{\infty} x_n e_n \in H$ , a computation shows

$$\begin{aligned} \sum_{n=1}^{\infty} x_n e_n &= (I + Y(z)X(z))(I + Y(z)X(z))^{-1}x \\ &= (I + Y(z)X(z)) \left( \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} x_j a_{i,j}(z) \right) e_i \right) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\delta_{k,n} + f_T^{(k,n)}(z)) \left\langle \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} x_j a_{i,j}(z) \right) e_i, e_k \right\rangle e_n \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\delta_{k,n} + f_T^{(k,n)}(z)) \sum_{j=1}^{\infty} x_j a_{k,j}(z) e_n \\ &= \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x_j a_{k,j}(z) (\delta_{k,n} + f_T^{(k,n)}(z)) \right) e_n. \end{aligned}$$

Matching the  $n$ -th Fourier coefficients for  $n \in \mathbb{N}$ , (3.9) follows and the proof of the Theorem 3.6 is finished.  $\square$

#### 4. Spectral idempotents: the key ingredient

In this section we introduce a family of non-trivial idempotents in the bicommutant  $\{T\}''$  such that their ranges are spectral subspaces for  $T$ .

**Theorem 4.1.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  be a bounded sequence not lying in any vertical line such that  $\Lambda'$  is not a singleton and  $\{u_k\}_{k \geq 1}$  and  $\{v_k\}_{k \geq 1}$  are non zero vectors in  $H$  satisfying (1.1). Assume that the trace-class perturbation  $T = D_\Lambda + \sum_{k=1}^{\infty} u_k \otimes v_k$  satisfies that both  $\sigma(D_\Lambda) = \bar{\Lambda}$  and  $\sigma(T)$  are contained in  $\mathbb{D}$ . Let  $\mathbf{x} \in \Delta(T)$  and  $X(\cdot)$ ,  $Y(\cdot)$  the operators defined on  $\gamma_{\mathbf{x}}^+$  given in (3.3) and write*

$$(I + Y(\cdot)X(\cdot))^{-1} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}(\cdot) e_i \otimes e_j.$$

Then the operator

$$\begin{aligned} J_{\mathbf{x}}^+ x &= \sum_{n \in N_{F_{\mathbf{x}}^+}} x_n e_n \\ &+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}}^+} \frac{\sum_{j=1}^{\infty} \frac{x_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{\lambda_n - \xi} d\xi \right) \alpha_n^{(k)} e_n, \end{aligned} \tag{4.1}$$

where  $x = \sum_{n=1}^{\infty} x_n e_n \in H$ , is well-defined and bounded in  $H$ .

As we mentioned in Section 3, for the sake of simplicity, we will deal with the curves  $\gamma_x^+$  and the analogous results will also hold when the curves  $\gamma_x^-$  are considered.

**Proof.** It is obvious that the map  $x = \sum_{n=1}^{\infty} x_n e_n \in H \mapsto \sum_{n \in N_{\gamma_x^+}} x_n e_n$  is bounded, so our task is dealing with the second summand of (4.1). Namely, the goal is finding a bound for

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \left( \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{x_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{\lambda_n - \xi} d\xi \right) \alpha_n^{(k)} \right|^2 \tag{4.2}$$

which is, clearly, less than or equal to

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |x_j| \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|}{|\lambda_j - \xi| |\lambda_n - \xi|} |d\xi| \left| \alpha_n^{(k)} \right| \right)^2 \tag{4.3}$$

Now, we apply the Cauchy-Schwarz inequality twice, first under the integral sign and then to series indexed in  $k \geq 1$ , having that (4.3) is less than or equal to

$$\begin{aligned} & \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |x_j| \left( \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2}{|\lambda_j - \xi|^2} |d\xi| \right)^{1/2} \left( \int_{\gamma_x^+} \frac{|d\xi|}{|\lambda_n - \xi|^2} \left| \alpha_n^{(k)} \right|^2 \right)^{1/2} \right) \leq \\ & \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} |x_j| \left( \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2}{|\lambda_j - \xi|^2} |d\xi| \right)^{1/2} \right|^2 \right) \left( \sum_{k=1}^{\infty} \int_{\gamma_x^+} \frac{|d\xi|}{|\lambda_n - \xi|^2} \left| \alpha_n^{(k)} \right|^2 \right) \end{aligned}$$

which is equal to

$$\underbrace{\left( \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} |x_j| \left( \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2}{|\lambda_j - \xi|^2} |d\xi| \right)^{1/2} \right|^2 \right)}_{(I)} \cdot \underbrace{\left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{\gamma_x^+} \frac{|d\xi|}{|\lambda_n - \xi|^2} \left| \alpha_n^{(k)} \right|^2 \right)}_{(II)} \tag{4.4}$$

First, we focus our attention on (II) in (4.4). As it is shown in the proof of [13, Theorem 2.8], there exists a constant  $C > 0$  such that

$$\int_{\gamma_{\mathbf{x}}^+} \frac{|d\xi|}{|\lambda_n - \xi|^2} \leq \frac{C}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|}. \tag{4.5}$$

Thus,

$$(II) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|d\xi|}{|\lambda_n - \xi|^2} |\alpha_n^{(k)}|^2 \leq C \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|},$$

which converges since  $\mathbf{x} \in \Delta(T)$ .

Now, let us bound the factor (I) in (4.4). Upon applying the Cauchy-Schwarz inequality to the series indexed in  $j \geq 1$ , we obtain

$$\begin{aligned} & \underbrace{\sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} |x_j| \left( \int_{\gamma_{\mathbf{x}}^+} \frac{|\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)|^2}{|\lambda_j - \xi|^2} |d\xi| \right)^{1/2} \right|^2}_{(I)} \\ & \leq \|x\|^2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)|^2}{|\lambda_j - \xi|^2} |d\xi| \\ & = \|x\|^2 \sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)|^2}{|\lambda_j - \xi|^2} |d\xi|, \end{aligned} \tag{4.6}$$

where last equality follows as consequence of the Monotone Convergence Theorem.

At this point, we note that for each  $j \geq 1$ , the vector  $\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} e_m \in H$ , so by (3.8) in Theorem 3.6, we deduce that

$$\sum_{k=1}^{\infty} \left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2 \leq C_{\mathbf{x}}^+ \sum_{m=1}^{\infty} |\beta_j^{(m)}|^2$$

for every  $\xi \in \gamma_{\mathbf{x}}^+$ . This along with (4.5) yields that

$$\sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)|^2}{|\lambda_j - \xi|^2} |d\xi| \leq C \cdot C_{\mathbf{x}}^+ \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\beta_j^{(m)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|},$$

which is finite, since  $\mathbf{x} \in \Delta(T)$ .

Consequently, there exists a positive constant  $M > 0$  given by

$$M := C^2 C_{\mathbf{x}}^+ \left( \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{|\beta_j^{(m)}|^2}{|\operatorname{Re}(\lambda_j) - \mathbf{x}|} \right) \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} \right), \tag{4.7}$$

such that

$$\|J_{\mathbf{x}}^+ x\| \leq \left\| \sum_{n \in N_{F_{\mathbf{x}}^+}} x_n e_n \right\| + M^{1/2} \|x\| \leq (1 + M^{1/2}) \|x\|,$$

for  $x \in H$ , which proves the theorem.  $\square$

The next step consists of applying Theorem 2.2 to show that the ranges of  $J_{\mathbf{x}}^{\pm}$  are contained in the spectral subspaces associated to the closed sets  $F_{\mathbf{x}}^{\pm}$  for every  $\mathbf{x}$  in the decomposability set of  $T$ . We state it for  $J_{\mathbf{x}}^+$ :

**Theorem 4.2.** *Under the hypotheses of Theorem 4.1, for each  $\mathbf{x} \in \Delta(T)$*

$$\operatorname{ran}(J_{\mathbf{x}}^+) \subseteq H_T(F_{\mathbf{x}}^+).$$

**Proof.** Let  $y = \sum_{n=1}^{\infty} y_n e_n \in H$  be fixed and denote  $x = J_{\mathbf{x}}^+ y$ . We will prove that  $x \in H_T(F_{\mathbf{x}}^+)$  applying Theorem 2.2: note that  $T$  enjoys the SVEP (because  $\sigma_p(T)$  is at most countable) and the series  $\sum_{k=1}^{\infty} (\|u_k\|^2 + \|v_k\|^2)$  is finite (as a consequence of (1.1)). Accordingly, our task is defining holomorphic functions  $g_k : \mathbb{C} \setminus F_{\mathbf{x}}^+ \rightarrow \mathbb{C}$  for  $k \geq 1$  satisfying conditions (i) – (iv) of the aforementioned theorem.

For  $k \geq 1$  let

$$g_k(z) = \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}}^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{z - \xi} d\xi \quad (z \in \mathbb{C} \setminus \gamma_{\mathbf{x}}^+).$$

We will show that  $g_k$  is a well-defined and holomorphic function in  $\mathbb{C} \setminus F_{\mathbf{x}}^+$ .

Fix  $K \subset \mathbb{C} \setminus \gamma_{\mathbf{x}}^+$  a compact set and denote  $d = \operatorname{dist}(K, \gamma_{\mathbf{x}}^+) > 0$ . Observe that, for every  $z \in K$ , Cauchy-Schwarz inequality applied twice yields

$$\begin{aligned} & \sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|y_j| \left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|}{|\lambda_j - \xi| |z - \xi|} |d\xi| \\ & \leq \frac{\|y\|}{d} \left( \sum_{j=1}^{\infty} \left( \int_{\gamma_{\mathbf{x}}^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2}{|\lambda_j - \xi|^2} |d\xi| \right)^{1/2} \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\|y\|}{d} \ell \cdot \left( \sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|^2}{|\lambda_j - \xi|^2} \right)^{1/2} \\
 &\leq \frac{\|y\|}{d} \ell \cdot (C_{\mathbf{x}}^+)^{1/2} \underbrace{\left( \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} |\beta_j^{(m)}|^2 \int_{\gamma_{\mathbf{x}}^+} \frac{|d\xi|}{|\lambda_j - \xi|^2} \right)^{1/2}}_{\text{(III)}}, \tag{4.8}
 \end{aligned}$$

where  $\ell := \text{length}(\gamma_{\mathbf{x}}^+)^{1/2}$  and the bound (3.8) of Theorem 3.6 has been applied in the last inequality.

Finally, (4.5) yields that (III) is finite since  $\mathbf{x} \in \Delta(T)$ , and hence the series in (4.8)

$$\sum_{j=1}^{\infty} \int_{\gamma_{\mathbf{x}}^+} \frac{|y_j| \left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|}{|\lambda_j - \xi| |z - \xi|} |d\xi|$$

converges uniformly in  $K$ . As a consequence, we have that

$$\begin{aligned}
 g_k(z) &= \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}}^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{z - \xi} d\xi \\
 &= \sum_{j=1}^{\infty} \frac{y_j}{2\pi i} \int_{\gamma_{\mathbf{x}}^+} \frac{\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)}{(\lambda_j - \xi)(z - \xi)} d\xi \tag{4.9}
 \end{aligned}$$

for  $z \in \mathbb{C} \setminus \gamma_{\mathbf{x}}^+$ .

Since the series in the right hand side of (4.9) converges uniformly on compact subsets of  $\mathbb{C} \setminus \gamma_{\mathbf{x}}^+$ , in order to prove that  $g_k$  is holomorphic it would be enough to show that for each  $j \in \mathbb{N}$  the map

$$z \in \mathbb{C} \setminus \gamma_{\mathbf{x}}^+ \mapsto \int_{\gamma_{\mathbf{x}}^+} \frac{\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi)}{(\lambda_j - \xi)(z - \xi)} d\xi \tag{4.10}$$

is holomorphic.

For such a task, fix  $j \in \mathbb{N}$  and define  $\beta_j := \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} e_m \in H$ . Observe that given  $\xi \in \gamma_{\mathbf{x}}^+$ ,

$$\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) = \langle (I + Y(\xi)X(\xi))^{-1} \beta_j, e_k \rangle,$$

which along with Proposition 3.7 yield that the map

$$\xi \in \gamma_x^+ \mapsto \frac{\sum_{m=1}^\infty \overline{\beta_j^{(m)}} a_{k,m}(\xi)}{(\lambda_j - \xi)(z - \xi)}$$

is continuous for every  $z \in \mathbb{C} \setminus \gamma_x^+$ .

Finally, for  $\xi \in \gamma_x^+$  the map

$$z \in \mathbb{C} \setminus \gamma_x^+ \mapsto \frac{1}{z - \xi}$$

is holomorphic, so for each  $j \in \mathbb{N}$  the map (4.10) is holomorphic and, hence,  $g_k$  is holomorphic as well.

Now, let us show that for each  $k \geq 1$  the function  $g_k$  meets the conditions (i) – (iv) of Theorem 2.2. To prove (i), let  $n \in N_{(F_x^+)^c}$  and note that

$$\langle x, e_n \rangle = \langle J_x^+ y, e_n \rangle = \sum_{k=1}^\infty g_k(\lambda_n) \alpha_n^{(k)}.$$

Thus, condition (i) is fulfilled.

In order to show (ii), observe that for every  $z \in \mathbb{C} \setminus \gamma_x^+, n \in \mathbb{N}$  and  $z \neq \lambda_n$ , we have

$$\Gamma(g_k)(z, \lambda_n) = \frac{-1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^\infty \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^\infty \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{(z - \xi)(\lambda_n - \xi)} d\xi.$$

If  $h = \sum_{n=1}^\infty h_n e_n \in H$ , we are required to prove that the map

$$z \in \mathbb{C} \setminus \gamma_x^+ \mapsto \sum_{n \in N_{(F_x^+)^c}} \sum_{k=1}^\infty \left( \frac{-1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^\infty \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^\infty \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{(z - \xi)(\lambda_n - \xi)} d\xi \right) \alpha_n^{(k)} \overline{h_n} \tag{4.11}$$

is holomorphic. Since the map  $z \in \mathbb{C} \setminus \gamma_x^+ \mapsto \Gamma(g_k)(z, \lambda_n) \alpha_n^{(k)}$  is holomorphic for every  $n \in N_{(F_x^+)^c}$  and  $k \in \mathbb{N}$ , it suffices to show that the double series in (4.11) converges uniformly on compact subsets of  $\mathbb{C} \setminus \gamma_x^+$ .

Fix  $K \subset \mathbb{C} \setminus \gamma_x^+$  a compact subset and denote  $d = \text{dist}(K, \gamma_x^+)$ . Note that

$$\begin{aligned}
 & \sum_{n \in N_{(F_x^+)^c}} \sum_{k=1}^{\infty} \frac{1}{2\pi} \left| \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{(z - \xi)(\lambda_n - \xi)} d\xi \alpha_n^{(k)} \overline{h_n} \right| \\
 & \leq \frac{1}{2\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |y_j| \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|}{|\lambda_j - \xi| |z - \xi| |\lambda_n - \xi|} |d\xi| |\alpha_n^{(k)}| |h_n| \\
 & \leq \frac{\|h\|}{2\pi d} \underbrace{\left( \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |y_j| \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right|}{|\lambda_j - \xi| |\lambda_n - \xi|} |d\xi| |\alpha_n^{(k)}| \right)^2 \right)^{1/2}}_{(IV)},
 \end{aligned} \tag{4.12}$$

where last inequality follows upon applying Cauchy-Schwarz inequality. But, observe that (IV) looks like (4.3) (where  $y_j$  plays the role of  $x_j$ ), and (4.3) was already shown to be finite. Consequently, (ii) follows as well.

Now, let us prove condition (iii). Having into account that  $g_k$  is holomorphic in  $\mathbb{C} \setminus \gamma_x^+$ , our task is proving that

$$\sum_{n \in N_{F_x^+}} \frac{y_n \overline{\beta_n^{(k)}}}{\lambda_n - z} = g_k(z) - \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \Gamma(g_\ell)(z, \lambda_n) \alpha_n^{(\ell)} \overline{\beta_n^{(k)}}, \tag{4.13}$$

for each  $z \in (F_x^+)^c$  and  $k \in \mathbb{N}$ .

Now,

$$\sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \Gamma(g_\ell)(z, \lambda_n) \alpha_n^{(\ell)} \overline{\beta_n^{(k)}} = \underbrace{\sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{-1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) \right)}{(z - \xi)(\lambda_n - \xi)} d\xi \alpha_n^{(\ell)} \overline{\beta_n^{(k)}}}_{(V)}, \tag{4.14}$$

for  $z \in (F_x^+)^c$  and  $k \in \mathbb{N}$ . But the series

$$\sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} |y_j| \int_{\gamma_x^+} \frac{\left| \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) \right|}{|\lambda_j - \xi| |z - \xi| |\lambda_n - \xi|} |d\xi| |\alpha_n^{(\ell)}| |\beta_n^{(k)}|$$

converges (analogously as (4.3)), so we can change the order of summation of  $n, \ell, j$  and the integral in (V) (note that we are not changing the series indexed in  $m$ ) and (4.14) turns out to be:

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \Gamma(g_{\ell})(z, \lambda_n) \alpha_n^{(\ell)} \overline{\beta_n^{(k)}} \\ &= \frac{-1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \sum_{\ell=1}^{\infty} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) \right) \sum_{n=1}^{\infty} \frac{\alpha_n^{(\ell)} \overline{\beta_n^{(k)}}}{\lambda_n - \xi}}{(z - \xi)} d\xi \\ &= \frac{-1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \sum_{\ell=1}^{\infty} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) f_T^{(\ell,k)}(\xi) \right)}{(z - \xi)} d\xi, \end{aligned}$$

for each  $z \in (F_x^+)^c$  and  $k \in \mathbb{N}$ .

Hence, equality (4.13) turns into

$$\begin{aligned} \sum_{n \in N_{F_x^+}} \frac{\overline{y_n \beta_n^{(k)}}}{\lambda_n - z} &= \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{z - \xi} d\xi + \\ &+ \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \sum_{\ell=1}^{\infty} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) f_T^{(\ell,k)}(\xi) \right)}{(z - \xi)} d\xi \\ &= \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \sum_{\ell=1}^{\infty} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{\ell,m}(\xi) (\delta_{\ell,k} + f_T^{(\ell,k)}(\xi)) \right)}{(z - \xi)} d\xi \\ &= \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \overline{\beta_j^{(k)}}}{z - \xi} d\xi, \end{aligned} \tag{4.15}$$

where the last equality follows upon applying Theorem 3.6, identity (3.9) with the vector  $\sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} e_m \in H$  for each  $j \in \mathbb{N}$ .

So, proving identity (4.13) is equivalent to prove the identity:

$$\sum_{n \in N_{F_x^+}} \frac{\overline{y_n \beta_n^{(k)}}}{\lambda_n - z} = \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \overline{\beta_j^{(k)}}}{z - \xi} d\xi, \tag{4.16}$$

for each  $z \in (F_x^+)^c$  and  $k \in \mathbb{N}$ . At this point, we argue as in [13, p. 24] to check that such equality holds.

Observe that

$$\int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \overline{\beta_j^{(k)}}}{z - \xi} d\xi = \sum_{j=1}^{\infty} y_j \overline{\beta_j^{(k)}} \int_{\gamma_x^+} \frac{d\xi}{(\lambda_j - \xi)(z - \xi)} \tag{4.17}$$

for every  $z \in (F_x^+)^c$  since

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{\gamma_x^+} \left| \frac{y_j \overline{\beta_j^{(k)}}}{(\lambda_j - \xi)(z - \xi)} \right| |d\xi| &= \sum_{j=1}^{\infty} |y_j \beta_j^{(k)}| \int_{\gamma_x^+} \frac{|d\xi|}{|\lambda_j - \xi| |z - \xi|} \\ &\leq \frac{\|y\|}{\text{dist}(z, \gamma_x^+)} \left( \sum_{j=1}^{\infty} \left( \int_{\gamma_x^+} \frac{|d\xi|}{|\lambda_j - \xi|} \right)^2 |\beta_j^{(k)}|^2 \right)^{1/2}, \end{aligned}$$

being the latter series convergent because  $x$  belongs to the decomposability set.

Accordingly, proving (4.16) turns into showing

$$\sum_{n \in N_{F_x^+}} \frac{y_n \overline{\beta_n^{(k)}}}{\lambda_n - z} = \sum_{j=1}^{\infty} y_j \overline{\beta_j^{(k)}} \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{d\xi}{(\lambda_j - \xi)(z - \xi)}$$

for each  $z \in (F_x^+)^c$  and  $k \in \mathbb{N}$ , which holds upon applying Cauchy’s integral formula.

Thus, condition (iii) is fulfilled.

It remains to show that condition (iv) holds. That is, we have to prove that, for each  $z \in (F_x^+)^c$ ,

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} \right|^2 < \infty.$$

So, let  $z \in (F_x^+)^c$ . Having in mind (4.9), it follows that

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} g_k(z) \alpha_n^{(k)} \right|^2 = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\gamma_x^+} \frac{\sum_{j=1}^{\infty} \frac{y_j}{\lambda_j - \xi} \left( \sum_{m=1}^{\infty} \overline{\beta_j^{(m)}} a_{k,m}(\xi) \right)}{z - \xi} d\xi \alpha_n^{(k)} \right|^2, \tag{4.18}$$

which turns out to be equivalent to the series (4.2) considered in the proof of Theorem 4.1, where  $z - \xi$  plays the role of  $\lambda_n - \xi$ . Hence, arguing analogously as in the proof of Theorem 4.1 and taking into account that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{\gamma_x^+} \frac{|d\xi|}{|z - \xi|^2} |\alpha_n^{(k)}|^2 \leq \text{length}(\gamma_x^+) \frac{1}{d^2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_n^{(k)}|^2 < \infty,$$

with  $d = \text{dist}(z, \gamma_x^+) > 0$ , it follows that (4.18) holds.

Finally, Theorem 2.2 yields that  $x = J_x^+ y \in H_T(F_x^+)$ , so  $\text{ran}(J_x^+) \subset H_T(F_x^+)$ , as we wished to prove.  $\square$

Now, the goal is proving that for almost every  $\mathbf{x}$  in decomposability set the operators  $J_{\mathbf{x}}^+$  are spectral idempotents. The proof is based on the following two lemmas:

**Lemma 4.3.** *Under the hypotheses of Theorem 4.1, for every  $\mathbf{x} \in \Delta(T)$*

$$J_{\mathbf{x}}^+ + J_{\mathbf{x}}^- = Id_H.$$

**Proof.** Let  $\mathbf{x} \in \Delta(T)$  be fixed. In order to prove  $J_{\mathbf{x}}^+ + J_{\mathbf{x}}^- = Id_H$ , it is enough to show that

$$(J_{\mathbf{x}}^+ + J_{\mathbf{x}}^-)e_N = e_N$$

for every  $N \in \mathbb{N}$ . Note that

$$(J_{\mathbf{x}}^+ + J_{\mathbf{x}}^-)e_N = e_N + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\frac{1}{\lambda_N - \xi} \sum_{m=1}^{\infty} \overline{\beta_N^{(m)}} a_{k,m}(\xi)}{\lambda_n - \xi} \alpha_n^{(k)} d\xi e_n, \tag{4.19}$$

so let us show that the second summand in the right-hand side of (4.19) is zero. Arguing as in (4.2) in the proof of Theorem 4.1, we can change the series indexed in  $n$  and  $k$  with the integral and prove that

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\frac{1}{\lambda_N - \xi} \sum_{m=1}^{\infty} \overline{\beta_N^{(m)}} a_{k,m}(\xi)}{\lambda_n - \xi} \alpha_n^{(k)} e_n d\xi = 0 \tag{4.20}$$

for every  $N \in \mathbb{N}$ . For this purpose, the strategy will be finding the inverse of the operator  $T - zI$  for  $z \in \mathbb{T}$  and applying an argument involving the Dunford Functional Calculus.

For each  $z \in \mathbb{T}$ , let  $B(z)$  be the operator

$$B(z) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}(z) (D_{\Lambda} - zI)^{-1} u_i \otimes (D_{\Lambda}^* - \bar{z}I)^{-1} v_j. \tag{4.21}$$

Since  $\bar{\Lambda} \subset \mathbb{D}$ , it follows that  $(D_{\Lambda} - zI)^{-1/2}$  and  $(D_{\Lambda} - zI)^{-1}$  are well defined bounded operators for every  $z \in \mathbb{T}$  and moreover, Proposition 3.3 yields that the operator

$$L(z) = X(z)(I + Y(z)X(z))^{-1}Y(z)$$

is bounded. Note that

$$\begin{aligned} L(z) &= X(z)(I + Y(z)X(z))^{-1}Y(z) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}(z) (D_{\Lambda} - zI)^{-1/2} u_i \otimes (D_{\Lambda}^* - \bar{z}I)^{-1/2} v_j, \end{aligned} \tag{4.22}$$

so  $B(z) = (D_\Lambda - zI)^{-1/2}L(z)(D_\Lambda - zI)^{-1/2}$  and hence, a bounded operator.

Now, we claim that for each  $z \in \mathbb{T}$

$$R(z) = (D_\Lambda - zI)^{-1} - B(z)$$

is the inverse of  $T - zI$ . Note that for each  $z \in \mathbb{T}$ , the operator  $R(z)$  is bounded since  $\sigma(D_\Lambda) \subset \mathbb{D}$ , so in order to prove the claim, it suffices to show that  $R(z)$  is the right inverse of  $(T - zI)$ .

Let  $z \in \mathbb{T}$  be fixed. Having in mind that  $(T - zI) = (D_\Lambda - zI)^{1/2}(I + X(z)Y(z))(D_\Lambda - zI)^{1/2}$  and (4.22), we deduce

$$\begin{aligned} &(T - zI)R(z) \\ &= (T - zI)((D_\Lambda - zI)^{-1} - B(z)) \\ &= (T - zI) \left( (D_\Lambda - zI)^{-1} - (D_\Lambda - zI)^{-1/2}L(z)(D_\Lambda - zI)^{-1/2} \right) \\ &= (T - zI)(D_\Lambda - zI)^{-1/2} (I - L(z)) (D_\Lambda - zI)^{-1/2} \\ &= (D_\Lambda - zI)^{1/2}(I + X(z)Y(z))(D_\Lambda - zI)^{1/2}(D_\Lambda - zI)^{-1/2} (I - L(z)) (D_\Lambda - zI)^{-1/2} \\ &= (D_\Lambda - zI)^{1/2}(I + X(z)Y(z))(I - L(z))(D_\Lambda - zI)^{-1/2} \\ &= (D_\Lambda - zI)^{1/2}(I + X(z)Y(z))(I - X(z)(I + Y(z)X(z))^{-1}Y(z))(D_\Lambda - zI)^{-1/2}. \end{aligned}$$

At this point, it is enough to notice that  $(I + X(z)Y(z))^{-1} = (I - X(z)(I + Y(z)X(z))^{-1}Y(z))$  (see, for instance, [4, p. 199]) to obtain that  $(T - zI)R(z) = Id_H$ , as claimed.

Now, by means of the Dunford functional calculus, we have that

$$\frac{-1}{2\pi i} \int_{\mathbb{T}} R(\xi)d\xi = Id_H \quad \frac{-1}{2\pi i} \int_{\mathbb{T}} (D_\Lambda - \xi I)^{-1}d\xi = Id_H,$$

so

$$\frac{1}{2\pi i} \int_{\mathbb{T}} B(\xi)d\xi = 0. \tag{4.23}$$

Having in mind (4.21), we observe that

$$B(z)e_N = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_N - z} \frac{\sum_{m=1}^{\infty} \overline{\beta_N^{(m)}} a_{k,m}(z)}{\lambda_n - z} \alpha_n^{(k)} e_n$$

for every  $z \in \mathbb{T}$ .

Thus, (4.20) turns into

$$0 = \frac{1}{2\pi i} \int_{\mathbb{T}} B(\xi) e_N d\xi,$$

which holds by (4.23) and concludes the proof.  $\square$

**Lemma 4.4.** *Under the hypotheses of Theorem 4.1, for almost every  $\mathbf{x} \in \Delta(T)$*

$$J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = J_{\mathbf{x}}^-(H_T(F_{\mathbf{x}}^+)) = \{0\}.$$

Consequently, for almost every  $\mathbf{x} \in \Delta(T)$ ,

$$J_{\mathbf{x}}^+ J_{\mathbf{x}}^- = J_{\mathbf{x}}^- J_{\mathbf{x}}^+ = 0.$$

This proof is based on that of [13, Lemma 2.14], so we will highlight the differences instead of repeating some arguments.

**Proof.** We are showing that  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = \{0\}$  and  $J_{\mathbf{x}}^+ J_{\mathbf{x}}^- = 0$ , the other equalities are equivalent.

First, following the computations of the proof of [13, Lemma 2.14] and using (3.9), it is easy to check that  $J_{\mathbf{x}}^+ \in \{T\}'$ , and as a consequence  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{y}}^-)) = \{0\}$  for every  $\mathbf{y} \in \Delta(T)$  and  $\mathbf{y} < \mathbf{x}$ .

We claim that for almost every  $\mathbf{x} \in \Delta(T)$ , there exists a sequence  $(\mathbf{x}_n)_{n \in \mathbb{N}} \subset \Delta(T)$  such that  $\mathbf{x}_n > \mathbf{x}$ ,  $\mathbf{x}_n \searrow \mathbf{x}$  and  $J_{\mathbf{x}_n}^+ \rightarrow J_{\mathbf{x}}^+$  in the weak operator topology. Note that, in particular, the claim implies the equality  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = \{0\}$  for almost every  $\mathbf{x} \in \Delta(T)$ .

In order to prove the claim, let  $p \in \mathbb{N}$ . Egorov Theorem yields the existence of a measurable subset  $\mathcal{X}_p \subset \Delta(T)$ , such that the Lebesgue measure of  $\Delta(T) \setminus \mathcal{X}_p$  is strictly smaller than  $1/p$  and the double series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} + \frac{|\beta_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}|} \right) \tag{4.24}$$

converges uniformly and is uniformly bounded on  $\mathcal{X}_p$ . Define

$$D_p := \{z \in \overline{\mathbb{D}} : \operatorname{Re}(z) \in \mathcal{X}_p\}.$$

Arguing as in Proposition 3.7 and having the uniform convergence and boundedness of (4.24), it follows that, indeed, the maps

$$z \in D_p \mapsto (I + Y(z)X(z))^{-1} \in \mathcal{L}(H)$$

are continuous for each  $p \in \mathbb{N}$ .

Likewise, reasoning as in the proof of [13, Lemma 2.14], it suffices to show the claim for those  $\mathbf{x} \in \mathcal{X}'_{p_0}$  where  $p_0 \in \mathbb{N}$ . In this situation, there exists a sequence  $(\mathbf{x}_n)_{n \in \mathbb{N}} \subset \mathcal{X}_{p_0}$  such that  $\mathbf{x}_n \nearrow \mathbf{x}$ . Let us prove that  $J_{\mathbf{x}_n}^+ \rightarrow J_{\mathbf{x}}^+$  in the weak operator topology.

Consider

$$\Gamma = \bigcup_{n \in \mathbb{N}} \gamma_{\mathbf{x}_n}^+ \cup \gamma_{\mathbf{x}}^+$$

and observe that it is a compact set. In particular, the map  $z \in \Gamma \mapsto \|(I + Y(z)X(z))^{-1}\|^2$  is continuous, so there exists  $A > 0$  such that the constants  $C_{\mathbf{x}_n}^+$  given in (3.8) are bounded by the constant  $A$ .

In order to show the weak convergence of  $(J_{\mathbf{x}_n}^+)_{n \geq 1}$ , we first check that the norm sequence  $(\|J_{\mathbf{x}_n}^+\|)_{n \geq 1}$  is uniformly bounded. Indeed, taking into account (4.7), it is enough to recall the uniform bound of  $C_{\mathbf{x}_n}^+$  by  $A$ , and the uniform bounds of the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}_n|} \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\beta_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}_n|}$$

given by the uniform convergence of (4.24) in  $\mathcal{X}_{p_0}$ . Finally, the constant  $C > 0$  given in (4.5) was obtained in the proof of [13, Theorem 2.8], and it does not depend on  $\mathbf{x}$ . Thus,  $\|J_{\mathbf{x}_n}^+\|$  are uniformly bounded.

Let us prove that  $J_{\mathbf{x}_n}^+ \rightarrow J_{\mathbf{x}}^+$  in the weak operator topology. Indeed, it is enough to show that

$$\langle J_{\mathbf{x}_n}^+ e_i, e_j \rangle \rightarrow \langle J_{\mathbf{x}}^+ e_i, e_j \rangle \tag{4.25}$$

for every  $i, j \in \mathbb{N}$ . Now,

$$\begin{aligned} \langle J_{\mathbf{x}_n}^+ e_i, e_j \rangle &= \tilde{\delta}_{n,i,j} + \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}_n}^+} \frac{\sum_{m=1}^{\infty} \overline{\beta_i^{(m)}} a_{k,m}(\xi)}{(\lambda_i - \xi)(\lambda_j - \xi)} d\xi \alpha_j^{(k)} \\ &= \tilde{\delta}_{n,i,j} + \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}_n}^+} \frac{\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \overline{\beta_i^{(m)}} a_{k,m}(\xi) \alpha_j^{(k)}}{(\lambda_i - \xi)(\lambda_j - \xi)} d\xi \\ &= \tilde{\delta}_{n,i,j} + \frac{1}{2\pi i} \int_{\gamma_{\mathbf{x}_n}^+} \frac{\langle (I + Y(\xi)X(\xi))^{-1} b_i, a_j \rangle}{(\lambda_i - \xi)(\lambda_j - \xi)} d\xi, \end{aligned}$$

where  $\tilde{\delta}_{n,i,j} = 1$  if  $i = j \in N_{F_{\mathbf{x}_n}^+}$  and 0 otherwise,  $b_i = \sum_{m=1}^{\infty} \overline{\beta_i^{(m)}} e_m$  and  $a_j = \sum_{k=1}^{\infty} \alpha_j^{(k)} e_k$  for every  $i, j \in \mathbb{N}$ . Finally, observe that the map  $\xi \in \Gamma \mapsto \langle (I + Y(\xi)X(\xi))^{-1} b_i, a_j \rangle$  is continuous, so a standard argument involving Lebesgue's Dominated Convergence Theorem shows the convergence of (4.25).

As a consequence, we have shown that  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = \{0\}$  for almost every  $\mathbf{x} \in \Delta(T)$ , which finishes the first part of the proof.

Finally, let  $\mathbf{x} \in \Delta(T)$  such that  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = \{0\}$  and observe that  $\text{ran}(J_{\mathbf{x}}^-) \subset H_T(F_{\mathbf{x}}^-)$ . It is trivial then that  $J_{\mathbf{x}}^+ J_{\mathbf{x}}^- = 0$ , and Lemma 4.4 is proved.  $\square$

We are now in position to prove the announced result of this section:

**Theorem 4.5.** *Under the hypotheses of Theorem 4.1, for almost every  $\mathbf{x} \in \Delta(T)$*

$$J_{\mathbf{x}}^+ + J_{\mathbf{x}}^- = Id_H, \quad J_{\mathbf{x}}^+ J_{\mathbf{x}}^- = J_{\mathbf{x}}^- J_{\mathbf{x}}^+ = 0. \tag{4.26}$$

*In particular, for almost every  $\mathbf{x} \in \Delta(T)$  the operators  $J_{\mathbf{x}}^+$  and  $J_{\mathbf{x}}^-$  are non-trivial spectral idempotents for  $T$ .*

**Proof.** First, Lemmas 4.3 and 4.4 yields that for almost every  $\mathbf{x} \in \Delta(T)$  the operators  $J_{\mathbf{x}}^+$  and  $J_{\mathbf{x}}^-$  are idempotents satisfying (4.26).

Let us show that they are spectral idempotents for  $T$ . Let us take  $\mathbf{x} \in \Delta(T)$  such that  $J_{\mathbf{x}}^+(H_T(F_{\mathbf{x}}^-)) = \{0\}$  and  $J_{\mathbf{x}}^+ + J_{\mathbf{x}}^- = 0$ . Following the arguments of [13, Corollary 2.15], it follows that  $J_{\mathbf{x}}^+ \in \{T\}''$  and  $\text{ran}(J_{\mathbf{x}}^+) = H_T(F_{\mathbf{x}}^+)$ . Accordingly,  $H_T(F_{\mathbf{x}}^+)$  is a closed subspace and  $J_{\mathbf{x}}^+$  is a spectral idempotent for  $T$ .

Finally, let us show that  $J_{\mathbf{x}}^+$  is non-trivial for every  $\mathbf{x} \in \Delta(T)$ . By the equality  $J_{\mathbf{x}}^+ + J_{\mathbf{x}}^- = Id_H$ , it is clear that if  $J_{\mathbf{x}}^+$  is non-trivial,  $J_{\mathbf{x}}^-$  is non-trivial as well.

Assume that  $J_{\mathbf{x}}^+ = Id_H$ . If  $J_{\mathbf{x}}^+ = 0$  then  $J_{\mathbf{x}}^- = Id_H$  and the argument is analogous. Since  $\text{ran}(J_{\mathbf{x}}^+) = H_T(F_{\mathbf{x}}^+)$ , it follows that  $H_T(F_{\mathbf{x}}^+) = H$ . In particular,  $\sigma(T) = \sigma(T|_{H_T(F_{\mathbf{x}}^+)}) \subseteq F_{\mathbf{x}}^+$ , which is a contradiction: recall that  $a < \mathbf{x}$ , where  $a = \min_{z \in \Lambda'} \text{Re}(z)$ . So, there exists  $\lambda \in \sigma(T)$  such that  $\text{Re}(\lambda) < \mathbf{x}$  and hence,  $\lambda \notin F_{\mathbf{x}}^+$ . This ends the proof.  $\square$

### 5. Proof of the Main Theorem

Finally, we are in position to prove the Main Theorem stated in the introductory section, which we recall again for the sake of clarity:

**Main Theorem.** *Let  $H$  be a separable, infinite dimensional complex Hilbert space,  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  a bounded sequence and  $\{u_k\}_{k \geq 1}, \{v_k\}_{k \geq 1}$  non zero vectors in  $H$ . Assume*

$$\sum_{(n,k) \in \mathcal{N}_u} |\alpha_n^{(k)}|^2 \log \left( 1 + \frac{1}{|\alpha_n^{(k)}|} \right) + \sum_{(n,k) \in \mathcal{N}_v} |\beta_n^{(k)}|^2 \log \left( 1 + \frac{1}{|\beta_n^{(k)}|} \right) < \infty, \tag{5.1}$$

where  $\mathcal{N}_u := \{(n, k) \in \mathbb{N} \times \mathbb{N} : \alpha_n^{(k)} \neq 0\}$  and  $\mathcal{N}_v := \{(n, k) \in \mathbb{N} \times \mathbb{N} : \beta_n^{(k)} \neq 0\}$ . Then, the trace-class perturbation of  $D_{\Lambda}$ ,  $T = D_{\Lambda} + \sum_{k=1}^{\infty} u_k \otimes v_k$ , acting on  $H$  by

$$Tx = \left( D_\Lambda + \sum_{k=1}^\infty u_k \otimes v_k \right) x = D_\Lambda x + \sum_{k=1}^\infty \langle x, v_k \rangle u_k, \quad (x \in H), \quad (5.2)$$

has non trivial closed hyperinvariant subspaces provided that it is not a scalar multiple of the identity operator. Moreover, if both point spectrum  $\sigma_p(T)$  and  $\sigma_p(T^*)$  are at most countable,  $T$  is decomposable.

**Proof.** First of all, assume that  $T$  is not a scalar multiple of the identity. For the first half, we may assume that  $\sigma_p(T) \cup \sigma_p(T^*)$  is empty, since otherwise  $T$  or  $T^*$  will have eigenvalues and then  $T$  will have non-trivial closed hyperinvariant subspaces. Moreover, one may assume that  $\Lambda'$  is not a singleton: if  $\Lambda' = \{\lambda\}$ , then  $D_\Lambda = \lambda Id_H + \hat{K}$ , where  $\hat{K}$  is a compact operator. Thus,  $T = \lambda I + \hat{K} + K$ , so  $T$  commutes with the non-zero compact operator  $\hat{K} + K$  and Lomonosov’s Theorem (see [17] or [21, Corollary 8.24]) provides a non-trivial closed hyperinvariant subspace for  $T$ . In such a case, the decomposability also follows since the spectrum of  $T$  is totally disconnected [16, Proposition 1.4.5].

Likewise, we also may assume that  $\sigma(D_\Lambda)$  and  $\sigma(T)$  are contained in  $\mathbb{D}$  and are not contained in a vertical line, since these properties may be achieved by translating and multiplying the operator by a scalar, which do not change the existence of non-trivial closed hyperinvariant subspaces and or the decomposability of  $T$ . In these conditions, we can apply Theorem 4.5 to obtain non-trivial spectral idempotents  $J_x^+$  and  $J_x^-$  for  $T$  for almost every  $x \in \Delta(T)$ . In particular, observe that  $H_T(F_x^+) = \text{ran}(J_x^+)$  and  $H_T(F_x^-) = \text{ran}(J_x^-)$  are non-trivial closed hyperinvariant subspaces for  $T$ .

Finally, for the second half, we may assume that  $\sigma_p(T) \cup \sigma_p(T^*)$  is at most countable. Then, in order to show that  $T$  is decomposable, it is enough to mimic the construction of [13, Section 3] in order to define the non-trivial spectral idempotents for  $T$  associated to horizontal lines (instead of vertical lines as  $J_x^+$ ) and generate a Boolean algebra of spectral idempotents associated to rectangles in  $\mathbb{C}$ . At this point, it remains to apply the arguments exposed in the proof of [9, Theorem 3.2] to obtain the decomposability of  $T$ , which yields the statement of the Main Theorem.  $\square$

### 5.1. A local version of the result

In this subsection, we prove a local version of the Main Theorem that will allow us to obtain non-trivial closed hyperinvariant subspaces replacing the assumption (5.1) by a weaker local summability condition. The result, which extends [12, Corollary 2.8] to trace class perturbations of diagonalizable normal operators, reads as follows:

**Theorem 5.1.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1} \subset \mathbb{C}$  be a bounded sequence not lying in any vertical line such that  $\Lambda'$  is not a singleton and denote  $a := \min_{z \in \Lambda'} \text{Re}(z)$  and  $b := \max_{z \in \Lambda'} \text{Re}(z)$ . Let  $u_k = \sum_{n=1}^\infty \alpha_n^{(k)} e_n$  and  $v_k = \sum_{n=1}^\infty \beta_n^{(k)} e_n$  be non-zero vectors in  $H$  for each  $k \geq 1$ . Assume that  $K = \sum_{k=1}^\infty u_k \otimes v_k$  is trace class and the compact perturbation  $T = D_\Lambda + K$*

satisfies both  $\sigma(D_\Lambda) = \overline{\Lambda}$  and  $\sigma(T)$  contained in  $\mathbb{D}$ . If there exist  $a < \mathbf{x}_1 < \mathbf{x}_2 < b$  such that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{|\alpha_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}_i|} + \frac{|\beta_n^{(k)}|^2}{|\operatorname{Re}(\lambda_n) - \mathbf{x}_i|} \right) < \infty \quad (i = 1, 2), \tag{5.3}$$

then  $T$  has non-trivial closed hyperinvariant subspaces.

**Proof.** We may assume that  $\sigma_p(T) \cup \sigma_p(T^*)$  is empty. A careful reading of the proofs of Theorems 4.1 and 4.2 yields that the operators  $J_{\mathbf{x}_2}^-$  and  $J_{\mathbf{x}_2}^+$  are well-defined bounded operators such that  $\operatorname{ran}(J_{\mathbf{x}_2}^-) \subset H_T(F_{\mathbf{x}_2}^-)$  and  $\operatorname{ran}(J_{\mathbf{x}_2}^+) \subset H_T(F_{\mathbf{x}_2}^+)$ .

Moreover, arguing as in the proof of Lemma 4.3, it follows that  $J_{\mathbf{x}_2}^+ + J_{\mathbf{x}_2}^- = Id_H$ . Likewise, the arguments shown in the proof of Theorem 4.5 yield that  $J_{\mathbf{x}_2}^+$  is a non-trivial operator. We remark that we cannot assure these operators are idempotent, since the uniform convergence of the series (4.24) is used to prove Lemma 4.4. Therefore, we know that  $H_T(F_{\mathbf{x}_2}^+)$  is a non-zero spectral subspace. At this point, observe that the same construction can be carried out for the adjoint  $T^*$ , since it also verifies (5.3). As a consequence,  $H_{T^*}(F_{\mathbf{x}_1}^-)$  is also a non-trivial spectral subspace.

Finally, by means of [16, Proposition 2.5.1], it follows that  $H_T(F_{\mathbf{x}_2}^+)$  is non-dense, since  $H_{T^*}(F_{\mathbf{x}_1}^-)$  is non-zero and  $F_{\mathbf{x}_1}^- \cap F_{\mathbf{x}_2}^+ = \emptyset$ . Accordingly,  $H_T(F_{\mathbf{x}_2}^+)$  is a non-trivial closed hyperinvariant subspace for  $T$  which concludes the proof.  $\square$

### An open question

In order to conclude, we note that the Main Theorem does not include every operator  $T$  in the class  $(\mathcal{N} + \mathcal{C}_1)$ , that is, the class of trace class perturbations of diagonalizable normal operators. Therefore, we close this manuscript with the following question:

**Question.** Let  $H$  be a separable, infinite dimensional complex Hilbert space. Does every operator  $T$  in the class  $(\mathcal{N} + \mathcal{C}_1)$  have non-trivial closed hyperinvariant subspaces? If so, if  $\sigma_p(T) \cup \sigma_p(T^*)$  is at most countable, is  $T$  decomposable?

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### Data availability

No data was used for the research described in the article.

## References

- [1] P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Springer, Netherlands, 2004.
- [2] E. Albrecht, B. Chevreau, Compact perturbations of scalar type spectral operators, *J. Oper. Theory* 86 (1) (2021) 163–188.
- [3] I. Colojoară, C. Foiaş, Theory of Generalized Spectral Operators, Gordon and Breach Sci. Publ., Inc., 1968.
- [4] J. Conway, A Course in Functional Analysis, second ed., Grad. Texts in Math., vol. 96, Springer-Verlag, New York, 1990.
- [5] N. Dunford, Schwartz, Linear Operators, Volume III, Interscience, New York, 1971.
- [6] C. Foiaş, Spectral maximal spaces and decomposable operators in Banach space, *Arch. Math.* 14 (1963) 341–349.
- [7] C. Foiaş, I.B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators I, *J. Funct. Anal.* 253 (2) (2007) 628–646.
- [8] C. Foiaş, I.B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators II, *Indiana Univ. Math. J.* 57 (6) (2008) 2745–2760.
- [9] C. Foiaş, I.B. Jung, E. Ko, C. Pearcy, Spectral decomposability of rank-one perturbations of normal operators, *J. Math. Anal. Appl.* 375 (2) (2011) 602–609.
- [10] Q. Fang, J. Xia, Invariant subspaces for certain finite-rank perturbations of diagonal operators, *J. Funct. Anal.* 263 (5) (2012) 1356–1377.
- [11] E.A. Gallardo-Gutiérrez, F.J. González-Doña, Finite rank perturbations of normal operators: spectral subspaces and Borel series, *J. Math. Pures Appl.* (9) 162 (2022) 23–75.
- [12] E.A. Gallardo-Gutiérrez, F.J. González-Doña, Finite rank perturbations of normal operators: hyperinvariant subspaces and a problem of Pearcy, *Indiana Univ. Math. J.* 73 (4) (2024) 1551–1578.
- [13] E.A. Gallardo-Gutiérrez, F.J. González-Doña, Finite rank perturbations of normal operators: spectral idempotents and decomposability, *J. Funct. Anal.* 285 (12) (2023) 110148, 50 pp.
- [14] D. Herrero, Indecomposable compact perturbations of the bilateral shift, *Proc. Am. Math. Soc.* 62 (2) (1977) 254–258.
- [15] H. Klaja, Hyperinvariant subspaces for some compact perturbations of multiplication operators, *J. Oper. Theory* 73 (1) (2015) 127–142.
- [16] K. Laursen, M. Neumann, An Introduction to Local Spectral Theory, London Math. Soc. Monographs, 2000.
- [17] V.I. Lomonosov, On invariant subspaces of families of operators commuting with a completely continuous operator, *Funkc. Anal. Prilozh.* 7 (1973) 55–56 (in Russian).
- [18] C.M. Pearcy, Some Recent Developments in Operator Theory, Regional Conference Series in Mathematics, vol. 36, American Mathematical Society, Providence, R.I., 1978.
- [19] M. Radjabalipour, On decomposability of compact perturbations of operators, *Proc. Am. Math. Soc.* 53 (1) (1975) 159–164.
- [20] M. Radjabalipour, H. Radjavi, On decomposability of compact perturbations of normal operators, *Can. J. Math.* 27 (3) (1975) 725–735.
- [21] H. Radjavi, P. Rosenthal, Invariant Subspaces, Springer-Verlag, New York, 1973.