

Supporting Information for  
 The surface tension of bulky colloids, capillarity under gravity  
 and the microscopic origin of the KPZ equation

by  
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This document contains supporting information on the derivation of results from the main paper. To facilitate cross referencing, this materials is written as an appendix section. The equation numbering and bibliography follow the original paper, with equation labels and references not in this document referring to those of the original paper.

**PROVE OF EQ. (7)**

In order to derive Eq. (7), we first rewrite Eq. (5) in condensed notation as:

$$H[h] = \int F(\mathbf{x}; h, h_{\mathbf{x}}) d\mathbf{x} \quad (10)$$

with

$$F(\mathbf{x}; h, h_{\mathbf{x}}) = \int \left[ V(z) \rho_{\pi} \left( \frac{z-h}{\sqrt{1+h_{\mathbf{x}}^2}} \right) \right] dz + \gamma_0 \sqrt{1+h_{\mathbf{x}}^2} - \Delta p h \quad (11)$$

Here, the subindex  $\mathbf{x}$  stands for differentiation with respect to  $\mathbf{x}$ .

The functional of Eq. (10) has an extremal that is given by the Euler-Lagrange equation:

$$\frac{\delta H}{\delta h(\mathbf{x})} = \frac{\partial F}{\partial h} - \frac{d}{d\mathbf{x}} \left( \frac{\partial F}{\partial h_{\mathbf{x}}} \right) \quad (12)$$

Differentiation of Eq. (11) with help of the chain rule yields:

$$\frac{\partial F}{\partial h} = - \frac{\tilde{\Pi}(h, h_{\mathbf{x}})}{\sqrt{1+h_{\mathbf{x}}^2}} - \Delta p \quad (13)$$

and

$$\frac{\partial F}{\partial h_{\mathbf{x}}} = \frac{\Delta \tilde{\gamma}(h, h_{\mathbf{x}}) h_{\mathbf{x}}}{(1+h_{\mathbf{x}}^2)^{3/2}} + \frac{\gamma_0 h_{\mathbf{x}}}{\sqrt{1+h_{\mathbf{x}}^2}} \quad (14)$$

where:

$$\tilde{\Pi}(h, h_{\mathbf{x}}) = \int \left[ V(z) \frac{d\rho_{\pi}}{dz} \left( \frac{z-h}{\sqrt{1+h_{\mathbf{x}}^2}} \right) \right] dz \quad (15)$$

and

$$\Delta \tilde{\gamma}(h, h_{\mathbf{x}}) = - \int \left[ (z-h) V(z) \frac{d\rho_{\pi}}{dz} \left( \frac{z-h}{\sqrt{1+h_{\mathbf{x}}^2}} \right) \right] dz \quad (16)$$

Replacing these results into Eq. (13), gives the following stationarity condition for  $h(\mathbf{x})$ :

$$- \frac{\tilde{\Pi}(h, h_{\mathbf{x}})}{\sqrt{1+h_{\mathbf{x}}^2}} - \Delta p = \frac{d}{dx} \left( \frac{\gamma_0 h_{\mathbf{x}}}{\sqrt{1+h_{\mathbf{x}}^2}} + \frac{\Delta \tilde{\gamma}(h, h_{\mathbf{x}}) h_{\mathbf{x}}}{(1+h_{\mathbf{x}}^2)^{3/2}} \right) \quad (17)$$

Notice that, whereas both  $\tilde{\Pi}$  and  $\Delta\tilde{\gamma}$  stem from the external field, the former plays the role of a disjoining pressure, while the latter effectively appears as a correction to the surface tension. The explicit dependence of these functions on  $h_{\mathbf{x}}$  is a consequence of the non-locality of the free energy functional, Eq. (10) with respect to  $h(\mathbf{x})$  (i.e. the non-local dependence of  $\tilde{\Pi}$  and  $\Delta\tilde{\gamma}$  on  $h$  can be cast approximately in terms of local functions of  $h(\mathbf{x})$  and  $h_{\mathbf{x}}(\mathbf{x})$ ).

The integrals of Eq. (15) and Eq. (16) cannot be evaluated in closed form without further assumptions. However, we notice that the derivative of the density profile can be considered to leading order as a sharp symmetrical distribution centered at  $z = h$ . For external fields varying smoothly in the scale of one correlation length, as is usually the case, we can therefore expand  $V(z)$  in the integrand about  $z = h$ . To leading order in the expansion, this yields:

$$\tilde{\Pi}(h, h_{\mathbf{x}}) = -V(h) \int \left[ \frac{d\rho_{\pi}}{dh} \left( \frac{z-h}{\sqrt{1+h_{\mathbf{x}}^2}} \right) \right] dz \quad (18)$$

A simple change of variables then leads to the convenient approximation:

$$\tilde{\Pi}(h, h_{\mathbf{x}}) \approx \Pi(h) \sqrt{1+h_{\mathbf{x}}^2} \quad (19)$$

where  $\Pi(h)$  is the disjoining pressure of a planar interface.

In order to evaluate  $\Delta\tilde{\gamma}$ , we notice that, to a good approximation  $t \frac{d\rho_{\pi}}{dz}(t) = -\xi^2 \frac{d^2\rho_{\pi}}{dt^2}(t)$ , where  $t$  is an arbitrary variable, and  $\xi$  is a measure of the interfacial width.[5, 16–18] By taking this into account, we can write:

$$\Delta\tilde{\gamma}(h, h_{\mathbf{x}}) = -(1+h_{\mathbf{x}}^2) \int V(z) \frac{d}{dh} \frac{d\rho_{\pi}}{dz} \left( \frac{z-h}{\sqrt{1+h_{\mathbf{x}}^2}} \right) dz \quad (20)$$

A simple rearrangement, followed by comparison with Eq. (15), leads to the convenient result:

$$\Delta\tilde{\gamma}(h, h_{\mathbf{x}}) \approx -(1+h_{\mathbf{x}}^2) \xi^2 \frac{d\tilde{\Pi}}{dh} \quad (21)$$

Finally, replacing Eq. (19) and Eq. (21) into Eq. (17), and taking the limit of small gradients, leads to:

$$\Pi(h) + \Delta p = -\frac{d}{d\mathbf{x}} \left( \gamma(h) h_{\mathbf{x}} \right) \quad (22)$$

which is the sought result.

Eq. (22) corrects the results from a preliminary version of this article (arXiv:2302.01959). The result also shows that the linearized form of the equilibrium condition published in Ref.[18] is inaccurate. In that paper, Eq. (17) was linearized, and it was assumed that the non-local functionals  $\tilde{\Pi}(h, h_{\mathbf{x}})$  and  $\Delta\tilde{\gamma}(h, h_{\mathbf{x}})$  could be approximated by their local forms for the flat profile, i.e.  $\Pi(h)$  and  $\Delta\gamma(h)$ , respectively. This appears to be incorrect in view of the above.

### PROVE OF EQ. (8)

To obtain Eq. (8), consider the one dimensional film profile,  $h(x)$  of a cylindrical drop or liquid wedge along the  $x$  direction. For this problem, the equilibrium condition, Eq. (7) simplifies to:

$$\Pi(h) + \Delta p = -\frac{d}{dx} (\gamma(h) h_x) \quad (23)$$

where  $h_x$  denotes derivation with respect to  $x$ .

Multiplying this result by  $dh$ , the equilibrium condition may be cast as:

$$(\Pi(h) + \Delta p) dh = -h_x d(\gamma(h) h_x) \quad (24)$$

The right hand side of this equation obeys:

$$h_x d(\gamma(h) h_x) = d(\gamma(h) h_x^2) - \frac{1}{2} \gamma(h) dh_x^2 \quad (25)$$

so that one can write exactly:

$$d(\gamma(h)h_x^2) + (\Pi(h) + \Delta p) dh = \frac{1}{2}\gamma(h)dh_x^2 \quad (26)$$

This result is now integrated from  $h(x) = h_e$  at  $x \rightarrow -\infty$ , where  $h_x(x) = 0$ , to  $h(x)$  at arbitrary  $x$ , leading to:

$$\gamma(h)h_x^2 - (\omega(h) - \omega(h_e)) = \frac{1}{2} \int_{h_e}^h \gamma(h) \frac{dh_x^2}{dh} dh \quad (27)$$

where  $\omega(h) = g(h) - \Delta p h$ .

This result remains also a complex integro-differential equation, but is now amenable to an approximate solution upon successive iteration.

To see this, first solve under the assumption that  $\gamma(h)$  is a constant equal to  $\gamma_0$ . This leads right away to:

$$h_x^2 = 2 \frac{\omega(h) - \omega(h_e)}{\gamma_0} \quad (28)$$

which corresponds to the exact first integral of the Derjaguin or augmented Young-Laplace equation (c.f. Ref.[28–30]).

Now, replacing this result back into the right hand side of Eq.(27), followed by a change of variables in the integrand of the right hand side, yields:

$$\gamma(h)h_x^2 - (g(h) - g(h_e)) = \int_{h_e}^h dg + \int_{h_e}^h \frac{1}{2} \frac{\xi^2}{\gamma_0} g' dg' \quad (29)$$

where it is assumed the system is exactly at coexistence, such that  $\Delta p = 0$  and  $\omega(h) = g(h)$ .

This equation leads readily to Eq. (8) upon integration. The film profile is then obtained by numerical quadrature.

A relation between the contact angle,  $\theta$  and  $g(h_e)$  may be obtained by noticing that for the choice  $\Delta p = 0$ , the droplet has zero curvature. Therefore, it is acknowledged that as  $x \rightarrow \infty$ ,  $h(x) \rightarrow \infty$ , and  $h_x(x) \rightarrow \tan \theta$ . Applying this condition in Eq. (8), one readily finds that  $\frac{1}{2}\gamma_0 \tan^2 \theta = -g(h_e)$ . In practice, to the order of small gradients that this result applies,  $\tan(\theta) = \theta$ , so the relation simplifies to  $\theta^2 = -\frac{2g(h_e)}{\gamma_0}$ .

## MODEL INTERFACE POTENTIAL

The results of Fig.2 are obtained for a model interface potential with a short range contribution and a long range tail favoring wetting:

$$g(h) = C_2 e^{-2\kappa h} - C_1 e^{-\kappa h} - \frac{A}{12\pi h^2} \quad (30)$$

where  $\kappa$  is the inverse correlation length,  $C_i$  are positive constants, and  $A$  is the Hamaker constant. In the explicit calculations, these parameters are set to  $C_2/\gamma_0 = 1$ ,  $C_1/\gamma_0 = 24$ ,  $\frac{A\kappa^2}{12\pi\gamma_0} = -12$ . This leads to a minimum at  $\kappa h_e \approx 1.88$ , with  $g(h_e)/\gamma_0 = -0.24$ , and a contact angle of  $\theta \approx 40$  degrees. For the calculation of  $\Delta\gamma(h)$ , a value of the interfacial width of  $\xi = \sqrt{3}\kappa^{-1}$  is assumed (based on comparison of Eq. (2) with results for an exact model.[17]).

## PROVE OF EQ. (9)

From the proof of Eq. (7), the first functional derivative of Eq. (5) is:

$$\frac{\delta H}{\delta h(\mathbf{x})} = -\Pi(h) - \Delta p - \frac{d}{dx} \left( \frac{\gamma(h)h_x}{\sqrt{1+h_x^2}} \right) \quad (31)$$

To quadratic order in  $h_x$ , this leads to:

$$\frac{\delta H}{\delta h(\mathbf{x})} = -\Pi(h) - \Delta p - \gamma(h)h_{xx} - \gamma'(h)h_x^2 \quad (32)$$

Replacing this result in the equation for non-conserved dynamics under the assumption of phase coexistence ( $\Delta p = 0$ ):

$$\frac{\partial h}{\partial t} = -\frac{\delta H}{\delta h(\mathbf{x})} \quad (33)$$

leads right away to Eq. (9).