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Additional symmetries and solutions of the dispersionless KP hierarchy

Luis Martínez Alonso[a] and Manuel Mañas[b]
Departamento de Física Teórica II, Universidad Complutense, E28040 Madrid, Spain

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The dispersionless KP hierarchy is considered from the point of view of the twistor formalism. A set of explicit additional symmetries is characterized and its action on the solutions of the twistor equations is studied. A method for dealing with the twistor equations by taking advantage of hodograph type equations is proposed. This method is applied for determining the orbits of solutions satisfying reduction constraints of Gelfand–Dikii type under the action of additional symmetries. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1587873]

I. INTRODUCTION

The so-called dispersionless hierarchies[1–9] provide an interesting type of nonlinear integrable model which cannot be studied by the standard schemes of the KP theory and require an entirely new approach. From the point of view of the Lax formalism, dispersionless hierarchies arise as the quasiclassical limits of Lax pair equations performed by replacing operators by phase space functions and commutators by Poisson brackets. In this way, when dealing with dispersionless hierarchies, instead of the associated auxiliary linear system of the standard formalism of integrable systems the underlying equations to be solved are of Hamilton–Jacobi type.

Several methods of solution of dispersionless hierarchies have been formulated. In Refs. 3 and 4 (see also Refs. 11 and 12) Kodama and Gibbons gave a direct method based on the use of reductions in which the dependent variables depend on a finite number of unknown functions. The corresponding reduced hierarchy becomes an infinite set of compatible hydrodynamic systems solvable by hodographic techniques. A $\overline{\partial}$ scheme has been proposed by Konopelchenko *et al.* in Refs. 13–15, which introduces an associated $\overline{\partial}$ equation of Hamilton–Jacobi type. In this article we deal with the twistorial method of Takasaki and Takebe,[9,10] Two important advantages of this method are the following.

(1) It provides a convenient scheme for describing the symmetries.
(2) All local solutions can be determined by means of the twistor method.

The main aim of this article is to present a technique for deriving explicit examples of both additional symmetries and solutions of dispersionless hierarchies within the framework of the twistor formalism. It requires a new formulation of the twistor equations which involves a certain type of generating function for canonical transformations of twistor data as well as the use of hodograph equations. To show our strategy, we concentrate on the dispersionless KP (dKP) hierarchy, which is the prototype of this kind of integrable hierarchy. Its Lax pair formulation involves a phase space with a canonical pair of coordinates (p,x) and an associated Poisson bracket

$$\{F,G\}=\frac{\partial F}{\partial p}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial p}.$$

[a]Electronic mail: luism@fis.ucm.es
[b]Electronic mail: manuel@darboux.fis.ucm.es

It is useful to introduce an enlarged Lax formalism with a pair of canonically conjugate variables $\mathcal{L}=\mathcal{L}(p,\mathbf{t})$ and $\mathcal{M}=\mathcal{M}(p,\mathbf{t})$ (i.e. $\{\mathcal{L},\mathcal{M}\}=1$) depending on p and an infinite set of time parameters

$$\mathbf{t}:=(t_1=x,t_2,\ldots,t_n,\ldots),$$

which are assumed to admit expansions of the form

$$\mathcal{L}=p+\sum_{n\geqslant 1}\frac{u_n(\mathbf{t})}{p^n},\quad \mathcal{M}=\sum_{n\geqslant 2} nt_n\mathcal{L}^{n-1}+x+\sum_{n\geqslant 1}\frac{v_n(\mathbf{t})}{\mathcal{L}^{n+1}}, \tag{1}$$

as $p\to\infty$ and $\mathcal{L}\to\infty$, respectively. The Lax equations of the dKP hierarchy are

$$\frac{\partial\mathcal{L}}{\partial t_n}=\{\mathcal{B}_n,\mathcal{L}\},\quad \frac{\partial\mathcal{M}}{\partial t_n}=\{\mathcal{B}_n,\mathcal{M}\},\quad n\geqslant 2, \tag{2}$$

where

$$\mathcal{B}_n:=(\mathcal{L}^n)_{\geqslant 0}.$$

Here $(\mathcal{F})_{\geqslant 0}$ denotes the projection of a Laurent series $\mathcal{F}$ in the variable p on the subspace generated by the non-negative powers of p [we will also use the notation $(\mathcal{F})_{\leqslant -1}:=\mathcal{F}-(\mathcal{F})_{\geqslant 0}$]. The system of compatibility equations

$$\frac{\partial\mathcal{B}_m}{\partial t_n}-\frac{\partial\mathcal{B}_n}{\partial t_m}+\{\mathcal{B}_m,\mathcal{B}_n\}=0,\quad m\neq n, \tag{3}$$

yields an infinite set of nonlinear equations for the coefficients u_n of the expansion (1) of $\mathcal{L}$. In particular for $(n,m)=(2,3)$ one gets the dKP equation (Zabolotskaya–Khokhlov equation)

$$(u_t-3uu_x)_x=\tfrac{3}{4}u_{yy},\quad u:=u_1,t:=t_3,y:=t_2. \tag{4}$$

This is an interesting nonlinear model with applications, in the study of quasi-plane sound beams,[17] quasi-transonic flows past thin wings[18] or Einstein-Weyl spaces.[19]

In the next section we first describe in brief the twistor approach to the solutions and symmetries of the dKP hierarchy. Then we present a class of additional symmetries depending on arbitrary functions of one variable, the action of which can be explicitly determined. As a particular case they include the symmetries of the dKP equation found by Dunajski, Mason and Tod in Ref. 19. The first part of Sec. III is devoted to a new formulation of twistor equations which is appropriate for dealing with the transformation laws of solutions under the action of symmetries. In the second part of Sec. III we show how solutions of the dKP hierarchy satisfying reduction constraints of Gelfand–Dikii type transform under the class of additional symmetries introduced in Sec. II. Finally, some explicit examples are worked out.

II. SYMMETRIES IN THE TWISTOR FORMALISM

A. Twistorial structure of the dKP hierarchy

The twistor formalism of the dKP hierarchy is based on the degenerate symplectic form[9]

$$\omega:=\mathrm{d}p\wedge\mathrm{d}x+\sum_{n\geqslant 2}\mathrm{d}\mathcal{B}_n\wedge\mathrm{d}t_n, \tag{5}$$

which plays the role of the Gindikin bundle[16] of curved twistor theory. The form ω encodes both the Lax equations and their compatibility conditions into the simple system

$$\omega = \mathrm{d}\mathcal{L} \wedge \mathrm{d}\mathcal{M}, \qquad \omega \wedge \omega = 0. \tag{6}$$

From the first equation we have that

$$\mathrm{d}\left(\mathcal{M}\mathrm{d}\mathcal{L} + p\,\mathrm{d}x + \sum_{n\geqslant 2} \mathcal{B}_n \mathrm{d}t_n \right) = 0,$$

so that there exists a generating function $S = S(\mathcal{L}, \mathbf{t})$ for the canonical transformation $(p,x) \mapsto (\mathcal{L}, \mathcal{M})$ satisfying

$$\mathrm{d}S = \mathcal{M}\mathrm{d}\mathcal{L} + p\,\mathrm{d}x + \sum_{n\geqslant 2} \mathcal{B}_n \mathrm{d}t_n,$$

or equivalently

$$\mathcal{M} = \frac{\partial S}{\partial \mathcal{L}}, \quad p = \frac{\partial S}{\partial x}, \quad \mathcal{B}_n = \frac{\partial S}{\partial t_n}, \quad n \geqslant 2. \tag{7}$$

Notice that from (1) and the first equation of (7) it follows that S can be defined as

$$S(\mathcal{L}, \mathbf{t}) = \sum_{n\geqslant 1} t_n \mathcal{L}^n - \sum_{n\geqslant 1} \frac{v_n(\mathbf{t})}{n} \mathcal{L}^{-n}.$$

The twistor scheme for solving the dKP hierarchy is based on the following result.[9]

Theorem 1: *Let (P(p,x),X(p,x)) be a pair of canonically conjugate variables (i.e. $\{P,X\}=1$). Then we have the following:*

(1) Given two functions $(\mathcal{L}(p,\mathbf{t}), \mathcal{M}(p,\mathbf{t}))$ of the form (1) such that the composite functions $(P(\mathcal{L},\mathcal{B}), X(\mathcal{L},\mathcal{B}))$ have Laurent series expansions in p satisfying the twistor equations

$$(P(\mathcal{L},\mathcal{M}))_{\leqslant -1} = 0, \quad (X(\mathcal{L},\mathcal{M}))_{\leqslant -1} = 0, \tag{8}$$

then $(\mathcal{L},\mathcal{M})$ gives a solution of the dKP hierarchy (2). The pair

$$(P(p,x), X(p,x))$$

is called the twistor data *of the solution $(\mathcal{L},\mathcal{M})$.*

(2) Each solution of the dKP hierarchy admits a set $(P(p,x), X(p,x))$ of twistor data.

In general, we cannot assume the existence of appropriate solutions $(\mathcal{L},\mathcal{M})$ of (8). For example, the canonical variables

$$P := p^2 x, \quad X := \frac{1}{p}, \tag{9}$$

determine the twistor equations

$$(\mathcal{L}^2 \mathcal{M})_{\leqslant -1} = 0, \quad \left(\frac{1}{\mathcal{L}} \right)_{\leqslant -1} = 0,$$

which obviously have no solutions satisfying (1).

B. Symmetry transformations

One of the main features of the twistor equations is that the symmetry properties of the dKP hierarchy can be formulated in a convenient way.[9] Indeed, the natural group acting on the set of twistor data $(P(p,x),X(p,x))$ is the group of canonical transformations generated by one-parameter groups of the form

$$\begin{aligned} &\exp(s\{F,\cdot\}):(P,X)\mapsto(P(s),X(s)), \quad F=F(p,x),\\ &P(s):=P(\exp(s\{F,\cdot\})p,\exp(s\{F,\cdot\})x),\\ &X(s):=X(\exp(s\{F,\cdot\})p,\exp(s\{F,\cdot\})x), \end{aligned} \tag{10}$$

where

$$\exp(s\{F,\cdot\})G:=G+s\{F,G\}+\frac{s^2}{2}\{F,\{F,G\}\}+\cdots.$$

The following theorem can be proved.[9]

Theorem 2: *A one-parameter group of canonical transformations (10) induces an action* $(\mathcal{L},\mathcal{M})\mapsto(\mathcal{L}(s),\mathcal{M}(s))$ *on the set of solutions of the dKP hierarchy determined by the flow*

$$\frac{\partial\mathcal{L}}{\partial s}=\{\mathcal{L},F(\mathcal{L},\mathcal{M})_{\leqslant -1}\}, \quad \frac{\partial\mathcal{M}}{\partial s}=\{\mathcal{M},F(\mathcal{L},\mathcal{M})_{\leqslant -1}\}. \tag{11}$$

Let us consider symmetries of the dKP hierarchy generated by double series of the form

$$F(\mathcal{L},\mathcal{M})=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}c_{ij}\mathcal{L}^i\mathcal{M}^j. \tag{12}$$

We will concentrate on the $(r+1)$th *truncated dKP hierarchies* defined as the sets of the first $r+1$ flows of the dKP hierarchy $(r\geqslant 2)$. Thus in order to analyze their symmetries we may set $t_n=0,\forall n\geqslant r+1$, and so we may write

$$\mathcal{M}=(r+1)t_{r+1}\mathcal{L}^r+rt_r\mathcal{L}^{r-1}+\cdots+x+\mathcal{O}\left(\frac{1}{\mathcal{L}^2}\right). \tag{13}$$

By substituting this expansion in (12), a series expansion of F in powers of $\mathcal{L}$ is obtained. Let us now investigate those symmetries of the $(r+1)$th truncated dKP hierarchy which do not involve the action of higher dKP flows. To this end, we have to avoid terms of the form $\{(\mathcal{L}^n)_{\geqslant 0},\mathcal{L}\}$ with $n>(r+1)$ on the right-hand side of the first equation in (11). Hence we impose $c_{ij}=0$ for $(i+jr)>(r+1)$, so that F can be expressed in the form

$$F(\mathcal{L},\mathcal{M})=\sum_{n\leqslant r+1}\alpha_n\left(\frac{\mathcal{M}}{(r+1)\mathcal{L}^r}\right)\mathcal{L}^n, \tag{14}$$

with $\alpha_n(t)$ being arbitrary smooth functions. Furthermore, Eq. (11) for $\mathcal{L}$ can be written as

$$\frac{\partial\mathcal{L}}{\partial s}=\frac{\partial F}{\partial\mathcal{M}}+\{F(\mathcal{L},\mathcal{M})_{\geqslant 0},\mathcal{L}\}, \tag{15}$$

and it is easy to see that only those terms in (14) with $n\geqslant 1$ contribute to $\partial u/\partial s$.

Therefore, we conclude that the symmetries of the $(r+1)$th truncated dKP hierarchy which do not involve higher dKP flows and define a nontrivial action on the coefficient u are of the form

$$F(\mathcal{L},\mathcal{M})=\sum_{n=1}^{r+1}\alpha_n\left(\frac{\mathcal{M}}{(r+1)\mathcal{L}^r}\right)\mathcal{L}^n. \tag{16}$$

This means that, under these conditions, there are essentially $r+1$ types of symmetry generators of the $(r+1)$th truncated dKP hierarchy given by

$$F_i(\mathcal{L},\mathcal{M}):=\alpha\left(\frac{\mathcal{M}}{(r+1)\mathcal{L}^r}\right)\mathcal{L}^i,\quad i=1,\ldots,r+1, \tag{17}$$

with $\alpha=\alpha(t)$ being an arbitrary function.

The action of the one-parameter groups generated by F_i on the coefficient u can be explicitly found. Indeed, by identifying the coefficients of $1/p$ in both members of (15) one gets a first-order *linear* partial differential equation for

$$u(s,\mathbf{t}):=\exp(s\{F_i,\cdot\})u(\mathbf{t}),$$

the integration of which provides the symmetry transformation

$$u=u(\mathbf{t})\mapsto\tilde{u}=u(s,\mathbf{t}).$$

Let us illustrate these facts by considering the case $r=2$. We observe that (13) implies that near points t in the region of analyticity of α

$$\alpha\left(\frac{\mathcal{M}}{3\mathcal{L}^2}\right)=\alpha(t)+\frac{2}{3}y\alpha'(t)\frac{1}{\mathcal{L}}+\left(\frac{1}{3}x\alpha'(t)+\frac{2}{9}y^2\alpha''(t)\right)\frac{1}{\mathcal{L}^2}+\mathcal{O}\left(\frac{1}{\mathcal{L}^3}\right). \tag{18}$$

One finds the following results for the corresponding three generators (17):

(1) F_1

From (15) we have

$$\frac{\partial\mathcal{L}}{\partial s}=\alpha'\left(\frac{\mathcal{M}}{3\mathcal{L}^2}\right)\frac{1}{3\mathcal{L}}+\alpha(t)\frac{\partial\mathcal{L}}{\partial x},$$

so that

$$\frac{\partial u}{\partial s}=\alpha(t)\frac{\partial u}{\partial x}+\frac{1}{3}\alpha'(t). \tag{19}$$

The solution of this equation is

$$u=U(x+s\alpha(t),y,t)+\tfrac{1}{3}s\alpha'(t),$$

where U is an arbitrary function. It leads to the symmetry

$$\tilde{u}=u(x+s\alpha(t),y,t)+\tfrac{1}{3}s\alpha'(t). \tag{20}$$

(2) F_2

In this case (15) becomes

$$\frac{\partial\mathcal{L}}{\partial s}=\frac{1}{3}\alpha'\left(\frac{\mathcal{M}}{3\mathcal{L}^2}\right)+\frac{2}{3}y\alpha'(t)\frac{\partial\mathcal{L}}{\partial x}+\alpha(t)\frac{\partial\mathcal{L}}{\partial y},$$

and the equation for u is

$$\frac{\partial u}{\partial s}=\frac{2}{3}y\alpha'(t)\frac{\partial u}{\partial x}+\alpha(t)\frac{\partial u}{\partial y}+\frac{2}{9}y\alpha''(t), \tag{21}$$

which has the solution

$$u=U(x+\tfrac{2}{3}sy\alpha'(t)+\tfrac{1}{3}s^2\alpha(t)\alpha'(t),y+s\alpha(t),t)+\tfrac{2}{9}sy\alpha''(t)+\tfrac{1}{9}s^2\alpha(t)\alpha''(t),$$

where U is an arbitrary function. The corresponding symmetry transformation of the dKP equation is

$$\tilde{u}=u(x+\tfrac{2}{3}sy\alpha'(t)+\tfrac{1}{3}s^2\alpha(t)\alpha'(t),y+s\alpha(t),t)+\tfrac{2}{9}sy\alpha''(t)+\tfrac{1}{9}s^2\alpha(t)\alpha''(t). \tag{22}$$

(3) F_3

Now Eq. (15) takes the form

$$\frac{\partial\mathcal{L}}{\partial s}=\frac{1}{3}\alpha'\left(\frac{\mathcal{M}}{3\mathcal{L}^2}\right)\mathcal{L}+\left(\frac{1}{3}x\alpha'(t)+\frac{2}{9}y^2\alpha''(t)\right)\frac{\partial\mathcal{L}}{\partial x}+\frac{2}{3}y\alpha'(t)\frac{\partial\mathcal{L}}{\partial y}+\alpha(t)\frac{\partial\mathcal{L}}{\partial t},$$

which implies

$$\frac{\partial u}{\partial s}=\left(\frac{1}{3}x\alpha'(t)+\frac{2}{9}y^2\alpha''(t)\right)\frac{\partial u}{\partial x}+\frac{2}{3}y\alpha'(t)\frac{\partial u}{\partial y}+\alpha(t)\frac{\partial u}{\partial t}+\frac{1}{3}\alpha'(t)u+\frac{1}{9}x\alpha''(t)+\frac{2}{27}y^2\alpha'''(t). \tag{23}$$

The solution of this equation is

$$u=(c'(t))^{2/3}U\left(x(c'(t))^{1/3}+\frac{2}{9}y^2\frac{c''(t)}{(c'(t))^{2/3}},y(c'(t))^{2/3},s+c(t)\right)$$
$$+\frac{1}{9}x\frac{c''(t)}{c'(t)}+\frac{2}{27}y^2\left(\frac{c'''(t)}{c'(t)}-\frac{4}{3}\left(\frac{c''(t)}{c'(t)}\right)^2\right),$$

where U is an arbitrary function and $c(t)$ is such that $c'(t)=1/\alpha(t)$. Hence, by defining $T:=T(s,t)$ through the implicit relation

$$c(T)=s+c(t),$$

and by taking into account that

$$T':=\frac{\partial T}{\partial t}=\frac{c'(t)}{c'(T)},$$

one finds that the symmetry transformation determined by (23) is

$$\tilde{u}=(T')^{2/3}u\left(x(T')^{1/3}+\frac{2}{9}y^2\frac{T''}{(T')^{2/3}},y(T')^{2/3},T\right)+\frac{1}{9}x\frac{T''}{T'}+\frac{2}{27}y^2\left(\frac{T'''}{T'}-\frac{4}{3}\left(\frac{T''}{T'}\right)^2\right). \tag{24}$$

The three symmetries (20), (22), and (24) coincide with the symmetries of the dKP equation found by Dunajski, Mason, and Tod.[19] by analyzing equivalence transformations of Einstein–Weyl spaces.

1. Transformation law of twistor data

According to (10) the dKP symmetry generated by (17) corresponds to a canonical transformation law of the twistor data determined by the Hamiltonian system

$$\frac{\mathrm{d}p}{\mathrm{d}s}=\{\alpha(\rho)p^i,p\},\quad \frac{\mathrm{d}x}{\mathrm{d}s}=\{\alpha(\rho)p^i,x\},\tag{25}$$

where we are denoting

$$\rho:=\frac{x}{(r+1)p^r}.$$

In terms of (p,ρ) this system becomes

$$\frac{\mathrm{d}p}{\mathrm{d}s}=-\frac{\alpha'(\rho)}{r+1}p^{i-r},\quad \frac{\mathrm{d}\rho}{\mathrm{d}s}=i\frac{\alpha(\rho)}{r+1}p^{i-r-1},\tag{26}$$

and by taking into account that the Hamiltonian function

$$h:=\alpha(\rho)p^i$$

is a constant of the motion it follows that the solution of (25) can be written as

$$p(s)=\frac{p}{(j_\rho)^{1/(r+1)}},\quad x(s)=(r+1)j\,p(s)^r.\tag{27}$$

Here $j=j(s,\rho,h)$ is the evolution law of the variable ρ. That is to say, it is the solution of the initial value problem

$$\frac{\partial j}{\partial s}=\beta(\rho,h),\quad j(0,\rho,h)=\rho,\tag{28}$$

where

$$\beta(\rho,h):=\frac{i}{r+1}\left(\frac{\alpha(\rho)}{h}\right)^{(r+1)/i}h.$$

The expressions (27) define the action of the additional symmetries (17) on the twistor data. It is important to observe that the solution of (28) satisfies

$$s=\int_\rho^{j(s,\rho,h)}\frac{\mathrm{d}\rho}{\beta(\rho,h)},$$

and, as a consequence, one deduces that the first-order derivatives of j with respect to ρ and h are

$$\begin{aligned} j_\rho&=\left(\frac{\alpha(j)}{\alpha(\rho)}\right)^{(r+1)/i},\\ j_h&=s\left(\frac{i}{r+1}-1\right)\left(\frac{\alpha(j)}{h}\right)^{(r+1)/i}=\left(\frac{i}{r+1}-1\right)\frac{s}{p(s)^{r+1}}. \end{aligned}\tag{29}$$

As we will see below, these relations will be useful for determining the action of the additional symmetries on the solutions of the twistor equations.

III. SOLUTIONS OF THE dKP HIERARCHY

A. Generating functions and hodograph equations

We are going to present a scheme for solving twistor equations which is particularly suitable to investigate the action of the additional symmetries introduced in the above section. An ingre-

dient of our analysis is the use of a type of generating functions for canonical transformations of twistor data,[20] which allows us to introduce hodograph type equations to formulate part of the constraints imposed by the twistor equations.

Let $(P(p,x),X(p,x))$ be a pair of canonically conjugate variables. Then for each positive integer r we have

$$\mathrm{d}P\wedge\mathrm{d}X=\mathrm{d}p\wedge\mathrm{d}x=\mathrm{d}(p^{r+1})\wedge\mathrm{d}\rho,\quad \rho:=\frac{x}{(r+1)p^r}.$$

Hence there exists an associated generating function $J_r:=J_r(P,\rho)$ of the canonical transformation $(p,x)\mapsto(P,X)$ such that

$$\mathrm{d}J_r=p^{r+1}\mathrm{d}\rho+X\mathrm{d}P,$$

or equivalently

$$p^{r+1}=\frac{\partial J_r(P,\rho)}{\partial\rho},\quad X=\frac{\partial J_r(P,\rho)}{\partial P}. \tag{30}$$

In this way by denoting

$$\mathcal{M}_r:=\frac{\mathcal{M}}{(r+1)\mathcal{L}^r},$$

we deduce

$$\frac{\partial}{\partial p}J_r(P(\mathcal{L},\mathcal{M}),\mathcal{M}_r)=\frac{\partial J_r}{\partial P}(P(\mathcal{L},\mathcal{M}),\mathcal{M}_r)\frac{\partial P(\mathcal{L},\mathcal{M})}{\partial p}+\frac{\partial J_r}{\partial\rho}(P(\mathcal{L},\mathcal{M}),\mathcal{M}_r)\frac{\partial\mathcal{M}_r}{\partial p}$$

$$=X(\mathcal{L},\mathcal{M})\frac{\partial P(\mathcal{L},\mathcal{M})}{\partial p}+\mathcal{L}^{r+1}\frac{\partial\mathcal{M}_r}{\partial p},$$

and by taking into account that

$$\mathcal{L}^{r+1}\frac{\partial\mathcal{M}_r}{\partial p}=\frac{1}{r+1}\frac{\partial(\mathcal{L}\mathcal{M})}{\partial p}-\frac{\partial S}{\partial p},$$

where S is the function introduced in (7), we deduce that

$$X(\mathcal{L},\mathcal{M})=\frac{(\partial/\partial p)\,(S+J_r(P(\mathcal{L},\mathcal{M}),\mathcal{M}_r)-[1/(r+1)]\mathcal{L}\mathcal{M})}{(\partial/\partial p)\,P(\mathcal{L},\mathcal{M})}. \tag{31}$$

This formula enables us to state the following:

Theorem 3: *In terms of the function*

$$\mathcal{S}_r:=S+J_r(P(\mathcal{L},\mathcal{M}),\mathcal{M}_r)-\frac{1}{r+1}\mathcal{L}\mathcal{M}, \tag{32}$$

the second twistor equation $(X(\mathcal{L},\mathcal{M}))_{\leqslant -1}=0$ *is equivalent to the following two conditions*

(1) The expansion of $\mathcal{S}_r$ *in powers of* p *satisfies*

$$(\mathcal{S}_r)_{\leqslant -1}=0. \tag{33}$$

(2) At each zero p_i *of* $\partial P(\mathcal{L},\mathcal{M})/\partial p$ *it is verified that*

$$\frac{\partial \mathcal{S}_r}{\partial p}(p_i,\mathbf{t})=0. \tag{34}$$

Henceforth we will refer to (34) as the *hodograph equations*.

A natural problem is to determine generating functions $J_r(P,\rho)$ leading to solvable twistor equations. In this sense, an important class arises when $P=P(p,x)$ is independent of x and has a finite-order expansion as $p\to\infty$:

$$P(p)=\sum_{n=-\infty}^{N} a_n p^n.$$

The corresponding generating function J_0 is of the form

$$J_0(P,x)=f(P)+g(P)x,$$

where $g(P)$ is the inverse function of $P=P(p)$. As a consequence

$$J_0(P(\mathcal{L},\mathcal{M}),\mathcal{M})=f(P(\mathcal{L}))+\mathcal{L}\mathcal{M},$$

$$\mathcal{S}_0=S+f(P(\mathcal{L})).$$

It can be shown that, provided $f(P(p))$ admits a Laurent expansion as $p\to\infty$, the twistor equations determined by J_0 have a solution. Moreover, it turns out that solving the hodograph equations for $\mathcal{S}_0$ is enough for computing $\mathcal{L}$. Let us illustrate these facts with the following important example

Gelfand–Dikii reductions: If we set

$$J_0(P,x)=f(P^{1/m})+P^{1/m}x, \quad f(P^{1/m}):=\sum_{n=-\infty}^{\infty} c_n P^{n/m}, \tag{35}$$

for a given integer $m>1$, the associated twistor data are

$$P=p^m, \quad X=\frac{1}{mp^{m-1}}(f'(p)+x). \tag{36}$$

Then, the first twistor equation is

$$\mathcal{L}^m=(\mathcal{L})_{\geqslant 0},$$

so that

$$\mathcal{L}^m=p^m+q_{m-2}(\mathbf{u})p^{m-2}+\cdots+q_1(\mathbf{u})p+q_0(\mathbf{u}), \tag{37}$$

where the functions $q_i(\mathbf{u})$ depend on the $(m-1)$ first coefficients $\mathbf{u}:=(u_1,\ldots,u_{m-1})$ of the expansion (1) of $\mathcal{L}$. This constraint defines the mth Gelfand–Dikii reduction of the dKP hierarchy.

For example the first few reductions are

$$m=2, \quad \mathcal{L}^2=p^2+2u_1,$$

$$m=3, \quad \mathcal{L}^3=p^3+3u_1p+3u_2,$$

$$m=4, \quad \mathcal{L}^4=p^4+4u_1p^2+4u_2p+6u_1^2+4u_3.$$

To determine $\mathcal{L}$ we must find the $(m-1)$ unknowns u_i as functions of $\mathbf{t}$ through the second twistor equation. Thus, according to Theorem 2, we impose

$$\mathcal{S}_0 = S + f(\mathcal{L}) = (S + f(\mathcal{L}))_{\geqslant 0} = \sum_{n\geqslant 1} (t_n + c_n)(\mathcal{L}^n)_{n\geqslant 0} + c_0 .$$

Hence, by using (37) we can express $\mathcal{S}_0$ as a function of $(p,\mathbf{t},\mathbf{u})$. If we now impose the hodograph equations (34), we get $(m-1)$ implicit equations

$$\left(\sum_{n\geqslant 1} (t_n + c_n) \frac{\partial}{\partial p} (\mathcal{L}^n)_{n\geqslant 0} \right) \Bigg|_{p=p_i(\mathbf{u})} = 0, \quad i = 1,\ldots,m-1, \tag{38}$$

which determine the functions $u_i(\mathbf{t})$ and, consequently, $\mathcal{L}$. Furthermore, by eliminating p in (37) we can express p as a function $p = p(\mathcal{L},\mathbf{t})$, which under substitution into

$$S = \sum_{n\geqslant 1} t_n \mathcal{L}^n - \left(\sum_{n\geqslant 1} t_n \mathcal{L}^n - f(\mathcal{L}) \right)_{\leqslant -1},$$

leads to $\mathcal{M} = \partial S/\partial \mathcal{L}$. Thus, it is easy to see that the functions $\mathcal{L}$ and $\mathcal{M}$ are solutions of the twistor equations which satisfy (1) and, therefore, they solve the dKP hierarchy Henceforth these solutions will be called *Gelfand–Dikii solutions* of the dKP hierarchy.

For instance, if $m = 2$ (dKdV reduction),

$$\mathcal{L}^2 = p^2 + 2u, \quad u := u_1,$$

and we set $t_n = 0$, $\forall n > 3$, one gets the hodograph relation

$$3ut + x = F(u), \tag{39}$$

which solves the dKdV equation $u_t = 3uu_x$. Here

$$F(u) := - = \frac{\partial}{\partial p} \sum_{n\geqslant 1} c_n \cdot (\mathcal{L}^n)_{n\geqslant 0}|_{p=0}$$

can be assumed to be an arbitrary smooth function of u. Some elementary solutions provided by (39) are

$$F(u) = cu, \quad u = -\frac{x}{3t-c},$$

$$F(u) = cu^2, \quad u = \frac{1}{2c}(3t + \sqrt{9t^2 + 4cx}), \tag{40}$$

$$F(u) = cu^3, \quad u = \frac{f}{2c} + \frac{2t}{f}, \quad f := \left(4x + 4c^2 \sqrt{x^2 - \frac{4t^3}{c}} \right)^{1/3}.$$

B. The action of additional symmetries on Gelfand–Dikii solutions

Our aim now is to characterize solutions of the dKP hierarchy by applying the symmetry transformations (17) to Gelfand–Dikii solutions. Obviously we may start from solutions of the hodograph equations (38) and then perform the corresponding symmetry transformation. However, in order to do it we need to know how the coefficients u_i of the expansion (1) of $\mathcal{L}$ transform under the symmetries (17), which requires us to solve a system of first-order linear partial differential equations. We are trying instead an alternative way consisting in determining the generating

functions $J_r(P,\rho)$ for the transformed twistor data and then solving the corresponding twistor equations according to the scheme of Theorem 3. In this alternative procedure the problem reduces to solving a system of implicit algebraic equations.

The dKP symmetry generated by (17) acts on twistor data according to the canonical transformation (27). In particular, the twistor data (36) for the Gelfand–Dikii reductions transform as

$$P(s)=\left(\frac{p}{(j_\rho)^{1/(r+1)}}\right)^m,$$
$$X(s)=\frac{P^{(m-1)/m}}{m}(f'(P^{1/m})+(r+1)j\,P^{r/m}). \tag{41}$$

Hence, by taking into account that j is a function of (s,ρ,h), it follows that

$$p^{r+1}=j_\rho P^{(r+1)/m}=\frac{\partial}{\partial\rho}(jP^{(r+1)/m})-\hat{h}_\rho j_h P^{(r+1)/m},$$

$$X=\frac{\partial}{\partial P}(f(P)^{1/m}+jP^{(r+1)/m})-\hat{h}_P j_h P^{(r+1)/m},$$

where

$$\hat{h}=\hat{h}(P,\rho):=h(p(P,\rho),\rho)=\alpha(\rho)p(P,\rho)^i.$$

By using now (29) we deduce

$$p^{r+1}=\frac{\partial J_r^{(i)}(P,\rho)}{\partial\rho},\quad X=\frac{\partial J_r^{(i)}(P,\rho)}{\partial P}, \tag{42}$$

where

$$J_r^{(i)}(s,P,\rho):=f(P^{1/m})+j(s,\rho,\hat{h})P^{(r+1)/m}+s\left(1-\frac{i}{r+1}\right)\hat{h}(P,\rho). \tag{43}$$

Wide families of solutions of the $(r+1)$th truncated dKP can be found by solving the twistor equations associated with the generating functions (43). The calculations are simple but long and require computer aid. To illustrate the strategy for computing these solutions let us consider the family of generating functions $J_r^{(i)}$ with

$$i=r+1\geqslant m. \tag{44}$$

The choice $i=r+1$ means that we are dealing with the orbits of Gelfand–Dikii solutions under the action of the symmetry generator

$$F_{r+1}(\mathcal{L},\mathcal{M}):=\alpha\left(\frac{\mathcal{M}}{(r+1)\mathcal{L}^r}\right)\mathcal{L}^{r+1}. \tag{45}$$

Thus, according to (29) the function j in (43) is determined from α through the solution of the initial value problem

$$\frac{\partial j}{\partial s}=\alpha(\rho),\quad j(0,\rho)=\rho. \tag{46}$$

Hence j is independent of h and by setting s to be a constant, we may take j as a function of ρ only. Therefore, the generating functions $J_r^{(i)}$ that we are considering are

$$J_r(P,\rho)=f\left(P^{1/m}\right)+j(\rho)\,P^{(r+1)/m}. \tag{47}$$

Notice that

$$P=\frac{p^m}{(j_\rho)^{m/(r+1)}}, \tag{48}$$

so that the first twistor equation reads

$$\mathbb{L}^m=(\mathbb{L}^m)_{\geqslant 0}, \tag{49}$$

where

$$\mathbb{L}:=\frac{\mathcal{L}}{j_\rho(\mathcal{M}_r)}, \quad \mathcal{M}_r:=\frac{\mathcal{M}}{(r+1)\mathcal{L}^r}. \tag{50}$$

From (1) one deduces at once that the integer powers of $\mathcal{L}$ have expansions of the form

$$\begin{aligned}\mathcal{L}^N&=p^N+\cdots+a_n(u_1,\ldots,u_{N-n-1})p^n+\cdots+b_n(u_1,\ldots,u_{N+n-1})\frac{1}{p^n}+\cdots,\\ \frac{1}{\mathcal{L}^N}&=\frac{1}{p^N}+\cdots+c_n(u_1,\ldots,u_{n-N-1})\frac{1}{p^n}+\cdots.\end{aligned} \tag{51}$$

Furthermore, (1) implies that for any smooth function $g=g(t)$ the composite function $g(\mathcal{M}_r)$ can be expanded in the form

$$\begin{aligned}g(\mathcal{M}_r)&=g\left(t_{r+1}+\frac{rt_r}{r+1}\frac{1}{\mathcal{L}}+\cdots+\frac{v_n(\mathbf{t})}{r+1}\frac{1}{\mathcal{L}^n}+\cdots\right)\\ &=g(t_{r+1})+\frac{rt_r}{r+1}g'(t_{r+1})\frac{1}{p}+\cdots+d_n(\mathbf{t},u_1,\ldots,u_{n-2},v_1,\ldots,v_{n-r-1})\frac{1}{p^n}+\cdots.\end{aligned} \tag{52}$$

Thus, from (51) and (52) and by taking into account (44), we deduce that $\mathbb{L}$ is of the form

$$\mathbb{L}=(q_m(\mathbf{t},\mathbf{u})p^m+\cdots+q_1(\mathbf{t},\mathbf{u})p+q_0(\mathbf{t},\mathbf{u}))^{1/m}, \tag{53}$$

where $\mathbf{u}:=(u_1,\ldots,u_{m-1})$.

Two different cases arise.

(1) $\mathbf{r}=\mathbf{m}-1,\mathbf{m}$. This is the simplest situation since from (51)–(53) it follows at once that

$$\mathcal{S}_r=\left(\sum_{s=1}^{r}\frac{r-s+1}{r+1}t_s\mathcal{L}^s+\gamma\mathbb{L}^{m+n}+j(\mathcal{M}_r)\mathbb{L}^{r+1}\right)_{\geqslant 0}$$

is a function depending of $(p,\mathbf{t})$ and $\mathbf{u}=(u_1,\ldots,u_{m-1})$. Therefore, the $(m-1)$ hodograph equations

$$\frac{\partial\mathcal{S}_r}{\partial p}(p_i,\mathbf{t})=0, \tag{54}$$

where $p_i=p_i(\mathbf{t},\mathbf{u})$ are the zeros of $\partial\mathbb{L}^m/\partial p$, are enough for determining $\mathbf{u}$.

(2) $\mathbf{r}\geqslant\mathbf{m}+\mathbf{1}$. The function $\mathcal{S}_r=(\mathcal{S}_r)_{\geqslant 0}$ depends on $(p,\mathbf{t})$ and $\tilde{\mathbf{u}}=(u_1,\ldots,u_{r-1})$, so that in addition to the $(m-1)$ hodograph equations (54) a set of $(r-m)$ new equations involving $\mathbf{t}$ and $\tilde{\mathbf{u}}$ are required. These additional equations are supplied by vanishing the coefficients of the negative powers $1/p^n$ $(n=1,\ldots,r-m)$ in

$$(\mathbb{L}^m)_{\leqslant -1}=0.$$

C. Examples

In the following examples we exhibit solutions u of the dKP equation (4) depending on an arbitrary function $j=j(\rho)$. They are orbits of Gelfand–Dikii solutions u_0 under the action of the symmetry generated by (45). Notice that according to (45)–(46) we can obtain u_0 by setting $j=\rho$ in the expression of u.

(1) For

$$r=m=2, \quad f(P^{1/2}):=\gamma P^{7/2},$$

the generating function (47) becomes

$$J_2(P,\rho)=\gamma P^{7/2}+j(\rho)P^{3/2}, \quad \rho:=\frac{x}{3p^2}, \tag{55}$$

and $\mathbb{L}^2$ takes the form

$$\begin{aligned}\mathbb{L}^2&=(\mathbb{L}^2)_{\geqslant 0}\\ &=\frac{p^2}{(j'(t))^{2/3}}-\frac{4}{9}\frac{yj''(t)}{(j'(t))^{5/3}}p+\frac{2u_1}{(j'(t))^{2/3}}-\frac{2}{9}\frac{xj''(t)}{(j'(t))^{5/3}}-\frac{4}{27}\frac{y^2j'''(t)}{(j'(t))^{5/3}}+\frac{20}{81}\frac{y^2(j''(t))^2}{(j'(t))^{8/3}}.\end{aligned} \tag{56}$$

Hence $\partial\mathbb{L}^2/\partial p$ has a unique zero given by

$$p_1=\frac{2}{9}y\frac{j''(t)}{j'(t)}.$$

Moreover, the expression of

$$\mathcal{S}_2=(\tfrac{1}{3}y\mathcal{L}^2+\tfrac{2}{3}x\mathcal{L}+\gamma\mathbb{L}^7+j(\mathcal{M}_2)\mathbb{L}^3)_{\geqslant 0}$$

as a function of p can be computed by using (57) and the expansion

$$\begin{aligned}j(\mathcal{M}_2)=j(t)+\frac{2}{3}yj'(t)\frac{1}{p}+\left(\frac{x}{3}j'(t)+\frac{2}{9}y^2j''(t)\right)\frac{1}{p^2}\\ +\left(-\frac{2}{3}yj'(t)u_1+\frac{4}{81}y^3j'''(t)+\frac{2}{9}xyj''(t)\right)\frac{1}{p^3}+\mathcal{O}\left(\frac{1}{p^4}\right).\end{aligned}$$

In this way the hodograph equation $(\partial\mathcal{S}_2/\partial p)|_{p=p_1}=0$ turns out to be an equation for $u=u_1$, which yields the following solution of the dKP equation

$$u=\frac{F}{105\gamma}-\frac{6j(t)j'(t)^{4/3}}{F}+\frac{9j'(t)j''(t)x+6j'(t)j'''(t)y^2-8j''(t)^2y^2}{81(j'(t))^2}, \tag{57}$$

where

$$F := \gamma^{2/3}(-7350 j'(t)^{4/3} j''(t) y^2 - 33075 j'(t)^{7/3} x + 105\sqrt{35\,G})^{1/3},$$

$$G := \frac{1}{\gamma}(648 j(t)^3 j'(t)^4 + 140\gamma j'(t)^{8/3} j''(t)^2 y^4 + 1260\gamma j'(t)^{11/3} j''(t) x y^2 + 2835\gamma j'(t)^{14/3} x^2).$$

(2) By setting

$$r = m = 3, \quad f(P^{1/3}) := \gamma P^{7/3},$$

in (47) one finds that the first two coefficients of the expansion (1) of $\mathcal{L}$ are given by

$$u = u_1 = -\frac{1}{1024 j_1^2}(90 j_2^2 t^2 - 72 j_1 j_3 t^2 - 128 j_1 j_2 y + Z^2), \tag{58}$$

$$u_2 = \frac{-21\,\gamma j_1^8 j_2 t\, Z^4 + F\, Z^2 + 8388608\, j_1^{59/4} y + 2359296\, j_1^{55/4} j_2 t^2}{114688\,\gamma j_1^{11} Z}, \tag{59}$$

where

$$j_i := \frac{\partial^i j}{\partial \rho^i}(t_4), \quad i \geqslant 0,$$

$$F := -16384\, j_0 j_1^{47/4} + 7168\,\gamma j_1^{10} j_2 x - 13440\,\gamma j_1^9 j_2^2 t y + 5670\,\gamma j_1^8 j_2^3 t^3 + 2016\,\gamma j_1^{10} j_4 t^3 - 7560\,\gamma j_1^9 j_2 j_3 t^3 + 10752\,\gamma j_1^{10} j_3 t y,$$

and $Z = Z(x, y, t, t_4)$ is a root of the equation

$$49 j_1^{30}\gamma^2 Z^{10} + (5\,637\,144\,576\,\gamma j_1^{151/4} x + 2\,113\,929\,216\,\gamma j_1^{147/4} j_2 t y - 1\,610\,612\,736\, j_0^2 j_1^{75/2} + 396\,361\,728\,\gamma j_1^{147/4} j_3 t^3 - 297\,271\,296\,\gamma j_1^{143/4} j_2^2 t^3) Z^4 + 422\,212\,465\,065\,984\, j_1^{87/2} y^2 + 33\,397\,665\,693\,696\, j_1^{83/2} j_2^2 t^4 + 237\,494\,511\,599\,616\, j_1^{85/2} j_2 t^2 y = 0.$$

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