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PRESENTADA POR

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Homotopy Theory and Geometric Structures

Dissertation submitted for the degree of Doctor of Philosophy by
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Under the supervision of
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INTRODUCTION

This thesis consists of four different papers that I have written, in collaboration with other authors, during my Ph.D. program. The papers deal with the classification of certain compact homogeneous spaces of nilpotent Lie groups and their use as explicit examples of differentiable manifolds endowed with specific geometric structures. The main goal of this introduction is to give a general taste of the ideas contained in the papers, and to discuss the motivations that inspired them. Indeed, I do not write many details or definitions, but try rather to give a survey of the state of the art and of how the results fit in it. I also give references to research articles, books as well as to the four papers, in order to put my work in the right context.

Let G be a connected, simply connected $n-$dimensional nilpotent Lie group and let $\mathfrak{g}$ denote its Lie algebra. It is well known that G is diffeomorphic to $\mathbb{R}^n$; indeed, the diffeomorphism is given by the exponential map $\exp : \mathfrak{g} \to G$. A discrete, co-compact subgroup $\Gamma \subset G$ is called a lattice. In [68], Mal'cev proves that a lattice $\Gamma \subset G$ exists if and only if the structure constants of $\mathfrak{g}$ are rational numbers. We call the quotient G/Γ a (compact) *nilmanifold*. Mal'cev also shows that Γ is torsion-free. A non-compact nilmanifold can always be written as the product $N \times \mathbb{R}^m$ for a compact nilmanifold N and a suitable integer $m \geq 0$. The torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a nilmanifold; here we consider $\mathbb{R}^n$ with its natural structure of abelian Lie group, and the standard embedding of $\mathbb{Z}^n$ in $\mathbb{R}^n$. If the nilpotent Lie group G is not abelian, and $\Gamma \subset G$ is a lattice, we say that $N = G/\Gamma$ is a non-abelian nilmanifold.

One of the reasons why nilmanifolds are studied extensively is that they are easy enough to be approached from many points of view (topologically, geometrically, group-theoretically), but also complicated enough to display all sorts of behaviours. As an example, let us consider the following problem: determine which nilmanifolds carry a Kähler structure. The answer is contained in the following theorem:

Theorem 0.1 (Benson and Gordon, [8]). *Let $N = G/\Gamma$ be a compact nilmanifold, and assume that N is endowed with a Kähler structure. Then N is diffeomorphic to a torus.*

Indeed, the first example of a compact symplectic manifold without a Kähler structure was given by Thurston in [91]: it is a compact, orientable $4-$dimensional manifold with the structure of a T^2-bundle over T^2. This manifold can also be described as a non-abelian nilmanifold; it was discovered independently by Kodaira, as a product of his work on the classification of compact complex surfaces (see [57]). It is then known as *Kodaira-Thurston manifold.*

Nilmanifolds can be studied using rational homotopy theory. In the foundational paper *Infinitesimal Computations in Topology* ([88]), Sullivan set the bases for a very ambitious project. In the words of Sullivan:

> This paper was written in the effort to understand the nature of the mathematical object presented by a diffeomorphism class of compact smooth manifold. Under suitable restrictions on the fundamental group and the dimension, we find a rather understandable and complete answer to the question posed with "finite ambiguity". Roughly speaking, our answer [...] is that this mathematical object behaves up to "finite ambiguity" like a finite dimensional real vector space with additional structure provided by tensors, lattices and canonical elements.

The type of spaces that rational homotopy theory deals with are CW complexes of finite type; the correspondence indicated by Sullivan works perfectly in the case of simply connected spaces. The "finite ambiguity" to which Sullivan refers is the torsion, in the context of homotopy and (co)homology groups of the space. Indeed, the process of passing from a CW complex X to an algebraic model, denoted $\mathcal{M}_X$, requires, roughly speaking, to tensor with $\mathbb{Q}$ the Postnikov tower of X, and the torsion information is lost. This process produces a space $X_{\mathbb{Q}}$, the *rationalization* of X, and a map $X \to X_{\mathbb{Q}}$ such that

- $\pi_i(X) \otimes \mathbb{Q} \cong \pi_i(X_{\mathbb{Q}})$ for every $i \geq 2$ or, equivalently,
- $H_i(X) \otimes \mathbb{Q} \cong H_i(X_{\mathbb{Q}}; \mathbb{Q})$ for every $i \geq 2$.

The algebraic model that Sullivan associates to such a space X is the *minimal model* $\mathcal{M}_X$ of X; this is a minimal (commutative) differential graded algebra (CDGA for short) defined over $\mathbb{Q}$ and unique up to isomorphism. The cohomology of the minimal model is isomorphic to the singular cohomology of the corresponding space (say, with rational coefficients). Further, there is an isomorphism between the degree k generators of $\mathcal{M}_X$ and the dual of the rationalized $k-$th homotopy group, $(\pi_k(X) \otimes \mathbb{Q})^*$. The lattices Sullivan talks about come from the generators of the minimal model which take integer values when evaluated on integral homotopy.

Let X be a topological space with the structure of a finite simplicial complex; the minimal model $\mathcal{M}_X$ of X is obtained from the so-called *piecewise linear* forms $(A^*_{PL}(X), d)$; this is a CDGA defined over $\mathbb{Q}$ which can be described in terms of a simplicial structure on X. For this CDGA there is a de Rham type theorem: the cohomology of $(A^*_{PL}(X), d)$ is isomorphic to the singular cohomology of X with rational coefficients. When X is a smooth manifold the real minimal model is built from the de Rham algebra $(\Omega^*(X), d)$ of X. Equivalently, it can be obtained by tensoring with $\mathbb{R}$ the rational minimal model.

When the rationalizations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ of two spaces X and Y have the same homotopy type, one says that X and Y have the same rational homotopy type. Sullivan proves that two simply connected spaces X and Y have the same rational homotopy type if and only if their minimal models $\mathcal{M}_X$ and $\mathcal{M}_Y$ are isomorphic.

In the non-simply connected case, one has to put some restrictions on the fundamental group $\pi = \pi_1(X)$ in order to have a nice theory. In particular, π has to be a nilpotent group, acting in a nilpotent way on the higher homotopy groups of X. For further references to rational homotopy theory and its connections with geometry, see for instance [31, 32, 47, 51, 80, 89].

The two conditions on the fundamental group which we referred to in the last paragraph are fulfilled when X is a nilmanifold. Indeed, suppose $N = G/\Gamma$ is a compact nilmanifold; as we remarked above, G is diffeomorphic to $\mathbb{R}^n$; the projection $G \to N$ is the universal cover and $\pi_1(N) \cong \Gamma$. The long exact homotopy sequence associated to the fibration $\Gamma \to G \to N$ shows that $\pi_i(N) = 0$ for every $i \geq 2$, so that N is an Eilenberg-MacLane space $K(\Gamma, 1)$. Since Γ is a subgroup of G, it is clearly nilpotent. As remarked in [3], the fundamental group of a nilmanifold determines its diffeomorphism type. Indeed, one has the following result:

Theorem 0.2 (Auslander, [3])**.** *Let $N_j = G_j/\Gamma_j$, $j = 1, 2$ be two nilmanifolds and assume that there exists an isomorphism $\alpha : \Gamma_1 \to \Gamma_2$. Then α extends to an isomorphism of Lie groups $G_1 \to G_2$.*

In [77], Nomizu proves the following result:

Theorem 0.3 (Nomizu, [77])**.** *Let $N = G/\Gamma$ be a compact nilmanifold and let $\mathfrak{g}$ be the Lie algebra of G; let $(\wedge\mathfrak{g}^*, d)$ denote the Chevalley-Eilenberg complex of $\mathfrak{g}$. Then the natural inclusion $(\wedge\mathfrak{g}^*, d) \hookrightarrow (\Omega^*(N), d)$ induces an isomorphism in cohomology.*

It is easy to see that the nilpotency of the Lie algebra $\mathfrak{g}$ is equivalent to the minimality condition, which we referred to in a previous paragraph, for the Chevalley-Eilenberg complex. Thus, from Theorem 0.3, we obtain the following corollary:

Corollary 0.1. *The Chevalley-Eilenberg complex* $(\wedge\mathfrak{g}^*, d)$ *of* $\mathfrak{g}$ *is the minimal model of the nilmanifold* $N = G/\Gamma$*.*

Corollary 0.1 can be rephrased in this way: the rational homotopy type of a compact nilmanifold is determined by the Chevalley-Eilenberg complex. Notice that, since G admits a lattice by hypothesis, then, by Mal'cev theorem, the structure constants of $\mathfrak{g}$ are rational numbers, so that the Chevalley-Eilenberg complex is defined over $\mathbb{Q}$.

Let X be a topological space (CW complex of finite type). Formality is an important geometric and topological property of X, which is characterized in terms of the minimal model $\mathcal{M}_X$. A minimal CDGA is *formal* if there exists a map $\mathcal{M}_X \to H^*(\mathcal{M}_X)$ which is a *quasi-isomorphism*, i.e. it induces an isomorphism on cohomology. $H^*(\mathcal{M}_X)$, the cohomology of $\mathcal{M}_X$, is taken to be a differential graded algebra with zero differential. A manifold M is formal if its minimal model is. Concerning formality for nilmanifolds, we have the following theorem:

Theorem 0.4 (Hasegawa, [52])**.** *Let N be a nilmanifold. If N is formal, then it is diffeomorphic to a torus.*

On the other hand, in [28], Deligne *et al.* prove that Kähler manifolds are formal. Hence, non-abelian symplectic nilmanifolds provide examples of symplectic non Kähler manifolds (as we already knew by Theorem 0.1).

Another feature of Kähler manifolds, which relies on the Hodge decomposition of harmonic forms, is that they satisfy the Lefschetz property. A symplectic manifold (M^{2n}, ω) is called *Hard Lefschetz* if the map

$$[\omega]^{n-p} : H^p(M) \to H^{2n-p}(M), \quad \nu \mapsto \omega \wedge \nu \tag{1}$$

is an isomorphism for each $p = 0, \ldots, n$; it is called *of Lefschetz type* if it satisfies (1) for $p = 1$. Kähler manifolds are Hard Lefschetz. There exist symplectic, non-Kähler manifolds which are Hard Lefschetz, but not in the category of nilmanifolds. Indeed, we have the following proposition

Proposition 0.1 (Benson and Gordon, [8])**.** *Let $N = G/\Gamma$ be a compact nilmanifold and assume that N is of Lefschetz type. Then N is diffeomorphic to a torus.*

Compact nilmanifolds are never simply connected. The first example of a compact, simply connected symplectic non-Kähler manifold was given by McDuff in [69]. She constructed a compact, 10 dimensional, simply connected, symplectic manifold X with $b_3(X) = 3$; hence X can not be Kähler, since the odd Betti numbers are even on a Kähler manifold. The Lefschetz property on a compact symplectic manifold also implies that the odd Betti numbers are even, hence X is not Hard Lefschetz. It is interesting to notice that McDuff example starts with the Kodaira-Thurston manifold KT. Then she considers a symplectic embedding $KT \to \mathbb{CP}^5$; the manifold X is the blow-up of $\mathbb{CP}^5$ along the image of KT. We would like to remark here that there are not many known techniques for constructing symplectic manifolds, the main examples often coming from Kähler and algebraic geometry. Techniques such as the symplectic blow-up (outlined by Gromov in [48] and studied in more detail by McDuff in [69]) or the fibre connected sum (Gompf, [42]) have been developed with the special goal of understanding the behaviour of symplectic manifolds compared with Kähler manifolds. For a nice investigation on the relation between symplectic blow-ups, Lefschetz property and formality, see [18]. There, Cavalcanti proves that, under certain hypothesis, the kernel of the Lefschetz map decreases under blow-up, while non-formality is preserved (more specifically, Massey products persist under blow-up). As a side remark, Merkulov (see [70]) proved that Lefschetz property for a symplectic manifold is equivalent to the so-called $d\delta-$lemma.

For a certain time, it was conjectured (see e.g. [66, 80]) that a compact, simply connected symplectic manifold had to be formal. This is the so-called *formalising tendency* of a symplectic structure, also known as *Lupton-Oprea conjecture*. This conjecture was proven false, first by Babenko-Taimanov ([4]) in dimension ≥ 10 and, later, by Fernández-Muñoz in [36], in dimension 8. Both constructions start with nilmanifolds and then, using different techniques, are able to kill the fundamental group, preserving non-formality. We remark that neither example is Hard Lefschetz; nevertheless, in [19], the authors use techniques similar to those in [36] to construct an example of a simply connected symplectic non-formal manifold of dimension 8 which is Hard Lefschetz. Notice that, by a result of Miller [72] (see also [37]), a compact, simply connected manifold of dimension ≤ 6 is automatically formal.

The first Chapter of this Thesis analyzes the Fernández-Muñoz construction of a simply connected, symplectic non-formal manifold in dimension 8. We describe the idea briefly. The authors start with a complexified Kodaira-Thurston manifold. This is the complex nilmanifold $KT = \Gamma\backslash G$; the complex Lie algebra of G has non-zero Lie bracket $[X_1, X_2] = X_4$ with respect to a basis $\{X_1, X_2, X_3, X_4\}$. This is a symplectic, complex, non-Kähler, $8-$dimensional manifold, but it is not simply connected. They consider then a symplectic $\mathbb{Z}_3-$action on KT; the quotient space $\widehat{KT} = KT/\mathbb{Z}_3$ turns out to be a simply connected symplectic *orbifold*: $\mathbb{Z}_3$ acts with some fixed points. They show how to resolve the singularities, by blowing up fixed points, and prove that the resulting manifold is non-formal, by showing the persistence of a (kind of) Massey product under the blow-up process (see also [19] and [20] for further details on the relation between blow-up and formality). In the first chapter we compare the *symplectic* resolution of Fernández-Muñoz with the *complex* resolution of the complex orbifold $\widehat{KT}$. Indeed, the complex structure with respect to which the two blow-ups are performed are different: the change of variables used by Fernández-Muñoz to obtain a local model for the singularity is not holomorphic with respect to the standard complex structure on KT. Nevertheless, we are able to prove the following result:

Proposition 0.2. *The symplectic and the complex resolution of the orbifold $\widehat{KT}$ are diffeomorphic.*

Proposition 0.2 shows that the manifold constructed by Fernández and Muñoz is an example of an $8-$dimensional, simply connected, symplectic and complex manifold which admits no Kähler structure, as it happens for the Kodaira-Thurston manifold in dimension 4.

The Fernández-Muñoz construction is inspired by a similar work of Guan (see [49]). Guan starts with the real Kodaira-Thurston manifold, which happens to have a left-invariant complex structure, and, applying a very sophisticated construction, is able to produce an infinite series of examples of simply connected, holomorphic symplectic non-Kähler manifolds. The obstruction to being Kähler relies on some known cohomological properties of compact Kähler manifolds. It is still an open question whether Guan examples are formal or not.

The second and third Chapter, which contain the first two published papers, deal with the *classification* of nilmanifolds up to rational and real homotopy type. The classification is accomplished up to dimension 6 in the first paper, and in dimension 7 in the second one, although we restrict to the case of $2-$step nilmanifolds. This classification is obtained by investigating the possible minimal models of these nilmanifolds. As we said above, the minimal model of a nilmanifold $N = G/\Gamma$ is the Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, d)$ of the Lie algebra $\mathfrak{g}$ of G. Part of the originality of the papers consists in the fact that this classification is accomplished over any field $\mathbf{k}$ of characteristic $\neq 2$, generalizing the previously published classifications (see [2, 23, 26, 43, 44, 45, 67, 82] for instance). Over such a field $\mathbf{k}$ there is a $1-1$ correspondence between nilpotent Lie algebras and minimal algebras generated in degree 1, defined over $\mathbf{k}$ (see Chapter 2 for all relevant definitions). Indeed, all the quoted papers have the classification of nilpotent Lie algebras as a goal, while our primary interest is the classification of nilmanifolds up to rational homotopy type. The relevant case in the geometric setting is, of course, when the field $\mathbf{k}$ are rational numbers; but we can consider real or complex homotopy type of nilmanifolds, by allowing $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$, or even $\mathbf{k}-$homotopy type for an algebraic extension $\mathbf{k}$ of $\mathbb{Q}$. An important corollary of this classification is the following (see Chapter 2, Remark 2.3):

1. There are nilmanifolds which have the same real homotopy type but different rational homotopy type.

2. There are nilmanifolds which have the same complex homotopy type but different real homotopy type.

3. There are nilmanifolds N_1, N_2 for which the de Rham algebras $(\Omega^*(N_1), d)$ and $(\Omega^*(N_2), d)$ are joined by chains of quasi-isomorphisms (i.e., they have the same *real* minimal model), but for which there is no $f : N_1 \to N_2$ inducing a quasi-isomorphism $f^* : (\Omega^*(N_2), d) \to (\Omega^*(N_1), d)$. Just consider N_1, N_2 not of the same rational homotopy type. If there was such f, then there is a map on the rational minimal models $f^* : \mathcal{M}_{N_2} \to \mathcal{M}_{N_1}$ such that $f^*_{\mathbb{R}} : \mathcal{M}_{N_2} \otimes \mathbb{R} \to \mathcal{M}_{N_1} \otimes \mathbb{R}$ is an isomorphism. Hence f^* is an isomorphism itself, and N_1, N_2 would be of the same rational homotopy type.

We would like to remark that the study of minimal algebras over fields other than $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ is important in order to compare different rational homotopy types and to establish formality. Extending the well known result on *real* formality of Kähler manifolds, Sullivan proves the following theorem:

Theorem 0.5 ([88], Theorem 12.1). *The notion of formality for a nilpotent minimal algebra is independent of the ground field. Therefore the rational model of a compact Kähler manifold is formal over $\mathbb{Q}$. In particular, one can deduce the (rational) model from the cohomology ring.*

Another aspect of the papers is the study of some geometric structure on these nilmanifolds. In dimension 4 and 6, we determine which nilmanifolds admit a (left-invariant) symplectic structure; such a symplectic structure can be read off in the Chevalley-Eilenberg complex. In Appendix A we determine which 5$-$dimensional and 7$-$dimensional 2$-$step nilmanifolds admit a (left-invariant) co-symplectic structure. We will talk about co-symplectic structures later on in the introduction; also refer to Chapter 4.

One might be interested in other geometric structure on nilmanifolds, for instance complex, strong Kähler with torsion (SKT), complex generalized, Hermitian symplectic in even dimension and contact or G_2-calibrated in odd dimension. We refer to [21, 22, 30, 38, 82, 92] for the first setting and [26, 80] for the second one.

In Chapters 4 and 5 we study two geometric structures which are stricly related to symplectic and Kähler structures, namely co-symplectic and co-Kähler structures on odd-dimensional manifolds. These structures were introduced first by Boothby-Wang in [14] and Gray in [46], and further explored, for instance, by Blair, Goldberg and Yano, Hatakeyama and Sasaki (see [9, 12, 41, 55, 83]).

Let us start with the notion of *almost contact metric structure*[1] on a manifold M of dimension $2n+1$; this is defined to be a $4-$tuple (J, ξ, η, g), where $J \in \mathrm{End}(TM)$, $\xi \in \mathfrak{X}(M)$, $\eta \in \Omega^1(M)$ and g is a Riemannian metric, satisfying

- $J^2 = -\mathrm{Id} + \eta \otimes \xi$;
- $\eta(\xi) = 1$;
- $g(JX, JY) = g(X, Y) - \eta(X)\eta(Y)$ for $X, Y \in \mathfrak{X}(M)$.

The vector field ξ is known as *Reeb vector field*. The *rank* of an almost contact metric structure is the maximum k, $0 \leq k \leq n$ such that

$$\eta \wedge (d\eta)^k \neq 0$$

at every point of M. The metric g is determined by (J, ξ, η); indeed, in [83] it is proved that, given (J, ξ, η), there exists a positive definite Riemannian metric g such that $g(JX, JY) = g(X, Y) - \eta(X)\eta(Y)$; also, one sees easily that $J^2 = -\mathrm{Id} + \eta \otimes \xi$ implies $J(\xi) = 0$, which gives in turn $\eta(X) = g(\xi, X)$. Given an almost contact metric structure on a manifold M, one can define a $2-$form ω, called the *Kähler form*, by

$$\omega(X, Y) = g(JX, Y).$$

An almost contact metric structure is called *co-symplectic* if $d\eta = 0$ and the associated Kähler form ω is closed; it is called *contact* if it has rank n and $\omega = d\eta$. In the co-symplectic case, the horizontal distribution $\ker(\eta)$ is integrable (the integrability condition $\eta \wedge d\eta = 0$ is trivially satisfied), while in the contact case it is as far as possible from being integrable.

To some extent, both co-symplectic and contact structures can be seen as odd dimensional analogues of symplectic structures. We give some evidences in both cases:

- let M be a co-symplectic manifold; then the products $M \times \mathbb{R}$ and $M \times S^1$ have natural structures symplectic manifolds: denoting by t the coordinate on the $\mathbb{R}$ (resp. S^1) factor, the symplectic form on the product is given by $\Omega = \omega + \eta \wedge dt$;

[1]The general reference for this is the book of Blair, [11].

- let M be a contact manifold and set $\tilde{M} = M \times S^1$; let $p : \tilde{M} \to M$ denote the projection, and write a point of $\tilde{M}$ as (m, t); then $\Omega = d(e^t(p^*\eta))$ is a symplectic form on $\tilde{M}$ (this can be rephrased in the following way: the cone $M \times \mathbb{R}^+$ on a contact manifold is symplectic).

Let T be a tensor of type $(1, 1)$ on a manifold M; its *Nijenhuis torsion* N_T is the tensor of type $(2, 1)$ defined by

$$N_T(X, Y) = -T^2[X, Y] + T[TX, Y] + T[X, TY] - [TX, TY].$$

Now let (J, ξ, η, g) be an almost contact metric structure on a manifold M; the structure is called *normal* if

$$N_J + d\eta \otimes \xi = 0.$$

Let M be an odd-dimensional manifold. A *co-Kähler* structure on M is a normal co-symplectic structure; a *Sasakian* structure on M is a normal contact structure (see [16]). Note that normality for a co-symplectic manifold is equivalent to the condition $N_J = 0$. As it happened with the symplectic versus co-symplectic/contact, there is a strong relationship between Kähler and co-Kähler/Sasakian manifolds:

- let M be a co-Kähler manifold; then the products $M \times \mathbb{R}$ and $M \times S^1$ have a natural Kähler structure (see [55]);

- let M be a Sasakian manifold; then the cone $M \times \mathbb{R}^+$ is Kähler (see [16]).

In this thesis we focus on co-symplectic and co-Kähler structures on odd dimensional manifolds. Historically, more interest has been devoted to research in the Sasakian direction (for instance, $5-$dimensional Sasaki-Einstein manifolds can be related to G_2-manifolds, see [17]). Nevertheless, co-symplectic and co-Kähler manifolds have attracted attention, for example as the proper setting for time-dependent mechanics (see [60]). In the compact case, there is a nice parallel between the Kähler and the co-Kähler situation, which we summarize in Table 1. We assume K to be a Kähler manifold of dimension $2n$ and M to be a co-Kähler manifold of dimension $2n + 1$.

There are other nice features of co-Kähler manifolds:

- the Reeb vector field is Killing and parallel;

- the $1-$form η and the Kähler form ω are parallel.

In both cases, we consider the Levi-Civita connection of the metric g. In Chapter 5 we obtain an alternative, more geometric proof of the first two

Table 1: Kähler versus co-Kähler manifolds

Kähler	co-Kähler
the odd Betti numbers are even	the first Betti number is odd
$b_1 \leq b_2 \leq \ldots \leq b_n$	$b_1 \leq b_2 \leq \ldots \leq b_n = b_{n+1}$
the Lefschetz map is an isomorphism	
they are formal in the sense of Sullivan	

properties of Table 1.

The definition of the Lefschetz map $\mathscr{L}$ on a co-Kähler manifold, due to Chinea, de León and Marrero, [24], uses harmonic theory with respect to the natural Riemannian metric g. Let us denote by $\mathcal{H}^*(M)$ the harmonic forms on M and suppose that $\nu \in \mathcal{H}^p(M)$; then the Lefschetz map is defined by

$$\mathscr{L}(\nu) = \omega^{n-p} \wedge (\imath_\xi(\omega \wedge \nu) + \eta \wedge \nu) \in \mathcal{H}^{2n-p+1}(M); \tag{2}$$

here $\imath_\xi$ denotes contraction with the vector field ξ. A co-Kähler manifold satisfies the Lefschetz property; by this we mean that (2) is an isomorphism for $p = 1, \ldots, n$. Notice that the definition of the Lefschetz map for co-Kähler manifolds heavily requires the use of harmonic forms; more precisely, if $\nu \in \Omega^p(M)$ is closed but not co-closed, then $\imath_\xi(\nu)$ need not be closed, hence the Lefschetz map is not well defined on closed forms; for this, it is usually impossible to extend the definition of the Lefschetz map to arbitrary co-symplectic manifolds; we will see an example of this in Appendix A. Notice that things are different in the symplectic and Kähler context, since the Lefschetz map is well defined on every symplectic manifolds.

In [62], Li proves a very nice structure theorem for co-symplectic and co-Kähler manifolds. To state the theorem, we need to recall the notion of *mapping torus*. Let X be a topological space and let $\varphi : X \to X$ be a homeomorphism. The mapping torus X_φ of φ is defined as the quotient space

$$\frac{X \times [0,1]}{((x,0) \sim (\varphi(x),1))}.$$

The mapping torus X_φ has a natural projection to the circle S^1, and indeed $X \to X_\varphi \to S^1$ is a fibre bundle. When X is a smooth manifold and $\varphi : X \to X$ is a diffeomorphism, then $X_\varphi \to S^1$ is a smooth fibre bundle.

Let (M, ω) be a symplectic manifold; a *symplectomorphism* of M is a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\omega = \omega$. When (M, h) is a Kähler manifold (with h the Hermitian metric), a *Hermitian isometry* of M is an automorphism $\varphi : M \to M$ which satisfies $\varphi^* h = h$. Thus φ is a biholomorphic map which respects the symplectic form ω and the Riemannian metric g on M, where $h = g - i\omega$. Now let (M, ω) (resp. (M, h)) be a symplectic (resp. Kähler) manifold and let $\varphi : M \to M$ be a symplectomorphism (resp. a Hermitian isometry); then M_φ is called a *symplectic* (resp. *Kähler*) *mapping torus.*

Theorem 0.6 (Li, [62])**.** *There is a* $1-1$ *correspondence between co-symplectic manifolds and symplectic mapping tori. There is a* $1-1$ *correspondence between co-Kähler manifolds and Kähler mapping tori.*

Theorem 0.6 gives a very explicit way to construct all co-symplectic and co-Kähler manifolds.

In Chapter 4 we focus on the non-formality aspects of co-symplectic manifolds. Notice that a compact co-symplectic manifold M always has $b_1(M) \geq 1$, as the $1-$form η defines a non-zero cohomology class. We prove the following theorem:

Theorem 0.7. *For every pair* $(2k+1, b)$*,* $k, b \geq 1$*, there exists examples of compact non-formal co-symplectic manifolds of dimension* $2k+1$ *and with* $b_1 = b$*, except for the pair* $(3, 1)$*.*

Previously, the same result had been obtained in [35] for compact non-formal manifolds. Theorem 0.7 can be interpreted as a *geographic* classification of non-formal co-symplectic manifolds. The construction of the examples is straightforward in high dimension: just take a compact, simply connected non-formal symplectic manifold M (which exists in dimension ≥ 8) and form the product $M \times S^1$. This produces examples in dimension ≥ 9. One quickly realizes that the interesting case is $(5, 1)$, i.e. the construction of a compact $5-$dimensional non-formal co-symplectic manifold with $b_1 = 1$. We are able to give two different examples; let us give a short description of one of them. The construction starts with the abelian Lie algebra $\mathfrak{g}$ in dimension 4, endowed with a symplectic form ω and a completely solvable derivation D which respects the symplectic form. Set $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$, with $\mathbb{R}-$factor generated by ξ. The bracket on $\mathfrak{h}$ is defined by $[\xi, v] = D(v)$ $\forall v \in \mathfrak{g}$. Then $\mathfrak{h}$ is a completely solvable Lie algebra with $b_1 = 1$. We prove that the corresponding simply connected, completely solvable Lie group H has a lattice Γ. Let S denote the compact homogeneous space H/Γ. We show that $b_1(S) = 1$ and prove that S is non-formal by showing that S has a non-zero triple Massey product and, also, that it is not $2-$formal, hence not formal (see [37]).

Let M be an oriented manifold, $\varphi : M \to M$ an orientation-preserving diffeomorphism and denote by M_φ the corresponding mapping torus. The cohomology of M_φ is easily computed out of the cohomology of M, using the Mayer-Vietoris sequence. This gives us a way to study the minimal model and the formality of a mapping torus (see Section 4.4, Theorem 4.3 and 4.4).

Chapter 5 is devoted to the proof of a nice structure theorem for co-Kähler manifolds. As remarked in [24], it is not true that every compact co-Kähler manifold is a global product of a Kähler manifold and a circle. Nevertheless, we are able to prove the following theorem:

Theorem 0.8. *A compact co-Kähler manifold $(M^{2n+1}, J, \xi, \eta, g)$ with integral structure and mapping torus bundle $K \to M \to S^1$ splits as $M \cong S^1 \times_{\mathbb{Z}_m} K$, where $S^1 \times K \to M$ is a finite cover with structure group $\mathbb{Z}_m$ acting diagonally and by translations on the first factor. Moreover, M fibres over the circle $S^1/(\mathbb{Z}_m)$ with finite structure group.*

Li shows that, given a co-Kähler structure (J, ξ, η, g) on a compact manifold M, we can always replace it with another structure $(\tilde{J}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ such that $\tilde{\eta}$ is an *integral* $1-$form. We call the latter an *integral co-Kähler structure.* We can interpret Li's fibre bundle as given by the map $\tilde{\eta} : M \to S^1$ under the correspondence $H^1(M;\mathbb{Z}) = [M, K(\mathbb{Z}, 1)] = [M, S^1]$. As we remarked above, Theorem 0.8 allows us to recover the topological properties of co-Kähler manifolds in a very geometric way.

Let (M, g) be a compact Riemannian manifold; let $\xi \in \mathfrak{X}(M)$ be a Killing and parallel vector field and let $\eta \in \Omega^1(M;\mathbb{R})$ be the dual $1-$form, defined by $\eta(X) = g(\xi, X)$. Then η is parallel and harmonic. Being ξ Killing, its flow generates a subtorus C of the isometry group of M. In [93] Welsh shows (using the Albanese torus) that one can find a subtorus $T \subset C$ such that $M = T \times_G F$ where G is a finite group and F is a manifold. We can perturb ξ to a non-vanishing vector field Y which generates an S^1-action on M. Using the fact that η is parallel, Sadowski ([81]) proves that the orbit map of this action is homologically injective. Since on a co-Kähler manifold the vector field ξ is parallel and Killing and the $1-$form η is parallel, we obtain

Proposition 0.3. *A compact co-Kähler manifold $(M^{2n+1}, J, \xi, \eta, g)$ with integral structure supports a smooth homologically injective S^1 action.*

Write our compact co-Kähler manifold as a fibre bundle $M \to S^1$. Sadowski introduces the notion of *transversally equivariant fibration.* For a fibration $M \xrightarrow{p} S^1$, this means the following: M is endowed with an S^1-action whose orbits are transversal to the fibres, and such that $p(t\cdot x) - p(x)$ depends only on $t \in S^1$. Sadowski proves two things:

- transversality of the fibres of p is equivalent to the map $p_* \circ \alpha_* : H_*(S^1;\mathbb{Z}) \to H_*(S^1;\mathbb{Z})$ being injective, where $\alpha : S^1 \to M$ is the orbit map;

- transversal equivariance for p (with respect to a suitable S^1-action) is equivalent to the structure group of p being reducible to the finite group $\mathbb{Z}_m = \pi_1(S^1)/\mathrm{im}(p_* \circ \alpha_*)$.

Using these results, we are able to prove Theorem 0.8. It says that, up to a finite cover, a compact co-Kähler manifold *is* the product of a Kähler manifold and a circle.

As a consequence of the above splitting, we are able to infer a quite surprising property of Hermitian isometries of Kähler manifolds.

Theorem 0.9. *Let K be a Kähler manifold; then the elements of the group $\mathscr{H}$ have finite order.*

Here $\mathscr{H}$ is the group of Hermitian isometries of K quotiented by the connected component of the identity. Theorem 0.9 can be seen a special case of a much deeper result, due to Lieberman (see [63] and Theorem 5.7). Lieberman's proof, though, is very complicated and uses heavy methods of algebraic geometry.

In Section 5.6 we show that Theorem 0.9 is not true in the context of symplectic manifolds by taking the torus T^2 and a symplectomorphism $\varphi : T^2 \to T^2$ which does not respect the standard Kähler structure of T^2. This φ has indeed infinite order in $\mathrm{Symp}(T^2)/\mathrm{Symp}_0(T^2)$.

In view of obtaining a nice splitting theorem for (a certain class of) co-symplectic manifolds, the following question is of interest:

Question 0.1. Let (M, ω) be a symplectic manifold; let $\mathrm{Symp}(M)$ denote the group of symplectomorphisms of M and let $\mathrm{Symp}_0(M)$ be the connected component of the identity. When is it true that $[\varphi] \in \mathrm{Symp}(M)/\mathrm{Symp}_0(M)$ has finite order?

For such φ, indeed, the arguments we use to prove Theorem 0.9 can be reversed and used to produce a splitting of the co-symplectic manifold M_φ up a finite $\mathbb{Z}_m-$cover, where $\mathbb{Z}_m = \langle[\varphi]\rangle$.

We also use the Theorem 0.8 to give a description of the fundamental group of a co-Kähler manifold and to describe, along the lines of [39], the case of aspherical co-Kähler manifolds with solvable fundamental group. We prove two results:

Theorem 0.10. *If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then $\pi_1(M)$ has a subgroup of the form $H \times \mathbb{Z}$, where H is the fundamental group of a compact Kähler manifold, such that the quotient*

$$\frac{\pi_1(M)}{H \times \mathbb{Z}}$$

is a finite cyclic group.

Theorem 0.11. *Let $(M^{2n+1}, J, \xi, \eta, g)$ be an aspherical co-Kähler manifold with integral structure and suppose $\pi_1(M)$ is a solvable group. Then M is a finite quotient of a torus.*

Finally, in the Appendix we consider co-symplectic nilmanifolds and solvmanifolds. In this contest, we study two questions: the Lefschetz property and formality. As we said above, the Lefschetz map (2) is not well defined for arbitrary co-symplectic manifolds. Nevertheless, we see that it is well defined for $p = 1$ in the case M is a co-symplectic completely solvable solvmanifold. For co-symplectic nilmanifolds we prove the following result:

Theorem 0.12. *Let $N = G/\Gamma$ be a compact co-symplectic nilmanifold which satisfies Lefschetz property (2) for $p = 1$. Then N is diffeomorphic to a torus.*

Using Hasegawa result (Theorem 0.4), we also prove

Theorem 0.13. *Let $N = G/\Gamma$ be a compact nilmanifold endowed with a co-Kähler structure. Then N is diffeomorphic to a torus.*

We show that the Lefschetz property (for $p = 1$) and formality are not related for co-symplectic completely solvable solvmanifolds.

Finally, we determine which nilmanifolds in dimension 3, 5 and 7 (only $2-$step nilmanifolds in the latter case) carry a left-invariant co-symplectic structure.

CHAPTER

ONE

COMPLEX STRUCTURE ON THE FERNÁNDEZ-MUÑOZ MANIFOLD

1.1 The Fernández-Muñoz example

In [36], the authors constructed the first example of an 8−dimensional, compact, simply connected, symplectic non formal manifold, completing the last step in the confutation of the Lupton-Oprea conjecture. As anticipated in the introduction, this construction starts with a complex nilmanifold. The authors consider the complex Heisenberg group $H_{\mathbb{C}}$, defined as

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix} \quad | \quad u_1, u_2, u_3 \in \mathbb{C} \right\}.$$

The map $H_{\mathbb{C}} \to \mathbb{C}^3$, $A \mapsto (u_1, u_2, u_3)$ gives a global system of holomorphic coordinates on $H_{\mathbb{C}}$. Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$. Define $G = H_{\mathbb{C}} \times \mathbb{C}$, with global coordinates (u_1, u_2, u_3, u_4). Also, let $\Gamma \leq G$ be the discrete subgroup of the matrices with entries in Λ. We let Γ act on G on the left and set $M = \Gamma \backslash G$. Then M is a compact parallelizable nilmanifold. Notice that M can be seen as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \hookrightarrow M \to T^6 = (\mathbb{C}/\Lambda)^3$$

using the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)$. Also, let $H_\Lambda \leq H_{\mathbb{C}}$ denote the subgroup of matrices whose elements belong to Λ. Then M is the product of the *Iwasawa manifold* $H_\Lambda \backslash H_{\mathbb{C}}$ and a torus $\Lambda \backslash \mathbb{C}$. As such, M can be seen as a complex version of the Kodaira-Thurston manifold.

The authors consider a right $\mathbb{Z}_3$-action on G, given by

$$(u_1, u_2, u_3, u_4) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4). \tag{1.1}$$

One has $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$, where $\cdot$ denotes the group operation of G. Since ρ preserves the lattice, it descends to an action on M. We call $\hat{M}$ the quotient $M/\mathbb{Z}_3$; $\hat{M}$ is not smooth: it has 81 isolated quotient singularities.

A basis for the left invariant $1-$forms on G is given by

$$\mu = du_1, \quad \nu = du_2, \quad \theta = du_3 - u_2 du_1 \quad \text{and} \quad \eta = du_4$$

with

$$d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu.$$

The action of $\mathbb{Z}_3$ on the left invariant $1-$forms is given by

$$\rho^*\mu = \zeta\mu, \quad \rho^*\nu = \zeta\nu, \quad \rho^*\theta = \zeta^2\theta \quad \text{and} \quad \rho^*\eta = \zeta\eta.$$

The $2-$form

$$\omega = i\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i\eta \wedge \bar{\eta} \tag{1.2}$$

on M satisfies $\bar{\omega} = \omega$, so it is real; it is closed and satisfies $\omega^4 \neq 0$. Thus ω is a symplectic form. Notice also that

$$\rho^*\omega = \zeta^3(i\mu \wedge \bar{\mu} + \nu \wedge \theta + i\eta \wedge \bar{\eta}) + \zeta^{-3}\bar{\nu} \wedge \bar{\theta} = \omega,$$

hence ω is $\mathbb{Z}_3-$invariant and descends to a symplectic form $\hat{\omega}$ on the quotient $\hat{M}$. Therefore $(\hat{M}, \hat{\omega})$ is a symplectic orbifold. In [36], the authors prove

Proposition 1.1. *There exists a smooth compact simply connected symplectic manifold $(\tilde{M}, \tilde{\omega})$ which is isomorphic to $(\hat{M}, \hat{\omega})$ outside a small neighborhood of the singular points.*

We call $(\tilde{M}, \tilde{\omega})$ the *Fernández-Muñoz manifold.* In the proof of Proposition 1.1 the authors use the following change of coordinates to obtain a local Kähler model in a small neighborhood of a fixed point of the action:

$$\left\{ \begin{array}{lcl} w_1 & = & u_1 \\ w_2 & = & \frac{1}{\sqrt{2}}(u_2 + i\bar{u}_3) \\ w_3 & = & \frac{1}{\sqrt{2}}(i\bar{u}_2 - u_3) \\ w_4 & = & u_4 \end{array} \right. \tag{1.3}$$

We remark that this change of coordinates is not holomorphic with respect to the natural complex structure on G. We will say more about it in the next section.

Also, the following result is proved:

Theorem 1.1. *The Fernández-Muñoz manifold $(\tilde{M}, \tilde{\omega})$ is non formal and does not satisfy the Lefschetz property. Hence $(\tilde{M}, \tilde{\omega})$ is not Kähler.*

Theorem 1.1 was the final step in the disproof of the so-called *Lupton-Oprea conjecture* about the formalising tendence of a symplectic structure. Roughly speaking, this conjecture says that a simply connected symplectic manifold is formal. The conjecture was proven false by Babenko and Taĭmanov in 2000 ([4]) for real dimension ≥ 10, leaving a remarkable gap in dimension 8. Indeed, a result of Miller (see [37, 72]) says that simply connected manifolds of dimension ≤ 6 are formal.

The aim of this chapter is to build a *complex* resolution of singularities of the *complex* orbifold $(\hat{M}, \hat{J})$ and to prove that the smooth manifolds underlying the complex and the symplectic resolutions are diffeomorphic. This will produce an example of a simply connected, $8-$dimensional complex and symplectic manifold without Kähler structure, along the lines of the Kodaira-Thurston example.

1.2 The complex structure

First of all, we describe the complex structure J on G in two equivalent ways, showing that it descends to the nilmanifold $M = \Gamma\backslash G$ and also to the orbifold $\hat{M} = (\Gamma\backslash G)/\mathbb{Z}_3$. Then we will describe the resolution of singularities, which will give a smooth complex $4-$fold $(\bar{M}, \bar{J})$. Finally, we will prove that the smooth manifolds $\bar{M}$ and $\tilde{M}$ which underly the two resolutions are diffeomorphic.

Let us consider the group $G = H_{\mathbb{C}} \times \mathbb{C}$ above. Notice that G can be realized as a complex Lie subgroup of $\mathrm{GL}(5, \mathbb{C})$ by sending the pair $(A, u_4) \in H_{\mathbb{C}} \times \mathbb{C}$ to the matrix

$$\begin{pmatrix} 1 & u_2 & u_3 & 0 & 0 \\ 0 & 1 & u_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{GL}(5, \mathbb{C})$ is an open subset of $\mathbb{C}^{25}$, hence each tangent space $T_X\mathrm{GL}(5, \mathbb{C}) \cong \mathbb{C}^{25}$, $X \in \mathrm{GL}(5, \mathbb{C})$, inherits the standard complex structure of the ambient space, which is the multiplication by $i = \sqrt{-1}$. As a complex submanifold of $\mathrm{GL}(5, \mathbb{C})$, G inherits the same complex structure on each tangent space. This says that the complex structure on G is the multiplication by i on each tangent space T_gG, $g \in G$. The left translations $L_g : G \to G$, $h \mapsto gh$, are holomorphic maps, since they are written as polynomials in local coordinates; this shows that G is a complex parallelizable Lie group: the differential of L_g is complex linear and a parallelization is given by moving T_eG around. Let J denote the complex structure on G induced by the inclusion $G \hookrightarrow \mathrm{GL}(5,\mathbb{C})$, which is the multiplication by i on each tangent space; the above considerations show that J is left invariant.

Let us consider the tangent space T_eG, where $e \in G$ is the identity; there is an identification between T_eG, the Lie algebra $\mathfrak{g}$ of G and the vector space of left invariant holomorphic vector field on G, endowed with the natural Lie bracket. The complex structure on $\mathfrak{g}$ is the multiplication by i and $\mathfrak{g}$ is a complex vector space of dimension 4, described as follows:

$$\mathfrak{g} = \{\langle Z_1, Z_2, Z_3, Z_4\rangle \mid [Z_1, Z_2] = -Z_3\}.$$

By identifying $\mathfrak{g}$ with T_eG, one has $T_gG = d_eL_g(\mathfrak{g})$, $\forall g \in G$. This shows again that the complex structure J_g on T_gG is multiplication by i, for every $g \in G$.

We go through the details of the construction of left invariant complex structure on G. Let J_e denote the complex structure (i.e. multiplication by i) on $\mathfrak{g}$ and let $g \in G$ be a point. Define the complex structure $J_g : T_gG \to T_gG$ as

$$J_g(X(g)) = d_eL_g(ix),$$

where X is a left invariant vector field on G and $x \in \mathfrak{g}$ is such that $d_eL_g(x) = X(g)$. This defines J as a smooth section of the bundle $\mathrm{End}(TG)$. Let us show that $J^2 = -\mathrm{Id}$. Indeed,

$$J_g^2(X(g)) = J_g(J_g(X(g))) = d_eL_g(i(ix)) = -d_eL_g(x) = -X(g).$$

Lemma 1.1. *The (almost) complex structure defined above is left invariant.*

Proof. We must prove that, for every $g \in G$, $(L_g)^*J = J$. So take $X(h) \in T_hG$. Then

$$J_h(X(h)) = d_eL_h(ix)$$

where $x \in \mathfrak{g}$ is the unique vector satisfying $d_eL_h(x) = X(h)$. On the other hand we have

$$\begin{aligned}((L_g)^*J)(X(h)) &= d_{gh}L_{g^{-1}} \circ (J_{gh}) \circ (d_hL_g(X(h))) = \\ &= d_{gh}L_{g^{-1}} \circ d_eL_{gh}(ix) = d_eL_h(ix) = \\ &= J_h(X(h)).\end{aligned}$$

□

Lemma 1.2. *The (almost) complex structure defined above is integrable.*

Proof. This is trivial. Since J is left invariant, it is enough to work in the Lie algebra $\mathfrak{g}$. But on $\mathfrak{g}$ the complex structure is multiplication by i, hence the Nijenhuis tensor

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad X, Y \in \mathfrak{g}$$

vanishes. □

Lemma 1.3. *The two complex structures on G coincide.*

Proof. It is enough to notice that the left translations are holomorphic maps, thus their differential is complex linear. Let $g \in G$ be a point and X a left invariant vector field on G, such that $X(g) = d_e L_g(x)$, $x \in \mathfrak{g}$. Then

$$iX(g) = i d_e L_g(x) = d_e L_g(ix) = J_g(X(g)).$$

$\square$

So far we know that the natural complex structure J on the Lie group $G = H_{\mathbb{C}} \times \mathbb{C}$ is left invariant and is multiplication by i on each tangent space. As above, let $\Gamma \subset G$ be the subgroup of matrices whose elements belong to the lattice $\Lambda = \{a + b\zeta \mid \zeta = e^{2\pi i/3}\} \subset \mathbb{C}$. Since J is left invariant, it defines a complex structure on the quotient $M = \Gamma \backslash G$, which we denote again by J. Hence (M, J) is a complex nilmanifold.

Next we show that J is compatible with the $\mathbb{Z}_3-$action defined by 1.1. The complex structure J on M is multiplication by i at each tangent space $T_p M$, $p \in M$, since it comes from the complex structure on G. Let $\varphi : M \to M$ denote the $\mathbb{Z}_3-$action, and consider the map

$$d_p\varphi : T_p M \to T_{\varphi(p)} M.$$

The map φ can be lifted to a holomorphic map $\tilde{\varphi} : G \to G$; by taking global coordinates (u_1, u_2, u_3, u_4) on G, $\tilde{\varphi}$ is represented by the diagonal matrix $\mathrm{diag}(\zeta, \zeta, \zeta^2, \zeta)$, where $\zeta = e^{2\pi i/3}$. Since $\tilde{\varphi}$ is linear, it coincides with its differential $d_g\tilde{\varphi} : T_g G \to T_{\tilde{\varphi}(g)} G$; this is clearly a complex linear map, i.e.

$$d_g\tilde{\varphi} \circ J_g = J_{\tilde{\varphi}(g)} \circ d_g\tilde{\varphi}. \tag{1.4}$$

The action $\varphi : M \to M$ can thus be lifted to a holomorphic (in the standard sense) action of $\mathbb{Z}_3$ on G; since the complex structure J on M is multiplication by i on each tangent space, 1.4 shows that we can write

$$d_p\varphi \circ J_p = J_{\varphi(p)} \circ d_p\varphi.$$

showing that the complex structure commutes with the $\mathbb{Z}_3-$action, hence descends to the quotient $\hat{M} = M/\mathbb{Z}_3$. We denote by $\hat{J}$ the complex structure on $\hat{M}$; thus $(\hat{M}, \hat{J})$ is a complex orbifold, and we proved

Proposition 1.2. *Let $M = \Gamma \backslash G$ be as above and denote by J the natural complex structure on M; then the quotient of M by the $\mathbb{Z}_3-$action (1.1) is a complex orbifold $(\hat{M}, \hat{J})$.*

Remark 1.1. The complex nilmanifold M is an example of an 8−dimensional non-simply connected complex, symplectic and non-Kähler manifold. Indeed, M is non-formal, hence it can not be Kähler. By investigating the action of $\mathbb{Z}_3$ on the fundamental group of M, one sees that $(\hat{M}, \hat{J}, \hat{\omega})$ is simply connected. Therefore $(\hat{M}, \hat{J}, \hat{\omega})$ is an example of an 8−dimensional simply connected complex and symplectic orbifold which is not Kähler. Indeed, one sees that $\hat{M}$ is not formal.

1.3 Complex resolution

In this section we construct a complex resolution of the complex orbifold $(\hat{M}, \hat{J})$, along the lines of [36].

Proposition 1.3. *There exists a smooth complex manifold $(\bar{M}, \bar{J})$ which is biholomorphic to $(\hat{M}, \hat{J})$ outside the singular locus.*

Proof. Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$−action. Translating with an element $g \in G$, we can suppose that $p = (0, 0, 0, 0)$ in our coordinates. Let $U \subset M$ be a neighborhood of p and let $\phi : U \to B$ be a holomorphic local chart, given by the exponential map (by holomorphic we mean with respect to the complex structure J); here $B = B_{\mathbb{C}^4}(0, \varepsilon) \subset \mathbb{C}^4$. In these coordinates, the action of $\mathbb{Z}_3$ can be written as

$$(u_1, u_2, u_3, u_4) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4).$$

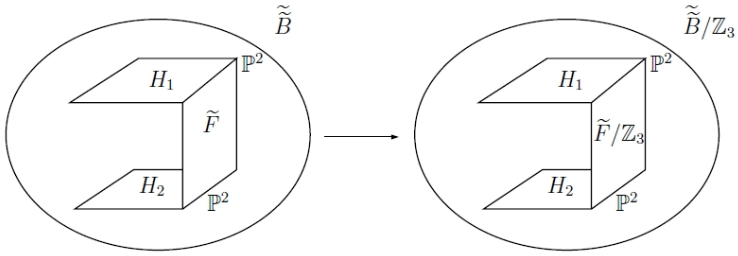
The local model for the singularity is thus $B/\mathbb{Z}_3$. From now on, the desingularization process is analogous to that in [36]. We blow up B at p to obtain $\tilde{B}$. The point p is replaced with a complex projective space $F = \mathbb{P}^3 = \mathbb{P}(T_pB)$ on which $\mathbb{Z}_3$ acts by

$$[u_1 : u_2 : u_3 : u_4] \mapsto [\zeta u_1 : \zeta u_2 : \zeta^2 u_3 : \zeta u_4] = [u_1 : u_2 : \zeta u_3 : u_4].$$

Thus $\mathbb{Z}_3$ acts on the exceptional divisor F with fixed locus $q \cup H$ where $q = [0 : 0 : 1 : 0]$ and $H = \{u_3 = 0\}$. Then one blows up $\tilde{B}$ at q and H to obtain $\tilde{\tilde{B}}$. The point q is replaced by a projective space $H_1 \cong \mathbb{P}^3$. The normal bundle to $H \subset F \subset \tilde{B}$ is the sum of the normal bundle of H in $\mathbb{P}^3$, which is $\mathcal{O}_{\mathbb{P}^2}(1)$, and the restriction to H of the normal bundle of F in $\tilde{B}$, which is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the second blow up replaces the projective plane H with a $\mathbb{P}^1$−bundle over $\mathbb{P}^2$ defined as $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1))$. The strict transform of $F \subset \tilde{B}$ under the second blow up is the blow up $\tilde{F}$ of F at q, which is a $\mathbb{P}^1$−bundle over $\mathbb{P}^2 = H$, actually $\tilde{F} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The resulting situation is depicted in the figure below (taken from [36]).

The fix-point locus of the $\mathbb{Z}_3$−action on $\tilde{\tilde{B}}$ consists of the two disjoint divisors H_1 and H_2. According to ([5], page 82), the quotient $\tilde{\tilde{B}}/\mathbb{Z}_3$ is a

Figure 1.1: The second blow-up and the $\mathbb{Z}_3-$action



smooth Kähler manifold. This provides a complex resolution of the singularity $B/\mathbb{Z}_3$. Notice that the blowing up is performed with respect to the natural complex structure inherited from the ambient space. By resolving every singular point, we obtain a smooth complex manifold $(\bar{M}, \bar{J})$. □

Proposition 1.4. *The complex manifold $(\bar{M}, \bar{J})$ is simply connected.*

Proof. The proof follows the same lines as in [36]. □

Notice that even though the $\mathbb{Z}_3-$action we have desingularized is written in the same way as in the case of the symplectic resolution, the two blow ups are performed with respect to *different* complex structures. In the complex resolution, one uses the natural complex structure $\hat{J}$ of $\hat{M}$, described in the previous section, while in the symplectic resolution one uses a Kähler model for a neighborhood of the fixed point which is not holomorphically equivalent to a local holomorphic chart for $\hat{J}$. This is because the change of variables (1.3) is not holomorphic with respect to the natural complex structure on M.

The local situation is as follows: on a small neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^4$ (which is a fixed point of the $\mathbb{Z}_3-$action in suitable coordinates) we have two complex structures, J_1 and J_2. The two complex structures are not holomorphically equivalent, because the change of variables which brings one to the other is not holomorphic. As a consequence, the two blow ups are different. In fact, the natural map that one would construct from one resolution to the other would not be even continuous. This becomes particularly clear when the blow up is interpreted as a symplectic cut, following Lerman and McDuff (see for instance [61]). The blow up of $\mathbb{C}^n$ at 0 can be thought of as removing a small ball of radius ε centered at the origin and then collapsing the fibers of the Hopf fibration in the boundary of the remaining set. But the fibers of the Hopf fibration (i.e. the intersections of the boundary of the ball, which is a S^{2n-1}, with a "complex" line) depend heavily on the complex structure of the ball.

On the other hand, we can prove that the following result:

Proposition 1.5. *The symplectic and the complex resolution of the orbifold* $(\hat{M}, \hat{J}, \hat{\omega})$ *are diffeomorphic.*

Proof. We work locally, in a small neighborhood of each fixed point. There we consider a smooth map which is the identity outside this small neighborhood and that does the right job inside the neighborhood. The local model is thus a small ball $B_{\mathbb{C}^4}(0,\delta) \subset \mathbb{C}^4$ endowed with two different complex structure J_1 and J_2. Define the map $\Theta : B_{\mathbb{C}^4}(0,\delta) \to B_{\mathbb{C}^4}(0,\delta)$ as the one that satisfies

$$\Theta^* J_1 = J_2.$$

If we take J_1 as the complex structure on the ball induced by the natural complex structure on $\hat{M}$ and J_2 to be the complex structure associated to the local Kähler model used for the symplectic resolution, then Θ is given by (1.3). We introduce real coordinates $u_k = x_k + iy_k$ and $w_k = s_k + it_k$, $k = 1,2,3,4$; in such coordinates, (1.3) is an automorphism of $\mathbb{R}^8$ written as

$$\left\{ \begin{array}{lcl} s_1 &=& x_1 \\ t_1 &=& y_1 \\ s_2 &=& \frac{1}{\sqrt{2}}(x_2 + y_3) \\ t_2 &=& \frac{1}{\sqrt{2}}(y_2 + x_3) \\ s_3 &=& \frac{1}{\sqrt{2}}(y_2 - x_3) \\ t_3 &=& \frac{1}{\sqrt{2}}(x_2 - y_3) \\ s_4 &=& x_4 \\ t_4 &=& y_4 \end{array} \right. \tag{1.5}$$

The associated matrix is

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix Θ belongs to $\mathrm{SO}(8,\mathbb{R})$. To construct the diffeomorphism we may try to find an isotopy $\{\Theta_t,\ t \in [0,1]\}$, such that Θ_0 is the identity $\mathrm{Id} \in \mathrm{SO}(8)$ and $\Theta_1 = \Theta$; in this way we get a path of complex structures $J_t = \Theta_t^* J_1$ connecting J_1 and J_2. To do this we must produce a smooth path in $\mathrm{SO}(8)$ between the identity matrix and Θ, which is equivariant with respect to the

$\mathbb{Z}_3-$action. In fact it is enough to find a smooth, $\mathbb{Z}_3-$equivariant path in SO(4) connecting the identity to the matrix

$$\theta = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

In the coordinates (s_2, t_2, s_3, t_3) spanning the $\mathbb{R}^4$ of interest, the $\mathbb{Z}_3-$action can be written

$$\Upsilon = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

under the natural inclusion $\mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$. We must check that the path $\{\Theta_s\} \subset \mathrm{SO}(4)$ satisfies $\Theta_s \circ \Upsilon = \Upsilon \circ \Theta_s$ for every $s \in [0,1]$. We do this explicitly. First notice that $\theta = P\theta'$, where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The matrix θ' is the image of the exponential map $\exp : \mathfrak{so}(4) \to \mathrm{SO}(4)$, computed at time $s = \pi/4$, of the matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

thus a smooth path in SO(4) between the identity and θ' is given by the image of the straight line in $\mathfrak{so}(4)$ joining the zero matrix with Q:

$$\begin{array}{rccc} \gamma : & [0, \pi/4] & \to & \mathrm{SO}(4) \\ & s & \mapsto & \exp(sQ) \end{array}$$

One sees that, for every $s \in [0, \pi/4]$, $\gamma(s) \circ \Upsilon = \Upsilon \circ \gamma(s)$, hence $\gamma(s)$ is $\mathbb{Z}_3-$equivariant. Now consider the matrix P; we juxtapose the following three paths in order to join P with the identity matrix:

$$P_1(s) = \begin{pmatrix} 0 & 0 & \sin(\pi s/2) & \cos(\pi s/2) \\ 0 & 0 & \cos(\pi s/2) & -\sin(\pi s/2) \\ \sin(\pi s/2) & \cos(\pi s/2) & 0 & 0 \\ \cos(\pi s/2) & -\sin(\pi s/2) & 0 & 0 \end{pmatrix},$$

$$P_2(s) = \begin{pmatrix} \sin(\pi s/2) & 0 & \cos(\pi s/2) & 0 \\ 0 & \sin(\pi s/2) & 0 & -\cos(\pi s/2) \\ \cos(\pi s/2) & 0 & -\sin(\pi s/2) & 0 \\ 0 & -\cos(\pi s/2) & 0 & -\sin(\pi s/2) \end{pmatrix},$$

$$P_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos(\pi s) & \sin(\pi s) \\ 0 & 0 & -\sin(\pi s) & -\cos(\pi s) \end{pmatrix}.$$

Again, a computation shows that $P_i(s) \circ \Upsilon = \Upsilon \circ P_i(s)$ $\forall s \in [0,1]$, $i = 1,2,3$. Hence the path $P(s) = P_1 * P_2 * P_3(s)$ satisfies $P(0) = \mathrm{Id}$, $P(1) = P$ and is $\mathbb{Z}_3-$equivariant. The path $\theta(s) = P(s)\theta'$ satisfies $\theta(0) = \theta'$ and $\theta(1) = \theta$, hence $\Psi = \theta * \gamma$ connects θ with the identity. The path Ψ is not globally smooth: in the concatenation points, it is only continuous. To smooth it, we proceed as follows. Let $0 = s_0 < s_1 < \ldots < s_{n-1} < s_1 = 1$ denote the cusps of the resulting path (including the starting and the ending point). Consider a smooth, increasing function $h : [0,1] \to [0,1]$ such that there exist intervals $\mathcal{J}_i = (t_i - \varepsilon, t_i + \varepsilon)$, $0 = t_0 < t_1 < \ldots < t_{n-1} < t_1 = 1$ with $h(t) = s_i$ for $s \in \mathcal{J}_i$. Define a new path $\Theta_t = \Psi_{h(t)}$. Clearly Ψ and Θ have the same image. Then Θ_t is a smooth, $\mathbb{Z}_3-$equivariant path in $\mathrm{SO}(4)$ connecting θ with the identity matrix. Viewing it as a path in $\mathrm{SO}(8)$ we obtain the isotopy Θ_t such that $\Theta_0 = \mathrm{Id}$ and $\Theta_1 = \Theta$; thus $\Theta_0^* J_1 = J_1$ and $\Theta_1^* J_1 = J_2$. We also need to define a $\mathbb{Z}_3$-invariant metric on the ball $B_{\mathbb{C}^4}(0,\delta)$. This can be done easily by averaging the standard metric of the ball over the elements of $\mathbb{Z}_3$.

We are ready to define the diffeomorphism between the two resolutions. Notice that the expression of the $\mathbb{Z}_3-$action is the same in the two sets of coordinates $(u_1,\ldots,u_4)$ and $(w_1,\ldots,w_4)$. Thus when we blow up we get, in both cases, an exceptional divisor $\mathbb{P}^3$ with one fixed point $q = [0:0:1:0]$ and one fixed hyperplane $H = \{u_3 = 0\} = \{w_3 = 0\}$; the differential of Θ at $0 \in B_{\mathbb{C}^4}(0,\delta)$, which we denote $d_0\Theta$, defines an automorphism of the exceptional divisor (when we projectivize the action), which fixes q and maps H to itself ($d_0\Theta$ is $(J_1,J_2)-$holomorphic, meaning that $d_0\Theta \circ J_1 = J_2 \circ d_0\Theta$). Thus $d_0\Theta$ also lifts to the second blow-up, hence to a map between the two exceptional divisors. Let $\rho : \mathbb{R} \to [0,1]$ be the standard cut-off function, i.e. a C^∞ function which is identically 0 on $(-\infty,0]$ and identically 1 on $[1,\infty)$. Using the $\mathbb{Z}_3$-invariant metric on the ball, the diffeomorphism f can then be defined as follows:

$$f(x) = \begin{cases} x & \text{if } |x| > \frac{2\delta}{3} \\ \Theta_t(x) & \text{if } \frac{\delta}{3} < |x| < \frac{2\delta}{3} \\ \Theta(x) & \text{if } |x| < \frac{\delta}{3} \end{cases}$$

where $t = \rho\big(\big(\frac{2\delta}{3} - |x|\big)\frac{3}{\delta}\big)$. $\square$

Corollary 1.1. *The Fernández-Muñoz manifold* $(\tilde{M}, \tilde{J}, \tilde{\omega})$ *is an example of simply connected,* $8-$*dimensional, complex and symplectic which does not admit any Kähler structure.*

CHAPTER

TWO

CLASSIFICATION OF MINIMAL ALGEBRAS OVER ANY FIELD UP TO DIMENSION 6

Giovanni Bazzoni and Vicente Muñoz

Abstract

We give a classification of minimal algebras generated in degree 1, defined over any field $\mathbf{k}$ of characteristic different from 2, up to dimension 6. This recovers the classification of nilpotent Lie algebras over $\mathbf{k}$ up to dimension 6. In the case of a field $\mathbf{k}$ of characteristic zero, we obtain the classification of nilmanifolds of dimension less than or equal to 6, up to $\mathbf{k}$-homotopy type. Finally, we determine which rational homotopy types of such nilmanifolds carry a symplectic structure.

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Key words: nilmanifolds, rational homotopy, nilpotent Lie algebras, minimal model.

2.1 Introduction and Main Results

Let X be a nilpotent space of the homotopy type of a CW-complex of finite type over $\mathbb{Q}$ (all spaces considered hereafter are of this kind). A space is nilpotent if $\pi_1(X)$ is a nilpotent group and it acts in a nilpotent way on $\pi_k(X)$ for $k > 1$. The rationalization of X (see [31], [47]) is a rational space $X_{\mathbb{Q}}$ (i.e. a space whose homotopy groups are rational vector spaces) together with a map $X \to X_{\mathbb{Q}}$ inducing isomorphisms $\pi_k(X)\otimes\mathbb{Q} \stackrel{\cong}{\to} \pi_k(X_{\mathbb{Q}})$ for $k \geq 1$ (recall that the rationalization of a nilpotent group is well-defined [47]). Two spaces X and Y have the same rational homotopy type if their rationalizations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ have the same homotopy type, i.e. if there exists a map $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ inducing isomorphisms in homotopy groups.

The theory of minimals models developed by Sullivan [88] allows to classify rational homotopy types algebraically. In fact, Sullivan constructed a $1-1$ correspondence between nilpotent rational spaces and isomorphism classes of minimal algebras over $\mathbb{Q}$:

$$X \leftrightarrow (\wedge V_X, d)\,. \tag{2.1}$$

Recall that, in general, a minimal algebra is a commutative differential graded algebra (CDGA henceforth) $(\wedge V, d)$ over a field $\mathbf{k}$ of characteristic different from 2 in which

1. $\wedge V$ denotes the free commutative algebra generated by the graded vector space $V = \oplus V^i$;
2. there exists a basis $\{x_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(x_\mu) \le \deg(x_\tau)$ if $\mu < \tau$ and each dx_τ is expressed in terms of preceding x_μ $(\mu < \tau)$. This implies that dx_τ does not have a linear part.

In the above formula (2.1), $(\wedge V_X, d)$ is known as the minimal model of X. Hence, X and Y have the same rational homotopy type if and only if they have isomorphic minimal models (as CDGAs over $\mathbb{Q}$).

The notion of *real* or *complex* homotopy type already appears in the literature (cf.[28] and [74]): two manifolds M_1, M_2 have the same real (resp. complex) homotopy type if the corresponding CDGAs of real (resp. complex) differential forms $(\Omega^*(M_1), d)$ and $(\Omega^*(M_2), d)$ have the same homotopy type, i.e. can be joined by a chain of morphisms inducing isomorphisms on cohomology (quasi-isomorphisms henceforth). This is equivalent to say that the two CDGAs have the same real (resp. complex) minimal model. It is convenient to remark ([31], §11(d)) that, if $(\wedge V, d)$ is the rational minimal model of M, then $(\wedge V \otimes_{\mathbb{Q}} \mathbb{R}, d)$ is the real minimal model of M. Recall that, given a CDGA A over a field $\mathbf{k}$, a minimal model of A is a minimal $\mathbf{k}$-algebra $(\wedge V, d)$ together with a quasi-isomorphism $(\wedge V, d) \xrightarrow{\simeq} A$. While the minimal model of a CDGA over a field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 0$ is unique up to isomorphism, the same result for arbitrary characteristic is unknown (see the appendix in which we prove uniqueness for the special case of minimal algebras treated in this paper).

We generalize this notion to an arbitrary field $\mathbf{k}$ of characteristic zero. Note that $\mathbb{Q} \subset \mathbf{k}$.

Definition 2.1. Let $\mathbf{k}$ be a field of characteristic zero. The $\mathbf{k}$-minimal model of a space X is $(\wedge V_X \otimes \mathbf{k}, d)$. We say that X and Y have the same $\mathbf{k}$-homotopy type if and only if the $\mathbf{k}$-minimal models $(\wedge V_X \otimes \mathbf{k}, d)$ and $(\wedge V_Y \otimes \mathbf{k}, d)$ are isomorphic.

Note that if $\mathbf{k}_1 \subset \mathbf{k}_2$, then the fact that X and Y have the same $\mathbf{k}_1$-homotopy type implies that X and Y have the same $\mathbf{k}_2$-homotopy type.

Recall that a nilmanifold is a quotient $N = G/\Gamma$ of a nilpotent connected Lie group by a discrete co-compact subgroup (i.e. the resulting quotient is compact). The minimal model of N is precisely the Chevalley-Eilenberg complex $(\wedge \mathfrak{g}^*, d)$ of the nilpotent Lie algebra $\mathfrak{g}$ of G (see [77]). Here, $\mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{Q})$ is assumed to be concentrated in degree 1 and the differential $d : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ reflects the Lie bracket via the pairing

$$dx(X, Y) = -x([X, Y]), \quad x \in \mathfrak{g}^*, \ X, Y \in \mathfrak{g}.$$

Indeed, consider a basis $\{X_i\}$ of $\mathfrak{g}$, such that

$$[X_j, X_k] = \sum_{i<j,k} a^i_{jk} \, X_i \, . \tag{2.2}$$

Let $\{x_i\}$ be the dual basis for $\mathfrak{g}^*$, so that $a^i_{jk} = x_i([X_j, X_k])$. Then the differential is expressed as

$$dx_i = -\sum_{j,k>i} a^i_{jk} \, x_j x_k \, . \tag{2.3}$$

Mal'cev proved that the existence of a basis $\{X_i\}$ of $\mathfrak{g}$ with *rational* structure constants a^i_{jk} in (2.2) is equivalent to the existence of a co-compact $\Gamma \subset G$. The minimal model of the nilmanifold $N = G/\Gamma$ is

$$(\wedge(x_1, \ldots, x_n), d),$$

where $V = \langle x_1, \ldots x_n \rangle = \oplus_{i=1}^n \mathbb{Q} x_i$ is the vector space generated by $x_1, \ldots, x_n$ over $\mathbb{Q}$, with $|x_i| = 1$ for every $i = 1, \ldots, n$ and dx_i is defined according to (2.3).

We prove the following:

Theorem 2.1. *Let $\mathbf{k}$ be a field of characteristic zero. The number of minimal models of 6-dimensional nilmanifolds, up to $\mathbf{k}$-homotopy type, is $26+4s$, where s denotes the cardinality of $\mathbb{Q}^*/((\mathbf{k}^*)^2 \cap \mathbb{Q}^*)$. In particular:*

- *There are 30 complex homotopy types of 6-dimensional nilmanifolds.*
- *There are 34 real homotopy types of 6-dimensional nilmanifolds.*
- *There are infinitely many rational homotopy types of 6-dimensional nilmanifolds.*

One of the consequences is the existence of pairs of nilmanifolds M_1, M_2 which have the same real homotopy type, but for which there is no map $f : M_1 \to M_2$ inducing an isomorphism in the real minimal models.

Theorem 2.1 is a consequence of the following classification of all minimal algebras generated in degree 1 by a vector space of dimension less than or equal to 6, in which we also give an explicit representative of each isomorphism class. (From now on, by the dimension of a minimal algebra $(\wedge V, d)$ we mean the dimension of V.)

Theorem 2.2. *Let* $\mathbf{k}$ *any field of any characteristic* $\mathrm{char}(\mathbf{k}) \neq 2$. *There are* $26+4r$ *isomorphism classes of* 6*-dimensional minimal algebras generated in degree* 1 *over* $\mathbf{k}$*, where* r *is the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$.

As the Chevalley-Eilenberg complex, defined as above over a nilpotent Lie algebra, gives a one-to-one correspondence between these objects and minimal algebras generated in degree 1, we obtain the following

Corollary 2.1. *There are* $26+4r$ *isomorphism classes of* 6*-dimensional nilpotent Lie algebras over* $\mathbf{k}$*, where* r *is the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$*. In particular:*

- *There are* 30 *isomorphism classes of* 6*-dimensional nilpotent real Lie algebras.*
- *There are* 34 *isomorphism classes of* 6*-dimensional nilpotent complex Lie algebras.*
- *For finite fields* $\mathbf{k} = \mathbb{F}_{p^n}$*, with* $p \neq 2$*, the cardinality of* $\mathbf{k}^*/(\mathbf{k}^*)^2$ *is* $r = 2$*. So there are* 34 *isomorphism classes of* 6*-dimensional nilpotent Lie algebras defined over* $\mathbb{F}_{p^n}$, $p \neq 2$.

This result is already known in the literature (see for instance [23] or [45]), but we obtain it from a new perspective: our starting point is the classification of minimal models.

Note that the classification of real homotopy types of 6-dimensional nilmanifolds already appears in the literature (see for instance [44] and [67]).

We end up the paper by determining which 6-dimensional nilmanifolds admit a symplectic structure. In particular, there are 27 real homotopy types of 6-dimensional nilmanifolds admitting symplectic forms. This appears already in [82], but we have decided to include it here for completeness, and to write down explicit symplectic forms in the cases where the nilmanifold does admit them.

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2.2 Preliminaries

Let $\mathbf{k}$ be a field of characteristic different from 2. Let $V = \langle x_1, \ldots x_n \rangle = \oplus_{i=1}^{n} \mathbf{k} x_i$ be a finite dimensional vector space over $\mathbf{k}$ with $\dim V \geq 2$. We want to analyse minimal algebras of the type

$$(\wedge(x_1, \ldots, x_n), d)$$

where $|x_i| = 1$, for every $i = 1, \ldots, n$, and dx_i is defined according to (2.3), with $a_{ij}^k \in \mathbf{k}$. Write $(\wedge V, d)$ with $V = V^1$ (i.e. $\wedge V$ is generated as an algebra by elements of degree 1). Set

$$\begin{aligned} W_1 &= \ker(d) \cap V \\ W_k &= d^{-1}(\wedge^2 W_{k-1}), \text{ for } k \geq 2\,. \end{aligned}$$

This is a filtration of V intrinsically defined. We see that $W_k \subset W_{k+1}$, for $k \geq 1$, as follows. First notice that $W_1 \subset W_2$ since $W_1 = d^{-1}(0)$. By induction, suppose that $W_{k-1} \subset W_k$; then we have

$$d(W_k) = d(d^{-1}(\wedge^2 W_{k-1})) \subset \wedge^2 W_{k-1} \subset \wedge^2 W_k\,.$$

This proves that $W_k \subset W_{k+1}$, as required.

Now define

$$\begin{aligned} F_1 &= W_1 \\ F_k &= W_k / W_{k-1} \text{ for } k \geq 2\,. \end{aligned}$$

Then, in a non-canonical way, one has $V = \oplus F_i$. The numbers $f_k = \dim(F_k)$ are invariants of V. Notice that $f_k = 0$ eventually. Under the splitting $W_k = W_{k-1} \oplus F_k$, the differential decomposes as[1]

$$d : W_{k+1} \longrightarrow \wedge^2 W_k = \wedge^2 W_{k-1} \oplus (W_{k-1} \otimes F_k) \oplus \wedge^2 F_k$$

If we project to the second and third summands, we have

$$d : W_{k+1} \longrightarrow \frac{\wedge^2 W_k}{\wedge^2 W_{k-1}} = (W_{k-1} \otimes F_k) \oplus \wedge^2 F_k$$

which vanishes on W_k, and hence induces a map

$$\bar{d} : F_{k+1} \longrightarrow (W_{k-1} \otimes F_k) \oplus \wedge^2 F_k = ((F_1 \oplus \ldots \oplus F_{k-1}) \otimes F_k) \oplus \wedge^2 F_k\,. \quad (2.4)$$

This map is injective, because $W_k = d^{-1}(\wedge^2 W_{k-1})$. Notice that the map (2.4) is not canonical, since it depends on the choice of the splitting.

[1] We use the notation $W_{k-1} \otimes F_k$ instead of $W_{k-1} \cdot F_k$, tacitly using the natural isomorphism $W_{k-1} \cdot F_k \cong W_{k-1} \otimes F_k$. We prefer this notation, as the other one could lead to some apparent incoherences along the paper.

The differential d also determines a well-defined map (independent of choice of splitting)

$$\hat{d} : F_{k+1} \to H^2(\wedge(F_1 \oplus \ldots \oplus F_k), d)\,,$$

which is also injective.

By considering $\bar{d} : F_2 \to \wedge^2 F_1$, we see that $f_1 \geq 2$. Moreover, if $f_1 = 2$ then $f_2 = 1$, and $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism.

We shall make extensive use of the following (easy) result.

Lemma 2.1. *Let W be a $\mathbf{k}$-vector space of dimension k, where $\mathbf{k}$ is a field of characteristic different from 2. Given any element $\varphi \in \wedge^2 W$, there is a (not unique) basis $x_1, \ldots, x_k$ of W such that $\varphi = x_1 \wedge x_2 + \ldots + x_{2r-1} \wedge x_{2r}$, for some $r \geq 0$, $2r \leq k$.*

The $2r$-dimensional space $\langle x_1, \ldots, x_{2r} \rangle \subset W$ is well-defined (independent of the basis).

Proof. Interpret φ as a antisymmetric bilinear map $W^* \times W^* \to \mathbb{Q}$. Let $2r$ be its rank, and consider a basis $e_1, \ldots, e_k$ of W^* such that $\varphi(e_{2i-1}, e_{2i}) = 1$, $1 \leq i \leq r$, and the other pairings are zero. Then the dual basis $x_1, \ldots, x_k$ does the job. $\square$

2.3 Classification in low dimensions

As we said in the introduction, a minimal algebra $(\wedge V, d)$ is of dimension k if $\dim V = k$. We start with the classification of minimal algebras over $\mathbf{k}$ of dimensions 2, 3 and 4.

Dimension 2

It should be $f_1 = 2$, so there is just one possibility:

$$(\wedge(x_1, x_2), dx_1 = dx_2 = 0)\,.$$

The corresponding Lie algebra is abelian.

For $\mathbf{k} = \mathbb{Q}$, where we are classifying 2-dimensional nilmanifolds, the corresponding nilmanifold is the 2-torus.

Dimension 3

Now there are two possibilities:

- $f_1 = 3$. Then the minimal algebra is $(\wedge(x_1, x_2, x_3), dx_1 = dx_2 = dx_3 = 0)$. The corresponding Lie algebra is abelian. In the case $\mathbf{k} = \mathbb{Q}$, the associated nilmanifold is the 3-torus.

- $f_1 = 2$ and $f_2 = 1$. Then $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism. We choose a generator $x_3 \in F_2$ such that $dx_3 = x_1x_2 \in \wedge^2 F_1$. The minimal algebra is $(\wedge(x_1, x_2, x_3), dx_1 = dx_2 = 0, dx_3 = x_1x_2)$. The corresponding Lie algebra is the Heisenberg Lie algebra. And for $\mathbf{k} = \mathbb{Q}$, the associated nilmanifold is known as the Heisenberg nilmanifold (see [80]).

We summarize the classification in the following table:

(f_i)	dx_1	dx_2	dx_3	$\mathfrak{g}$
(3)	0	0	0	A_3
$(2,1)$	0	0	x_1x_2	L_3

In the last column we have the corresponding Lie algebra: the abelian one, A_3, and the Lie algebra of the Heisenberg group, which we denote by L_3.

Dimension 4

The minimal algebra is of the form $(\wedge(x_1, x_2, x_3, x_4), d)$. We have to consider the following cases:

- $f_1 = 4$. Then the 4 elements x_i have zero differential. The corresponding Lie algebra is abelian.
- $f_1 = 3$, $f_2 = 1$. As the map $\bar{d} : F_2 \to \wedge^2 F_1$ is injective, there is a non-zero element in the image $\varphi_4 \in \wedge^2 F_1$. Using Lemma 2.1, we can choose a basis x_1, x_2, x_3 for F_1 such that $\varphi_4 = x_1x_2$. Then choose $x_4 \in F_2$ such that $dx_4 = \varphi_4 = x_1x_2$. Obviously, $dx_1 = dx_2 = dx_3 = 0$.
- $f_1 = 2$, $f_2 = 1$, $f_3 = 1$. In this case, we have a basis for $F_1 \oplus F_2$ such that $dx_1 = 0, dx_2 = 0$ and $dx_3 = x_1x_2$. The map

  $$\bar{d} : F_3 \to F_1 \otimes F_2$$

  is injective, hence the image determines a line $\ell \subset F_1$ such that $\bar{d}(F_3) = \ell \otimes F_2$. As $d(F_1 \oplus F_2) = \wedge^2 F_1$, we can choose $F_3 \subset W_3$ such that $d(F_3) = \ell \otimes F_2$. We choose the basis as follows: let $x_1 \in F_1$ be a vector spanning ℓ; x_2 another vector so that x_1, x_2 is a basis of F_1; let $x_3 \in F_2$ so that $dx_3 = x_1x_2$; finally choose x_4 such that $dx_4 = x_1x_3$.

The results are collected in the following table:

(f_i)	dx_1	dx_2	dx_3	dx_4	$\mathfrak{g}$
(4)	0	0	0	0	A_4
$(3,1)$	0	0	0	x_1x_2	$L_3 \oplus A_1$
$(2,1,1)$	0	0	x_1x_2	x_1x_3	L_4

The n-dimensional abelian Lie algebra is A_n; L_4 denotes the (unique) irreducible 4-dimensional nilpotent Lie algebra.

2.4 Classification in dimension 5

The minimal algebra is of the form $(\wedge(x_1, x_2, x_3, x_4, x_5), d)$. The possibilities for the numbers f_k are the following: $(f_1) = (5)$, $(f_1, f_2) = (4, 1)$, $(f_1, f_2) = (3, 2)$, $(f_1, f_2, f_3) = (3, 1, 1)$, $(f_1, f_2, f_3) = (2, 1, 2)$, $(f_1, f_2, f_3, f_4) = (2, 1, 1, 1)$ (noting that $f_1 \geq 2$ and that $f_1 = 2 \implies f_2 = 1$). We study all these possibilities in detail:

Case (5)

All the elements have zero differential.

Case $(4, 1)$

Then F_1 is a 4-dimensional vector space. Now the image of $\bar{d} : F_2 \to \wedge^2 F_1$ defines a line generated by some non-zero element $\varphi_5 \in \wedge^2 F_1$. By Lemma 2.1, we have two cases, according to the rank of φ_5 (by the *rank* of φ_5, we mean henceforth its rank as a bivector):

1. There is a basis $F_1 = \langle x_1, x_2, x_3, x_4 \rangle$ such that $dx_5 = \varphi_5 = x_1 x_2$.
2. There is a basis $F_1 = \langle x_1, x_2, x_3, x_4 \rangle$ such that $dx_5 = \varphi_5 = x_1 x_2 + x_3 x_4$.

Case $(3, 2)$

Now F_1 is a 3-dimensional vector space, and $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$. By Lemma 2.1, every non-zero element $\varphi \in \wedge^2 F_1$ is of the form $\varphi = x_1 x_2$ for a suitable basis x_1, x_2, x_3 of F_1, and determines a well-defined plane $\pi = \langle x_1, x_2 \rangle \subset F_1$.

Now $F_2 \subset \wedge^2 F_1$ is a two-dimensional vector space. Consider two linearly independent elements of F_2, which give two different planes in F_1, and let x_1 be a vector spanning their intersection. Now take a vector x_2 completing a basis for the first plane and a vector x_3 completing a basis for the second plane. Then we get the differentials $dx_4 = x_1 x_2$, $dx_5 = x_1 x_3$.

Case $(3, 1, 1)$

F_1 is 3-dimensional, and the image of $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$ determines a plane $\pi \subset F_1$. Now

$$\bar{d} : F_3 \hookrightarrow F_1 \otimes F_2$$

determines a line $\ell \subset F_1$ (such that $\bar{d}(F_3) = \ell \otimes F_2$). We easily compute

$$H^2(\wedge(F_1 \oplus F_2), d) = \frac{\ker(d : \wedge^2(F_1 \oplus F_2) \to \wedge^3(F_1 \oplus F_2))}{\operatorname{im}(d : F_1 \oplus F_2 \to \wedge^2(F_1 \oplus F_2))} = (\wedge^2 F_1 / d(F_2)) \oplus (\pi \otimes F_2)\,. \tag{2.5}$$

(The map $d : F_1 \otimes F_2 \hookrightarrow F_1 \otimes \wedge^2 F_1 \to \wedge^3 F_1$ sends $v \otimes F_2 \mapsto 0$ if and only if $v \in \pi$).

Hence $\ell \subset \pi$. We can arrange a basis x_1, x_2, x_3, x_4, x_5 with $\ell = \langle x_1 \rangle$, $\pi = \langle x_1, x_2 \rangle$, $F_1 = \langle x_1, x_2, x_3 \rangle$, so that $\varphi_4 = dx_4 = x_1x_2$, $\varphi_5 = dx_5 = x_1x_4 + v$, where $v \in \wedge^2 F_1$. Recall that F_2, F_3 are not well-defined (only $W_1 \subset W_2 \subset W_3$ is a well-defined filtration). In particular, this means that φ_4 is well-defined, but φ_5 is only well defined up to $\varphi_5 \mapsto \varphi_5 + \mu\varphi_4$. But then $\varphi_5^2 \in \wedge^4 W_2$ is well-defined, so we can distinguish cases according to the rank (as a bilinear form) of $\varphi_5 \in \wedge^2(F_1 \oplus F_2)$:

1. φ_5 is of rank 2. This determines a plane $\pi' \subset W_2 = F_1 \oplus F_2$. The intersection of π' with F_1 is the line ℓ. Take an element $x_4 \in \pi'$ not in the line, and declare $F_2 \subset W_2$ to be the span of x_4. Therefore $dx_5 = x_1x_4$.

2. φ_5 is of rank 4. The vector v is well-defined in $\wedge^2 F_1/d(F_2)$. Thus $v = ax_1x_3 + bx_2x_3$ with $b \neq 0$. We do the change of variables $x_4' = x_4 + ax_3$, $x_3' = bx_3$. Then $x_1, x_2, x_3', x_4', x_5$ is a basis with $dx_4' = x_1x_2$, $dx_5' = x_1x_4' + x_2x_3'$.

Case $(2, 1, 2)$

Now F_1 is 2-dimensional; then $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism and $\bar{d} : F_3 \to F_1 \otimes F_2$ is an isomorphism. Therefore there is a basis x_1, x_2, x_3, x_4, x_5 such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, and $dx_5 = x_2x_3$.

Case $(2, 1, 1, 1)$

Now $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism and the image of $\bar{d} : F_3 \to F_1 \otimes F_2$ produces a line $\ell \subset F_1$. Write $\ell = \langle x_1 \rangle$, $F_1 = \langle x_1, x_2 \rangle$, $F_2 = \langle x_3 \rangle$ and $F_3 = \langle x_4 \rangle$ so that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$.

For studying F_4, compute

$$H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3). \tag{2.6}$$

(Clearly $d(F_1 \otimes F_2) = 0$, $d : F_1 \otimes F_3 \to \wedge^2 F_1 \otimes F_2$ has kernel equal to $\ell \otimes F_3$, and $d : F_2 \otimes F_3 \to \wedge^2 F_1 \otimes F_3$ is injective, so $\ker d = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus (\ell \otimes F_3)$; on the other hand $\operatorname{im} d = \wedge^2 F_1 \oplus (\ell \otimes F_2)$.) Recall that the element φ_5 generating $d(F_4)$ should have non-zero projection to $\ell \otimes F_3$. Also, φ_5 can be understood as a bivector in $W_3 = F_1 \oplus F_2 \oplus F_3$. This is well-defined up to the addition of elements in $d(W_3) = \wedge^2 F_1 \oplus (\ell \otimes F_2)$; so $\varphi_5^2 \in \wedge^2 W_3$ is well-defined, and hence we can talk about the rank of φ_5. We have two cases:

1. φ_5 is of rank 2. This determines a plane $\pi' \subset W_3$, which intersects $F_1 \oplus F_2$ in a line. Let v span this line and x_4 be another generator of π'. Write $\varphi_5 = vx_4$. It must be $\langle v \rangle = \ell$, so $v = x_1$. Then $dx_3 = x_1x_2$, $dx_4 = x_1x_3$ and $dx_5 = x_1x_4$.

2. φ_5 is of rank 4. Then the projection of φ_5 to the first summand in (2.6) must be non-zero. So there is a choice of basis so that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$ and $dx_5 = x_1x_4 + x_2x_3$.

Summary of results

We gather all the results in the following table; the first 3 columns display the nonzero differentials. The fourth one gives the corresponding Lie algebras, and the last one refers to the list contained in [23]:

Table 2.1: Minimal algebras in dimension 5 over any field $\mathbf{k}$

(f_i)	dx_3	dx_4	dx_5	$\mathfrak{g}$	[23]
(5,0)	0	0	0	A_5	–
(4,1)	0	0	x_1x_2	$L_3 \oplus A_2$	–
	0	0	$x_1x_2 + x_3x_4$	$L_{5,1}$	$\mathcal{N}_{5,6}$
(3,2)	0	x_1x_2	x_1x_3	$L_{5,2}$	$\mathcal{N}_{5,5}$
(3,1,1)	0	x_1x_2	x_1x_4	$L_4 \oplus A_1$	–
	0	x_1x_2	$x_1x_4 + x_2x_3$	$L_{5,3}$	$\mathcal{N}_{5,4}$
(2,1,2)	x_1x_2	x_1x_3	x_2x_3	$L_{5,5}$	$\mathcal{N}_{5,3}$
(2,1,1,1)	x_1x_2	x_1x_3	x_1x_4	$L_{5,4}$	$\mathcal{N}_{5,2}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{5,6}$	$\mathcal{N}_{5,1}$

As before, $L_{5,k}$ denote the non-split 5-dimensional nilpotent Lie algebras.

Recall that this classification works over any field $\mathbf{k}$. In the case $\mathbf{k} = \mathbb{Q}$, this means in particular that there are 9 nilpotent Lie algebras of dimension 5 over $\mathbb{Q}$ and, as a consequence, 9 rational homotopy types of 5-dimensional nilmanifolds.

2.5 Classification in dimension 6

Now we move to study minimal algebras of the form $(\wedge(x_1, \ldots, x_6), d)$, where $|x_i| = 1$. The numbers $\{f_k\}$ can be the following: $(f_1) = (6)$, $(f_1, f_2) = (5, 1)$, $(f_1, f_2) = (4, 2)$, $(f_1, f_2, f_3) = (4, 1, 1)$, $(f_1, f_2) = (3, 3)$, $(f_1, f_2, f_3) = (3, 2, 1)$, $(f_1, f_2, f_3) = (3, 1, 2)$, $(f_1, f_2, f_3, f_4) = (3, 1, 1, 1)$, $(f_1, f_2, f_3, f_4) = (2, 1, 2, 1)$, $(f_1, f_2, f_3, f_4) = (2, 1, 1, 2)$ and $(f_1, f_2, f_3, f_4) = (2, 1, 1, 1, 1)$.

The case $(2, 1, 3)$ does not appear due to the injectivity of the differential $\bar{d} : F_3 \to W_1 \otimes F_2$. Also the case $(2, 1, 1, 2)$ does not show up, as we will see at the end of this section. Now we consider all the cases in detail.

Case (6)

In this case we have $F_1 = V$, $d(F_1) = 0$. This corresponds to the abelian Lie algebra.

Case $(5, 1)$

Here F_1 is a 5-dimensional vector space and F_2 is 1-dimensional, $F_2 = \langle x_6 \rangle$; $\bar{d}(F_2) \subset \wedge^2 F_1$. Let $\varphi_6 = dx_6 \in \wedge^2 F_1$ be a generator of $d(F_2)$. By Lemma 2.1, we have the following cases:

1. $\mathrm{rank}(\varphi_6) = 2$. Then there exists a basis of F_1 such that $dx_6 = x_1 x_2$.

2. $\mathrm{rank}(\varphi_6) = 4$. Then there exists a basis of F_1 such that $dx_6 = x_1 x_2 + x_3 x_4$.

Case $(4, 2)$

Here F_1 is a 4-dimensional vector space and $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$. This defines a projective line ℓ in $\mathbb{P}(\wedge^2 F_1) = \mathbb{P}^5$.

The skew-symmetric matrices of dimension 4 with rank ≤ 2 are given as the zero locus of the single quadratic homogeneous equation

$$a_1 a_6 - a_2 a_5 + a_3 a_4 = 0\,,$$

where

$$A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}$$

is a skew-symmetric matrix. This defines a smooth quadric $\mathcal{Q}$ in $\mathbb{P}^5$.

Now we have to look at the intersection of ℓ with $\mathcal{Q}$. Here it is where the field of definition matters.

1. $\ell \cap \mathcal{Q} = \{p_1, p_2\}$, two different points. Choose $\varphi_5, \varphi_6 \in \wedge^2 F_1$ so that they correspond to the points $p_1, p_2 \in \mathbb{P}(\wedge^2 F_1)$. Accordingly, choose x_5, x_6 generators of F_2 so that $\varphi_5 = dx_5$, $\varphi_6 = dx_6$. Note that both are bivectors of F_1 of rank 2, but the elements $a\varphi_5 + b\varphi_6$, $ab \neq 0$ are of rank 4. By Lemma 2.1, a rank 2 element determines a plane in F_1. The two planes corresponding to φ_5, φ_6 intersect transversally (otherwise, we are in case (2) below). Thus we can choose a basis x_1, x_2, x_3, x_4 for F_1 so that $dx_5 = x_1 x_2$ and $dx_6 = x_3 x_4$. Note that the elements $a x_1 x_2 + b x_3 x_4$ are of rank 4 when $ab \neq 0$.

2. $\ell \subset \mathcal{Q}$. We choose a basis x_5, x_6 so that both $\varphi_5 = dx_5$, $\varphi_6 = dx_6$ have rank 2. All linear combinations $a dx_5 + b dx_6$ are also of rank 2. The

planes determined by φ_5, φ_6 do not intersect transversally (otherwise we are in case (1) above), so they intersect in a line. Then we can choose a basis x_1, x_2, x_3, x_4 for F_1 so that $dx_5 = x_1x_2$ and $dx_6 = x_1x_3$, the line being $\langle x_1 \rangle$. Note that all elements $a\varphi_5 + b\varphi_6 = x_1(ax_2 + bx_3)$ are of rank 2.

3. $\ell \cap \mathcal{Q} = \{p\}$. This means that ℓ is tangent to $\mathcal{Q}$. Let $\varphi_5 \in \wedge^2 F_1$ corresponding to p. This is of rank 2, so it determines a plane $\pi \subset F_1$. The plane π is described by some equations $e_3 = e_4 = 0$, where $e_3, e_4 \in F_1^*$. Now consider $\varphi_6 \in \wedge^2 F_1$ giving another point $q \in \ell$. So φ_6 is of rank 4 (see Lemma 2.1). If $\varphi_6(e_3, e_4) = 1$, then choose e_1, e_2 so that $\varphi_6 = x_1x_2 + x_3x_4$, but then $\varphi_5 = \lambda x_1 x_2$, with $\lambda \neq 0$, and $\varphi_6 - \lambda\varphi_5$ is also of rank 2, which is a contradiction.
   Therefore $\varphi_6(e_3, e_4) = 0$, and so $\langle e_3, e_4 \rangle$ is Lagrangian in (F_1^*, φ_6). We can complete the basis to e_1, e_2, e_3, e_4 so that $dx_6 = \varphi_6 = x_1x_3 + x_2x_4$. Normalize φ_5 so that $dx_5 = \varphi_5 = x_1x_2$. All forms $dx_6 + a\, dx_5$ are of rank 4.

4. $\ell \cap \mathcal{Q} = \emptyset$. This means that ℓ and $\mathcal{Q}$ intersect in two points with coordinates in the algebraic closure of $\mathbf{k}$. As this intersection is invariant by the Galois group, there must be a quadratic extension $\mathbf{k}' \supset \mathbf{k}$ where the coordinates of the two points lie; the two points are conjugate by the Galois automorphism of $\mathbf{k}'|\mathbf{k}$. Therefore, there is an element $a \in \mathbf{k}^*$ such that $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, a is not a square in $\mathbf{k}$, and the differentials

$$dx_5 = x_1x_2, \qquad dx_6 = x_3x_4.$$

   satisfy that the planes $\pi_1 = \langle x_1, x_2 \rangle$ and $\pi_2 = \langle x_3, x_4 \rangle$ are conjugate under the Galois map $\sqrt{a} \mapsto -\sqrt{a}$. Write:

$$\begin{aligned} x_1 &= y_1 + \sqrt{a} y_2, \\ x_2 &= y_3 + \sqrt{a} y_4, \\ x_3 &= y_1 - \sqrt{a} y_2, \\ x_4 &= y_3 - \sqrt{a} y_4, \\ x_5 &= y_5 + \sqrt{a} y_6, \\ x_6 &= y_5 - \sqrt{a} y_6, \end{aligned}$$

   where $y_1, \ldots, y_6$ are defined over $\mathbf{k}$. Then $dy_5 = y_1y_3 + ay_2y_4$, $dy_6 = y_1y_4 + y_2y_3$.
   This is the "canonical" model. Two of these minimal algebras are not isomorphic over $\mathbf{k}$ for different quadratic field extensions, since the equivalence would be given by a $\mathbf{k}$-isomorphism, therefore commuting with the action of the Galois group.

The quadratic field extensions are parametrized by elements $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. Note that for $a = 1$, we recover case (1), where $dy_5 + dy_6 = (y_1 + y_2)(y_3 + y_4)$ and $dy_5 - dy_6 = (y_1 + y_2)(y_3 - y_4)$ are of rank 2.

Remark 2.1. If $\mathbf{k} = \mathbb{C}$ (or any algebraically closed field) then case (4) does not appear.

For $\mathbf{k} = \mathbb{R}$, we have that $\mathbb{R}^*/(\mathbb{R}^*)^2 - \{1\} = \{-1\}$, and there is only one minimal algebra in this case, given by $dy_5 = y_1y_3 - y_2y_4$, $dy_6 = y_1y_4 + y_2y_3$.

The case $\mathbf{k} = \mathbb{Q}$ is very relevant, as it corresponds to the classification of rational homotopy types of nilmanifolds. Note that in this case the classes in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ are parametrized bijectively by elements $\pm p_1p_2 \dots p_k$, where p_i are different primes, and $k \geq 0$. In particular, if a is a square in $\mathbb{Q}$ then we fall again in (1) above.

Remark 2.2. Note that we get examples of distinct rational homotopy types of nilmanifolds which have the same real homotopy type. Also, we get nilmanifolds with different real homotopy types but the same complex homotopy type.

Case $(4, 1, 1)$

Now F_1 is 4-dimensional, and $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$ determines an element $\varphi_5 \in \wedge^2 F_1$. Clearly, $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2)$. The differential $d : F_1 \otimes F_2 \to \wedge^3 F_1$ is given as wedge by φ_5. So if φ_5 is of rank 4, then this map is an isomorphism and

$$\ker(d : \wedge^2(F_1 \oplus F_2) \to \wedge^3(F_1 \oplus F_2)) = \wedge^2 F_1.$$

So there cannot be an injective map $\bar{d} : F_3 \to F_1 \otimes F_2$. This shows that φ_5 must be of rank 2, and therefore it determines a plane $\pi \subset F_1$. Now the closed elements are given as $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. The differential $\bar{d} : F_3 \to \pi \otimes F_2$ determines a line $\ell \subset \pi$. Let x_1 be a generator for ℓ, and $\pi = \langle x_1, x_2 \rangle$. Then there is a basis x_1, x_2, x_3, x_4 such that $dx_5 = x_1x_2$ and $dx_6 = x_1x_5 + \varphi'$, where $\varphi' \in \wedge^2 F_1$. We are allowed to change x_5 by $x_5' = x_5 + v$ with $v \in F_1$. This has the effect of changing dx_6 by adding $x_1 v$. This means that we may assume that φ' does not contain x_1, so $\varphi' \in \wedge^2(F_1/\ell)$. Actually, wedging $\varphi_6 = dx_6 \in \wedge^2 F_1 \oplus (\pi \otimes F_2)$ by x_1, we get an element $\varphi_6\, x_1 \in \wedge^3 F_1$ which is the image of φ' under the map $\wedge^2(F_1/\ell) \overset{x_1}{\hookrightarrow} \wedge^3 F_1$. It is then easy to see then that φ' is well-defined (independent of the choices of F_2, F_3).

We have the following cases:

1. $\varphi' = 0$. So $dx_6 = x_1x_5$.

2. φ' is non-zero, so it is of rank 2. Therefore it determines a plane π' in F_1/ℓ. If this is transversal to the line π/ℓ, then $\varphi' = x_3x_4$ and we have that $dx_6 = x_1x_5 + x_3x_4$.

3. If π' contains π/ℓ, then $\varphi' = x_2x_3$ and we have $dx_6 = x_1x_5 + x_2x_3$.

Case $(3,3)$

This case is very easy, since F_1 is three-dimensional, and $\bar{d}: F_2 \to \wedge^2 F_1$ must be an isomorphism. So there exists a basis such that $dx_4 = x_1x_2$, $dx_5 = x_1x_3$ and $dx_6 = x_2x_3$.

Case $(3,2,1)$

We have a three-dimensional space F_1. Then there is a two-dimensional space F_2 with a map $\bar{d}: F_2 \to \wedge^2 F_1$. Note that any element in F_2 determines a plane in F_1. Intersecting those planes, we get a line $\ell \subset F_1$. Then the differential gives an isomorphism $h: F_2 \xrightarrow{\cong} F_1/\ell$ (defined up to a non-zero scalar). Choosing $\ell = \langle x_1 \rangle$, we take basis such that $h(x_4) = x_2$ and $h(x_5) = x_3$. So

$$dx_4 = x_1x_2, \quad dx_5 = x_1x_3\,.$$

Let us compute the closed elements in $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus \wedge^2 F_2$. Clearly, $d: \wedge^2 F_2 \hookrightarrow \wedge^2 F_1 \otimes F_2$. Also the map $d: F_1 \otimes F_2 \cong F_1 \otimes (F_1/\ell) \to \wedge^3 F_1$ is the map $(u,v) \mapsto u \wedge v \wedge x_1$. As $\operatorname{im} d = d(F_2)$, we have that

$$H^2(\wedge(F_1 \oplus F_2), d) = \wedge^2(F_1/\ell) \oplus \ker(F_1 \otimes F_2 \to \wedge^3 F_1),$$

and F_3 determines an element φ_6 in that space. Let π_4, π_5 be the planes in F_1 corresponding to dx_4, dx_5. There are vectors $v_2 \in \pi_4$, $v_3 \in \pi_5$ and $\lambda \in \mathbf{k}$ so that $\varphi_6 = \lambda x_2x_3 + v_2x_4 + v_3x_5$. We have the following cases:

1. Suppose that $\varphi_6^2 x_1 \neq 0$ (this condition is well-defined, independently of the choices of F_2, F_3). This is an element in $\wedge^3 F_1 \otimes \wedge^2 F_2 \cong x_1 \otimes \wedge^2(F_1/\ell) \otimes \wedge^2 F_2 \cong (\wedge^2 F_2)^2$. Taking an isomorphism $\wedge^2 F_2 \cong \mathbf{k}$, we have that the class of $\varphi_6^2 x_1 \in (\wedge^2 F_2)^2 \cong \mathbf{k}$ gives a well-defined element in $\mathbf{k}^*/(\mathbf{k}^*)^2$.
   The condition $\varphi_6^2 x_1 \neq 0$ translates into v_2, v_3, x_1 being linearly independent. So we can arrange $x_2 = a_2v_2$, $x_3 = a_3v_3$, with $a_2, a_3 \neq 0$. Normalizing x_6, we can assume $a_2 = 1$. So $dx_4 = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = \lambda x_2x_3 + x_2x_4 + ax_3x_5$. Note that the class defined by $\varphi_6^2 x_1$ is $-2a \in \mathbf{k}^*/(\mathbf{k}^*)^2$. (If we change the basis $x_3' = \mu x_3$, $x_5' = \mu x_5$ we obtain $dx_6 = x_2x_4 + a\mu^{-2}x_3'x_5'$. We see again that $-2a$ is defined in $\mathbf{k}^*/(\mathbf{k}^*)^2$).
   Changing the basis as $x_4' = x_4 + \lambda x_3$, we get $dx_4' = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = x_2x_4' - \frac{a}{2}x_3x_5$.

2. Now suppose $\varphi_6^2 x_1 = 0$, $\varphi_6 x_1 \notin \wedge^3 F_1$ and $\varphi_6^2 \notin \wedge^3 F_1 \otimes F_2$ (again these conditions are independent of the choices of F_2, F_3). Then $v_2v_3x_1 = 0$ and $v_2v_3 \neq 0$. We can choose the coordinates x_2, x_3 (and x_4, x_5 accordingly through h) so that $v_2 = x_2$, $v_3 = x_1$. Therefore $\varphi_6 =$

$\lambda x_2x_3 + x_2x_4 + x_1x_5$. Now the change of variable $x_4' = x_4 + \lambda x_3$ gives the form $dx_4' = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = x_2x_4' + x_1x_5$.

3. Suppose that $\varphi_6^2 \in \wedge^3 F_1 \otimes F_2$ and $\varphi_6 x_1 \notin \wedge^3 F_1$. Then $v_2v_3 = 0$ but x_1 is linearly independent with $\langle v_2, v_3\rangle$. Choose coordinates so that $v_2 = x_2$ and $v_3 = 0$. So $\varphi_6 = \lambda x_2x_3 + x_2x_4$. The change of variable $x_4' = x_4 + \lambda x_3$ gives the form $dx_4' = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = x_2x_4'$.

4. Suppose that $\varphi_6 x_1 \in \wedge^3 F_1$, $\varphi_6^2 \neq 0$. So that we can choose $v_2 = x_1$, $v_3 = 0$. We have $dx_4 = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = \lambda x_2x_3 + x_1x_4$, where $\lambda \neq 0$. Now take $x_3' = \lambda x_3$ and $x_5' = \lambda x_5$. So $dx_4 = x_1x_2$, $dx_5' = x_1x_3'$, $dx_6 = x_2x_3' + x_1x_4$

5. Finally, we have $\varphi_6 x_1 \in \wedge^3 F_1$, $\varphi_6^2 = 0$ and this gives the minimal algebra $dx_4 = x_1x_2$, $dx_5 = x_1x_3$, $dx_6 = x_1x_4$.

Case $(3, 1, 2)$

We have a 3-dimensional vector space F_1. Then $\bar{d} : F_2 \to \wedge^2 F_1$ determines a well-defined plane $\pi \subset F_1$. Looking at $\wedge^2(F_1 \oplus F_2) = \wedge^2 F_1 \oplus (F_1 \otimes F_2)$, we see that the closed elements are $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. The differential is defined by

$$\hat{d} : F_3 \to H^2(\wedge(F_1 \oplus F_2), d) = (\wedge^2 F_1/d(F_2)) \oplus (\pi \otimes F_2)\,, \qquad (2.7)$$

where the projection $\bar{d} : F_3 \to \pi \otimes F_2$ is injective, hence an isomorphism. So we identify $F_3 \cong \pi \otimes F_2$. Let x_1, x_2 be a basis for π, and x_5, x_6 the corresponding basis of F_3 through the above isomorphism. So $dx_4 = x_1x_2$, $dx_5 = x_1x_4 + v_5$, $dx_6 = x_2x_4 + v_6$, where $v_5, v_6 \in \wedge^2 F_1/d(F_2)$.

The map (2.7) together with $\bar{d}^{-1} : \pi \otimes F_2 \to F_3$ gives a map $\phi : \pi \otimes F_2 \to (\wedge^2 F_1/d(F_2))$. It is easy to see that the pairing $F_1 \otimes \wedge^2 F_1 \to \wedge^3 F_1$ induces a non-degenerate pairing $\pi \otimes (\wedge^2 F_1/d(F_2)) \to \wedge^3 F_1$, and hence an isomorphism $(\wedge^2 F_1/d(F_2)) \cong \pi^* \otimes \wedge^3 F_1$. Hence $\phi : \pi \otimes F_2 \to \pi^* \otimes \wedge^3 F_1$, and using that $\pi^* \cong \pi \otimes \wedge^2 \pi^*$, we finally get a map

$$\phi : \pi \to \pi \otimes (\wedge^2 \pi^* \otimes \wedge^3 F_1 \otimes F_2^*).$$

This gives an endomorphism of π defined up to a constant.

Now let us see the indeterminacy of ϕ. With the change of variables $x_4' = x_4 + \mu x_3 + \nu x_2 + \eta x_1$ we get $dx_5 = x_1x_4' + v_5'$, $dx_6 = x_2x_4' + v_6'$, where $v_5' = v_5 - \mu x_1x_3$, $v_6' = v_6 - \mu x_2x_3$. Therefore the corresponding map $\phi' = \phi - \mu\,\mathrm{Id}$. So ϕ is defined up to addition of a multiple of the identity.

We get the following classification:

1. Suppose that ϕ is zero (or a scalar multiple of the identity). Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4$.

2. Suppose that ϕ is diagonalizable. Adding a multiple of the identity, we can assume that one of the eigenvalues is zero and the other is not. Let x_2 generate the image and x_1 be in the kernel. Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4 + x_2x_3$.

3. Suppose that ϕ is not diagonalizable. Adding a multiple of the identity, we can assume that the eigenvalues are zero. Let x_1 generate the image, so that x_1 is in the kernel. Then $dx_4 = x_1x_2$, $dx_5 = x_1x_4$, $dx_6 = x_2x_4 + x_1x_3$.

4. Finally, ϕ can be non-diagonalizable if $\mathbf{k}$ is not algebraically closed. To diagonalize ϕ we need a quadratic extension of $\mathbf{k}$. Let $a \in \mathbf{k}^*$ so that ϕ diagonalizes over $\mathbf{k}' = \mathbf{k}(\sqrt{a})$. If we arrange ϕ to have zero trace (by adding a multiple of the identity), then the minimum polynomial of ϕ is $T^2 - a$. So we can choose a basis such that $\phi(x_1) = x_2$, $\phi(x_2) = ax_1$. Thus $dx_4 = x_1x_2$, $dx_5 = x_1x_4 + x_2x_3$, $dx_6 = x_2x_4 + ax_1x_3$. The minimal algebras are parametrized by $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. (The value $a = 1$ recovers case (2)).

Case $(3, 1, 1, 1)$

Now F_1 is of dimension 3. We have a one-dimensional space given as the image of $\bar{d} : F_2 \hookrightarrow \wedge^2 F_1$, which determines a plane $\pi \subset F_1$. The closed elements in $\wedge^2(F_1 \oplus F_2)$ are $\wedge^2 F_1 \oplus (\pi \otimes F_2)$. Therefore, $\varphi_5 = dx_5$ determines a line $\ell \subset \pi$. But it also determines an element in $\wedge^2 F_1$, up to $d(F_2)$ and up to $\ell \wedge F_1$, i.e. in $\wedge^2(F_1/\ell)$. Then

1. $dx_4 = x_1x_2$, $dx_5 = x_1x_4$. Now we compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$ to be $\wedge^2 F_1 \oplus (\pi \otimes F_2) \oplus (\ell \otimes F_3)$. The element $\varphi_6 = dx_6$ has non-zero last component in $\ell \otimes F_3$. It is well-defined up to $\ell \wedge F_1$ and up to $\ell \otimes F_2$. There are several cases:

   (a) $\varphi_6 \in \ell \otimes F_3$. Then $dx_6 = x_1x_5$.
   (b) $\varphi_6 \in (\pi \otimes F_2) \oplus (\ell \otimes F_3)$. Then $dx_6 = x_2x_4 + x_1x_5$.
   (c) $\varphi_6 \in \wedge^2 F_1 \oplus (\ell \otimes F_3)$, then $dx_6 = x_2x_3 + x_1x_5$.
   (d) φ_6 has non-zero components in all summands. Then $dx_6 = \lambda x_2x_3 + x_2x_4 + x_1x_5$. We can arrange $\lambda = 1$ by choosing $x_3' = \lambda x_3$.

   (We can check that these cases are not equivalent: the first one is characterised by $\varphi_6 x_1 = 0$; the second one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 = 0$; the third one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 \neq 0$, $\varphi_6\varphi_4 = 0$; the last one by $\varphi_6 x_1 \neq 0$, $\varphi_6\varphi_5 \neq 0$, $\varphi_6\varphi_4 \neq 0$).

2. $dx_4 = x_1x_2$, $dx_5 = x_1x_4 + x_2x_3$. Then the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$ are those in

$$\wedge^2 F_1 \oplus (\pi \otimes F_2) \oplus \langle x_1x_5 + x_4x_3 \rangle.$$

So $\varphi_6 = ax_1x_3 + bx_2x_3 + cx_1x_4 + dx_2x_4 + x_1x_5 + x_4x_3$. The change of variables $x_6' = x_6 - bx_5$ arranges $b = 0$. Then the change of variables $x_3' = -dx_2 + x_3$ and $x_5' = ax_3 + x_5$ arranges $a = 0$ and $d = 0$. Thus $\varphi_6 = cx_1x_4 + x_1x_5 + x_4x_3$. Finally $x_3' = -\frac{c}{2}x_1 + x_3$, $x_5' = \frac{c}{2}x_4 + x_5$ arranges $c = 0$. Hence $\varphi_6 = x_1x_5 - x_3x_4$.

Case $(2, 1, 2, 1)$

Now we have a 2-dimensional space F_1, and an isomorphism $\bar{d} : F_2 \to \wedge^2 F_1$. Also $\bar{d} : F_3 \to \wedge^2(F_1 \oplus F_2)/\wedge^2 F_1 = F_1 \otimes F_2$ is an isomorphism. Then there is a basis for $F_1 \oplus F_2 \oplus F_3$ such that

$$dx_3 = x_1x_2,\ dx_4 = x_1x_3 \text{ and } dx_5 = x_2x_3\,.$$

Let us compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3)$. First, $d : F_2 \otimes F_3 \to \wedge^2 F_1 \otimes F_3$ is an isomorphism; second $d : \wedge^2 F_3 \hookrightarrow F_1 \otimes F_2 \otimes F_3$ is an injection; finally, $d : F_1 \otimes F_3 \cong F_1 \otimes F_1 \otimes F_2 \to \wedge^2 F_1 \otimes F_2$. So the kernel of d is isomorphic to $\wedge^2(F_1 \oplus F_2) \oplus (s^2 F_1 \otimes F_2)$. Then

$$\varphi_6 \in H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = s^2 F_1 \subset F_1 \otimes F_1 \cong F_1 \otimes F_3$$

determines a non-zero quadratic form on F_1 up to multiplication by scalar, call it A. (Here we use the natural identification $F_3 \cong F_1$, $x_4 \mapsto x_1$, $x_5 \mapsto x_2$, defined up to scalar).

We have the following cases:

1. If $\operatorname{rank}(A) = 1$, then A has non-zero kernel. We get a basis such that $dx_6 = x_1x_4$.
2. If $\operatorname{rank}(A) = 2$ then $\det(A) \neq 0$. This determines a 2×2-matrix A defined up to conjugation $A \mapsto M^T AM$ and up to $A \mapsto \lambda A$. Note that the class of the determinant $a = \det(A) \in \mathbf{k}^*/(\mathbf{k}^*)^2$ is well-defined. Take a basis diagonalizing A. We can arrange that $A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. So $dx_6 = x_1x_4 + ax_2x_5$. (Note that for $a = 0$ we recover case (1)).

Case $(2, 1, 1, 2)$

Now F_1 is 2-dimensional, and $\bar{d} : F_2 \to \wedge^2 F_1$ is an isomorphism. F_3 is one-dimensional and $\bar{d} : F_3 \to \wedge^2(F_1 \oplus F_2)/\wedge^2 F_1 = F_1 \otimes F_2$. Therefore there exists a line $\ell \subset F_1$ such that $d(F_3) = \ell \otimes F_2$.

We compute the closed elements in $\wedge^2(F_1 \oplus F_2 \oplus F_3) = \wedge^2 F_1 \oplus (F_1 \otimes F_2) \oplus (F_1 \otimes F_3) \oplus (F_2 \otimes F_3)$. As $d : F_1 \otimes F_3 \to \wedge^2 F_1 \otimes F_2$ has kernel $\ell \otimes F_3$ and $d : F_2 \otimes F_3 \hookrightarrow \wedge^2 F_1 \otimes F_3$, we have that

$$H^2(\wedge(F_1 \oplus F_2 \oplus F_3), d) = ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3)\,.$$

As $\bar{d} : F_4 \to \wedge^2(F_1 \oplus F_2 \oplus F_3)/\wedge^2 (F_1 \oplus F_2)$ is injective, and $\dim(\ell \otimes F_3) = 1$, it cannot be that $f_4 = 2$.

Case $(2, 1, 1, 1, 1)$

We work as in the previous case. Now $\bar{d} : F_4 \to ((F_1/\ell) \otimes F_2) \oplus (\ell \otimes F_3)$ produces an isomorphism $F_4 \cong \ell \otimes F_3$ and hence a map

$$\phi : \ell \otimes F_3 \to (F_1/\ell) \otimes F_2.$$

Note that this map is well-defined, independent of the choice of F_3 satisfying $W_2 \oplus F_3 = W_3$. We have the following cases

1. Suppose that $\phi = 0$. So there is a basis such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, $dx_5 = x_1x_4$, where we have chosen $\ell = \langle x_1 \rangle$, $F_1 = \langle x_1, x_2 \rangle$. We can easily compute

$$H^2(\wedge(x_1, x_2, x_3, x_4, x_5), d) = \langle x_1x_5, x_2x_3, x_2x_5 - x_3x_4 \rangle .$$

Then

$$\varphi_6 = dx_6 = ax_1x_5 + bx_2x_3 + c(x_2x_5 - x_3x_4) . \tag{2.8}$$

We have

(a) If $\varphi_6 x_1 = 0$ then $b = c = 0$. We can choose generators so that $dx_6 = x_1x_5$.

(b) If $\varphi_6 x_1 \neq 0$ and $\varphi_6 x_1 x_2 = 0$, then $c = 0$ and $a, b \neq 0$. We can arrange $a = 1$ by normalizing x_6 and then do the change of variables $x_2' = bx_2$ $x_3' = bx_3$, $x_4' = bx_4$, $x_5' = bx_5$, $x_6' = bx_6$. This produces an equation as (2.8) with $b = 1$. Hence $dx_6 = x_1x_5 + x_2x_3$.

(c) If $\varphi_6 x_1 x_2 \neq 0$, then $c \neq 0$. We can arrange $c = 1$ by normalizing x_6. Now put $x_2' = x_2 + ax_1$ to arrange $a = 0$. Finally take $x_5' = x_5 + bx_3$, $x_4' = x_4 + bx_2$ to be able to put $b = 0$. So $dx_6 = x_2x_5 - x_3x_4$.

2. Suppose that $\phi \neq 0$. Then there is a basis for $F_1 \oplus F_2 \oplus F_3 \oplus F_4$ such that $dx_3 = x_1x_2$, $dx_4 = x_1x_3$, $dx_5 = x_1x_4 + x_2x_3$. We can easily compute

$$H^2(\wedge(x_1, x_2, x_3, x_4, x_5), d) = \langle x_1x_4, x_1x_5 + x_2x_4, x_2x_5 - x_3x_4 \rangle .$$

Then

$$\varphi_6 = dx_6 = ax_1x_4 + b(x_1x_5 + x_2x_4) + c(x_2x_5 - x_3x_4) .$$

We have

(a) If $\varphi_6 x_1 x_2 = 0$ then $c = 0$. We can suppose $b = 1$, and put $x_2' = x_2 + \frac{a}{2}x_1$, $x_5' = x_5 + \frac{a}{2}x_4$, to arrange $a = 0$. So $dx_6 = x_1x_5 + x_2x_4$.

(b) If $\varphi_6 x_1 x_2 \neq 0$ then we can suppose $c = 1$. Put $x_2' = bx_1 + x_2$ and $x_5' = bx_4 + x_5$ to eliminate b. Finally do the change of variables $x_4' = x_4 - \frac{a}{2}x_2$, $x_5' = x_5 - \frac{a}{2}x_3$ and $x_6' = -ax_5 + x_6$ to arrange $a = 0$. Hence $dx_6 = x_2x_5 - x_3x_4$.

Classification of minimal algebras over $\mathbf{k}$

Let $\mathbf{k}$ be any field of characteristic different from 2. The above work can be summarized in Table 2.2.

The first 4 columns display the non-zero differentials, and the fifth one is a labelling of the corresponding Lie algebra. Denote $\Lambda = \mathbf{k}^*/(\mathbf{k}^*)^2$. There are 4 families which are indexed by a parameter a: $L^a_{6,2}$ and $L^a_{6,12}$, which are indexed by $a \in \Lambda - \{1\}$; $L^a_{6,8}$ and $L^a_{6,17}$, which are indexed by $a \in \Lambda$. Thus, if we denote by r the cardinality of Λ, we obtain $28 + 2(r-1) + 2r = 26 + 4r$ minimal algebras.

If $\mathbf{k}$ is algebraically closed (e.g. $\mathbf{k} = \mathbb{C}$), then there are 30 minimal models over $\mathbf{k}$. We can assume $a = 1$ in lines $L^a_{6,8}$ and $L^a_{6,17}$, while lines $L^a_{6,2}$ and $L^a_{6,12}$ disappear (actually, they are equivalent to lines $L_3 \oplus L_3$ and L_{10} respectively).

Notice that when we set $a = 0$, the minimal algebra $L^a_{6,2}$ reduces to $L_{6,1}$; the minimal algebra $L^a_{6,8}$ reduces to $L_{6,6}$; the minimal algebra $L^a_{6,12}$ reduces to $L_{6,9}$; and the minimal algebra $L^a_{6,17}$ reduces to $L_{6,16}$.

Finally, recall that this classification yields the classification of nilpotent Lie algebras of dimension 6 over $\mathbf{k}$.

2.6 k-homotopy types of 6-dimensional nilmanifolds

In the case $\mathbf{k} = \mathbb{Q}$, the classification in Table 2.2 gives all rational homotopy types of 6-dimensional nilmanifolds. Note that $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is indexed by rational numbers up to squares, hence by $a = \pm p_1 p_2 \dots p_k$, where p_i are different primes, and $k \geq 0$.

Let us explicitly give the classification of real homotopy types of 6-dimensional nilmanifolds. Note that $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{\pm 1\}$. Therefore there are 34 real homotopy types, and we have Table 2.3.

Notice that all these minimal algebras do actually correspond to nilmanifolds, since they are defined over $\mathbb{Q}$.

The fifth column is a labeling of the nilpotent Lie algebra corresponding to the associated minimal algebra; for instance, when we write $L_{5,1} \oplus A_1$ we mean that the 6-dimensional nilpotent Lie algebra splits as the sum of a 5-dimensional nilpotent Lie algebra with an abelian Lie algebra of dimension 1. In geometric terms, the corresponding 6-dimensional nilmanifold is the product of the corresponding 5-dimensional nilmanifold with S^1.

The sixth column refers to the list contained in [23]. In [23], the problem of classifying 6-dimensional nilmanifolds is treated in a different way. Cerezo classifies 6-dimensional nilpotent Lie algebras over $\mathbb{R}$. Let us explain how we derived the correspondence between our list and his. Consider, for example, the nilmanifold with real minimal model associated to the Lie algebra $L_{6,14}$.

The 6-dimensional nilpotent Lie algebra $\mathcal{N}_{6,10}$ considered by Cerezo has generators $\langle X_1, \dots, X_6 \rangle$ and commutators

$$[X_1, X_2] = X_4, \quad [X_1, X_4] = X_5, \quad [X_1, X_5] = X_6,$$
$$[X_2, X_3] = X_6 \quad \text{and} \quad [X_2, X_4] = X_6.$$

Using the correspondence between nilpotent Lie algebras and minimal algebras, according to formula (2.3), we associate the Lie algebra $\mathcal{N}_{6,10}$ to the nilmanifold $L_{6,14}$. To check the other correspondences, it might be necessary to switch variables.

The last columns contain the Betti numbers of the nilmanifolds, and the total dimension of the cohomology. The computation of the Betti numbers has been perfomed using the following facts:

- Thanks to Poincaré duality, we have $b_0 = b_6$, $b_1 = b_5$ and $b_2 = b_4$, where $b_i = \dim H^i(N)$.
- Nilmanifolds are parallelizable and parallelizable manifolds have Euler characteristic zero, so
$$\sum_{i=0}^{n} (-1)^i \, b_i = 0\,. \tag{2.9}$$
- to compute b_3 we use Poincaré duality and (2.9); we obtain
$$b_3 = 2(b_0 - b_1 + b_2). \tag{2.10}$$
- $b_0 = 1$ and $b_1 = f_1$.

Thus it is enough to compute b_2 to obtain the whole information. As an example, we compute the Betti numbers of the nilmanifold $N = L_{6,12}$. We have $b_0 = b_6 = 1$ and $b_1 = b_5 = f_1 = 3$. The computation of b_2 goes as follows: a basis for $\ker d \cap \wedge^2 V$ is given by

$$\langle x_1x_2, x_1x_3, x_1x_4, x_1x_5 + x_2x_6, x_1x_6 - x_2x_5, x_2x_3, x_2x_4, x_3x_4 + x_2x_6 \rangle\,,$$

and $\ker d \cap \wedge^2 V$ is 8-dimensional. On the other hand, $\dim(\operatorname{im} d \cap \wedge^2 V) = n - f_1 = 3$. Thus $b_2 = \dim H^2(N) = 8 - 3 = 5 = b_4$. This gives, according to (2.10), $b_3 = 6$ and $\sum_i b_i = 24$.

Note that $\min \dim H^*(N) = 12$. This agrees with [65], proposition 3.3.

We end up with the proof of Theorem 2.1.

Proof. of Theorem 2.1 If $(\wedge V, d)$ is a minimal model of a nilmanifold, then it is defined over $\mathbb{Q}$. So it is a minimal algebra in Table 1, with the condition that $a \in \mathbb{Q}^*$ if we are dealing with any of the four cases with parameter. (This element a is an invariant of the minimal algebra.)

Now, two nilmanifolds with minimal models $(\wedge V_1, d)$, $(\wedge V_2, d)$ are of the same $\mathbf{k}$-homotopy type if $(\wedge V_1 \otimes \mathbf{k}, d)$ and $(\wedge V_2 \otimes \mathbf{k}, d)$ are isomorphic (over $\mathbf{k}$). Then, first they should be in the same line in Table 1; second, if they correspond to a parameter case, with respective parameters $a_1, a_2 \in \mathbb{Q}^*$, then the $\mathbf{k}$-minimal models are isomorphic if and only if there exists $\lambda \in \mathbf{k}^*$ with $a_1 = \lambda^2 a_2$. Therefore a_1, a_2 define the same class in $\mathbb{Q}^*/((\mathbf{k}^*)^2 \cap \mathbb{Q}^*)$. $\square$

Remark 2.3. A consequence of Theorem 2.1 is that:

1. There are nilmanifolds which have the same real homotopy type but different rational homotopy type.

2. There are nilmanifolds which have the same complex homotopy type but different real homotopy type.

3. There are nilmanifolds M_1, M_2 for which the CDGAs $(\Omega^*(M_1), d)$ and $(\Omega^*(M_2), d)$ are joined by chains of quasi-isomorphisms (i.e., they have the same *real* minimal model), but for which there is no $f : M_1 \to M_2$ inducing a quasi-isomorphism $f^* : (\Omega^*(M_2), d) \to (\Omega^*(M_1), d)$. Just consider M_1, M_2 not of the same rational homotopy type. If there was such f, then there is a map on the rational minimal models $f^* : (\wedge V_2, d) \to (\wedge V_1, d)$ such that $f^*_{\mathbb{R}} : (\wedge V_2 \otimes \mathbb{R}, d) \to (\wedge V_1 \otimes \mathbb{R}, d)$ is an isomorphism. Hence f^* is an isomorphism itself, and M_1, M_2 would be of the same rational homotopy type.

Remark 2.4. The fact that there exist nilpotent Lie algebras that are isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$ was noticed already by Lehmann in [59]. He gave a particular example of two nilpotent 6-dimensional Lie algebras that are isomorphic over $\mathbb{R}$ but not over $\mathbb{Q}$.

2.7 Symplectic nilmanifolds

In this section we study which of the above rational homotopy types of nilmanifolds admit a symplectic structure. The subject is important because symplectic nilmanifolds which are not a torus supply a large source of examples of symplectic non-Kähler manifolds (see for instance [80]).

In the 2-dimensional case we have only the torus T^2 which carries the symplectic area form $\omega = x_1 x_2$.

The three 4-dimensional examples are symplectic. We recall them:

1. $dx_i = 0$ for $i = 1, 2, 3, 4$. Here a symplectic for is given, for instance, by $\omega = x_1 x_2 + x_3 x_4$;

2. $dx_i = 0$ for $i = 1, 2, 3$ and $dx_4 = x_1 x_2$. Here we can take for example $\omega = x_1 x_3 + x_2 x_4$;

3. $dx_i = 0$ for $i = 1, 2$, $dx_3 = x_1 x_2$ and $dx_4 = x_1 x_3$. Take $\omega = x_1 x_4 + x_2 x_3$.

In the 6-dimensional case our approach is based on the following simple remark: if there is a symplectic form, then there is an invariant symplectic form. Let $\omega \in \wedge^2(x_1, \ldots, x_6)$. We can assume that it has rational coefficients, i.e.

$$\omega = \sum_{i<j} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Q}. \tag{2.11}$$

In order for it to be a symplectic form, ω must be closed ($d\omega = 0$) and non-degenerate ($\omega^3 \neq 0$). The second condition implies that ω must be of the form

$$\omega = a_{i_1 i_2} x_{i_1} x_{i_2} + a_{i_3 i_4} x_{i_3} x_{i_4} + a_{i_5 i_6} x_{i_5} x_{i_6} + \omega', \tag{2.12}$$

where $i_1, \ldots, i_6$ is a permutation of $1, 2, 3, 4, 5, 6$. If this is not possible then there is no symplectic form ω and hence no symplectic structure on the associated nilmanifold. We list the symplectic 6-dimensional nilmanifolds in Table 2.4. In the first column we mention the Lie algebra of Table 2.3 associated to the rational homotopy type of the nilmanifold. In the second column either we produce an explicit symplectic form for the type, or we say that there does not exist symplectic structures on it.

As an example of computations, we show that the nilmanifold $L_{5,5} \oplus A_1$ is not symplectic and also how we constructed one possible symplectic form on $L_{6,9}$. The minimal model of $L_{5,5} \oplus A_1$ is $(\wedge V, d)$ with

$$dx_4 = x_1 x_2, \; dx_5 = x_1 x_4 \; \text{ and } dx_6 = x_2 x_4.$$

It is easy to see that the space of closed elements of degree 2 is generated by

$$x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5 + x_1x_6, x_2x_6\,,$$

so ω is a linear combination of these terms. But now, according to (2.12), the subindices $5, 6$ do not go together, and 5 goes either with 1 or 2, whereas 6 goes either with 1 or 2. This implies that $3, 4$ should form a pair, which it is impossible.

To show that some nilmanifold admits some symplectic structure is much easier: it is enough to find a symplectic form. If we take $L_{6,9}$ we have the minimal model $(\wedge V, d)$ with the following differentials:

$$dx_4 = x_1 x_2, \; dx_5 = x_1 x_3 \; \text{ and } dx_6 = x_1 x_4 + x_2 x_3.$$

Now $d(x_1x_6) = d(x_3x_4) = -x_1x_2x_3$ and $d(x_2x_5) = x_1x_2x_3$. Therefore

$$\omega = x_1 x_6 + 2x_2 x_5 + x_3 x_4$$

is closed and we easily see that $\omega^3 = 12\, x_1x_2x_3x_4x_5x_6 \neq 0$. Thus ω is symplectic.

Appendix

This appendix is devoted to the study of the minimal model of commutative differential graded algebras defined over fields of characteristic $p \neq 2$. Let $\mathbf{k}$ be a field of arbitrary characteristic $p \neq 2$.

Theorem 2.3. *Any CDGA (A, d) has a Sullivan model: there exist a minimal algebra $(\wedge V, d)$ (in the sense of the definition given in the introduction) and a quasi-isomorphism $(\wedge V, d) \to (A, d)$.*

Proof. The proof of the existence is the same as in the case of characteristic zero, given in ([31], chapter 14). □

Now we want to study the issue of uniqueness of the minimal model. It is not known in general whether if $(\wedge V, d) \to (A, d)$ and $(\wedge W, d) \to (A, d)$ are two minimal models, then $(\wedge W, d) \cong (\wedge V, d)$ necessarily. This is known in characteristic zero ([88]), but it is an open question in positive characteristic $p \neq 2$ (see [50]).

Here we give a positive answer for the case of CDGAs with a minimal model generated in degree 1. However, some of the results which follow are valid in full generality.

Lemma 2.2. *Let $(\wedge V, d)$ be a minimal algebra and let (A, d) and (B, d) be two CDGAs. Suppose that $f : (\wedge V, d) \to (A, d)$ is a CDGA morphism and that $\pi : (B, d) \to (A, d)$ is a surjective quasi-isomorphism. Then f can be lifted to a CDGA map $g : (\wedge V, d) \to (B, d)$ such that the following diagram is commutative:*

$$\begin{tikzcd} & (B,d) \arrow[d, "\pi"] \\ (\wedge V, d) \arrow[ur, dashed, "g"] \arrow[r, "f"] & (A,d) \end{tikzcd}$$

Moreover, if f is a quasi-isomorphism, then so is g.

Proof. We work inductively. By minimality, there is an increasing filtration $\{V_\mu\}$ of V such that d maps V_μ to $\wedge(V_{<\mu})$ (V_μ is the span of those generators x_τ with $\tau \leq \mu$). Suppose that g has been constructed on $V_{<\mu}$ and consider $x = x_\mu$. Since $dx \in \wedge(V_{<\mu})$, $g(dx)$ is well defined. We want to solve

$$\begin{cases} g(dx) = dy \\ f(x) = \pi(y), \end{cases} \tag{2.13}$$

so that we can set $g(x) = y$.

There is some $b \in B$ such that $\pi(b) = f(x)$. Then $\pi(g(dx)) = f(dx) = d(f(x)) = d(\pi(b)) = \pi(db)$, so $c = g(dx) - db \in \ker \pi$. We compute $dc = d(g(dx)) = 0$, so c is closed. But $[c] \in H^*(B) \cong H^*(A)$ and $\pi(c) = 0$, so $[c] = 0$, i.e. there is some $e \in B$ such that $c = de$. Now $d\pi(e) = \pi(c) = 0$, so $\pi(e)$ is closed and $[\pi(e)] \in H^*(A) \cong H^*(B)$. Hence there is some closed $\beta \in B$ and $\alpha \in A$ such that $\pi(e) = \pi(\beta) + d\alpha$. Using the surjectivity of π again, $\alpha = \pi(\psi)$, for some $\psi \in B$. So $\pi(e) = \pi(\beta + d\psi)$. Now take $y = b + e - \beta - d\psi$. Clearly $\pi(y) = \pi(b) = f(x)$ and $dy = db + de = g(dx)$. Now suppose that f is a quasi-isomorphism and denote f_* and π_* the maps induced by f and π respectively at cohomology level. One has $f = \pi \circ g$, hence $f_* = \pi_* \circ g_*$; thus $g_* = \pi_*^{-1} \circ f_*$ is also an isomorphism. □

Now we particularise to minimal algebras generated in degree 1. In this case, we do not need surjectivity to prove a lifting property.

Theorem 2.4. *Let $(\wedge V, d)$ be a minimal algebra generated in degree 1 (i.e. $V = V^1$), and let (A, d) and (B, d) be two CDGAs. Suppose that $A^0 = \mathbf{k}$. If $f : (\wedge V, d) \to (A, d)$ is a CDGA morphism and $\psi : (B, d) \to (A, d)$ is a quasi-isomorphism, then there exists a CDGA map $g : (\wedge V, d) \to (B, d)$ such that $\psi \circ g = f$.*

Moreover, if f is a quasi-isomorphism, then so is g.

Proof. We work as in the proof of lemma 2.2. Consider generators $\{x_\tau\}$ of $V = V^1$. Assume that g has been defined for $V_{<\mu}$, and let $x = x_\mu$. Since $dx \in \wedge^2(V_{<\mu})$, $g(dx)$ is well defined. As before, we want to solve (2.13).

Now $d(g(dx)) = g(dd(x)) = 0$, so $[g(dx)] \in H^2(B, d)$. But $\psi_*[g(dx)] = [\psi(g(dx))] = [f(dx)] = [d(f(x))] = 0$, so $[g(dx)] = 0$. Therefore, there exists $\xi \in B^1$ such that $g(dx) = d\xi$. Now $d(\psi(\xi)) = \psi(g(dx)) = f(dx) = d(f(x))$, so $\psi(\xi) - f(x) \in A^1$ is closed. As $A^0 = \mathbf{k}$, we have that $H^1(A, d) = Z^1(A, d) = \ker(d : A^1 \to A^2)$. Clearly the quasi-isomorphism $\psi : (B, d) \to (A, d)$ gives a surjective map $Z^1(B, d) \to Z^1(A, d)$. Therefore, there exists $b \in Z^1(B, d) \subset B^1$ such that $\psi(\xi) - f(x) = \psi(b)$. Take $y = \xi - b$, to solve (2.13).

□

Lemma 2.3. *Suppose $\varphi : (\wedge V, d) \to (\wedge W, d)$ is a quasi-isomorphism between minimal algebras. Then φ is an isomorphism.*

Proof. We can assume inductively that $\wedge(V^{<n}) \cong \wedge(W^{<n})$. We first show that $\varphi : \wedge(V^{\leq n}) \to \wedge(W^{\leq n})$ is injective. It is enough to see that the composition $\bar{\varphi} : V^n \to (\wedge W^{\leq n})^n \to W^n$ is injective. Suppose $v \in V^n$ satisfies $\bar{\varphi}(v) = 0$. Then there exists $v' \in \wedge(W^{<n}) \cong \wedge(V^{<n})$ such that $\varphi(v) = \varphi(v')$. Then $\varphi(v'') = 0$, where $v'' = v - v'$. Then

$$0 = d(\varphi(v'')) = \varphi(dv'').$$

Thus $dv'' = 0$. Since φ is a quasi-isomorphism and $\varphi^*[v''] = 0$, we have that $v'' = d(v''')$ for some $v''' \in (\wedge V)^{n-1}$; but this is impossible since $\wedge V$ is a minimal algebra.

Now we prove the surjectivity of $\varphi : \wedge(V^{\leq n}) \to \wedge(W^{\leq n})$. First note that the minimality condition means the existence of an increasing filtration V_i^n such that $d(V_i^n) \subset \wedge(V^{<n} \oplus V_{i-1}^n)$ (and an analogous filtration W_i^n for W^n). We assume by induction that $\wedge(V^{<n} \oplus V_{i-1}^n) \cong \wedge(W^{<n} \oplus W_{i-1}^n)$. Consider

$$\mathcal{V}_i = V_i^n \oplus \wedge(V^{<n} \oplus V_{i-1}^n).$$

These are differential vector subspaces. Write $\mathcal{V}_i \hookrightarrow \wedge V \to C$, where C is the cokernel. Then C has only terms of degree $\geq n$. Moreover if we take the filtration with V_i^n maximal (i.e. $\mathcal{V}_i = d^{-1}(\wedge(V^{<n} \oplus V_{<i}^n))$, then $H^n(C) = 0$. This implies that $H^{\leq n}(\mathcal{V}_i) \cong H^{\leq n}(\wedge V)$ and $H^{n+1}(\mathcal{V}_i) \hookrightarrow H^{n+1}(\wedge V)$.

We define analogously $\mathcal{W}_i = W_i^n \oplus \wedge(W^{<n} \oplus W_{i-1}^n)$. Clearly $\varphi : \mathcal{V}_i \to \mathcal{W}_i$. We have an exact sequence $0 \to \mathcal{V}_i \to \mathcal{W}_i \to Q \to 0$, where $Q = W_i^n / V_i^n$ is the cokernel. Again, Q does not have terms of degree $< n$. Also d on Q^n is zero, so $H^n(Q) = Q^n$. Note that the isomorphism $H^*(\wedge V) \cong H^*(\wedge W)$ implies that $H^{\leq n}(\mathcal{V}_i) \cong H^{\leq n}(\mathcal{W}_i)$ and $H^{n+1}(\mathcal{V}_i) \hookrightarrow H^{n+1}(\mathcal{W}_i)$. The long exact sequence in cohomology gives $H^n(Q) = Q^n = 0$, and hence $\mathcal{V}_i \cong \mathcal{W}_i$, which completes the induction. □

This gives us the uniqueness of the minimal model for the CDGAs that we are interested in.

Theorem 2.5. *Let (A, d) be a CDGA, defined over a field $\mathbf{k}$ of characteristic $p \neq 2$, such that $A^0 = \mathbf{k}$. Suppose that its minimal model $\varphi : (\wedge V, d) \to (A, d)$ satisfies that $(\wedge V, d)$ is a minimal algebra generated in degree 1. If $(\wedge W, d) \to (A, d)$ is another minimal model for (A, d), then $(\wedge W, d) \cong (\wedge V, d)$.*

Proof. By Theorem 2.4, there exists a quasi-isomorphism $g : (\wedge V, d) \to (\wedge W, d)$. By Lemma 2.3, g is an isomorphism. □

We have the following refinement.

Corollary 2.2. *Consider the category of CDGAs (A, d) with $A^0 = \mathbf{k}$ and whose minimal model is generated in degree 1. If two of such CDGAs (A, d) and (B, d) are quasi-isomorphic, then they have the same minimal model.*

Proof. Without loss of generality, we may assume that there is a quasi-isomorphism $\psi : (B, d) \to (A, d)$. If $\varphi : (\wedge V, d) \to (A, d)$ is a minimal model for (A, d) then there exists a quasi-isomorphism $g : (\wedge V, d) \to (B, d)$. Any other minimal model of (B, d) is isomorphic to $(\wedge V, d)$ by Theorem 2.5. □

Table 2.2: Classification of minimal algebras over $\mathbf{k}$

(f_i)	dx_3	dx_4	dx_5	dx_6	$\mathfrak{g}$
(6,0)	0	0	0	0	A_6
(5,1)	0	0	0	x_1x_2	$L_3 \oplus A_3$
	0	0	0	$x_1x_2 + x_3x_4$	$L_{5,1} \oplus A_1$
(4,2)	0	0	x_1x_2	x_1x_3	$L_{5,2} \oplus A_1$
	0	0	x_1x_2	x_3x_4	$L_3 \oplus L_3$
	0	0	x_1x_2	$x_1x_3 + x_2x_4$	$L_{6,1}$
	0	0	$x_1x_3 + ax_2x_4$	$x_1x_4 + x_2x_3$	$L^a_{6,2}, a \in \Lambda - \{1\}$
(4,1,1)	0	0	x_1x_2	x_1x_5	$L_4 \oplus A_2$
	0	0	x_1x_2	$x_1x_5 + x_3x_4$	$L_{6,3}$
	0	0	x_1x_2	$x_1x_5 + x_2x_3$	$L_{5,3} \oplus A_1$
(3,3)	0	x_1x_2	x_1x_3	x_2x_3	$L_{6,4}$
(3,2,1)	0	x_1x_2	x_1x_3	x_1x_4	$L_{6,5}$
	0	x_1x_2	x_1x_3	x_2x_4	$L_{6,6}$
	0	x_1x_2	x_1x_3	$x_1x_5 + x_2x_4$	$L_{6,7}$
	0	x_1x_2	x_1x_3	$x_2x_4 + ax_3x_5$	$L^a_{6,8}, a \in \Lambda$
	0	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{6,9}$
(3,1,2)	0	x_1x_2	x_1x_4	x_2x_4	$L_{5,5} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_2x_3 + x_2x_4$	$L_{6,10}$
	0	x_1x_2	x_1x_4	$x_1x_3 + x_2x_4$	$L_{6,11}$
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_3 + ax_2x_4$	$L^a_{6,12}, a \in \Lambda - \{1\}$
(3,1,1,1)	0	x_1x_2	x_1x_4	x_1x_5	$L_{5,4} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,13}$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_4$	$L_{5,6} \oplus A_1$
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3 + x_2x_4$	$L_{6,14}$
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_5 - x_3x_4$	$L_{6,15}$
(2,1,2,1)	x_1x_2	x_1x_3	x_2x_3	x_1x_4	$L_{6,16}$
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 + ax_2x_5$	$L^a_{6,17}, a \in \Lambda$
(2,1,1,1,1)	x_1x_2	x_1x_3	x_1x_4	x_1x_5	$L_{6,18}$
	x_1x_2	x_1x_3	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,19}$
	x_1x_2	x_1x_3	x_1x_4	$x_2x_5 - x_3x_4$	$L_{6,20}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_1x_5 + x_2x_4$	$L_{6,21}$
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_2x_5 - x_3x_4$	$L_{6,22}$

Table 2.3: Real homotopy types of 6-dimensional nilmanifolds

(f_i)	dx_3	dx_4	dx_5	dx_6	$\mathfrak{g}$	[23]	b_1	b_2	b_3	$\sum_i b_i$
(6,0)	0	0	0	0	A_6	–	6	15	20	64
(5,1)	0	0	0	x_1x_2	$L_3 \oplus A_3$	–	5	11	14	48
	0	0	0	$x_1x_2 + x_3x_4$	$L_{5,1} \oplus A_1$	–	5	9	10	40
(4,2)	0	0	x_1x_2	x_1x_3	$L_{5,2} \oplus A_1$	–	4	9	12	40
	0	0	x_1x_2	x_3x_4	$L_3 \oplus L_3$	–	4	8	10	36
	0	0	x_1x_2	$x_1x_3 + x_2x_4$	$L_{6,1}$	$\mathcal{N}_{6,24}$	4	8	10	36
	0	0	$x_1x_3 - x_2x_4$	$x_1x_4 + x_2x_3$	$L_{6,2}$	$\mathcal{N}_{6,23}$	4	8	10	36
(4,1,1)	0	0	x_1x_2	x_1x_5	$L_4 \oplus A_2$	–	4	7	8	32
	0	0	x_1x_2	$x_1x_5 + x_3x_4$	$L_{6,3}$	$\mathcal{N}_{6,22}$	4	6	6	28
	0	0	x_1x_2	$x_1x_5 + x_2x_3$	$L_{5,3} \oplus A_1$	–	4	7	8	32
(3,3)	0	x_1x_2	x_1x_3	x_2x_3	$L_{6,4}$	$\mathcal{N}_{6,21}$	3	8	12	36
(3,2,1)	0	x_1x_2	x_1x_3	x_1x_4	$L_{6,5}$	$\mathcal{N}_{6,20}$	3	6	8	28
	0	x_1x_2	x_1x_3	x_2x_4	$L_{6,6}$	$\mathcal{N}_{6,18}$	3	6	8	28
	0	x_1x_2	x_1x_3	$x_1x_5 + x_2x_4$	$L_{6,7}$	$\mathcal{N}_{6,17}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_2x_4 + x_3x_5$	$L_{6,8}^+$	$\mathcal{N}_{6,15}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_2x_4 - x_3x_5$	$L_{6,8}^-$	$\mathcal{N}_{6,16}$	3	5	6	24
	0	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$L_{6,9}$	$\mathcal{N}_{6,19}$	3	6	8	28
(3,1,2)	0	x_1x_2	x_1x_4	x_2x_4	$L_{5,5} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_2x_3 + x_2x_4$	$L_{6,10}$	$\mathcal{N}_{6,12}$	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_3 + x_2x_4$	$L_{6,11}$	$\mathcal{N}_{6,13}$	3	5	6	24
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_3 - x_2x_4$	$L_{6,12}$	$\mathcal{N}_{6,14}$	3	5	6	24
(3,1,1,1)	0	x_1x_2	x_1x_4	x_1x_5	$L_{5,4} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,13}$	$\mathcal{N}_{6,11}$	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_4$	$L_{5,6} \oplus A_1$	–	3	5	6	24
	0	x_1x_2	x_1x_4	$x_1x_5 + x_2x_3 + x_2x_4$	$L_{6,14}$	$\mathcal{N}_{6,10}$	3	5	6	24
	0	x_1x_2	$x_1x_4 + x_2x_3$	$x_1x_5 - x_3x_4$	$L_{6,15}$	$\mathcal{N}_{6,9}$	3	4	4	20
(2,1,2,1)	x_1x_2	x_1x_3	x_2x_3	x_1x_4	$L_{6,16}$	$\mathcal{N}_{6,8}$	2	4	6	20
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 + x_2x_5$	$L_{6,17}^+$	$\mathcal{N}_{6,6}$	2	4	6	20
	x_1x_2	x_1x_3	x_2x_3	$x_1x_4 - x_2x_5$	$L_{6,17}^-$	$\mathcal{N}_{6,7}$	2	4	6	20
(2,1,1,1,1)	x_1x_2	x_1x_3	x_1x_4	x_1x_5	$L_{6,18}$	$\mathcal{N}_{6,5}$	2	3	4	16
	x_1x_2	x_1x_3	x_1x_4	$x_1x_5 + x_2x_3$	$L_{6,19}$	$\mathcal{N}_{6,4}$	2	3	4	16
	x_1x_2	x_1x_3	x_1x_4	$x_2x_5 - x_3x_4$	$L_{6,20}$	$\mathcal{N}_{6,2}$	2	2	2	12
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_1x_5 + x_2x_4$	$L_{6,21}$	$\mathcal{N}_{6,3}$	2	3	4	16
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	$x_2x_5 - x_3x_4$	$L_{6,22}$	$\mathcal{N}_{6,1}$	2	2	2	12

Table 2.4: Symplectic 6-dimensional nilmanifolds

Type	**Symplectic form**	**Type**	**Symplectic form**
A_6	$x_1x_2 + x_3x_4 + x_5x_6$	$L_{5,5} \oplus A_1$	Not symplectic
$L_3 \oplus A_3$	$x_1x_6 + x_2x_3 + x_4x_5$	$L_{6,10}$	$x_1x_6 + x_2x_5 - x_3x_4$
$L_{5,1} \oplus A_1$	Not symplectic	$L_{6,11}$	$x_1x_5 + x_2x_6 + x_3x_4$
$L_{5,2} \oplus A_1$	$x_1x_5 + x_2x_4 + x_3x_6$	$L_{6,12}$	$x_1x_6 + 2x_2x_5 + x_3x_4$
$L_3 \oplus L_3$	$x_1x_5 + x_3x_6 + x_2x_4$	$L_{5,4} \oplus A_1$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{6,1}$	$x_1x_3 + x_2x_6 + x_3x_5$	$L_{6,13}$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{6,2}$	$x_1x_6 + x_2x_5 + x_3x_4$	$L_{5,6} \oplus A_1$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_4 \oplus A_2$	$x_1x_6 + x_2x_5 + x_3x_4$	$L_{6,14}$	$x_1x_3 + x_2x_6 - x_4x_5$
$L_{5,3} \oplus A_1$	$x_1x_6 + x_2x_4 - x_3x_5$	$L_{6,15}$	$x_1x_4 + x_2x_6 + x_3x_5$
$L_{6,3}$	Not symplectic	$L_{6,16}$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,4}$	$x_1x_4 + x_2x_6 + x_3x_5$	$L_{6,17}^+$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,5}$	$x_1x_6 + x_2x_4 + x_3x_5$	$L_{6,17}^-$	$x_1x_6 + x_1x_5 + x_2x_4 + x_3x_5$
$L_{6,6}$	$x_1x_4 + x_2x_6 + x_3x_5$	$L_{6,18}$	$x_1x_6 + x_2x_5 - x_3x_4$
$L_{6,7}$	Not symplectic	$L_{6,19}$	$x_1x_6 + x_2x_4 + x_2x_5 - x_3x_4$
$L_{6,8}^+$	Not symplectic	$L_{6,20}$	Not symplectic
$L_{6,8}^-$	Not symplectic	$L_{6,21}$	$2x_1x_6 + x_2x_5 + x_3x_4$
$L_{6,9}$	$x_1x_6 + 2x_2x_5 + x_3x_4$	$L_{6,22}$	Not symplectic

CHAPTER
THREE

MINIMAL ALGEBRAS AND 2−STEP NILPOTENT LIE ALGEBRAS IN DIMENSION 7

Giovanni Bazzoni

Abstract

We use the methods of [7] to give a classification of 7−dimensional minimal algebras, generated in degree 1, over any field $\mathbf{k}$ of characteristic $\mathrm{char}(\mathbf{k}) \neq 2$, whose characteristic filtration has length 2. Equivalently, we classify 2−step nilpotent Lie algebras in dimension 7. This classification also recovers the real homotopy type of 7−dimensional 2−step nilmanifolds.

MSC classification [2010]: Primary 55P62, 17B30; Secondary 22E25, 11E04.

Key words: Nilmanifolds, rational homotopy, nilpotent Lie algebras, minimal model.

3.1 Introduction and Main Theorem

In this paper we classify some minimal algebras of dimension 7 generated in degree 1 over a field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2$. More specifically, we focus on minimal algebras whose characteristic filtration has length 2. This recovers the classification of 2−step nilpotent Lie algebras over $\mathbf{k}$ in dimension 7. This classification had already been obtained over the fields $\mathbb{C}$ and $\mathbb{R}$ (see for instance [26], [43], [44] or [67]), but the result over arbitrary fields is original (see [84] for partial results over finite fields). When the field $\mathbf{k}$ has characteristic zero, we obtain a classification of 2−step nilmanifolds in dimension 7, up to $\mathbf{k}$−homotopy type. The approach to this classification problem is different from others. Indeed, the starting point is the classification of minimal algebras as examples of homotopy types of nilmanifolds. A similar approach, though, was used in the beautiful paper [87].

Table 3.1: Minimal algebras of dimension 7 and length 2 over any field

(f_0, f_1)	dx_5	dx_6	dx_7
(6,1)	0	0	x_1x_2
(6,1)	0	0	$x_1x_2 + x_3x_4$
(6,1)	0	0	$x_1x_2 + x_3x_4 + x_5x_6$
(5,2)	0	x_1x_2	x_1x_3
(5,2)	0	x_1x_2	x_3x_4
(5,2)	0	x_1x_2	$x_1x_3 + x_2x_4$
(5,2)	0	x_1x_2	$x_1x_3 + x_4x_5$
(5,2)	0	$x_1x_2 + x_3x_4$	$x_1x_3 + x_2x_5$
(5,2)	0	$x_1x_3 + \alpha x_2x_4$	$x_1x_4 + x_2x_3$
(4,3)	x_1x_2	x_1x_3	x_1x_4
(4,3)	x_1x_2	x_1x_3	x_2x_3
(4,3)	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$
(4,3)	x_1x_2	x_3x_4	x_1x_3
(4,3)	x_1x_2	x_3x_4	$x_1x_3 + x_2x_4$
(4,3)	$x_1x_4 + x_2x_3$	$\alpha x_1x_3 + x_2x_4$	x_1x_2
(4,3)	$x_1x_4 + x_2x_3$	$ax_1x_3 + x_2x_4$	$x_1x_2 - bx_3x_4$

The main theorem is stated in terms of $7-$dimensional minimal algebras generated in degree 1 of length 2.

Theorem 3.1. *There are $10 + 2r + s$ isomorphism classes of minimal algebras of dimension 7 and length 2, generated in degree 1, over a field $\mathbf{k}$ of characteristic different from two; r is the cardinality of the square class group $\mathbf{k}^*/(\mathbf{k}^*)^2$ and s is the number of non-isomorphic quaternion algebras over $\mathbf{k}$. In particular, when $\mathbf{k}$ is algebraically closed, $r = s = 1$ and there 13 non-isomorphic minimal algebras; when $\mathbf{k} = \mathbb{R}$, $r = s = 2$ and there are 16.*

Table 3.1 above contains a list of $7-$dimensional minimal algebras of length 2, generated in degree 1, over any field $\mathbf{k}$. Every line contains one isomorphism class of minimal algebra. The legend is as follows; let V be the vector space which generates the minimal algebra and let $\{x_1, \ldots, x_7\}$ be a set of generators. Then f_0 is the dimension of the $\ker(d)$, $d : V \to \wedge^2 V$, and f_1 is the dimension of a complementary subspace. The other three columns contain the expression of the non-zero differential in terms of the chosen basis. See section 3.2 for details.

The parameter α varies in $\mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. The pair (a, b) which appears in the last algebra varies in $\mathbf{k}^* \times \mathbf{k}^*$ and two pairs give the same minimal algebra if and only if the corresponding quaternion algebras are isomorphic (see section 3.6 below).

This paper is organized as follows. In the section 3.2 we recall all the relevant algebraic and topological definitions (minimal algebras, nilpotent Lie algebras, nilmanifolds). In the following sections we proceed with the classification, which is accomplished by a case-by-case study.

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3.2 Preliminaries

A *commutative differential graded algebra* (CDGA, for short) over a field $\mathbf{k}$ (of characteristic $\mathrm{char}(\mathbf{k}) \neq 2$) is a graded $\mathbf{k}$-algebra $A = \oplus_{k\geq 0} A^k$ such that $xy = (-1)^{|x||y|} yx$, for homogeneous elements x, y, where $|x|$ denotes the degree of x, and endowed with a differential $d : A^k \to A^{k+1}$, $k \geq 0$, satisfying the graded Leibniz rule

$$d(xy) = (dx)y + (-1)^{|x|} x(dy) \tag{3.1}$$

for homogeneous elements x, y. Given a CDGA (A, d), one can compute its cohomology, and the cohomology algebra $H^*(A)$ is itself a CDGA with zero differential. A CDGA is said to be *connected* if $H^0(A) \cong \mathbf{k}$. A *CDGA morphism* between CDGAs (A, d) and (B, d) is an algebra morphism which preserves the degree and commutes with the differential.

A *minimal algebra* is a CDGA (A, d) of the following form:

1. A is the free commutative graded algebra $\wedge V$ over a graded vector space $V = \oplus V^i$,

2. there exists a collection of generators $\{x_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(x_\mu) \leq \deg(x_\tau)$ if $\mu < \tau$ and each dx_τ is expressed in terms of preceding x_μ ($\mu < \tau$). This implies that dx_τ does not have a linear part.

We have the following fundamental result: every connected CDGA (A, d) has a *minimal model*; this means that there exists a minimal algebra $(\wedge V, d)$

together with a CDGA morphism

$$\varphi : (\wedge V, d) \to (A, d)$$

which induces an isomorphism on cohomology. The minimal model of a CDGA over a field $\mathbf{k}$ of characteristic zero is unique up to isomorphism. The corresponding result for fields of arbitrary characteristic is not known: in fact, existence is proved in exactly in the same way as for characteristic zero, but the uniqueness is an open question. For a study of minimal models over fields of arbitrary characteristic, see for instance [50]. In [7], uniqueness is proved for minimal algebras generated in degree 1.

The *dimension* of a minimal algebra is the dimension over $\mathbf{k}$ of the graded vector space V. We say that a minimal algebra is *generated in degree k* if the vector space V is concentrated in degree k. In this paper we will focus on minimal algebras of dimension 7 generated in degree 1.

We turn to nilpotent Lie algebras; there is a precise correspondence between minimal algebras generated in degree 1 and nilpotent Lie algebras.

Given a Lie algebra $\mathfrak{g}$, we define the lower central series of $\mathfrak{g}$ as follows:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \text{and} \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}].$$

A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there exists a positive integer n such that $\mathfrak{g}^{(n)} = \{0\}$. In particular, the nilpotency condition implies that $\mathfrak{g}^{(1)} \subset \mathfrak{g}^{(0)}$.

Lemma 3.1. *If $\mathfrak{g}$ is a nilpotent Lie algebra then $\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \ldots \supset \mathfrak{g}^{(n)} = \{0\}$.*

Proof. As we noticed above, $\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)}$. We suppose inductively that $\mathfrak{g}^{(k-1)} \supset \mathfrak{g}^{(k)}$ and show that $\mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)}$: in fact,

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}] \subset [\mathfrak{g}, \mathfrak{g}^{(k-1)}] = \mathfrak{g}^{(k)}.$$

□

One can form the quotients

$$E_k = \mathfrak{g}^{(k)} / \mathfrak{g}^{(k+1)} \tag{3.2}$$

and write $\mathfrak{g} = \oplus_k E_k$, but the splitting is not canonical. Nevertheless the numbers $e_k := \dim(E_k)$ are invariants of the lower central series. Notice that $e_k = 0$ eventually.

A nilpotent Lie algebra is called *$m-$step nilpotent* if $\mathfrak{g}^{(m)} = \{0\}$ and $\mathfrak{g}^{(m-1)} \neq \{0\}$. Notice that if $\mathfrak{g}$ is $m-$step nilpotent then the last nonzero term of the central series, $\mathfrak{g}^{(m-1)}$, is contained in the center of $\mathfrak{g}$. In this paper we classify nilpotent Lie algebras in dimension 7 which are 2−step nilpotent. For more details, see [44].

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. It is possible to choose a basis $\{X_1, \dots, X_n\}$ for $\mathfrak{g}$, called Mal'cev basis, such that the Lie brackets can be written as follows:

$$[X_i, X_j] = \sum_{k>i,j} a_{ij}^k X_k. \tag{3.3}$$

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\{X_1, \dots, X_n\}$ be a Mal'cev basis. Consider the dual vector space $\mathfrak{g}^*$ with the dual basis $\{x_1, \dots, x_n\}$, i.e., $x_i(X_j) = \delta_j^i$. We can endow $\mathfrak{g}^*$ with a differential d, defined according to the Lie bracket structure of $\mathfrak{g}$. Namely, we define

$$dx_k = -\sum_{k>i,j} a_{ij}^k x_i \wedge x_j\,. \tag{3.4}$$

We will usually omit the exterior product sign. $\wedge\mathfrak{g}^*$ is the exterior algebra of $\mathfrak{g}^*$, which we assume to be a vector space concentrated in degree 1; we extend the differential d to $\wedge\mathfrak{g}^*$ by imposing the graded Leibnitz rule (3.1). The CDGA $(\wedge\mathfrak{g}^*, d)$ is the Chevalley-Eilenberg complex associated to $\mathfrak{g}$. When $\mathfrak{g}$ is nilpotent, the formula for the differential (3.4) shows that $(\wedge\mathfrak{g}^*, d)$ is a minimal algebra, according to the above definition. Therefore, the Chevalley-Eilenberg complex of a nilpotent Lie algebra is a minimal algebra generated in degree 1.

Let $(\wedge V, d)$ be a minimal algebra generated in degree 1; in particular, the case of our interest is when $(\wedge V, d) = (\wedge\mathfrak{g}^*, d)$ is the Chevalley-Eilenberg complex associated to a nilpotent Lie algebra $\mathfrak{g}$. We define the following subsets of V:

$$\begin{aligned} W_0 &= \ker(d) \cap V \\ W_k &= d^{-1}(\wedge^2 W_{k-1}),\ \text{for } k \geq 1\,. \end{aligned}$$

Lemma 3.2. *For any $k \geq 0$, $W_k \subset W_{k+1}$.*

Proof. First notice that $W_0 \subset W_1$ since $W_0 = d^{-1}(0)$. By induction, suppose that $W_{k-1} \subset W_k$; then we have

$$d(W_k) = d(d^{-1}(\wedge^2 W_{k-1})) \subset \wedge^2 W_{k-1} \subset \wedge^2 W_k\,.$$

This proves that $W_k \subset W_{k+1}$, as required. □

In particular, $W_0 \subset W_1 \subset \ldots \subset W_m = V$ is an increasing filtration of V, which we call *characteristic filtration.* The *length* of the filtration is, by definition, the least k such that $W_{k-1} = V$. In general, we will say that a minimal algebra generated in degree 1, $(\wedge V, d)$, has length n if its characteristic filtration has length n. Define

$$\begin{aligned} F_0 &= W_0 \\ F_k &= W_k / W_{k-1} \text{ for } k \geq 1\,. \end{aligned}$$

Then one can write $V = \oplus_k F_k$, although not in a canonical way. Nevertheless, the numbers $f_k = \dim(F_k)$ are invariants of V. Notice that $f_k = 0$ eventually, and the length of the filtration coincides with the least k such that $f_k = 0$. In case $(\wedge V, d) = (\wedge \mathfrak{g}^*, d)$ one has $F_k = E_k^*$, where the E_k are defined in (3.2).

The differential

$$d : W_{k+1} \longrightarrow \wedge^2(F_0 \oplus \ldots \oplus F_k)$$

can be decomposed according to the following diagram:

$$\begin{tikzcd}
W_{k+1} \arrow[r, "d"] \arrow[d, two heads] & \wedge^2 W_k \arrow[r, "\simeq"] \arrow[d, two heads] & \wedge^2(W_k) \simeq \wedge^2(W_{k-1}) \oplus ((W_{k-1}) \otimes F_k) \arrow[d, two heads] \\
F_{k+1} \arrow[r, "\bar{d}"] & \wedge^2 W_k / \wedge^2 W_{k-1} \arrow[r, "\simeq"] & (F_0 \oplus \ldots \oplus F_{k-1}) \otimes F_k
\end{tikzcd}$$

where the map

$$\bar{d} : F_{k+1} \to (F_0 \oplus \ldots \oplus F_{k-1}) \otimes F_k$$

is injective.

Lemma 3.3. *A nilpotent Lie algebra* $\mathfrak{g}$ *is* $n-$*step nilpotent if and only if the characteristic filtration* $\{W_k\}$ *of* $\mathfrak{g}^*$ *has length* n.

Proof. We argue by induction. Suppose that $\mathfrak{g}$ is $1-$step nilpotent. Then $\mathfrak{g}$ is abelian and formula (3.4), which relates brackets in $\mathfrak{g}$ with differential in $\mathfrak{g}^*$, says that the differential d is identically zero on $\mathfrak{g}^*$. Therefore $W_0 = \mathfrak{g}^*$ and the characteristic filtration has length 1. The converse is also clear. Now assume that $\mathfrak{g}$ is $n-$step nilpotent. Set $\tilde{\mathfrak{g}} := \mathfrak{g}/\mathfrak{g}^{(n-1)}$; then $\tilde{\mathfrak{g}}$ is an $(n-1)-$step nilpotent Lie algebra, thus the characteristic filtration of $\tilde{\mathfrak{g}}^*$ has length $n-1$ by the inductive hypothesis. One has then

$$\tilde{\mathfrak{g}}^* = \left(\mathfrak{g}/\mathfrak{g}^{(n-1)}\right)^* = \mathrm{Ann}(\mathfrak{g}^{(n-1)})$$

and $\mathfrak{g}^* = \tilde{\mathfrak{g}}^* \oplus (\mathfrak{g}^{(n-1)})^*$. As we remarked above, this splitting is not canonical, but shows that the length of the characteristic filtration of $\tilde{\mathfrak{g}}^*$ is n. The other way is similar and straightforward. □

To sum up, in order to classify $2-$step nilpotent Lie algebras in dimension 7 we can classify minimal algebras in dimension 7, generated in degree 1, such that the corresponding filtration has length 2.

If $(\wedge \mathfrak{g}^*, d)$ is a minimal algebra generated in degree 1, of length 2, one can write $\mathfrak{g}^* = F_0 \oplus F_1$, where d is identically zero on F_0 and $d : F_1 \hookrightarrow \wedge^2 F_0$. Given a vector $v \in F_1$, we say that $dv \in \wedge^2 F_0$ is a *bivector*. When $\mathfrak{g}^*$ is 7 dimensional, we must handle the following pairs of numbers:

$$(f_0, f_1) = (6, 1),\ (5, 2) \text{ and } (4, 3).$$

There are no other possibilities; for instance $(3,4)$ can not be because $\dim(\wedge^2 F_0) = 3 \le 4 = \dim(F_1)$ and there can be no injective map $F_1 \to \wedge^2 F_0$.

We will make systematic use of the following result:

Lemma 3.4. *Let W be a vector space of dimension k over a field $\mathbf{k}$ whose characteristic is different from 2. Given any element $\varphi \in \wedge^2 W$, there is a (not unique) basis $x_1, \ldots, x_k$ of W such that $\varphi = x_1 \wedge x_2 + \ldots + x_{2r-1} \wedge x_{2r}$, for some $r \ge 0$, $2r \le k$. The $2r$-dimensional space $\langle x_1, \ldots, x_{2r} \rangle \subset W$ is well-defined (independent of the basis).*

Proof. Interpret φ as a skew-symmetric bilinear map $W^* \times W^* \to \mathbf{k}$. Let $2r$ be its rank, and consider a basis $e_1, \ldots, e_k$ of W^* such that $\varphi(e_{2i-1}, e_{2i}) = 1$, $1 \le i \le r$, and the other pairings are zero. Then the dual basis $x_1, \ldots, x_k$ does the job. $\square$

Finally, we relate our algebraic classification to the classification of rational homotopy types of 7−dimensional 2−step nilmanifolds. The bridge from algebra to topology is provided by rational homotopy theory. In the seminal paper [88], Sullivan showed that it is possible to associate to any nilpotent CW-complex X a CDGA, defined over the rational numbers $\mathbb{Q}$, which encodes the rational homotopy type of X.

More precisely, let X be a nilpotent space of the homotopy type of a CW-complex of finite type over $\mathbb{Q}$ (all spaces considered in this paper are of this kind). A space is *nilpotent* if $\pi_1(X)$ is a nilpotent group and it acts in a nilpotent way on $\pi_k(X)$ for $k > 1$. The *rationalization* of X (see [47]) is a rational space $X_{\mathbb{Q}}$ (i.e., a space whose homotopy groups are rational vector spaces) together with a map $X \to X_{\mathbb{Q}}$ inducing isomorphisms $\pi_k(X) \otimes \mathbb{Q} \to \pi_k(X_{\mathbb{Q}})$ for $k \ge 1$ (recall that the rationalization of a nilpotent group is well-defined - see for instance [47]). Two spaces X and Y have the same *rational homotopy type* if their rationalizations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ have the same homotopy type, i.e. if there exists a map $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ inducing isomorphisms in homotopy groups. Sullivan constructed a $1-1$ correspondence between nilpotent rational spaces and isomorphism classes of minimal algebras over $\mathbb{Q}$:

$$X \leftrightarrow (\wedge V_X, d)\,.$$

The minimal algebra $(\wedge V_X, d)$ is the *minimal model* of the space X.

We recall the notion of $\mathbf{k}-$homotopy type for a field $\mathbf{k}$ of characteristic 0, given in [7]. The $\mathbf{k}-$minimal model of a space X is $(\wedge V_X \otimes \mathbf{k}, d)$. We say that X and Y have the same $\mathbf{k}-$homotopy type if and only if the $\mathbf{k}-$minimal models $(\wedge V_X \otimes \mathbf{k}, d)$ and $(\wedge V_Y \otimes \mathbf{k}, d)$ are isomorphic.

A *nilmanifold* is a quotient $N = G/\Gamma$ of a nilpotent, simply connected Lie group by a discrete co-compact subgroup Γ, such that the resulting quotient is compact ([80]). According to Nomizu theorem ([77]), the minimal model of

N is precisely the Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, d)$ of the nilpotent Lie algebra $\mathfrak{g}$ of G. Here, $\mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{Q})$. Mal'cev proved that the existence of a basis $\{X_i\}$ of $\mathfrak{g}$ with *rational* structure constants a^i_{jk} in (3.3) is equivalent to the existence of a co-compact $\Gamma \subset G$. The minimal model of the nilmanifold $N = G/\Gamma$ is

$$(\wedge(x_1, \ldots, x_n), d),$$

where $V = \langle x_1, \ldots, x_n\rangle = \oplus_{i=1}^n \mathbb{Q}x_i$ is the vector space generated by $x_1, \ldots, x_n$ over $\mathbb{Q}$, with $|x_i| = 1$ for every $i = 1, \ldots, n$ and dx_i is defined according to (3.4). We say that $N = G/\Gamma$ is an $m-$step nilmanifold if $\mathfrak{g}$ is an $m-$step nilpotent Lie algebra.

From this we see that the algebraic classification of $7-$dimensional minimal algebras generated in degree 1 of length 2 over a field $\mathbf{k}$ of characteristic 0 gives the classification of $2-$step nilmanifolds of dimension 7 up to $\mathbf{k}-$homotopy type. It is important here to remark that the knowledge of explicit examples of nilmanifolds is useful when one wants to endow nilmanifolds with extra geometrical structures; for instance, in dimension 7, one may think of nilmanifolds with a G_2 structure (see [26]).

3.3 Case $(6, 1)$

The space F_0 is $6-$dimensional and the differential $d : F_1 \to \wedge^2 F_0$ gives a bivector $\varphi_7 \in \wedge^2 F_0$; its only invariant is the rank, which can be 2, 4 or 6. We choose a generator x_7 for F_1 and generators $x_1, \ldots, x_6$ for F_0. According to the above lemma 3.4, we have 3 cases:

rank 2 $dx_7 = x_1x_2$;

rank 4 $dx_7 = x_1x_2 + x_3x_4$;

rank 6 $dx_7 = x_1x_2 + x_3x_4 + x_5x_6$;

We remark that this description is valid over any field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2$.

3.4 Case $(5, 2)$

The space F_0 has dimension 5 and F_1 has dimension 2. The differential is an injective map $d : F_1 \hookrightarrow \wedge^2 F_0$; the latter is a $10-$dimensional vector space. The image of d gives two linearly indipendent bivectors φ_6, φ_7 spanning a plane in $\wedge^2 F_0$ or, equivalently, a line ℓ in $\mathbb{P}^9 = \mathbb{P}(\wedge^2 F_0)$. The rank of the bivectors can be 2 or 4. The indecomposable (i.e., rank 2) bivectors in $\wedge^2 F_0$ are parametrized by the Grassmannian $\mathrm{Gr}(2, F_0)$ of $2-$planes in F_0. Under the Plücker embedding, this Grassmannian is sent to a $6-$dimensional subvariety $\mathscr{X} \subset \mathbb{P}^9$ of degree 5. The algebraic classification problem leads us to the geometric study of the mutual position of a line ℓ and the smooth projective variety $\mathscr{X}$ in $\mathbb{P}^9$. The next proposition describes the possible cases, assuming that $\mathbf{k}$ is algebraically closed. The case in which $\mathbf{k}$ is not algebraically closed will be treated separately.

Proposition 3.1. *Let V be a vector space of dimension 5 over an algebraically closed field $\mathbf{k}$. Let $\mathscr{X}$ denote the Plücker embedding of the Grassmannian $\mathrm{Gr}(2, V)$ in $\mathbb{P}^9 = \mathbb{P}(\wedge^2 V)$ and let $\ell \subset \mathbb{P}^9$ be a projective line. Then one and only one of the following possibilities occurs:*

1. *the line ℓ and $\mathscr{X}$ are disjoint;*
2. *the line ℓ is contained in $\mathscr{X}$;*
3. *the line ℓ is tangent to $\mathscr{X}$;*
4. *the line ℓ is bisecant to $\mathscr{X}$.*

Proof. As we said before, $\mathscr{X}$ is a 6−dimensional smooth subvariety of $\mathbb{P}^9$ of degree 5; by degree and dimension, a generic $\mathbb{P}^3$ cuts $\mathscr{X}$ in 5 points, but a generic $\mathbb{P}^2$ need not meet it. The same is also clearly true for a generic line ℓ. Thus there are lines in $\mathbb{P}^9$ disjoint from $\mathscr{X}$.

Let $W \subset V$ be a 4 dimensional vector subspace. This gives embeddings $\mathbb{P}^5 = \mathbb{P}(\wedge^2 W) \hookrightarrow \mathbb{P}^9 = \mathbb{P}(\wedge^2 V)$ and $\mathrm{Gr}(2, W) \hookrightarrow \mathscr{X}$. The Grassmannian $\mathrm{Gr}(2, W)$ is a smooth quadric in $\mathbb{P}^5$, and has the property that through any point there are two 2−planes contained in it. In particular, $\mathrm{Gr}(2, W)$ contains a line ℓ, and so does $\mathscr{X}$. On the other hand, if ℓ is contained in this $\mathbb{P}^5$ then, by dimension and degree reasons, it cuts the quadric $\mathrm{Gr}(2, W)$, and hence $\mathscr{X}$, in two points.

Let $p \in \mathscr{X}$ be a point and consider the projective tangent space to $\mathbb{T}_p\mathscr{X}$. If the line ℓ is contained in this $\mathbb{P}^6$, and $p \in \ell$, but ℓ is not contained in $\mathscr{X}$ (such a line exists because $\mathscr{X}$ is not linear), then ℓ is tangent to $\mathscr{X}$.

To conclude, we show that there are no trisecant lines to $\mathscr{X}$. Indeed, suppose that a line $\ell \subset \mathbb{P}^9$ cuts the Grassmannian in three points. We may assume that ℓ is the projectivization of a vector subspace $U \subset \wedge^2 V$ of dimension 2, spanned by bivectors ϕ_1 and ϕ_2 such that $\mathbb{P}(\phi_1)$ and $\mathbb{P}(\phi_2)$ are two of the three points of intersection of ℓ with $\mathscr{X}$; then the rank of the bivectors ϕ_1 and ϕ_2 is 2 and they give two 2−planes π_1 and π_2 in V. The fact that there is a third intersection point between ℓ and $\mathscr{X}$ means that there exists exactly one linear combination $a\phi_1 + b\phi_2$, with $a, b \in \mathbf{k}^*$, which has rank 2, while all the other linear combination have rank 4. But the planes π_1 and π_2 either meet in the origin or they intersect in a line. In the first case, all linear combinations $a\phi_1 + b\phi_2$, $a, b \in \mathbf{k}^*$, have rank 4, in the second one they have all rank 2. □

3.4.1 $\ell \cap \mathscr{X} = \emptyset$

The two bivectors have rank 4. If $\langle \varphi_6, \varphi_7 \rangle$ is a basis of $\mathrm{im}(d) \subset \wedge^2 F_0$, then φ_j is a symplectic form on some 4−plane $H_j \subset F_0$, $j = 6, 7$ (here we are somehow identifying F_0 with its dual, but this is not a problem, since all the vectors are defined modulo scalars). Suppose first that $H_6 = H_7$; then we have two rank 4 bivectors on a 4−dimensional vector space $H := H_6$; consider the inclusion $H \hookrightarrow F_0$, which gives $\wedge^2 H \hookrightarrow \wedge^2 F_0$ and, projectivizing, $\mathbb{P}(\wedge^2 H) \hookrightarrow \mathbb{P}(\wedge^2 F_0)$. The rank 2 bivectors in $\wedge^2 H$ are parametrized by the Grassmannian $\mathrm{Gr}(2, H)$ which, as we noticed above, is a quadric hypersurface in $\mathbb{P}(\wedge^2 H)$. The two bivectors φ_6 and φ_7 give a projective line ℓ contained in $\mathbb{P}(\wedge^2 H)$. For dimension reasons, any line in $\mathbb{P}(\wedge^2 H)$ meets this quadric hypersurface[1]; therefore we can always choose coordinates in H in such a way that at least on bivector has rank 2. But our hypothesis is that both bivectors have rank 4 and this implies that $H_6 \neq H_7$. We set $V = H_6 \cap H_7$; the Grassmann formula says that $\dim(V) = 3$. Notice that (H_6, φ_6) and (H_7, φ_7) are 4−dimensional symplectic vector spaces.

Lemma 3.5. *If (W, ω) is a symplectic vector space and $U \subset W$ is a codimension 1 subspace, then U is coisotropic, i.e., the symplectic orthogonal U^ω of U is contained in U.*

Proof. The dimension of U^ω is 1. If $U^\omega \not\subseteq U$ we can write $W = U \oplus U^\omega$ for dimension reasons. But this is impossible, because ω would descend to a symplectic form on U^ω. □

This shows that V is a coisotropic subspace of both H_6 and H_7. The differential d gives a map $h : F_1 \to F_0/V$, defined up to nonzero scalars; we choose vectors v_6 and v_7 spanning F_0/V and set $x_j = h^{-1}(v_j)$, $j = 6, 7$. We choose generators x_1, x_2 and x_3 for V and rename $v_6 = x_4$, $v_7 = x_5$. Thus we get

$$H_6 = \langle x_1, x_2, x_3, x_4 \rangle, \quad H_7 = \langle x_1, x_2, x_3, x_5 \rangle.$$

We can write $\varphi_6 = x_1 x_2 + x_3 x_4$. This choice implies that the plane $\pi = \langle x_1, x_2 \rangle \subset V$ is symplectic for φ_6. If it was also symplectic for φ_7, we could write

$$dx_6 = \varphi_6 = x_1 x_2 + x_3 x_4 \quad \text{and} \quad dx_7 = \varphi_7 = x_1 x_2 + x_3 x_5.$$

But then setting $x_4' = x_4 - x_5$, the bivector $\varphi' = \varphi_6 - \varphi_7 = x_3(x_4 - x_5) = x_3 x_4'$ would have rank 2, and this is not possible. The plane π must therefore be Lagrangian for φ_7 and consequently $\varphi_7 = x_1 x_3 + x_2 x_5$. This gives finally

$$\begin{cases} dx_6 &= x_1 x_2 + x_3 x_4 \\ dx_7 &= x_1 x_3 + x_2 x_5 \end{cases}$$

[1] Here we are using the fact that $\mathbf{k}$ is algebraically closed

3.4.2 $\ell \subset \mathscr{X}$

This means that both φ_6 and φ_7 have rank 2. They give two planes π_6 and π_7 in F_0, which can not coincide: either their intersection is just the origin, or they share a line. But the first case does not show up; indeed, in that case we could take coordinates $\{x_1, \ldots, x_5\}$ in F_0 so that $dx_6 = x_1x_2$ and $dx_7 = x_3x_4$. Then all bivectors $a\varphi_6 + b\varphi_7$, $ab \in \mathbf{k}^*$, would have rank 4, contradicting the assumption that $\ell \subset \mathscr{X}$. This implies that $\pi_6 \cap \pi_7$ is a line, which we suppose spanned by a vector x_1. We complete this to a basis $\langle x_1, x_2\rangle$ of π_6 and $\langle x_1, x_3\rangle$ of π_7, giving at the end

$$\begin{cases} dx_6 &= x_1x_2 \\ dx_7 &= x_1x_3 \end{cases}$$

3.4.3 $\ell \cap \mathscr{X} = \{p, q\}$

This case is complementary to case $\ell \subset \mathscr{X}$ above. In fact, we still have two rank 2 bivectors φ_6 and φ_7, but every linear combination $a\varphi_6 + b\varphi_7$, $ab \in \mathbf{k}^*$, must now have rank 4. Thus, arguing as we did there, we exclude the case in which the 2−planes associated by φ_6 and φ_7 intersect in a line and conclude that they intersect in the origin. Then the expression of the differentials is

$$\begin{cases} dx_6 &= x_1x_2 \\ dx_7 &= x_3x_4 \end{cases}$$

3.4.4 $\ell \cap \mathscr{X} = \{p\}$

In this case the line ℓ is tangent to $\mathscr{X}$. The point p identifies a rank 2 bivector in $\wedge^2 F_0$, while all the other bivectors on ℓ have rank 4. This gives a symplectic 2−plane (π_6, φ_6) and a symplectic 4−plane (π_7, φ_7) in F_0. π_6 can not be contained in π_7 as a symplectic subspace; in fact, if this was the case, we could choose coordinates $\{x_1, \ldots, x_4\}$ in π_7 in such a way that $\pi_6 = \langle x_1, x_2\rangle$, $\varphi_6 = x_1x_2$ and $\varphi_7 = x_1x_2 + x_3x_4$; but then the bivector $\varphi' = \varphi_7 - \varphi_6$ would belong to ℓ and have rank 2, which is impossible since ℓ containes only one rank 2 bivector. Then either $\pi_6 \subset \pi_7$ as a Lagrangian subspace, or Grassmann's formula says that $\dim(\pi_6 \cap \pi_7) = 1$ and the subspaces meet along a line. In the first case we choose vectors x_1, x_2, x_3, x_4 spanning π_7; then we can write

$$\begin{cases} dx_6 &= x_1x_2 \\ dx_7 &= x_1x_3 + x_2x_4 \end{cases}$$

In the second case, call x_1 a generator of this line. We can complete this to a basis of π_6 and to a basis of π_7. In particular, we set

$$\pi_6 = \langle x_1, x_2\rangle \quad \text{and} \quad \pi_7 = \langle x_1, x_3, x_4, x_5\rangle$$

and we obtain the following expression for the differentials:

$$\begin{cases} dx_6 &= x_1x_2 \\ dx_7 &= x_1x_3 + x_4x_5 \end{cases}$$

3.4.5 k non algebraically closed

Finally we discuss the case in which the field $\mathbf{k}$ is non-algebraically closed. Going through the above list, one sees that there are two points where the field comes into play. More specifically, in case (4) of proposition 3.1 above, it could happen that ℓ and $\mathscr{X}$ intersect in two points with coordinates in the algebraic closure of $\mathbf{k}$. As this intersection is invariant by the Galois group, there must be a quadratic extension $\mathbf{k}' \supset \mathbf{k}$ where the coordinates of the two points lie; the two points are conjugate by the Galois automorphism of $\mathbf{k}'|\mathbf{k}$. Therefore, there is an element $a \in \mathbf{k}^*$ such that $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, a is not a square in $\mathbf{k}$, and the differentials

$$dx_6 = x_1x_2, \qquad dx_7 = x_3x_4.$$

satisfy that the planes $\pi_6 = \langle x_1, x_2 \rangle$ and $\pi_7 = \langle x_3, x_4 \rangle$ are conjugate under the Galois map $\sqrt{a} \mapsto -\sqrt{a}$. Write:

$$\begin{aligned} x_1 &= y_1 + \sqrt{a}y_2, \\ x_2 &= y_3 + \sqrt{a}y_4, \\ x_3 &= y_1 - \sqrt{a}y_2, \\ x_4 &= y_3 - \sqrt{a}y_4, \\ x_5 &= y_5 \\ x_6 &= y_6 + \sqrt{a}y_7, \\ x_7 &= y_6 - \sqrt{a}y_7, \end{aligned}$$

where $y_1, \ldots, y_7$ are defined over $\mathbf{k}$. Then $dy_6 = y_1y_3 + ay_2y_4$, $dy_7 = y_1y_4 + y_2y_3$. This is the canonical model. Two of these minimal algebras are not isomorphic over $\mathbf{k}$ for different quadratic field extensions, since the equivalence would be given by a $\mathbf{k}$-isomorphism, therefore commuting with the action of the Galois group. The quadratic field extensions are parametrized by elements $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. Note that for $a = 1$, setting $z_6 = y_6 + y_7$ and $z_7 = y_6 - y_7$, we recover case (4) of proposition 3.1, where $dz_6 = (y_1 + y_2)(y_3 + y_4)$ and $dz_7 = (y_1 - y_2)(y_3 - y_4)$ are of rank 2. The model in this case is

$$\begin{cases} dx_6 &= x_1x_3 + ax_2x_4 \\ dx_7 &= x_1x_4 + x_2x_3 \end{cases}, \quad a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}. \tag{3.5}$$

The other point where the field comes into play is in subsection (3.4.3). There, in order to exclude the possibility $H_6 = H_7$, we used the fact that $\mathbf{k}$ is algebraically closed. Again, if $\mathbf{k}$ is not algebraically closed, we can argue as above and deduce that there exists a quadratic extension $\mathbf{k}'' = \mathbf{k}(\sqrt{b})$ with b a nonsquare in $\mathbf{k}$, such that the two intersection points are interchanged by the action of the Galois automorphism of $\mathbf{k}''|\mathbf{k}$. The model in this case coincides with (3.5).

3.5 Case $(4, 3)$

In this case F_0 has dimension 4, F_1 has dimension 3 and the differential $d : F_1 \hookrightarrow \wedge^2 F_0$ determines three linearly independent bivectors φ_5, φ_6 and φ_7 in $\wedge^2 F_0$, spanning a $3-$dimensional vector subspace $d(F_1) \subset \wedge^2 F_0$; the rank of the bivectors can be 2 or 4. Taking the projectivization, we obtain a projective plane $\pi = \mathbb{P}(d(F_1)) \subset \mathbb{P}(\wedge^2 F_0)$. The indecomposable bivectors in $\wedge^2 F_0$ are parametrized by the Grassmannian $\mathrm{Gr}(2, F_0)$ of $2-$planes in F_0. Under the Plücker embedding, this Grassmannian is sent to a quadric hypersurface $\mathcal{Q} \subset \mathbb{P}^5$, known as Klein quadric. As it happened in the previous section, the algebraic classification problem leads us to the geometric study of the mutual position of a plane π and the Klein quadric $\mathcal{Q}$ in projective space $\mathbb{P}^5$. In the next lemmas we study this geometry, assuming that $\mathbf{k}$ is algebraically closed. The case in which $\mathbf{k}$ is non-algebraically closed will be treated separately.

In what follows, we fix a $4-$dimensional vector space V over an algebraically closed field $\mathbf{k}$ and we denote by $\mathcal{Q}$ the Plücker embedding of the Grassmannian $\mathrm{Gr}(2, 4)$ in projective space $\mathbb{P}^5 = \mathbb{P}(\wedge^2 V)$.

Lemma 3.6. *Let $p \in \mathcal{Q}$ be a point; there exist two planes π_1 and π_2 such that $\pi_1 \cap \pi_2 = \{p\}$ and contained in $\mathcal{Q}$.*

Proof. We take homogeneous coordinates $[X_0 : \ldots : X_5]$ in $\mathbb{P}^5$. The Klein quadric $\mathcal{Q}$ is given as the zero locus of the homogeneous quadratic equation $X_0X_5 - X_1X_4 + X_2X_3$. Since $\mathcal{Q}$ is homogeneous, we can assume that p is the point $[1 : 0 : 0 : 0 : 0 : 0] \in \mathcal{Q}$; the planes π_1 and π_2 have equations $X_2 = X_4 = X_5 = 0$ and $X_1 = X_3 = X_5 = 0$. □

Lemma 3.7. *Let $p \in \mathcal{Q}$ be a point and let $\mathbb{T}_p\mathcal{Q} \cong \mathbb{P}^4$ be the projective tangent space to $\mathcal{Q}$ at p. Let $\pi \subset \mathbb{T}_p\mathcal{Q}$ be a $2-$plane, with $p \in \pi$. Then one of the following possibilities occurs:*

1. *$\pi \subset \mathcal{Q}$;*
2. *$\pi \cap \mathcal{Q}$ is a double line;*
3. *$\pi \cap \mathcal{Q}$ is a pair lines.*

Proof. Take homogeneous coordinates $[X_0 : \ldots : X_5]$ in $\mathbb{P}^5$; as above, the Klein quadric is the zero locus of the quadratic equation $X_0X_5 - X_1X_4 +$

X_2X_3. We can assume again that $p = [1 : 0 : \ldots : 0]$. The tangent space $\mathbb{T}_p\mathcal{Q} \cong \mathbb{P}^4$ has equation $X_5 = 0$ and intersects $\mathcal{Q}$ along the quadric $X_1X_4 - X_2X_3 = 0$. Its rank is 4, thus it is a cone over a smooth quadric $\mathcal{C}$ in $\mathbb{P}^3$, with vertex in p. The equation of $\mathcal{C}$ is $X_1X_4 - X_2X_3 = 0$ in this $\mathbb{P}^3 = \{X_0 = X_5 = 0\}$; then $\mathcal{C} \cong \mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding, and it contains a line. The plane π intersects this $\mathbb{P}^3$ in a line ℓ, which can be contained in $\mathcal{C}$, or tangent to $\mathcal{C}$ or bisecant to $\mathcal{C}$. In the first case, the whole plane π is contained in the quadric $\mathcal{Q}$, since $\mathcal{Q}$ contains ℓ, the point p and all the lines joining p to ℓ. In the second case $\pi \cap \mathcal{Q}$ is a double line; indeed, the cone over $\mathcal{C}$ intersected with π is just one line, counted with multiplicity. In the third case, π contains the cone over two points, which is a pair of lines. □

These two lemmas cover the cases in which the 2−plane is in special position. The general case (i.e., the case of a *generic* projective plane in $\mathbb{P}^5$) is that the intersection between the plane and the Klein quadric is a smooth conic. We collect these results in the next proposition:

Proposition 3.2. *Let V be a vector space of dimension 4 over an algebraically closed field* $\mathbf{k}$. *Let $\mathcal{Q}$ denote the Plücker embedding of the Grassmannian $\mathrm{Gr}(2, V)$ in $\mathbb{P}^5 = \mathbb{P}(\wedge^2 V)$ and let $\pi \subset \mathbb{P}^5$ be a projective plane. Then one and only one of the following possibilities occurs:*

1. *the plane π is contained in $\mathcal{Q}$;*
2. *the plane π is tangent to $\mathcal{Q}$, and $\pi \cap \mathcal{Q}$ is either a double line or two lines;*
3. *the plane π cuts $\mathcal{Q}$ along a smooth conic.*

According to this proposition, we study the various cases.

3.5.1 $\pi \subset \mathcal{Q}$

Let V be a vector space of dimension 4 over the field $\mathbf{k}$. Recall that the Plücker embedding maps $\mathrm{Gr}(2, V)$ onto the Klein quadric $\mathcal{Q} \subset \mathbb{P}(\wedge^2 V)$. In the previous section we proved that given a point $p \in \mathcal{Q}$ there exist two skew planes $\mathbb{P}^2$ contained in $\mathcal{Q}$ and such that p belongs to both. Now we describe these planes more precisely.

Lemma 3.8. *Let $\ell \subset V$ be a line and denote by $\Sigma_\ell \subset \mathrm{Gr}(2, V)$ the locus of 2−planes in V containing ℓ; given a hyperplane $W \subset V$, we denote with $\Sigma_W \subset \mathrm{Gr}(2, V)$ the locus of 2−planes in V contained in W. Under the Plücker embedding Σ_ℓ and Σ_W are carried to projective 2−planes $\mathbb{P}^2 \subset \mathcal{Q}$; conversely, every projective 2−plane $\mathbb{P}^2 \subset \mathcal{Q}$ is equal to the image under the Plücker embedding of either Σ_ℓ or Σ_W.*

Proof. Let us start with the first case. We fix a line $\ell \subset V$ and take a hyperplane U such that $\ell \oplus U = V$. A 2−plane must intersect U along a line is a line r and then Σ_ℓ is in bijection with the space of lines in U, which is a projective plane $\mathbb{P}^2$. The other case is easier: we have an inclusion $\Sigma_W \hookrightarrow \mathrm{Gr}(2, V)$, and Σ_W is a projective plane $\mathbb{P}^2$ (more precisely, $(\mathbb{P}^2)^*$). The converse is also easy to see. □

If the projective plane π is contained in the quadric, the three bivectors φ_5, φ_6 and φ_7 have rank 2 and any linear combination of them also has rank 2. They give three planes π_5, π_6 and π_7 in F_0. According to lemma 3.8, we have two possibilities:

- π is associated to 2−dimensional vector subspaces of F_0 containing a given line $r \subset F_0$. In this case, we choose a vector x_1 spanning r and complete it to a basis of each plane, obtaining $\pi_5 = \langle x_1, x_2 \rangle$, $\pi_6 = \langle x_1, x_3 \rangle$ and $\pi_7 = \langle x_1, x_4 \rangle$. In term of differentials,

$$\begin{cases} dx_5 &= x_1x_2 \\ dx_6 &= x_1x_3 \\ dx_7 &= x_1x_4 \end{cases}$$

- π is associated to 2−dimensional vector subspaces of F_0 contained in a given hyperplane $W \subset F_0$. We can take coordinates so that $W = \langle x_1, x_2, x_3 \rangle$ and set $\pi_5 = \langle x_1, x_2 \rangle$, $\pi_6 = \langle x_1, x_3 \rangle$ and $\pi_7 = \langle x_2, x_3 \rangle$. This gives the model

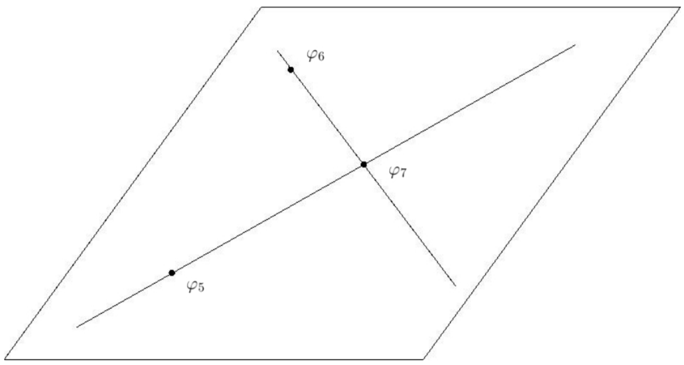
$$\begin{cases} dx_5 &= x_1x_2 \\ dx_6 &= x_1x_3 \\ dx_7 &= x_2x_3 \end{cases}$$

3.5.2 $\pi \cap \mathcal{Q}$ is a double line

We can suppose that φ_5 and φ_6 are on ℓ, but φ_7 is not (recall that the three points can not be collinear). Then every linear combination $a\varphi_5 + b\varphi_6$ has rank 2 and, arguing as above, the corresponding planes π_5 and π_6 in F_0 intersect along some line $r \subset F_0$. Since $\varphi_7 \notin \ell$, it has rank 4 and it is then a symplectic form in F_0. The lines ℓ_5 and ℓ_6, joining φ_7 with φ_5 and φ_6 respectively, are tangent to the Klein quadric $\mathcal{Q}$, thus their points are bivectors of rank 4 except for φ_5 and φ_6. Arguing as in case $\ell \cap \mathscr{X} = \{p\}$ of section (3.4.1), we deduce that the planes π_5 and π_6 are Lagrangian for the symplectic form φ_7 and we can choose coordinates in F_0 to arrange $\varphi_5 = x_1x_2$, $\varphi_6 = x_1x_3$ and $\varphi_7 = x_1x_4 + x_2x_3$. This gives the model

$$\begin{cases} dx_5 &= x_1x_2 \\ dx_6 &= x_1x_3 \\ dx_7 &= x_1x_4 + x_2x_3 \end{cases}$$

Figure 3.1: The two incident lines in the tangent plane



3.5.3 $\pi \cap \mathcal{Q}$ is a pair of lines

We call p the intersection point of the two lines and we assume that $\varphi_7 = p$, so that φ_7 has rank 2. Notice that φ_5, φ_6 and φ_7 span π, thus they can not be collinear. We change the basis in F_1 so that φ_5, φ_6 and φ_7 are as in figure (3.1); the three bivectors have rank 2 and give three 2−planes π_5, π_6 and π_7 in F_0. The projective lines ℓ_5 and ℓ_6, joining φ_7 with φ_5 and φ_6 respectively, are contained in $\mathcal{Q}$, but the line $r = \langle \varphi_5, \varphi_6 \rangle$ is not. This means that any linear combination $a_5\varphi_5 + a_7\varphi_7$ and $b_6\varphi_6 + b_7\varphi_7$ has rank 2 ($a_5, a_7, b_6, b_7 \in \mathbf{k}$) while any combination $c_5\varphi_5 + c_6\varphi_6$, $c_5 \cdot c_6 \in \mathbf{k}^*$ has rank 4. Going back to F_0, we get $\pi_5 \cap \pi_7 = \ell_1$ and $\pi_6 \cap \pi_7 = \ell_2$, while $\pi_5 \oplus \pi_6 = F_0$. We choose vectors x_1 spanning ℓ_1 and x_3 spanning ℓ_2, so that $\pi_7 = \langle x_1, x_3 \rangle$; then we complete x_1 to a basis $\langle x_1, x_2 \rangle$ of π_5 and x_3 to a basis $\langle x_3, x_4 \rangle$ of π_6. This gives the model

$$\left\{ \begin{array}{lcl} dx_5 & = & x_1x_2 \\ dx_6 & = & x_3x_4 \\ dx_7 & = & x_1x_3 \end{array} \right. \tag{3.6}$$

3.5.4 $\pi \cap \mathcal{Q}$ is a smooth conic

We call $\mathcal{C}$ this conic and we choose the points φ_5, φ_6 on $\mathcal{C}$. φ_7 is chosen as the intersection point between the tangent lines to the conic $\mathcal{C}$ at φ_5 and φ_6. The bivectors φ_5 and φ_6 have rank 2, while φ_7 has rank 4. We denote π_5 and π_6 the planes in F_0 associated to φ_5 and φ_6 respectively. The projective line $\ell = \langle \varphi_5, \varphi_6 \rangle$ contains rank 4 bivectors, except for φ_5 and φ_6: any form $a\varphi_5 + b\varphi_6$, $a \cdot b \neq 0$ has rank 4. We take coordinates in F_0 so that $\varphi_5 = x_1x_2$ and $\varphi_6 = x_3x_4$. Using these coordinates we can write

$$\varphi_7 = x_1x_3 + \alpha x_1x_4 + \beta x_2x_3 + \mathfrak{g}x_2x_4 = x_1(x_3 + \alpha x_4) + x_2(\beta x_3 + \mathfrak{g}x_4);$$

consider the change of variables $y_3 = x_3 + \alpha x_4$, $y_4 = \beta x_3 + \mathfrak{g}x_4$; then, scaling x_6, one sees that the resulting model is

$$\begin{cases} dy_5 &= & y_1y_2 \\ dy_6 &= & y_3y_4 \\ dy_7 &= & y_1y_3 + y_2y_4. \end{cases}$$

A generic point φ in the plane π may be written as $\varphi = X\varphi_5 + Y\varphi_6 + Z\varphi_7$ for $X, Y, Z \in \mathbf{k}$. Then φ has rank 2 if and only if $\varphi \wedge \varphi = 0$. Computing, we obtain the conic $XY - Z^2 = 0$ in $\mathbb{P}^2$. In our models, we have taken φ_5 and φ_6 as the intersection between the conic $XY - Z^2 = 0$ and the line $Z = 0$; the points φ_7 has been chosen as intersection point between the two tangent lines to the conic at φ_5 and φ_6. Notice that over an algebraically closed field, all smooth conics are equivalent.

3.6 Case $(4,3)$ when k is not algebraically closed

In this section we study the case $(4,3)$ when the ground field $\mathbf{k}$ is not algebraically closed. In what follows, $\mathbb{P}^n$ will always denote $\mathbb{P}^n_{\mathbf{k}}$.

When the plane π is contained in the Klein quadric, the fact that $\mathbf{k}$ is not algebraically closed does not matter. But it does matter when the π cuts the Klein quadric in a (not necessarily smooth) conic. Indeed, the classification of conics over non-algebraically closed fields is nontrivial.

This section is organized as follows: first, we find a normal form for a conic in $\mathbb{P}^2$. Then, according to this normal form, we show that any conic may be obtained as intersection between a plane $\pi \subset \mathbb{P}^5$ and the Klein quadric $\mathcal{Q}$; we also show how to recover the minimal algebra from the conic. Finally we give a criterion to decide whether two conics are isometric.

We start with the classification of conics. Fix a $3-$dimensional vector space W over $\mathbf{k}$ such that $\mathbb{P}^2 = \mathbb{P}(W)$. If $\mathcal{C} \subset \mathbb{P}^2$ is a conic, taking coordinates $[X_0 : X_1 : X_2]$ in $\mathbb{P}^2$ we can write $\mathcal{C}$ as the zero locus of a quadratic homogeneous polynomial

$$P(X_0, X_1, X_2) = \sum_{i \leq j} a_{ij} X_i X_j.$$

To $\mathcal{C}$ we may associate the quadratic form Q defined on W by the matrix $A = (a_{ij})$. A very well known theorem in linear algebra asserts that every quadratic form can be diagonalized by congruency. This means that there exists a basis of W such that the matrix $B = (b_{ij})$ associated to Q in this basis is diagonal and $B = P^t AP$ for an invertible matrix P. In this basis we can write the quadratic form as

$$Q(Y_0, Y_1, Y_2) = \alpha Y_0^2 - \beta Y_1^2 - \gamma Y_2^2$$

for suitable coefficients α, β and $\mathfrak{g}$ in $\mathbf{k}$ ($\pm$ the eigenvalues of the matrix A). Suppose that Q_1 and Q_2 are two quadratic forms with associated matrices A_1 and A_2; then Q_1 and Q_2 are are *isometric* if there exists a nonsingular matrix P such that $A_2 = P^t A_1 P$. Since we exclude the case $Q \equiv 0$, we may assume $\alpha \neq 0$. The conic $\mathcal{C}$ is the zero locus of the polynomial λQ for every $\lambda \in \mathbf{k}^*$; multiplying Q by α^{-1}, we may assume that $\mathcal{C}$ is given as zero locus of the polynomial

$$P(Y_0, Y_1, Y_2) = Y_0^2 - aY_1^2 - bY_2^2, \quad a, b \in \mathbf{k}. \tag{3.7}$$

We take this to be canonical form of a conic. Two conics written in the canonical form are isomorphic if and only if the corresponding quadratic forms are isometric. A first step in the classification is given by the *rank* of the conic, which is defined as the rank of the associated symmetric matrix.

rank 1 If $a = b = 0$ we obtain the double line $Y_0^2 = 0$;

rank 2 if $b = 0$ but $a \neq 0$ we obtain the "pair of lines" $Y_0^2 - aY_1^2 = 0$;

rank 3 if $a \cdot b \neq 0$ we obtain the smooth conic $Y_0^2 - aY_1^2 - bY_2^2 = 0$.

Lemma 3.9. *Any conic can be obtained as the intersection in $\mathbb{P}^5$ between the Klein quadric and a suitable plane π.*

Proof. Le $\mathcal{C}$ be the conic defined by the equation $X^2 - aY^2 - bZ^2 = 0$, where $[X : Y : Z]$ are homogeneous coordinates in $\mathbb{P}^2$ and $a, b \in \mathbf{k}$. If we take coordinates $[X_0 : \ldots : X_5]$ in $\mathbb{P}^5$, the Klein quadric $\mathcal{Q}$ is given by the equation $X_0X_5 - X_1X_4 + X_2X_3 = 0$. Consider in $\mathbb{P}^5$ the plane $\pi \cong \mathbb{P}^2$ of equations

$$X_0 - X_5 = 0, \quad X_1 - aX_4 = 0 \quad \text{and} \quad X_2 + bX_3 = 0.$$

Then $\pi \cap \mathcal{Q}$ is given by $X^2 - aY^2 - bZ^2 = 0$. $\square$

When $\mathbf{k}$ is algebraically closed (or in case every element of $\mathbf{k}$ is a square), the geometry of the Klein quadric $\mathcal{Q}$ and of the projective plane determines the minimal algebras. In fact, the differential $d : F_1 \to \wedge^2 F_0$ gives a plane $\pi \subset \mathbb{P}^5$ and proposition 3.2 gives all the possible positions of π with respect to $\mathcal{Q}$. From each of these positions we have deduced the corresponding minimal algebra. As we said above, over a non algebraically closed field the classification of conics is more complicated and more care is needed. In particular, it is not anymore true that every conic has *rational* points, where by rational point we mean points in $\mathbf{k}$. The problem of determining which conics have rational points is equivalent to the problem of determining whether the quadratic form Q associated to $\mathcal{C}$ is isotropic, this is, if there exists a vector $v \in W$ such that $Q(v) = 0$. If a conic defined over $\mathbf{k}$ has no rational points, it might not be possible to choose representatives of rank 2 for the bivectors. Notice however that the rank 1 conic always has rational points. We need to discuss the rank 2 and rank 3 cases.

3.6.1 Rank 2 conics

Any rank 2 conic can be put in the form $X^2 - aY^2 = 0$. If a is not a square in $\mathbf{k}^*$ the conic has just one rational point, $p = [0 : 0 : 1]$. There is a quadratic extension $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, such that $X^2 - aY^2 = (X - \sqrt{a}Y)(X + \sqrt{a}Y) = 0$ and the conic splits as two intersecting lines in $\mathbb{P}^2_{\mathbf{k}'}$. The quadratic field extensions are parametrized by elements $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. The Galois group $\mathrm{Gal}(\mathbf{k}' : \mathbf{k})$ permutes the two lines; the intersection point is fixed by this action, and thus already in $\mathbb{P}^2$; it is the point p above.

We set $F_0' = F_0 \otimes \mathbf{k}'$; the plane $\pi' = \pi \otimes \mathbf{k}'$ is spanned by three bivectors φ_5, φ_6 and φ_7 in $\wedge^2 F_0'$. We choose $\varphi_7 = p$ and suppose that φ_5 and φ_6 are conjugated by the action of the Galois group. These points represent rank 2 bivectors, hence planes π_5, π_6 and π_7 in F_0'. We take vectors $x_1, \dots, x_4$ so that $\pi_5 = \langle x_1, x_2 \rangle$, $\pi_6 = \langle x_3, x_4 \rangle$ and $\pi_7 = \langle x_1, x_3 \rangle$, see (3.5.3). The model over $\mathbf{k}'$ is

$$\begin{cases} dx_5 &= x_1 x_2 \\ dx_6 &= x_3 x_4 \\ dx_7 &= x_1 x_3 \end{cases}$$

Now write

$$\begin{cases} x_1 &= \sqrt{a} y_1 + y_2, \\ x_2 &= \sqrt{a} y_3 + y_4, \\ x_3 &= -\sqrt{a} y_1 + y_2, \\ x_4 &= -\sqrt{a} y_3 + y_4, \\ x_5 &= \sqrt{a} y_5 + y_6, \\ x_6 &= -\sqrt{a} y_5 + y_6, \\ x_7 &= -2\sqrt{a} y_7 \end{cases}$$

where the y_i are now defined over $\mathbf{k}$. This gives the model

$$\begin{cases} dy_5 &= y_1 y_4 + y_2 y_3 \\ dy_6 &= a y_1 y_3 + y_2 y_4 \\ dy_7 &= y_1 y_2 \end{cases} \tag{3.8}$$

with $a \in \mathbf{k}^*/(\mathbf{k}^*)^* - \{1\}$; this is canonical: two of these minimal algebras are not isomorphic over $\mathbf{k}$ for different quadratic field extensions, since the equivalence would be given by a $\mathbf{k}-$isomorphism, therefore commuting with the action of the Galois group. Note that for $a = 1$, we recover case (3.6), where $dy_5 + dy_6 = (y_1 + y_2)(y_3 + y_4)$ and $dy_5 - dy_6 = (y_1 - y_2)(y_3 - y_4)$ are of rank 2.

3.6.2 Smooth conics

Let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth conic; then $\mathcal{C}$ can be written as $X^2 - aY^2 - bZ^2$ for suitable coefficients $a, b \in \mathbf{k}^*$.

Lemma 3.10. *Let $p \in \mathcal{C}$ be a rational point. Then $\mathcal{C}$ is isomorphic to the projective line $\mathbb{P}^1$.*

Proof. Fix a line $\mathbb{P}^1 \subset \mathbb{P}^2$ not passing through p and consider the set of lines in $\mathbb{P}^2$ through p. Each line ℓ meets the conic in some point p_ℓ with coordinates in $\mathbf{k}$. In fact, the coordinates of p_ℓ are given as solution to a quadratic equation with coefficients in $\mathbf{k}$ and with one root in $\mathbf{k}$. The map sending p_ℓ to the intersection of ℓ with the fixed projective line $\mathbb{P}^1$ is defined on $\mathcal{C} - \{p\}$, but can be extended to the whole $\mathcal{C}$ by sending p to the intersection of the tangent line at p with the fixed line. This map is birational, hence an isomorphism. □

By considering the inverse map, we see that every conic with a rational point can be parametrized by a projective line $\mathbb{P}^1$; by this we mean that there exists an isomorphism $\mathbb{P}^1 \to \mathcal{C}$ of the form

$$[X_0 : X_1] \to [q_0(X_0, X_1) : q_1(X_0, X_1) : q_2(X_0, X_1)]$$

where $q_i = a_{i0}X_0^2 + a_{i1}X_0X_1 + a_{i2}X_1^2$ is a quadratic homogeneous polynomial. By letting the parametrization vary, we obtain all possible conics with rational points. This proves the following lemma:

Lemma 3.11. *Let $\mathcal{C}$ be a conic in $\mathbb{P}^2$ with one rational point. Then $\mathcal{C}$ is projectively equivalent to the conic $\mathcal{C}_0$ of equation $X^2 + Y^2 - Z^2$.*

Proof. It is clear that $\mathcal{C}_0$ has rational points; for instance, $[1 : 0 : 1] \in \mathcal{C}_0$. According to the previous discussion, we can find a change of coordinates of $\mathbb{P}^2$ sending $\mathcal{C}$ to $\mathcal{C}_0$. □

Remark 3.1. It is well known that five points $p_1, \ldots, p_5 \in \mathbb{P}^2$, such that no three of them are colinear, determine a conic in $\mathbb{P}^2$. There is a remarkable exception: the projective space $\mathbb{P}^2_{\mathbb{Z}_3}$ contains 13 points, but no matter how one chooses five of them, there will be at least three on a line. Indeed, a conic in $\mathbb{P}^2_{\mathbb{Z}_3}$ only has 4 points.

The previous lemma allows us to divide conics in two classes:

- conics with rational points; all of them are equivalent to $\mathcal{C}_0$;

- conics without rational points.

A conic $\mathcal{C}$ with equation $X^2 - aY^2 - bZ^2$ defined over $\mathbf{k}$ without rational points has points in many quadratic extension of $\mathbf{k}$, for instance

- in $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, $p = [\sqrt{a} : 1 : 0]$;
- in $\mathbf{k}' = \mathbf{k}(\sqrt{b})$, $p = [\sqrt{b} : 0 : 1]$;
- in $\mathbf{k}' = \mathbf{k}(\sqrt{-a/b})$, $p = [0 : 1 : \sqrt{-a/b}]$.

These quadratic extensions are not necessarily isomorphic if the square class group has more than two elements.

Suppose $\mathcal{C} = X^2 - aY^2 - bZ^2$ is a conic without rational points. Then we consider a quadratic extension $\mathbf{k}' = \mathbf{k}(\sqrt{a})$, where $\mathcal{C}$ has rational points. Over $\mathbf{k}'$,

$$X^2 - aY^2 - bZ^2 = (X - \sqrt{a}Y)(X + \sqrt{a}Y) - bZ^2 = \bar{X}\bar{Y} - b\bar{Z}^2.$$

We set $\bar{\mathcal{C}} \subset \mathbb{P}^2_{\mathbf{k}'}$ and we argue as in section (3.5.4). We choose φ_5 and φ_6 on $\bar{\mathcal{C}}$ conjugated under the action of the Galois group $\mathrm{Gal}(\mathbf{k}' : \mathbf{k})$ (notice that this action does not fix any point of $\bar{\mathcal{C}}$); also, we choose φ_7 as the intersection point between the tangent lines to $\bar{\mathcal{C}}$ at φ_5 and φ_6; hence φ_7 is already in $\mathbf{k}$, thus fixed by the action of the Galois group. We can write, in $F_0' = F_0 \otimes \mathbf{k}'$,

$$\begin{cases} dx_5 &=& x_1x_2 \\ dx_6 &=& x_3x_4 \\ dx_7 &=& a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4. \end{cases}$$

and consider

$$\begin{cases} x_1 &=& \sqrt{a}y_1 + y_2, \\ x_2 &=& \sqrt{a}y_3 + y_4, \\ x_3 &=& -\sqrt{a}y_1 + y_2, \\ x_4 &=& -\sqrt{a}y_3 + y_4, \\ x_5 &=& \sqrt{a}y_5 + y_6, \\ x_6 &=& -\sqrt{a}y_5 + y_6, \\ x_7 &=& y_7 \end{cases}$$

where the y_i are defined over $\mathbf{k}$. Then, if σ is a generator of $\mathrm{Gal}(\mathbf{k}' : \mathbf{k})$, $\sigma(x_1) = x_3$ and $\sigma(x_2) = x_4$. Thus

$$\begin{aligned} [\sigma(dx_7)] &= a_{13}x_3x_1 + a_{14}x_2x_3 + a_{23}x_4x_1 + a_{24}x_4x_2 = \\ &= -(a_{13}x_1x_3 + a_{23}x_1x_4 + a_{14}x_2x_3 + a_{24}x_2x_4) = \\ &= [dx_7] \Leftrightarrow a_{14} = a_{23}, \end{aligned}$$

where the brackets denote the equivalence class of dx_7 in $\mathbb{P}^2_{\mathbf{k}'}$. Then we can write

$$dx_7 = a_{13}x_1x_3 + a_{24}x_2x_4 + a_{14}(x_1x_4 + x_2x_3).$$

Performing the change of variables we obtain

$$\begin{cases} dy_5 &= y_1y_4 + y_2y_3 \\ dy_6 &= ay_1y_3 + y_2y_4 \\ dy_7 &= b_{12}y_1y_2 + b_{34}y_3y_4 + c(y_1y_4 - y_2y_3). \end{cases}$$

If $b_{12} \neq 0$ we can substitute $y_1 \mapsto y_1 + \frac{c}{b_{12}}y_3$ and $y_2 \mapsto y_2 + \frac{c}{b_{12}}y_4$ to get rid of the term $c(y_1y_4 - y_2y_3)$. Scaling φ_7, the model becomes

$$\begin{cases} dy_5 &= y_1y_4 + y_2y_3 \\ dy_6 &= ay_1y_3 + y_2y_4 \\ dy_7 &= y_1y_2 + \alpha y_3y_4. \end{cases} \tag{3.9}$$

From this model we must be able to recover the conic $\bar{X}\bar{Y} - b\bar{Z}^2 = 0$ in $\mathbb{P}^2_{\mathbf{k}'}$; take a generic $\varphi = [X : Y : Z] \in \mathbb{P}^2$, where the reference system in $\mathbb{P}^2$ is $\langle dy_5, dy_6, dy_7 \rangle$; then φ has rank 2 if and only if $\varphi \wedge \varphi = 0$, which gives the equation

$$X^2 - aY^2 + \alpha Z^2 = (X - \sqrt{a}Y)(X + \sqrt{a}Y) + \alpha Z^2;$$

this must be equal to $\bar{X}\bar{Y} - b\bar{Z}^2 = 0$, which forces $\alpha = -b$. Finally, the model is

$$\begin{cases} dy_5 &= y_1y_4 + y_2y_3 \\ dy_6 &= ay_1y_3 + y_2y_4 \\ dy_7 &= y_1y_2 - by_3y_4. \end{cases} \tag{3.10}$$

Going back to (3.9), one sees easily that if $b_{12} = 0$ but $b_{34} \neq 0$, there is a change of variables that gives again (3.10). If $b_{12} = b_{34} = 0$, then a linear combination of dy_5 and dy_7 has rank 2, giving some point of intersection with the conic. But this is impossible.

To sum up, the discussion in rank 1, 2 and 3, gives the following proposition:

Proposition 3.3. *There is a $1-1$ correspondence between minimal algebras of type $(4,3)$ over $\mathbf{k}$ such that the plane π determined by the differential $d : F_1 \to \wedge^2 F_0$ is not contained in the Klein quadric $\mathcal{Q}$ and the set of conics in $\mathbb{P}^2_{\mathbf{k}}$.*

The last step is to give a criterion to say when two conics are equivalent. If the conic has rational points, it can be put in the form $\mathcal{C}_0$ under a suitable change of variables. So we assume that the conic has no rational points. As we remarked above, two conics are equivalent if and only if the corresponding quadratic forms are isometric up to a scalar factor (which allows to write the conic in the normal form (3.7). The problem of establishing when two quadratic forms are isometric is quite complicated and a complete answer requires a lot of algebra. Here we need an answer only the $3-$dimensional case. We refer to [78] for all the details.

Let W be a vector space of dimension 3 over $\mathbf{k}$. A quadratic form on W is *regular* if its matrix in any basis is nonsingular. Equivalently, if the associated conic $\mathcal{C} \subset \mathbb{P}^2$ is smooth.

Theorem 3.2. *Let Q_1 and Q_2 be two regular quadratic forms on W. Then Q_1 and Q_2 are isometric if and only if*

$$\det(Q_1) = \det(Q_2) \qquad \textit{and} \qquad S(Q_1) \sim S(Q_2).$$

The *determinant* of a quadratic form Q is the determinant of any matrix representing Q; it is well defined as an element of $\mathbf{k}^*/(\mathbf{k}^*)^2$. $S(Q)$ denotes the *Hasse algebra* of Q and $\sim$ denotes similarity. We define the Hasse algebra and explain what it means for two Hasse algebras to be similar.

Let V be a $4-$dimensional vector space over $\mathbf{k}$ and let a, b be two nonzero scalars. We fix a basis $\{\mathbf{1}, x_1, x_2, x_3\}$ of V and define a multiplication on these basis elements according to the rules of table 3.2. This multiplication is extended to the whole algebra using linearity. We denote this algebra by (a, b) and call it *quaternion algebra*. When $\mathbf{k} = \mathbb{R}$ and $(a, b) = (-1, -1)$, we obtain the usual Hamilton quaternions $\mathbb{H}$. As another example, we can take the algebra $M_2(\mathbf{k})$ of 2×2 matrices with entries in $\mathbf{k}$. It is easy to see that $M_2(\mathbf{k}) \cong (1, -1)$.

Table 3.2: Multiplication table of the quaternion algebra (a, b)

	$\mathbf{1}$	x_1	x_2	x_3
$\mathbf{1}$	$\mathbf{1}$	x_1	x_2	x_3
x_1	x_1	$a\mathbf{1}$	x_3	ax_2
x_2	x_2	$-x_3$	$b\mathbf{1}$	$-bx_1$
x_3	x_3	$-ax_2$	bx_1	$-ab\mathbf{1}$

Quaternion algebras have the following properties:

1. $(1, a) \cong (1, -1) \cong (b, -b) \cong (c, 1 - c)$, where $c \neq 1$;
2. $(b, a) \cong (a, b) \cong (a\lambda^2, b\mu^2)$ for $\lambda, \mu \in \mathbf{k}$;
3. $(a, ab) \cong (a, -b)$;
4. $(a, b) \otimes_{\mathbf{k}} (a, c) = (a, bc) \otimes_{\mathbf{k}} (1, -1)$.

We may write a quaternion $q \in (a, b)$ as $\xi_0\mathbf{1} + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$ with $\xi_i \in \mathbf{k}$; its conjugate is $\bar{q} = \xi_0\mathbf{1} - \xi_1 x_1 - \xi_2 x_2 - \xi_3 x_3$. The *norm* of a quaternion q is

$$N(q) = q\bar{q} = \xi_0^2 - a\xi_1^2 - b\xi_2^2 + ab\xi_3^2.$$

The elements of $(a,b)^0 = \langle x_1, x_2, x_3 \rangle \subset (a,b)$ are called purely imaginary quaternion. The norm on $(a,b)^0$ is the restriction of the norm on (a,b) and is given by

$$N(q^0) = -a\xi_1^2 - b\xi_2^2 + ab\xi_3^2.$$

Therefore $N : (a,b)^0 \to \mathbf{k}$ defines a quadratic form on $(a,b)^0 \cong \mathbf{k}^3$ and hence a conic in $\mathbb{P}((a,b)^0)$. Since we can multiply a conic by any $\lambda \in \mathbf{k}^*$, we see that $-a\xi_1^2 - b\xi_2^2 + ab\xi_3^2$ is equivalent to $-b\xi_1^2 - a\xi_2^2 + \xi_3^2$ (multiplying it by ab); this is the normal form of a conic we found at the beginning of this section. This explains the relation between quaternion algebras and plane conics. We associate to the conic $-b\xi_1^2 - a\xi_2^2 + \xi_3^2$ the quaternion algebra (a,b).

Quaternion algebras are a special example of *central simple algebras*. A central simple algebra is a finite dimensional algebra A over $\mathbf{k}$ with unit $\mathbf{1}_A$, satisfying two conditions

- the center of A can be identified with $\mathbf{k}$ under the inclusion $\lambda \mapsto \lambda \cdot \mathbf{1}_A$;
- A contains no two-sided ideals other than 0 and A itself.

The tensor product (over $\mathbf{k}$) of two central algebras is again a central algebra. Two central algebras A and B are *similar*, written $A \sim B$, if there exist matrix algebras $M_p(\mathbf{k})$ and $M_q(\mathbf{k})$ such that

$$A \otimes M_p(\mathbf{k}) \cong B \otimes M_q(\mathbf{k}).$$

Let $(\mathcal{A}, \otimes)/\sim$ denote the set of all central simple algebras over $\mathbf{k}$; one can prove that this is indeed an abelian group, called the *Brauer group* $\mathrm{Br}(\mathbf{k})$ of $\mathbf{k}$. For more details about the Brauer group, see for instance [73]. Property (4) above says that $(a,b) \otimes (a,c) = (a,bc)$ in $\mathrm{Br}(\mathbf{k})$ since $M_2(\mathbf{k}) \cong (1,-1)$. Also, $(a,b) \otimes (a,b) = (a,b^2) = (a,1) = 1$ in $\mathrm{Br}(\mathbf{k})$. This proves that quaternion algebras give elements of order 2 in the Brauer group.

Suppose that Q is a quadratic form on an $n-$dimensional vector space W over $\mathbf{k}$. In a suitable basis, the matrix of Q is $\mathrm{diag}(a_1, \ldots, a_n)$ with $a_i \in \mathbf{k}\ \forall\ i$. Define $d_j = \prod_{i=1}^{j} a_i$. The *Hasse algebra* associated to Q is

$$S(Q) = \bigotimes_{1 \le j \le n} (a_j, d_j),$$

where (a_j, d_j) denotes a quaternion algebra. Notice that the Hasse algebra is an element of the Brauer group $\mathrm{Br}(\mathbf{k})$.

Since we are working with conics, the determinant is not an invariant; indeed, we can multiply the equation of $\mathcal{C}$ by $\lambda \in \mathbf{k}^*$ so that $\det(\mathcal{C}) = 1$ in $\mathbf{k}^*/(\mathbf{k}^*)^2$. On the other hand, the Hasse algebra is an invariant, i.e. it remains unchanged when we scale the quadratic form.

Lemma 3.12. *The Hasse algebras of the quadratic forms Q and λQ are similar.*

Proof. Assume that Q has been diagonalized and normalized, so that $Q = X^2 - aY^2 - bZ^2$. Then

$$a_1 = d_1 = 1, \quad a_2 = d_2 = -a, \quad a_3 = -b, \quad d_3 = ab$$

and

$$\begin{aligned} S(Q) = (1,1) \otimes (-a,-a) \otimes (-b,ab) \sim (-a,-1) \otimes (-a,a) \otimes (-b,b) \otimes (-b,a) \sim \\ \sim (-a,-1) \otimes (-b,a) \sim (-1,-1) \otimes (a,-1) \otimes (a,-b) \sim (-1,-1) \otimes (a,b). \end{aligned}$$

Now $\lambda Q = \lambda X^2 - a\lambda Y^2 - b\lambda Z^2$, with

$$a_1' = d_1' = \lambda, \quad a_2' = -\lambda a, \quad d_2' = -\lambda^2 a, \quad a_3' = -\lambda b \quad \text{and} \quad d_3' = \lambda^3 ab.$$

One gets

$$\begin{aligned} S(\lambda Q) &= (\lambda,\lambda) \otimes (-\lambda a, -\lambda^2 a) \otimes (-\lambda b, \lambda^3 ab) \sim \\ &\sim (\lambda,\lambda) \otimes (-\lambda a, -a) \otimes (-\lambda b, \lambda ab) \sim \\ &\sim (\lambda,\lambda) \otimes (\lambda,-a) \otimes (\lambda, \lambda ab) \otimes (-b,\lambda) \otimes (-a,-a) \otimes (-b,ab) \sim \\ &\sim (\lambda, \lambda^2 a^2 b^2) \otimes (-a,-a) \otimes (-b,ab) \sim (-1,-1) \otimes (a,b). \end{aligned}$$

□

Then we may associate to the conic $\mathcal{C}$ two elements of the Brauer group: the quaternion algebra (a,b) and the Hasse algebra $(-1,-1) \otimes (a,b)$. Theorem 3.2 says that two conics $\mathcal{C}_1$ and $\mathcal{C}_2$, with equations $X^2 - aY^2 - bZ^2$ and $X^2 - \alpha Y^2 - \beta Z^2$ are equivalent if and only if $S(\mathcal{C}_1) \sim S(\mathcal{C}_2)$, that is, if and only if

$$(-1,-1) \otimes (a,b) \sim (-1,-1) \otimes (\alpha,\beta).$$

Since the Brauer group is a group, this is equivalent to $(a,b) \sim (\alpha,\beta)$. Then we see that two conics are isomorphic if and only if the corresponding quaternion algebras are isomorphic and we get as many non-isomorphic conics as non-isomorphic quaternion algebras over $\mathbf{k}$. Recall that the conic determines the minimal algebra; then we have shown the following proposition:

Proposition 3.4. *Let $(\wedge V, d)$ be a minimal algebra of dimension 7 and type $(4,3)$ and suppose that the differential $d : F_1 \hookrightarrow \wedge^2 F_0$ determines a plane π which cuts the Klein quadric $\mathcal{Q}$ in a smooth conic. The number of non isomorphic minimal algebras of this type is equal to number of isomorphism classes of quaternion algebras over $\mathbf{k}$.*

Remark 3.2. We saw above that quaternionic algebras over $\mathbf{k}$ define order two elements in the Brauer group $\mathrm{Br}(\mathbf{k})$. The converse is partially true. In fact, Merkurjev ([71]) proves that any element of order two in the Brauer group is equal (in the Brauer group) to a product of quaternion algebras. To avoid technicalities, we prefer to state the result in term of quaternion algebras.

Theorem 3.3. *A quadratic form Q is isotropic if and only if $S(Q) \sim (-1,-1)$.*

3.6.3 Examples

We end this section with some examples.

Assume $\mathbf{k} = \mathbb{R}$; in the rank 2 case we have two conics, $X^2 - Y^2$ and $X^2 + Y^2$. The first is the product of two real lines, and the second one is the product of two imaginary lines and gives the model (3.8) with $a = -1$. For the rank 3 case, we use the fact that $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}_2$, generated by the quaternion algebra $(-1,-1)$. We get two quadratic forms, $Q_0 = X^2 + Y^2 + Z^2$, which is not isotropic, and $Q_1 = X^2 + Y^2 - Z^2$, which is isotropic. The last case has already been studied, while the first one gives the model (3.10) with $a = b = -1$. The Hasse algebras are $S(Q_0) \sim (1,1)$ and $S(Q_1) \sim (-1,-1)$.

Suppose $\mathbf{k} = \mathbb{F}_{p^n}$ is a finite field. It is possible to show (see for instance [85]) that any quadratic form over a $3-$dimensional vector space over $\mathbb{F}_{p^n}$ is isotropic (indeed, the Brauer group of any finite field is trivial). Then any smooth conic in $\mathbb{P}^2$ has rational points: when the conic is smooth, there is no new minimal algebra with respect to the algebraically closed case. On the other hand, in the rank 2 case we obtain the model (3.8), with $a \in \mathbf{k}^*/(\mathbf{k}^*)^2 - \{1\}$. Since for finite fields $|\mathbf{k}^*/(\mathbf{k}^*)^2| = 2$, we get only one further minimal algebra.

Finally, we treat the case $\mathbf{k} = \mathbb{Q}$, which is very relevant on the rational homotopy side. The rank 2 case is straightforward: we get as many models as elements in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, all of them of the form (3.8). In the rank 3 case, we have the following exact sequence

$$0 \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_{p \in \mathcal{P}} \mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where $\mathcal{P} = \{2,3,5,\ldots,\infty\}$ is the set of all prime numbers and ∞, $\mathbb{Q}_p$ is the field of $p-$adic numbers and, by definition, $\mathbb{Q}_\infty = \mathbb{R}$. We remarked above that quaternion algebras are related to the $2-$torsion in the Brauer group of $\mathbf{k}$. Every $p-$adic field $\mathbb{Q}_p$ has two non isomorphic quaternionic algebras, one isotropic and one non isotropic. The above exact sequence shows that $\mathbb{Q}$ has an infinite number of non-isomorphic quaternionic algebras.

We give another method to establish whether a conic $\mathcal{C}$ defined over $\mathbb{Q}$ has rational points or not; we refer to [85] for further details. Since $\mathbb{Q} \subset \mathbb{Q}_p$ for every $p \in \mathcal{P}$, $\mathcal{C}$ can be interpreted as a quadratic form over $\mathbb{Q}_p$. If $\mathcal{C}$ is a conic in $\mathbb{P}^2_{\mathbb{Q}_p}$, zero locus of $X^2 - aY^2 - bZ^2$ with $a, b \in \mathbb{Q}$, we define its *Hilbert symbol* as

$$(a,b)_p = \begin{cases} 1 & \text{if } X^2 - aY^2 - bZ^2 \text{ is isotropic} \\ -1 & \text{otherwise.} \end{cases}$$

The Hilbert symbol satisfies the following properties (compare with the properties of the Hasse algebra):

1. $(a,b)_p = (b,a)_p$ and $(a,c^2)_p = 1$;

2. $(a,-a)_p = 1$ and $(a, 1-a)_p = 1$ $(a \neq 0,1)$;

3. $(aa', b)_p = (a,b)_p(a',b)_p$ (bilinearity);

4. $(a,b)_p = (a,-ab)_p = (a,(1-a)b)_p$.

It can be easily computed according to the following rules; suppose first that $p \neq \infty$; write $a = p^\alpha u$, $b = p^\beta v$ for $\alpha, \beta \in \mathbb{Z}$ and $u, v \in \mathbb{Q}_p^*$; then

$$(a,b)_p = \begin{cases} (-1)^{\alpha\beta\varepsilon(p)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha & \text{if } p \neq 2 \\ \\ (-1)^{\varepsilon(u)\varepsilon(u)+\alpha\omega(v)+\beta\omega(u)} & \text{if } p = 2. \end{cases}$$

where $\varepsilon(p)$ is the class $\frac{p-1}{2}$ mod 2 and $\omega(u)$ is the class $\frac{u^2-1}{8}$ mod 2. The case $p = \infty$ is straightforward: $(a,b)_\infty = -1$ if and only if the conic is $X^2 + Y^2 + Z^2$. The Hasse- Minkovski theorem says that a quadratic form defined over $\mathbb{Q}$ is isotropic if and only if it is isotropic over $\mathbb{Q}_p$ for every $p \in \mathcal{P}$.

We end this section proving the main theorem.

Proof. (*of the main theorem*) The theorem is a consequence of the case by case analysis of the previous sections. Case $(6,1)$ gives 3 isomorphism classes. Case $(5,2)$ gives $5+(r-1)$ isomorphism classes, where $r = |\mathbf{k}^*/(\mathbf{k}^*)^2|$. Finally, case $(4,3)$ gives $5+(r-1)+(s-1)$ isomorphism classes, where s is the number of isomorphism classes of quaternion algebras over $\mathbf{k}$. Summing the three numbers yields the thesis. □

3.7 Real homotopy types of 7−dimensional 2−step nilmanifolds

In this section we collect the results on minimal algebras over $\mathbb{R}$. Each of these minimal algebras is defined over $\mathbb{Q}$, hence the corresponding nilpotent Lie algebra $\mathfrak{g}$ has rational structure constants; Mal'cev theorem implies that each of these Lie algebras has an associated (rational homotopy type of) nilmanifold. We use Nomizu theorem to compute its real cohomology and real homotopy type. Columns 5 to 8 display the Betti numbers of the various nilmanifolds. Column 9 gives a labelling of the minimal algebras when interpreted as Lie algebras (the notation refers to [7]). Finally, column 10 gives the dimension of the center of the corresponding Lie algebra. We remark that this list coincides with the one contained in the paper [26], which, in turn, relies on [43].

Table 3.3: Minimal algebras of dimension 7 and length 2 over $\mathbb{R}$

(f_0, f_1)	dx_5	dx_6	dx_7	b_1	b_2	b_3	$\sum_i b_i$	$\mathfrak{g}$	$\dim Z(\mathfrak{g})$
(6,1)	0	0	x_1x_2	6	16	25	71	$L_3 \oplus A_4$	5
	0	0	$x_1x_2 + x_3x_4$	6	14	19	61	$L_{5,1} \oplus A_2$	3
	0	0	$x_1x_2 + x_3x_4 + x_5x_6$	6	14	14	56	$L_{7,1}$	1
(5,2)	0	x_1x_2	x_1x_3	5	13	21	59	$L_{5,2} \oplus A_2$	4
	0	x_1x_2	x_3x_4	5	12	18	54	$L_3 \oplus L_3 \oplus A_1$	3
	0	x_1x_2	$x_1x_3 + x_2x_4$	5	12	18	54	$L_{6,1} \oplus A_1$	3
	0	x_1x_2	$x_1x_3 + x_4x_5$	5	10	16	48	$L_{7,2}$	2
	0	$x_1x_2 + x_3x_4$	$x_1x_3 + x_2x_5$	5	9	15	45	$L_{7,3}$	2
	0	$x_1x_3 - x_2x_4$	$x_1x_4 + x_2x_3$	5	12	18	54	$L_{6,2} \oplus A_1$	3
(4,3)	x_1x_2	x_1x_3	x_1x_4	4	12	18	52	$L_{7,4}$	3
	x_1x_2	x_1x_3	x_2x_3	4	11	20	52	$L_{6,4} \oplus A_1$	4
	x_1x_2	x_1x_3	$x_1x_4 + x_2x_3$	4	11	17	49	$L_{7,5}$	3
	x_1x_2	x_3x_4	x_1x_3	4	11	16	48	$L_{7,6}$	3
	x_1x_2	x_3x_4	$x_1x_4 + x_2x_3$	4	11	14	46	$L_{7,7}$	3
	$x_1x_4 + x_2x_3$	$-x_1x_3 + x_2x_4$	x_1x_2	4	11	16	48	$L_{7,8}$	3
	$x_1x_4 + x_2x_3$	$-x_1x_3 + x_2x_4$	$x_1x_2 + x_3x_4$	4	11	14	46	$L_{7,9}$	3

CHAPTER
FOUR

NON-FORMAL CO-SYMPLECTIC MANIFOLDS

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Abstract

We study the formality of the mapping torus of an orientation-preserving diffeomorphism of a manifold. In particular, we give conditions under which a mapping torus has a non-zero Massey product. As an application we prove that there are non-formal compact co-symplectic manifolds of dimension m and with first Betti number b if and only if $m = 3$ and $b \geq 2$, or $m \geq 5$ and $b \geq 1$. Explicit examples for each one of these cases are given.

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Key words: co-symplectic manifold, mapping torus, minimal model, formal manifold.

4.1 Introduction

In this paper we follow the nomenclature of [62], where *co-symplectic* manifolds are the odd-dimensional counterpart to symplectic manifolds. In terms of differential forms, a *co-symplectic structure* on a $(2n+1)$-dimensional manifold M is determined by a pair (F, η) of closed differential forms, where F is a 2-form and η is a 1-form such that $\eta \wedge F^n$ is a volume form, so that M is orientable. In this case, we say that (M, F, η) is a *co-symplectic manifold*. Such a manifold was called earlier cosymplectic manifold by Libermann [64], or almost-cosymplectic by Goldberg and Yano [41].

The simplest examples of co-symplectic manifolds are the manifolds called *co-Kähler* by Li in [62], or *cosymplectic* by Blair [9]. Such a manifold is locally a product of a Kähler manifold with a circle or a line. In fact, a *co-Kähler structure* on a $(2n+1)$-dimensional manifold M is a normal almost contact metric structure (ϕ, η, ξ, g) on M, that is, a tensor field ϕ of type $(1,1)$, a 1-form η, a vector field ξ (the Reeb vector field) with $\eta(\xi) = 1$, and a Riemannian metric g satisfying certain conditions (see section 4.3 for details) such that the 1-form η and the fundamental 2-form F given by $F(X, Y) = g(\phi X, Y)$, for any vector fields X and Y on M, are closed.

The topological description of co-symplectic and co-Kähler manifolds is due to Li [62]. There he proves that a compact manifold M has a co-symplectic structure if and only if M is the mapping torus of a symplectomorphism of a symplectic manifold, while M has a co-Kähler structure if and only if M is a Kähler mapping torus, that is, M is the mapping torus of a Hermitian isometry on a Kähler manifold. This result may be considered an extension to co-symplectic and co-Kähler manifolds of Tischler's Theorem [90] that asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form.

The existence of a co-Kähler structure on a manifold M imposes strong restrictions on the underlying topology of M. Indeed, since co-Kähler manifolds are odd-dimensional analogues of Kähler manifolds, several known results from Kähler geometry carry over to co-Kähler manifolds. In particular, any compact co-Kähler manifold is formal. Another similarity is the monotone property for the Betti numbers of compact co-Kähler manifolds [24].

Intuitively, a simply connected manifold is formal if its rational homotopy type is determined by its rational cohomology algebra. Simply connected compact manifolds of dimension less than or equal to 6 are formal [76, 37]. We shall say that M is *formal* if its minimal model is formal or, equivalently, the de Rham complex $(\Omega M, d)$ of M and the algebra of the de Rham cohomology $(H^*(M), d = 0)$ have the same minimal model (see section 4.2 for details).

It is well-known that the existence of a non-zero Massey product is an obstruction to the formality. In [37] the concept of formality is extended to a weaker notion called s-formality. There, the second and third authors prove that an orientable compact connected manifold, of dimension $2n$ or $(2n-1)$, is formal if and only if it is $(n-1)$-formal.

The importance of formality in symplectic geometry stems from the fact that it allows to distinguish symplectic manifolds which admit Kähler structures from those which do not [27, 36, 80]. It seems thus interesting to analyze what happens for co-symplectic manifolds. In this paper we consider the following problem on the *geography* of co-symplectic manifolds:

> For which pairs $(m = 2n+1, b)$, with $n, b \geq 1$, are there compact co-symplectic manifolds of dimension m and with $b_1 = b$ which are non-formal?

We address this question in section 4.5. It will turn out that the answer is the same as for compact manifolds [35], i.e, that there are always non-formal examples except for $(m, b) = (3, 1)$.

On any compact co-symplectic manifold M, the first Betti number must satisfy $b_1(M) \geq 1$, since the $(2n+1)$-form $\eta \wedge F^n$ defines a non-zero cohomology class on M, and hence η defines a cohomology class $[\eta] \neq 0$. It is known that any orientable compact manifold of dimension ≤ 4 and with first Betti number $b_1 = 1$ is formal [35].

The main problem in order to answer the question above is to construct examples of non-formal compact co-symplectic manifolds of dimension $m = 3$ with $b_1 \geq 3$ as well as examples of dimension $m = 5$ with $b_1 = 1$. The other cases are covered in section 4.5, using essentially the 3-dimensional Heisenberg group to obtain non-formal co-symplectic manifolds of dimension $m \geq 3$ and with $b_1 = 2$ as well as non-formal co-symplectic manifolds of dimension $m \geq 5$ and with $b_1 \geq 2$, or from the non-formal compact simply connected symplectic manifold of dimension 8 given in [36] to exhibit non-formal co-symplectic manifolds of dimension $m \geq 9$ and with $b_1 = 1$.

To fill those gaps, we study in section 4.4 the formality of a mapping torus N_φ (not necessarily symplectic) that is, N_φ is the differentiable manifold obtained from $N \times [0, 1]$ with the ends identified by a diffeomorphism φ of a manifold N. The description of a minimal model for a mapping torus can be very complicated even for low degrees. Nevertheless, in Theorem 4.4 we determine a minimal model of N_φ up to some degree $p \geq 2$ when φ satisfies some extra conditions, namely the map induced on cohomology $\varphi^* : H^k(N) \to H^k(N)$ does not have the eigenvalue $\lambda = 1$, for any $k \leq (p-1)$, but $\varphi^* : H^p(N) \to H^p(N)$ has the eigenvalue $\lambda = 1$ with multiplicity $r \geq 1$. In particular (see Corollary 4.1), we show that if $r = 1$, N_φ is p-formal in the sense mentioned above.

Moreover, in Theorem 4.3 we prove that N_φ has a non-zero (triple) Massey product if there exists $p > 0$ such that $\lambda = 1$ is a double eigenvalue of the map

$$\varphi^* : H^p(N) \to H^p(N).$$

In fact, we show that the Massey product $\langle [dt], [dt], [\tilde{\alpha}] \rangle$ is well-defined on N_φ and it does not vanish, where dt is the 1-form defined on N_φ by the volume form on S^1, and $[\tilde{\alpha}] \in H^p(N_\varphi)$ is the cohomology class induced on N_φ by a certain cohomology class $[\alpha] \in H^p(N)$ fixed by φ^*.

Regarding symplectic mapping torus manifolds, first we notice that if N is a compact symplectic $2n$-manifold, and $\varphi : N \to N$ is a symplectomorphism, then the map induced on cohomology $\varphi^* : H^2(N) \to H^2(N)$ always has the eigenvalue $\lambda = 1$. As a consequence of Theorem 4.4, we get that if N_φ is a symplectic mapping torus such that the map $\varphi^* : H^1(N) \to H^1(N)$ does not have the eigenvalue $\lambda = 1$, then N_φ is 2-formal if and only if the eigenvalue $\lambda = 1$ of $\varphi^* : H^2(N) \to H^2(N)$ has multiplicity $r = 1$. Thus, in these conditions, the co-symplectic manifold N_φ is formal when N has dimension four.

In section 4.5, using Theorem 4.3, we solve the case $m = 3$ with $b_1 \geq 3$ taking the mapping torus of a symplectomorphism of a surface of genus $k \geq 2$ (see Proposition 4.7). For $m = 5$ and $b_1 = 1$ we consider the mapping torus of a symplectomorphism of a 4-torus (see Proposition 4.8).

Let G be a connected, simply connected solvable Lie group, and let $\Gamma \subset G$ be a discrete, cocompact subgroup. Then $M = \Gamma\backslash G$ is a solvmanifold. The manifold constructed in Proposition 4.8 is not a solvmanifold according to our definition. However, it is the quotient of a solvable Lie group by a closed subgroup. In section 4.6 we present an explicit example of a non-formal compact co-symplectic 5-dimensional manifold S, with first Betti number is $b_1(S) = 1$, which is a solvmanifold. We describe S as the mapping torus of a symplectomorphism of a 4-torus, so this example fits in the scope of Proposition 4.8.

4.2 Minimal models and formality

In this section we recall some fundamental facts of the theory of minimal models. For more details, see [28] and [31].

We work over the field $\mathbb{R}$ of real numbers. Recall that a *commutative differential graded algebra* (A, d) (CDGA for short) is a graded algebra $A = \oplus_{k\geq 0} A^k$ which is graded commutative, i.e. $x \cdot y = (-1)^{|x||y|} y \cdot x$ for homogeneous elements x and y, together with a differential $d : A^k \to A^{k+1}$ such that $d^2 = 0$ and $d(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy$ (here $|x|$ denotes the degree of the homogeneous element x).

Morphisms of CDGAs are required to preserve the degree and to commute with the differential. Notice that the cohomology of a CDGA is an algebra which can be turned into a CDGA by endowing it with the zero differential. A CDGA is said to be *connected* if $H^0(A, d) \cong \mathbb{R}$. The main example of CDGA is the de Rham complex of a smooth manifold M, $(\Omega^*(M), d)$, where d is the exterior differential.

A CDGA (A, d) is said to be *minimal* (or *Sullivan*) if the following happens:

- $A = \bigwedge V$ is the free commutative algebra generated by a graded (real) vector space $V = \oplus_k V^k$;
- there exists a basis $\{x_i,\ i \in \mathcal{J}\}$ of V, for a well-ordered index set $\mathcal{J}$, such that $|x_i| \leq |x_j|$ if $i < j$ and the differential of a generator x_j is expressed in terms of the preceding x_i $(i < j)$; in particular, dx_j does not have linear part.

We have the following fundamental result:

Proposition 4.1. *Every connected CDGA (A, d) has a unique minimal model: there exist a minimal algebra $(\bigwedge V, d)$ together with a morphism of CDGAs $\varphi : (\bigwedge V, d) \to (A, d)$ which induces an isomorphism $\varphi^* : H^*(\bigwedge V, d) \to H^*(A, d)$.*

The (real) *minimal model* of a differentiable manifold M is by definition the minimal model of its de Rham algebra $(\Omega^*(M), d)$.

Recall that a minimal algebra $(\bigwedge V, d)$ is *formal* if there is a morphism of differential algebras $\psi\colon (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ that induces the identity on cohomology. Also a differentiable manifold M is *formal* if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, homogeneous spaces, flag manifolds, and compact Kähler manifolds.

In [28], the formality of a minimal algebra is characterized as follows.

Proposition 4.2. *A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space V can be decomposed as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [37].

Definition 4.1. A minimal algebra $(\bigwedge V, d)$ is s-formal ($s > 0$) if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

1. $d(C^i) = 0$,
2. the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective,
3. any closed element in the ideal $I_s = I(\bigoplus\limits_{i\leq s} N^i)$, generated by the space $\bigoplus\limits_{i\leq s} N^i$ in the free algebra $\bigwedge(\bigoplus\limits_{i\leq s} V^i)$, is exact in $\bigwedge V$.

A differentiable manifold M is s-formal if its minimal model is s-formal. Clearly, if M is formal then M is s-formal, for any $s > 0$. The main result of [37] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 4.1. *Let M be a connected and orientable compact differentiable manifold of dimension $2n$, or $(2n-1)$. Then M is formal if and only if is $(n-1)$-formal.*

In order to detect non-formality, instead of computing the minimal model, which usually is a lengthy process, we can use Massey products, which are obstructions to formality. Let us recall their definition. The simplest type of Massey product is the triple Massey product. Let (A, d) be a CDGA and suppose $a, b, c \in H^*(A)$ are three cohomology classes such that $a \cdot b = b \cdot c = 0$. Take cocycles x, y and z representing these cohomology classes and let s, t be elements of A such that

$$ds = (-1)^{|x|} x \cdot y, \quad dt = (-1)^{|y|} y \cdot z.$$

Then one checks that

$$w = (-1)^{|x|} x \cdot t + (-1)^{|x|+|y|-1} s \cdot z$$

is a cocyle. The choice of different representatives gives an indeterminacy, represented by the space

$$\mathcal{I} = a \cdot H^{|y|+|z|-1}(A) + H^{|x|+|y|-1}(A) \cdot c.$$

We denote by $\langle a, b, c \rangle$ the image of the cocycle w in $H^*(A)/\mathcal{I}$. As it is proven in [28] (and essentially equivalent to Proposition 4.2), if a minimal CDGA is formal, then one can make uniform choices of cocyles so that the classes representing (triple) Massey products are exact. In particular, if the real minimal model of a manifold contains a non-zero Massey product, then the manifold is not formal.

4.3 Co-symplectic manifolds

In this section we recall some definitions and results about co-symplectic manifolds, and we extend to co-symplectic Lie algebras the result of Fino-Vezzoni [39] for co-Kähler Lie algebras.

Definition 4.2. Let M be a $(2n + 1)$-dimensional manifold. An *almost contact metric structure* on M consists of a quadruplet (ϕ, ξ, η, g), where ϕ is an endomorphism of the tangent bundle TM, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M satisfying the conditions

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (4.1)$$

for $X, Y \in \Gamma(TM)$.

Thus, ϕ maps the distribution $\ker(\eta)$ to itself and satisfies $\phi(\xi) = 0$. We call (M, ϕ, η, ξ, g) an *almost contact metric* manifold. The *fundamental* 2-form F on M is defined by

$$F(X, Y) = g(\phi X, Y),$$

for $X, Y \in \Gamma(TM)$.

Therefore, if (ϕ, ξ, η, g) is an almost contact metric structure on M with fundamental 2-form F, then $\eta \wedge F^n \neq 0$ everywhere. Conversely (see [10]), if M is a differentiable manifold of dimension $2n+1$ with a 2-form F and a 1-form η such that $\eta \wedge F^n$ is a volume form on M, then there exists an almost contact metric structure (ϕ, ξ, η, g) on M having F as the fundamental form.

There are different classes of structures that can be considered on M in terms of F and η and their covariant derivatives. We recall here those that are needed in the present paper:

- M is *co-symplectic* iff $dF = d\eta = 0$;
- M is *normal* iff the Nijenhuis torsion N_ϕ satisfies $N_\phi = -2d\eta \otimes \xi$;
- M is *co-Kähler* iff it is normal and co-symplectic or, equivalently, ϕ is parallel,

where the Nijenhuis torsion N_ϕ is given by

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for $X, Y \in \Gamma(TM)$.

In the literature, co-symplectic manifolds are often called *almost cosymplectic*, while co-Kähler manifolds are called *cosymplectic* (see [9, 11, 24, 39]).

The following result shows that co-symplectic manifolds are really the odd dimensional analogue of symplectic manifolds. Let us recall that a *symplectic* manifold (M, ω) is a pair consisting of a $2n$-dimensional differentiable manifold M with a closed 2-form ω which is non-degenerate (that is, ω^n never vanishes). The form ω is called symplectic. The following result is well-known and a proof of it can be found in Proposition 1 of [62].

Proposition 4.3. *A manifold M admits a co-symplectic structure if and only if the product $M \times S^1$ admits an S^1-invariant symplectic form.*

A theorem by Tischler [90] asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form. This result was extended recently to co-symplectic manifolds by Li [62]. Let us recall first some definitions.

Let N be a differentiable manifold and let $\varphi : N \to N$ be a diffeomorphism. The *mapping torus* N_φ of φ is the manifold obtained from $N \times [0, 1]$ with the ends identified by φ, that is

$$N_\varphi = \frac{N \times [0, 1]}{(x, 0) \sim (\varphi(x), 1)}.$$

It is a differentiable manifold, because it is the quotient of $N \times \mathbb{R}$ by the infinite cyclic group generated by $(x, t) \to (\varphi(x), t + 1)$. The natural map $\pi \colon N_\varphi \to S^1$ defined by $\pi(x, t) = e^{2\pi i t}$ is the projection of a locally trivial fiber bundle.

Definition 4.3. Let N_φ be a mapping torus of a diffeomorphism φ of N. We say that N_φ is a *symplectic* mapping torus if (N, ω) is a symplectic manifold and $\varphi : N \to N$ a symplectomorphism, that is, $\varphi^*\omega = \omega$.

Theorem 4.2 (Theorem 1, [62])**.** *A compact manifold M admits a co-symplectic structure if and only if it is a symplectic mapping torus $M = N_\varphi$.*

Notice that if M is a symplectic mapping torus $M = N_\varphi$, then the pair (F, η) defines a co-symplectic structure on M, where F is the closed 2-form on M defined by the symplectic form on N, and

$$\eta = \pi^*(\theta),$$

with θ the volume form on S^1. Moreover, notice that any 3-dimensional mapping torus is a symplectic mapping torus if the corresponding diffeomorphism preserves the orientation, since such diffeomorphism is isotopic to an area preserving one. However, in higher dimensions, there exist mapping tori with no co-symplectic structure, that is, they are not symplectic mapping tori (see Remark 4.3 in section 4.5 and [62]).

Next, we consider a Lie algebra $\mathfrak{g}$ of dimension $2n + 1$ with an *almost contact metric* structure, that is, with a quadruplet (ϕ, ξ, η, g) where ϕ is an endomorphism of $\mathfrak{g}$, ξ is a non-zero vector in $\mathfrak{g}$, $\eta \in \mathfrak{g}^*$ and g is a scalar product in $\mathfrak{g}$, satisfying (4.1). Then, $\mathfrak{g}$ is said to be *co-symplectic* iff $dF = d\eta = 0$; and $\mathfrak{g}$ is called *co-Kähler* iff it is normal and co-symplectic, where $d : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$ is the Chevalley-Eilenberg differential.

In [39] it is proved the following:

Proposition 4.4. *Co-Kähler Lie algebras in dimension $2n+1$ are in one-to-one correspondence with $2n$-dimensional Kähler Lie algebras endowed with a skew-adjoint derivation D which commutes with its complex structure.*

In order to extend this correspondence to co-symplectic Lie algebras we need to recall the following. Let (V, ω) be a symplectic vector space. An element $A \in \mathfrak{gl}(V)$ is an infinitesimal symplectic transformation if $A \in \mathfrak{sp}(V)$, that is, if

$$A^t\omega + \omega A = 0.$$

A scalar product g on (V, ω) is said to be compatible with ω if the endomorphism $J : V \to V$ defined by $\omega(u, v) = g(u, Jv)$ satisfies $J^2 = -\mathrm{id}$. We prove the following:

Proposition 4.5. *Co-symplectic Lie algebras of dimension $2n + 1$ are in one-to-one correspondence with $2n$-dimensional symplectic Lie algebras endowed with a compatible metric and a derivation D which is an infinitesimal symplectic transformation.*

Proof. Let (ϕ, ξ, η, g) be a co-symplectic structure on a Lie algebra $\mathfrak{g}$ of dimension $2n+1$. Set $\mathfrak{h} = \ker(\eta)$. For $u, v \in \mathfrak{h}$ we compute

$$\eta([u, v]) = -d\eta(u, v) = 0,$$

since η is closed. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ inherits an almost complex structure J and a metric g which are compatible. From ϕ and g we obtain the 2-form ω which is closed and nondegenerate by hypothesis. Thus $(\mathfrak{h}, \omega)$ is a symplectic Lie algebra.

Actually $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Indeed, the fact that $\eta(\xi) = 1$ implies that ξ does not belong to $[\mathfrak{g}, \mathfrak{g}]$, and then one has

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \quad \text{and} \quad [\xi, \mathfrak{h}] \subseteq \mathfrak{h}.$$

Thus one can write

$$\mathfrak{g} = \mathbb{R}\xi \oplus_{\mathrm{ad}_\xi} \mathfrak{h}.$$

Since ω is closed, we obtain

$$\begin{aligned} 0 = d\omega(\xi, u, v) &= -\omega([\xi, u], v) + \omega([u, v], \xi) - \omega([v, \xi], u) = \\ &= -\omega(\mathrm{ad}_\xi(u), v) - \omega(u, \mathrm{ad}_\xi(v)). \end{aligned} \tag{4.2}$$

The correspondence $X \mapsto \mathrm{ad}_\xi(X)$ gives a derivation D of $\mathfrak{h}$ (this follows from the Jacobi identity in $\mathfrak{g}$) and the above equality shows that D is an infinitesimal symplectic transformation.

Next suppose we are given a symplectic Lie algebra $(\mathfrak{h}, \omega)$ endowed with a metric g and a derivation $D \in \mathfrak{sp}(\mathfrak{h})$. Set

$$\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{h}$$

and define the following Lie algebra structure on $\mathfrak{g}$:

$$[u, v] := [u, v]_{\mathfrak{h}}, \quad [\xi, u] := D(u), \quad u, v \in \mathfrak{h}.$$

Since D is a derivation of $\mathfrak{h}$, the Jacobi identity holds in $\mathfrak{g}$. Let J denote the almost complex structure compatible with ω and g. Extend J to an endomorphism ϕ of $\mathfrak{g}$ setting $\phi(\xi) = 0$ and extend g so that ξ has length 1 and ξ is orthogonal to $\mathfrak{h}$. Also, let η be the dual 1-form with respect to the metric g. It is immediate to see that $d\eta = 0$. On the other hand, equation (4.2) shows that $d\omega = 0$ as D is an infinitesimal symplectic transformation. Thus $\mathfrak{g}$ is a co-symplectic Lie algebra. □

Remark 4.1. If one wants to obtain a co-symplectic *nilpotent* Lie algebra, then the initial data in Proposition 4.5 must be a symplectic nilpotent Lie algebra and a nilpotent derivation D. This gives a way to classify co-symplectic nilpotent Lie algebras in dimension $2n+1$ starting from nilpotent symplectic Lie algebras in dimension $2n$ and a nilpotent derivation.

4.4 Minimal models of mapping tori

In this section we study the formality of the mapping torus of a orientation-preserving diffeomorphism of a manifold. We start with some useful results.

Lemma 4.1. *Let N be a differentiabe manifold and let $\varphi : N \to N$ be a diffeomorphism. Let $M = N_\varphi$ denote the mapping torus of φ. Then the cohomology of M sits in an exact sequence*

$$0 \to C^{p-1} \to H^p(M) \to K^p \to 0,$$

where K^p is the kernel of $\varphi^* - \mathrm{id}\colon H^p(N) \to H^p(N)$, and C^p is its cokernel.

Proof. This is a simple application of the Mayer-Vietoris sequence. Take U, V two open intervals covering S^1, where $U \cap V$ is the disjoint union of two intervals. Let $U' = \pi^{-1}(U)$, $V' = \pi^{-1}(V)$. Then $H^p(U') \cong H^p(N)$, $H^p(V') \cong H^p(N)$ and $H^p(U' \cap V') \cong H^p(N) \oplus H^p(N)$. The Mayer-Vietoris sequence associated to this covering becomes

$$\begin{aligned} \ldots \to H^p(M) \to H^p(N) \oplus H^p(N) \xrightarrow{F} H^p(N) \oplus H^p(N) \to H^{p+1}(M) \to \\ \to H^{p+1}(N) \oplus H^{p+1}(N) \to \ldots \end{aligned} \tag{4.3}$$

where the map F is $([\alpha], [\beta]) \mapsto ([\alpha] - [\beta], [\alpha] - \varphi^*[\beta])$.

Write

$$K = \ker\Big(\varphi^* - \mathrm{id} : H^*(N) \to H^*(N)\Big)$$
$$C = \mathrm{coker}\Big(\varphi^* - \mathrm{id} : H^*(N) \to H^*(N)\Big).$$

These are graded vector spaces $K = \bigoplus K^p$, $C = \bigoplus C^p$. The exact sequence (4.3) then yields an exact sequence $0 \to C^{p-1} \to H^p(M) \to K^p \to 0$. □

Let us look more closely to the exact sequence in Lemma 4.1. First take $[\beta] \in C^{p-1}$. Then $[\beta]$ can be thought as an element in $H^{p-1}(N)$ modulo $\mathrm{im}(\varphi^* - \mathrm{id})$. The map $C^{p-1} \to H^p(M)$ in Lemma 4.1 is the connecting homomorphism δ^*. This is worked out as follows [15]: take a smooth function $\rho(t)$ on U which equals 1 in one of the intervals of $U \cap V$ and zero on the other. Then

$$\delta^*[\beta] = [d\rho \wedge \beta].$$

Write $\tilde\beta = d\rho \wedge \beta$. If we put the point $t = 0$ in $U \cap V$, then clearly $\tilde\beta(x, 0) = \tilde\beta(x, 1) = 0$, so $\tilde\beta$ is a well-defined closed p-form on M. (Note that $[d\rho] = [\eta] \in H^1(S^1)$, where $\eta = \pi^*(\theta) = dt$, so morally $[\tilde\beta] \in H^p(M)$ is $[\eta \wedge \beta]$.)

On the other hand, if $[\alpha] \in K^p$, then $\varphi^*[\alpha] = [\alpha]$. So $\varphi^*\alpha = \alpha + d\theta$, for some $(p-1)$-form θ. Let us take a function $\rho : [0, 1] \to [0, 1]$ such that $\rho \equiv 0$ near $t = 0$ and $\rho \equiv 1$ near $t = 1$. Then, the closed p-form $\tilde\alpha$ on $N \times [0, 1]$ given by

$$\tilde\alpha(x, t) = \alpha(x) + d(\rho(t)\theta(x)), \tag{4.4}$$

where $x \in N$ and $t \in [0,1]$, defines a closed p-form $\tilde{\alpha}$ on M. Indeed, $\varphi^*\tilde{\alpha}(x,0) = \varphi^*\alpha = \alpha + d\theta = \tilde{\alpha}(x,1)$. Moreover, the class $[\tilde{\alpha}] \in H^p(M)$ restricts to $[\alpha] \in H^p(N)$. This gives a splitting

$$H^p(M) \cong C^{p-1} \oplus K^p.$$

Theorem 4.3. *Let N be an oriented compact differentiable manifold of dimension n, and let $\varphi : N \to N$ be an orientation-preserving diffeomorphism. Let $M = N_\varphi$ be the mapping torus of φ. Suppose that for some $p > 0$, the eigenvalue $\lambda = 1$ of*

$$\varphi^* : H^p(N) \to H^p(N)$$

is double. Then M is non-formal since there exists a non-zero (triple) Massey product. More precisely, let $[\alpha] \in K^p \subset H^p(N)$ such that

$$[\alpha] \in \operatorname{im}\Big(\varphi^* - \mathrm{id} : H^p(N) \to H^p(N)\Big),$$

then the Massey product $\langle [\eta], [\eta], [\tilde{\alpha}] \rangle$ does not vanish.

Proof. First, we notice that if the eigenvalue $\lambda = 1$ of $\varphi^* : H^p(N) \to H^p(N)$ is double, then there exists $[\alpha] \in H^p(N)$ satisfying the conditions mentioned in Theorem 4.3. In fact, denote by

$$E = \ker{(\varphi^* - \mathrm{id})^2}$$

the eigenspace corresponding to $\lambda = 1$, which is graded $E = \bigoplus E^p$. Then $K = \ker(\varphi^* - \mathrm{id}) \subset E$ is a proper subspace. Take

$$[\beta] \in E^p \setminus K^p \subset H^p(N) \quad \text{and} \quad [\alpha] = \varphi^*[\beta] - [\beta]. \tag{4.5}$$

Thus $[\alpha] \in K^p \cap \operatorname{im}\Big(\varphi^* - \mathrm{id} : H^p(N) \to H^p(N)\Big)$, and Lemma 4.1 says that the Massey product $\langle [\eta], [\eta], [\tilde{\alpha}] \rangle$ is well-defined. In order to prove that it is non-zero we proceed as follows. Clearly,

$$C \cong E/I, \quad \text{where} \quad I = \operatorname{im}(\varphi^* - \mathrm{id}) \cap E.$$

As φ is an orientation-preserving diffeomorphism, the Poincaré duality pairing satisfies that $\langle \varphi^*(u), \varphi^*(v) \rangle = \langle u, v \rangle$, for $u \in H^p(N)$, $v \in H^{n-p}(N)$. Therefore the λ-eigenspace of φ^*, E_λ, pairs non-trivially only with $E_{1/\lambda}$. In particular, Poincaré duality gives a perfect pairing

$$E^p \times E^{n-p} \to \mathbb{R}.$$

Now $K^p \times I^{n-p}$ is sent to zero: if $x \in \ker(\varphi^* - \mathrm{id})$ and $y = \varphi^*(z) - z$, then $\langle x, y \rangle = \langle x, \varphi^*(z) - z \rangle = \langle x, \varphi^*(z) \rangle - \langle x, z \rangle = \langle \varphi^*(x), \varphi^*(z) \rangle - \langle x, z \rangle = 0$. Therefore there is a perfect pairing

$$E^p/K^p \times I^{n-p} \to \mathbb{R}.$$

Take $[\beta]$ and $[\alpha]$ as in (4.5). By the discussion above about Poincaré duality, there is some $[\xi] \in I^{n-p}$ such that

$$\langle [\beta], [\xi] \rangle \neq 0.$$

Note that in particular, $[\xi]$ pairs trivially with all elements in K^p.

Consider now the form $\tilde{\alpha}$ on M corresponding to α as in (4.4), $[\tilde{\alpha}] \in H^p(M)$. Let us take the p-form γ on N defined by

$$\gamma = \int_0^1 \tilde{\alpha}(x, s) ds.$$

Then $[\gamma] = [\alpha] = \varphi^*[\beta] - [\beta]$ on N. Hence we can write

$$\gamma = \varphi^* \beta - \beta + d\sigma,$$

for some $(p-1)$-form σ on N. Now let us set

$$\tilde{\gamma}(x, t) = \left(\int_0^t \tilde{\alpha}(x, s) ds \right) + \beta + d(\zeta(t)(\varphi^*)^{-1}\sigma),$$

where $\zeta(t)$, $t \in [0, 1]$, equals 1 near $t = 0$, and equals 0 near $t = 1$. Then

$$\varphi^*(\tilde{\gamma}(x, 0)) = \varphi^*(\beta + d((\varphi^*)^{-1}\sigma)) = \varphi^*\beta + d\sigma = \gamma + \beta = \tilde{\gamma}(x, 1),$$

so $\tilde{\gamma}$ is a well-defined p-form on M. Moreover,

$$d(\tilde{\gamma}(x, t)) = dt \wedge \tilde{\alpha}(x, t)$$

on the mapping torus M. Therefore we have the Massey product

$$\langle [dt], [dt], [\tilde{\alpha}] \rangle = [dt \wedge \tilde{\gamma}]. \tag{4.6}$$

We need to see that this Massey product is non-zero. For this, we multiply against $[\tilde{\xi}]$, where $\tilde{\xi}$ is the $(n-p)$-form on M naturally associated to ξ. Recall that $[\xi] \in I^{n-p} \subset K^{n-p} \subset H^{n-p}(M)$. We have

$$\langle [dt \wedge \tilde{\gamma}], [\tilde{\xi}] \rangle = \int_M dt \wedge \tilde{\gamma} \wedge \tilde{\xi} = \int_0^1 \left(\int_{N \times \{t\}} \tilde{\gamma} \wedge \tilde{\xi} \right) dt\,.$$

Restricting to the fibers, we have $[\tilde{\gamma}|_{N \times \{t\}}] = t[\alpha] + [\beta]$ and $[\tilde{\xi}|_{N \times \{t\}}] = [\xi]$. Moreover, $\langle [\alpha], [\xi] \rangle = 0$ and $\langle [\beta], [\xi] \rangle = \kappa \neq 0$. So $\int_{N \times \{t\}} \tilde{\gamma} \wedge \tilde{\xi} = \kappa \neq 0$. Therefore

$$\langle [dt \wedge \tilde{\gamma}], [\tilde{\xi}] \rangle = \kappa \neq 0\,.$$

Now the indeterminacy of the Massey product is in the space

$$\mathcal{I} = [\tilde{\alpha}] \wedge H^1(M) + [\eta] \wedge H^p(M).$$

To see that the Massey product (4.6) does not live in $\mathcal{I}$, it is enough to see that the elements in $\mathcal{I}$ pair trivially with $[\tilde{\xi}]$. On the one hand, $\tilde{\alpha} \wedge \tilde{\xi}$ is exact in every fiber (since $\langle [\alpha], [\xi] \rangle = 0$ on N). Therefore $[\tilde{\alpha}] \wedge [\tilde{\xi}] = 0$. On the other hand, $H^p(M) \cong C^{p-1} \oplus K^p$. The elements corresponding to C^{p-1} have all a dt-factor. Hence the elements in $[\eta] \wedge H^p(M)$ are of the form $[dt \wedge \tilde{\delta}]$, for some $[\delta] \in K^p \subset H^p(N)$. But then $\langle [dt \wedge \tilde{\delta}], [\tilde{\xi}] \rangle = \int_M dt \wedge \tilde{\delta} \wedge \tilde{\xi} = \langle [\delta], [\xi] \rangle = 0$. □

Remark 4.2. The non-formality of the mapping torus M is proved in [34, Proposition 9] when $p = 1$ and the eigenvalue $\lambda = 1$ has multiplicity $r \geq 2$, by a different method.

We finish this section with the following result, which gives a partial computation of the minimal model of M.

From now on we write

$$\varphi_k^* : H^k(N) \to H^k(N),$$

for each $1 \leq k \leq n$, the induced morphism on cohomology by a diffeomorphism $\varphi : N \to N$.

Theorem 4.4. *Suppose that there is some $p \geq 2$ such that φ_k^* does not have the eigenvalue $\lambda = 1$ (i.e. $\varphi_k^* - \mathrm{id}$ is invertible) for any $k \leq (p-1)$, and that φ_p^* does have the eigenvalue $\lambda = 1$ with some multiplicity $r \geq 1$. Denote*

$$K_j = \ker \Big((\varphi_p^* - \mathrm{id})^j : H^p(N) \to H^p(N) \Big),$$

for $j = 0, \ldots, r$. So $\{0\} = K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_r$. Write $G_j = K_j / K_{j-1}$, $j = 1, \ldots, r$. The map $F = \varphi_p^* - \mathrm{id}$ induces maps $F : G_j \to G_{j-1}$, $j = 1, \ldots, r$ (here $G_0 = 0$).

Then the minimal model of M is, up to degree p, given by the following generators:

$$\begin{aligned}
&W^1 = \langle a \rangle, \qquad da = 0, \\
&W^k = 0, \qquad k = 2, \ldots, p-1, \\
&W^p = G_1 \oplus G_2 \oplus \ldots \oplus G_r, \qquad dw = a \cdot F(w),\ w \in G_j.
\end{aligned}$$

Proof. We need to construct a map of differential algebras

$$\rho : (\bigwedge (W^1 \oplus W^p), d) \to (\Omega^*(M), d)$$

which induces an isomorphism in cohomology up to degree p and an injection in degree $p+1$ (see [28]). By Lemma 4.1, we have that

$$\begin{aligned}
H^1(M) &= \langle [dt] \rangle, \\
H^k(M) &= 0, \qquad 1 \leq k \leq p-1, \\
H^p(M) &= \ker(\varphi_p^* - \mathrm{id}) = K_1, \\
H^{p+1}(M) &= \big([dt] \wedge \mathrm{coker}(\varphi_p^* - \mathrm{id}) \big) \oplus \ker(\varphi_{p+1}^* - \mathrm{id})
\end{aligned}$$

We start by setting $\rho(a) = dt$, where t is the coordinate of $[0,1]$ in the description

$$M = (N \times [0,1])/(x,0) \sim (\varphi(x),1)\,.$$

This automatically gives that ρ induces an isomorphism in cohomology up to degree $p-1$. Now let us go to degree p. Take a Jordan component of φ_p^* for the eigenvalue $\lambda = 1$. Let $1 \le j_0 \le r$ be its size. Then we may take $v \in K_{j_0} \setminus K_{j_0-1}$ in it. First, this implies that $v \notin I = \mathrm{im}(\varphi_p^* - \mathrm{id})$. Set

$$v_j = (\varphi_p^* - \mathrm{id})^{j_0-j} v \in K_j\,,$$

for $j = 1, \ldots, j_0$. Now let b_j denote the class of v_j on $G_j = K_j/K_{j-1}$. Then $d(b_j) = a \cdot b_{j-1}$. We want to define ρ on $b_1, \ldots, b_{j_0}$. For this, we need to construct forms $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{j_0} \in \Omega^p(M)$ such that $[\tilde{\alpha}_1]$ represents $v_1 \in K_1 = H^p(M)$, and

$$d\tilde{\alpha}_j = dt \wedge \tilde{\alpha}_{j-1}\,.$$

Then we set $\rho(b_j) = \tilde{\alpha}_j$, and ρ is a map of differential algebras.

We work inductively. Let $v_j = [\alpha_j] \in H^p(N)$. Here $\varphi^*[\alpha_j] - [\alpha_j] = [\alpha_{j-1}]$. As $\varphi^*[\alpha_1] - [\alpha_1] = 0$, we have that $\varphi^*\alpha_1 = \alpha_1 + d\theta_1$. Set

$$\tilde{\alpha}_1(x,t) = \alpha_1(x) + d(\zeta(t)\theta_1(x)),$$

where $\zeta : [0,1] \to [0,1]$ is a smooth function such that $\zeta \equiv 0$ near $t = 0$ and $\zeta \equiv 1$ near $t = 1$. Clearly, $[\tilde{\alpha}_1] = [\alpha_1] = v_1$.

Assume by induction that $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j$ have been constructed, and moreover satisfying that

$$[\tilde{\alpha}_k|_{N\times\{t\}}] = [\alpha_k] + \sum_{i=1}^{k-1} c_{ik}(t)[\alpha_i],$$

for some polynomials $c_{ik}(t)$, $k = 1, \ldots, j$. Note that the result holds for $k = 1$. To construct $\tilde{\alpha}_{j+1}$, we work as follows. We define

$$\gamma_j(x) = \int_0^1 \left(\tilde{\alpha}_j - \sum_{i=1}^{j-1} c_i \tilde{\alpha}_i \right) dt$$

This is a closed form on N. The constants c_i are adjusted so that $[\gamma_j] = [\alpha_j] = v_j = \varphi^*[\alpha_{j+1}] - [\alpha_{j+1}]$. So we can write

$$\gamma_j = \varphi^*\alpha_{j+1} - \alpha_{j+1} - d\theta_{j+1}$$

for some $(p-1)$-form θ_{j+1} on N. Write

$$\hat{\alpha}_{j+1} = \int_0^t \tilde{\alpha}_j(x,s)ds + \alpha_{j+1} + d(\zeta(t)\theta_{j+1}(x))\,.$$

This is a p-form well-defined in M since $\varphi_p^*(\hat{\alpha}_{j+1}(x,0)) = \varphi_p^*(\alpha_{j+1}) = \gamma_j + \alpha_{j+1} + d\theta_{j+1} = \hat{\alpha}_{j+1}(x,1)$. Set

$$\tilde{\alpha}_{j+1} = \hat{\alpha}_{j+1} + \sum_{i<j} c_i \tilde{\alpha}_{i+1}$$

Then

$$d\tilde{\alpha}_{j+1} = dt \wedge \tilde{\alpha}_j\,.$$

Finally,

$$[\tilde{\alpha}_{j+1}|_{N\times\{t\}}] = [\alpha_{j+1}] + \sum_{i=1}^{j} c_i(t)[\alpha_i]\,,$$

for some $c_i(t)$, as required.

Repeating this procedure with all Jordan components, we finally get

$$\rho : (\bigwedge(W^1 \oplus W^p), d) \to (\Omega^*(M), d).$$

Clearly $H^p(\bigwedge(W^1 \oplus W^p)) = K_1$, so ρ^* is an isomorphism on degree p. For degree $p+1$, $H^{p+1}(\bigwedge(W^1 \oplus W^p))$ is generated by the elements $a \cdot b$, where $b \in G_{j_0}$ corresponds to some $v \in K_{j_0}$ generating a Jordan block (equivalently, $v \notin I$). These elements generate $\mathrm{coker}(\varphi_p^* - \mathrm{id})$, i.e.

$$H^{p+1}(\bigwedge(W^1 \oplus W^p)) \cong \mathrm{coker}(\varphi_p^* - \mathrm{id}).$$

An element $v = v_{j_0}$ is sent, by ρ, to a p-form $\tilde{\alpha}_{j_0}$ on M, which satisfies

$$[\tilde{\alpha}_{j_0}|_{N\times\{t\}}] = [\alpha_{j_0}] + \sum_{i=1}^{j_0-1} c_i[\alpha_i]\,,$$

for some $c_i = c_i(t)$, following the previous notations. Therefore the class $[dt \wedge \tilde{\alpha}_{j_0}]$ corresponds to $[dt] \wedge [\alpha_{j_0}]$, in the notation of Lemma 4.1. So

$$\rho^* : H^{p+1}(\bigwedge(W^1 \oplus W^p)) \to H^{p+1}(M)$$

is the injection into the subspace $[dt] \wedge \mathrm{coker}(\varphi_p^* - \mathrm{id})$. This completes the proof of the theorem. □

Note that, in the notation of Proposition 4.2, we have that $C^1 = W^1$, $C^p = G_1$ and $N^p = G_2 \oplus \ldots \oplus G_r$. Also take $w \in G_r$. Then $a \cdot w \in I(N)$, $d(a \cdot w) = 0$, but $a \cdot w$ is not exact. Hence

Corollary 4.1. *In the conditions of Theorem 4.4, if $r \geq 2$, then M is non-formal. Moreover, if $r = 1$, then M is p-formal (in the sense of Definition 4.1).*

Applying this to symplectic mapping tori, we have the following. Let N be a compact symplectic $2n$-manifold, and assume that $\varphi : N \to N$ is a symplectomorphism such that the map induced on cohomology $\varphi_1^* : H^1(N) \to H^1(N)$ does not have the eigenvalue $\lambda = 1$. As $\varphi_2^* : H^2(N) \to H^2(N)$ always has the eigenvalue $\lambda = 1$ (φ^* fixes the symplectic form), then we have that N_φ is 2-formal if and only if the eigenvalue $\lambda = 1$ of φ_2^* has multiplicity $r = 1$.

If $n = 2$, then N_φ is a 5-dimensional co-symplectic manifold with $b_1 = 1$. In dimension 5, Theorem 4.1 says that 2-formality is equivalent to formality. Therefore we have the following result:

Corollary 4.2. *5-dimensional non-formal co-symplectic manifolds with $b_1 = 1$ are given as mapping tori of symplectomorphisms $\varphi : N \to N$ of compact symplectic 4-manifolds N where φ_1^* does not have the eigenvalue $\lambda = 1$ and φ_2^* has the eigenvalue $\lambda = 1$ with multiplicity $r \geq 2$.*

Finally, let us mention that an analogue of Theorem 4.4 for $p = 1$ is harder to obtain. However, at least we can still say that if $\lambda = 1$ is an eigenvalue of φ_1^* with multiplicity $r \geq 2$, then $M = N_\varphi$ is non-formal (by Remark 4.2).

4.5 Geography of non-formal co-symplectic compact manifolds

In this section we consider the following problem:

> For which pairs $(m = 2n+1, b)$, with $n, b \geq 1$, there are compact co-symplectic manifolds of dimension m and with $b_1 = b$ which are non-formal?

It will turn out that the answer is the same as for compact smooth manifolds [35], i.e, that there are non-formal examples if and only if $m = 3$ and $b \geq 2$, or $m \geq 5$ and $b \geq 1$. We start with some straightforward examples:

- For $b = 1$ and $m \geq 9$, we may take a compact non-formal symplectic manifold N of dimension $m - 1 \geq 8$ and simply-connected. Such manifold exists for dimensions ≥ 10 by [4], and for dimension equal to 8 by [36]. Then consider $M = N \times S^1$.

- For $m = 3$, $b = 2$, we may take the 3-dimensional nilmanifold M_0 defined by the structure equations $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$. This is non-formal since it is not a torus. The pair $\eta = e^1$, $F = e^2 \wedge e^3$ defines a co-symplectic structure on M_0 since $d\eta = dF = 0$ and $\eta \wedge F \neq 0$.

- For $m \geq 5$ and $b \geq 2$ even take the co-symplectic compact manifold $M = M_0 \times \Sigma_k \times (S^2)^\ell$, where Σ_k is the surface of genus $k \geq 0$, $\ell \geq 0$, and $(S^2)^\ell$ is the product of ℓ copies of S^2. Then $\dim M = m = 5 + 2\ell$ and $b_1(M) = 2 + 2k$.

- For $m = 5$ and $b = 3$, we can take $M_1 = N \times S^1$, where N is a compact 4-dimensional symplectic manifold with $b_1 = 2$. For example, take N the compact nilmanifold defined by the equations $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$, $de^4 = e^1 \wedge e^3$, which is non-formal and symplectic with $\omega = e^1 \wedge e^4 + e^2 \wedge e^3$.

- For $m \geq 7$ and $b \geq 3$ odd, take $M = M_1 \times \Sigma_k \times (S^2)^\ell$, $k, \ell \geq 0$.

Other examples with $b_1 = 2$ and $m = 5$ can be obtained from the list of 5-dimensional compact nilmanifolds. According to the classification in [7, 67] of nilpotent Lie algebras of dimension < 7, there are 9 nilpotent Lie algebras $\mathfrak{g}$ of dimension 5, and only 3 of them satisfy $\dim H^1(\mathfrak{g}^*) = 2$, namely

$$(0, 0, 12, 13, 14 + 23), \quad (0, 0, 12, 13, 14), \quad (0, 0, 12, 13, 23).$$

In the description of the Lie algebras $\mathfrak{g}$, we are using the structure equations with respect to a basis $e^1, \ldots, e^5$ of the dual space $\mathfrak{g}^*$. For instance, $(0, 0, 12, 13, 14 + 23)$ means that there is a basis $\{e^j\}_{j=1}^5$ satisfying $de^1 = de^2 = 0$, $de^3 = e^1 \wedge e^2$, $de^4 = e^1 \wedge e^3$ and $de^5 = e^1 \wedge e^4 + e^2 \wedge e^3$; equivalently, the Lie bracket is given in terms of its dual basis $\{e_j\}_{j=1}^5$ by $[e_1, e_2] = -e_3$, $[e_1, e_3] = -e_4$, $[e_1, e_4] = [e_2, e_3] = -e_5$. Also, from now on we write $e^{ij} = e^i \wedge e^j$.

Proposition 4.6. *Among the* 3 *nilpotent Lie algebras* $\mathfrak{g}$ *of dimension* 5 *with* $\dim H^1(\mathfrak{g}^*) = 2$*, those that have a co-symplectic structure are*

$$(0, 0, 12, 13, 14 + 23), \quad (0, 0, 12, 13, 14).$$

Proof. Clearly the forms η and F given by

$$\eta = e^1, \quad F = e^{25} - e^{34}$$

satisfy $d\eta = dF = 0$ and $\eta \wedge F^2 \neq 0$, and so they define a co-sympectic structure on each of those Lie algebras.

To prove that the Lie algebra $(0, 0, 12, 13, 23)$ does not admit a co-sympectic structure, one can check it directly or using that the direct sum of $(0, 0, 12, 13, 23)$ with the 1-dimensional Lie algebra has no symplectic forms [7]. $\square$

Remark 4.3. Let N denote the 5-dimensional compact nilmanifold associated to the Lie algebra $\mathfrak{n}$ with structure $(0,0,12,13,23)$. Then N has a closed 1-form; indeed, $de_1 = de_2 = 0$. By Tischler's theorem [90], N is a mapping torus. However, it is not a *symplectic* mapping torus, since it is not co-symplectic. We describe this mapping torus explicitly. Since N is a nilmanifold, we can describe the structure at the level of Lie algebras. The map $\mathfrak{n} \to \mathbb{R}$, $(e_1, \ldots, e_5) \to e_1$ gives an exact sequence

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{n} \longrightarrow \mathbb{R} \longrightarrow 0 \tag{4.7}$$

of Lie algebras, and one sees immediately that $\mathfrak{k}$ is a 4-dimensional nilpotent Lie algebra, spanned by $e_2, \ldots, e_5$, with structure $(0,0,0,23)$ (with respect to the dual basis of $\mathfrak{k}^*$). $\mathfrak{k}$ is symplectic, a symplectic form being, for instance, $\omega = e^{25} + e^{34}$. The fiber of the corresponding bundle over S^1 is the Kodaira-Thurston manifold KT. Taking into account the proof of Proposition 4.5, the Lie algebra extension (4.7) is associated to the derivation $D = \mathrm{ad}(e_1)$ of $\mathfrak{k}$. In other words, $\mathfrak{n} = \mathbb{R} \oplus_D \mathfrak{k}$. A computation shows that this derivation is not symplectic and Proposition 4.5 implies that $\mathfrak{n}$ is not co-symplectic. The map $\varphi := \exp(D)$ is a diffeomorphism of KT which does not preserve the symplectic structure of KT, and $N = KT_\varphi$.

The previous examples leave some gaps, notably the cases $m = 3$, $b \geq 3$, and $m = 5$, $b = 1$. By [35], we know that there are compact non-formal manifolds with these Betti numbers and dimensions. Let us see that there are also non-formal co-symplectic manifolds in these cases.

Proposition 4.7. *There are non-formal compact co-symplectic manifolds with $m \geq 3$, $b_1 \geq 2$.*

Proof. We consider the symplectic surface Σ_k of genus $k \geq 1$. Consider a symplectomorphism $\varphi : \Sigma_k \to \Sigma_k$ such that $\varphi^* : H^1(\Sigma_k) \to H^1(\Sigma_k)$ has the form

$$\varphi^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with respect to a symplectic basis $\xi_1, \xi_2, \ldots, \xi_{2k-1}, \xi_{2k}$ of $H^1(\Sigma_k)$. Consider the mapping torus M of φ. The symplectic form of Σ_k induces a closed 2-form F on M. The pull-back η of the volume form of S^1 under $M \to S^1$ is closed and satisfies that $\eta \wedge F > 0$. Therefore M is co-symplectic.

Now $\varphi^*\xi_1 = \xi_1 + \xi_2$ and $\varphi^*\xi_i = \xi_i$, for $2 \leq i \leq 2k$. By Lemma 4.1, the cohomology of M is

$$\begin{aligned} H^1(M) &= \langle a, \xi_2, \ldots, \xi_{2k-1}, \xi_{2k} \rangle, \\ H^2(M) &= \langle F, a\,\xi_1, a\,\xi_3, \ldots, a\,\xi_{2k-1}, a\,\xi_{2k} \rangle, \end{aligned}$$

where $a = [\eta]$. So $b_1 = 2k \geq 2$. By Theorem 4.3, the Massey product $\langle a, a, \xi_2 \rangle$ does not vanish and so M is non-formal.

Similarly, taking Σ_k where $k \geq 2$. We consider a symplectomorphism $\psi : \Sigma_k \to \Sigma_k$ such that $\psi^* : H^1(\Sigma_k) \to H^1(\Sigma_k)$ has the form

$$\psi^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the mapping torus M of ψ has $b_1 = 2k-1 \geq 3$ and odd, and M is co-symplectic and non-formal.

For higher dimensions, take $M \times (S^2)^\ell$, $\ell \geq 0$. □

Remark 4.4. Notice that the case $k = 1$ in the first part of the previous proposition yields another description of the Heisenberg manifold.

Proposition 4.8. *There are non-formal compact co-symplectic manifolds with $m \geq 5$, $b_1 = 1$.*

Proof. It is enough to construct an example for $m = 5$. Take the torus T^4 and the mapping torus T^4_φ of the symplectomorphism $\varphi : T^4 \to T^4$ such that

$$\varphi^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

on $H^1(T^4)$. Taking η the pull-back of the 1-form θ on S^1 and $F = e^1 \wedge e^2 + e^3 \wedge e^4$, we have that T^4_φ is co-symplectic. The map φ^* on $H^2(T^4)$ satisfies:

$$\begin{aligned}
\varphi^*(e^1 \wedge e^2) &= e^1 \wedge e^2 \\
\varphi^*(e^1 \wedge e^3) &= e^1 \wedge e^3 - e^1 \wedge e^4 \\
\varphi^*(e^1 \wedge e^4) &= e^1 \wedge e^4 \\
\varphi^*(e^2 \wedge e^3) &= e^2 \wedge e^3 - e^2 \wedge e^4 \\
\varphi^*(e^2 \wedge e^4) &= e^2 \wedge e^4 \\
\varphi^*(e^3 \wedge e^4) &= e^3 \wedge e^4
\end{aligned}$$

Then $b_1(T^4_\varphi) = 1$ as $H^1(T^4_\varphi) = \langle a \rangle$, with $a = [\eta]$. Also $H^2(T^4_\varphi) = \langle e^{12}, e^{14}, e^{24}, e^{34} \rangle$. In particular, notice that $\mathrm{im}(\varphi^* - \mathrm{id}) = \langle e^{14}, e^{24} \rangle$. Then $e^{14} \in \ker(\varphi^* - \mathrm{id})$ and $e^{14} \in \mathrm{im}(\varphi^* - \mathrm{id})$. So Theorem 4.3 gives us the non-formality of T^4_φ.

For higher dimensions, take $M = N \times (S^2)^\ell$, where $\ell \geq 0$. Then $\dim M = 5 + 2\ell$ and $b_1(M) = 1$. □

Remark 4.5. Let us show that the 5-manifold T^4_φ is not a solvmanifold, that is, it cannot be written as a quotient of a simply-connected solvable Lie group by a discrete cocompact subgroup[1]. The fiber bundle

$$T^4 \longrightarrow T^4_\varphi \longrightarrow S^1$$

[1]If we define solvmanifold as a quotient $\Gamma \backslash G$, where G is a simply-connected solvable Lie group and $\Gamma \subset G$ is a closed (not necessarily discrete) subgroup, then any mapping torus N_φ, where N is a nilmanifold is of this type (see [75]).

gives a short exact sequence at the level of fundamental groups,

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{4.8}$$

where $H = \pi_1(T^4_\varphi)$. Since $\mathbb{Z}$ is free and $\mathbb{Z}^4$ is abelian, one has $H = \mathbb{Z} \ltimes \mathbb{Z}^4$. Now suppose that T^4_φ is a solvmanifold of the form $\Gamma\backslash G$. According to [75], we have a fibration

$$N \longrightarrow T^4_\varphi \longrightarrow T^k$$

where N is a nilmanifold and T^k is a k-torus. Since $b_1(T^4_\varphi) = 1$, we have $k = 1$ and N is a 4-dimensional nilmanifold. This gives another short exact sequence of groups

$$0 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\Delta = \pi_1(N)$. Looking at the abelianizations, we have $H_1(\Gamma) = \mathbb{Z} \oplus T$, where T is a torsion group. Hence there is a unique morphism $H_1(\Gamma) \to \mathbb{Z}$ and a unique epimorphism $\Gamma \to \mathbb{Z}$ which has to coincide with that of (4.8). Therefore $\Delta = \mathbb{Z}^4$. The Mostow fibration of $\Gamma\backslash G = T^4_\varphi$ coincides with the mapping torus bundle. At the level of Lie groups, it must be $G = \mathbb{R} \ltimes \mathbb{R}^4$ with semidirect product

$$(t, x) \cdot (t', x') = (t + t', x + f(t)x')$$

with f a 1-parameter subgroup in $\mathrm{GL}(4, \mathbb{R})$, i.e. $f(t) = \exp(tg)$ for some matrix g. Moreover, $f(1) = \exp(g) = \varphi^*$. But φ^* can not be the exponential of a matrix. Indeed, if g has real eigenvalues, then φ^* has positive eigenvalues. If g has purely imaginary eigenvalues and diagonalizes, so does φ^*. And if g has complex conjugate eigenvalues but does not diagonalize, then φ^* has two Jordan blocks. None of these cases occur.

4.6 A non-formal solvmanifold of dimension 5 with $b_1 = 1$

In this section we show an example of a non-formal compact co-symplectic 5-dimensional solvmanifold S with first Betti number $b_1(S) = 1$. Actually, S is the mapping torus of a certain diffeomorphism φ of a 4-torus preserving the orientation, so this example fits in the scope of Proposition 4.8.

Let $\mathfrak{g}$ be the abelian Lie algebra of dimension 4. Suppose $\mathfrak{g} = \langle e_1, e_2, e_3, e_4\rangle$, and take the symplectic form $\omega = e^{14} + e^{23}$ on $\mathfrak{g}$, where $\langle e^1, e^2, e^3, e^4\rangle$ is the dual basis for the dual space $\mathfrak{g}^*$ such that the first cohomology group $H^1(\mathfrak{g}^*) = \langle [e^1], [e^2], [e^3], [e^4]\rangle$. Consider the endomorphism of $\mathfrak{g}$ represented, with respect to the chosen basis, by the matrix

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

It is immediate to see that D is an infinitesimal symplectic transformation. Since $\mathfrak{g}$ is abelian, it is also a derivation. Applying Proposition 4.5 we obtain a co-symplectic Lie algebra

$$\mathfrak{h} = \mathbb{R}\xi \oplus \mathfrak{g}$$

with brackets defined by

$$[\xi, e_1] = -e_1 - e_3, \quad [\xi, e_2] = e_2 - e_4, \quad [\xi, e_3] = -e_3 \quad \text{and} \quad [\xi, e_4] = e_4.$$

One can check that $\mathfrak{h} = \langle e_1, e_2, e_3, e_4, e_5 = \xi \rangle$ is a completely solvable non-nilpotent Lie algebra. *Completely solvable* means that, for every $X \in \mathfrak{h}$, the eigenvalues of the map ad_X are real. We denote by $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ the dual basis for $\mathfrak{h}^*$. The Chevalley-Eilenberg complex of $\mathfrak{h}^*$ is

$$(\wedge(\alpha_1, \ldots, \alpha_5), d)$$

with differential d defined by

$$\begin{aligned}
d\alpha_1 &= -\alpha_1 \wedge \alpha_5, \\
d\alpha_2 &= \alpha_2 \wedge \alpha_5, \\
d\alpha_3 &= -\alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_5, \\
d\alpha_4 &= -\alpha_2 \wedge \alpha_5 + \alpha_4 \wedge \alpha_5, \\
d\alpha_5 &= 0.
\end{aligned}$$

Let H be the simply connected and completely solvable Lie group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix} e^{-x_5} & 0 & 0 & 0 & 0 & x_1 \\ 0 & e^{x_5} & 0 & 0 & 0 & x_2 \\ -x_5 e^{-x_5} & 0 & e^{-x_5} & 0 & 0 & x_3 \\ 0 & -x_5 e^{x_5} & 0 & e^{x_5} & 0 & x_4 \\ 0 & 0 & 0 & 0 & 1 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_i \in \mathbb{R}$, for $1 \leq i \leq 5$. Then a global system of coordinates $\{x_i, 1 \leq i \leq 5\}$ for H is defined by $x_i(a) = x_i$, and a standard calculation shows that a basis for the left invariant 1-forms on H consists of

$$\alpha_1 = e^{x_5} dx_1, \qquad \alpha_2 = e^{-x_5} dx_2, \qquad \alpha_3 = x_5 e^{x_5} dx_1 + e^{x_5} dx_3,$$
$$\alpha_4 = x_5 e^{-x_5} dx_2 + e^{-x_5} dx_4, \qquad \alpha_5 = dx_5.$$

This means that $\mathfrak{h}$ is the Lie algebra of H. We notice that the Lie group H may be described as a semidirect product $H = \mathbb{R} \ltimes_\rho \mathbb{R}^4$, where $\mathbb{R}$ acts on $\mathbb{R}^4$ via the linear transformation $\rho(t)$ of $\mathbb{R}^4$ given by the matrix

$$\rho(t) = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 \\ -te^{-t} & 0 & e^{-t} & 0 \\ 0 & -te^{t} & 0 & e^{t} \end{pmatrix}.$$

Thus the operation on the group H is given by

$$\mathbf{a}\cdot\mathbf{x} = (a_1+x_1e^{-a_5}, a_2+x_2e^{a_5}, a_3+x_3e^{-a_5}-a_5x_1e^{-a_5}, a_4+x_4e^{a_5}-a_5x_2e^{a_5}, a_5+x_5).$$

where $\mathbf{a} = (a_1, \ldots, a_5)$ and similarly for $\mathbf{x}$. Therefore $H = \mathbb{R} \ltimes_\rho \mathbb{R}^4$, where $\mathbb{R}$ is a connected abelian subgroup, and $\mathbb{R}^4$ is the nilpotent commutator subgroup.

Now we show that there exists a discrete subgroup Γ of H such that the quotient space $\Gamma\backslash H$ is compact. To construct Γ it suffices to find some real number t_0 such that the matrix defining $\rho(t_0)$ is conjugate to an element A of the special linear group $\mathrm{SL}(4,\mathbb{Z})$ with distinct real eigenvalues λ and λ^{-1}. Indeed, we could then find a lattice Γ_0 in $\mathbb{R}^4$ which is invariant under $\rho(t_0)$, and take $\Gamma = (t_0\mathbb{Z}) \ltimes_\rho \Gamma_0$. To this end, we choose the matrix $A \in \mathrm{SL}(4,\mathbb{Z})$ given by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{4.9}$$

with double eigenvalues $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. Taking $t_0 = \log(\frac{3+\sqrt{5}}{2})$, we have that the matrices $\rho(t_0)$ and A are conjugated. Indeed, put

$$P = \begin{pmatrix} 1 & \frac{-2(2+\sqrt{5})}{3+\sqrt{5}} & 0 & 0 \\ 1 & \frac{1+\sqrt{5}}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \log(\frac{2}{3+\sqrt{5}}) & \frac{2(2+\sqrt{5})\log(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \\ 0 & 0 & \log(\frac{2}{3+\sqrt{5}}) & -\frac{(1+\sqrt{5})\log(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \end{pmatrix}, \tag{4.10}$$

then a direct calculation shows that $PA = \rho(t_0)P$. So the lattice Γ_0 in $\mathbb{R}^4$ defined by

$$\Gamma_0 = P(m_1, m_2, m_3, m_4)^t,$$

where $m_1, m_2, m_3, m_4 \in \mathbb{Z}$ and $(m_1, m_2, m_3, m_4)^t$ is the transpose of the vector (m_1, m_2, m_3, m_4), is invariant under the subgroup $t_0\mathbb{Z}$. Thus $\Gamma = (t_0\mathbb{Z}) \ltimes_\rho \Gamma_0$ is a cocompact subgroup of H.

We denote by $S = \Gamma\backslash H$ the compact quotient manifold. Then S is a 5-dimensional (non-nilpotent) completely solvable manifold.

Alternatively, S may be viewed as the total space of a T^4-bundle over the circle S^1. In fact, let $T^4 = \Gamma_0\backslash\mathbb{R}^4$ be the 4-dimensional torus and $\varphi\colon \mathbb{Z} \to \mathrm{Diff}(T^4)$ the representation defined as follows: $\varphi(m)$ is the transformation of T^4 covered by the linear transformation of $\mathbb{R}^4$ given by the matrix

$$\rho(mt_0) = \begin{pmatrix} e^{-mt_0} & 0 & 0 & 0 \\ 0 & e^{mt_0} & 0 & 0 \\ -mt_0e^{-mt_0} & 0 & e^{-mt_0} & 0 \\ 0 & -mt_0e^{mt_0} & 0 & e^{mt_0} \end{pmatrix}.$$

So $\mathbb{Z}$ acts on $T^4 \times \mathbb{R}$ by

$$((x_1, x_2, x_3, x_4), x_5) \mapsto (\rho(mt_0) \cdot (x_1, x_2, x_3, x_4)^t, x_5 + m),$$

and S is the quotient $(T^4 \times \mathbb{R})/\mathbb{Z}$. The projection π is given by

$$\pi[(x_1, x_2, x_3, x_4), x_5] = [x_5].$$

Remark 4.6. We notice that S is a mapping torus associated to a certain symplectomorphism $\Phi : T^4 \to T^4$. Indeed, since D is an infinitesimal symplectic transformation, its exponential $\exp(tD)$ is a 1-parameter group of symplectomorphisms of $\mathbb{R}^4$. Notice that $\exp(tD) = \rho(t)$. We saw that there exists a number $t_0 \in \mathbb{R}$ such that $\rho(t_0)$ preserves a lattice $\Gamma_0 \cong \mathbb{Z}^4 \subset \mathbb{R}^4$. Therefore the symplectomorphism $\rho(t_0)$ descends to a symplectomorphism Φ of the 4-torus $\Gamma_0 \backslash \mathbb{R}^4$, whose mapping torus is precisely $\Gamma \backslash H$.

Next, we compute the real cohomology of S. Since S is completely solvable we can use Hattori's theorem [56] which asserts that the de Rham cohomology ring $H^*(S)$ is isomorphic with the cohomology ring $H^*(\mathfrak{h}^*)$ of the Lie algebra $\mathfrak{h}$ of H. For simplicity we denote the left invariant forms $\{\alpha_i\}$, $i = 1, \ldots, 5$, on H and their projections on S by the same symbols. Thus, we obtain

- $H^0(S) = \langle 1 \rangle$,
- $H^1(S) = \langle [\alpha_5] \rangle$,
- $H^2(S) = \langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3] \rangle$,
- $H^3(S) = \langle [\alpha_3 \wedge \alpha_4 \wedge \alpha_5], [(\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3) \wedge \alpha_5 \rangle$,
- $H^4(S) = \langle [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4] \rangle$,
- $H^5(S) = \langle [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5] \rangle$.

The product $H^1(S) \otimes H^2(S) \to H^3(S)$ is given by

$$[\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3] \wedge [\alpha_5] = [(\alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3) \wedge \alpha_5] \quad \text{and} \quad [\alpha_1 \wedge \alpha_2] \wedge [\alpha_5] = 0.$$

Theorem 4.5. *S is a compact co-symplectic 5-manifold which is non-formal and with first Betti number $b_1(S) = 1$.*

Proof. Take the 1-form $\eta = \alpha_5$, and let F be the 2-form on S given by

$$F = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.$$

Then (F, η) defines a co-symplectic structure on S since $dF = d\eta = 0$ and $\eta \wedge F^2 \neq 0$.

We prove the non-formality of S from its minimal model [80]. The minimal model of S is a differential graded algebra $(\mathcal{M}, d)$, with

$$\mathcal{M} = \bigwedge(a) \otimes \bigwedge(b_1, b_2, b_3, b_4) \otimes \bigwedge V^{\geq 3},$$

where the generator a has degree 1, the generators b_i have degree 2, and d is given by $da = db_1 = db_2 = 0$, $db_3 = a \cdot b_2$, $db_4 = a \cdot b_3$. The morphism $\rho\colon \mathcal{M} \to \Omega^*(S)$, inducing an isomorphism on cohomology, is defined by

$$\begin{aligned}
\rho(a) &= \alpha_5, \\
\rho(b_1) &= \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3, \\
\rho(b_2) &= \alpha_1 \wedge \alpha_2, \\
\rho(b_3) &= \frac{1}{2}(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3), \\
\rho(b_4) &= \frac{1}{2}\alpha_3 \wedge \alpha_4.
\end{aligned}$$

Following the notations in Definition 4.1, we have $C^1 = \langle a \rangle$ and $N^1 = 0$, thus *S is 1-formal.* We see that *S is not 2-formal.* In fact, the element $b_4 \cdot a \in N^2 \cdot V^1$ is closed but not exact, which implies that $(\mathcal{M}, d)$ is not 2-formal. Therefore, $(\mathcal{M}, d)$ is not formal. □

Remark 4.7. It can be seen that S is non-formal by computing a *quadruple* Massey product [80] $\langle [\alpha_1 \wedge \alpha_2], [\alpha_5], [\alpha_5], [\alpha_5] \rangle$. As $\alpha_1 \wedge \alpha_2 \wedge \alpha_5 = \frac{1}{2} d(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3)$ and $(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3) \wedge \alpha_5 = d(\alpha_3 \wedge \alpha_4)$, we have

$$\langle [\alpha_1 \wedge \alpha_2], [\alpha_5], [\alpha_5], [\alpha_5] \rangle = \frac{1}{2}[\alpha_3 \wedge \alpha_4 \wedge \alpha_5].$$

This is easily seen to be non-zero modulo the inderterminacies.

Remark 4.8. The non-formality of S in Theorem 4.5 can be also proved with the techniques of section 4.5. By Remark 4.6, S is the mapping torus of a diffeomorphism $\rho(t_0)$ of $T^4 = \Gamma_0 \backslash \mathbb{R}^4$. Conjugating by the matrix P in (4.10), we have that S is the mapping torus of A in (4.9) acting on the standard 4-torus $T^4 = \mathbb{Z}^4 \backslash \mathbb{R}^4$. The action of A on 1-forms leaves no invariant forms, so $b_1(S) = 1$. The action of A on 2-forms is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 \\ -1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

with respect to the basis $\{e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34}\}$. This matrix has eigenvalues $\lambda = \frac{1}{2}(7 \pm 3\sqrt{5})$ (simple) and $\lambda = 1$, with multiplicity 3 (one block of size 1 and another of size 3). Theorem 4.4 implies the non-formality of S.

Remark 4.9. We notice that the previous example S may be generalized to dimension $2n+1$ with $n \geq 2$. For this, it is enough to consider the $(2n+1)$-dimensional completely solvable Lie group H^{2n+1} defined by the structure equations

- $d\alpha_j = (-1)^j \alpha_j \wedge \alpha_{2n+1}$, $j = 1, \ldots, 2n-2$;
- $d\alpha_{2n-1} = -\alpha_1 \wedge \alpha_{2n+1} - \alpha_{2n-1} \wedge \alpha_{2n+1}$;
- $d\alpha_{2n} = -\alpha_2 \wedge \alpha_{2n+1} + \alpha_{2n} \wedge \alpha_{2n+1}$;
- $d\alpha_{2n+1} = 0$.

The co-symplectic structure (η, F) is defined by $\eta = \alpha_{2n+1}$, and $F = \alpha_1 \wedge \alpha_{2n} + \alpha_2 \wedge \alpha_{2n-1} + \alpha_3 \wedge \alpha_4 + \cdots + \alpha_{2n-3} \wedge \alpha_{2n-2}$.

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CHAPTER
FIVE

ON THE STRUCTURE OF CO-KÄHLER MANIFOLDS

Giovanni Bazzoni and John Oprea

Abstract

By the work of Li, a compact co-Kähler manifold M is a mapping torus K_φ, where K is a Kähler manifold and φ is a Hermitian isometry. We show here that there is always a finite cyclic cover $\overline{M}$ of the form $\overline{M} \cong K \times S^1$, where $\cong$ is equivariant diffeomorphism with respect to an action of S^1 on M and the action of S^1 on $K \times S^1$ by translation on the second factor. Furthermore, the covering transformations act diagonally on S^1, K and are translations on the S^1 factor. In this way, we see that, up to a finite cover, all compact co-Kähler manifolds arise as the product of a Kähler manifold and a circle.

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5.1 Recollections on Co-Kähler Manifolds

In [62], H. Li recently gave a structure result for compact co-Kähler manifolds stating that such a manifold is always a Kähler mapping torus. By a *Kähler mapping torus* we mean the mapping torus K_φ of a Hermitian isometry $\varphi : K \to K$ of a Kähler manifold K. A *Hermitian isometry* is a holomorphic map $\varphi : K \to K$ such that $\varphi^* h = h$, where h is the Hermitian metric of K. Note that φ preserves both the Riemannian metric and the symplectic form associated to h.

In this paper, using Li's characterization, we give another type of structure theorem for co-Kähler manifolds based on classical results in [25, 79, 81, 93]. As such, much of this paper is devoted to showing how an unknown interplay between the known geometry and the known topology of co-Kähler manifolds creates beautiful structure. Basic results on co-Kähler manifolds themselves come from [24] (see also [39]) [1].

Let $(M^{2n+1}, J, \xi, \eta, g)$ be an *almost contact metric manifold* given by the conditions

$$J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X, Y) - \eta(X)\eta(Y), \tag{5.1}$$

for vector fields X and Y, I the identity transformation on TM and g a Riemannian metric. Here, ξ is a vector field as well, η is a 1-form and J is a tensor of type $(1,1)$. A local J-basis $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi\}$ may be found with $\eta(X_i) = 0$ for $i = 1, \ldots, n$. The *fundamental* 2*-form* on M is given by

$$\omega(X, Y) = g(JX, Y),$$

and if $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \eta\}$ is a local 1-form basis dual to the local J-basis, then

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Note that $\imath_\xi \omega = 0$.

The geometric structure $(M^{2n+1}, J, \xi, \eta, g)$ is a **co-Kähler** structure on M if

$$[J, J] + 2\, d\eta \otimes \xi = 0 \text{ and } d\omega = 0 = d\eta$$

or, equivalently, J is parallel with respect to the metric g.

A crucial fact that we use in our result is that, on a co-Kähler manifold, the vector field ξ is Killing and parallel and the 1-form η is harmonic. This fact is well known, but the authors were not able to find a direct proof of this fact. It seems worth it to include a proof here.

Lemma 5.1. *On a co-Kähler manifold, the vector field ξ is Killing and parallel. Furthermore, the 1-form η is a harmonic form.*

Proof. The normality condition implies that $L_\xi J = 0$ (see [11]); in particular, $[\xi, JX] = J[\xi, X]$ for every vector field X on M. Compatibility of the metric g with J is expressed by the right-hand relation in (5.1); with $\omega(X, Y) = g(JX, Y)$, it yields

$$g(X, Y) = \omega(X, JY) + \eta(X)\eta(Y). \tag{5.2}$$

[1]The authors of [24] use the term *cosymplectic* for Li's co-Kähler because they view these manifolds as odd-dimensional versions of symplectic manifolds — even as far as being a convenient setting for time-dependent mechanics [60]. Li's characterization, however, makes clear the true underlying Kähler structure, so we have chosen to follow his terminology.

By definition,

$$(L_\xi g)(X,Y) = \xi g(X,Y) - g([\xi,X],Y) - g(X,[\xi,Y]). \tag{5.3}$$

Substituting (5.2) in (5.3), we obtain

$$\begin{aligned}(L_\xi g)(X,Y) = {} & \xi\omega(X,Y) + \xi(\eta(X)\eta(Y)) - \omega([\xi,X],JY) - \eta([\xi,X])\eta(Y) + \\ & - \ \omega(X,J[\xi,Y]) - \eta(X)\eta([\xi,Y]) = \\ = {} & \xi\omega(X,Y) - \omega([\xi,X],JY) - \omega(X,[\xi,JY]) + (\xi\eta(X))\eta(Y) + \\ & + \eta(X)(\xi\eta(Y)) - \eta([\xi,X])\eta(Y) - \eta(X)\eta([\xi,Y]) = \\ = {} & (L_\xi\omega)(X,JY) + \eta(X)(\xi\eta(Y) - \eta([\xi,Y])) + \\ & + \ \eta(Y)(\xi\eta(X) - \eta([\xi,X])) = \\ = {} & \eta(X)(d\eta(\xi,Y) + Y\eta(\xi)) + \eta(Y)(d\eta(\xi,X) + X\eta(\xi)) = \\ = {} & 0.\end{aligned}$$

The last equalities follow from these facts:

- since ω is closed and $\imath_\xi\omega = 0$, $L_\xi\omega = 0$ by Cartan's magic formula;
- $d\eta = 0$;
- as $\eta(\xi) \equiv 1$, one has $X\eta(\xi) = Y\eta(\xi) = 0$.

This proves that ξ is a Killing vector field. In order to show that ξ is parallel, we use the following formula for the covariant derivative ∇ of the Levi-Civita connection of g; for vector fields X, Y, Z on M, one has

$$\begin{aligned}2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + \\ & + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).\end{aligned} \tag{5.4}$$

Setting $Y = \xi$ in (5.4) and recalling that, on any almost contact metric manifold, $g(X,\xi) = \eta(X)$, we obtain

$$\begin{aligned}2g(\nabla_X\xi, Z) = {} & Xg(\xi,Z) + \xi g(X,Z) - Zg(X,\xi) + g([X,\xi],Z) + \\ & + \ g([Z,X],\xi) - g([\xi,Z],X) = \\ = {} & \xi g(X,Z) - g([\xi,X],Z) - g([\xi,Z],X) + X\eta(Z) + \\ & - \ Z\eta(X) - \eta([X,Z]) = \\ = {} & (L_\xi g)(X,Z) + d\eta(X,Z) = \\ = {} & 0.\end{aligned}$$

Since X and Z are arbitrary it follows that $\nabla\xi = 0$.

To prove that η is harmonic, we rely on the following result: a vector field on a Riemannian manifold (M,g) is Killing if and only if the dual 1-form is co-closed. For a proof, see for instance [40, page 107]. Applying this to ξ, we see that η co-closed; since it is closed, it is harmonic. □

Lemma 5.1 will be a key point in our structure theorem below. In fact, in [62], it is shown that we can replace η by a harmonic integral form η_θ with dual parallel vector field ξ_θ and associated metric g_θ, $(1,1)$-tensor J_θ and closed 2-form ω_θ with $i_{\xi_\theta}\omega_\theta = 0$. Then we have the following.

Theorem 5.1 ([62])**.** *With the structure $(M^{2n+1}, J_\theta, \xi_\theta, \eta_\theta, g_\theta)$, there is a compact Kähler manifold (K,h) and a Hermitian isometry $\psi\colon K \to K$ such that M is diffeomorphic to the mapping torus*

$$K_\psi = \frac{K \times [0,1]}{(x,0) \sim (\psi(x),1)}$$

with associated fibre bundle $K \to M = K_\psi \to S^1$.

An important ingredient in Li's theorem is a result of Tischler (see [90]) stating that a compact manifold admitting a non-vanishing closed 1-form fibres over the circle. The above result indicates that co-Kähler manifolds are very special types of manifolds. However it can be very difficult to see whether a manifold is such a mapping torus. In this paper, we will give another characterization of co-Kähler manifolds which we hope will allow an easier identification.

5.2 Parallel Vector Fields

From now on, when we write a co-Kähler structure $(M^{2n+1}, J, \xi, \eta, g)$, we shall mean Li's associated integral and parallel structures. Let's now employ an argument that goes back to [93], but which was resurrected in [81]. Consider the parallel vector field ξ and its associated flow ϕ_t. Because ξ is Killing, each ϕ_t is an isometry of (M,g). Therefore, in the isometry group $\mathrm{Isom}(M,g)$, the subgroup generated by ξ, C, is singly generated. Since M is compact, so is $\mathrm{Isom}(M,g)$ and this means that C is a torus. Using harmonic forms and the Albanese torus, Welsh [93] actually shows that there is a subtorus $T \subseteq C$ such that $M = T \times_G F$ where $G \subset T$ is finite and F is a manifold. Following Sadowski [81], we can modify the argument as follows.

Let $S^1 \subseteq C \subset \mathrm{Isom}(M,g)$ have associated vector field Y. Because S^1 acts on (M,g) by isometries, the vector field Y is Killing. Now, we can choose Y as close to ξ as we like, so at some point $x_0 \in M$, since $\eta(\xi)(x_0) \neq 0$, then $\eta(Y)(x_0) \neq 0$ as well. But η is harmonic and Y is Killing, so this means that $\eta(Y)(x) \neq 0$ for all $x \in M$. Hence, we may take $\eta(Y)(x) > 0$ for all $x \in M$. Now let σ be an orbit of the S^1 action. Then

$$\int_\sigma \eta = \int_0^1 \eta\left(\frac{d\sigma}{dt}\right)\, dt = \int \eta(Y)\, dt > 0.$$

This says that the orbit map $\alpha\colon S^1 \to M$ defined by $g \mapsto g \cdot x_0$ induces a non-trivial composition of homomorphisms

$$H_1(S^1;\mathbb{R}) \stackrel{\alpha_*}{\to} H_1(M;\mathbb{R}) \stackrel{\eta}{\to} H_1(S^1;\mathbb{R}),$$

where $d\eta = 0$ defines an integral cohomology class $\eta \in H^1(M;\mathbb{Z}) \cong [M, S^1]$. Here we use the standard identification of degree 1 cohomology with homotopy classes of maps from M to S^1. Since $H_1(S^1;\mathbb{Z}) = \mathbb{Z}$, this means that the integral homomorphism $\alpha_*\colon H_1(S^1;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is *injective.* Such an action is said to be *homologically injective* (see [25]). Hence, we have

Proposition 5.1. *A co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure supports a smooth homologically injective* S^1 *action.*

In fact, it can be shown that there is a homologically injective T^k action on M, where T^k is Welsh's torus T. However, we shall focus on the S^1-case since this will allow a connection to Li's mapping torus result.

5.3 Sadowski's Transversally Equivariant Fibrations

Homologically injective actions were first considered by P. Conner and F. Raymond in [25] (also see [58]) and were shown to lead to topological product splittings up to finite cover (also see [79]). Homological injectivity for a circle action is very unusual and this points out the extremely special nature of co-Kähler manifolds. Here we want to make use of the results in [81] to achieve smooth splittings for co-Kähler manifolds up to a finite cover. We will state the results of [81] only for the case we are interested in: namely, a mapping torus bundle $M \to S^1$.

Let's begin by recalling that a bundle map $p\colon M \to S^1$ is a *transversally equivariant fibration* if there is a smooth S^1-action on M such that the orbits of the action are transversal to the fibres of p and $p(t\cdot x) - p(x)$ depends on $t \in S^1$ only. This latter condition is simply the usual equivariance condition if we take an appropriate action of S^1 on itself (see [81, Remark 1.1]). Sadowski's key lemma is the following.

Lemma 5.2 ([81, Lemma 1.3])**.** *Let* $p\colon M \to S^1$ *be a smooth* S^1*-equivariant bundle map. Then the following are equivalent;*

1. *The orbits of the* S^1*-action are transversal to the fibres of* p*:*
2. $p_* \circ \alpha_*\colon \pi_1(S^1) \to \pi_1(S^1)$ *is injective, where* $\alpha\colon S^1 \to M$ *is the orbit map;*
3. *One orbit of the* S^1*-action is transversal to a fibre of* p *at a point* $x_0 \in M$*.*

Remark 5.1. Note the following.

1. Lemmas 1.1 and 1.2 of [81] show that, in the situation of Proposition 5.1, $\eta\colon M \to S^1$ is a transversally equivariant bundle map.

2. Note also that, because $\pi_1(S^1) \cong H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, the second condition of Lemma 5.2 is really saying that the action is homologically injective.

As pointed out in [62], *every smooth fibration $K \to M \xrightarrow{p} S^1$ can be seen as a mapping torus of a certain diffeomorphism $\varphi\colon K \to K$*, (also see Proposition 5.3 below). The following is a distillation of [81, Proposition 2.1 and Corollary 2.1] in the case of a circle action.

Theorem 5.2. *Let $M \xrightarrow{p} S^1$ be a smooth bundle projection from a smooth closed manifold M to the circle. The following are equivalent:*

1. *The structure group of p can be reduced to a finite cyclic group $G = \mathbb{Z}_m \subseteq \pi_1(S^1)/(\mathrm{Im}(p_* \circ \alpha_*))$ (i.e. the diffeomorphism φ associated to the mapping torus $M \xrightarrow{p} S^1$ has finite order);*

2. *The bundle map p is transversally equivariant with respect to an S^1-action on M, $A\colon S^1 \times M \to M$.*

Moreover, assuming (1) and (2), there is a finite G-cover $K \times S^1 \to M$ given by the action $(k,t) \mapsto A_t(k)$, where G acts diagonally and by translations on S^1.

Sketch of Proof ([81]). $(1 \Rightarrow 2)$ The bundle is classified by a map $S^1 \to BG$ or, equivalently, by an element of $\pi_1(BG) = G = \mathbb{Z}_m$ (since G is abelian). Now M may be written as a mapping torus K_φ for some diffeomorphism $\varphi \in \mathrm{Diffeo}(K)$ of order m. (So G is the structure group of a mapping torus bundle). Define an S^1-action by $A\colon S^1 \times M \to M$, $A(t,[k,s]) = [k, s+mt]$. (Geometrically, the action is simply winding around the mapping torus m times until we are back to the identity φ^m). Clearly, the action is transversally equivariant.

$(2 \Rightarrow 1)$ Let $A_t\colon M \to M$ be the S^1-action such that p is transversally equivariant. Let K be the fibre of p and let

$$G = \{g \in S^1 \,|\, A_g(K) = K\}.$$

Now, because orbits of the action are transversal to the fibre, G is a proper closed subgroup of S^1. Hence, $G = \mathbb{Z}_m = \langle g \,|\, g^m = 1\rangle$ for some positive integer m. Also note that the transversally equivariant condition saying $p(A_t(x)) - p(x)$ only depends on t implies that the action carries fibres of p to fibres of p. Moreover, fibres are then mapped back to themselves by G. Hence, letting G act diagonally on $K \times S^1$ and by translations on S^1, we see that the action is free and its restriction $A|\colon K \times S^1 \to M$ is a finite G-cover. Now, if we take the piece of the orbit from $x_0 \in K$ to $A_g(x_0)$ for fixed x_0 and $g \in G$, the projection to S^1 gives an element in $\pi_1(S^1) = \mathbb{Z}$. Because the full orbit is strictly longer than this piece, we see that the corresponding element in $\pi_1(S^1) = \mathbb{Z}$ can only be in $\mathrm{Im}(p_* \circ \alpha_*)$ if $g = 1$. Hence, $G \subseteq \pi_1(S^1)/(\mathrm{Im}(p_* \circ \alpha_*))$ which is finite due to homological injectivity. $\square$

We then have the following consequence for co-Kähler manifolds from Proposition 5.1 and Theorem 5.2.

Theorem 5.3. *A compact co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure and mapping torus bundle* $K \to M \to S^1$ *splits as* $M \cong S^1 \times_{\mathbb{Z}_m} K$*, where* $S^1 \times K \to M$ *is a finite cover with structure group* $\mathbb{Z}_m$ *acting diagonally and by translations on the first factor. Moreover,* M *fibres over the circle* $S^1/(\mathbb{Z}_m)$ *with finite structure group.*

Note that Theorem 5.3 provides the following.

Corollary 5.1. *For a compact co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure and mapping torus bundle* $K \to M \to S^1$*, there is a commutative diagram of fibre bundles:*

$$\begin{tikzcd} K \arrow[r] \arrow[d, "="'] & S^1 \times K \arrow[r] \arrow[d, "\times m"'] & S^1 \arrow[d, "\times m"'] \\ K \arrow[r] & K_\psi \arrow[r] & S^1 \end{tikzcd} .$$

where $K_\psi \cong M$ *according to Theorem 5.1 and the notation* $\times m$ *denotes an* $\mathbb{Z}_m$*-covering.*

Remark 5.2. Although we have used the very special results of [81] above, observe that a version of Theorem 5.3 may be proved in the continuous case using the Conner-Raymond Splitting Theorem [25]. In this case, we obtain a finite cover $S^1 \times Y \to M$, where $Y \to K$ is a homotopy equivalence. This type of result affords a possibility of weakening the stringent assumptions on co-Kähler manifolds with a view towards homotopy theory rather than geometry.

5.4 Betti Numbers

A main result of [24] was the fact that the Betti numbers of co-Kähler manifolds increase up to the middle dimension: $b_1 \leq b_2 \leq \ldots \leq b_n = b_{n+1}$ for M^{2n+1}. The argument in [24] was difficult, involving Hodge theory and a type of Hard Lefschetz Theorem for co-Kähler manifolds. In [62], the mapping torus description of co-Kähler manifolds yielded the result topologically through homology properties of the mapping torus. Here, we would like to see the Betti number result as a natural consequence of Theorem 5.3. Recall a basic result from covering space theory.

Lemma 5.3. *If* $\overline{X} \to X$ *is a finite* G*-cover, then*

$$H^*(X; \mathbb{Q}) = H^*(\overline{X}; \mathbb{Q})^G,$$

where the designation H^G *denotes the fixed algebra under the action of the covering transformations* G*.*

In order to see the Betti number relations, we need to know that the "Kähler class" on K is invariant under the covering transformations. The following result guarantees that such a class exists.

Lemma 5.4. *There exists a class $\bar{\omega} \in H^2(K;\mathbb{R})^G \subset H^2(S^1 \times K;\mathbb{R})$ which pulls back to $\omega \in H^2(K;\mathbb{R})$ via the inclusion $K \to S^1 \times K$ contained in Corollary 5.1.*

Proof. Let $\theta\colon S^1 \times K \to M$ denote the $G = \mathbb{Z}_m$- cover of Theorem 5.3 and Corollary 5.1. Then $\theta^*\omega = \eta \times \alpha + \bar{\omega}$, where η generates $H^1(S^1;\mathbb{R})$, $\alpha \in H^1(K;\mathbb{R})$ and $\bar{\omega} \in H^2(K;\mathbb{R})$. Note that $\bar{\omega}$ pulls back to $\omega \in H^2(K;\mathbb{R})$. Also, $\theta^*\omega$ is G-invariant, so for each $g \in G$, we have

$$\begin{aligned} \alpha \times \eta + \bar{\omega} &= g^*(\alpha \times \eta + \bar{\omega}) \\ &= g^*(\alpha) \times g^*(\eta) + g^*(\bar{\omega}) \\ &= g^*(\alpha) \times \eta + g^*(\bar{\omega}), \end{aligned}$$

using the fact that G acts on $K \times S^1$ diagonally and homotopically trivially on S^1. We then get

$$(\alpha - g^*(\alpha)) \times \eta = g^*(\bar{\omega}) - \bar{\omega}.$$

This also means that $g^*(\bar{\omega}) - \bar{\omega} \in H^2(K;\mathbb{R})$ and $(\alpha - g^*(\alpha)) \times \eta \in H^1(K;\mathbb{R}) \otimes H^1(S^1)$. Thus, the only way the equality above can hold is that both sides are zero. Hence, $\bar{\omega}$ is G-invariant. □

Theorem 5.4. *If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then*

$$H^*(M;\mathbb{R}) = H^*(K;\mathbb{R})^G \otimes H^*(S^1;\mathbb{R}),$$

where $G = \mathbb{Z}_m$. Hence, the Betti numbers of M satisfy:

(1) $b_s(M) = \overline{b}_s(K) + \overline{b}_{s-1}(K)$, where $\overline{b}_s(K)$ denotes the dimension of G-invariant cohomology $H^s(K;\mathbb{R})^G$;

(2) $b_1(M) \leq b_2(M) \leq \ldots \leq b_n(M) = b_{n+1}(M)$.

Proof. Lemma 5.3 and the fact that G acts by translations (so homotopically trivially) on S^1 produce $H^*(M;\mathbb{R}) = H^*(K;\mathbb{R})^G \otimes H^*(S^1;\mathbb{R})$. If we denote the Betti numbers of the G-invariant cohomology by $\overline{b}$, then the tensor product splitting gives

$$b_s(M) = \overline{b}_s(K) + \overline{b}_{s-1}(K),$$

using the fact that $\widetilde{H}^1(S^1;\mathbb{R}) = \mathbb{R}$ and vanishes otherwise.

Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for $H^{s-2}(K;\mathbb{R})^G$. According to Lemma 5.4, the class $\omega \in H^2(M;\mathbb{R})$, which comes from $H^2(K;\mathbb{R})$, provides a G-invariant class in $H^2(K;\mathbb{R})$. Furthermore, since K is compact Kähler, $H^*(K;\mathbb{R})$ obeys the Hard Lefschetz Property with respect to ω. Namely, for $j \leq n$, multiplication by powers of ω,

$$\cdot\, \omega^{n-j} \colon H^j(K;\mathbb{R}) \to H^{2n-j}(K;\mathbb{R}),$$

is an isomorphism. In particular, this means that multiplication by each power ω^s, $s \leq n-j$, must be injective. Therefore, for any $s \leq n$, we have an injective homomorphism $\cdot\, \omega \colon H^{s-2}(K;\mathbb{R}) \to H^s(K;\mathbb{R})$. Thus, since $\omega \in H^2(K;\mathbb{R})^G$, we obtain a linearly independent set $\{\omega\alpha_1, \ldots, \omega\alpha_k\} \subset H^s(K;\mathbb{R})^G$. But then we see that, for all $s \leq n$,

$$\overline{b}_{s-2}(K) \leq \overline{b}_s(K).$$

Now, let's compare Betti numbers of M. We obtain

$$\begin{aligned} b_s(M) - b_{s-1}(M) &= \overline{b}_s(K) + \overline{b}_{s-1}(K) - \overline{b}_{s-1}(K) - \overline{b}_{s-2}(K) \\ &= \overline{b}_s(K) - \overline{b}_{s-2}(K) \\ &\geq 0, \end{aligned}$$

by the argument above. Hence, the Betti numbers of M increase up to the middle dimension. □

In [24] it was shown that the first Betti number of a co-Kähler manifold is always odd. (Indeed, it was shown later that, for M co-Kähler, $S^1 \times M$ is Kähler, so this also follows by Hard Lefschetz). Here, we can infer this as a simple consequence of our splitting. Now, K is a Kähler manifold, so $\dim(H^1(K;\mathbb{R}))$ is even and there is a non-degenerate skew symmetric bilinear (i.e. symplectic) form $b \colon H^1(K;\mathbb{R}) \otimes H^1(K;\mathbb{R}) \to H^{2n}(K;\mathbb{R}) \cong \mathbb{R}$ defined by

$$b(\alpha, \beta) = \alpha \cdot \beta \cdot \omega^{n-1}.$$

Let $G = \mathbb{Z}_m = \langle \varphi \,|\, \varphi^m = 1 \rangle$, note that invariance of ω implies $\varphi^*\omega = \omega$ and compute:

$$\begin{aligned} \varphi^*(b)(\alpha, \beta) &= b(\varphi^*\alpha, \varphi^*\beta) \\ &= \varphi^*\alpha \cdot \varphi^*\beta \cdot \omega^{n-1} \\ &= \varphi^*\alpha \cdot \varphi^*\beta \cdot \varphi^*\omega^{n-1} \\ &= \varphi^*(\alpha \cdot \beta \cdot \omega^{n-1}) \\ &= \alpha \cdot \beta \cdot \omega^{n-1} \\ &= b(\alpha, \beta), \end{aligned}$$

where the second last line comes from the fact that $\alpha \cdot \beta \cdot \omega^{n-1} = k \cdot \omega^n$ and $\varphi^*\omega^n = \omega^n$. Hence, φ^* is a symplectic linear transformation on the symplectic vector space $H^1(K;\mathbb{R})$. But now the Symplectic Eigenvalue

Theorem says that the eigenvalue $+1$ occurs with even multiplicity. Thus $\bar{b}_1(K) = \dim(H^1(K;\mathbb{R})^G)$ is even. Hence, by Theorem 5.4 (1), we have the following result.

Corollary 5.2. *The first Betti number of a compact co-Kähler manifold is odd.*

5.5 Fundamental Groups of Co-Kähler Manifolds

An important question about compact Kähler manifolds is exactly what groups arise as their fundamental groups. For instance, every finite group is the fundamental group of a Kähler manifold, while a free group on more than one generator cannot be the fundamental group of a Kähler manifold (see [1] for more properties of these groups). Li's mapping torus result shows that the fundamental group of a compact co-Kähler manifold is always a semidirect product of the form $H \rtimes_\psi \mathbb{Z}$, where H is the fundamental group of a Kähler manifold. As an alternative, note that Theorem 5.3 implies the following.

Theorem 5.5. *If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then $\pi_1(M)$ has a subgroup of the form $H \times \mathbb{Z}$, where H is the fundamental group of a compact Kähler manifold, such that the quotient*

$$\frac{\pi_1(M)}{H \times \mathbb{Z}}$$

is a finite cyclic group.

5.6 Automorphisms of Kähler manifolds

In this section, we connect our results above with certain facts about compact Kähler manifolds and their automorphisms. In order to do this, we first need some general results about mapping tori. Let M be a smooth manifold and let $\varphi\colon M \to M$ be a diffeomorphism. Let M_φ denote the mapping torus of φ. We have the following result.

Proposition 5.2. *The mapping torus M_φ is trivial as a bundle over S^1 (i.e. $M_\varphi \cong M \times S^1$ over S^1) if and only if $\varphi \in \mathrm{Diff}_0(M)$, where $\mathrm{Diff}_0(M)$ denotes the connected component of the identity of the group $\mathrm{Diff}(M)$.*

Proof. First assume the mapping torus is trivial over S^1. We have the following commutative diagram with top row a diffeomorphism.

$$\begin{tikzcd} M_\varphi \arrow[rr, "f"] \arrow[dr, "p"'] & & M \times S^1 \arrow[dl, "\mathrm{pr}_2"] \\ & S^1 & \end{tikzcd}$$

where $\mathrm{pr}_2(f([x,t])) = [t] = p([x,t])$. This means that f maps level-wise, so we have $f([x,t]) = (g_t(x), t)$, where each $g_t\colon M \to M$ is a diffeomorphism. The mapping torus relation $(k,0) \sim (\varphi(k), 1)$ gives

$$(g_0(x), [0]) = f([x,0]) = f(\varphi(x), 1) = (g_1(\varphi(x)), [1]) = (g_1(\varphi(x)), [0]),$$

and then we have $g_0(x) = g_1(\varphi(x))$.

Define an isotopy $F\colon M \times I \to M$ by $F(x,t) = g_0^{-1} g_t(\varphi(x))$. Then $F(x,0) = g_0^{-1} g_0(\varphi(x)) = \varphi(x)$ and $F(x,1) = g_0^{-1} g_1(\varphi(x)) = g_0^{-1} g_0(x) = x$. Hence, φ is isotopic to the identity.

Conversely, suppose that $\varphi \in \mathrm{Diff}_0(M)$. Then there exists a smooth map $H\colon M \times [0,1] \to M$ such that

$$H(m,0) = m \qquad \text{and} \qquad H(m,1) = \varphi(m)$$

and $H(\cdot, t)$ is a diffeomorphism for all $t \in [0,1]$; in particular, for all $t \in [0,1]$, there exists a diffeomorphism $H^{-1}(\cdot, t)$. Define a map $f\colon M \times S^1 \to M_\varphi$ by

$$f(m, [t]) = [H(m,t), t];$$

where we identify $M \times S^1 = \frac{M \times [0,1]}{(m,0) \sim (m,1)}$. It is enough to check that f is well defined, as it is clearly smooth, but this is guaranteed by our definition of H. Next we define an inverse $g\colon M_\varphi \to M \times S^1$ by setting

$$g([m,t]) = (H^{-1}(m,t), [t]).$$

Again, g is smooth, and we must prove that it is well defined. Indeed, we have

$$g([m,0]) = (H^{-1}(m,0), [0]) = (m, [0])$$

and

$$g([\varphi(m), 1]) = (H^{-1}(\varphi(m), 1), [1]) = (\varphi^{-1}(\varphi(m)), [1]) = (m, [1]).$$

But $[m,[0]] = [m,[1]]$ in $M \times S^1$, so g is well-defined and is an inverse for f. $\square$

Remark 5.3. For reference, we make the simple observation that, for a diffeomorphism $\varphi \in \mathrm{Diff}_0(M)$, which is isotopic to the identity, the induced map on cohomology $\varphi^*\colon H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is the identity map.

The proposition suggests that, in order to obtain non-trivial examples of mapping tori, one should consider diffeomorphisms that do not belong to the identity component of the group of diffeomorphisms. It is then interesting to look at the groups $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ or $\mathrm{Diff}_+(M)/\mathrm{Diff}_0(M)$, the latter in case one is interested in orientation-preserving diffeomorphisms.

Remark 5.4. In case M is a compact complex manifold, one can replace $\mathrm{Diff}(M)$ by the group $\mathrm{Aut}(M)$ of holomorphic diffeomorphisms of M. Further, when M is compact Kähler, one may consider the subgroup $\mathrm{Aut}_\omega(M)$ of elements which preserve the Kähler class (but not necessarily the Kähler form). In each case, the corresponding mapping torus is trivial if and only if the automorphism belongs to the identity component.

Now let's consider the structure group of a mapping torus. Let M be a smooth manifold and let $\varphi\colon M \to M$ be a diffeomorphism. Then the mapping torus M_φ is a fibre bundle over S^1 with fibre M. In general, the structure group of a fibre bundle $F \to E \to B$ is a subgroup G of the homeomorphism group of F such that the transition functions of the bundle take values in G.

Proposition 5.3. *The structure group G of a mapping torus M_φ is the cyclic group $\langle\varphi\rangle \subset \mathrm{Diff}(M)$.*

Sketch of Proof (see [86, Section 18]). The mapping torus M_φ is a fibre bundle over S^1 with fiber the manifold M. We can cover S^1 by two open sets U, V such that $U \cap V = \{U_0, U_1\}$ consists of two disjoint open sets. Then $M_\varphi\big|_U = M \times U$ and $M_\varphi\big|_V = M \times V$, and the mapping torus is trivial over U and V. To describe M_φ it is sufficient to give the transition function $g\colon U \cap V \to \mathrm{Diff}(M)$. We can assume that g is the identity on U_0 and $g = \varphi$ on U_1. Then φ generates G.

□

Remark 5.5. Another way to describe the mapping torus of a diffeomorphism $\varphi\colon M \to M$ is as the quotient of $M \times \mathbb{R}$ by the group $\mathbb{Z}$ acting on $M \times \mathbb{R}$ by

$$(m, (p, t)) \mapsto (\varphi^m(p), t - m).$$

It is then clear that the structure group of M_φ is isomorphic to the group generated by φ.

Let (K, h, ω) be a compact Kähler manifold, where h denotes the Hermitian metric and ω is the Kähler form. Let $\mathrm{Isom}(K, h) \leq \mathrm{Aut}(K)$ denote the group of Hermitian isometries of K and let $\psi \in \mathrm{Isom}(K, h)$. Then ψ is a holomorphic diffeomorphism of K which preserves the Hermitian metric h. In particular, $\psi^*\omega = \omega$. Li's theorem [62] says that the mapping torus of ψ, denoted by K_ψ is a compact co-Kähler manifold. If K_ψ is non-trivial, then according to Proposition 5.2, ψ defines a non-zero element in

$$H := \mathrm{Isom}(K, h)/\mathrm{Isom}_0(K, h).$$

Our results prove that, up to a finite covering, $K_\psi \cong K \times_{\mathbb{Z}_m} S^1$ (Theorem 5.3), and the $\mathbb{Z}_m$ action is by translations on the S^1 factor. Furthermore, we get a fibre bundle $K_\psi \to S^1$ with structure group the finite group $\mathbb{Z}_m$. Notice that when we display K_ψ as a fibre bundle with fibre K, the structure group of this bundle is $\langle\psi\rangle$, the cyclic group generated by ψ in H. We then have the following theorem.

Theorem 5.6. *Let K be a Kähler manifold; then all elements of the group H have finite order.*

Proof. Pick an element $\psi \in H$ and form the mapping torus K_ψ. The discussion above proves that ψ has finite order in H. Since ψ is arbitrary, the result follows. □

Indeed, Lieberman [63] proves a much more general result, but in a much harder way.

Theorem 5.7 ([63, Proposition 2.2])**.** *Let K be a Kähler manifold and let $\mathrm{Aut}_\omega(K)$ denote the group of automorphisms of K preserving a Kähler class (but not necessarily the Kähler form). Let $\mathrm{Aut}_0(K)$ be the identity component. Then the quotient*

$$\mathrm{Aut}_\omega(K)/\mathrm{Aut}_0(K)$$

is a finite group.

Remark 5.6. In [62], Li also shows that the almost cosymplectic manifolds of [24] arise as symplectic mapping tori. That is, if M is almost cosymplectic in the terminology of [24], then there is a symplectic manifold S and a symplectomorphism $\varphi\colon S \to S$ such that $M \cong S_\varphi$. Li calls these manifolds *co-symplectic*. By the discussion in Section 5.3 and the results above, we see that there is a version of Theorem 5.3 for Li's co-symplectic manifolds *when the defining symplectomorphism φ is of finite order in*

$$\mathrm{Symp}(S)/\mathrm{Symp}_0(S).$$

Thus, knowledge about when this can happen would be very interesting.

In general, one can not expect a non-zero element in $\mathrm{Symp}(S)/\mathrm{Symp}_0(S)$ to have finite order. As an example, consider the torus T^2 with the standard symplectic structure and let $\varphi : T^2 \to T^2$ be the diffeomorphism covered by the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Then φ is an area-preserving diffeomorphism of T^2, hence a symplectomorphism. Notice that the action of φ on $H^1(T^2;\mathbb{R})$, which is represented by the matrix A, is nontrivial. Hence the symplectic mapping torus T^2_φ is not diffeomorphic to $T^3 = T^2 \times S^1$; according to Proposition 5.2, φ is non-zero in $\mathrm{Symp}(T^2)/\mathrm{Symp}_0(T^2)$. Clearly φ has infinite order.

5.7 Examples

A basic example was constructed in [24] to show that a co-Kähler manifold need not be a global product of a Kähler manifold and S^1. Of course, from what we have said above, this *is* true *up to a finite cover*. Here, we will analyze the CDM example from both Li's mapping torus and our finite cover splitting points of view.

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $GL(\mathbb{Z}, 2)$ and note that it defines a Kähler isometry of T^2 which we can write as $A(x, y) = (y, -x)$. Li's approach says to form the mapping torus

$$T^2_A = \frac{T^2 \times [0, 1]}{(x, y, 0) \sim (A(x, y), 1)},$$

and then T^2_A is a co-Kähler manifold with associated fibre bundle $T^2 \to T^2_A \to S^1$ given by the projection

$$[x, y, t] \mapsto [t].$$

Now, A has order 4, so the picture is quite simple: namely, a central circle winds around the mapping torus 4 times before closing up. Therefore, we see that we have a circle action on T^2_A given by

$$S^1 \times T^2_A \to T^2_A, \qquad ([s], [x, y, t]) \mapsto [x, y, t + 4s].$$

When the orbit map $S^1 \to T^2_A$, $[s] \mapsto [x_0, y_0, 0]$ is composed with the projection map $T^2_A \to S^1$, we get

$$S^1 \to S^1, \qquad [s] \mapsto [4s]$$

which induces multiplication by 4 on $H_1(S^1; \mathbb{Z})$. Hence, the S^1-action is homologically injective and Theorem 5.3 then gives a finite cover of T^2_A of the form $T^2 \times S^1$. Hence, T^2_A is finitely covered by a torus. Now let's look at the Betti numbers of T^2_A using Theorem 5.4.

The diffeomorphism A acts on $H^1(T^2; \mathbb{R})$ by the matrix $P_* = A^t$, $P_*(x, y) = (-y, x)$, and on $H^2(T^2; \mathbb{R})$ by the identity; hence the Kähler class is invariant (as we know in general). Otherwise, there are no invariant classes in degrees greater than zero. To see this, suppose $P_*(ax + by) = -ay + bx = ax + by$. Thus, $a = b$ and $a = -b$, so $a = b = 0$. Now we have the following.

- $b_1(T^2_A) = \bar{b}_1(T^2) + 1 = 0 + 1 = 1;$
- $b_2(T^2_A) = \bar{b}_2(T^2) + \bar{b}_1(T^2) = 1 + 0 = 1;$

- $b_3(T^2_A) = \bar{b}_3(T^2) + \bar{b}_2(T^2) = 0 + 1 = 1$.

As noted in [24], this shows that T^2_A is not a global product. For, as an orientable 3-manifold with first Betti number 1, there is no other choice but $S^1 \times S^2$ and this is ruled out since the fibre bundle $T^2 \to T^2_A \to S^1$ shows that T^2_A is aspherical.

The CDM example also fits in the scope of Theorem 5.5. To see this, we compute the fundamental group of T^2_A explicitly. The fibre bundle $T^2 \to T^2_A \to S^1$ shows that we have a short exact sequence of groups

$$0 \to \mathbb{Z}^2 \to \Gamma \to \mathbb{Z} \to 0,$$

where $\Gamma = \pi_1(T^2_A)$. Since $\mathbb{Z}$ is free, Γ is a semidirect product $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$. The action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given by the group homomorphism $\phi\colon \mathbb{Z} \to \mathrm{SL}(2,\mathbb{Z})$ sending $1 \in \mathbb{Z}$ to $\phi(1) = A \in \mathrm{SL}(2,\mathbb{Z})$. As we remarked above, T^2_A is covered $4:1$ by a torus T^3 and this covering gives a map $\psi\colon \mathbb{Z}^3 \to \Gamma$. The map ψ sends $(m,n,p) \in \mathbb{Z}^3$ to $(m,n,4p) \in \Gamma$, hence the quotient $\Gamma/\mathbb{Z}^3$ is isomorphic to $\mathbb{Z}_4$.

For a higher dimensional example, we can take the torus T^4 and consider the mapping torus T^4_B associated to the Kähler isometry given by the matrix

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then Li's theorem says T^4_B is a co-Kähler manifold, which can be displayed as a fibre bundle $T^4 \to T^4_B \to S^1$ given by the projection $[x,y,z,w,t] \mapsto [t]$. Notice that B has order 2 in $\mathrm{SL}(4,\mathbb{Z})$. We then have a circle action $S^1 \times T^4_B \to T^4_B$ given by

$$([s],[x,y,z,w,t]) \mapsto [x,y,z,w,t+2s].$$

Composing the orbit map $S^1 \to T^4_B$ with the projection $T^4_B \to S^1$ we get a map $S^1 \to S^1$, $[s] \mapsto [2s]$ which induces multiplication by 2 on $H_1(S^1;\mathbb{Z})$. Applying Theorem 5.3 we get a finite cover of T^4_B of the form $T^4 \times S^1$, showing that T^4_B is covered by a torus. Now Theorem 5.4 gives the Betti numbers of T^4_B:

- $b_1(T^4_B) = \bar{b}_1(T^4) + 1 = 2 + 1 = 3$;
- $b_2(T^4_B) = \bar{b}_2(T^4) + \bar{b}_1(T^4) = 2 + 2 = 4$;
- $b_3(T^4_B) = \bar{b}_3(T^4) + \bar{b}_2(T^4) = 2 + 2 = 4$;

- $b_4(T^4_B) = 1 + \bar{b}_3(T^4) = 1 + 2 = 3$.

Notice that, however, in this case, T^4_B *is* a product, even though it is not true that $T^4_B = T^4 \times S^1$. Indeed, we can start with T^2 and the Kähler mapping torus $T^2_{B'}$ where

$$B' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and then $T^4_B = T^2 \times T^2_{B'}$. Yet another description of T^4_B is as the product of a hyperelliptic surface by a circle (see [53] for more details). Concerning the fundamental group Γ of T^4_B, it can be described a semidirect product $\Gamma = \mathbb{Z}^4 \rtimes_\rho \mathbb{Z}$ where $1 \in \mathbb{Z}$ acts on $\mathbb{Z}^4$ by the matrix B. The covering $T^5 = T^4 \times S^1 \to T^4_B$ gives a map $\mathbb{Z}^5 \to \Gamma$ at the level of fundamental groups. The image of $(j, k, m, n, p) \in \mathbb{Z}^5$ is $(j, k, m, n, 2p) \in \Gamma$ and $\Gamma/\mathbb{Z}^5 \cong \mathbb{Z}_2$.

5.8 Co-Kähler manifolds with solvable fundamental group

There has been much work done in the past 20 years regarding the question of whether Kähler solvmanifolds are tori. In [53], for instance, it is shown that such a manifold is a finite quotient of a complex torus which is also the total space of a complex torus bundle over a complex torus. In [39], Hasegawa's result was applied to show the following.

Theorem 5.8. *A solvmanifold has a co-Kähler structure if and only if it is a finite quotient of torus which has a structure of a torus bundle over a complex torus. As a consequence, a solvmanifold $M = G/\Gamma$ of completely solvable type has a co-Kähler structure if and only if it is a torus.*

Note that we have changed the terminology of [39] to match ours. We can use Theorem 5.3 to contribute something in this vein.

Theorem 5.9. *Let $(M^{2n+1}, J, \xi, \eta, g)$ be an aspherical co-Kähler manifold with integral structure and suppose $\pi_1(M)$ is a solvable group. Then M is a finite quotient of a torus.*

Proof. We know that every aspherical solvable Kähler group contains a finitely generated abelian subgroup of finite index (see [6, section 1.5] for instance). Now, if $M = K_\varphi$ is the Li mapping torus description of M, we see that K is Kähler and aspherical with solvable fundamental group (as a subgroup of $\pi_1(M)$). Hence, K is finitely covered by a torus. By Theorem 5.3, there is a finite $\mathbb{Z}_m$-cover $\cong K \times S^1 \to M$ and this then displays M itself as a finite quotient of a torus. $\square$

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APPENDIX

A

CO-SYMPLECTIC NILMANIFOLDS AND SOLVMANIFOLDS

In this appendix we would like to collect some results about co-symplectic nilmanifolds and solvmanifolds. The appendix is divided in three section. In the first section we address the Lefschetz property for co-symplectic nilmanifolds. In the second section, we study formality and its relationship with the Lefschetz property for co-symplectic nilmanifolds and completely solvable solvmanifolds. In the third section, finally, we study which nilmanifolds in dimension 3, 5 and 7 (only $2-$step nilmanifolds in the latter case) admit a left-invariant co-symplectic structure. We give explicit examples in case such a structure exists, or explain why it does not exist.

A.1 The Lefschetz property

In [24] the authors define a Lefschetz map for co-Kähler manifolds and prove that it is an isomorphism. As we remarked in the introduction, this definition is not quite the usual definition of Lefschetz map for Kähler manifolds, since it involves the Kähler form ω, the vector field ξ and the dual $1-$form η. The authors prove first a Hodge-type decomposition on a co-Kähler manifold, then they define the Lefschetz map on *harmonic* forms. So given a co-Kähler manifold (M, J, ξ, η, g) of dimension $2n+1$, let $\mathcal{H}^p(M)$ denote the space of harmonic $p-$forms on M. Being M co-Kähler, the Kähler form ω is closed. Let $\nu \in \mathcal{H}^p(M)$ be a harmonic $p-$form, $p \leq n$. The Lefschetz map $\mathscr{L} : \mathcal{H}^p(M) \to \mathcal{H}^{2n+1-p}(M)$ is

$$\mathscr{L}(\nu) = \omega^{n-p} \wedge (\imath_\xi(\omega \wedge \nu) + \eta \wedge \nu) \in \mathcal{H}^{2n+1-p}(M). \tag{A.1}$$

If we assume that ν is only closed, but not co-closed, then we obtain

$$d\mathscr{L}(\nu) = d(\omega^{n-p} \wedge (\imath_\xi(\omega \wedge \nu) + \eta \wedge \nu)) = \omega^{n-p+1} \wedge d(\imath_\xi \nu);$$

hence the Lefschetz map does not send closed forms to closed forms. Let ν be a closed form on M; then

$$d\mathscr{L}(\nu) = 0 \Leftrightarrow \omega^{n-p+1} \wedge d(\imath_\xi \nu) = 0.$$

Cartan's magic formula gives

$$L_\xi \nu = \imath_\xi(d\nu) + d(\imath_\xi \nu) = d(\imath_\xi \nu),$$

where L_ξ is the Lie derivative in the direction of ξ; hence $d\mathscr{L}(\nu) = 0$ if and only if $\omega^{n-p+1} \wedge L_\xi \nu = 0$. In [24], the authors prove that a harmonic form ν on a co-Kähler manifold satisfies $L_\xi \nu = 0$.

On the other hand, it is easy to come up with examples of co-symplectic non co-Kähler manifolds on which this latter condition is not satisfied. Consider for instance the $5-$dimensional nilpotent Lie algebra $\mathfrak{g}$ given by the following brackets:

$$[X_1, X_2] = -X_4, \quad [X_1, X_5] = -X_3.$$

The Chevalley-Eilenberg complex is then:

$$(\wedge(x_1, \ldots, x_5), dx_3 = x_1 x_5,\ dx_4 = x_1 x_2).$$

A co-symplectic structure on $\mathfrak{g}$ is given as follows:

- $J(X_1) = X_3$, $J(X_2) = X_4$, $J(X_5) = 0$;
- $\xi = X_5$, $\eta = x_5$;
- g the standard euclidean metric;
- $\omega = x_1 x_3 + x_2 x_4$.

The Lie algebra $\mathfrak{g}$ is defined over $\mathbb{Q}$, hence the corresponding simply connected nilpotent Lie group G has a lattice Γ, and the nilmanifold $N = G/\Gamma$ admits a co-symplectic structure. Let us study the Lefschetz map on this co-symplectic nilmanifold. Let $\nu = x_3 x_5$; then $\imath_\xi \nu = x_3$ and $\mathscr{L}(\nu) = x_2 x_3 x_4$, so that $d\mathscr{L}(\nu) = -x_1 x_2 x_4 x_5 \neq 0$. This proves that the Lefschetz map is not well defined on co-symplectic manifolds and also that the co-symplectic manifold N is not co-Kähler.

Let $N = G/\Gamma$ be a co-symplectic nilmanifold of dimension $2n+1$ and let $\mathfrak{g}$ denote the Lie algebra of G. Suppose further that the co-symplectic structure is left-invariant. Denote by (J, ξ, η, g) the corresponding co-symplectic structure on $\mathfrak{g}$. The Chevalley-Eilenberg $(\wedge\mathfrak{g}^*, d)$ computes the cohomology of N, then we can define the Lefschetz map directly on it. In general, it will not take value on closed forms, but $\mathscr{L} : \mathfrak{g}^* \to \wedge^{2n}\mathfrak{g}^*$ does take value on closed elements; indeed, the differential d is identically zero on $\wedge^{2n}\mathfrak{g}^*$: if some element of $\wedge^{2n}\mathfrak{g}^*$ was not closed, its differential would kill the volume form, giving a contradiction to Poincaré duality or orientabilty of the nilmanifold. Thus we obtain a well defined map

$$\mathscr{L} : H^1(\mathfrak{g}^*) \to H^{2n}(\mathfrak{g}^*). \tag{A.2}$$

Definition A.1. A co-symplectic nilmanifold satisfies the $1-$Lefschetz property if the map (A.2) is an isomorphism.

Theorem A.1. *Let $N = G/\Gamma$ be a nilmanifold endowed with a left invariant co-symplectic structure. If the $1-$Lefschetz map (A.2) is an isomorphism, then M is diffeomorphic to a torus*[1].

Proof. The minimal model of N is the Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, d)$, with differential defined according to the Lie algebra structure of $\mathfrak{g}$. Notice that since the $\eta \in \Omega^1(N)$ is closed on a co-symplectic manifold, we can assume that η appears as one of the generators of $\mathfrak{g}^*$, maybe performing some change of variables. Assume that d is nonzero; then we can write

$$\mathfrak{g}^* = \langle x_1, \dots, x_s, \eta, x_{s+1}, \dots, x_{2n}\rangle$$

for some $1 \leq s \leq 2n$, in such a way that $dx_k = 0$ for $k \leq s$ while $dx_\ell \neq 0$ for $\ell \geq s+1$. Clearly, also $d\eta = 0$. The symplectic form ω can be written in the following form:

$$\omega = \sum_{1\leq i<j<2n} a_{ij}x_ix_j + zx_{2n},$$

where $a_{ij} \in \mathbb{Q}$ and we have put together all the summands which contain x_{2n}; since $\imath_\xi\omega = 0$, η does not appear in the expression for ω. Notice that z is a cocyle; indeed, when one computes the differential of ω, which must be zero, the term $dz \cdot x_{2n}$ pops up. But according to the ordering of the basis of $\mathfrak{g}^*$, x_{2n} can not appear in the expression for the differential of any other x_k, and this forces dz to be zero. Define a derivation λ on the generators of $\mathfrak{g}^*$ by setting

$$\lambda(\eta) = 0, \quad \lambda(x_i) = 0 \text{ for } i < 2n \quad \text{and} \quad \lambda(x_{2n}) = 1.$$

This gives $\lambda(\omega) = z$. The Lefschetz $1-$map is

$$\mathscr{L}(\nu) = \omega^n \wedge (i_\xi(\nu)) + \omega^{n-1}(\eta \wedge \nu),$$

where ν is a closed $1-$form. Assume it is an isomorphism and apply it to the cocycle z above:

$$\mathscr{L}(z) = \omega^n(\imath_\xi(z)) + \omega^{n-1}(\eta \wedge z) = \omega^{n-1} \wedge \eta \wedge z.$$

[1]The proof of Theorem A.1 is analogous to the proof of the corresponding result for symplectic nilmanifolds contained in [80] and was suggested to me by J. Oprea.

For degree reasons, given any $1-$cocycle y, we have $y \wedge \eta \wedge \omega^n = 0$. Then

$$0 = \lambda(y \wedge \eta \wedge \omega^n) = \lambda(y) \wedge \eta \wedge \omega^n - y \wedge \lambda(\eta) \wedge \omega^n + y \wedge \eta \wedge \lambda(\omega^n) = \\ = ny \wedge \eta \wedge \lambda(\omega) \wedge \omega^{n-1} = ny \wedge \eta \wedge z \wedge \omega^{n-1}.$$

This must be true for every cocycle y, and $z \wedge \eta \wedge \omega^{n-1}$ is non-zero by the Lefschetz condition. This contradicts Poincaré duality since

$$H^1(\mathfrak{g}^*) \times H^{2n}(\mathfrak{g}^*) \to H^{2n+1}(\mathfrak{g}^*) \cong \mathbb{Q}$$

is non-degenerate. So this contradicts dx_{2n} non-zero, because we defined the derivation λ to be zero on all cocycles. Hence $d = 0$. □

Let $N = G/\Gamma$ be a co-symplectic nilmanifold of dimension $2n+1$. The $1-$form η is closed, so it defines a cohomology class in degree 1. This means that we can use η as a generating vector of the minimal model of N. In other words, we can assume that

$$\mathcal{M}_N = (\wedge(x_1, \ldots, x_{2n}, \eta), d).$$

The existence of η gives a splitting of the space $\mathfrak{g}^*$ of generators; indeed, we can write

$$\mathfrak{g}^* = W \oplus \langle \eta \rangle$$

and consequently

$$\wedge^p \mathfrak{g}^* = \wedge^p W \oplus \wedge^{p-1} W \otimes \langle \eta \rangle = \wedge^{p,0} \oplus \wedge^{p,1},$$

where the latter is only notation. Every $p-$form ν can thus be decomposed as $\nu = \nu_0 + \nu_1$ with $\nu_i \in \wedge^{p,i}$, $i = 0, 1$.

Lemma A.1. *Let $\nu \in \wedge^p \mathfrak{g}^*$; then $\nu \in \wedge^{p,0}$ if and only if $\imath_\xi \nu = 0$.*

Proof. Since ξ is the dual vector field to η, if $\nu \in \wedge^{p,0}$ then clearly $\imath_\xi \nu = 0$. Conversely, let $\nu \in \wedge^p \mathfrak{g}^*$ with $\imath_\xi \nu = 0$. Write $\nu = \nu_0 + \nu_1$; then from $\imath_\xi \nu = 0$ we get $\imath_\xi \nu_1 = 0$. We can write $\nu_1 = \eta \wedge \nu_1'$ with $\nu_1' \in \wedge^{p-1,0}$; being $\imath_\xi$ a derivation, we get

$$0 = \imath_\xi \nu_1 = \imath_\xi (\eta \wedge \nu_1') = \nu_1' - \eta \wedge \imath_\xi \nu_1'.$$

Cleary $\imath_\xi \nu_1' = 0$, so we obtain $\nu_1' = 0$. □

Lemma A.2. *Let $\nu \in \wedge^p \mathfrak{g}^*$; then $\nu \in \wedge^{p,1}$ if and only if $\eta \wedge \nu = 0$.*

Proof. Sufficiency is clear. So suppose ν is a $p-$form with $\eta \wedge \nu = 0$. Decomposing $\nu = \nu_0 + \eta \wedge \nu_1'$ we get

$$0 = \eta \wedge \nu = \eta \wedge (\nu_0 + \eta \wedge \nu_1') = \eta \wedge \nu_0.$$

Then we can apply $\imath_\xi$ to both sides, getting

$$0 = \imath_\xi (\eta \wedge \nu_0) = \nu_0 + \eta \wedge \imath_\xi (\nu_0) = \nu_0.$$

□

Using these two lemmas we obtain the following result:

Proposition A.1. *Let $N = G/\Gamma$ be a nilmanifold endowed with a left invariant co-symplectic structure (J, ξ, η, g). Then $\mathfrak{g}^* = W \oplus \langle\eta\rangle$ and the Chevalley-Eilenberg complex of $\mathfrak{g}^*$ admits a splitting*

$$\wedge^p \mathfrak{g}^* = \wedge^p W \oplus \langle\eta\rangle \otimes \wedge^{p-1} W = \wedge^p_\xi \oplus \wedge^p_\eta, \tag{A.3}$$

where $\wedge^p_\xi = \{\nu \in \wedge^p \mathfrak{g}^* \mid \imath_\xi \nu = 0\}$ and $\wedge^p_\eta = \{\nu \in \wedge^p \mathfrak{g}^* \mid \eta \wedge \nu = 0\}$.

Notice that, for $\nu \in \wedge^p \mathfrak{g}^*$, $\eta \wedge \nu = 0 \Leftrightarrow \nu = \eta \wedge \nu'$ for some $\nu' \in \wedge^{p-1}_\xi$; therefore we can rewrite (A.3) as

$$\wedge^p \mathfrak{g}^* = \wedge^p_\xi \oplus \left(\eta \otimes \wedge^{p-1}_\xi\right).$$

Let $d : \wedge^p \mathfrak{g}^* \to \wedge^{p+1} \mathfrak{g}^*$ denote the differential. Then we decompose d as $d = d + \bar{d} + \delta$ with

$$d : \wedge^p_\xi \to \wedge^{p+1}_\xi, \quad \bar{d} : \wedge^p_\xi \to \eta \otimes \wedge^p_\xi \quad \text{and} \quad \delta : \eta \otimes \wedge^{p-1}_\xi \to \eta \otimes \wedge^p_\xi$$

Clearly, d and δ are the same differential in practice, d acting on $p-$forms and δ acting on $(p-1)-$forms. We keep a formal difference to avoid abuse of notation.

Lemma A.3. *In the above notation, $\bar{d} = 0$ if and only if $\mathfrak{g}$ admits a Lie algebra splitting $\mathfrak{g} = U \oplus \langle\xi\rangle$ with $U = W^*$.*

Proof. Let $\{X_1, \ldots, X_{2n}, X_{2n+1} = \xi\}$ be a basis of $\mathfrak{g}$ and let $\{x_1, \ldots, x_{2n}, x_{2n+1} = \eta\}$ be the dual basis of $\mathfrak{g}^*$. Recall that the differential of $\mathfrak{g}^*$ is related to the bracket in $\mathfrak{g}$ as follows: if

$$[X_i, X_j] = \sum_k c^k_{ij} X_k$$

then

$$dx_k = -\sum_{i,j} c^k_{ij} x_i \wedge x_j.$$

If $\bar{d} = 0$, the differential of an element x_k with $k \neq 2n+1$ contains only products $x_i \wedge x_j$ with $i, j \neq 2n+1$. Correspondingly, every commutator of the form $[X_i, X_{2n+1}]$ is zero, and $U \oplus \langle\xi\rangle$ is a Lie algebra splitting. In the other direction, the same argument works for the differential of the generators of $\mathfrak{g}^*$. Since the differential is extended to the whole Chevalley-Eilenberg complex by requiring Leibnitz rule to hold, the thesis follows. □

Remark A.1. If $\bar{d} = 0$ then the following facts are true:

- the differential commutes with the derivation $\imath_\xi$; indeed, let us write a $p-$form ν as $\nu_0 + \eta \wedge \nu_1 \in \wedge^p_\xi \oplus \left(\eta \otimes \wedge^{p-1}_\xi\right)$; then

$$\imath_\xi(d\nu) = \imath_\xi(d\nu_0) + \imath_\xi((\delta(\eta \wedge \nu_1)) = \imath_\xi(\eta \wedge \delta\nu_1) = \delta\nu_1;$$
$$d(\imath_\xi \nu) = d(\imath_\xi(\nu_0 + \eta \wedge \nu_1)) = d(\imath_\xi(\eta \wedge \nu_1)) = \delta\nu_1.$$

  In particular, the Lefschetz map sends closed forms to closed forms, and it is thus well defined.

- the corresponding co-symplectic nilmanifold is the produt of a symplectic nilmanifold of dimension $2n$ and a circle;

- putting together the two facts above, we obtain that for co-symplectic *product* nilmanifolds the Lefschetz map is well defined in any degree. Theorem (A.1) applies and shows that a co-symplectic product nilmanifold which satisfies the Lefschetz $1-$map is diffeomorphic to a torus.

A.2 Formality

As we said in the introduction (Theorem 0.4), a formal nilmanifold is diffeomorphic to a torus. As a consequence, we get the following proposition:

Proposition A.2. *Let $N = G/\Gamma$ be a compact nilmanifold. If N admits a co-Kähler structure, then N is diffeomorphic to a torus.*

Proof. The proof is a simple consequence of the fact, proved in [24], that co-Kähler manifolds are formal. The thesis follows applying the same argument as in [80]. □

We would like to remark that Proposition A.2 (as well as Theorem A.1) can not be extended to *solvmanifolds*, that is, compact homogeneous spaces of solvable Lie groups. Indeed, a characterization of Kähler solvmanifolds has been achieved only recently by Hasegawa [54], and says the following:

Theorem A.2. *A compact solvmanifold admits a Kahler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. In particular, a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.*

Using the result of Hasegawa, Fino and Vezzoni ([39]) obtained a characterization of co-Kähler solvmanifolds:

Theorem A.3. *A solvmanifold has a co-Kähler structure if and only if it is a finite quotient of torus which has a structure of a torus bundle over a complex torus. In particular, a solvmanifold $M = G/\Gamma$ of completely solvable type has a co-Kähler structure if and only if it is a torus.*

We would like to give an example of a co-symplectic non co-Kähler solvmanifold which is formal and satisfies the Lefschetz property. Let $\mathfrak{g}$ denote the abelian Lie algebra of dimension 2 and set $\mathfrak{g} = \langle e_1, e_2 \rangle$. We take the symplectic form $\omega = e^1 \wedge e^2$ on $\mathfrak{g}$ and the standard metric $g = (e^1)^2 + (e^2)^2$. Consider the linear map $D : \mathfrak{g} \to \mathfrak{g}$ given by $D(e_1) = e_1$ and $D(e_2) = -e_2$. It is easy to see that D is an infinitesimal symplectic transformation. According to proposition Proposition 4.5, the Lie algebra $\mathfrak{h} = \mathfrak{g} \oplus \langle e_3 \rangle$ has brackets

$$[e_3, e_1] = e_1 \quad \text{and} \quad [e_3, e_2] = -e_2$$

and a co-symplectic structure on $\mathfrak{h}$ is given by:

- $J(e_1) = e_2,\ J(e_2) = -e_1,\ J(e_3) = 0;$
- $\eta = e^3 \in \mathfrak{h}^*$.

One sees that $\mathfrak{h}$ is completely solvable. Also, the above co-symplectic structure on $\mathfrak{h}$ is not co-Kähler: indeed, one has $N_J(e_1, e_3) = 2e_1 \neq 0$. In [80] it is proved that the simply connected completely solvable Lie group H such that $\mathfrak{h} = \mathrm{Lie}(H)$ has a lattice Γ, therefore there exists a solvmanifold $M = \Gamma \backslash H$. The Chevalley-Eilenberg complex of $\mathfrak{h}$ is

$$(\wedge \mathfrak{h}^*, d) = (\wedge(e^1, e^2, e^3), de^1 = e^{13}, de^2 = e^{23}),$$

where, as usual, e^{ij} stands for $e^i \wedge e^j$. Since $\mathfrak{h}$ is completely solvable, Hattori's theorem applies and we can use $(\wedge \mathfrak{h}^*, d)$ to compute the cohomology of M. One gets

$$H^1(M) = \langle e^3 \rangle,\ H^2(M) = \langle e^{12} \rangle \text{ and } H^3(M) = \langle e^{123} \rangle.$$

The minimal model of M is

$$\mathcal{M}_M = (\wedge(x_1, x_2, x_3), dx_3 = x_2^2)$$

where $\deg(x_i) = i$. Then M has the same minimal model as the co-Kähler manifold $S^1 \times S^2$ and is formal. Clearly, these two manifolds do not have the same rational homotopy type: M is aspherical while $S^1 \times S^2$ is not. M does not have a nilpotent fundamental group, and the minimal model contains no rational homotopic information other than the cohomology. Also, M does not admit any co-Kähler structure; to see this, we can either apply Theorem A.3 or Theorem 2 of [33] to see that the product $M \times S^1$ does not admit a complex structure.

We should remark that the argument we used above to show that the Lefschetz map is well defined on $1-$forms for nilmanifolds (i.e. it maps closed forms to closed forms) also works for *completely solvable solvmanifolds* $S = \Gamma \backslash G$. Indeed, Hattori theorem says that the Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, d)$, where $\mathfrak{g}$ is the Lie algebra of G, is a model for the cohomology of S. Thus, for completely solvable co-symplectic solvmanifolds, it makes sense to ask whether formality and the $1-$Lefschetz property are related or not. We would like to show that, as it happens in the symplectic case, the two properties are not related. For this we describe two examples. The first one is a $5-$dimensional co-symplectic non-formal solvmanifold which satisfies the Lefschetz property for $i = 1$. The second one is a $7-$dimensional solvmanifold which is formal but not $1-$Lefschetz.

Proposition A.3. *Let M be a compact co-symplectic manifold of dimension $2n + 1$; suppose $b_1(M) = 1$; then M satisfies the Lefschetz property (A.1) with $p = 1$.*

Proof. Since M is compact, the $1-$form η defines a non-zero cohomology class. Hence the first cohomology of M is generated by η and the Lefschetz map for η is

$$\mathscr{L}(\eta) = \omega^{n-1} \wedge (i_\xi(\omega \wedge \eta) + \eta \wedge \eta) = \omega^n.$$

Since the Kähler form of a $(2n+1)-$dimensional co-symplectic manifold has rank n, ω^n is nowhere zero and the thesis follows. □

Let us consider again the solvmanifold of Theorem 4.5; we started with the connected, simply connected $5-$dimensional solvable Lie group H with Lie algebra $\mathfrak{h}$ given by

$$[e_1, e_5] = e_1 + e_3, \quad [e_2, e_5] = -e_2 + e_4, \quad [e_3, e_5] = e_3 \quad \text{and} \quad [e_4, e_5] = -e_4.$$

We proved that H admitted a lattice Γ and set $S = \Gamma\backslash H$. Theorem 4.5 shows that S is a compact, co-symplectic non-formal manifold with $b_1(S) = 1$. Applying Proposition A.3, we see that S is $1-$Lefschetz.

Finally, let us give an example of a co-symplectic manifold in dimension 7 which is formal but not $1-$Lefschetz. We consider the simply connected, $6-$dimensional, solvable Lie group G with Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, d)$, where $\mathfrak{g}^* = \langle e^1, \ldots, e^6\rangle$ and

$$de^1 = e^{16} - e^{25}, \quad de^2 = -e^{45}, \quad de^3 = -e^{24} - e^{36} - e^{46},$$
$$de^4 = -e^{45}, \quad de^5 = e^{56} \quad \text{and} \quad de^6 = 0.$$

Then G is the group $G_{6.78}$ considered by Bock in ([13], Theorem 8.3.4). There, he proves that $G = \mathbb{R} \ltimes_\mu N$, where N is a $5-$dimensional simply connected nilpotent Lie group and μ is a $1-$parameter group of diffeomorphisms of N. The $\mathbb{R}-$factor is associated to the generator e^6 of $\mathfrak{g}^*$. The change of variables

$$x^1 = e^5, \quad x^2 = e^4, \quad x^3 = e^2, \quad x^4 = e^1 \quad \text{and} \quad x^5 = e^3$$

in $\mathfrak{g}^*/\langle e^6\rangle$ shows that N is the nilpotent Lie group with Lie algebra $L_{5,5}$ of our classification (see Table 2.1). Bock proves ([13], Theorem 8.3.4) that $G = G_{6.78}$ is a completely solvable Lie group, that it admits a lattice Γ and that quotient $M = \Gamma\backslash G$ is a formal symplectic solvmanifold with $b_1(M) = b_2(M) = 1$. Since G is completely solvable, the cohomology of M can be computed using Hattori's theorem. We get

- $H^1(M) = \langle e^6\rangle$;
- $H^2(M) = \langle e^{14} + e^{26} + e^{35}\rangle$;
- $H^3(M) = \langle e^{124}, e^{146} + e^{356}\rangle$;
- $H^4(M) = \langle e^{1246}\rangle$;
- $H^5(M) = \langle e^{12345}\rangle$;
- $H^6(M) = \langle e^{123456}\rangle$.

We see that $\omega := e^{14} + e^{26} + e^{35}$ is a symplectic form on M. Let us construct the minimal model of M, $\mathcal{M}_M = (\wedge V, d)$ and the quasi-isomorphism $\varphi : \mathcal{M}_M \to (\wedge\mathfrak{g}^*, d)$. In degree 1, we have one generator, which is clearly closed. Thus $V^1 = \langle a\rangle$ and $\varphi(a) = e^6$. Since $a^2 = 0$, there are no other generators in degree 1. Then we add a degree 2 closed generator b mapping to ω; then $\varphi(ab) = e^{146} + e^{356}$ which is closed and not exact. Thus there are no other generators in degree 2. This shows that $\mathcal{M}_M$ is $2-$formal, hence formal by [37].

The manifold we need is $P = M \times S^1$. Then P has a left-invariant co-symplectic structure given by $\eta = e^7$ and ω as above, where e^7 generates the Lie algebra of S^1. Clearly P is formal. An application of Künneth formula shows that $H^1(P) = \langle e^6, e^7\rangle$; we prove that P is not $1-$Lefschetz by showing that $\mathscr{L}(e^6) = 0$ in cohomology. Indeed, one has

$$\begin{aligned}\mathscr{L}(e^6) &= \omega^2 \wedge (\imath_\xi(\omega \wedge e^6) + e^7 \wedge e^6) = \\ &= 2(-e^{1246} - e^{1345} + e^{2356}) \wedge (-e^{67}) = \\ &= e^{134567} = \\ &= d(e^{12367}).\end{aligned}$$

We thus have proved the following result:

Theorem A.4. *For completely solvable co-symplectic solvmanifolds, the $1-$Lefschetz property and formality are independent from each other.*

A.3 Co-symplectic nilmanifolds in low dimension

In this section we would like to see which low-dimensional nilmanifolds admits co-symplectic structures. For this we use the classification of low dimensional nilmanifolds that we obtained in Chapters 2 and 3.

In dimension 3, we have two nilmanifolds:

- the torus T^3, which is obviously co-Kähler;
- the Heisenberg manifold, whose Lie algebra has the only non-zero bracket $[X_1, X_2] = X_3$, with respect to a Mal'cev basis $\{X_1, X_2, X_3\}$. A co-symplectic structure is given, for example, by choosing

  $$\xi = X_2, \quad \eta = x_2, \quad \omega = x_1 x_3$$

  and g the standard euclidean metric; here $\{x_1, x_2, x_3\}$ is the dual basis. The tensor J is determined by ω and g and is easily seen to be $J(X_1) = -X_3$, $J(X_2) = 0$.

In dimension 5 there are 9 rational homotopy types of nilmanifolds. We use the labeling of the fifth column of Table 2.1. The metric g is always assumed to be the standard one; we also fix a Mal'cev basis $\{X_1, \ldots, X_5\}$ of $\mathfrak{g}$ with dual basis $\{x_1, \ldots, x_5\}$. For the last three nilmanifolds, also check Proposition 4.6. We describe the left invariant co-symplectic structure by giving the Kähler form ω and the distinguished $1-$form η. Notice that $\xi = X_k$ when $\eta = x_k$. The tensor J can be recovered using g and ω on the ideal $\ker(\eta)$ and by setting $J(\xi) = 0$. The results are contained in Table A.1.

We finish with the study of co-symplectic structures on $7-$dimensional $2-$step nilmanifolds. We refer to Table 3.3 for the labelling. We use the same conventions as in the $5-$dimensional case, except for $L_{7,9}$, where we take $g = \mathrm{diag}(1, 1, \sqrt{2}, 1, 1, 1, \sqrt{2})$. The results are contained in Table A.2.

As we did in Chapter 2 for symplectic structures on $6-$dimensional nilmanifolds, we show how to construct a co-symplectic structure on a nilmanifold and how to prove that a nilmanifold does not admit any co-symplectic structure. We take the standard euclidean metric on $\mathfrak{g}^*$[2]. We can assume that the closed $1-$form η of the co-symplectic structure is one of the generators of $\mathfrak{g}^*$. The co-symplectic structure will then be determined by η and ω, where $\omega \in \wedge^2\mathfrak{g}^*$ is such that $\eta \wedge \omega^n \neq 0$ (here $\dim \mathfrak{g}^* = 2n+1$). The condition $\imath_\xi \omega = 0$ implies that ω can not contain the generator η in its expression. We take for η a closed generator x_j and check whether there exists an element $\omega \in \wedge^2(\mathfrak{g}^*/\langle x_j \rangle)$ such that $d\omega = 0$ and $\omega^n \neq 0$. If we find such an ω, then

[2] Except in the case $L_{7,9}$, as we remarked above.

the co-symplectic structure is determined, otherwise, the nilmanifold has no co-symplectic structure. Let us start with the nilmanifold $L_{7,3}$; here $n = 3$; if $\langle x_1, \ldots, x_7 \rangle$ is a basis of $\mathfrak{g}^*$, then the differential is

$$dx_i = 0,\ i = 1, \ldots, 5,\ dx_6 = x_1x_2 + x_3x_4 \text{ and } dx_7 = x_1x_3 + x_2x_5.$$

We try setting $\eta = x_2$. Then one sees that the element

$$\omega = x_1x_7 + x_3x_4 - x_5x_6 \in \wedge^2(\mathfrak{g}^*/\langle x_2 \rangle)$$

satisfies $d\omega = 0$ and $\omega^3 \neq 0$.

Next we show that $L_{7,2}$ does not admit a co-symplectic structure; it is easy to see that for every choice of a closed generator x_i of $\mathfrak{g}^*$ as our η, the space $\wedge^2(\mathfrak{g}^*/\langle x_i \rangle)$ does not contain any closed element ω such that $\omega^3 \neq 0$. Alternatively, we can show that the 8$-$dimensional nilmanifold $L_{7,2} \times S^1$ does not admit a symplectic structure.

Table A.1: Co-symplectic 5$-$dimensional nilmanifolds

Nilmanifold	co-symplectic structure	
	ω	η
A_5	$x_1x_2 + x_3x_4$	x_5
$L_3 \oplus A_2$	$x_2x_5 + x_3x_4$	x_1
$L_4 \oplus A_1$	$x_1x_5 + x_2x_4$	x_3
$L_{5,1}$	not co-symplectic	
$L_{5,2}$	$x_1x_5 + x_2x_4$	x_3
$L_{5,3}$	$x_1x_5 - x_3x_4$	x_2
$L_{5,4}$	$x_2x_5 - x_3x_4$	x_1
$L_{5,5}$	not co-symplectic	
$L_{5,6}$	$x_2x_5 - x_3x_4$	x_1

Table A.2: Co-symplectic 7−dimensional 2−step nilmanifolds

Nilmanifold	co-symplectic structure	
	ω	η
A_7	$x_1x_2 + x_3x_4 + x_5x_6$	x_7
$L_3 \oplus A_4$	$x_1x_7 + x_2x_4 + x_5x_6$	x_3
$L_3 \oplus L_3 \oplus A_1$	$x_1x_6 + x_3x_7 + x_4x_5$	x_2
$L_{5,1} \oplus A_2$	not co-symplectic	
$L_{5,2} \oplus A_2$	$x_1x_6 + x_3x_7 + x_2x_4$	x_5
$L_{6,1} \oplus A_1$	$x_1x_4 + x_2x_7 + x_3x_6$	x_5
$L_{6,2} \oplus A_1$	$x_1x_6 - x_2x_7 + x_3x_4$	x_5
$L_{6,4} \oplus A_1$	$x_1x_5 + x_3x_6 + x_2x_7$	x_4
$L_{7,1}$	not co-symplectic	
$L_{7,2}$	not co-symplectic	
$L_{7,3}$	$x_1x_7 + x_3x_4 - x_5x_6$	x_2
$L_{7,4}$	$x_2x_6 + x_3x_5 + x_4x_7$	x_1
$L_{7,5}$	$x_1x_6 + x_2x_7 - x_3x_5$	x_4
$L_{7,6}$	$x_1x_5 + x_3x_7 + x_4x_6$	x_2
$L_{7,7}$	$x_1x_7 - x_3x_5 + x_4x_6$	x_2
$L_{7,8}$	$x_1x_7 - x_3x_6 + x_4x_5$	x_2
$L_{7,9}$	$x_1x_5 + x_2x_6 - 2x_3x_7$	x_4

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