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Integrable coupled massive Thirring model with field values in a Grassmann algebra

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ABSTRACT: A coupled massive Thirring model of two interacting Dirac spinors in $1+1$ dimensions with fields taking values in a Grassmann algebra is introduced, which is closely related to a $\mathrm{SU}(1,1)$ version of the Grassmannian Thirring model also introduced in this work. The Lax pair for the system is constructed, and its equations of motion are obtained from a zero curvature condition. It is shown that the system possesses several infinite hierarchies of conserved quantities, which strongly confirms its integrability. The model admits a canonical formulation and is invariant under space-time translations, Lorentz boosts and global $\mathrm{U}(1)$ gauge transformations, as well as discrete symmetries like parity and time reversal. The conserved quantities associated to the continuous symmetries are derived using Noether's theorem, and their relation to the lower-order integrals of motion is spelled out. New nonlocal integrable models are constructed through consistent nonlocal reductions between the field components of the general model. The Lagrangian, the Hamiltonian, the Lax pair and several infinite hierarchies of conserved quantities for each of these nonlocal models are obtained substituting its reduction in the expressions of the analogous quantities for the general model. It is shown that, although the Lorentz symmetry of the general model breaks down for its nonlocal reductions, these reductions remain invariant under parity, time reversal, global $\mathrm{U}(1)$ gauge transformations and space-time translations.

KEYWORDS: Integrable Field Theories, Integrable Hierarchies

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1 Introduction

The massive Thirring model (MTM) has attracted considerable attention in the literature as a rare example of an exactly solvable relativistic model in $1+1$ dimensions, which is relevant in different branches of physics. The MTM was originally introduced as a toy model in the realm of quantum field theory, with the purpose of understanding the non-perturbative physical phenomena arising in realistic $(3+1)$-dimensional systems [1]. Further developments such as the Coleman correspondence [2] between the quantum sine-Gordon model and the zero charge sector of the quantum MTM stimulated the study of the Thirring model as an integrable quantum field theory. In fact, subsequent work established the complete integrability of the MTM both in its quantum [3–5] and classical variants [6–9]. More precisely, at the classical level there exist two versions of the MTM, depending on whether the field variables take values in the complex numbers field or in a Grassmann algebra. The first version, often referred to as the bosonic massive Thirring model (BMTM), has important physical applications including pulse propagation in Bragg nonlinear optical media [10–14] and dipole-dipole interactions among many-body bosonic atoms [15]. A great deal of research has been devoted to the BMTM applying well-known techniques such as the inverse scattering transform, Bäcklund or Darboux transformations, and various types of soliton solutions and rogue waves thereof have been obtained [6, 7, 16–25]. Further interesting results include the proof of the existence of soliton solutions in a nonvanishing background [26–28] and the construction of general bright and dark soliton solutions via KP hierarchy reductions [29], or the study of the integrability of the model in the presence of defects [30, 31] and of balanced loss and gain [32].

The study of the integrability of the classical MTM with fields taking values in a Grassmann algebra (GMTM for short) is important in the context of quantization, in which case the fermionic fields are taken as operators satisfying canonical anticommutation relations. The integrability of the GMTM at the classical level was established early on in refs. [8, 9]. The Lax pair for this model was constructed in the latter reference, and an infinite number of conserved quantities were obtained. Furthermore, by employing the inverse scattering transform it was shown that the usual soliton solution is absent in this model. A study of the GMTM in the presence of defects was also undertaken in refs. [30, 31].

A coupled BMTM describing the interaction between two independent self-interacting Dirac spinors was recently introduced in ref. [33], where the model's Lagrangian and Hamiltonian were constructed. It was shown that the action of the system is invariant under parity, time reversal, global U(1) gauge transformations and Lorentz boosts, as is the case for the original MTM. The new model possesses a Lax pair which yields the equations of motion as a zero curvature condition. The linear equations associated with this Lax pair can be used to construct an infinite number of conserved quantities, which confirms the integrability of the system. A detailed study of the complete integrability (in the Liouville sense) of the coupled BMTM was also carried out in ref. [33], and some novel nonlocal nonlinear integrable systems related to different reductions of the coupled BMTM were constructed and analyzed.

The integrability of both the bosonic and Grassmannian versions of the original MTM, as well as the new coupled MTM introduced in ref. [33], strongly suggest that this might also be the case for the Grassmannian version of the latter model. In fact, the main aim of the present paper is to investigate the integrability properties of the classical version of the coupled MTM with fields taking values in a Grassmann algebra. To this end, we first construct the model's Lax pair, which consists of 3×3 matrices instead of 2×2 matrices as is the case in its bosonic version. We then show that the compatibility condition between the two linear Lax equations yields the equations of motion for the coupled GMTM via a zero curvature condition.

Another interesting outcome of the present investigation is that the original Lagrangian for this coupled GMTM can be written in a new form by introducing a pair of new Grassmann fields related to the old ones through a rotation by an angle $-\pi/4$ in the internal space of the field variables. We show that in this form the model is closely related to a $\mathrm{SU}(1,1)$ generalization of the GMTM model that, to the best of our knowledge, has not been previously discussed in the literature. On the other hand, the quantized version of this new form of the Lagrangian of the coupled GMTM may also be interpreted as describing the mutual interaction between a self-interacting Thirring fermion and a ghost Thirring fermion of the same mass but with opposite sign of the coupling constant. It should be noted in this respect that this type of ghost fields has recently been considered in the literature in the context of the bosonic field theory of quantized gravity [34].

Apart from the Lax pair, a fundamental aspect for the integrability of a given system is the existence of conserved quantities. The construction of the conserved quantities of the new model is accomplished by exploiting the linear equations coming from the Lax pair and the zero curvature condition. In this way several infinite hierarchies of conserved quantities are obtained, which ensure the integrability of the system. As in the case of the ordinary GMTM, some of the conserved quantities of the new coupled GMTM are local whereas others turn out to be nonlocal (by contrast, all the conserved quantities of the coupled BMTM are local). The Hamiltonian formulation of the coupled GMTM is presented following the standard canonical formalism for Grassmann-valued field theories [35, 36]. The action for the coupled GMTM is shown to be manifestly invariant under space-time translations and U(1) global gauge transformations, and the corresponding conserved charges are obtained through Noether's theorem. These conserved charges, which may readily be identified as the system's Hamiltonian, total momentum and charge, are in fact related to the local integrals of motion constructed using the Lax pair formalism. Finally, the Noether current arising from the invariance of the system under Lorentz boosts is also obtained. Apart from these continuous symmetries, the system is also invariant under several discrete symmetries, namely parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$), and hence under their composition ($\mathcal{PT}$). Interestingly, the lower-order integrals of motion of the coupled GMTM are found to Poisson commute among themselves, a fact strongly suggesting that the system might be completely integrable in Liouville's sense as in the case of the coupled BMTM. However, a full proof of the complete integrability for the coupled GMTM is not considered in the present study and will be the subject of future work.

A systematic study of the reductions of the field components of the coupled GMTM shows that its Lax pair can be used to generate new integrable systems. Apart from the ordinary GMTM, we obtain two nonlocal reductions with real reverse space and real reverse time, producing new integrable systems. These two models are nonlocal in the sense that the value of the interaction potential at (x, t) depends on the value of the fields at $(-x, t)$ (space inversion) or at $(x, -t)$ (time inversion). It should be mentioned here that the type of nonlocal integrable models discussed in the present paper were first introduced in the context of the nonlinear Schrödinger equation (NLSE) with a space inversion interaction [37]. Since then, a vast amount of research has been carried out on several types of nonlocal integrable dynamical systems, including different generalized versions of the NLSE [38–44], nonlocal derivative NLSE [45, 46] and nonlocal sine-Gordon model [47]. To the best of our knowledge, however, such nonlocal integrable models had not been constructed so far for field theories with Grassmann-valued variables. In our present study, the Lax pair and integrals of motion for the new nonlocal integrable models are simply constructed by reduction of the analogous expressions for the coupled GMTM. The Lagrangian and the corresponding Hamiltonian are also derived in the same way. It is also shown that the new nonlocal reductions break the Lorentz invariance of the coupled GMTM. However, they remain invariant under global U(1) gauge transformations and discrete space-time symmetries (parity, time reversal and their composition).

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The paper's organization is as follows. In the next section we introduce the coupled MTM with field values in a Grassmann algebra. The Lax pair of the model is constructed, and the equations of motion are derived as the corresponding zero curvature condition. Sections 3 and 4 respectively deal with the construction of several infinite hierarchies of nonlocal and local conserved quantities by using the Lax equations and the zero curvature condition. In section 5 the canonical formulation of the model is presented. The continuous symmetries of the system —space-time translations, global U(1) gauge transformations and Lorentz boosts— are discussed, and the corresponding conserved quantities are obtained by applying Noether's theorem. The invariance of the system under discrete symmetries —parity and time reversal— is also discussed in this section. The nonlocal integrable models obtained by considering different reductions between the field components of the coupled GMTM are presented in section 6. Finally, in the last section we make a brief summary of the results obtained and outline some future developments.

2 The model

2.1 Definition and Lax pair

A (bosonic) coupled MTM in $1+1$ dimensions incorporating the interaction between two Lorentz spinors ψ and ϕ was recently introduced in ref. [33]. The Lagrangian density for this model is given by

$$\mathcal{L} = \frac{1}{2}\bar{\phi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\phi - \frac{1}{2}\bar{\phi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m)\psi - \frac{1}{2}\bar{\psi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m)\phi$$
$$+ \frac{g}{2}[(\bar{\phi}\gamma_\mu\psi)(\bar{\phi}\gamma^\mu\psi) + (\bar{\psi}\gamma_\mu\phi)(\bar{\psi}\gamma^\mu\phi)], \tag{2.1}$$

where $\phi = (\phi_1, \phi_2)^\mathsf{T}$, $\bar{\phi} = \phi^\dagger\gamma^0$ (and similarly for ψ), the two-dimensional gamma matrices are taken as $\gamma^0 = \sigma_x$, $\gamma^1 = -\mathrm{i}\sigma_y$, and the spacetime metric tensor is $\mathrm{diag}(1, -1)$. The last term in square brackets represents the interaction between the fields ψ and ϕ, g being the (real) coupling constant. In the present study we consider the fields ϕ_i, ψ_j (with $i, j = 1, 2$) as functions taking values in the odd (fermionic) sector of an (infinite-dimensional) Grassmann algebra $\mathbf{G}$. In other words, the fields ϕ_i and ψ_j anticommute among themselves and with their conjugates ϕ_i^* and ψ_j^*, where "$*$" denotes the involution (complex conjugation) in $\mathbf{G}$ [48]. The Euler-Lagrange equations of motion for the fields ϕ and ψ are thus

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\phi + g(\bar{\psi}\gamma_\mu\phi)\gamma^\mu\phi = 0, \tag{2.2}$$
$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi + g(\bar{\phi}\gamma_\mu\psi)\gamma^\mu\psi = 0, \tag{2.3}$$

while the fields ϕ^* and ψ^* obey the conjugate equations

$$\bar{\phi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m) - g\bar{\phi}\gamma^\mu(\bar{\phi}\gamma_\mu\psi) = 0, \tag{2.4}$$
$$\bar{\psi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m) - g\bar{\psi}\gamma^\mu(\bar{\psi}\gamma_\mu\phi) = 0. \tag{2.5}$$

These equations are clearly invariant under the exchange $\phi \leftrightarrow \psi$. Moreover, for $g = 0$ the fields ϕ and ψ both satisfy the free Dirac equation, while for $\psi = \phi$ eqs. (2.2) and (2.3) reduce to the equation of motion of the original (Grassmannian) MTM.

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Before proceeding further it is convenient to perform the change of scale

$$x \to \frac{1}{m}x, \quad t \to \frac{1}{m}t, \quad \phi_j \to \sqrt{\frac{m}{2|g|}}\phi_j, \quad \psi_j \to \sqrt{\frac{m}{2|g|}}\psi_j, \qquad j = 1, 2, \tag{2.6}$$

which transforms eqs. (2.2)–(2.3) into

$$\begin{aligned} &\mathrm{i}(\partial_t + \partial_x)\phi_1 - \phi_2 - 2\varepsilon(\psi_2^*\phi_2)\phi_1 = 0, \\ &\mathrm{i}(\partial_t - \partial_x)\phi_2 - \phi_1 - 2\varepsilon(\psi_1^*\phi_1)\phi_2 = 0 \\ &\mathrm{i}(\partial_t + \partial_x)\psi_1 - \psi_2 - 2\varepsilon(\phi_2^*\psi_2)\psi_1 = 0, \\ &\mathrm{i}(\partial_t - \partial_x)\psi_2 - \psi_1 - 2\varepsilon(\phi_1^*\psi_1)\psi_2 = 0 \end{aligned} \tag{2.7}$$

with $\varepsilon := -\operatorname{sgn} g$. The component form of the conjugate equations (2.4)–(2.5) is easily obtained by taking the complex conjugate of eqs. (2.7). Moreover, since the change of variables $\phi \to -\phi$ (or $\psi \to -\psi$) reverses the sign of the term proportional to ε in eqs. (2.7), we shall take from now on $\varepsilon = 1$ without loss of generality.

In order to study the integrability properties of the coupled GMTM (2.7) we shall employ the zero curvature formulation, which relies on constructing a Lax pair for the system. The U, V matrices in this Lax pair can be expressed as $U = U_+$ and $V = U_-$, with

$$U_\pm = \begin{pmatrix} \mathrm{i}\,\rho_\mp + \frac{\mathrm{i}}{2}(\lambda^2 \mp \lambda^{-2}) & 0 & -r_1^\mp \\ 0 & -\mathrm{i}\,\rho_\mp + \frac{\mathrm{i}}{2}(\lambda^2 \mp \lambda^{-2}) & -r_2^\pm \\ r_2^\mp & r_1^\pm & \mathrm{i}(\lambda^2 \mp \lambda^{-2}) \end{pmatrix}, \tag{2.8}$$

where

$$\rho_\pm = \phi_2^*\psi_2 \pm \phi_1^*\psi_1, \qquad r_1^\pm = -\mathrm{i}(\lambda\phi_2^* \pm \lambda^{-1}\phi_1^*), \qquad r_2^\pm = \mathrm{i}(\lambda\psi_2 \pm \lambda^{-1}\psi_1) \tag{2.9}$$

and $\lambda \in \mathbb{C}$ is a spectral parameter independent of x and t. The linear equations of the Lax pair associated to the latter matrices are

$$w_x = Uw, \qquad w_t = Vw, \tag{2.10}$$

where $w(x,t)$ is the three-component column vector $w = (w_1, w_2, w_3)^\mathsf{T}$, $w_x = \frac{\partial w}{\partial x}$, and $w_t = \frac{\partial w}{\partial t}$. Note that from the form of the matrices U and V it follows that the auxiliary field components (w_1, w_2) and w_3 must have opposite parity. The compatibility condition $w_{xt} = w_{tx}$ of the equations in (2.10) yields the zero curvature condition

$$U_t - V_x + [U, V] = 0, \tag{2.11}$$

where $[U, V]$ is the usual commutator. It can be readily checked that, when written in component form, eq. (2.11) reduces to equations (2.7). This shows that the equations of motion of the coupled GMTM are indeed obtained as the zero curvature condition (2.11) of the Lax pair (2.10).

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2.2 Connection to the SU(1,1) Thirring model

The non-standard form of the kinetic energy terms in the Lagrangian (2.1) can be easily remedied by performing a rotation of angle $-\pi/4$ in internal (field) space, i.e., introducing a pair of new two-component fermionic fields (Φ, Ψ) through the relation

$$\phi = \frac{1}{\sqrt{2}}(\Phi - \Psi), \qquad \psi = \frac{1}{\sqrt{2}}(\Phi + \Psi). \tag{2.12}$$

Indeed, in terms of the new fields (Φ, Ψ) the Lagrangian (2.1) becomes

$$\mathcal{L} = \mathcal{L}_{\mathrm{Th}}(\Phi; m, g/2) - \mathcal{L}_{\mathrm{Th}}(\Psi; m, -g/2) + \mathcal{L}_{\mathrm{I}}, \tag{2.13}$$

where

$$\mathcal{L}_{\mathrm{Th}}(\chi; m, g) = \frac{1}{2}\bar{\chi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\chi - \frac{1}{2}\bar{\chi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m)\chi + \frac{g}{2}(\bar{\chi}\gamma_\mu\chi)(\bar{\chi}\gamma^\mu\chi) \tag{2.14}$$

is the usual (Grassmannian) Thirring Lagrangian, and the interaction Lagrangian $\mathcal{L}_{\mathrm{I}}$ is given by

$$\mathcal{L}_{\mathrm{I}} = \frac{g}{4}\Big[(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Phi}\gamma^\mu\Psi) + (\bar{\Psi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Phi) - 2(\bar{\Phi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Psi) - 2(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Phi)\Big]. \tag{2.15}$$

The minus sign in front of the second Thirring Lagrangian in eq. (2.14) suggests a connection of the present model with an SU(1,1) version of the massive Thirring model that we shall now make explicit. To this end, we regard the pair $F = (\Phi, \Psi)^{\mathsf{T}}$ as a (Grassmann-valued) vector field whose components $F^1 = \Phi$ and $F^2 = \Psi$ transform under the fundamental representation of the SU(1,1) group. This is of course motivated by the fact that the kinetic energy and mass terms of the Lagrangian (2.13)–(2.15) can be written as

$$\frac{1}{2}\eta_{ab}\Big[\bar{F}^a(\mathrm{i}\gamma_\mu\partial_\mu - m)F^b - \bar{F}^a(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m)F^b\Big] \equiv \mathcal{K}_{\mathrm{SU}(1,1)}, \tag{2.16}$$

where (as in what follows) we are implicitly summing over repeated indices, $\bar{F}^a = (F^a)^\dagger\gamma^0$ and $\eta_{11} = -\eta_{22} = 1$, $\eta_{12} = \eta_{21} = 0$. This term is easily seen to be invariant under the global SU(1,1) transformation $F \mapsto AF$, where A is a 2×2 complex matrix satisfying $A^\dagger\eta A = \eta$ and $\det A = 1$. More generally, inspired by the definition of the SU(N) (massless) Thirring model in ref. [49], we define the SU(1,1) massive Thirring model Lagrangian as

$$\mathcal{L}_{\mathrm{SU}(1,1)} = \mathcal{K}_{\mathrm{SU}(1,1)} + g_0 J^0_\mu J^{0\mu} + g_1\eta_{bc}\eta_{ad}(\bar{F}^a\gamma_\mu F^b)(\bar{F}^c\gamma^\mu F^d), \tag{2.17}$$

where

$$J^0_\mu = \eta_{ab}\bar{F}^a\gamma_\mu F^b, \qquad \mu = 0, 1,$$

is the U(1,1) current and g_0, g_1 are arbitrary real parameters. It is straightforward to check that $\mathcal{L}_{\mathrm{SU}(1,1)}$ is real, due to the identities $(\gamma^0)^* = \gamma^0$ and $\gamma^0\gamma^*_\mu\gamma^0 = \gamma_\mu$ satisfied by the gamma matrices. The Lagrangian (2.17) is also readily seen to be invariant under the global SU(1,1) transformation

$$F \mapsto AF, \qquad A \in \mathrm{SU}(1,1),$$

since both interaction terms in eq. (2.17) are in fact separately invariant. The Lagrangian (2.17) can also be written in terms of the three currents

$$J^i_\mu = \bar{F}^a(\eta t^i)_{ab}\gamma_\mu F^b, \qquad i = 1,2,3, \tag{2.18}$$

associated to the su(1,1) generators[1] t^i ($i = 1,2,3$) in the fundamental representation, which we shall take as

$$t^1 = \mathrm{i}\sigma^x, \qquad t^2 = \mathrm{i}\sigma^y, \qquad t^3 = \sigma^z.$$

Note that t^1 and t^2 are anti-Hermitian while t^3 is Hermitian, and that

$$\operatorname{tr} t^i = 0, \quad (t^i)^\dagger \eta - \eta t^i = 0, \qquad i = 1,2,3.$$

Indeed, the trace formulas

$$\operatorname{tr}(t^i t^j) = 2\varepsilon_i \delta_{ij}, \qquad \text{with} \quad \varepsilon_3 = -\varepsilon_1 = -\varepsilon_2 = 1,$$

imply the identity

$$\varepsilon_i \eta_{a'a}\eta_{c'c} t^i_{ab} t^i_{cd} = 2\eta_{a'd}\eta_{bc'} - \eta_{a'b}\eta_{c'd},$$

and hence

$$\varepsilon_i J^i_\mu J^{i\mu} = 2\eta_{a'd}\eta_{bc'}(\bar{F}^{a'}\gamma_\mu F^b)(\bar{F}^{c'}\gamma^\mu F^d) - J^0_\mu J^{0\mu}. \tag{2.19}$$

From this equation it immediately follows that the SU(1,1) Thirring model Lagrangian (2.17) can be written as

$$\mathcal{L}_{\mathrm{SU}(1,1)} = \mathcal{K}_{\mathrm{SU}(1,1)} + \left(g_0 + \frac{g_1}{2}\right) J^0_\mu J^{0\mu} + g_1 \varepsilon_i J^i_\mu J^{i\mu}. \tag{2.20}$$

Note that, as is the case for the SU(N) Thirring model studied in ref. [49], the last term has the same structure as the su(1,1) Casimir $\varepsilon_i t^i t^i$.

In order to elucidate the precise relation between the SU(1,1) Thirring model (2.20) (or (2.17)) and our model (cf. eqs. (2.13)–(2.15)), it suffices to expand the interaction terms in eq. (2.17), namely:

$$\begin{aligned} J^0_\mu J^{0\mu} &= (\bar{\Phi}\gamma_\mu\Phi - \bar{\Psi}\gamma_\mu\Psi)(\bar{\Phi}\gamma^\mu\Phi - \bar{\Psi}\gamma^\mu\Psi) \\ &= (\bar{\Phi}\gamma_\mu\Phi)(\bar{\Phi}\gamma^\mu\Phi) + (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Psi) - 2(\bar{\Phi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Psi), \end{aligned} \tag{2.21}$$

$$\eta_{bc}\eta_{ad}(\bar{F}^a\gamma_\mu F^b)(\bar{F}^c\gamma^\mu F^d) = (\bar{\Phi}\gamma_\mu\Phi)(\bar{\Phi}\gamma^\mu\Phi) + (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Psi) - 2(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Phi). \tag{2.22}$$

We thus find that the SU(1,1) Thirring model Lagrangian (2.17) —or, equivalently, (2.20)— can be written as

$$\begin{aligned} \mathcal{L}_{\mathrm{SU}(1,1)} = \mathcal{K}_{\mathrm{SU}(1,1)} &+ (g_0 + g_1)\left[(\bar{\Phi}\gamma_\mu\Phi)(\bar{\Phi}\gamma^\mu\Phi) + (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Psi)\right] \\ &- 2g_0(\bar{\Phi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Psi) - 2g_1(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Phi). \end{aligned} \tag{2.23}$$

[1] Here, and in what follows, we shall use the term "generator" adhering to the physicists' convention, according to which the one-parameter group generated by a generator X is $\mathrm{e}^{\mathrm{i}tX}$ instead of e^{tX} (with $t \in \mathbb{R}$ the group parameter).

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It is apparent that the terms

$$(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Phi}\gamma^\mu\Psi)+(\bar{\Psi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Phi)$$

in the Lagrangian (2.13)–(2.15) of the coupled GMTM do not appear in eq. (2.23), nor is it possible to reproduce the coefficients of the remaining terms in (2.15) for any choice of g_0 and g_1. We thus conclude that the Lagrangian (2.13)–(2.15) is *not* of the $\mathrm{SU}(1,1)$ invariant form (2.17). On the other hand, (2.13)–(2.15) can be expressed in terms of $\mathrm{su}(1,1)$ currents in a remarkably simple way. Indeed, eq. (2.21) and the identity

$$\begin{aligned}J^1_\mu J^{1\mu} &= -(\bar{\Phi}\gamma_\mu\Psi-\bar{\Psi}\gamma_\mu\Phi)(\bar{\Phi}\gamma^\mu\Psi-\bar{\Psi}\gamma^\mu\Phi)\\ &= -(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Phi}\gamma^\mu\Psi)-(\bar{\Psi}\gamma_\mu\Phi)(\bar{\Psi}\gamma^\mu\Phi)+2(\bar{\Phi}\gamma_\mu\Psi)(\bar{\Psi}\gamma_\mu\Phi),\end{aligned}$$

imply that

$$\mathcal{L}=\mathcal{K}_{\mathrm{SU}(1,1)}+\frac{g}{4}\Big(J^0_\mu J^{0\mu}-J^1_\mu J^{1\mu}\Big). \tag{2.24}$$

It is straightforward to check that each $\mathrm{su}(1,1)$ current J^k_μ is invariant under the one-parameter group $\mathrm{e}^{\mathrm{i}\theta t^k}$ (with $\theta\in\mathbb{R}$) generated by its corresponding generator $t^k\in\mathrm{su}(1,1)$. Indeed, under a general $\mathrm{SU}(1,1)$ transformation $F\mapsto AF$ the current J^k_μ is mapped to

$$\bar{F}^a(A^\dagger\eta t^kA)_{ab}\gamma_\mu F^b,$$

and we have

$$\mathrm{e}^{-\mathrm{i}\theta(t^k)^\dagger}\eta t^k\mathrm{e}^{\mathrm{i}\theta t^k}=\eta t^k,\qquad k=1,2,3.$$

For instance, for the current J^1_μ

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\theta(t^1)^\dagger}\eta t^1\mathrm{e}^{\mathrm{i}\theta t^1} &= \mathrm{e}^{-\theta\sigma^x}(\mathrm{i}\sigma^z\sigma^x)\mathrm{e}^{-\theta\sigma^x}=-\mathrm{e}^{-\theta\sigma^x}\sigma^y\mathrm{e}^{-\theta\sigma^x}\\ &= -(\cosh\theta-\sinh\theta\,\sigma^x)\sigma^y(\cosh\theta-\sinh\theta\,\sigma^x)=-\sigma^y=\mathrm{i}\sigma^z\sigma^x=\eta t^1.\end{aligned}$$

Since the first two terms in the Lagrangian (2.24) are obviously $\mathrm{SU}(1,1)$ invariant, it follows from the previous remark that the latter Lagrangian is invariant under the one-parameter group of transformations $F\mapsto\mathrm{e}^{\mathrm{i}\theta t^1}F$ (with $\theta\in\mathbb{R}$) generated by the $\mathrm{su}(1,1)$ generator t^1. From the fact that eq. (2.12) can be written in matrix form as

$$(\phi,\psi)^\mathsf{T}=\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\sigma^y}(\Phi,\Psi)^\mathsf{T},$$

it follows that the original Lagrangian (2.1) is invariant under the one-parameter group

$$(\phi,\psi)^\mathsf{T}\mapsto\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\sigma^y}\mathrm{e}^{-\theta\sigma^x}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma^y}(\phi,\psi)^\mathsf{T}=\mathrm{e}^{\theta\sigma^z}(\phi,\psi)^\mathsf{T},$$

i.e., the obvious scaling symmetry

$$\phi\mapsto\mathrm{e}^\theta\phi,\qquad\psi\mapsto\mathrm{e}^{-\theta}\psi. \tag{2.25}$$

The relation between the coupled GMTM model studied in this work and the $\mathrm{SU}(1,1)$ Thirring model (2.17) is thus akin to the relation between the quantum XXX Heisenberg model (invariant under the full rotation group $\mathrm{SO}(3)$) and its XXZ counterpart (only invariant under rotations around an axis). It should be noted, in this respect, that this residual invariance of the XXZ spin chain plays a key role in the construction of the Bethe eigenstates of this model.

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Remark 1. In the quantized version of the Lagrangian (2.14)–(2.15), the minus sign in front of the second Thirring Lagrangian could perhaps be explained by regarding the Ψ field as a kind of fermionic ghost field. With this interpretation, the quantum counterpart of (2.14) would describe the mutual interaction of a (self-interacting) Thirring fermion with a ghost Thirring fermion, with the same mass m and opposite coupling constants $\pm g/2$. This type of ghost fields has recently been discussed in bosonic field theories in the context of quantized gravity, where the propagator of the (four-derivative) graviton can be effectively expressed as the sum of the standard propagators of a massless graviton field and a massive ghost field with negative kinetic energy [34].

3 Nonlocal conserved quantities

One of the key aspects in the study of the integrability of a given system is the construction of an infinite set of conserved quantities. In this section we shall exploit the zero curvature formulation of the equations of motion of the coupled GMTM developed above in order to obtain several infinite hierarchies of conserved quantities. Following ref. [31], we shall derive these conserved quantities from the quotients

$$\Gamma_{ij} = w_j^{-1} w_i = w_i w_j^{-1} \equiv \frac{w_i}{w_j},$$

where $i \neq j$ and w_j is assumed to be bosonic (even) and with nonvanishing body (complex number part) to guarantee that w_j^{-1} exists and is also even [48]. From the identity

$$\partial_t(w_j^{-1}\partial_x w_j) = \partial_x(w_j^{-1}\partial_t w_j)$$

and the Lax pair equations (2.10) it then follows that

$$\partial_t\Big(U_{jj} + \sum_{i;i\neq j} U_{ji}\Gamma_{ij}\Big) = \partial_x\Big(V_{jj} + \sum_{i;i\neq j} V_{ji}\Gamma_{ij}\Big). \tag{3.1}$$

If the fields vanish sufficiently fast as $|x| \to \infty$, from these equations we deduce that[2]

$$I_j(t) = \int \mathrm{d}x\Big(U_{jj} + \sum_{i;i\neq j} U_{ji}\Gamma_{ij}\Big), \tag{3.2}$$

is the generating function for an infinite hierarchy of conserved quantities [31]. It is also immediate to show that the auxiliary functions Γ_{ij} (with $i \neq j$) satisfy the following pair of coupled Riccati equations:

$$\partial_x\Gamma_{ij} = U_{ij} - U_{jj}\Gamma_{ij} + \sum_{k\neq j}\Big(U_{ik} - \Gamma_{ij}U_{jk}\Big)\Gamma_{kj}, \tag{3.3}$$

$$\partial_t\Gamma_{ij} = V_{ij} - V_{jj}\Gamma_{ij} + \sum_{k\neq j}\Big(V_{ik} - \Gamma_{ij}V_{jk}\Big)\Gamma_{kj}, \tag{3.4}$$

[2]Here and in what follows, unless otherwise stated we shall assume that the integration range is the whole real line.

where U_{ij} and V_{ij} denote the elements of the matrices U and V. It can be readily checked that the compatibility condition $\partial_x \partial_t \Gamma_{ij} = \partial_t \partial_x \Gamma_{ij}$ for the latter system is automatically satisfied once the equations of motion for the fields ϕ, ψ are taken into account.

In this section we shall assume that w_1 and w_2 are even (and with nonvanishing body), so that w_3 is odd, and shall compute the conserved quantities obtained from the quotients Γ_{i1} with $i = 2, 3$. Note that the conserved quantities constructed from Γ_{i2} with $i = 1, 3$ can be derived from the former by the substitutions

$$\phi_1^* \leftrightarrow \psi_1, \qquad \phi_2^* \leftrightarrow -\psi_2. \tag{3.5}$$

Indeed, under this mapping the functions in eq. (2.9) transforms as

$$\rho_\pm \leftrightarrow -\rho_\pm, \qquad r_1^\pm \leftrightarrow r_2^\mp,$$

which by eq. (2.8) is equivalent to reversing the roles of w_1 and w_2 in the Lax pair. In fact, from the previous argument it follows that (3.5) is a symmetry of the field equations (2.7) themselves, and thus maps any conserved quantity of these equations into another conserved quantity.

3.1 Expansion in negative powers of λ

The equations (3.3)–(3.4) satisfied by the quotients Γ_{31} and Γ_{21} can be more explicitly written as the system of coupled equations

$$\begin{aligned}
\partial_x \Gamma_{31} &= r_2^- + r_1^+ \Gamma_{21} + \mathrm{i}\left[\frac{1}{2}(\lambda^2 - \lambda^{-2}) - \rho_-\right]\Gamma_{31}, \\
\partial_x \Gamma_{21} &= -2\mathrm{i}\rho_- \Gamma_{21} - r_2^+ \Gamma_{31} + r_1^- \Gamma_{21}\Gamma_{31}, \\
\partial_t \Gamma_{21} &= -2\mathrm{i}\rho_+ \Gamma_{21} - r_2^- \Gamma_{31} + r_1^+ \Gamma_{21}\Gamma_{31}.
\end{aligned} \tag{3.6}$$

We have omitted the equation for $\partial_t \Gamma_{31}$, since we shall see below that it is not actually needed. In order to solve this system, we first expand Γ_{31} and Γ_{21} in negative powers of the spectral parameter λ, namely

$$\Gamma_{i1}(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} \Gamma_{i1}^{(n)}(x,t), \qquad i = 2, 3. \tag{3.7}$$

The expansion coefficients $\Gamma_{i1}^{(n)}$ can be computed substituting the previous expansions in the Riccati equations (3.6). In this way we obtain the equations

$$\begin{aligned}
\Gamma_{31}^{(n)} &= 2(\psi_1 \delta_{n3} - \psi_2 \delta_{n1}) + \Gamma_{31}^{(n-4)} + 2(-\mathrm{i}\partial_x + \rho_-)\Gamma_{31}^{(n-2)} \\
&\quad + 2\Big(\phi_2^* \Gamma_{21}^{(n-1)} + \phi_1^* \Gamma_{21}^{(n-3)}\Big),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathrm{i}\,\partial_x \Gamma_{21}^{(n)} &= 2\rho_- \Gamma_{21}^{(n)} + \psi_2 \Gamma_{31}^{(n+1)} + \psi_1 \Gamma_{31}^{(n-1)} \\
&\quad + \phi_2^* \sum_{k=1}^{n} \Gamma_{21}^{(n+1-k)} \Gamma_{31}^{(k)} - \phi_1^* \sum_{k=1}^{n-2} \Gamma_{21}^{(n-1-k)} \Gamma_{31}^{(k)},
\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathrm{i}\,\partial_t\Gamma_{21}^{(n)} =& 2\rho_+\Gamma_{21}^{(n)} + \psi_2\Gamma_{31}^{(n+1)} - \psi_1\Gamma_{31}^{(n-1)}\\ &+ \phi_2^*\sum_{k=1}^{n}\Gamma_{21}^{(n+1-k)}\Gamma_{31}^{(k)} + \phi_1^*\sum_{k=1}^{n-2}\Gamma_{21}^{(n-1-k)}\Gamma_{31}^{(k)},\end{aligned}\tag{3.10}$$

where $n \geqslant 1$ and $\Gamma_{31}^{(n)} = \Gamma_{21}^{(n)} = 0$ for $n \leqslant 0$. The previous equations can be recursively solved as follows. To begin with, eq. (3.8) with $n = 1, 2$ yields

$$\Gamma_{31}^{(1)} = -2\psi_2, \qquad \Gamma_{31}^{(2)} = 2\phi_2^*\Gamma_{21}^{(1)}. \tag{3.11}$$

Substituting the previous expressions for $\Gamma_{31}^{(1)}$ and $\Gamma_{31}^{(2)}$ into eq. (3.9) we then obtain a first-order homogeneous equation for $\Gamma_{21}^{(1)}$, namely

$$\partial_x\Gamma_{21}^{(1)} = -2\mathrm{i}\rho_-\Gamma_{21}^{(1)} - 2\mathrm{i}\psi_2\phi_2^*\Gamma_{21}^{(1)} + 2\mathrm{i}\phi_2^*\psi_2\Gamma_{21}^{(1)} = 2\mathrm{i}\rho_+\Gamma_{21}^{(1)}\,.$$

Integrating this equation we obtain

$$\Gamma_{21}^{(1)} = A_1 R(x,t), \tag{3.12}$$

where

$$R(x,t) := \mathrm{e}^{2\mathrm{i}\int_{-\infty}^{x}\mathrm{d}y\,\rho_+(y,t)} \tag{3.13}$$

and A_1 is a bosonic function of t alone. We still must enforce eq. (3.10) with $n = 1$, which reads

$$\partial_t\Gamma_{21}^{(1)} = 2\mathrm{i}\rho_-\Gamma_{21}^{(1)}.$$

Using the explicit expression (3.12) for $\Gamma_{21}^{(1)}$ and the relation

$$\partial_t\rho_+ = \partial_x\rho_-, \tag{3.14}$$

which easily follows from the equations of motion, we obtain (taking into account that A_1 is even and denoting the derivative with respect to t by a dot)

$$R(x,t)^{-1}\partial_t\Gamma_{(21)}^{(1)} = 2\mathrm{i}\rho_-A_1 = \dot A_1 + 2\mathrm{i}A_1\int_{-\infty}^{x}\mathrm{d}y\,\partial_t\rho_+(y,t) = \dot A_1 + 2\mathrm{i}A_1\rho_-.$$

Thus A_1 is a constant. This determines $\Gamma_{21}^{(1)}$, which yields $\Gamma_{31}^{(2)}$ through eq. (3.11).

In general, assume that $\Gamma_{21}^{(k)}$ (with $k \leqslant n-1$) and $\Gamma_{31}^{(k)}$ (with $k \leqslant n$) have been computed. Combining eqs. (3.8) (with $n+1$ instead of n) and (3.9)–(3.10) we easily obtain a linear inhomogeneous system of the form

$$\partial_x\Gamma_{21}^{(n)} = 2\mathrm{i}\rho_+\Gamma_{21}^{(n)} + B_n, \qquad \partial_t\Gamma_{21}^{(n)} = 2\mathrm{i}\rho_-\Gamma_{21}^{(n)} + C_n, \tag{3.15}$$

where B_n and C_n depend on the *known* functions $\Gamma_{21}^{(1)}, \dots, \Gamma_{21}^{(n-1)}$ and $\Gamma_{31}^{(1)}, \dots, \Gamma_{31}^{(n)}$ by the induction hypothesis. Note that these equations are automatically compatible, due to the compatibility of eqs. (3.3)–(3.4). From eq. (3.14) and the compatibility condition for the system (3.15) we arrive at the identity

$$(\partial_x - 2\mathrm{i}\rho_+)C_n = (\partial_t - 2\mathrm{i}\rho_-)B_n. \tag{3.16}$$

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Integrating the first equation in (3.15) with respect to x we easily obtain:

$$\Gamma_{21}^{(n)} = R(x,t)\left(A_n(t) + \int_{-\infty}^{x} \mathrm{d}y\, R(y,t)^{-1} B_n(y,t)\right). \tag{3.17}$$

The (bosonic) function $A_n(t)$ is then determined (up to an arbitrary constant) substituting this expression into the second eq. (3.15), which yields the differential equation

$$\dot{A}_n(t) = R^{-1} C_n - \int_{-\infty}^{x} \mathrm{d}y\, R^{-1}(y,t)\Big(\partial_t - 2\mathrm{i}\rho_-(y,t)\Big) B_n(y,t). \tag{3.18}$$

Note that the right-hand side (r.h.s.) of this equation is indeed a function of t alone, since its derivative with respect to x,

$$R^{-1}\Big[(-2\mathrm{i}\rho_+ C_n + \partial_x C_n - (\partial_t - 2\mathrm{i}\rho_-) B_n\Big],$$

vanishes on account of eq. (3.16). In this way we can determine $\Gamma_{21}^{(n)}$, which in turn yields $\Gamma_{31}^{(n+1)}$ through eq. (3.8) with n replaced by $n+1$.

By eq. (3.2) with $j=1$, the generating function for the conserved quantities constructed from Γ_{31} and Γ_{21} reads

$$I_1 = \int \mathrm{d}x \Big(U_{11} + U_{13}\Gamma_{31}\Big) = \mathrm{i} \int \mathrm{d}x \Big[\rho_- + \Big(\lambda\phi_2^* - \lambda^{-1}\phi_1^*\Big)\Gamma_{31}\Big], \tag{3.19}$$

where we have discarded the trivial constant term in U_{11}. Substituting the expansion (3.7) of Γ_{31} into the above equation we obtain a corresponding expansion

$$I_1 = \mathrm{i} \sum_{n=0}^{\infty} \lambda^{-n} I_1^{(n)},$$

where each coefficient $I_1^{(n)}$ is a conserved quantity. The general expression for these conserved quantities is given by

$$I_1^{(n)} = \int \mathrm{d}x \Big(\rho_-\delta_{n0} + \phi_2^*\Gamma_{31}^{(n+1)} - \phi_1^*\Gamma_{31}^{(n-1)}\Big), \qquad n = 0, 1, \ldots, . \tag{3.20}$$

The explicit form of the conserved quantities (3.20) can be obtained recursively from eq. (3.8). The first two nontrivial conserved quantities turn out to be local, and are given by

$$I_1^{(0)} = -\int \mathrm{d}x\, (\phi_1^*\psi_1 + \phi_2^*\psi_2), \tag{3.21}$$

$$I_1^{(2)} = 2\int \mathrm{d}x\, \Big(2\mathrm{i}\, \phi_2^*\psi_{2,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\phi_1^*\psi_1\phi_2^*\psi_2\Big), \tag{3.22}$$

while $I_1^{(1)}$ vanishes identically. On the other hand, the conserved quantities $I_1^{(n)}$ with $n \geqslant 3$ are nonlocal. For instance, from eq. (3.20) with $n=3$ and the expressions for Γ_{31}^2, $\Gamma_{31}^{(4)}$ we readily obtain

$$I_1^{(3)} = 4\int \mathrm{d}x\, \phi_2^*\Big(\phi_1^* - \mathrm{i}\phi_{2,x}^*\Big)\Gamma_{21}^{(1)} = 4A_1 \int \mathrm{d}x\, \phi_2^*\Big(\phi_1^* - \mathrm{i}\phi_{2,x}^*\Big) R(x,t), \tag{3.23}$$

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where A_1 is a bosonic constant and $R(x,t)$ is defined by eq. (3.13) (cf. eq. (3.12)). Proceeding in the same way, a long but straightforward calculation yields

$$I_1^{(4)} = -2\int \mathrm{d}x\Big(2\mathrm{i}(\phi_1^*\psi_{2,x}+\phi_2^*\psi_{1,x}) - 4\phi_2^*\psi_{2,xx} + 4\mathrm{i}\,\phi_1^*\phi_2^*(\psi_2\psi_{1,x}-2\psi_1\psi_{2,x}) + 4\mathrm{i}\phi_2^*\psi_1\psi_2\phi_{1,x}^* + \rho_+ + 2\phi_2^*(\mathrm{i}\phi_{2,x}^*-\phi_1^*)\Gamma_{21}^{(2)}\Big). \tag{3.24}$$

The bosonic function $\Gamma_{21}^{(2)}$ is obtained integrating eqs. (3.9)–(3.10) with $n=2$, which in this case can be seen to reduce to the system (3.15) with

$$B_2 = 4\psi_2(\psi_{2,x}-\mathrm{i}\psi_1), \qquad C_2 = 4\psi_2\psi_{2,x}.$$

From eq. (3.17) with $n=2$ we then obtain

$$\Gamma_{21}^{(2)} = R(x)\Big[A_2(t) + 4\int_{-\infty}^{x}\mathrm{d}y\, R(y)^{-1}\psi_2(y)\Big(\psi_{2,x}(y)-\mathrm{i}\,\psi_1(y)\Big)\Big], \tag{3.25}$$

where for the sake of simplicity, we have dropped the t dependence of the fields and the function R. The bosonic function $A_2(t)$ is determined (up to a constant) by eq. (3.18) with $n=2$, namely

$$\dot{A}_2(t) = 4R^{-1}\psi_2\psi_{2,x} - 4\int_{-\infty}^{x}\mathrm{d}y\, R^{-1}(y)\Big(\partial_t - 2\mathrm{i}\rho_-(y)\Big)\Big(\psi_2(y)\psi_{2,x}(y)+\mathrm{i}\,\psi_1(y)\psi_2(y)\Big). \tag{3.26}$$

As explained above, it is guaranteed that the r.h.s. of the latter equation is independent of x. This can also be explicitly checked by noting that the x derivative of the r.h.s. of eq. (3.26) multiplied by $R(x,t)/4$ is explicitly given by

$$\begin{aligned} &-2\mathrm{i}\,\rho_+\psi_2\psi_{2,x} + \psi_2\psi_{2,xx} - (\partial_t - 2\mathrm{i}\,\rho_-)(\psi_2\psi_{2,x}+\mathrm{i}\psi_1\psi_2) \\ &\qquad = 4\mathrm{i}\,\phi_1^*\psi_1\psi_2\psi_{2,x} + \psi_2\psi_{2,xx} - \partial_t(\psi_2\psi_{2,x}+\mathrm{i}\psi_1\psi_2), \end{aligned}$$

which is readily seen to vanish using the equations of motion of the fields.

By the remark at the beginning of this section, the conserved quantities $I_2^{(n)}$ constructed from Γ_{i2} with $i \neq 2$ can be derived from the ones obtained above through the mapping (3.5). In this way we obtain the explicit formulas

$$I_2^{(0)} = \int \mathrm{d}x\,(\phi_1^*\psi_1 + \phi_2^*\psi_2), \tag{3.27}$$

$$I_2^{(2)} = 2\int \mathrm{d}x\,\Big(2\mathrm{i}\,\psi_2\phi_{2,x}^* + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\phi_1^*\psi_1\phi_2^*\psi_2\Big), \tag{3.28}$$

$$I_2^{(3)} = -4A_1\int \mathrm{d}x\,\psi_2\Big(\psi_1 + \mathrm{i}\psi_{2,x}\Big)R(x,t)^{-1}, \tag{3.29}$$

$$I_2^{(4)} = -2\int \mathrm{d}x\Big(2\mathrm{i}(-\psi_1\phi_{2,x}^* - \psi_2\phi_{1,x}^*) - 4\psi_2\phi_{2,xx}^* + 4\mathrm{i}\,\psi_1\psi_2(\phi_2^*\phi_{1,x}^* - 2\phi_1^*\phi_{2,x}^*) + 4\mathrm{i}\phi_1^*\phi_2^*\psi_2\psi_{1,x} - \rho_+ + 2\psi_2(\mathrm{i}\psi_{2,x}+\psi_1)\Gamma_{12}^{(2)}\Big), \tag{3.30}$$

where A_1 is again an arbitrary bosonic constant,

$$\Gamma_{12}^{(2)} = R(x)^{-1}\Big[A_2(t) + 4\int_{-\infty}^{x}\mathrm{d}y\, R(y)\phi_2^*(y)\Big(\phi_{2,x}^*(y)+\mathrm{i}\,\phi_1^*(y)\Big)\Big], \tag{3.31}$$

and $A_2(t)$ is a bosonic function determined (up to a constant) by the equation

$$\dot{A}_2(t) = 4R\phi_2^*\phi_{2,x}^* - 4\int_{-\infty}^{x}\mathrm{d}y\, R(y)\Big(\partial_t + 2\mathrm{i}\rho_-(y)\Big)\Big(\phi_2^*(y)\phi_{2,x}^*(y) - \mathrm{i}\,\phi_1^*(y)\phi_2^*(y)\Big). \tag{3.32}$$

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3.2 Expansion in positive powers of λ

A second hierarchy of conserved quantities can be obtained by expanding Γ_{i1} with $i \neq 1$ in positive powers of λ as follows:

$$\Gamma_{i1}(x,t;\lambda) = \sum_{k=1}^{\infty} \widetilde{\Gamma}_{i1}^{(k)}(x,t)\lambda^k, \qquad 1 = 2,3. \tag{3.33}$$

Using the procedure described above, the expansion coefficients $\widetilde{\Gamma}_{i1}^{(k)}$ can easily be obtained in a recursive way from the Riccati equations (3.6), which yield system

$$\begin{aligned}\widetilde{\Gamma}_{31}^{(n)} &= 2(\psi_2\delta_{n3} - \psi_1\delta_{n1}) + \widetilde{\Gamma}_{31}^{(n-4)} + 2(\mathrm{i}\partial_x - \rho_-)\widetilde{\Gamma}_{31}^{(n-2)} \\ &\quad - 2\Big(\phi_2^*\widetilde{\Gamma}_{21}^{(n-3)} + \phi_1^*\widetilde{\Gamma}_{21}^{(n-1)}\Big),\end{aligned} \tag{3.34}$$

$$\begin{aligned}\mathrm{i}\,\partial_x\widetilde{\Gamma}_{21}^{(n)} &= 2\rho_-\widetilde{\Gamma}_{21}^{(n)} + \psi_2\widetilde{\Gamma}_{31}^{(n-1)} + \psi_1\widetilde{\Gamma}_{31}^{(n+1)} \\ &\quad + \phi_2^*\sum_{k=1}^{n-2}\widetilde{\Gamma}_{21}^{(n-1-k)}\widetilde{\Gamma}_{31}^{(k)} - \phi_1^*\sum_{k=1}^{n}\widetilde{\Gamma}_{21}^{(n+1-k)}\widetilde{\Gamma}_{31}^{(k)},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\mathrm{i}\,\partial_t\widetilde{\Gamma}_{21}^{(n)} &= 2\rho_+\widetilde{\Gamma}_{21}^{(n)} + \psi_2\widetilde{\Gamma}_{31}^{(n-1)} - \psi_1\widetilde{\Gamma}_{31}^{(n+1)} \\ &\quad + \phi_2^*\sum_{k=1}^{n-2}\widetilde{\Gamma}_{21}^{(n-1-k)}\widetilde{\Gamma}_{31}^{(k)} + \phi_1^*\sum_{k=1}^{n}\widetilde{\Gamma}_{21}^{(n+1-k)}\widetilde{\Gamma}_{31}^{(k)},\end{aligned} \tag{3.36}$$

where $n \geqslant 1$ and $\widetilde{\Gamma}_{31}^{(n)} = \widetilde{\Gamma}_{21}^{(n)} = 0$ for $n \leqslant 0$. Substituting the expansion (3.33) of $\widetilde{\Gamma}_{31}$ into eq. (3.19) we obtain an expansion of the form

$$I_1 = \mathrm{i}\sum_{n=0}^{\infty}\lambda^n \widetilde{I}_1^{(n)},$$

whose coefficients

$$\widetilde{I}_1^{(n)} = \int \mathrm{d}x\Big(\rho_-\delta_{n0} + \phi_2^*\widetilde{\Gamma}_{31}^{(n-1)} - \phi_1^*\widetilde{\Gamma}_{31}^{(n+1)}\Big), \qquad n = 0,1,\dots, \tag{3.37}$$

are conserved quantities. The explicit form of the first few nontrivial conserved quantities is as follows:

$$\begin{aligned}\widetilde{I}_1^{(0)} &= \int \mathrm{d}x\,(\phi_1^*\psi_1 + \phi_2^*\psi_2), \\ \widetilde{I}_1^{(2)} &= 2\int \mathrm{d}x\,\Big(2\mathrm{i}\phi_1^*\psi_{1x} - \phi_1^*\psi_2 - \phi_2^*\psi_1 - 2\phi_1^*\psi_1\phi_2^*\psi_2\Big), \\ \widetilde{I}_1^{(3)} &= 4\widetilde{A}_1\int \mathrm{d}x\,\phi_1^*(\phi_2^* + \mathrm{i}\phi_{1x}^*)R(x,t)^{-1},\end{aligned} \tag{3.38}$$

where $\widetilde{A}_1$ is a bosonic constant. As was the case with the conserved quantities $I_1^{(n)}$ discussed above, all the conserved quantities $\widetilde{I}_1^{(n)}$ with $n \geqslant 3$ are nonlocal. The corresponding conserved quantities constructed from Γ_{i2} with $i \neq 2$ are obtained from the above through the mapping (3.5), namely

$$\begin{aligned}\widetilde{I}_2^{(0)} &= -\int \mathrm{d}x\,(\phi_1^*\psi_1 + \phi_2^*\psi_2), \\ \widetilde{I}_2^{(2)} &= 2\int \mathrm{d}x\,\Big(2\mathrm{i}\psi_1\phi_{1x}^* + \phi_1^*\psi_2 + \phi_2^*\psi_1 - 2\phi_1^*\psi_1\phi_2^*\psi_2\Big), \\ \widetilde{I}_2^{(3)} &= 4\widetilde{A}_1\int \mathrm{d}x\,\psi_1(\psi_2 + \mathrm{i}\psi_{1x})R(x,t),\end{aligned} \tag{3.39}$$

where again $\widetilde{A}_1$ is an arbitrary bosonic constant.

Remark 2. As is also the case with the ordinary GMTM [31], the conserved quantities $I_j^{(k)}$, $\widetilde{I}_j^{(k)}$ with $k \leqslant 2$ are local, while the ones with $k \geqslant 3$ are nonlocal. Note, however, that all the conserved quantities of the coupled BMTM introduced in ref. [33] are local.

Remark 3. There are some obvious relations among the lowest-order conserved quantities derived above, namely

$$I_1^{(0)} = \widetilde{I}_2^{(0)} = -\widetilde{I}_1^{(0)} = -I_2^{(0)}, \qquad I_1^{(2)} = I_2^{(2)}, \qquad \widetilde{I}_1^{(2)} = \widetilde{I}_2^{(2)},$$

where for the last two equalities we have taken into account that the integral of a total x derivative vanishes due to the boundary conditions imposed on the fields. Note, however, that no such relations are apparent for the conserved quantities of order greater than 2.

Remark 4. Setting

$$c = \frac{\Gamma_{31}}{\mathrm{i}\lambda}, \qquad b = \Gamma_{21},$$

the Riccati system (3.6) can be rewritten as follows:

$$\begin{aligned}
\partial_x c &= \psi_2 - \lambda^{-2}\psi_1 - (\phi_2^* + \lambda^2\phi_1^*)bc,\\
\partial_x b &= -2\mathrm{i}\rho_- b + (\lambda^2\psi_2 + \psi_1)c + (\lambda^2\phi_2^* - \phi_1^*)bc,\\
\partial_t b &= -2\mathrm{i}\rho_+ b + (\lambda^2\psi_2 - \psi_1)c + (\lambda^2\phi_2^* + \phi_1^*)bc.
\end{aligned}$$

Since the r.h.s. of the latter equations depends on λ only through λ^2, it is clear that it admits solutions in which c and b are functions of λ^2. Hence the original equations (3.6) also admit solutions in which Γ_{31} is odd in λ and Γ_{21} is even in λ. By eqs. (3.20)–(3.37), the conserved quantities $I_1^{(n)}$ and $\widetilde{I}_1^{(n)}$ constructed from these solutions vanish identically when n is odd. Applying the symmetry transformation (3.5) we deduce that the same is true for the functions Γ_{i2} with $i \neq 2$ and their corresponding conserved quantities $I_2^{(n)}$, $\widetilde{I}_2^{(n)}$ of odd order.

4 Local conserved quantities

In this section we shall regard the auxiliary fields $w_{1,2}$ as fermionic and w_3 as bosonic, and construct the conserved quantities derived from the quotients Γ_{i3} with $i \neq 3$. According to the general equation (3.2), the generating function for these quantities is

$$I_3 = \int \mathrm{d}x\Big(U_{31}\Gamma_{13} + U_{32}\Gamma_{23}\Big) = \mathrm{i}\int \mathrm{d}x\Big[(\lambda\psi_2 - \lambda^{-1}\psi_1)\Gamma_{13} - (\lambda\phi_2^* + \lambda^{-1}\phi_1^*)\Gamma_{23}\Big], \tag{4.1}$$

where we have again dropped the trivial constant term U_{33}. The differential equations for the functions Γ_{i3} (with $i \neq 3$) read

$$\begin{aligned}
\partial_x\Gamma_{13} &= -r_1^- + \mathrm{i}\Big[\rho_- - \frac{1}{2}(\lambda^2 - \lambda^{-2})\Big]\Gamma_{13} + r_1^+\Gamma_{13}\Gamma_{23},\\
\partial_t\Gamma_{13} &= -r_1^+ + \mathrm{i}\Big[\rho_+ - \frac{1}{2}(\lambda^2 + \lambda^{-2})\Big]\Gamma_{13} + r_1^-\Gamma_{13}\Gamma_{23},\\
\partial_x\Gamma_{23} &= -r_2^+ - \mathrm{i}\Big[\rho_- + \frac{1}{2}(\lambda^2 - \lambda^{-2})\Big]\Gamma_{23} - r_2^-\Gamma_{13}\Gamma_{23},\\
\partial_t\Gamma_{23} &= -r_2^- - \mathrm{i}\Big[\rho_+ + \frac{1}{2}(\lambda^2 + \lambda^{-2})\Big]\Gamma_{23} - r_2^+\Gamma_{13}\Gamma_{23}.
\end{aligned}$$

The structure of these equations makes it convenient to introduce light-cone coordinates

$$\xi = \frac{1}{2}(t+x), \qquad \eta = \frac{1}{2}(t-x),$$

so that

$$\partial_\xi = \partial_t + \partial_x, \qquad \partial_\eta = \partial_t - \partial_x \,.$$

Setting

$$\frac{\Gamma_{13}}{2\lambda} = c\,, \qquad \frac{\Gamma_{23}}{2\lambda} = b$$

we obtain the system

$$\partial_\xi c = \mathrm{i}\phi_2^* + (2\mathrm{i}\rho_2 - \mu)c + 4\mu\phi_2^* bc, \tag{4.2}$$

$$\partial_\xi b = -\mathrm{i}\psi_2 - (2\mathrm{i}\rho_2 + \mu)b + 4\mu\psi_2 bc, \tag{4.3}$$

$$\partial_\eta c = -\mu^{-1}\phi_1^* + (2\mathrm{i}\rho_1 + \mu^{-1})c - 4\mathrm{i}\phi_1^* bc, \tag{4.4}$$

$$\partial_\eta b = -\mu^{-1}\psi_1 - (2\mathrm{i}\rho_1 - \mu^{-1})b + 4\mathrm{i}\psi_1 bc, \tag{4.5}$$

where $\mu = \mathrm{i}\lambda^2$ is a new (complex) spectral parameter and

$$\rho_i = \phi_i^* \psi_i\,, \qquad i = 1, 2\,.$$

Using the field equations (2.7), it can be readily checked that the compatibility conditions for the previous system are automatically verified. The main difference with the procedure followed in the previous section is that, as we shall show below, when the functions c and b are expanded in powers of the spectral parameter μ either eqs. (4.2)–(4.3) (when expanding in negative powers) or (4.4)–(4.5) (when expanding in positive powers) can be used to generate a pure recursion relation for the expansion coefficients of the latter functions. In this way these coefficients can be recursively determined in terms of the fields without having to perform any integrations.

Remark 5. Equations (4.2)–(4.4) are easily seen to be invariant under the transformation

$$c \leftrightarrow b^*, \qquad \phi_i \leftrightarrow \psi_i.$$

Since our model reduces to two identical copies of the original Thirring mode when $\phi = \psi$, it follows that (with suitable boundary conditions) one can take $b = c^*$ in the latter model. This simplification, actually used in ref. [9], is of course impossible in our case.

4.1 Negative powers of μ

Let us start by expanding the (fermionic) functions c and b in negative powers of the spectral parameter μ, namely

$$c = \sum_{n\geqslant 1} c_n \mu^{-n}, \qquad b = \sum_{n\geqslant 1} b_n \mu^{-n}.$$

Substituting into eqs. (4.2)–(4.3) we immediately obtain the coupled recursion relations

$$c_{n+1} = \mathrm{i}\phi_2^*\delta_{n0} + (2\mathrm{i}\rho_2 - \partial_\xi)c_n + 4\phi_2^* \sum_{k=1}^{n} b_{n+1-k}c_k, \tag{4.6}$$

$$b_{n+1} = -\mathrm{i}\psi_2\delta_{n0} - (2\mathrm{i}\rho_2 + \partial_\xi)b_n + 4\psi_2 \sum_{k=1}^{n} b_{n+1-k}c_k \tag{4.7}$$

with $n = 0, 1, \ldots$. Since the r.h.s. of these equations contain only functions c_k and b_k with $1 \leqslant k \leqslant n$, it is obvious that they allow the recursive computation of c_n and b_n for all $n \geqslant 1$. Once this is done, the corresponding conserved quantities $I_3^{(n)}$ are obtained expanding the generating function (4.1), which can be written as

$$I_3 = 2\int \mathrm{d}x \left[(\mu\psi_2 - \mathrm{i}\psi_1)c - (\mu\phi_2^* + \mathrm{i}\phi_1^*)b\right] \tag{4.8}$$

in powers of μ. Setting

$$I_3 = 2\sum_{n\geqslant 0} \mu^{-n} I_3^{(n)}$$

we thus obtain the explicit expression

$$I_3^{(n)} = \int \mathrm{d}x \left[\psi_2 c_{n+1} - \phi_2^* b_{n+1} - \mathrm{i}(\psi_1 c_n + \phi_1^* b_n)\right], \qquad n \geqslant 0\,. \tag{4.9}$$

From the latter expression it is obvious that these conserved quantities are all local. The first of these quantities, $I_3^{(0)}$, is easily seen to vanish, since from eqs. (4.6)–(4.7) we have

$$c_1 = \mathrm{i}\phi_2^*, \qquad b_1 = -\mathrm{i}\psi_2\,.$$

The first nontrivial conserved density is easily obtained setting $n = 1$ in the recursion relations (4.6)–(4.7), which yields

$$c_2 = (2\mathrm{i}\rho_2 - \partial_\xi)c_1 + 4\phi_2^* b_1 c_1 = -\mathrm{i}(2\mathrm{i}\rho_2 + \partial_\xi)\phi_2^* = -\mathrm{i}\phi_{2,\xi}^*.$$

Although b_2 can be computed in a similar way from eq. (4.7) with $n = 1$, it is easier to note that $c = \Gamma_{13}/(2\lambda)$ is mapped into $b = \Gamma_{23}/(2\lambda)$ under the symmetry transformation (3.5). In this way we obtain

$$b_2 = \mathrm{i}\psi_{2,\xi},$$

and from eq. (4.9) with $n = 1$ we have

$$I_3^{(1)} = -\int \mathrm{d}x \Big[\mathrm{i}(\psi_2\phi_{2,\xi}^* + \phi_2^*\psi_{2,\xi}) + \phi_1^*\psi_2 + \phi_2^*\psi_1\Big].$$

Using the field equations (2.7) we can express the ξ derivatives of ψ_2 and ϕ_2^* in terms of their x derivatives as follows:

$$\psi_{2,\xi} = 2\psi_{2,x} - \mathrm{i}\psi_1 - 2\mathrm{i}\rho_1\psi_2, \qquad \phi_{2,\xi}^* = 2\phi_{2,x}^* + \mathrm{i}\phi_1^* + 2\mathrm{i}\rho_1\phi_2^*.$$

Substituting into the previous expression for $I_3^{(1)}$ we finally obtain the explicit formula

$$I_3^{(1)} = -2\int \mathrm{d}x \Big[\mathrm{i}\Big(\phi_2^*\psi_{2,x} - \phi_{2,x}^*\psi_2\Big) + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\rho_1\rho_2\Big]. \tag{4.10}$$

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Integrating by parts we obtain the equivalent expression

$$I_3^{(1)} = -2\int \mathrm{d}x\Big(2\mathrm{i}\phi_2^*\psi_{2,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\rho_1\rho_2\Big). \tag{4.11}$$

Similarly, from eqs. (4.6)–(4.7) with $n = 2$ we obtain

$$c_3 = \mathrm{i}\phi^*_{2,\xi\xi} - 2\rho_2\phi^*_{2,\xi}, \qquad b_3 = -\mathrm{i}\psi_{2,\xi\xi} - 2\rho_2\psi_{2,\xi}.$$

However, the corresponding conserved quantity $I_3^{(2)}$ is trivial, since using the field equations it can be shown that

$$I_3^{(2)} = 2\int \mathrm{d}x\,\partial_x\Big(4\mathrm{i}\phi_2^*\psi_{2,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 4\rho_1\rho_2\Big) = 0$$

is the integral of a total x derivative, which vanishes on account of the boundary conditions at infinity.

The calculation of the conserved quantities of order higher than 2 becomes increasingly more involved. For example, for $n = 3$ we have

$$c_4 = -\mathrm{i}\phi^*_{2,\xi\xi\xi} + 4\rho_2\phi^*_{2,\xi\xi} - 6\phi_2^*\phi^*_{2,\xi}\psi_{2,\xi}, \qquad b_4 = \mathrm{i}\psi_{2,\xi\xi\xi} + 4\rho_2\psi_{2,\xi\xi} + 6\psi_2\psi_{2,\xi}\phi^*_{2,\xi},$$

and hence

$$\begin{aligned} I_3^{(3)} = \int \mathrm{d}x\,\Big(& -\mathrm{i}\phi_2^*\psi_{2,\xi\xi\xi} - \mathrm{i}\psi_2\phi^*_{2,\xi\xi\xi} - \phi_1^*\psi_{2,\xi\xi} + \psi_1\phi^*_{2,\xi\xi} \\ & + 2\rho_2(\mathrm{i}\phi_1^*\psi_{2,\xi} + \mathrm{i}\psi_1\phi^*_{2,\xi} + 6\phi^*_{2,\xi}\psi_{2,\xi})\Big), \end{aligned}$$

where the ξ derivatives must be expressed in terms of x derivatives using the field equations (2.7). When this is done we obtain the explicit expression

$$\begin{aligned} I_3^{(3)} = 2\int \mathrm{d}x\Big[& 4\phi^*_{1,x}\psi_{2,x} + 4\phi^*_{2,x}\psi_{1,x} - 8\phi_1^*\phi_2^*\psi_{1,x}\psi_{2,x} + 24(\rho_1+\rho_2)\phi^*_{2,x}\psi_{2,x} \\ & -8\phi_1^*\psi_2\phi^*_{2,x}\psi_{1,x} - 8\phi_2^*\psi_1\phi^*_{1,x}\psi_{2,x} - 8\psi_1\psi_2\phi^*_{1,x}\phi^*_{2,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 18\rho_1\rho_2 \\ & +2\mathrm{i}\Big(4\phi^*_{2,x}\psi_{2,xx} + \phi_1^*\psi_{1,x} + 2\phi_2^*\psi_{2,x} + 2\rho_1(\psi_2\phi^*_{1,x} + \phi_2^*\psi_{1,x}) + 10\rho_2(\phi_1^*\psi_{2,x} + \psi_1\phi^*_{2,x})\Big)\Big], \end{aligned}$$

where we have discarded a total derivative with respect to x in the integrand. We have verified with *Mathematica*™ that the even-order conserved quantities $I_3^{(4)}$ and $I_3^{(6)}$ are trivial, as their densities are a total x derivative. In fact, we conjecture that this is the case for all conserved quantities of even order.

4.2 Positive powers of μ

We shall next expand the fermionic functions c and b in positive powers of the spectral parameter μ, namely

$$c = \sum_{n\geqslant 0}\mu^n\widetilde{c}_n, \qquad b = \sum_{n\geqslant 0}\mu^n\widetilde{b}_n,$$

where the inclusion of the term independent of μ is justified by the structure of eqs. (4.4)–(4.5). Substituting this expansion into the latter equations we obtain the coupled recursion relation

$$\widetilde{c}_{n+1} = \phi_1^* \delta_{n,-1} + (\partial_\eta - 2\mathrm{i}\rho_1)\widetilde{c}_n + 4\mathrm{i}\phi_1^* \sum_{k=0}^{n} \widetilde{b}_{n-k}\widetilde{c}_k, \tag{4.12}$$

$$\widetilde{b}_{n+1} = \psi_1 \delta_{n,-1} + (\partial_\eta + 2\mathrm{i}\rho_1)\widetilde{b}_n - 4\mathrm{i}\psi_1 \sum_{k=0}^{n} \widetilde{b}_{n-k}\widetilde{c}_k. \tag{4.13}$$

Likewise, from the expansion

$$I_3 = 2\sum_{n\geqslant 0} \mu^n \widetilde{I}_3^{(n)}$$

and eq. (4.8) it follows that

$$\widetilde{I}_3^{(n)} = \int \mathrm{d}x \Big[\psi_2 c_{n-1} - \phi_2^* b_{n-1} - \mathrm{i}(\psi_1 c_n + \phi_1^* b_n)\Big], \qquad n \geqslant 0\,. \tag{4.14}$$

The calculation of the expansion coefficients $\widetilde{c}_n$, $\widetilde{c}_n$ from the recursion relations (4.12)–(4.13) and the corresponding first integrals $\widetilde{I}_3^{(n)}$ from (4.14) proceeds along the same line as in the previous subsection. The first nontrivial conserved quantities are

$$\begin{aligned}
\widetilde{I}_3^{(1)} &= -2\int \mathrm{d}x \left(-\mathrm{i}\phi_1^*\psi_{1,x} - \mathrm{i}\psi_1\phi_{1,x}^* + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\rho_1\rho_2\right) \\
&= -2\int \mathrm{d}x \left(- 2\mathrm{i}\phi_1^*\psi_{1,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\rho_1\rho_2\right),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\widetilde{I}_3^{(3)} = 2\int \mathrm{d}x \Big[&8\rho_2(\psi_1\phi_{1,xx}^* - \phi_1^*\psi_{1,xx} + \phi_{1,x}^*\psi_{1,x}) + 24\rho_1\phi_{1,x}^*\psi_{1,x} + 4\phi_{1,x}^*\psi_{2,x} + 4\phi_{2,x}^*\psi_{1,x} \\
&+ \phi_1^*\psi_2 + \phi_2^*\psi_1 + 18\rho_1\rho_2 - 2\mathrm{i}\Big(4\phi_{1,x}^*\psi_{1,xx} + 2\psi_1\phi_{1,x}^* + \psi_2\phi_{2,x}^* \\
&+ 10\rho_1(\phi_2^*\psi_{1,x} + \psi_2\phi_{1,x}^*) + 2\rho_2(\phi_1^*\psi_{2,x} + \psi_1\phi_{2,x}^*)\Big)\Big],
\end{aligned} \tag{4.16}$$

where in the last equation we have discarded total x derivatives in the integrand to simplify the expression for the corresponding conserved quantity. As before, the conserved quantity $\widetilde{I}_3^{(0)}$ is identically zero, while $\widetilde{I}_3^{(2)}$ is easily seen to vanish on account of the boundary conditions at spatial infinity:

$$I_3^{(2)} = 2\int \mathrm{d}x\, \partial_x\Big(-4\mathrm{i}\phi_1^*\psi_{1,x} + \phi_1^*\psi_2 + \phi_2^*\psi_1 + 4\rho_1\rho_2\Big) = 0.$$

We again conjecture that all the even-order integrals $\widetilde{I}_3^{(2n)}$ with $n = 0, 1, \ldots$ are trivial; in fact, we have verified this conjecture for the additional cases $n = 2, 3$ with the help of *Mathematica*™.

Remark 6. The conserved quantities of order 1 are obviously related to the local conserved quantities obtained in the previous section, namely

$$I_3^{(1)} = -I_{1,2}^{(2)}, \qquad \widetilde{I}_3^{(1)} = \widetilde{I}_{1,2}^{(2)}.$$

Note, however, that the zeroth order local conserved quantity $\int \mathrm{d}x\, \rho_+$ does not appear among the conserved quantities obtained in this section.

Remark 7. None of the conserved quantities computed above are real. Note, however, that if I is a conserved quantity so is its complex conjugate I^*. It follows that the complex conjugates of the quantities $I_i^{(n)}$, $\widetilde{I}_i^{(n)}$ (with $i = 1, 2, 3$) are also conserved, and can be used to obtain the real conserved quantities

$$\operatorname{Re} I_i^{(n)} = \frac{1}{2}\Big(I_i^{(n)} + (I_i^{(n)})^*\Big), \qquad \operatorname{Im} I_i^{(n)} = \frac{1}{2\mathrm{i}}\Big(I_i^{(n)} - (I_i^{(n)})^*\Big),$$

and similarly for $\widetilde{I}_i^{(n)}$. For instance, from the complex conserved quantity $\int \mathrm{d}x\, \rho_+$ we obtained the real conserved quantities

$$\frac{1}{2}\int \mathrm{d}x\, \big(\phi_1^*\psi_1 + \phi_2^*\psi_2 + \psi_1^*\phi_1 + \psi_2^*\phi_2\big), \qquad \frac{1}{2\mathrm{i}}\int \mathrm{d}x\, \big(\phi_1^*\psi_1 + \phi_2^*\psi_2 - \psi_1^*\phi_1 - \psi_2^*\phi_2\big).$$

The first of these quantities reduces to the fermion number of the standard (Grassmannian) Thirring model if we set $\phi = \psi$, while the second one vanishes.

5 Canonical formulation

In this section we shall present the canonical (Hamiltonian) formulation of the coupled massive Thirring model defined by the field equations (2.7). To begin with, the Lagrangian generating these equations is given by eq. (2.1) with $m = -g = 1$, or more explicitly

$$\mathcal{L} = \frac{\mathrm{i}}{2}\Big[\phi_a^*\dot{\psi}_a + \psi_a^*\dot{\phi}_a - (-1)^a\Big(\phi_a^*\psi_{a,x} + \psi_a^*\phi_{a,x}\Big)\Big] - \phi_1^*\psi_2 - \phi_2^*\psi_1 - 2\phi_1^*\psi_1\phi_2^*\psi_2 + \text{c.c.},$$

where the dot denotes partial derivative with respect to the time t, c.c. stands for the complex conjugate, and (as in what follows) summation over repeated indices $a = 1, 2$ is understood. In fact, in order to derive the canonical formulation of the field equations it is more convenient to work with the equivalent (complex) Lagrangian

$$\mathcal{L} = \mathrm{i}\Big[\phi_a^*\dot{\psi}_a + \psi_a^*\dot{\phi}_a - (-1)^a\Big(\phi_a^*\psi_{a,x} + \psi_a^*\phi_{a,x}\Big)\Big] - \Big(\phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\phi_1^*\psi_1\phi_2^*\psi_2 + \text{c.c}\Big), \tag{5.1}$$

which differs from the previous one by a trivial spacetime divergence. We shall next construct the model's Hamiltonian following the canonical formalism for Grassmann-valued field theories outlined in refs. [35, 36]. We start by defining the canonical momenta associated to the field variables (χ, χ^*), with

$$\chi := (\phi, \psi) = (\chi_\alpha)_{1 \leqslant \alpha \leqslant 4},$$

as

$$\pi_{\chi_\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_\alpha}, \qquad \pi_{\chi_\alpha^*} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_\alpha^*},$$

where the partial derivatives, as in the sequel, are *left* derivatives. With this convention the Hamiltonian is defined as

$$\mathcal{H} = \dot{\chi}_\alpha \pi_{\chi_\alpha} + \dot{\chi}_\alpha^* \pi_{\chi_\alpha^*} - \mathcal{L}, \tag{5.2}$$

where summation over repeated indices $\alpha = 1, \ldots, 4$ is again understood. In our case we have

$$\pi_{\phi_a} = -\mathrm{i}\psi_a^*, \qquad \pi_{\psi_a} = -\mathrm{i}\phi_a^*, \qquad \pi_{\phi_a^*} = \pi_{\psi_a^*} = 0, \tag{5.3}$$

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and thus the Hamiltonian of the model is simply

$$\mathcal{H} = \dot{\chi}_\alpha \pi_{\chi_\alpha} - \mathcal{L} = \mathrm{i}(-1)^a \Big(\phi_a^* \psi_{a,x} + \psi_a^* \phi_{a,x}\Big) + \Big(\phi_1^* \psi_2 + \phi_2^* \psi_1 + 2\phi_1^* \psi_1 \phi_2^* \psi_2 + \text{c.c.}\Big). \tag{5.4}$$

At this point it is important to note that eqs. (5.3) cannot be used (as is usual with bosonic field theories) to express the generalized velocities $\dot{\chi}_\alpha, \dot{\chi}_\alpha^*$ in terms of the corresponding canonical momenta, but are rather *constraints* relating the canonical momenta to the field variables. Thus in this case the canonical formalism should be developed using Dirac's method for constrained systems [50] as presented, e.g., in ref. [51]. However, the vanishing of the canonical momenta associated to the conjugate field variables χ_α^* suggests that we can simply regard the fields χ_α and their conjugate momenta π_{χ_α} as the fundamental canonical variables, using the first two equations in (5.3) to relate the complex conjugate fields χ_α^* to the canonical momenta π_{χ_α} (see the appendix for a detailed justification of this statement). Taking into account our convention of using left derivatives, it is then easily checked that the field equations (2.7) adopt indeed the canonical form

$$\dot{\chi}_\alpha = -\frac{\delta \mathcal{H}}{\delta \pi_{\chi_\alpha}}, \qquad \dot{\pi}_{\chi_\alpha} = -\frac{\delta \mathcal{H}}{\delta \chi_\alpha}, \tag{5.5}$$

where $(\frac{\delta \mathcal{H}}{\delta \chi_\alpha}, \frac{\delta \mathcal{H}}{\delta \pi_{\chi_\alpha}})$ are variational derivatives defined by the relation[3]

$$\delta H = \int \mathrm{d}x \Big(\delta\chi_\alpha \frac{\delta \mathcal{H}}{\delta \chi_\alpha(x)} + \delta\pi_{\chi_\alpha} \frac{\delta \mathcal{H}}{\delta \pi_{\chi_\alpha}(x)} \Big), \qquad \text{with} \quad H := \int \mathrm{d}x \, \mathcal{H}.$$

Following refs. [35, 36], we define the Poisson bracket of two dynamical variables $F[\chi_\alpha, \pi_{\chi_\alpha}] = \int \mathrm{d}x \, \mathcal{F}$ and $G[\chi_\alpha, \pi_{\chi_\alpha}] = \int \mathrm{d}x \, \mathcal{G}$ by

$$\{F, G\} = (-1)^{|F|} \int \mathrm{d}x \left(\frac{\delta \mathcal{F}}{\delta \chi_\alpha(x)} \frac{\delta \mathcal{G}}{\delta \pi_{\chi_\alpha}(x)} + \frac{\delta \mathcal{F}}{\delta \pi_{\chi_\alpha}(x)} \frac{\delta \mathcal{G}}{\delta \chi_\alpha(x)} \right) =: \int \mathrm{d}x \, \{\mathcal{F}, \mathcal{G}\}, \tag{5.6}$$

where $|F|$ is the grading of F (i.e., $|F| = 0$ if F is even and $|F| = 1$ if F is odd). With this definition the fundamental Poisson brackets are given by[4]

$$\begin{aligned} \{\chi_\alpha(x), \chi_\beta(y)\} &= \{\pi_{\chi_\alpha}(x), \pi_{\chi_\beta}(y)\} = 0, \\ \{\chi_\alpha(x), \pi_{\chi_\beta}(y)\} &= \{\pi_{\chi_\alpha}(x), \chi_\beta(y)\} = -\delta_{\alpha\beta}\delta(x-y). \end{aligned} \tag{5.7}$$

(Note the minus sign, which is due to our choice of left derivatives.) Using eqs. (5.3) we obtain the fundamental Poisson brackets

$$\{\phi_a(x), \psi_b^*(y)\} = \{\psi_b^*(y), \phi_a(x)\} = \{\psi_a(x), \phi_b^*(y)\} = \{\phi_b^*(y), \psi_a(x)\} = -\mathrm{i}\,\delta_{ab}\delta(x-y). \tag{5.8}$$

In particular, the local first integrals computed in the previous sections are of the form

$$F = \int \mathrm{d}x \, \mathcal{F}(\chi_\alpha, \chi_\alpha^*, \partial_x \chi_\alpha, \partial_x \chi_\alpha^*, \dots, \partial_x^n \chi_\alpha, \partial_x^n \chi_\alpha^*), \tag{5.9}$$

[3] In this section we shall often drop the time dependence of the fields and the canonical momenta, writing for instance $\chi_\alpha(x)$ instead of $\chi_\alpha(x,t)$.

[4] Recall that the Poisson bracket between two odd dynamical functions is symmetric.

where as remarked above the complex conjugates of the fields are related to the canonical momenta by the first two equations in (5.3). Taking this into account, the Poisson bracket between two (even) dynamical variables F_1 and F_2 of the latter type is explicitly given by eq. (5.6) with

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \mathrm{i}\Big(\frac{\delta \mathcal{F}_1}{\delta \phi_a(x)}\frac{\delta \mathcal{F}_2}{\delta \psi_a^*(x)} + \frac{\delta \mathcal{F}_1}{\delta \psi_a(x)}\frac{\delta \mathcal{F}_2}{\delta \phi_a^*(x)} + \frac{\delta \mathcal{F}_1}{\delta \phi_a^*(x)}\frac{\delta \mathcal{F}_2}{\delta \psi_a(x)} + \frac{\delta \mathcal{F}_1}{\delta \psi_a^*(x)}\frac{\delta \mathcal{F}_2}{\delta \phi_a(x)}\Big), \tag{5.10}$$

where

$$\frac{\delta \mathcal{F}_i}{\delta \chi_a} := \frac{\partial \mathcal{F}_i}{\partial \chi_a} + \sum_{k=1}^{n} (-1)^k \partial_x^k \frac{\partial \mathcal{F}_i}{\partial(\partial_x^k \chi_a)}, \qquad i = 1, 2 \tag{5.11}$$

and similarly for $\frac{\delta \mathcal{F}_i}{\delta \chi_a^*}$.

The action $\int \mathrm{d}t\,\mathrm{d}x\,\mathcal{L}$ of the Lagrangian (5.1) is manifestly invariant under constant spacetime translations

$$x^\mu \mapsto x'^\mu = x^\mu + \varepsilon^\mu, \qquad \varepsilon^\mu \in \mathbb{R}, \quad \mu = 0, 1, \tag{5.12}$$

as well as under global U(1) gauge transformations

$$\chi \mapsto \mathrm{e}^{\mathrm{i}\varepsilon}\chi, \qquad \chi^* \mapsto \mathrm{e}^{-\mathrm{i}\varepsilon}\chi^*, \qquad \varepsilon \in \mathbb{R}, \tag{5.13}$$

and global SU(1,1) scaling transformation (2.25). Moreover, in view of eq. (2.1) the action is also invariant under Lorentz boosts

$$(x, t) \mapsto (x', t') = (x\cosh\varepsilon - t\sinh\varepsilon, -x\sinh\varepsilon + t\cosh\varepsilon), \qquad \varepsilon \in \mathbb{R}. \tag{5.14}$$

Indeed, in light-cone coordinates $\xi = (t + x)/2$, $\eta = (t - x)/2$ the Lorentz boost (5.14) becomes

$$(\xi, \eta) \mapsto (\xi', \eta') = (\mathrm{e}^{-\varepsilon}\xi, \mathrm{e}^{\varepsilon}\eta)$$

and the Lagrangian (5.1) reads

$$\begin{aligned}\mathcal{L} &= \mathrm{i}\Big(\phi_1^*\psi_{1,\xi} + \phi_2^*\psi_{2,\eta} + \psi_1^*\phi_{1,\xi} + \psi_2^*\phi_{2,\eta}\Big) - \Big(\phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\phi_1^*\psi_1\phi_2^*\psi_2 + \text{c.c.}\Big)\\ &= \mathrm{i}\mathrm{e}^{-\varepsilon}\Big(\phi_1^*\psi_{1,\xi'} + \psi_1^*\phi_{1,\xi'}\Big) + \mathrm{i}\mathrm{e}^{\varepsilon}\Big(\phi_2^*\psi_{2,\eta'} + \psi_2^*\phi_{2,\eta'}\Big) - \Big(\phi_1^*\psi_2 + \phi_2^*\psi_1 + 2\phi_1^*\psi_1\phi_2^*\psi_2 + \text{c.c.}\Big).\end{aligned}$$

Hence the action will be invariant under the Lorentz boost (5.14) provided that the fields transform as

$$\phi(x,t) \mapsto \phi'(x',t') = \mathrm{e}^{-\frac{\varepsilon}{2}\sigma_z}\phi(x,t), \qquad \psi(x,t) \mapsto \psi'(x',t') = \mathrm{e}^{-\frac{\varepsilon}{2}\sigma_z}\psi(x,t), \tag{5.15}$$

and similarly for (ϕ^*, ψ^*). Note, in particular, that each field component transforms under a *different* irreducible representation $(\mathrm{e}^{\pm\varepsilon})$ of the Lorentz group.[5]

The conserved current associated to the invariance under spacetime translations by Noether's theorem is the energy-momentum tensor

$$T^\mu{}_\nu = \chi_{\alpha,\nu}\frac{\partial \mathcal{L}}{\partial \chi_{\alpha,\mu}} - \delta^\mu{}_\nu \mathcal{L}, \qquad \mu, \nu = 0, 1,$$

[5]Note that, since the Lorentz group in two spacetime dimensions is one-dimensional, all its irreducible representations are necessarily one-dimensional.

where we have used the fact that the Lagrangian (5.1) does not contain derivatives of the complex conjugates of the fields. The corresponding conserved quantities are

$$\int \mathrm{d}x\, T^0{}_0 = H, \qquad \int \mathrm{d}x\, T^0{}_1 = \mathrm{i}\int \mathrm{d}x\Big(\psi_a^*\phi_{a,x} + \phi_a^*\psi_{a,x}\Big) =: -P,$$

where P is the total linear momentum of the fields. Note that H and P can be easily expressed in terms of the lower-order local conserved quantities computed in the previous section as follows:

$$H = -\frac{1}{2}\operatorname{Re}\Big(I_3^{(1)} + \widetilde{I}_3^{(1)}\Big), \qquad P = \frac{1}{2}\operatorname{Re}\Big(I_3^{(1)} - \widetilde{I}_3^{(1)}\Big).$$

Likewise, from the global U(1) gauge invariance of the Lagrangian we deduce the conservation of the current

$$j^\mu = -\mathrm{i}\chi_\alpha\frac{\partial\mathcal{L}}{\partial\chi_{\alpha,\mu}}, \qquad \mu = 0, 1.$$

The corresponding conserved quantity is the system's total charge (or fermion number)

$$Q_R := \int \mathrm{d}x\, j^0 = \int \mathrm{d}x\,(\phi_a^*\psi_a + \psi_a^*\phi_a) = 2\operatorname{Re}\int \mathrm{d}x\,\rho_+.$$

Similarly, the invariance of the Lagrangian under the SU(1,1) transformation (2.25) yields the conserved current

$$j^\mu = \phi_a\frac{\partial\mathcal{L}}{\partial\phi_{\alpha,\mu}} - \psi_a\frac{\partial\mathcal{L}}{\partial\psi_{\alpha,\mu}}, \qquad \mu = 0, 1,$$

whose conserved quantity is given by

$$Q_I := \int \mathrm{d}x\,\mathrm{i}(\phi_a^*\psi_a - \psi_a^*\phi_a) = -2\operatorname{Im}\int \mathrm{d}x\,\rho_+.$$

Finally, from the invariance of the action under the Lorentz boosts (5.14)–(5.15) it follows that the current

$$j^\mu = xT^\mu{}_0 + tT^\mu{}_1 + \frac{(-1)^a}{2}\Big(\phi_a\frac{\partial\mathcal{L}}{\partial\phi_{a,\mu}} + \psi_a\frac{\partial\mathcal{L}}{\partial\psi_{a,\mu}}\Big),$$

is conserved (see, e.g., ref. [52]). The corresponding conserved quantity is given by

$$\int \mathrm{d}x\, j^0 = \int \mathrm{d}x\Big[xH + \frac{\mathrm{i}}{2}(-1)^a(\phi_a^*\psi_a + \psi_a^*\phi_a)\Big] - tP. \tag{5.16}$$

Note that the time-dependent conservation law (5.16) amounts to the equation of motion

$$\int \mathrm{d}x\Big[xH + \frac{\mathrm{i}}{2}(-1)^a(\phi_a^*\psi_a + \psi_a^*\phi_a)\Big] = tP + \mathrm{const.}$$

In particular, the left-hand side of the latter relation is conserved in the system's center of momentum frame, in which $P = 0$.

Remark 8. Apart from the continuous symmetries discussed above, the Lagrangian (2.1) is manifestly invariant under the following transformations:

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i) Parity $\mathcal{P}$: $(x,t)\to(-x,t)$, $\phi(x,t)\to\sigma_x\phi(-x,t)$, $\psi(x,t)\to\sigma_x\psi(-x,t)$.

ii) Time reversal $\mathcal{T}$: $(x,t)\to(x,-t)$, $\phi(x,t)\to\sigma_x\phi^*(x,-t)$, $\psi(x,t)\to-\sigma_x\psi^*(x,-t)$.

In particular, the field equations (2.7) are invariant under the $\mathcal{PT}$ mapping

$$(x,t)\to(-x,-t),\quad (\phi(x,t),\psi(x,t))\to(\phi^*(-x,-t),-\psi^*(-x,-t)).$$

Using eqs. (5.10)–(5.11), it is straightforward to compute the Poisson brackets of the lowest-order local conserved quantities $\int \mathrm{d}x\,\rho_+ = I_2^{(0)}$, $I_3^{(1)}$, and $\widetilde{I}_3^{(1)}$ derived in the previous section. To begin with, if F is of the form (5.9) we have

$$\left\{I_2^{(0)},F\right\}=\mathrm{i}\int\mathrm{d}x\left(\psi_a(x)\frac{\delta\mathcal{F}}{\delta\psi_a(x)}-\phi_a^*(x)\frac{\delta\mathcal{F}}{\delta\phi_a^*(x)}\right).$$

For $F=I_3^{(1)}$, the integrand in the r.h.s. of this equation is (omitting the argument of the fields)

$$\begin{aligned}&-2\Big[\psi_1(-\phi_2^*-2\phi_1^*\phi_2^*\psi_2)+\psi_2(2\mathrm{i}\phi_{2,x}^*-\phi_1^*-2\phi_1^*\phi_1\phi_2^*)\\&-\phi_1^*(\psi_2+2\psi_1\phi_2^*\psi_2)-\phi_2^*(2\mathrm{i}\psi_{2,x}+\psi_1+2\phi_1^*\psi_1\psi_2)\Big]=4\mathrm{i}\Big(\phi_2^*\psi_{2,x}-\psi_2\phi_{2,x}^*\Big)=4\mathrm{i}\,\partial_x\rho_2,\end{aligned}$$

which vanishes upon integration. Thus

$$\left\{I_2^{(0)},I_3^{(1)}\right\}=0,$$

and a similar calculation shows that $\{I_2^{(0)},\widetilde{I}_3^{(1)}\}$ vanishes as well. Finally, if we denote by $\mathcal{I}_3^{(1)}$ and $\widetilde{\mathcal{I}}_3^{(1)}$ the densities of the first integrals $I_3^{(1)}$ and $\widetilde{I}_3^{(1)}$ from eqs. (4.11)–(4.15) it easily follows that

$$\begin{aligned}\left\{\mathcal{I}_3^{(1)},\widetilde{\mathcal{I}}_3^{(1)}\right\}&=8\mathrm{i}\Big\{\phi_a^*\psi_{a,x},\phi_1^*\psi_2+\phi_2^*\psi_1+2\phi_1^*\psi_1\phi_2^*\psi_2\Big\}\\&=-8\Big[\psi_{1,x}(-\phi_2^*-2\phi_1^*\phi_2^*\psi_2)+\psi_{2,x}(-\phi_1^*-2\phi_1^*\psi_1\phi_2^*)+\phi_{1,x}^*(\psi_2+2\psi_1\phi_2^*\psi_2)\\&\quad+\phi_{1,x}^*(\psi_2+2\psi_1\phi_2^*\psi_2)+\phi_{2,x}^*(\psi_1+2\phi_1^*\psi_1\psi_2)\Big]=-8\partial_x\Big(\phi_1^*\psi_2+\phi_2^*\psi_1+2\rho_1\rho_2\Big),\end{aligned}$$

and hence

$$\left\{I_3^{(1)},\widetilde{I}_3^{(1)}\right\}=0.$$

The previous explicit calculations show that the conserved quantities $I_2^{(0)}$, $I_3^{(1)}$, $\widetilde{I}_3^{(1)}$ derived in the previous section Poisson commute among each other. In fact, with the help of *Mathematica*™ we have checked that $I_3^{(3)}$ and $\widetilde{I}_3^{(3)}$ are also in involution with the latter quantities and among themselves. This strongly suggests that the coupled Thirring model (2.7) is completely integrable in Liouville's sense, as was shown to be the case for its bosonic counterpart [33]. The analysis of this conjecture will in fact be the subject of a forthcoming publication.

6 Nonlocal reductions and their symmetries

In the previous sections we have regarded the fields ϕ and ψ as independent, obeying the coupled field equations (2.7). However, as is the case with the bosonic version of the model studied in ref. [33], these equations admit several interesting reductions that we shall now analyze in detail.

As mentioned in section 2, the simplest reduction of eqs. (2.7) is obtained setting $\phi = \psi$, in which case the ψ (or ϕ field) obeys the equations of the original (Grassmannian) Thirring model. More precisely, when $\phi = \psi$ the Lagrangian (2.1) reduces to (twice) the Lagrangian of the ordinary Grassmannian MTM, and similarly for the corresponding Hamiltonian. Likewise, the Lax pair and the integrals of motion of the coupled GMTM turn into the Lax pair and the integrals of motion of the ordinary GMTM under the replacement of ϕ by ψ. Apart from this trivial reduction, only two of the five nonlocal reductions of the coupled bosonic MTM studied in ref. [33] survive in our case, namely the real space/time reflections defined by

I) Real space reflection: $x \mapsto -x, \quad \phi(x,t) = \sigma_x \psi(-x,t)$

II) Real time reflection: $t \mapsto -t, \quad \phi(x,t) = -\mathrm{i}\sigma_y \psi(x,-t)$

The equations of motion of the ψ field in each of these reductions are respectively

$$\begin{aligned} &\mathrm{i}(\partial_t + \partial_x)\psi_1 - \psi_2 + 2\psi_1^*(-x,t)\psi_1\psi_2 = 0, \\ &\mathrm{i}(\partial_t - \partial_x)\psi_2 - \psi_1 + 2\psi_2^*(-x,t)\psi_2\psi_1 = 0 \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &\mathrm{i}(\partial_t + \partial_x)\psi_1 - \psi_2 + 2\psi_1^*(x,-t)\psi_1\psi_2 = 0, \\ &\mathrm{i}(\partial_t - \partial_x)\psi_2 - \psi_1 - 2\psi_2^*(x,-t)\psi_2\psi_1 = 0, \end{aligned} \tag{6.2}$$

where for the sake of simplicity we have suppressed the usual argument (x,t) wherever appropriate. In both cases, the Lax pair and the integrals of motion are obtained from those of the coupled GMTM replacing the ϕ field by its expression in terms of $\psi(-x,t)$ or $\psi(x,-t)$, and the same is true for the Lagrangian and the corresponding Hamiltonian. For instance, the Lax pair for eqs. (6.1)–(6.2) is given by eq. (2.8)–(2.10) with $\rho_\pm$ and $r_1^\pm$ defined by

I) $\rho_\pm = \psi_1^*(-x,t)\psi_2 \pm \psi_2^*(-x,t)\psi_1, \qquad r_1^\pm = -\mathrm{i}\big[\lambda\psi_1^*(-x,t) \pm \lambda^{-1}\psi_2^*(-x,t)\big],$

II) $\rho_\pm = \psi_1^*(-x,t)\psi_2 \mp \psi_2^*(-x,t)\psi_1, \qquad r_1^\pm = -\mathrm{i}\big[\lambda\psi_1^*(-x,t) \mp \lambda^{-1}\psi_2^*(-x,t)\big].$

The above reductions clearly remain invariant under spacetime translations (5.12) and global U(1) gauge transformations (5.13). It should be noted, however, that these reductions break the Lorentz invariance of the original coupled model (2.1). This is ultimately due to the fact that in both of them the second (resp. first) component of the ϕ field is related to the first (resp. second) component of the ψ field, whereas according to eq. (5.15) these components transform under different irreducible representations of the Lorentz group. The breakdown of Lorentz invariance can also be directly checked from the field equations, which in light-cone coordinates can be written in terms of the transformed variables $(\xi', \eta') = (\mathrm{e}^{-\varepsilon}\xi, \mathrm{e}^{\varepsilon}\eta)$ as

$$\text{I)}\quad \mathrm{i}e^{-\varepsilon}\psi_{1,\xi'} - \psi_2 + 2\psi_1^*(\eta,\xi)\psi_1\psi_2 = 0, \qquad \mathrm{i}e^{\varepsilon}\psi_{2,\eta'} - \psi_1 + 2\psi_2^*(\eta,\xi)\psi_2\psi_1 = 0,$$
$$\text{II)}\quad \mathrm{i}e^{-\varepsilon}\psi_{1,\xi'} - \psi_2 + 2\psi_1^*(-\eta,-\xi)\psi_1\psi_2 = 0, \qquad \mathrm{i}e^{\varepsilon}\psi_{2,\eta'} - \psi_1 - 2\psi_2^*(-\eta,-\xi)\psi_2\psi_1 = 0.$$

It is clear that no scaling transformation $(\psi_1, \psi_2) = (\lambda_1\psi_1', \lambda_2\psi_2')$ can eliminate the $e^{\pm\varepsilon}$ factors in both equations of motion.

On the other hand, it is straightforward to show that both nonlocal reductions (6.1)–(6.2) preserve the invariance under space/time reflections of the original equations (2.7). Indeed, the type I) field equations (6.1) are easily seen to be invariant under the transformations

$$\begin{aligned} \mathcal{P}: &\quad x \to -x, &\quad \psi(x,t) \to \sigma_x\psi(-x,t),\\ \mathcal{T}: &\quad t \to -t, &\quad \psi(x,t) \to -\mathrm{i}\sigma_y\psi(x,-t). \end{aligned} \tag{6.3}$$

Likewise, the type II) reduction equations of motion (6.2) are invariant under

$$\begin{aligned} \mathcal{P}: &\quad x \to -x, &\quad \psi(x,t) \to \mathrm{i}\sigma_y\psi^*(-x,t),\\ \mathcal{T}: &\quad t \to -t, &\quad \psi(x,t) \to \sigma_x\psi^*(x,-t). \end{aligned} \tag{6.4}$$

In particular, both reductions (6.1)–(6.2) are $\mathcal{PT}$-symmetric, i.e., invariant under the composition of the mappings $\mathcal{P}$ and $\mathcal{T}$, which in both cases is explicitly given by

$$\mathcal{PT}: \qquad (x,t) \to (-x,-t), \qquad \psi(x,t) \to \sigma_z\psi(-x,-t). \tag{6.5}$$

7 Conclusions and outlook

In this paper we introduce a model of two interacting Dirac fermions with Thirring self-interactions whose fields take values in a Grassmann algebra, which can be regarded as the fermionic version of the bosonic model recently studied in ref. [33]. The model is relativistically invariant, and is in addition symmetric under space and time reflections and U(1) global gauge transformations. It contains as particular cases the free Dirac equation (when its coupling constant vanishes) or the Grassmannian massive Thirring model (when its two independent field variables are set to be equal). We start by constructing a Lax pair for the model, from which the field equations are derived as a zero curvature condition. We also show that the model is closely related to an SU(1, 1) version of the Grassmannian Thirring model that we introduce in this work. Following the approach of ref. [31] for the Grassmannian Thirring model, from the Riccati-type equations satisfied by the quotient of suitable components of the auxiliary vector variable in the Lax pair we obtain four infinite hierarchies of conserved quantities. All of these quantities turn out to be nonlocal, with the exception of the two lowest-order ones. A variant of this method, going back to refs. [8, 9], is then used to construct four additional infinite hierarchies of local conserved quantities.

Using Dirac's method for constrained systems, we develop in detail the Hamiltonian formulation of the model. As is the case with the Dirac equation, the definitions of the canonical momenta turn out to be second-class constraints allowing the determination of all the Lagrange multipliers associated to the constraints. A consistent Poisson bracket for the system can then be constructed by eliminating the complex conjugates of the fields and the corresponding canonical momenta. Using this bracket, we show that the first few lower-order local first integrals previously computed Poisson commute with each other, which strongly suggests that the model is completely integrable in Liouville's sense.

Following the work in ref. [37] on nonlocal integrable reductions of the nonlinear Schrödinger equation, and the analogous results for the bosonic coupled Thirring model in ref. [33], we investigate the existence of such reductions for the present model. We construct two nonlocal integrable reductions based on real space and time reflections, and investigate their symmetries. Although the Lorentz invariance of the general model is not preserved, we show that both of these reductions are invariant under parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$), and are thus $\mathcal{PT}$-invariant. We also show that the Lax pair, Lagrangian, Hamiltonian and hierarchies of conserved quantities of the nonlocal reductions can be easily constructed by suitable reductions of their counterparts for the general model.

The present work suggests several lines for future work. In the first place, it should be of interest to study the complete integrability of the model by a suitable generalization of the method used in ref. [33] to establish this property for its bosonic counterpart. Another natural problem is to investigate the existence of solitonic solutions —which the standard Grassmannian Thirring model does not possess [9]— using the inverse scattering method. Likewise, it would be of interest to construct particular solutions of the model using Bäcklund transformations, as done in ref. [53] for the Grassmannian Thirring model. A natural continuation of the present work would be the construction of the quantized version of the model and the analysis of its properties (integrability, spectrum, symmetries, etc), including its unitarity. Finally, it would also be of interest to investigate the integrability properties of the SU(1, 1) generalization of the GMTM introduced in this paper. Work on some of this topics is currently going on and will appear in future publications.

A Derivation of the fundamental Poisson brackets (5.7) through Dirac's method for constrained systems

In this appendix we shall apply Dirac's method for systems with constraints to justify eq. (5.7) for the fundamental Poisson brackets of the coupled Thirring model used in section 5. Our starting point is the Lagrangian (5.1), which leads to the expressions (5.3) for the conjugate momenta (π_χ, π_{χ^*}) of the field variables $(\chi, \chi^*) \equiv (\phi, \psi, \phi^*, \psi^*)$. Since these expressions do not involve the generalized velocities $(\dot{\chi}, \dot{\chi}^*)$, they give rise to the set of primary constraints

$$\Gamma \equiv (\Gamma_A)_{1 \leqslant A \leqslant 8} = (\pi_\phi + \mathrm{i}\psi^*, \pi_\psi + \mathrm{i}\phi^*, \pi_{\phi^*}, \pi_{\psi^*}). \tag{A.1}$$

Following Dirac's method, we then define the primary Hamiltonian density

$$\mathcal{H}_P = \mathcal{H} + \lambda_A \Gamma_A,$$

where $\mathcal{H}$ is the canonical Hamiltonian (5.4) obtained from eq. (5.2), and the λ_A are (odd) Lagrange multipliers depending on the spacetime coordinates (x, t) but independent of the field variables and their conjugate momenta. The canonical equations of motion generated by the primary Hamiltonian

$$H_P = \int \mathrm{d}x \left(\mathcal{H}(x) + \lambda_A(x)\Gamma_A(x)\right) = H + \int \mathrm{d}x \, \lambda_A(x)\Gamma_A(x)$$

are the most general equations of motion compatible with variations of the fields and their conjugate momenta respecting the constraints. As is the case for the Dirac equation (see, e.g., ref. [51]), all the primary constraints turn out to be second class. In other words, imposing that the primary constraints be consistent with the time evolution, i.e., that[6]

$$\{\Gamma_A, H_P\} \approx 0, \qquad 1 \leqslant A \leqslant 8, \tag{A.2}$$

determines the Lagrange multipliers λ_A. It should be noted that the Poisson bracket in eq. (A.2) is the *canonical* one, computed regarding the fields (χ, χ^*) and their conjugate momenta (π_χ, π_{χ^*}) as *independent* (anticommuting) variables satisfying the standard canonical relations[7]

$$\{\chi_\alpha(x), \pi_{\chi_\beta}(y)\} = \{\pi_{\chi_\alpha}(x), \chi_\beta(y)\} = \{\chi^*_\alpha(x), \pi_{\chi^*_\beta}(y)\} = \{\pi_{\chi^*_\alpha}(x), \chi^*_\beta(y)\} = -\delta_{\alpha\beta}\delta(x-y)$$

(all other Poisson brackets vanishing identically). For instance, from the Poisson bracket

$$\begin{aligned}\{\Gamma_1(x), H_P\} &= \Big\{\pi_{\phi_1}(x) + \mathrm{i}\psi_1^*(x), H + \int \mathrm{d}y\, \lambda_A(y)\Gamma_A(y)\Big\} \\ &= \{\pi_{\phi_1}(x), H\} - \int \mathrm{d}y\, \lambda_7(y)\Big\{\pi_{\phi_1}(x) + \mathrm{i}\psi_1^*(x), \pi_{\psi_1^*}(y)\Big\} \\ &= -\mathrm{i}\psi^*_{1,x}(x) - \psi_2^*(x) - 2\psi_1^*(x)\psi_2^*(x)\phi_2(x) + \mathrm{i}\lambda_7(x)\end{aligned}$$

we obtain

$$\lambda_7 = \psi^*_{1,x} - \mathrm{i}\Big(\psi_2^* + 2\psi_1^*\psi_2^*\phi_2\Big).$$

Proceeding in this way we arrive at the following expressions for the Lagrange multipliers $\lambda_A(x)$:

$$\begin{aligned}\lambda_1 &= -\phi_{1,x} - \mathrm{i}\Big(\phi_2 + 2\phi_1\psi_2^*\phi_2\Big), & \lambda_2 &= \phi_{2,x} - \mathrm{i}\Big(\phi_1 + 2\phi_2\psi_1^*\phi_1\Big), \\ \lambda_3 &= -\psi_{1,x} - \mathrm{i}\Big(\psi_2 + 2\psi_1\phi_2^*\psi_2\Big), & \lambda_4 &= \psi_{2,x} - \mathrm{i}\Big(\psi_1 + 2\psi_2\phi_1^*\psi_1\Big), \\ \lambda_5 &= \phi^*_{1,x} - \mathrm{i}\Big(\phi_2^* + 2\phi_1^*\phi_2^*\psi_2\Big), & \lambda_6 &= -\phi^*_{2,x} - \mathrm{i}\Big(\phi_1^* + 2\phi_2^*\phi_1^*\psi_1\Big), \\ \lambda_7 &= \psi^*_{1,x} - \mathrm{i}\Big(\psi_2^* + 2\psi_1^*\psi_2^*\phi_2\Big), & \lambda_8 &= -\psi^*_{2,x} - \mathrm{i}\Big(\psi_1^* + 2\psi_2^*\psi_1^*\phi_1\Big).\end{aligned}$$

As expected, the Lagrange multipliers coincide with the generalized velocities $(\dot\chi, \dot\chi^*)$ expressed in terms of the fields and their space derivatives (cf. eqs. (2.7)). Furthermore, since eqs. (A.2) have not produced any new constraints the primary constraints (A.1) are the only constraints in the system. In addition, since all the Lagrange multipliers have been determined the Dirac matrix with elements

$$C_{AB}(x,y) := \{\Gamma_A(x), \Gamma_B(y)\}, \qquad 1 \leqslant A, B \leqslant 8,$$

[6]As is customary, we shall use the symbol $\approx$ to denote an equality modulo the constraints.

[7]Throughout this appendix we shall omit the time variable, whose value is the same for all fields and momenta.

is invertible, and the Dirac bracket of the densities $\mathcal{F}(x), \mathcal{G}(y)$ of any two dynamical variables is given by the standard formula

$$\{\mathcal{F}(x), \mathcal{G}(y)\}_D = \{\mathcal{F}(x), \mathcal{G}(y)\} - \int \mathrm{d}z\mathrm{d}w\, \{\mathcal{F}(x), \Gamma_A(z)\}(C^{-1})_{AB}(z,w)\{\Gamma_B(w), \mathcal{G}(y)\}. \tag{A.3}$$

In our case the Dirac matrix is easily computed, with the result[8]

$$C(x,y) = -\mathrm{i}\delta(x,y) \begin{pmatrix} 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix},$$

where $\mathbb{1}$ denotes the 2×2 identity matrix. For instance,

$$\{\Gamma_1(x), \Gamma_B(y)\} = \{\pi_{\phi_1}(x) + \mathrm{i}\psi_1^*(x), \Gamma_B(y)\} = \delta_{B7}\{\mathrm{i}\psi_1^*(x), \pi_{\psi_1^*}(y)\} = -\mathrm{i}\delta_{B7}\delta(x-y).$$

Since $C^{-1}(x,y) = -C(x,y)$, from eq. (A.3) we immediately obtain the following explicit expression for the Dirac bracket:

$$\begin{aligned} \{\mathcal{F}(x), \mathcal{G}(y)\}_D = \{\mathcal{F}(x), \mathcal{G}(y)\} - \mathrm{i}\int \mathrm{d}z \Big[&\{\mathcal{F}(x), \pi_{\phi_a}(z) + \mathrm{i}\psi_a^*(z)\}\{\pi_{\psi_a^*}(z), \mathcal{G}(y)\} \\ &+ \{\mathcal{F}(x), \pi_{\psi_a}(z) + \mathrm{i}\phi_a^*(z)\}\{\pi_{\phi_a^*}(z), \mathcal{G}(y)\} \\ &+ \{\mathcal{F}(x), \pi_{\phi_a^*}(z)\}\{\pi_{\psi_a}(z) + \mathrm{i}\phi_a^*(z), \mathcal{G}(y)\} \\ &+ \{\mathcal{F}(x), \pi_{\psi_a^*}(z)\}\{\pi_{\phi_a}(z) + \mathrm{i}\psi_a^*(z), \mathcal{G}(y)\}\Big]. \end{aligned} \tag{A.4}$$

Using eq. (A.4) it is straightforward to show that the only nonvanishing Dirac brackets between the fields (χ, χ^*) appearing in the canonical Hamiltonian (5.4) turn out to be

$$\{\phi_a(x), \psi_b^*(y)\}_D = \{\psi_a(x), \phi_b^*(y)\}_D = \{\phi_a^*(x), \psi_b(y)\}_D = \{\psi_a^*(x), \phi_b(y)\}_D = -\mathrm{i}\delta_{ab}\delta(x-y). \tag{A.5}$$

Indeed,

$$\begin{aligned} \{\phi_a(x), \psi_b^*(y)\}_D &= \{\phi_a(x), \psi_b^*(y)\} - \mathrm{i}\int \mathrm{d}z \{\phi_a(x), \pi_{\phi_c}(z) + \mathrm{i}\psi_c^*(z)\}\{\pi_{\psi_c^*}(z), \psi_b^*(y)\} \\ &= -\mathrm{i}\int \mathrm{d}z(-\delta_{ac}\delta(z-x))(-\delta_{bc}\delta(z-y)) = -\mathrm{i}\delta_{ab}\delta(x-y), \end{aligned}$$

etc. Comparing with eqs. (5.8) we conclude that the prescription used in section 5 of replacing χ_α^* by its expression (5.3) in terms of the canonical momenta π_{χ_α} in the computation of Poisson brackets of dynamical variables of the form (5.9) —which amounts to using the constraints to eliminate the canonical variables (χ^*, π_{χ^*}) — is justified if we interpret the Poisson bracket as a Dirac bracket. Note, in this respect, that by construction all the constraints $\Gamma_A(x)$ have vanishing Dirac bracket with any dynamical variable $\mathcal{F}(x)$, i.e.,

$$\{\mathcal{F}(x), \Gamma_A(y)\}_D = 0,$$

[8]Note that in this case the Dirac matrix is not antisymmetric but rather *symmetric*, as the Poisson bracket of two odd variables is symmetric.

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and therefore

$$\{\mathcal{F}(x), H_P\}_D = \{\mathcal{F}(x), H\}_D.$$

Thus we can use the canonical Hamiltonian H instead of the primary one H_P to compute the time evolution of dynamical variables, as we did in section 5. In particular, the fields' equations of motion can be written in the canonical form (5.5).

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References

[1] W.E. Thirring, *A Soluble relativistic field theory*, *Annals Phys.* **3** (1958) 91 [INSPIRE].

[2] S.R. Coleman, *The Quantum Sine-Gordon Equation as the Massive Thirring Model*, *Phys. Rev. D* **11** (1975) 2088 [INSPIRE].

[3] H.B. Thacker, *Exact Integrability in Quantum Field Theory and Statistical Systems*, *Rev. Mod. Phys.* **53** (1981) 253 [INSPIRE].

[4] P.P. Kulish and E.K. Sklyanin, *Quantum spectral transform method. Recent developments*, in *Integrable Quantum Field Theories*, J. Hietarinta and C. Montonen eds., Springer (1982), pp. 61–119 [DOI:10.1007/3-540-11190-5_8].

[5] T. Bhattacharyya, *Quantum integrability of bosonic massive Thirring model in continuum*, *J. Math. Phys.* **46** (2005) 012301 [hep-th/0406090] [INSPIRE].

[6] A.V. Mikhailov, *Integrability of the two-dimensional Thirring model*, *JETP Lett.* **23** (1976) 320.

[7] S.J. Orfanidis, *Soliton Solutions of the Massive Thirring Model and the Inverse Scattering Transform*, *Phys. Rev. D* **14** (1976) 472 [INSPIRE].

[8] A.G. Izergin and P.P. Kulish, *Massive Thirring model with field values in a Grassmann algebra*, *J. Sov. Math.* **22** (1983) 1586.

[9] A.G. Izergin and P.P. Kulish, *On the Inverse Scattering Method for the Classical Massive Thirring Model with Anticommuting Variables*, *Lett. Math. Phys.* **2** (1978) 297 [INSPIRE].

[10] H.G. Winful and G.D. Cooperman, *Self-pulsing and chaos in distributed feedback bistable optical devices*, *Appl. Phys. Lett.* **40** (1982) 298.

[11] D.N. Christodoulides and R.I. Joseph, *Slow Bragg solitons in nonlinear periodic structures*, *Phys. Rev. Lett.* **88** (1989) 1747.

[12] A.B. Aceves and S. Wabnitz, *Self-induced transparency solitons in nonlinear refractive periodic media*, *Phys. Lett. A* **141** (1989) 37.

[13] B.J. Eggleton, R.E. Slusher, C.M. de Sterke, P.A. Krug and J.E. Sipe, *Bragg grating solitons*, *Phys. Rev. Lett.* **76** (1996) 1627.

[14] B.J. Eggleton, C.M. de Sterke and R.E. Slusher, *Bragg solitons in the nonlinear Schrödinger limit: experiment and theory*, *J. Opt. Soc. Am. B* **16** (1999) 587.

[15] G. Lenz, P. Meystre and E.M. Wright, *Nonlinear atom optics*, *Phys. Rev. Lett.* **71** (1993) 3271.

[16] E.A. Kuznetsov and A.V. Mikhailov, *On the complete integrability of the two-dimensional classical Thirring model*, *Theor. Math. Phys.* **30** (1977) 193.

[17] T. Kawata, T. Morishima and H. Inoue, *Inverse scattering method for the two-dimensional massive Thirring model*, *J. Phys. Soc. Jap.* **47** (1979) 1327 [INSPIRE].

[18] D.J. Kaup and A.C. Newell, *On the Coleman Correspondence and the Soliton of the Massive Thirring Model*, *Lett. Nuovo Cim.* **20** (1977) 325 [INSPIRE].

[19] M. Wadati, *General solution and Lax pair for 1-D classical massless Thirring model*, *J. Phys. Soc. Jap.* **52** (1983) 1084 [INSPIRE].

[20] D.E. Pelinovsky and A. Saalman, *Inverse scattering for the massive Thirring model*, in *Nonlinear Dispersive Partial Differential Equations and Inverse Scattering*, P.D. Miller, P.A. Perry, J.-C. Saut and C. Sulem eds., Springer (2019), pp. 497–528 [DOI:10.1007/978-1-4939-9806-7_11].

[21] A.K. Prikarpatsky, *Geometric Structure and Bäcklund Transformations of Nonlinear Evolution Equations*, *Theor. Math. Phys.* **46** (1981) 249 [INSPIRE].

[22] A. Degasperis, *Darboux polynomial matrices: the classical massive Thirring model as a study case*, *J. Phys. A* **48** (2015) 235204 [arXiv:1411.7965] [INSPIRE].

[23] A. Degasperis, S. Wabnitz and A.B. Aceves, *Bragg grating rogue wave*, *Phys. Lett. A* **379** (2015) 1067.

[24] L. Guo, L. Wang, Y. Cheng and J. He, *High-order rogue wave solutions of the classical massive Thirring model equations*, *Commun. Nonlinear Sci.* **52** (2017) 11.

[25] Y. Ye, L. Bu, C. Pan, S. Chen, D. Mihalache and F. Baronio, *Super rogue wave states in the classical massive Thirring model system*, *Rom. Rep. Phys.* **73** (2021) 117(16).

[26] I.V. Barashenkov and B.S. Getmanov, *Multi-Soliton Solutions in the Scheme for Unified Description of Integrable Relativistic Massive Fields. Nondegenerate $sl(2,C)$ Case*, *Commun. Math. Phys.* **112** (1987) 423 [INSPIRE].

[27] I.V. Barashenkov, B.S. Getmanov and V.E. Kovtun, *The Unified approach to integrable relativistic equations: Soliton solutions over nonvanishing backgrounds. I*, *J. Math. Phys.* **34** (1993) 3039 [INSPIRE].

[28] I.V. Barashenkov and B.S. Getmanov, *The Unified approach to integrable relativistic equations: Soliton solutions over nonvanishing backgrounds. II*, *J. Math. Phys.* **34** (1993) 3054 [INSPIRE].

[29] J. Chen and B.-F. Feng, *General bright and dark soliton solutions to the massive Thirring model via KP hierarchy reductions*, arXiv:2111.05718 [INSPIRE].

[30] A.R. Aguirre, J.F. Gomes, L.H. Ymai and A.H. Zimerman, *Grassmannian and bosonic Thirring models with jump defects*, *JHEP* **02** (2011) 017 [arXiv:1012.1537] [INSPIRE].

[31] A.R. Aguirre, *Inverse scattering approach for massive Thirring models with integrable type-II defects*, *J. Phys. A* **45** (2012) 205205 [arXiv:1111.5249] [INSPIRE].

[32] N.V. Alexeeva, I.V. Barashenkov and A. Saxena, *Spinor solitons and their PT-symmetric offspring*, *Annals Phys.* **403** (2019) 198 [arXiv:1812.02423].

[33] B. Basu-Mallick and D. Sinha, *Integrable coupled bosonic massive Thirring model and its nonlocal reductions*, arXiv:2307.00351 [INSPIRE].

[34] C. Gross, A. Strumia, D. Teresi and M. Zirilli, *Is negative kinetic energy metastable?*, *Phys. Rev. D* **103** (2021) 115025 [arXiv:2007.05541] [INSPIRE].

[35] R. Casalbuoni, *On the Quantization of Systems with Anticommutating Variables*, *Nuovo Cim. A* **33** (1976) 115 [INSPIRE].

[36] J. Maharana, *Canonical structure and quantization of Grassmann variables*, *Fortsch. Phys.* **33** (1985) 645 [INSPIRE].

[37] M.J. Ablowitz and Z.H. Musslimani, *Integrable Nonlocal Nonlinear Schrödinger Equation*, *Phys. Rev. Lett.* **110** (2013) 064105.

[38] A.K. Sarma, M.-A. Miri, Z.H. Musslimani and D.N. Christodoulides, *Continuous and discrete Schrödinger systems with parity-time-symmetric nonlinearities*, *Phys. Rev. E* **89** (2014) 052918 [arXiv:1310.7399] [INSPIRE].

[39] D. Sinha and P.K. Ghosh, *Integrable nonlocal vector nonlinear Schrödinger equation with self-induced parity-time-symmetric potential*, *Phys. Lett. A* **381** (2017) 124 [arXiv:1506.03036] [INSPIRE].

[40] M.J. Ablowitz and Z.H. Musslimani, *Integrable Nonlocal Nonlinear Equations*, *Stud. Appl. Math.* **139** (2016) 7.

[41] A. Khare and A. Saxena, *Periodic and hyperbolic soliton solutions of a number of nonlocal nonlinear equations*, *J. Math. Phys.* **56** (2015) 032104 [arXiv:1405.5267] [INSPIRE].

[42] D. Sinha and P.K. Ghosh, *Symmetries and exact solutions of a class of nonlocal nonlinear Schrödinger equations with self-induced parity-time-symmetric potential*, *Phys. Rev. E* **91** (2015) 042908 [arXiv:1408.0954] [INSPIRE].

[43] V.S. Gerdjikov and A. Saxena, *Complete integrability of nonlocal nonlinear Schrödinger equation*, *J. Math. Phys.* **58** (2017) 013502 [INSPIRE].

[44] D. Sinha, *Integrable local and non-local vector Non-linear Schrödinger Equation with balanced loss and gain*, *Phys. Lett. A* **448** (2022) 128338 [arXiv:2112.11926] [INSPIRE].

[45] Y. Shi, S.-F. Shen and S.-L. Zhao, *Solutions and connections of nonlocal derivative nonlinear Schrödinger equations*, *Nonlinear Dynamics* **95** (2018) 1257.

[46] Z.-X. Zhou, *Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation*, *Commun. Nonlinear Sci. Numer. Simul.* **62** (2018) 480.

[47] X.-B. Xiang, S.-L. Zhao and Y. Shi, *Solutions and continuum limits to nonlocal discrete sine-Gordon equations: bilinearization reduction method*, arXiv:2208.12007.

[48] B. DeWitt, *Supermanifolds*, 2nd edition, Cambridge University Press, Cambridge, U.K. (1992) [DOI:10.1017/CBO9780511564000].

[49] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th edition, Clarendon Press, Oxford, U.K. (2002). [ISBN: 9780198509233].

[50] P.A.M. Dirac, *Generalized Hamiltonian dynamics*, *Can. J. Math.* **2** (1950) 129 [INSPIRE].

[51] A. Das, *Lectures on Quantum Field Theory*, 2nd edition, World Scientific, Singapore (2021).

[52] M. Maggiore, *A Modern introduction to quantum field theory*, Oxford University Press (2005) [INSPIRE].

[53] A. Izergin and J. Stehr, *A Bäcklund Transformation for the Classical Anticommuting Massive Thirring Model in One Space Dimension*, DESY-76-60 (1976) [DOI:10.3204/PUBDB-2017-12688] [INSPIRE].