SOME CLASSES OF MULTILINEAR OPERATORS ON $C(K)$ SPACES

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Abstract. We obtain a classification of the projective tensor product of $C(K)$ spaces according to the fact that none, exactly one or more than one of the factors contain copies of ℓ_1, in terms of the behaviour of certain classes of multilinear operators on the product of the spaces or the verification of certain Banach space properties on the corresponding tensor product. The main tool is an improvement of some results of G. Emmanuele and Hensgen about the Reciprocal Dunford-Pettis and Pelczynski's V properties on the projective tensor product of Banach spaces. We also study the relationship between several classes of multilinear operators and the corresponding linear associated operators.

1. Introduction

In the past years much research has been done in the theory of multilinear operators and polynomials between Banach spaces. In particular, different classes of multilinear operators or polynomials have been defined which extended the corresponding notions for linear operators, and the relations between some of these classes have been studied. If E, F and X are Banach spaces and $T : E \times F \longrightarrow X$ is a bilinear operator, it is well known that there exists only one linear operator $\hat{T} : E\hat{\otimes}_\pi F \longrightarrow X$ canonically associated to T, where $E\hat{\otimes}_\pi F$ is the projective tensor product of E and F. In Section 2 we improve some results of G. Emmanuele and Hensgen to establish, under suitable conditions, some non trivial relationships between several classes of bilinear operators. In Section 3 we use the results of Section 2 to obtain a classification of the projective tensor product of several $C(K)$ spaces, according to the fact that none, exactly one o more than one of the factors contain copies of ℓ_1, in terms of the behaviour of certain classes of multilinear operators on the product of the spaces or the verification of certain Banach space properties on the corresponding tensor product. Finally, in Section 4 we study the relation between $\hat{T}$ belonging

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to certain operator ideals and T belonging to certain classes of multilinear operators.

The notations and terminology used along the paper will be the standard in Banach space theory, as for instance in [9]. However, before going any further, we shall clear out some terminology: $\mathcal{L}^k(E_1 \dots, E_k; X)$ will be the Banach space of all the continuous k-linear mappings from $E_1 \times \cdots \times E_k$ into X and $\mathcal{L}^k_{wc}(E_1 \dots, E_k; X)$ will be the closed subspace of it formed by the weakly compact multilinear operators. When $X = \mathbb{K}$ or $k = 1$, we omit them. We write $\mathcal{K}(E; X)$ for the space of compact operators from E into X. As usual, $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k$ stands for the (complete) projective tensor product of the Banach spaces $E_1, \dots, E_k$. It $T \in \mathcal{L}^k(E_1 \dots, E_k; X)$ we denote by $\hat{T} : E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k \to X$ its linearization.

We say that $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is *completely continuous*, and we write $T \in \mathcal{L}^k_{cc}(E_1, \dots, E_k; X)$, if, given weak Cauchy sequences $(x_i^n)_{n \in \mathbb{N}} \subset E_i$ $(1 \le i \le k)$, the sequence $(T(x_1^n, \dots, x_k^n))_n$ is norm convergent in X. This definition may be adapted to polynomials in an obvious way. The space of completely continuous polynomials is denoted by $\mathcal{P}_{cc}({}^k E; X)$. By the polarization formula [17, Theorem 1.10], a polynomial is completely continuous if and only if so is its associated symmetric multilinear operator. If $X = \mathbb{K}$, i.e., if T is a multilinear *form*, we will use the notation *weakly sequentially continuous* instead of *completely continuous*.

If $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ we denote by T_i $(1 \le i \le k)$ the operator $T_i \in \mathcal{L}(E_i; \mathcal{L}^{k-1}(E_1, \overset{[i]}{\dots}, E_k; X))$ defined by

$$T_i(x_i)(x_1, \overset{[i]}{\dots}, x_k) := T(x_1, \dots, x_k),$$

We shall say that T is *regular* if all the maps $T_i, 1 \le i \le k$ defined above, are weakly compact.

Recall that E has the *Dunford-Pettis property* (*DPP*, for short) if, for every X, $\mathcal{L}_{wc}(E; X) \subseteq \mathcal{L}_{cc}(E; X)$. Examples of spaces with the DPP are $C(K)$ and $L_1(\mu)$ spaces. E has the *reciprocal Dunford-Pettis property* (*RDPP*, for short) if, for every X, $\mathcal{L}_{cc}(E; X) \subseteq \mathcal{L}_{wc}(E; X)$. The spaces containing no copy of ℓ_1, and $C(K)$ spaces have the RDPP. Both properties were introduced in [14].

A formal series $\sum x^n$ in a Banach space E is *weakly unconditionally Cauchy* (*w.u.C.*, for short) if there is $C > 0$ such that, for any finite subset Δ of $\mathbb{N}$ and any signs $\pm$, we have $\| \sum_{n \in \Delta} \pm x^n \| \le C$. For other equivalent definitions, see [8, Theorem V.6]. The series $\sum x^n$ is *unconditionally convergent* if every subseries is norm convergent. Other equivalent definitions may be seen in [9, Theorem 1.9].

A linear operator between Banach spaces is *unconditionally converging* if it takes w.u.C. series into unconditionally convergent series. A Banach space E is said to have Pełczyński's *property (V)* if every unconditionally converging linear operator on E is weakly compact. This property was introduced in [18], where it is shown that $C(K)$ spaces have property (V), and that the dual of a space with property (V) is weakly sequentially complete.

Following [12], we say that $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is *unconditionally converging* if, given w.u.C. series $\sum_{n\in\mathbb{N}} x_i^n$ in E_i $(1 \le i \le k)$, the sequence

$$(T(s_1^m, \dots, s_k^m))_m$$

is norm convergent in X, where $s_i^m = \sum_{n=1}^m x_i^n$. This definition may be adapted to polynomials in an obvious way. Since a linear operator fails to be unconditionally converging if and only if it preserves a copy of c_0 [8, Exercise V.8], it is clear that the definition of unconditionally converging k-linear operators agrees for $k = 1$ with that of unconditionally converging linear operators.

By the polarization formula, a polynomial is unconditionally converging if and only if so is its associated symmetric multilinear operator.

Since $B_{E\hat{\otimes}_\pi F} = \overline{coe}(B_E \otimes B_F)$, it follows that T is (weakly) compact if and only in $\hat{T}$ is (weakly) compact.

2. Some properties of Bilinear operators

In [10] and [11], the following results are proved:

Theorem 2.1. *Let E be a Banach space not containing ℓ_1 and F a Banach space with the RDPP. If $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$, then $E\hat{\otimes}_\pi F$ has the RDPP.*

Theorem 2.2. *Let E, F be Banach spaces with the RDPP such that E^* and F^* are weakly sequentially complete. If $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$, then $E\hat{\otimes}_\pi F$ has the RDPP.*

Theorem 2.3. *Let E, F be Banach spaces with Property (V) such that $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$. Then $E\hat{\otimes}_\pi F$ has property (V).*

Taking advantage of the ideas in those papers, we prove now a strengthening of these results. First we will need some definitions and lemmas.

Definition 2.4. *Let E be a Banach space. A set $M \subset E^*$ is an L-set (respectively a V-set), if for every weakly null sequence $(x_n) \subset E$ (resp. every w.u.C. series $\sum_n x_n \subset E$), we have*

$$\lim_{n\to\infty} \sup\{|x^*(x_n)| : x^* \in M\} = 0.$$

The following result is well known.

Proposition 2.5. *Let E be a Banach space. Then:*

(a) E has RDPP if and only if every L-set is relatively weakly compact ([16]*. see also* [2]*).*

(b) E has property (V) if and only if every V-set is relatively weakly compact ([18]*).*

We will also need some results concerning the Aron-Berner extension of a multilinear operator: if $T : E \times F \longrightarrow X$ is a bilinear operator, then we can define its Aron-Berner extension,

$$AB(T) : E^{**} \times F^{**} \longrightarrow X^{**}$$

by

$$AB(T)(z_1, z_2) = \lim_{\alpha} \lim_{\beta} T(x_\alpha, y_\beta),$$

where $(x_\alpha) \subset E$ is a net weak-star converging to z_1 and $(y_\beta) \subset F$ is a net weak-star converging to z_2. Related to this extension we will use the following results from [15].

Lemma 2.6. *Let E, F be Banach spaces with the RDPP, and X any Banach space. If $T : E \times F \longrightarrow X$ is a completely continuous bilinear operator, then its Aron-Berner extension $AB(T)$ takes values in X.*

Lemma 2.7. *Let E, F be Banach spaces such that their duals E^* and F^* have the Dunford-Pettis Property and such that $\mathcal{L}(E; F^*) = \mathcal{L}_{wc}(E; F^*)$. For any Banach space X, if $T : E \times F \longrightarrow X$ is a bilinear operator such that its Aron-Berner extension $AB(T)$ is X-valued, then $AB(T) : E^{**} \times F^{**} \longrightarrow X$ is completely continuous.*

Lemma 2.8. *Let E, F be Banach spaces with property V, and X any Banach space. If $T : E \times F \longrightarrow X$ is an unconditionally converging bilinear operator, then its Aron-Berner extension $AB(T)$ takes values in X.*

Lemma 2.9. *Let E, F be Banach spaces such that $\mathcal{L}(E; F^*) = \mathcal{L}_{wc}(E; F^*)$. For any Banach space X, if $T : E \times F \longrightarrow X$ is a bilinear operator such that its Aron-Berner extension $AB(T)$ is X-valued, then $AB(T) : E^{**} \times F^{**} \longrightarrow X$ is unconditionally converging.*

Now we can prove the following.

Proposition 2.10. *Let E be a Banach space not containing ℓ_1 and F a Banach space with the RDPP. Assume further that $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$ and that E^* and F^* have the Dunford-Pettis Property. For every Banach space X, if $T : E \times F \longrightarrow X$ is a completely continuous bilinear operator, then T is weakly compact.*

Proof. Let T be as in the hypothesis and let $\hat{T} : E\hat{\otimes}_\pi F \longrightarrow X$ be the operator canonically associated to T. Since T is weakly compact if and only if $\hat{T}$ is weakly compact, it suffices to prove that $\hat{T}^*$ is weakly compact. Let then $M = \hat{T}^*(B_{X^*}) \subset (E\hat{\otimes}_\pi F)^* = \mathcal{K}(E; F^*)$. Let $(h_n)_n \subset M$ and let $(\varphi_n)_n \subset B_{X^*}$ be such that $\hat{T}^*(\varphi_n) = h_n$ for every $n \in \mathbb{N}$. Define H by $H = \overline{\text{span}}[h_n(x) : x \in E, n \in \mathbb{N}]$. Then H is a closed subspace of F^* and H is separable, because, for every $n \in \mathbb{N}$, $h_n : E \longrightarrow F^*$ is compact. Let now $Y \subset F$ be a countable norming set of H and let $y \in Y$.

Claim 1: The set $\{h_n^*(y); n \in \mathbb{N}\} \subset E^*$ is an L-set.

Proof of the claim: Let $(x_m)_m \subset E$ be a weakly converging to 0 sequence. We have

$$h_n^*(y)(x_m) = h_n(x_m)(y) = \hat{T}^*(\varphi_n)(x_m \otimes y) = \langle \hat{T}(x_m \otimes y), \varphi_n\rangle = \langle T(x_m, y), \varphi_n\rangle.$$

Therefore

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} |h_n^*(y)(x_m)| \leq \lim_{m\to\infty} \|T(x_m, y)\| = 0,$$

and the claim is proved.

So $\{h_n^*(y); n \in \mathbb{N}\} \subset E^*$ is relatively weakly compact and therefore we can suppose (using the fact that Y is countable and considering subsequences if necessary) that, for every $y \in Y$, $(h_n^*(y))_n$ is a weakly Cauchy sequence. Let now $x^{**} \in E^{**}$.

Claim 2: The set $\{h_n^{**}(x^{**}); n \in \mathbb{N}\} \subset F^*$ is an L-set.

Proof of the claim: If we think of $h_n \in \mathcal{K}(E; F^*)$ as a bilinear form, $h_n : E \times F \longrightarrow \mathbb{K}$, it is clear that $h_n^{**}(x^{**})(y) = AB(h_n)(x^{**}, y)$, where $AB(h_n)$ denotes any of the two Aron-Berner extensions of h_n. Let then $(y_m)_m \subset F$ be a weakly converging to 0 sequence. Then

$$h_n^{**}(x^{**})(y_m) = AB(\hat{T}^*(\varphi_n)(x^{**}, y_m).$$

Let us see now that $AB(\hat{T}^*(\varphi_n)(x^{**}, y_m) = \langle AB(T)(x^{**}, y_m), \varphi_n\rangle$: let $(x_\alpha)_\alpha \subset E$ be a bounded net weak-star convergent to x^{**}. Then

$$AB(\hat{T}^*(\varphi_n)(x^{**}, y_m) = \lim_\alpha \hat{T}^*(\varphi_n)(x_\alpha, y_m) =$$

$$= \lim_\alpha \langle \hat{T}(x_\alpha \otimes y_m), \varphi_n\rangle = \lim_\alpha \langle T(x_\alpha, y_m), \varphi_n\rangle = \langle AB(T)(x^{**}, y_m), \varphi_n\rangle.$$

Therefore,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} |h_n^{**}(x^{**})(y_m)| \leq \lim_{m\to\infty} \|AB(T)(x^{**}, y_m)\| = 0$$

The last limit is 0 because $AB(T)$ is completely continuous, as follows from Lemmas 2.6 and 2.7. So, the claim is proved.

Now we can proceed as in [10] to obtain $h \in \mathcal{K}(E; F^*)$ such that h_n weakly converges to h, which finishes the proof. □

Corollary 2.11. *Let E, F be Banach spaces with the RDPP such that E^* and F^* are weakly sequentially complete and have the Dunford-Pettis Property. Assume further that $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$. For every Banach space X, if $T : E \times F \longrightarrow X$ is a completely continuous bilinear operator, then T is weakly compact.*

Proof. The beginning of the proof runs as in Proposition 2.10 to prove that $(h_n)_n$ is weakly Cauchy. To finish it, we must reason as in the proof of Theorem 2.2 (see [10, Cor. 4]). □

Since compact operators are completely continuous, Proposition 4.1 below implies that our results are indeed a strengthening (under the additional hypothesis that E^* and F^* have the DP property) of Theorems 2.1 and 2.2. For instance, note that, since $c_0\hat{\otimes}_\pi\ell_\infty$ does not have the DP property ([6]), there are completely continuous bilinear operators defined on $c_0 \times \ell_\infty$ such that their linear associated operator defined on $c_0\hat{\otimes}_\pi\ell_\infty$ is not completely continuous.

We have a similar result for unconditionally converging bilinear operators. This time we do not need additional hypothesis on E and F.

Proposition 2.12. *Let E, F be Banach spaces with Property (V). Assume further that $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$. For every Banach space X, if $T : E \times F \longrightarrow X$ is an unconditionally converging bilinear operator, then T is weakly compact.*

Proof. As in the proof of Proposition 2.10, it suffices to prove that $\hat{T}^*(B_{X^*}) = M \subset (E\hat{\otimes}_\pi F)^* = \mathcal{K}(E; F^*)$ is relatively weakly compact. Let then h_n, φ_n, H and Y be as in the proof of Proposition 2.10.

Claim 1: The set $\{h_n^*(y); n \in \mathbb{N}\} \subset E^*$ is a V-set.

Proof of the claim: Let $\sum_m x_m \subset E$ be a w.u.C. series. As in the proof of Proposition 2.10,

$$h_n^*(y)(x_m) = h_n(x_m)(y) = \langle T(x_m, y), \varphi_n\rangle.$$

It is very easy to see that T is separately unconditionally converging, so

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} |h_n^*(y)(x_m)| \leq \lim_{m\to\infty} \|T(x_m, y)\| = 0,$$

and the claim is proved

So $\{h_n^*(y); n \in \mathbb{N}\} \subset E^*$ is relatively weakly compact and again as in the proof of Proposition 2.10 we can suppose that, for every $y \in Y$, $(h_n^*(y))_n \subset E^*$ is a weakly Cauchy sequence. Let now $x^{**} \in E^{**}$.

Claim 2: The set $\{h_n^{**}(x^{**}); n \in \mathbb{N}\} \subset F^*$ is a V-set.

Proof of the claim: Let us observe that it follows from Lemmas 2.8 and 2.9 that $AB(T)$ is unconditionally converging, hence separately unconditionally converging. So, proceeding as in the proof of Proposition 2.10 we get

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}|h_n^{**}(x^{**})(y_m)| \leq \lim_{m\to\infty}\|AB(T)(x^{**},y_m)\| = 0,$$

and the claim is proved.

Now we can proceed as in [11] to obtain $h \in \mathcal{K}(E;F^*)$ such that h_n weakly converges to h, which finishes the proof. $\square$

From Theorem 4.2 below it follows that Proposition 2.12 is strictly stronger than Theorem 2.3.

3. The projective tensor product of $C(K)$ spaces

This section was the original motivation of this paper. We apply now the results of the previous sections to obtain a classification of the projective tensor products of $C(K)$ spaces in terms of some of the classical Banach space properties.

It is known that the projective tensor product of Banach spaces is associative, that is, if E, F, G are Banach spaces, then $E\hat{\otimes}_\pi F\hat{\otimes}_\pi G = E\hat{\otimes}_\pi(F\hat{\otimes}_\pi G) = (E\hat{\otimes}_\pi F)\hat{\otimes}_\pi G$. We will make frequent use of this fact.

Recall that a compact Hausdorff space K is said to be *scattered* (or *dispersed*) if it does not contain any non void perfect set. In [19] it is proved, among other interesting results, that K is scattered if and only if $C(K)$ contains no copy of ℓ_1. In this case, $C(K)^*$ can be identified with $\ell_1(\Gamma)$ for some Γ and, consequently, it is a Schur space.

Theorem 3.1. *Let $k \geq 2$ and $K_1, \ldots, K_k$ be infinite compact Hausdorff spaces. Then, the following assertions are equivalent:*

(a_1) For every $i \in \{1,\ldots,k\}$, K_i is scattered.

(b_1) $C(K_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(K_k)$ has properties DP, RDP and V.

(c_1) $C(K_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(K_k)$ does not contain any isomorphic copy of ℓ_1.

(d_1) For any Banach space X, and any k-linear operator $T : C(K_1)\times\cdots\times C(K_k) \longrightarrow X$ the following are equivalent:

(1) T is completely continuous.

(2) T is unconditionally converging.

(3) T is weakly compact.

(4) T is regular.

(5) T is compact.

Proof. We will first prove that (a_1) implies all of the others. By a standard argument (see, for instance, the proof of [5, Theorem 3.1]), it can be proved that

$$(C(K_1)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(K_k))^* = C(K_1)^*\check{\otimes}_\epsilon\cdots\check{\otimes}_\epsilon C(K_k)^* =$$
$$= \mathcal{K}(C(K_1);(C(K_2)\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi C(K_k))^*)$$

Hence, $(C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*$ is a Schur space, and so, by a well known result of Pethe and Thakare, $X := C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ has the DPP and contains no copy of ℓ_1. Also, from the associativity of projective tensor products and Theorem 2.3 it follows that X has property V, and hence it also has the RDPP. So (b$_1$) and (c$_1$) hold.

Let us check (d$_1$): Compact multilinear operators are always weakly compact, and weakly compact multilinear operators are completely continuous on a product of spaces with the DP property. On $C(K)$ spaces, multilinear unconditionally converging and completely continuous operators, coincide ([15]). Since no $C(K_i)$ $(1 \leq i \leq k)$ contains copies of ℓ_1 by hypothesis, every completely continuous multilinear operator on $C(K_1) \times \cdots \times C(K_k)$ is compact. On $C(K)$ spaces, every regular multilinear operator is completely continuous ([5, Lemma 2.6]). Finally, reasoning as in [1] we can prove that, under the assumption (a$_1$), if T is completely continuous, it is weakly continuous on bounded sets ([1, Proposition 2.12]), hence regular ([1, Theorem 2.9]).

For the converse implications, let us notice that one and only one of the conditions (a$_1$), or (a$_2$), (a$_3$) (in Theorems 3.3 and 3.4 below) hold. Then, by exclusion, it is enough to prove that conditions (a$_i$) $(i = 1, 2, 3)$ imply all the others in Theorems 3.1, 3.3 and 3.4. □

Thanks are given to Joaquín Gutiérrez for his help on shortening the proof of (a$_1$) ⇒(d$_1$).

Corollary 3.2. *Let K be an infinite compact Hausdorff space and $2 \leq k \in \mathbb{N}$. Then, the following assertions are equivalent:*

(a) K is scattered.

(b) $\hat{\otimes}^k_{\pi,s} C(K)$ has properties DP, RDP and V.

(c) $\hat{\otimes}^k_{\pi,s} C(K)$ does not contain any isomorphic copy of ℓ_1.

(d) Any k-homogeneous unconditionally converging polynomial on $C(K)$ is weakly compact.

Proof. Reasoning as in the last part of the proof of Theorem 3.1, it suffices to prove that condition (a) implies all of the others in this Corollary and in Corollary 3.5. Hence, suppose (a) holds. Since $\hat{\otimes}^k_{\pi,s} C(K)$ is complemented in $\hat{\otimes}^k_{\pi} C(K)$, (b) and (c) follow. If a polynomial $P : C(K) \longrightarrow X$ is unconditionally converging, then its associated symmetric multilinear form T is also unconditionally converging. By Theorem 3.1, T is weakly compact, hence P is weakly compact. □

Theorem 3.3. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces. Then the following assertions are equivalent*

(a_2) There exists precisely one $i \in \{1, \dots, k\}$ such that K_i is not scattered (i.e., $C(K_i) \supset \ell_1$).

(b_2) $C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ has properties RDP and V, but it does not have the DP property.

(c_2) $C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ contains ℓ_1, but not complemented.

(d_2) For any Banach space X and any k-linear operator $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$, T is unconditionally converging (equivalently completely continuous) if and only if T is weakly compact, but there are weakly compact multilinear operators on $C(K_1) \times \cdots \times C(K_k)$ which are neither compact nor regular.

Proof. As mentioned before, we only need to show that (a_2) implies all of the others. Let us prove (b_2): The statement about the DP property can be seen in [6]. As for properties V and RDP, we will do it by induction on k. For $k = 2$ the result follows from Theorems 2.1 and 2.3. Let us suppose it true for $k - 1$, and let us suppose that $C(K_k) \supset \ell_1$. From the induction hypothesis it follows that $C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ has property V, hence it can not contain complemented copies of ℓ_1. Therefore, $(C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*$ does not contain copies of c_0. Then, every operator from $C(K_1)$ into $(C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*$ is compact. Now we use the associativity of the projective tensor product and Theorem 2.3.

Clearly $C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ contains a copy of ℓ_1. But since it has property V, none of such copies can be complemented. So, (b_2) implies (c_2).

Let us now see that (a_2) implies (d_2). Let us first see that, under such assumption, unconditionally converging multilinear operators on $C(K_1) \times \cdots \times C(K_k)$ are weakly compact. The proof is a refinement of the proof of Proposition 2.12. We apply induction on k. For $k = 2$ the result has already been proved. Let us suppose it true for $k - 1$, and let $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ be an unconditionally converging multilinear operator and let $\hat{T}$ its linearization. We define

$$S : C(K_1) \times (C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)) \longrightarrow X$$

by

$$S(f_1, y) := \hat{T}((f_1 \otimes y).$$

Clearly, S is bilinear and continuous, with $\|S\| = \|\hat{T}\| = \|T\|$. Let

$$\hat{S} : C(K_1)\hat{\otimes}_\pi C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k) \longrightarrow X$$

be the linear operator associated to S. Clearly, we just have to check that $\hat{S}^* : X^* \longrightarrow \mathcal{K}(C(K_1); (C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*)$ is weakly compact. As before, let $M = \hat{S}^*(B_{X^*})$, let $(\varphi_n)_n \subset B_{X^*}$ and let $h_n = \hat{S}^*(\varphi_n)$. We just need to extract a weakly converging subsequence from $(h_n)_n$. Let $H = \overline{\text{span}}[h_n(f_1) : f_1 \in C(K_1), n \in \mathbb{N}]$. Then H is a separable closed subspace of $(C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*$ As before, let $Y \subset C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ be a countable norming set of H and let $y \in Y$.

Let us prove that $S_y = S(\cdot, y) : C(K_1) \longrightarrow X$ is unconditionally converging. Since T is separately unconditionally converging, this is clear when $y = f_2 \otimes \cdots \otimes f_k$, and it follows readily for $y = \sum_{i=1}^n f_2^i \otimes \cdots \otimes f_k^i$. For the general case it suffices to take into account the density of $C(K_2) \otimes \cdots \otimes C(K_k)$ is $C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ and the fact that the canonical continuous linear map

$$C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k) \ni y \mapsto S_y \in \mathcal{L}(C(K_1), X)$$

takes values in the closed subspace of the unconditionally converging operators when $y \in C(K_2) \otimes \cdots \otimes C(K_k)$.

Using this, we can reason as in the proof of Proposition 2.12 to establish that the set $\{h_n^*(y); n \in \mathbb{N}\} \subset C(K_1)^*$ is a V-set.

So $\{h_n^*(y); n \in \mathbb{N}\} \subset C(K_1)^*$ is relatively weakly compact and we can suppose that, for every $y \in Y$, $(h_n^*(y))_n \subset C(K_1)^*$ is a weakly Cauchy sequence. Let now $z \in C(K_1)^{**}$.

<u>Claim</u> : The set $\{h_n^{**}(z); n \in \mathbb{N}\} \subset (C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k))^*$ is a V-set.

<u>*Proof of the claim*</u>: Let $\sum_n y_n \subset C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ be a w.u.C series. As in the proof of Proposition 2.12, we get that

$$|h_n^{**}(z)(y_m)| \leq \|AB(\hat{S})(z, y_m)\|.$$

T is unconditionally converging, hence so is $AB(T)$ (Lemma 2.9). Therefore $AB(T)_z : C(K_2) \times \ldots \times C(K_k) \longrightarrow X$ defined by

$$AB(T)_z(f_2, \ldots, f_k) = AB(T)(z, f_2, \ldots, f_k)$$

is unconditionally converging. Let $\hat{AB(T)}_z : C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k) \longrightarrow X$ be the linear operator associated to it. By the induction hypothesis, $\hat{AB(T)}_z$ is weakly compact, hence unconditionally converging. Clearly we have that, for every $y \in C(K_2)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$, $AB(S)(z, y) = \hat{AB(T)}_z(y)$ and so the claim follows.

Now we can again proceed as in [11] to finish the proof that unconditionally converging multilinear operators are weakly compact.

For a weakly compact, neither regular nor compact multilinear operator on $C(K_1) \times \ldots \times C(K_k)$ we proceed similarly to the proof of the main result of [6]: suppose that $C(K_1) \supset \ell_1$. Then, there exists a surjective operator $q : C(K_1) \longrightarrow \ell_2$ ([9, Corollary 4.16]). Let $(x_2^n) \subset C(K_2)$ and $(\mu_2^n) \subset B_{C(K_2)^*}$ be two sequences such that (x_2^n) converges weakly to 0 and so that $\mu_2^n(x_2^n) = 1$ for every $n \in \mathbb{N}$. Let us choose norm one elements $\mu_i \in C(K_i)^*$, $x_i^0 \in C(K_i)$ such that $\mu_i(x_i^0) = 1$ $(3 \leq i \leq k)$. Then we can consider the multilinear operator

$$T : C(K_1) \times \ldots \times C(K_k) \longrightarrow \ell_2$$

defined by

$$T(x_1, \ldots, x_k) = \left(q(x_1)_n \mu_2^n(x_2) \prod_{i=3}^{k} \mu_i(x_i) \right)_n$$

T is clearly weakly compact. Let $(x_1^n) \subset \|q\| B_{C(K_1)}$ be a sequence such that $q(x_1^n) = e_n$. The sequence $(x_1^n \otimes x_2^n \otimes x_3^0 \otimes \cdots \otimes x_k^0)_n \subset C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ converges weakly to 0, since $(x_1^n \otimes x_2^n)_n$ converges weakly to 0 in $C(K_1) \hat{\otimes}_\pi C(K_2)$ ([6, Lemma 2.1]). However,

$$T(x_1^n, x_2^n, x_3^0, \ldots, x_k^0) = 1,$$

for every n. So, $\hat{T}$ is not completely continuous. Hence, $\hat{T}$, and consequently T, can not be compact.

Moreover, the operator $T_1 : C(K_2) \longrightarrow \mathcal{L}^{k-1}(C(K_1), C(K_3), \ldots, C(K_k); \ell_2)$ associated to T is not completely continuous, because

$$\|T_1(x_2^n)\| \geq \|q\| \|T_1(x_2^n)(x_1^n, x_3^0, \ldots, x_k^0)\| = \|q\|.$$

So, T_1 is not weakly compact, i.e., T is not regular. $\square$

Finally, we consider the remaining possibility:

Theorem 3.4. *Let $K_1, \ldots, K_k$ be infinite compact Hausdorff spaces. Then, the following assertions are equivalent:*

(a$_3$) At least two of the spaces $K_1, \ldots, K_k$ are not scattered.

(b$_3$) $C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ does not have any of the properties DP, RDP and V.

(c$_3$) $C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ contains a complemented copy of ℓ_1.

(d$_3$) There exists a Banach space X and an unconditionally converging, multilinear operator $T : C(K_1) \times \cdots \times C(K_k) \longrightarrow X$ which is not weakly compact.

Proof. We just show that (a$_3$) implies all of the others. [13, Proposition 13] states that, if $E \supset \ell_1$, then the space of two homogeneous polynomials $\mathcal{P}(^2E)$ contains a copy of ℓ_∞. That proof can be easily modified to show that, if both E and F contain copies of ℓ_1, then $\mathcal{L}^2(E, F) \supset \ell_\infty$. These facts imply that, in that case, $E \hat{\otimes}_{\pi,s} E$ and $E \hat{\otimes}_\pi F$ contain complemented copies of ℓ_1. To finish the proof of (c$_3$) we just need to observe that $C(K_i) \hat{\otimes}_\pi C(K_j)$ is complemented in $C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$.

Let us suppose that (c$_3$) (and (a$_3$)) hold. Since ℓ_1 is Schur and not reflexive, the projection $\pi : C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k) \longrightarrow \ell_1$ is an example of a completely continuous (hence unconditionally converging) operator not weakly compact. So, $C(K_1)\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_k)$ has neither property V nor RDPP. The statement about the DP property can be seen in [6].

The multilinear operator $\tilde{\pi} : C(K_1) \times \cdots \times C(K_k) \longrightarrow \ell_1$ associated to π above, proves (d_3). □

Corollary 3.5. *Let K be an infinite compact Hausdorff space and $2 \leq k \in \mathbb{N}$. Then the following assertions are equivalent:*

(a) K is not scattered.

(b) $\hat{\otimes}^k_{\pi,s} C(K)$ does not have properties DP, RDPP and V.

(c) $\hat{\otimes}^k_{\pi,s} C(K)$ contains a complemented copy of ℓ_1.

(d) There exists a k-homogeneous unconditionally converging polynomial on $C(K)$ which is not weakly compact.

Proof. As we already said in Corollary 3.2, we only have to show that (a) implies all of the others. As above, (c) follows from [13, Proposition 13] and the fact that $\hat{\otimes}^2_{\pi,s} C(K)$ is complemented in $\hat{\otimes}^k_{\pi,s} C(K)$.

Let us prove (b): The statement about the DP property can again be seen in [6]. As before, the projection $\pi : \hat{\otimes}^k_{\pi,s} C(K) \longrightarrow \ell_1$ is a completely continuous and unconditionally converging operator not weakly compact.

The multilinear symmetric operator $\tilde{\pi} : C(K) \times \overset{(k)}{\cdots} \times C(K) \longrightarrow \ell_1$ associated to π immediately above proves (d). □

4. Relationships between some classes of multilinear operators and their linearization

Let us start this section with a simple and essentially known result which relates the complete continuity of a multilinear operator T and its linearization $\hat{T}$:

Proposition 4.1. *Let $E_1, \dots, E_k$ be Banach spaces. Then the following assertions are equivalent:*

(a) For every Banach space X, if $\hat{T} : E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k \longrightarrow X$ is completely continuous then its multilinear associated operator $T : E_1 \times \cdots \times E_k \longrightarrow X$ is completely continuous.

(b) Every multilinear form $\varphi \in \mathcal{L}^k(E_1, \dots, E_k)$ is weakly sequentially continuous.

(c) If $(x_i^n)_n \subset E_i$ $(1 \leq i \leq k)$ is a weakly Cauchy sequence, then $(x_1^n \otimes \cdots \otimes x_k^n)_n \subset E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k$ is weakly Cauchy.

Moreover, if $k = 2$,

(d) $\mathcal{L}(E_1; E_2^*) = \mathcal{L}_{cc}(E_1; E_2^*)$

implies all of the others.

Proof. The equivalence between (a), (b) and (c) is very easy and can be left to the reader. Under the hypothesis (d), if $(x_1^n) \subset E_1$ is a weakly Cauchy sequence and $(x_2^n) \subset E_2$ is bounded, then $x_1^n \otimes x_2^n \subset E_1 \hat{\otimes}_\pi E_2$ is weakly Cauchy, which is (c). $\square$

So we see that if $\hat{T}$ is completely continuous, T does not need to be completely continuous. Conversely if T is completely continuous, $\hat{T}$ need not be completely continuous, even if E and F have DP (see [6]).

We study now the relation between T being an unconditionally converging multilinear operator and $\hat{T}$ being an unconditionally converging linear operator. From the previous section it follows that, for every Banach space X and $k \in \mathbb{N}$, $T \in \mathcal{L}^k(c_0; X)$ is unconditionally converging if and only if $\hat{T} \in \mathcal{L}(c_0 \hat{\otimes}_\pi \overset{(k)}{\cdots} \hat{\otimes}_\pi c_0; X)$ is unconditionally converging (if and only if both of them are weakly compact).

As a consequence we have

Theorem 4.2. *Let $E_1, \dots, E_k$ and X be Banach spaces. If the operator $\hat{T} : E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k \longrightarrow X$ is unconditionally converging, then $T : E_1 \times \cdots \times E_k \longrightarrow X$ is also unconditionally converging.*

Proof. Let $\hat{T} : E \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k \longrightarrow X$ be unconditionally converging. If T is not unconditionally converging, then there exist w.u.C series $\sum_n x_1^n \subset E_1$, $\dots$, $\sum_n x_k^n \subset E_k$ such that $(T(\sum_{n=1}^m x_1^n, \dots, \sum_{n=1}^m x_k^n))_m$ is not a Cauchy sequence. Let $i_i : c_0 \longrightarrow E_i$ be defined by $i_i(e_n) = x_i^n$ $(1 \leq i \leq k)$. Then, the multilinear operator

$$V : c_0 \times \overset{(k)}{\cdots} \times c_0 \longrightarrow X$$

defined by

$$V(y_1, \dots, y_k) = T(i_1(y_1), \dots, i_k(y_k))$$

is not unconditionally converging. Hence, $\hat{V}$ is not unconditionally converging. On the other hand, it is easy to check that $\hat{V} = \hat{T} \circ (i_1 \otimes \cdots \otimes i_k)$ which is unconditionally converging by the hypothesis. This contradiction finishes the proof. $\square$

The converse of Theorem 4.2 is not true, as the following example shows.

Example 4.3. In [7], the authors provide an example of a Banach space X with the RNP (hence it does not contain c_0) such that $X \hat{\otimes}_\pi X$ contains c_0. Let us consider the operator

$$\gamma : X \times X \longrightarrow X \hat{\otimes}_\pi X$$

defined by

$$\gamma(x, y) = x \otimes y.$$

Since $X \not\supset c_0$, γ is unconditionally converging (see [4, Proposition 2.10]), but $\hat{\gamma}$, which is the identity on $X\hat{\otimes}_\pi X$, is not unconditionally converging, since it fixes a copy of c_0.

We have not been able to find a less exotic example. Let us observe that one can prove that every bilinear operator $T : \ell_\infty \times \ell_\infty \longrightarrow c_0$ is unconditionally converging, so any not unconditionally converging operator (for example any projection) $P : \ell_\infty\hat{\otimes}_\pi\ell_\infty \longrightarrow c_0$ would provide a more familiar counterexample. We have not been able to find such object, in particular we do not know if $\ell_\infty\hat{\otimes}_\pi\ell_\infty$ contains complemented copies of c_0.

From Theorem 4.2 it follows easily that, if every unconditionally converging bilinear operator on $E \times F$ is weakly compact, then $E\hat{\otimes}_\pi F$ has property V. We do not believe the converse to be true, but we do not have a counterexample. For a wide variety of spaces the converse *is* true. For $C(K)$ spaces, this follows from Section 3, but more generally we have

Proposition 4.4. *Let E and F be Banach spaces such that $E\hat{\otimes}_\pi F$ has property V. Assume further that at least one of the following conditions holds:*

1. *E^* or F^* has the metric approximation property*
2. *E or F has an unconditional compact expansion of the identity*
3. *E^* or F^* has the compact approximation property and is a subspace of a Banach space Z possessing an unconditional compact expansion of the identity*

Then every unconditionally converging bilinear operator on $E \times F$ is weakly compact.

Proof. Clearly, both E and F have property V. Moreover, [11, Theorem 7] states that, in the hypothesis, $\mathcal{L}(E; F^*) = \mathcal{K}(E; F^*)$ (or $\mathcal{L}(F; E^*) = \mathcal{K}(F; E^*)$). Now, Proposition 2.12 applies. □

For ℓ_p spaces we can precise this last result a little more

Proposition 4.5. *Let $1 < p_i < \infty$. Then the following are equivalent:*

(a) $\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_k}$ *has property V*

(b) $\mathcal{L}(\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_{k-1}}; \ell_{q_k}) = \mathcal{K}(\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_{k-1}}; \ell_{q_k})$*, where* $\ell_{p_k}^* = \ell_{q_k}$.

(c) $\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_k}$ *is reflexive*

(d) $\sum_{i=1}^k \frac{1}{p_i} < 1$

(e) Every unconditionally converging multilinear operator on $\ell_{p_1} \times \cdots \times \ell_{p_k}$ *is weakly compact*

Proof. If (a) holds, then (b) follows from [11, Theorem 7]. The equivalence between (b), (c) and (d) can be seen in [3, Section 4].

Let us observe that, in general, if $E_1, \ldots, E_k$ are reflexive spaces, then $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k$ is reflexive if and only if every unconditionally converging multilinear operator defined on $E_1 \times \cdots \times E_k$ is weakly compact. For the non trivial implication of this statement, it suffices to realize that the multilinear operator

$$T : E_1 \times \cdots \times E_k \longrightarrow E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_k$$

given by

$$T(x_1, \ldots x_k) = x_1 \otimes \cdots \otimes x_k$$

is unconditionally converging (this can be seen, for instance, applying [4, Proposition 2.10] as in Example 4.3). Therefore, (c) and (e) are equivalent. We already mentioned before that (e) implies (a) always, i.e., not only for ℓ_p spaces. □

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