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Exact results on quantum many-body systems in one dimensión

Resultados exactos sobre modelos cuánticos de muchos cuerpos en una dimensión

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under the supervision of

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Exact Results on Quantum Many-Body Systems in one Dimension
Resultados Exactos sobre Modelos Cuánticos de Muchos Cuerpos en una Dimensión

y dirigida por: Federico Finkel Morgenstern, Artemio González López, Piergiulio Tempesta

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Esta DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD debe ser insertada en la primera página de la tesis presentada para la obtención del título de Doctor.

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Acknowledgments

This dissertation is built from the joint work referred below as List of publications. I want to stress my gratitude to each of the coauthors for their respective contributions.

Summary

In the course of the completion of this thesis we have addressed several problems concerning quantum many-body systems and one-dimensional lattice models. Before getting into details, let me briefly outline some of them together with our main contributions, and sketch out some central concepts we shall develop later.

To begin with, we have studied the entanglement properties of the ground state of several one-dimensional systems (see Refs. [**P1**]–[**P4**]), and investigated several exact spectral properties of lattice models in one dimension, obtaining an analytic expression for the partition function of certain models with long-range interactions (see [**P4**] and [**P5**] for a detailed description of the models involved). We have completely characterized the spectrum of the so-called long range t-J model for the first time and, in particular, rigorously proved a conjecture proposed by Saiga and Kuramoto in the nineties for the spin 1/2 case. We first showed that the spectrum can be obtained from the partition function of the Hamiltonian of the supersymmetric long range Haldane–Shastry spin chain restricted to a specific subspace [**P6**]. The latter equivalence was then fully exploited in [**P7**] to obtain the complete asymptotic series for the free energy per unit length. As a byproduct, a new conjecture arises from the relation of our approach with that of Kuramoto and Kato in the thermodynamic limit. Futhermore, in [**P8**] we have studied non-relativistic quantum nay-body systems in one dimension admitting a Jastrow-like graund-state function, where we showed how they can be classified together with their parent Hamiltonians. This result is the short-range counterpart of the well-known classification of long-range models of Calogero–Sutherland type carried out by Koprucki and Wagner. Finally, we have also worked on some problems related to new algebraic structures associated to entanglement detection from a mathematiccal point of view. More precisely:

1. For a free fermion system, the entanglement entropy depends essentially on two sets, namely the set u of sites of the subsystem considered

and the set v of excited momentum modes. In Ref. [**P1**] we make use of a general duality principle establishing the invariance of the entanglement entropy under exchange of the sets u and v to tackle complex problems by studying their dual counterparts. The duality principle is also a key ingredient in the formulation of a novel conjecture for the asymptotic behavior of the entanglement entropy of a free fermion system in the general case in which both sets u and v consist of an arbitrary number of blocks. In Refs. [**P2**]-[**P3**] we study a large class of su(1|1) supersymmetric spin chains with a general (not necessarily monotonic) dispersion relation. We compute the Rényi entanglement entropy of the ground state and deduce that the models considered have a critical phase in the same universality class of a $(1+1)$-dimensional conformal field theory (CFT) with central charge equal to the number of connected components of the Fermi sea. Our results confirm the widely believed conjecture that the critical behavior of fermionic lattice models is completely determined by the topology of their Fermi surface.

2. A new class of generalized Lipkin–Meshkov–Glick (gLMG) [**56**, **89**, **92**] models is introduced in Ref. [**P4**] with su(m) spin and long range nonconstant interactions, whose nondegenerate ground state is a Dicke state of su(m) type for an arbitrary integer $m \geq 2$, thus generalizing the original su(2) isotropic LMG model. We evaluate in closed form the reduced density matrix of a block of spins when the whole system is in its ground state, and study the corresponding von Neumann and Rényi entanglement entropy. In particular, our results show that none of these gLMG models are critical. We also prove that the region in parameter (chemical potential) space in which all the magnon densities in the ground state are nonvanishing is an m-simplex in $\mathbf{R}^{m-1}$, whose vertices are the weights of the fundamental representation of su(m). In Ref. [**P5**] we study a large class of gLMG models with su(m) interactions of Haldane–Shastry type [**62**, **112**]. We compute the partition function of these models in closed form by exactly evaluating the partition function of the restriction of the Haldane–Shastry type Hamiltonian to subspaces with well-defined magnon numbers. As a byproduct of our analysis, we obtain numerical evidence of the Gaussian character of the level density of eigenvalues also for the restriction of the Hamiltonian to the latter subspaces, and study the distribution of the spacings of consecutive unfolded levels therein.

3. We consider a one-dimensional lattice model with N equally spaced sites, each of which can be either empty or occupied by a single su(m) fermion. When $m = 2$ and the interactions among the fermions and

their hopping amplitude involve only nearest-neighbor sites we recover the original t-J model introduced in [**109**]. In the same way, the long-range supersymmetric model introduced by Kuramoto and Yokoyama [**86**, **87**] is obtained when the interaction strength and the hopping amplitude between any two particles i and j are proportional to $(\sin u_{ij})^{-2}$ with $u_{ij} = \pi|i-j|/N$. We have introduced and studied [**P6**] more general su(m) t-J models with long-range interactions for arbitrary $m > 2$, and in particular, obtained a description of the system in terms of an appropriate supersymmetric spin chain. The results of [**52**] can then be used to compute the partition function of the system and determine its thermodynamic behavior. Indeed, in [**P7**] we follow this approach and obtain the whole asymptotic series expansion of the relevant thermodynamic quantities of the model. This enables us to corroborate for the first time the presence of strong spin-charge separation to all orders and to prove the Saiga–Kuramoto conjecture [**107**] on the spectrum of these models. Our method provides exact results for a finite number of particles, and its advantage over the usual method based on the asymptotic Bethe ansatz in the thermodynamic limit relies on the existence and uniqueness of the Perron–Frobenius eigenvalue of a certain site-dependent transfer matrix. It also clarifies the heuristic previous attempts to describe the spectrum of the KY model for a large number of particles, due essentially to Kuramoto and Kato [**81**]. In particular, the intuition of the latter authors is intriguingly vindicated and actually led us to propose a novel conjecture which we have verified in some cases.

4. In Ref. [**P8**] we consider the Schrödinger operator for a quantum many-body problem in one dimension (both on the circle and the real line) and look for translation-invariant potentials involving only interactions between nearest- and next-to-nearest-neighbors such that the ground state of the system is a Jastrow-like wave function (i.e., a product of the values of a fixed function evaluated at the successive differences of the coordinates of nearest-neightbor particles). We classify all such potentials and their respective Jastrow-like ground states in terms of an essentially arbitrary function and a real parameter, thus providing the short range counterpart of the work of Koprucki and Wagner [**84**]. The class of Jastrow-like ground states can be thought of as continuous matrix product states (cMPS) of rank one, and their associated Schrödinger operator as a parent Hamiltonian (see [**126**] for the definition of cMPS). These parent Hamiltonians

turn out to coincide with that of the truncated[1] versions of the scalar models of Calogero–Sutherland type (see, e.g., [**32**, **113**, **114**]).

5. Finally, we investigate [**P9**] whether the logarithmic negativity, a quantity which is well-defined and meaningful in general mixed states [**127**] (not only for pure states), admits a generalization as a fundamental information measure. In fact, a new multi-parametric class of generalized negativity measures can be introduced. These generalized negativities turn out to be measures of entanglement possessing a group-theoretical structure (see [**119**, **120**]).

[1] We tried to restrict them to just nearest-neighbors interactions and found it impossible. Our results rigorously show that such a solution cannot exist, thus explaining why all previously known potentials of this type contain terms involving next-to-nearest neighbors interactions.

Resumen

Durante el desarrollo de esta tesis, hemos abordado diferentes problemas relacionados con sistemas cuánticos de muchos cuerpos y cadenas de espines o modelos discretos en una dimensión. Intentaremos resumir brevemente las palabras e ideas clave asociadas a cada uno de ellos así como nuestras contribuciones a la resolución de los mismos.

En primer lugar, hemos estudiado el entrelazamiento del estado fundamental de varios sistemas unidimensionales (ver [**P1**]–[**P4**]) e investigado distintas propiedades exactas relativas al espectro de cadenas de espines, obteniendo una expresión analítica para la función de partición de ciertos modelos con interacciones de largo alcance (ver [**P4**] y [**P5**] para una descripción detallada de los modelos tratados). Hemos conseguido caracterizar completamente el espectro de energías del modelo t-J de largo alcance introducido por Kuramoto y Yokoyama, probando en particular por primera vez de manera rigurosa una conjetura propuesta por Saiga y Kuramoto en los años noventa. Para ello primero demostramos en [**P6**] que el espectro se puede obtener a partir de la función de partición asociada a una modificación de la cadena de espines de Haldane–Shastry cuando su hamiltoniano es restringido a un subespacio específico. Esta última equivalencia es aprovechada en [**P7**] para obtener la energía libre por unidad de longitud en forma de serie asintótica completa y, de manera adicional, formular una nueva conjetura que surge al comparar nuestro enfoque con el de Kuramoto y Kato en el límite termodinámico. También fueron estudiadas en [**P8**] las funciones de onda del estado fundamental en un sistema cuántico no relativista de muchos cuerpos en una dimensión, tanto en la recta real como en la circunferencia unidad. Clasificamos todos los hamiltonianos cuyo estado fundamental es de tipo Jastrow (el producto de una función fija evaluada en las sucesivas diferencias de las coordenadas asociadas a cada par de partículas inmediatamente adyacentes). Esta clasificación de modelos de corto alcance es análoga a la llevada a cabo por Calogero, Inozemtsev, Koprucki y Wagner para modelos de largo

alcance de tipo Calogero o Sutherland. Por último, hemos trabajado en problemas relacionados con nuevas estructuras algebraicas asociadas a la detección y medida de entrelazamiento desde un punto de vista formal. Concretamente:

1. Para un sistema de fermiones libres, la entropía de entrelazamiento depende esencialmente de dos conjuntos, a saber, el conjunto de posiciones u del subsistema considerado y el conjunto v de los modos de excitación activos en el espacio de momentos. En [**P1**] hacemos uso de un principio general de dualidad que establece la invariancia de la entropía de entrelazamiento ante el intercambio de los conjuntos u y v para abordar problemas complejos mediante el estudio de sus equivalentes duales. El principio de dualidad es también un ingrediente clave en la formulación de una nueva conjetura para el comportamiento asintótico de la entropía de entrelazamiento de un sistema de fermiones libres en el caso general en que tanto u como v constan de un número arbitrario de bloques (componentes conexas). En las referencias [**P2**]-[**P3**] estudiamos una clase amplia de cadenas de espines supersimétricas, asociadas a la superálgebra de Lie su(1|1), con una relación de dispersión genérica (no necesariamente monotóna creciente o monótona decreciente). Calculamos la entropía de entrelazamiento de Rényi del estado fundamental y deducimos que los modelos considerados tienen una fase crítica en la misma clase de universalidad que las teorías de campo con simetría conforme (CFT) en dimensión $1+1$ y con carga central igual al número de componentes conexas del mar de Fermi. Nuestros resultados confirman la conjetura, ampliamente aceptada, de que el comportamiento crítico de modelos fermiónicos en una dimensión está completamente determinado por la topología de su superficie de Fermi.

2. Una nueva clase de modelos de Lipkin–Meshkov–Glick [**56**, **89**, **92**] generalizados (modelos gLMG) para partículas de espín su(m) con interacciones no constantes de largo alcance y cuyo estado fundamental no degenerado es un estado de Dicke de tipo su(m) para enteros arbitrarios $m \geq 2$, generalizando por lo tanto el modelo su(2) original. Obtenemos expresiones en forma cerrada para la matriz densidad reducida de un bloque de espines cuando el sistema completo se halla en el estado fundamental. En particular, nuestros resultados muestran que estos modelos gLMG no son críticos, aunque su entropía de von Neumann escala logarítmicamente con el tamaño del bloque considerado. Demostramos que la región del espacio de parámetros en la cual ninguna de las densidades de magnones es nula se corresponde con un m-símplice en $\mathbf{R}^{m-1}$ cuyos vértices son los

pesos de la representación fundamental de su(m). En [**P5**] estudiamos una clase de modelos su(m) gLMG que pueden ser considerados una deformación de la famosa cadena de espines introducida independientemente por Haldane [**62**] y Shastry [**112**] a finales de los años ochenta. La evaluación exacta de las funciones de partición de las diferentes restricciones del hamiltoniano de esta cadena de espines a subespacios con número de magnones bien definido, nos permite calcular la función de partición de tales modelos gLMG en forma cerrada. Como corolario de nuestro análisis, obtenemos evidencia númerica del carácter gaussiano de la densidad de niveles del espectro de la cadena de Haldane–Shastry restringido a subespacios con contenido fijo de magnones, y estudiamos la distribución normalizada de espaciados entre niveles de energía consecutivos en dichos subespacios.

3. En [**P6**] y [**P7**] se parte de modelos cuánticos en una red unidimensional con N nodos equiespaciados, cada uno de los cuales puede estar o bien vacío o bien ocupado por un único fermión de tipo su(m). Cada uno de estos fermiones puede saltar de su posición a otra cualquiera, si ésta está vacía, con una cierta amplitud de probabilidad, y además los fermiones interactúan debido tanto a su espín como a su carga. Cuando $m = 2$, y tanto las interacciones como las amplitudes de salto entre dos nodos involucran solo el par de sitios inmediatamente vecinos, recuperamos el modelo t-J original introducido en [**109**]. De la misma manera, el modelo supersimétrico de largo alcance de Kuramoto y Yokoyama [**86**,**87**] (modelo KY) se obtiene cuando la interacción y la amplitud correspondientes a las posiciones i y j son proporcionales a $(\sin u_{ij})^{-2}$, donde $u_{ij} = \pi|i-j|/N$. Hemos estudiado modelos t-J de tipo su(m) para enteros arbitrarios $m \geq 2$ obteniendo en particular una descripción del sistema en términos de operadores de intercambio de espines supersimétricos. A partir de aquí se pueden usar los resultados de [**52**] para calcular la función de partición del sistema y determinar su comportamiento en el límite termodinámico. De hecho, en [**P7**] seguimos este enfoque y obtenemos las cantidades termodinámicas relevantes en forma de series asintóticas completas. Esto nos ha permitido corroborar por primera vez la presencia de separación carga-espín en sentido estricto para todos los órdenes del desarrollo asintótico así como demostrar de manera rigurosa la conjetura de Saiga y Kuramoto [**107**] acerca del espectro de estos modelos. Nuestro método no solo proporciona resultados exactos para un número finito N de posiciones, sino que además es especialmente eficiente en el límite termodinámico, donde la energía libre es calculada con ventaja debido esencialmente a la

existencia y unicidad del autovalor de Perron–Froebenius de una matriz de transferencia apropiada. En estudios previos se había intentado derivar el espectro del modelo KY para un número grande de partículas. Especialmente relevante a este respecto es el trabajo [**81**] de Kuramoto y Kato (KK) donde, mediante argumentos heurísticos e ingenio, consiguen deducir las cantidades termodinámicas. Nuestros resultados confirman los de KK, cuya intuición resulta absolutamente correcta de manera casi inexplicable. Tanto es así que esta coincidencia nos llevó a proponer una nueva conjetura basada en la relación de nuestro método con el de KK, que hemos verificado ya en algún caso.

4. En [**P8**] consideramos el operador de Schrödinger para el problema de muchos cuerpos en una dimensión y buscamos potenciales invariantes bajo traslaciones que involucren interacciones a lo sumo entre pares de partículas contiguas, de manera tal que el estado fundamental del sistema está dado por una función de onda de tipo Jastrow, obtenida como el producto de los valores que toma una función dada, cuando es evaluada en las sucesivas diferencias de las coordenadas de cada par de partículas inmediatamente vecinas. Clasificamos todos esos potenciales y sus respectivos estados fundamentales de tipo Jastrow en términos de una función que resulta ser esencialmente arbitraria y un parámetro real, proporcionando así la contrapartida de corto alcance del famoso resultado debido a Calogero, Sutherland, Koprucki y Wagner [**84**] relativo a la clasificación de modelos de largo alcance de tipo Calogero–Sutherland. Es interesante observar que los estados tipo Jastrow que sustancian nuestra clasificación pueden ser considerados como el caso de rango uno desde el punto de vista de los estados introducidos en [**126**] conocidos como cMPS[2]. Cada cMPS de rango uno es pues el estado fundamental de un operador de Schrödinger que coincide con la versión de corto alcance, o truncada, de los modelos clásicos de tipo Calogero y Sutherland (ver, e.g., [**32**, **113**, **114**]).

5. Finalmente, también hemos investigado [**P9**] bajo qué circunstancias la negatividad logarítmica, que está bien definida no solo para estados puros sino también para estados mezcla en general, admite generalizaciones como medida de la información a nivel fundamental. Introducimos nuevas negatividades generalizadas que resultan ser útiles como medidas

[2]cMPS son las siglas de “continuous matrix product state”, es decir, estados construidos como la traza del producto de ciertas matrices que cambian de un punto a otro de manera continua.

de entrelazamiento y con características propias de la teoría de grupos formales (ver [**119**, **120**]).

Introduction

The main purpose of this dissertation is the analysis of quantum many-body models characterized by a certain degree of integrability, in order to formulate some exact predictions and relate them with other fields of current interest in theoretical physics or modern mathematics like random matrices, quantum entanglement or group representation theory. The relevance of solvable or integrable models is, on the one hand, due to their universality. Indeed, a large class of physical systems can be approximately described through suitable solvable models. On the other hand, the wealth of mathematical properties associated to the notion of solvability (both classical and quantum) makes these interesting enough to explain their presence in multiple areas of physics and mathematics.

A class of solvable models which plays a central role in this thesis is that of spin chains of Haldane–Shastry type [**50**, **62**, **102**, **112**]. They are related to superconductivity at high temperature via the one-dimensional Hubbard model, and are also important in the characterization of quantum chaos versus integrability, via the Berry–Tabor conjecture for generic integrable quantum systems [**20**] or that of Bohigas–Giannoni–Schmit for completely chaotic ones described by an appropriate ensamble of random matrices [**25**]. In general, spin chains of this type can be obtained from Calogero–Sutherland [**32**, **33**, **102**, **113**, **114**, **116**] (CS) spin dynamical models in the strong coupling limit. In the latter limit, spin degrees of freedom of the CS model decouple from dynamical ones due to the fact that particles become frozen at the coordinates of the equilibrium point of the scalar (dominant) part of the interaction potential [**100**]. In other words, in this limit the kinetic energy eventually becomes negligible compared to the potential one, so that the particles freeze at the points that will become the corresponding chain sites. Due to this close relation, most of the exceptional properties of CS models regarding their solvability or integrability are inherit by spin chains of Haldane–Shastry type.

As pointed out by Haldane [63], the spinon excitations of the Haldane–Shastry spin chain provide one of the simplest examples of a quantum system featuring fractional statistics (see also [57, 58, 65]). On the other hand, the Haldane–Shastry chain is closely connected to important conformal field theories (CFTs) like the $k = 1$ Wess–Zumino–Novikov–Witten model [14, 63], and has recently been related to infinite matrix product states [37]. Integrable extensions of the Haldane–Shastry chain with long-range interactions involving more than two spins also play a key role for describing non-perturbatively the spectrum of planar $\mathcal{N} = 4$ gauge theory in the context of the AdS-CFT correspondence [9, 10].

The interest in exactly solvable spin chains of Haldane–Shastry type has further increased due to the recent developments in quantum simulation, as witnessed by the proposal of an experimental realization of the Haldane–Shastry spin chain using two internal states of trapped atoms in a photonic crystal waveguide [72]. In fact, quantum many-body systems are becoming increasingly popular in the field of quantum computation and simulation, due to the their wide range of practical applications. In particular, the amount of entanglement in the ground state has proved to be a valuable resource, and not just a theoretical question, in every quantum many-body system. The (bipartite) entanglement entropy of a pure state defined as the entropy of the density matrix of a subsystem, is widely used as an entanglement measure in this case [94]. Indeed, many one-dimensional quantum critical systems are effectively described by CFTs in $1 + 1$ dimensions. Near the critical point correlations become long-ranged and consequently the entanglement entropy diverges. However, its asymptotic scaling behavior is universal, and thus characteristic of critical many-body systems in low dimensions [41, 124]. Rather unexpectedly, this behaviour can be observed and its knowledge has inspired new ideas in important open problems such as the validation and/or verification of a quantum computation [78, 79, 90].

Quantum simulations are possible in some specific cases. Even though the exponential growth of the needed resources makes it a really challenging, when not unfeasible, task there are important physical phenomena whose properties can be efficiently unraveled with quantum computations, namely quantum phase transitions [106]. Efficiency here essentially means that a property of a quantum system of, say, n spin-$\frac{1}{2}$ particles is simulated in a quantum computation on a system with less than n qubits. For the Ising [85] or XY [26, 27] models, for instance, it has been possible

to observe their respective phase transitions and measure some physical quantities at the critical point, just by making computations with a number of ~ 10 qubits. The latter experiments are quite impressive considering that critical phenomena appear only in the thermodynamic limit. Indeed, no classical computer would be able to do the same calculation with the same resources (space or memory and time). In fact, the study of exact properties of quantum many-body systems even if they do not exist in nature is useful in order to design protocols to test the quantumness of a system [77].

On the other hand, the conceptual simplicity of exactly solvable quantum many-body systems in one dimension often makes it possible to derive exact analytical expressions for the relevant physical quantities. This fact has caused these systems to be widely used as proving grounds for new ideas, not only in the theory of critical phenomena or, more recently, in quantum computation, but also in other areas of science, most notably in condensed matter physics in general. This explains in part the reason why most of the work in this field historically has concentrated on systems with short-range interactions, like the already mentioned Heisenberg, Ising or short-range Hubbard models. In the last few years, however, it has become feasible to realize in the laboratory quantum spin chains featuring various types of long range interactions through different experiments involving, e.g., optical lattices of ultracold Rydberg atoms and trapped ions, or neutral atoms in optical cavities [80, 82, 103, 104, 108]. In particular, with the help of the so-called hyperfine "clock states" of trapped ions it has been possible to simulate quantum spin chains for which the coupling between two sites is inversely proportional to a power $\alpha \in (0, 3)$ of their distance [103, 104]. The interactions of the Haldane–Shastry chain [62, 112] are precisely of this type, since its spins are equispaced on the unit circle, with a coupling inversely proportional to the square of the (chord) distance. In point of fact, this chain is a limiting case of a more general model due to Inozemtsev, for which the coupling h_{ij} between particles i and j is an elliptic function of the difference $i - j$ with real period N [73].

Though in the original spin chain introduced by Haldane and Shastry each particle carried spin $\frac{1}{2}$, the model was shortly generalized to su(m) spin without losing its integrability properties [65]. This fact suggests a deep relation between this chain and the representation theory of the symmetric group [18]. In fact, the su($m|p$) supersymmetric version of it

originally introduced by Haldane [64], has also been studied in the literature [12, 14]. Particularly interesting is the su(1|m) case (with $m > 1$) due to its equivalence to a t-J model [8, 109, 116] with su(m) fermions. For any pair of sites within the circle in the t-J model, the exchange and hopping amplitudes are both inversely proportional to the square chord distance between sites. We shall see that this chain, first introduced by Kuramoto and Yokoyama in the su(2) case [87], is an exactly solvable model providing one of the simplest realizations of spin-charge separation.

We start by introducing and studying a wide class of su(1|1) supersymmetric spin chains with general translation-invariant couplings $h_{ij} > 0$ and a chemical potential term. For zero chemical potential $\mu \in \mathbf{R}$, these models include in particular the supersymmetric elliptic chain studied in Ref. [51] and its two limiting cases, namely the su(1|1) Haldane–Shastry chain and the XX model. These models can be transformed into a system of free spinless fermions in a non-standard way using the properties of the supersymmetric spin exchange operator. This fact, which is a characteristic property of the su(1|1) models, makes it possible to examine a number of key properties in the theory of quantum critical systems in an analytic fashion.

More precisely, we study whether these models are quantum critical for suitable values of the chemical potential, and determine the central charge of the associated Virasoro algebra. As is well known, a characteristic feature of $(1+1)$-dimensional CFTs is the fact that at low temperature T their free energy per unit length is approximately given by

$$f = f_0 - \frac{\pi c}{6 v_{\mathrm{s}}} T^2$$

where f_0 is a constant, v_{s} is the effective speed of the conformal fields of the theory [2, 21] (usually called just the speed of sound) and c is the central charge associated to the Virasoro algebra. Since the low temperature behavior of f is determined by the low-lying states of the spectrum, the latter equation should also hold for any one-dimensional quantum system whose low energy levels are described by a CFT in $1+1$ dimensions. In particular, the determination of the low temperature behavior of the free energy of a one-dimensional critical model provides an efficient way of determining the central charge of its underlying CFT. In this way we have been able to show that if the dispersion relation $\mathcal{E}(p)$ is monotonic when p is in the range $[0, \pi]$, then the su(1|1) models under study are critical

when the chemical potential μ belongs to the open interval $(0, \mathcal{E}(\pi))$, with central charge $c = 1$ (free boson CFT).

As further confirmation of this result, we have studied the ground state entanglement entropy, i.e., the entropy of the reduced density matrix of a block of L consecutive spins when the whole chain is in its ground state. Indeed, it is well known that in a CFT in two dimensions the Rényi and von Neumann entanglement entropies scale as $\frac{c}{6}(1 + \frac{1}{n}) \log L$ and $\frac{c}{3} \log L$, respectively, where n is the Rényi parameter [28,29,69]. Thus the entanglement entropy of a critical quantum system in one dimension should be proportional to $\log L$ for $L \gg 1$, the proportionality constant fixing the central charge. Again, we have verified that when the chemical potential belongs to the open critical interval $(0, \mathcal{E}(\pi))$ the entanglement entropy of the models under consideration scales as that of a CFT in $1 + 1$ dimensions with $c = 1$. We have also examined the behavior of the entanglement entropy and the zero-temperature fermion density as μ approaches the endpoints of the critical interval, showing that it is consistent with a quantum phase transition at both ends. For the su(1|1) chain with elliptic interactions we have studied numerically the fermion density at finite temperature, finding that its behavior is far more complex: when the chemical potential lies in the critical interval, the fermion density as a function of the temperature can present up to two extrema.

One realizes that the entanglement entropy crucially depends on the Fermi sea (the set of excited modes in the ground state); precisely, it depends on whether this set is connected or not. This fact motivated us to consider dispersion relations allowing for more complex Fermi "surfaces" (boundary of the Fermi sea) which, as we show, can appear in systems whose interactions are both short- and long-ranged whose dispersion relation is not monotonic. In general, the entanglement entropy of the ground state of these models grows logarithmically with the size L of the subsystem, with a constant prefactor determined by the number of boundary points of the Fermi surface in $[0, 2\pi)$. This logarithmic scaling is a manifestation of the so-called area law, which is believed to hold for critical fermionic systems in an arbitrary number of dimensions [41]. We have shown that the su(1|1) supersymmetric chains studied do indeed satisfy the area law. More precisely, by analyzing the low-temperature behavior of the free energy we first show that the models under consideration are critical for $\mathcal{E}_{\min} < \mu < \mathcal{E}_{\max}$, where $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ respectively denote

the minimum and maximum values of the dispersion relation[3]. From the latter analysis it also follows that the central charge of these models is equal to the number of disjoint intervals that make up the Fermi sea. For instance, the Rényi entanglement entropy behaves as $k_n \log L + C_n$ when $L \gg 1$. We shall explicitly compute the (non-universal) constant C_n, and prove that the prefactor k_n is equal to $\frac{1}{6}(1 + \frac{1}{n})\tilde{c}$ where $\tilde{c}$ is the number of connected components of the Fermi sea. This is in agreement with the value of the central charge deduced from the low-temperature analysis of the free energy, and once again confirms the conjecture that the entanglement properties of critical fermion models are entirely determined by the topology of their Fermi surface [**41**].

As the final step in our analysis of su(1|1) spin chains (or equivalently free fermion systems) initiated with the elliptic chain and then extended to general (not necessarily monotonic) dispersion relations, we have extended the previous investigations to the more general case in which both the state of the system and the topology of its subsystem (of size L) considered are respectively made up of several blocks of consecutive excited momentum modes and sites respectively. This makes it possible to treat position and momentum space on a more equal footing, and turns out to reveal symmetries that remained unnoticed so far. This approach naturally leads to a novel duality principle for the behavior of the entanglement entropy under the exchange of the position and momentum space block configurations, which in fact can be exploited to solve problems that up until now had defied an analytic treatment [**4**] with standard techniques like the Fisher–Hartwig conjecture [**53**]. We have applied this duality principle to propose a new conjecture on the composability of the entanglement entropy in the multi-block case, which yields a closed asymptotic formula for the Rényi entanglement entropy of a free fermion system in the most general multi-block configuration, both in position and momentum space. This formula, which we have numerically verified for a wide range of configurations both for $0 < n < 1$ and $n \geq 1$, reduces to the known ones when the configuration in momentum space consists of a single block[4]. It also

[3]Strictly speaking, this is only true if the roots of the equation $\mathcal{E}(p) = \mu$ are all simple. We shall implicitly make this assumption in what follows.

[4]The use of n for the Rényi parameter is typical in CFT literature, since the approach used in that community is based on the so-called replica trick, which consists in computing the trace of any power $n = 1, 2, \ldots$ of the reduced density matrix and, under some assumptions, obtain an analytic continuation which yields the von Neumann entanglement entropy as the $n \to 1$ limit of the logarithmic derivative of the Rényi

leads to closed asymptotic formulas for the mutual and the tripartite [36] (or r-partite [38]) information, which again agree with the general CFT predictions. The previous results are presented in Chapter 1.

Regarding the contents of Chapter 2, recall that Inozemtsev's elliptic su(1|1) spin chain reduces to the Heisenberg XX model via the Jordan–Wigner transformation [106] in an appropriate limit. Since the latter chain is long-ranged in general, one may wonder whether this connection can be used to study the long-ranged counterpart of the XX model, namely the Lipkin–Meshkov–Glick (LMG) model introduced by the latter authors in the sixties [56, 89, 92]. In the original LMG model, the interaction strength among the spins is constant (like in the Heisenberg XX chain) but involves any pair of particles and not just nearest-neighbors. We study generalizations of the LMG model with su(m) spin and arbitrary (positive) long-ranged interactions. To this end, we take advantage of the representation of the Hamiltonians associated to the latter generalized models in terms of permutation operators of su(m) type. We introduce a large family of su(m) spin chains which, like the Haldane–Shastry type chains, feature variable long range interactions, and whose ground state entanglement properties are similar to those of the LMG model. The models that we construct admit as non-degenerate ground state a generalized Dicke state of su(m) type, i.e., a state totally symmetric under permutations and with well-defined numbers of particles of each type (magnons). We call these new models su(m) generalized LMG (gLMG) models.

More precisely, the original LMG model contains a term quadratic in the total spin operator s^z, whereas in the gLMG models this term is replaced by a sum of similar terms in each of the generators of a Cartan subalgebra of su(m). The resulting models can thus be considered a natural generalization of the original LMG model, to which they actually reduce when $m = 2$ and all the two-body interactions are constant. One of the main results in this respect is the computation in closed form of the reduced density matrix for any quantum system whose ground state is an su(m)-type Dicke state. The formula we have obtained is in fact a generalization to m variables of the hypergeometric distribution derived in Ref. [88] for the original spin-$\frac{1}{2}$ LMG model. We shall then show that in the thermodynamic limit, when the total number of particles $N \gg 1$ and the particles $L \gg 1$ in a subsystem satisfy $L/N \to \alpha$ (finite), the

entropy with respect to n. We do not assume that n is an integer in what follows and thus our expressions do not need to be analytically continued.

eigenvalues of the reduced density matrix can be well approximated by multivariate Gaussian distributions with suitable parameters. Hence in this limit the entanglement spectrum can be derived from the covariance matrix of the Gaussian distribution which can be explicitly computed. From this point on, it is a straightforward matter to obtain closed-form asymptotic expressions for the von Neumann and Rényi entropies in the thermodynamic limit. Remarkably, in the region of parameter space for which all the ground state magnon densities are non-vanishing, both of these entropies scale as $\frac{m-1}{2}\log L$ as L tends to infinity. This behavior of the von Neumann entropy is that of a critical model whose low-lying energy levels are effectively described by a CFT with $c = \frac{3(m-1)}{2}$. This may not seem surprising at first sight, taking into account that many critical one-dimensional spin chains, including the Heisenberg (XXX) and the su(m) Haldane–Shastry chains, are effectively described by theories of this type (see, e.g., [**1**, **65**, **110**]). Here, however, the situation is more subtle. Indeed, the Rényi entropies R_n, though still proportional to $\log L$ for large L, are independent of n for $L \to \infty$, and as a consequence the family of gLMG models cannot contain any critical instances[5].

It is well known that a crucial requirement of classical thermodynamics is the extensivity of the (Maxwell–Boltzmann) entropy of a given system, i.e., that $S \propto L^d$, where L is a characteristic length of the system and d is the number of space dimensions. In a quantum context this requirement is violated in many cases, at least for the von Neumann entropy, as for instance in black hole thermodynamics [**16**, **17**, **66**, **67**]. In contrast to classical thermodynamics, however, it is not that clear why the quantum entropy should be extensive. In fact, the area law mentioned above strongly suggests a non-extensive behavior of von Neumann's entropy in strongly correlated quantum systems. On the other hand, it is very natural to inquire whether this behavior holds for all quantum entropies available in the literature. Interestingly enough, this is not the case. For instance, as already noted in Ref. [**35**], the quantum Tsallis entropy [**121**, **122**] can be

[5]Explicitly, recall that $R_n^{\mathrm{CFT}} = \frac{c}{6}(1+\frac{1}{n})\log L$ whereas we obtain $R_n^{\mathrm{LMG}} = \frac{m-1}{2}\log L$. Note that this conclusion is a consequence of the fact that for a $(1+1)$-dimensional CFT it is possible to evaluate the traces of successive powers $n = 1, 2, \ldots$ of the reduced density. We will come back to this point by the end of the dissertation, but let us agree now that the Rényi entropy functional satisfies two properties, namely (a) it admits an information-theoretic interpretation, and (b) it reduces to the von Neumann entropy for a particular value of its parameter. Thus checking that $R_n^{\mathrm{CFT}} = R_n$ for all n is a much stronger test of criticality than simply verifying that $S \equiv R_1$ coincides with S^{CFT}.

extensive for a suitable value of its parameter when the von Neumann one is not in several one- and two-dimensional strongly correlated systems, including the Heisenberg XY model. One finds a similar behavior for the Tsallis entanglement entropy of the ground state of generalized su(m) LMG models with $m \geq 4$, while for $m = 2, 3$ the Tsallis entropy is not extensive for any value of its parameter.

The ground state of the original LMG model has two quantum phases (entangled and non-entangled), respectively determined by the values of the (suitably normalized) magnetic field strength being less or greater than 1 in absolute value. For the gLMG models proposed, the phase diagram is more intricate since in this case the ground state can be in exactly m quantum phases, characterized by the number of vanishing of a certain number of magnon densities. In particular, in the phase with k vanishing magnon densities both the von Neumann and Rényi entropies scale as $\frac{(m-k-1)}{2} \log L$, implying again that none of these phases can contain any critical models. We have performed a detailed analysis of the ground state phases in the su(3) case, completely identifying the corresponding regions in parameter space. Remarkably, these regions are entirely determined in a geometric way by the weights of the fundamental representation of su(3) associated to the particular choice of the Cartan generators. A similar result holds in the general su(m) case; for instance, we show that the region for which all the magnon densities are non-vanishing is an m-simplex in $\mathbf{R}^{m-1}$ whose vertices are the weights of the fundamental representation of su(m).

As we have already mentioned, we take advantage of the representation of the Hamiltonians associated to the latter generalized models in terms of permutation operators of type su(m). At this point one may wonder whether it is possible to use the techniques of [**50**] in order to compute the spectrum of gLMG models by evaluating their partition function when the Hamiltonian is restricted to subspaces with fixed magnon content. Indeed, one of the key features of spin chains of Haldane–Shastry type is the fact that their partition functions can be exactly computed for any number of spins [**7**, **50**, **101**] by exploiting their connection with a corresponding spin dynamical model of Calogero–Sutherland type [**32**, **61**, **74**, **93**, **113**, **114**] by the so-called Polychronakos "freezing trick" [**101**]. This has made it possible to check the validity of several fundamental conjectures on the characterization of quantum chaos vs. integrability [**20**, **25**]. In particular, it has been shown that spin chains of Haldane–Shastry

type do not behave as expected for a generic integrable system, in the sense that the distribution of the spacings between consecutive (unfolded) levels is not Poissonnian [6, 7, 50].

There is a large subclass of gLMG models that can be regarded as a deformation of the three families of su(m) spin chains of Haldane–Shastry type, respectively known as rational, trigonometric and hyperbolic. We shall introduce this models more formally in due time and refer globally to them under the generic name of "spin chains of Haldane–Shastry type". The rational version was introduced by Polychronakos and Frahm, the trigonometric one is the very chain introduced by Haldane and Shastry in the late eighties, and the hyperbolic version was studied by Frahm and Inozemtsev for the first time. The subclass of gLMG models mentioned above is constructed by adding to the Hamiltonians of the latter models a term depending on the generators of the standard su(m) Cartan subalgebra. When this extra term is linear in the Cartan generators it can be interpreted just as an su(m) external magnetic field, and the corresponding models were studied in Ref. [45]. We shall see that the Hilbert space of a general gLMG model decomposes as a direct sum of subspaces with fixed magnon numbers, which are separately invariant under the action of both the original Hamiltonians and the added term. By suitably adapting the freezing trick, we shall be able to compute the partition function of the restrictions of the Hamiltonians of the three spin chains of Haldane–Shastry type to subspaces with fixed magnon content. This in turn yields the partition function of the corresponding gLMG Hamiltonian, since the Cartan generators are proportional to the identity on these subspaces.

The knowledge of the partition function of the gLMG models of Haldane–Shastry type as well as the restricted partition functions of the corresponding spin chains enables one to study several statistical properties of the spectrum of the latter models. In particular, we have obtained strong numerical evidence that the level density of the restriction of the HS-type chain Hamiltonians to subspaces with fixed magnon numbers follows a Gaussian distribution in the large N limit, as is known to be the case for the full spectrum of these models [43, 44]. We have also studied the distribution of the spacings between consecutive unfolded levels of the restrictions of these models to the invariant subspaces, showing that it follows the characteristic law for an approximately equispaced spectrum with normally distributed energy levels [6, 7]. Finally, we have numerically computed the thermodynamic functions of gLMG models of Haldane–Shastry type whose extra term is quadratic in the Cartan generators, comparing

them with the exact results for the original chains in the thermodynamic limit derived in Ref. [45].

It is natural at this point to consider supersymmetric versions of the models described above, defined in terms of representations of su($m|p$) permutation operators with positive integers m and p, as in Chapter 3 where the supersymmetric su(m) t-J model with long-range inverse-square interactions introduced by Kuramoto and Yokoyama [86,87] is considered. The Kuramoto–Yokoyama (KY) model consists of a one-dimensional equispaced circular lattice with N sites, each of which can be either empty or occupied by a single charged fermion with m internal degrees of freedom. The fermions interact pairwise with one another through their spin and charge and can hop among any two sites, and both the interaction strength and the hopping amplitude are inversely proportional to the (chord) distance between sites. The supersymmetric character of the su(m) KY model can be established by mapping it to a suitable modification of the su($1|m$) HS spin chain [64]. This connection shall in fact be exploited to fully determine the spectrum of the former model in terms of supersymmetric motifs and their corresponding Young tableaux [13,83].

The thermodynamics of the supersymmetric KY model has been actively investigated ever since its introduction. In fact, in the original reference [87] the low-temperature asymptotic behavior of the magnetic and charge susceptibilities was determined by means of the asymptotic Bethe ansatz (ABA). A few years later, the thermodynamics of the su(m) KY model at arbitrary temperature in the $N \to \infty$ limit was derived by Kato and Kuramoto [81] applying Polychronakos's freezing trick [100] to the su($1|m$) supersymmetric Sutherland model. This method, which is rather involved, requires first establishing the equivalence of the latter model to a system of non-interacting su($1|m$) particles and then modding out the contribution of the dynamical degrees of freedom. Moreover, it essentially relies on specific properties of the HS chain such as its equivalence to a model of free particles with generalized momenta obeying fractional statistics. On a more practical level, the formula for the grand potential obtained by Kato and Kuramoto depends on a function which must be determined by solving an implicit algebraic equation with an appropriate choice of branch.

Chapter 3 contains the description of the decomposition of the Hilbert space of general su($1|m$) models with (complete or broken) Yangian symmetry into invariant subspaces. This allows one to fully characterize the spectrum and the magnon numbers of all the eigenfunctions. We show

how a formula for the grand potential of these models, akin to Kato and Kuramoto's for the long-range supersymmetric t-J model, emerges in a transparent way from their partition function without requiring that they be described by generalized pseudo-momenta or fractional statistics. In the simplest and most interesting case $m = 2$, the corresponding implicit equation is quadratic and can therefore be explicitly solved, which leads to a new closed-form expression for the grand potential of the spin-$\frac{1}{2}$ KY model.

Finally, Chapter 4 contains an exhaustive classification of short-range Calogero–Sutherland (CS) models in one dimension with a Jastrow-like ground state. The introduction of the latter type of Hamiltonians and their corresponding ground states is motivated also from the appearance of generalized ensembles in random matrix theory. As we shall comment later, they are candidates to describe the spectrum of models that are neither integrable nor fully chaotic. This is analogous to the situation of the ground state of the scalar (long-range) Sutherland model whose square is the joint probability density of eigenvalues of the homogeneous[6] matrix ensembles introduced by Dyson [**40**]. We solve the problem of classifying all one-dimensional quantum Hamiltonians with nearest- and next-to-nearest-neighbors (translation invariant) interactions admitting a Jastrow-like ground state, both for motion on the real line and on a circle. This is the simplest near-neighbors analogue of the well-known problem for Calogero–Sutherland models with long-range interactions proposed shortly after their introduction [**32**, **113**, **114**] and completely solved in Ref. [**84**]. Our solution differs in two fundamental ways from its long-range counterpart. In the first place, we show that the potential must necessarily contain a three-body interaction term, which by construction is absent in the long-range solution. Secondly, the near-neighbors solution depends on an essentially arbitrary function of one variable (and, for motion on the line, on an additional positive parameter). The general solution contains a potential featuring elliptic interactions, which yields the (rational and trigonometric) particular solutions considered so far (see, e.g. [**76**]) as limiting cases.

[6] They yield for $\beta = 2, 1, 4$ the unitary, orthogonal and symplectic circular ensembles, respectively defined as the unitary group $\mathrm{U}(N)$, the quotient $\mathrm{U}(N)/\mathrm{O}(N)$ and the quotient $\mathrm{U}(2N)/\mathrm{Sp}(2N)$, where $\mathrm{O}(N)$ and $\mathrm{Sp}(2N)$ are the orthogonal group and the symplectic group, respectively.

Several lines of work for further research are suggested in view of these results. The first one that comes to mind is the construction and analysis of the spin-dynamical versions of the near-neighbors models considered (see Refs. [46, 47] for the rational and trigonometric models), and their associated short-range spin chains (as was done in Ref. [48] for the rational model). It would also be of interest to study in detail the potentials contained in the general solution, and in particular determine whether one can exactly compute other eigenfunctions besides the ground state. This is known to be the case for the previously known rational and trigonometric models. Another possible line for future work is to study the extension of our results to ground states factorized over other root systems like BC_N [5, 49].

Another topic worth investigating is the explicit computation of the correlation functions of the eigenvalue probability densities given by the special choices of the Jastrow-like ground states considered, particularly the case involving Weierstrass elliptic functions. This can be done in principle with the techniques of Refs. [60, 125], although the evaluation of the resulting integrals could be far from trivial in this case. In fact, an analogous problem[7] has already been solved in Ref. [24]. The corresponding distribution of the spacings of consecutive eigenvalues has been shown in the latter reference to be a good approximation to this statistic for certain pseudo-integrable billiards and for the Anderson model at the transition point. The above facts, together with the connection to continuous matrix product states (cMPS) established in Chapter 4 for the first time, may be helpful in the search for parent Hamiltonians for the states of Ref. [126]. Indeed, we shall identify a subclass of cMPS which is strictly included in the Jastrow-like ground states considered in our classification, thus providing the first examples of parent Hamiltonians in this case. If there exist examples of greater rank, our research clearly suggests pseudo-integrable systems as good places to look for them.

In Chapter 5 we summarize our analysis of new information-theoretical entanglement measures [P9]. They are based on formal groups as a consequence of the group-theoretical structure of their entropic counterparts (see Refs. [119, 120]). This allows us to construct composable generalized negativities with prescribed composition laws, as shown for instance in

[7] More precisely, for the density given by Eq. (23) below with ρ identically one and $\chi(x) = |x|^{\beta/2}$.

Eqs. (28) below, where Eq. (28a) defines a generalized negativity associated to the composition law given by Eq (28b).

CHAPTER 1

Free fermion systems and supersymmetric spin chains

The Hamiltonian of a translation-invariant free fermion system (Eqs. (8) below) is given by $H = -\sum_{ij} h_N(i-j)\, a_i^\dagger a_j$ in terms of a function $h_N(x)$ satisfying $h_N(x) = h_N(-x) = h_N(N-x)$ where a_i (respectively $a_i^\dagger$) denotes the operator that annihilates (resp. creates) a fermion localized at site i, i.e., they satisfy canonical anticommutation relations (CAR), and the values of i and j in the sum runs over $\{0, \dots, N-1\}$. The condition $h_N(x) = h_N(N-x)$, or equivalently the translation invariance of the system, effectively means that we are considering periodic boundary conditions $0 \equiv N$. The invariance of the Hamiltonian under translations $i \mapsto i+1$ (with $N \equiv 0$) means that the Fourier transform $(a_i) \mapsto (\hat{a}_k)$ provides a natural basis of common eigenvactors of the Hamiltonian and the total momentum operator. Indeed, under this transformation the Hamiltonian can be written as

$$H = \sum_k \varepsilon_N(k)\, \hat{a}_k^\dagger \hat{a}_k$$

where $\varepsilon_N(k)$ is called the dispersion relation and is given in terms of h_N by

$$\varepsilon_N(k) = -\sum_k \big(1 - \cos(2\pi j\, l/N)\big) h_N(j)\,. \tag{1}$$

The (Fermionic) operators $\hat{a}_k^\dagger$ create a mode with momentum $2k\pi/N$ mod 2π. If we write $|0\rangle$ for the vacuum, the states $\hat{a}_{u_1}^\dagger \cdots \hat{a}_{u_L}^\dagger |0\rangle$ are then a basis of common eigenstates of the Hamiltonian H and the total momentum operator, with respective eigenvalues $\sum_j \varepsilon_N(u_j)$ and $\sum_j 2\pi\, u_j/N$ mod 2π.

Thus, for free fermion systems, the Fourier transform provides a natural basis of eigenvectors. The elements of the computational Fourier-basis are common eigenstates of the Hamiltonian and the total momentum, the

energy of the eigenstate containing a set of excited modes being the sum of the energies of each excited mode. There are both short- and long-range well-known models that are actually equivalent to translation-invariant free fermion systems, for instance the XX model and the su(1|1) Haldane–Shastry chain [**62**, **112**] and its generalization, respectively. Indeed, these models are two limiting cases of the supersymmetric elliptic chain introduced in Ref. [**51**], which is included itself in the more general models that we studied in Refs. [**P2**]-[**P3**].

Given a bipartition of a system of free fermions, we would like to quantitatively measure the entanglement between its parts. In this case any entropy functional defines a quantitative measure of entanglement for pure states, the entanglement entropy. This measure is defined as the (classical) entropy of the reduced density matrix of one of the subsystems (it can be easily shown that the result is independent of the subsystem chosen). Although the number of eigenvalues of the reduced density matrix grows exponential with the size of the subsystem, the complexity of this problem can be reduced drastically. Indeed, there exists a Hermitian matrix whose spectrum determines that of the reduced density matrix, the so-called correlation matrix, whose order is just the number of elements of the subsystem. For additive entropies the following property holds: the entanglement entropy is given by the sum of the values of a certain function evaluated at the eigenvalues of the correlation matrix. Using the Cauchy integral theorem the previous sum is rewritten as an appropriate contour integral in the complex plane for a suitable function that involves the characteristic polynomial of the correlation matrix. If the subsystem is simply connected then the correlation matrix is Toeplitz and there is an asymptotic expression for its characteristic polynomial proved by Basor [**11**]. In principle, integrating along the appropriate contour one obtains the entanglement entropy.

In Refs. [**P2**]-[**P3**] we follow the previous approach to explicitly calculate the entanglement entropy of the ground state of several supersymmetric spin chains for a bipartition into two simply connected pieces. The models considered allow for different kind of Fermi seas (i.e., set of excited modes in the ground state) and our results confirm the widely believe conjecture that the critical behavior of fermionic lattices is completely determined by the topology of their Fermi surface (the boundary of the Fermi sea). Finally, in Ref. [**P1**] we study the entanglement entropy of a general free fermion system when the subsystem considered is not simply connected. In the approach described above, the connectedness of the

considered subsystem is crucial. If the correlation matrix is not Toepliz one has to explicitly diagonalize it, since the formula for the entanglement entropy (Eq. (3) below) in terms of its eigenvalues is useless without an explicit expression for its characteristic polynomial. This makes it impossible in practice to obtain any relevant quantity in the critical regime, since they are all characterized by their leading behavior as the number of elements in the subsystem tends to infinity. However, we have established a duality symmetry in the system under the exchange of the set of sites of the subsystem with the set of excited modes. We can then use the Basor's result and compute the entanglement entropy for subsystems which are not simply connected provided the Fermi sea is so. Of course, when both sets (the subsystem and the Fermi sea) are not simply connected at the same time, the previous approach is not applicable. However, on the basis of the previous observations we have formulated a conjecture for the entanglement entropy in this general case which not only reproduces the above results, but is also in excellent agreement with numerical computations in a wide variety of situations.

In this chapter we shall present the results [**P1**]–[**P3**] starting with the above mentioned duality. This allows one to establish from the beginning that the entanglement entropy is a symmetric function, so that previous results can be analized taking this important property into account.

1. Duality

Consider a one-dimensional lattice of spin-$\frac{1}{2}$ particles whose sites are labeled by the set of integers $\{0,\dots,N-1\}$ (with periodic boundary conditions $N\equiv 0$). It is well known that the Jordan–Wigner transformation

$$a_i=\sigma_1^z\cdots\sigma_{i-1}^z\frac{1}{2}(\sigma_i^x-\mathrm{i}\sigma_i^y)$$

maps this system into a system of fermions[8]. In many cases (as, for instance, for the XX or XY Heisenberg chains, or for the supersymmetric chains studied in [**P1**]–[**P3**]), the latter system is a (translation-invariant) free fermion system, which as mentioned above can be diagonalized through the Fourier transform. Moreover, under the above mapping a state of the chain with certain coefficients $c(s_1,\dots,s_N)$ in the basis of elements $|s_1\cdots s_N\rangle$ with $s_i=0,1$ is mapped to a state of the

[8]Here σ^α for $\alpha=x,y,z$ are the Pauli matrices and $\sigma_i^\alpha=(\otimes^{i-1}1)\otimes\sigma^\alpha\otimes(\otimes^{N-i}1)$. So that $\sigma_i^x=\prod_{j<i}(1-2a_j^\dagger a_j)(a_i+a_i^\dagger)$, $\sigma_i^y=-\mathrm{i}\prod_{j<i}(1-2a_j^\dagger a_j)(a_i^\dagger-a_i)$ and $\sigma_i^z=1-2a_i^\dagger a_i$.

fermion system with the same coefficients in the basis of states of the form $(\hat{a}_1^\dagger)^{s_1}\cdots(\hat{a}_N^\dagger)^{s_N}|0\rangle$, where $|0\rangle$ is the vacuum. It follows that the entanglement entropy of the original state of the spin system coincides with that of the corresponding fermionic state, which, as we shall see, is much easier to evaluate.

Given two subsets u, v consider the pure state $\rho(u) = |u\rangle\langle u|$ obtained by acting with $\hat{a}_{u_1}^\dagger \ldots \hat{a}_{u_L}^\dagger$ in the vaccum if $u_1, \ldots, u_L$ are the elements of u and write $\rho_v(u) = \mathrm{tr}_{\setminus v}\, \rho(u)$ for the reduced density matrix describing the state of the subsystem of particles labeled by the elements $v_1, \ldots, v_M$ of v. The states $\rho_u(v)$ and $\rho_v(u)$ defined in this way are closely connected: their non zero eigenvalues are the same. More precisely, if we denote by $\mathrm{spec}_0\,\rho$ the spectrum of a density matrix (i.e., the set of its distinct eigenvalues, counted with their respective multiplicities) excluding the zero eigenvalue, i.e., $\mathrm{spec}_0\,\rho = \mathrm{spec}(\rho|_{(\ker\rho)^\perp})$, then we say that ρ and σ are similar up to zero eigenvalues if $\mathrm{spec}_0\,\rho = \mathrm{spec}_0\,\sigma$. The following basic result is easy to prove [**P1**] and its consequences are explored in the next sections:

Given two subsets u and v of $\{0, \ldots, N-1\}$, the reduced density matrices $\rho_u(v)$ and $\rho_v(u)$ are similar up to zero eigenvalues.

If $S(\cdot)$ is an entropy functional satisfying the Shannon-Khinchin axioms, the previous result automatically imply the *duality principle*

$$S(\rho_v(u)) = S(\rho_u(v))\,. \tag{2}$$

Hence, for a free fermion system the entanglement entropy of pure states is a symmetric function of two arguments which are subsets of $\{0, \ldots, N-1\}$. Alternatively, the latter statement is equivalent to the following: given a bipartition $\mathcal{H}_u \otimes \mathcal{H}_{\setminus u}$ and a state $\rho(v) : \mathcal{H} \to \mathcal{H}$, its entanglement content coincides exactly with that of the state $\rho(u) : \mathcal{H} \to \mathcal{H}$ with respect to the bipartition $\mathcal{H}_v \otimes \mathcal{H}_{\setminus v}$ for all subsets u, v.

2. The correlation matrix method

Given u, v, the eigenvalues of $\rho_u(v) = \mathrm{tr}_{\setminus u}\,\rho(v)$ can be obtained from those of a Hermitian positive matrix A of dimension $|u|$, the correlation matrix, whose entries are defined as $A_{ij} = \langle v|a_{u_i}^\dagger a_{u_j}|v\rangle$ Indeed, it can be shown that $\rho_u(v)$ factorizes and the quantitative value of the entanglement entropy is the sum of the entropies of each factor if $S(\cdot)$ is additive. Note that the eigenvalues of $\rho_v(u)$ are obtained from a correlation matrix A' of dimension $|v|$ with entries $A'_{ij} = \langle u|a_{v_i}^\dagger a_{v_j}|u\rangle$ in the same way as those of

$\rho_u(v)$ are obtained from A. This implies that the matrix A can be replaced by the matrix A' in the final expression for the entanglement entropy. This is relevant because important physical examples (non connected bipartitions) can be solved only in terms of A' and not A, as we shall explain below. First, one needs the following result:

Given $u, v \subset \mathbf{Z}/N\mathbf{Z}$ with $|u| = L$, the reduced density matrix $\rho_u(v)$ can be written as $\rho_u(v) = \sigma_1 \cdots \sigma_L$ where $\sigma_i = \nu_i c_i^\dagger c_i + (1-\nu_i) c_i c_i^\dagger$ and the operators (c_i) satisfy CAR.

After applying the (inverse) Jordan–Wigner transformation to the operators (c_i) one has $\sigma_1 \cdots \sigma_L = \tau_1 \otimes \cdots \otimes \tau_L$ where $\tau_l = \frac{1}{2}1 + (\nu_l - \frac{1}{2})z$ is the density matrix of a spin-$\frac{1}{2}$ particle and we have written 1 and z to denote respectively the square matrices of order two $\mathrm{diag}(1,1)$ and $\mathrm{diag}(1,-1)$. If we now assume that the considered entropy functional $S(\cdot)$ is additive, the entanglement entropy is given by the sum $S = s(\nu_1) + \cdots + s(\nu_i)$ where $s(\nu_l) = S(\tau_l)$ is the (classical) entropy of the binary distribution of probabilities ν_l and $1 - \nu_l$ since $\tau_l = \mathrm{diag}(\nu_l, 1-\nu_l)$. The entanglement entropy S can then be expressed as

$$S = \frac{1}{2\pi \mathrm{i}} \oint_\gamma s(z) \sum_j (z - \nu_j)^{-1} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \oint_\gamma s(z) \frac{\mathrm{d} \log P_A(z)}{\mathrm{d}z} \mathrm{d}z \tag{3}$$

for a contour $\gamma \in \mathbf{C}$ around all the eigenvalues ν_i of the correlation matrix A, with P_A the corresponding characteristic polynomial[9]. It can be shown [**P1**] that the eigenvalues ν_i are all real numbers $0 \leq \nu_i \leq 1$ so that γ is a contour around the unit interval $[0,1] \in \mathbf{C}$.

We shall say that a set of integers is connected if it consists of consecutive integers (modulo N). Consider a proper subset u which is the disjoint union of $r \geq 1$ connected components $\bar{u}_i$, which we shall denote by $u = \bar{u}_1 \uplus \cdots \uplus \bar{u}_r$. We shall also write $\bar{u}_i = \{u_i^0, u_i^0 + 1, \ldots, u_i^1\}$, with $|\bar{u}_i| = u_i^1 - u_i^0 + 1$ and $u_{i+1}^0 - u_i^1 > 1$. If the set $u = \bar{u}$ is simply connected then the correlation matrix A is a Toepliz matrix and we write $A_{i-j} = A_{ij}$. If $f_A(\theta) = \sum A_k \mathrm{e}^{\mathrm{i}k\theta}$ satisfies certain technical requirements,

[9] In the latter expression one should actually replace the function $s(z)$ by an appropriate deformation $s_\delta(z)$ such that $\lim_{\delta \to 0} s_\delta(z) = s(z)$, in order to push the branch cuts that $s(z)$ usually has along the half-lines $\mathrm{Re}\, z \leq 0$ and $\mathrm{Re}\, z \geq 1$ to the left of 0 and to the right of 1, and then let $\delta \to 0$. For instance, for the von Neumann entropy one can take $s_\delta(z) = -z \log(z+\delta) - (1-z)\log(1+\delta-z)$. We have omitted this technicality for the sake of conciseness.

the asymptotic behavior of P_A can be described in terms of (the discontinuities of) f_A when $1 \ll |\bar{u}| \ll N$. The formula for the asymptotic behavior of the determinant of a Toeplitz matrix is known as the Fisher–Hartwig conjecture [**53**] for historical reasons. All the physical cases treated here satisfy the hypotheses of the work of Basor [**11**], and thus the expressions for the determinants used in what follows come from proved theorems and not conjectures. Nevertheless, it is common in the literature to refer to all this kind of results as the "Fisher–Hartwig conjecture".

By virtue of the duality principle, Eq. (3) also hols with A' instead of A. Since the behavior of the characteristic polynomial of A' is known when $v = \bar{v}$ is simply connected and $1 \ll |\bar{v}| \ll N$, Eq. (3) can be used to determine the asymptotic behavior of the entanglement entropy in two regimes: (a) when the set $u = \bar{u}$ is simply connected and $1 \ll |\bar{u}| \ll N$, and (b) when the set $v = \bar{v}$ is simply connected and $1 \ll |\bar{v}| \ll N$.

3. The simplest scenario

Suppose now that $u = \bar{u}$ and $v = \bar{v}$ are two sets of consecutive integers modulo N and let us write R_α for the Rényi entropies of the reduced density matrices $\rho_u(v)$ and $\rho_v(u)$. According to the previous discussion, we know the behavior of R_α in the regimes $1 \ll |\bar{u}| \ll N$ and $1 \ll |\bar{v}| \ll N$. There must exist an extension of these formulas that is valid in the more general case in which both $|\bar{u}|/N$ and $|\bar{v}|/N$ tend to a nonzero limit as $N \to \infty$. We shall write $\varkappa_u = \lim_{N\to\infty} |\bar{u}|/N$ and $\varkappa_v = \lim_{N\to\infty} |\bar{v}|/N$.

Let us write down the results obtained when the approach described in the previous section is followed (see [**P2**]-[**P3**]). Assume first that Eq. (3) is used in terms of the characteristic polynomial of A. In this case the asymptotic behavior of P_A for fixed $\bar{u}$ is known for all ratios $\varkappa_v$. We obtain

$$R_\alpha(\bar{u},\bar{v}) \equiv R_\alpha(\rho_u(v)) = c_\alpha + b_\alpha \log\left(2|\bar{u}|\sin(\pi\varkappa_v)\right) + \cdots \tag{4a}$$

where the dots stand for terms that tend to zero when $|\bar{u}| \to \infty$, the coefficient c_α was computed in [**P3**], with the result

$$c_\alpha = \int_0^\infty \mathrm{d}t \Big(\alpha(1-\alpha)^{-1} \operatorname{csch}^2 t - (1-\alpha)^{-1} \operatorname{csch}(t/\alpha) \operatorname{csch} t - b_\alpha \, \mathrm{e}^{-2t} \Big) ,$$

and $b_\alpha = \frac{1}{6}(1+\alpha^{-1})$. Alternatively, we can use Eq. (3) with the characteristic polynomial of A'. In this case we fix $\bar{v}$ and use, for all $\varkappa_u$, the known asymptotic behavior of $P_{A'}$ to obtain

$$R_\alpha(\bar{u},\bar{v}) = R_\alpha(\rho_v(u)) = c_\alpha + b_\alpha \log\left(2|\bar{v}|\sin(\pi\varkappa_u)\right) + \cdots \tag{4b}$$

where now the dots stand for terms that tend to zero when $|\bar{v}| \to \infty$. Expressions (4) give us the value of the sought for function R_α in two different regimes. In particular they coincide in the (rather uninteresting) case in which both $\varkappa_u \ll 1$ and $\varkappa_v \ll 1$. On the other hand, it should be noted that neither Eq. (4a) nor (4b) are valid in the general situation in which both $\varkappa_u$ and $\varkappa_v$ tend to nonzero limits as $N \to \infty$. Indeed, they are not invariant under complements, a property that holds for an arbitrary bipartition of the system when it is described by a pure state. In this context it implies $R_\alpha(\backslash u, v) = R_\alpha(u, v)$. By duality, it also follows that

$$R_\alpha(\backslash u, \backslash v) = R_\alpha(\backslash u, v) = R_\alpha(u, \backslash v) = R_\alpha(u, v)\,. \tag{5}$$

Taking into account the previous discussion, without loss of generality, the sought for function can be written as

$$R_\alpha(\bar{u}, \bar{v}) = b_\alpha \log N + \eta_\alpha(\varkappa_u, \varkappa_v) + \cdots$$

where the dots stand for terms that tend to zero when $N \to \infty$. The previous expression is justified by the known critical behavior since by translation invariance it can only depend on the sizes $|\bar{u}| = N\varkappa_u$ and $|\bar{v}| = N\varkappa_v$ of the sets involved. In addition, the function η_α must satisfy the following three properties: (a) $\eta_\alpha(x, y) = \eta_\alpha(y, x)$ on account of the duality principle. (b) $\eta_\alpha(1 - x, y) = \eta_\alpha(x, y)$ due to the invariance under complements of the entanglement entropy[10]. (c) Its behaviour when $x \ll 1$ or $y \ll 1$ is fixed by the previous observations. For instance[11], $\eta_\alpha(x, y) = c_\alpha + b_\alpha \log\big(2x \sin(\pi y)\big) + o_x(1)$ for small values of x follows from from Eq. (4a).

The simplest function satisfying the previous requirements is obviously given by $\eta_\alpha(x, y) = c_\alpha + b_\alpha \log\big((2/\pi)\sin(\pi x)\sin(\pi y)\big)$. We thus conjecture that

$$R_\alpha(\bar{u}, \bar{v}) = c_\alpha + b_\alpha \log\big((2N/\pi)\sin(\pi\varkappa_u)\sin(\pi\varkappa_v)\big) + \cdots \tag{6}$$

where the dots stand for terms that tend to zero when $N \to \infty$. The last equation was obtained without assuming that the cardinal of neither the subsystem considered $|u|$ (or its complement) nor the Fermi sea $|v|$ is small compared with the total number of sites N. Up to our knowledge, the behaviour in this intermediate regime has not previously been reported in the literature.

[10] We automatically have $\eta_\alpha(x, 1 - y) = \eta_\alpha(x, y)$ by using $\eta_\alpha(x, y) = \eta_\alpha(y, x)$.

[11] We have written $o_z(1)$ to denote that $\lim_{z\to 0} o_z(1) = 0$, and also noted that $\eta_\alpha(x, y) = c_\alpha + b_\alpha \log\big(2y \sin(\pi x)\big) + o_y(1)$ for small values of y by Eq. (4b).

4. Multi-block entanglement entropy

We next compute the entanglement entropy between a non-simply connected subsystem $u = \bar{u}_1 \uplus \cdots \uplus \bar{u}_r$ consisting of r connected components and its complement $\setminus u$, assuming that the set of excited modes $v = \bar{v}$ is simply connected. To this end, we can use Eq. (3) with the correlation matrix A' which is Toeplitz in this case[12].

Applying Basor's result we then obtained an asymptotic expression for $P_{A'}$ for fixed $\bar{v}$ which is valid for all $\varkappa_u$. The important point here is that such an exact asymptotic formula is expressed in terms of the quantities $R_\alpha(\bar{u}_i, \bar{v})$ given by Eq. (4b), i.e., the limit $\varkappa_v \to 0$ of Eq. (6). More precisely,

$$R_\alpha(\bar{u}_1 \uplus \cdots \uplus \bar{u}_r\,, \bar{v}) = \sum R_\alpha(\bar{u}_i, \bar{v}) - I_\alpha(\bar{u}_1 \uplus \cdots \uplus \bar{u}_r) + \cdots \tag{7a}$$

where the dots stand for terms that tend to zero when $N \to \infty$, $I_\alpha(\cdot)$ is the so-called mutual information defined bellow and $R_\alpha(\bar{u}_i, \bar{v})$ are given by the limit $\varkappa_v \to 0$ of Eq. (6). Up to our knowledge, this is the first rigorous derivation of the entanglement entropy between a non-simply connected subsystem u consisting of an arbitrary number r of connected components and its complement when the set of excited modes $v = \bar{v}$ is simply connected and $\lim \varkappa_v \to 0$. We conjecture the validity of Eq. (7a) with $R(\bar{u}_i, \bar{v})$ given by Eq. (6) for general $\varkappa_v$. The dual expression giving the exact asymptotic behavior of the entanglement entropy between a simply connected subsystem u and its complement when $\lim \varkappa_u \to 0$ for a general set of excited modes $v = \bar{v}_1 \uplus \cdots \uplus \bar{v}_s$ consisting of an arbitrary number s of connected components was obtained in [**P3**] "from first principles", i.e., without using the duality principle, and reads

$$R_\alpha(\bar{u}\,, \bar{v}_1 \uplus \cdots \uplus \bar{v}_s) = \sum R_\alpha(\bar{u}, \bar{v}_j) - I_\alpha(\bar{v}_1 \uplus \cdots \uplus \bar{v}_s) + \cdots \tag{7b}$$

where the dots stand for terms that tend to zero when $N \to \infty$ and $R_\alpha(\bar{u}, \bar{v}_j)$ are given by the limit $\varkappa_u \to 0$ of Eq. (6). Again, we conjecture the validity of Eq. (7b) with $R(\bar{u}, \bar{v}_j)$ given by Eq. (6) for general $\varkappa_u$.

Finally, for any subset w the mutual information $I_\alpha(w)$ is a function of the boundaries of the connected components of w that we shall now describe. To this end, let us assume that w has t connected components and write $w = \bar{w}_1 \uplus \cdots \uplus \bar{w}_t$ with, according to the notation previously

[12] Recall that the matrix elements of A' are $A'_{ij} = \langle u | a^\dagger_{v_i} a_{v_j} | u \rangle$

defined, $\bar{w}_i = \{w_i^0, w_i^0 + 1, \dots, w_i^1\}$. We have $I_\alpha(w) = -b_\alpha \log I(w)$ with

$$I(\bar{w}_1 \uplus \cdots \uplus \bar{w}_t) = \prod_{i<j} \frac{(w_i^0 : w_j^0 : w_j^1)}{(w_i^1 : w_j^0 : w_j^1)} \equiv \prod_{i<j} (w_i^0 : w_i^1 : w_j^0 : w_j^1)\,.$$

Here $(a : b : c)$, with $a, b, c \in \{0, \dots, N-1\}$, is defined as the ratio

$$(a : b : c) = \frac{d(z_a - z_b)}{d(z_a - z_c)}\,,$$

where $z_k = \mathrm{e}^{2\pi \mathrm{i} k/N}$ and $d(z_j - z_k) = \mathrm{sgn}(j-k)|z_j - z_k|$ denotes the signed distance between the points z_j and z_k in the complex plane. Note that $I_\alpha(w) = I_\alpha(\backslash w)$, so that Eqs. (7) satisfy the duality principle and are invariant under complements, i.e., they are consistent with Eqs. (5).

To the best of our knowledge, all previous expressions involving the entanglement entropy or the mutual information of free fermion systems that have appeared in the literature can be obtained from the previous formulas[13] in one of the two limits $\varkappa_v \to 0$ and $\varkappa_u \to 0$. Note that in these regimes, as we have already pointed out, Eqs. (7) are exact. In particular, Eq. (7b) when $\varkappa_u \to 0$ was rigorously obtained in [**P3**]. Equation (7a) when $\varkappa_v \to 0$ follows from Eq. (7b) by using the duality principle, as we explicitly showed in [**P1**].

We want to finally address in this section the general problem of determining $S(\rho_u(v)) = S(\rho_v(u))$ when both subsets u, v have several connected components. Let us denote by $S(u, v)$ this entanglement entropy. In Ref. [**P1**] we propose for this case in which the standard procedure based on the Fisher–Hartwig conjecture is not applicable. Our approach relies on the duality principle and the following assumption: when the different connected components $\bar{u}_i$ of u are far apart, the entanglement entropy $S(u, v)$ should be asymptotic to the sum $\sum S(\bar{u}_i, v)$. Of course, by duality $S(u, v)$ should also be asymptotic to the sum $\sum S(u, \bar{v}_j)$ if the components $\bar{v}_j$ of v are far apart. The conjecture then reads

$$S(\bar{u}_1 \uplus \cdots \uplus \bar{u}_r, \bar{v}_1 \uplus \cdots \uplus \bar{v}_s) = \sum_{i=1}^{r} S(\bar{u}_i, v) + \sum_{j=1}^{s} S(u, \bar{v}_j) - \sum_{i=1}^{r} \sum_{j=1}^{s} S(\bar{u}_i, \bar{v}_j) + \cdots$$

where the dots stand for terms that tend to zero when $N \to \infty$. The previous expression only involves terms in which at least one of the arguments of the entropy is a connected subset. For instance, for the Rényi entropy the

[13] More precisely, the limits $\varkappa_v \to 0$ of $R_\alpha(\bar{u}_i, \bar{v})$ in (7a) and $\varkappa_u \to 0$ of $R_\alpha(\bar{u}, \bar{v}_j)$ in (7b) with both $R_\alpha(\bar{u}_i, \bar{v})$ and $R_\alpha(\bar{u}, \bar{v}_j)$ given by Eq. (6).

function S is explicitly given by the RHS of Eqs. (6)-(7). The previous conjecture and its consequences were numerically tested respectively in Ref. [**P1**] and, for a wide class of models, [**P2**]-[**P3**]. We refer to these works for figures, tables with errors and, in general, visual presentations of the results. In particular, the central charge of the underlying CFT in the critical phases can be obtained through the above way. For instance, in the case of a bipartition into two connected parts, it is simply given by the number of connected components of the Fermi sea.

5. A class of supersymmetric spin chains

In the next sections we construct specific models with Fermi seas having different number of connected components. In fact, as we shall see bellow bi-parametric Hamiltonians can be introduced in terms of a chemical potential μ and a second parameter J such that: (a) the model is critical or gapped according to whether μ belongs to a certain range, and (b) in the critical regime, it is described by a CFT whose central charge varies with J.

We introduce a wide class of translation-invariant su(1|1) supersymmetric spin chains featuring both long- and short-range interactions. The class include as particular instances the su(1|1) supersymmetric versions of the Inozemtsev (elliptic) and Haldane–Shastry (trigonometric) chains, as well as the celebrated XX model. In Ref. [**P2**] we show that this kind of models are indeed equivalent to free fermion systems and study the entanglement entropy of their ground states. If the dispersion relation of the system is monotonic the ground state consists of a set of excited modes which is simply connected and then Eq. (7a) can be directly used. In Ref. [**P3**] we study the more general case in which the dispersion relation can be nonmonotonic thus allowing for ground states determined by subsets of excited modes with several connected components, so that Eq. (7b) for the entanglement entropy of a connected bipartition can be used in this case.

5.1. Hamiltonian

Consider a (closed) spin chain with translationally invariant interactions whose N sites are occupied by either a boson or a hopping (spinless) fermion. As usual, we shall write (f_i) and (b_i) respectively for the fermionic and bosonic operators. Bosonic operators commute with the fermionic

ones and satisfy canonical commutation relations (CCR) among themselves.

The Hilbert space $\mathcal{H} = \mathcal{P}\mathcal{F}$ of the model is a subspace of the infinite dimensional Fock space $\mathcal{F}$ given by the image under the projector onto single-occupancy states $\mathcal{P}$. A basis of this subspace consists of elements of the form $O_1 \cdots O_N|0\rangle$ where $|0\rangle$ is the vacuum and with each O_i equal to either $b_i^\dagger$ or $f_i^\dagger$. The Hamiltonian of the models we shall be interested in is given in terms of the graded permutation operators $P_{ij}^{(1|1)}$ related to the Lie superalgebra su(1|1). Their action differs from regular (not graded) permutations by a suitable sign, and can be defined in terms of (graded) adjacent transpositions $P_i^{(1|1)} = P_{i,i+1}^{(1|1)}$ due to the relation $P_{ij}^{(1|1)} = P_{j-1}^{(1|1)} P_{i,j-1}^{(1|1)} P_{j-1}^{(1|1)}$. They act on the basis elements in the standard way, namely as

$$P_i^{(1|1)}(\cdots O_i O_{i+1} \cdots |0\rangle) = (-1)^\pi (\cdots O_{i+1} O_i \cdots |0\rangle)$$

where $\pi = \pi(O_i)\pi(O_{i+1})$ with $\pi(b_i^\dagger) = 0$ and $\pi(f_i^\dagger) = 1$.

Finally we are in position to introduce the Hamiltonians $H = H_0 + H_1$ considered in the rest of this chapter. The first term contains the su(1|1) permutation operators and is written as $H_0 = -\sum_{i<j} h_N(j-i)(1 - P_{ij}^{(1|1)})$ where $h_N(-x) = h_N(x) = h_N(N-x)$. Note that the latter function describes the strength of the interactions in a closed (circular) chain which is translationally invariant. Finally, H_1 only depends on the total number of fermions and is given by $H_1 = -\mu \sum f_i^\dagger f_i$, where $\mu \in \mathbf{R}$ is a parameter interpreted as the chemical potential of the fermions.

It was shown by Haldane [64] that the new set of operators (a_i) defined in terms of the old bosonic and fermionic ones as $a_i = b_i^\dagger f_i$ satisfy CAR. Moreover, the subspace $\mathcal{H}$ can be identified with the (2^N-dimensional) complete Fock space of the operators a_i, and the supersymmetric su(1|1) exchange operators can be expressed in terms of the latter operators when restricted to the subspace $\mathcal{H}$. Along the same lines, it is easy to show that $f_i^\dagger f_i = a_i^\dagger a_i$. In terms of the new fermionic operators (a_i) the Hamiltonian of the system still can be written as a sum $H = H_0 + H_1$ where now $H_0 = -\sum_{ij} h_N(|i-j|)\, a_i^\dagger a_j$ and $H_1 = -\mu \sum a_i^\dagger a_i$. Introducing the Fourier-transformed operators $(\hat{a}_i)$, which also satisfy CAR, the Hamiltonian is diagonalized as $H = \sum (\varepsilon_N(i) - \mu)\, \hat{a}_i^\dagger \hat{a}_i$ with $\varepsilon_N(i)$ given by Eq. (1) and thus describes a free fermion system. Let us summarize the three

different transformations performed on the original Hamiltonian:

$$H = -\sum_{i<j} h_N(j-i)\,(1 - P_{ij}^{(1|1)}) - \mu \sum f_i^\dagger f_i \qquad (b_i \text{ and } f_i)$$

$$= -\sum_{ij} h_N(|i-j|)\; a_i^\dagger a_j - \mu \sum a_i^\dagger a_i \qquad (a_i = b_i^\dagger f_i) \tag{8a}$$

$$= \sum (\varepsilon_N(l) - \mu)\, \hat{a}_l^\dagger\, \hat{a}_l \qquad (\text{Fourier-basis } \hat{a}_i) \tag{8b}$$

where the one-mode energies $\varepsilon_N(i)$ in the final expression of the Hamiltonian are given in terms of h_N by Eq. (1).

Let $H = \sum \varepsilon_N(i)\, \hat{a}_i^\dagger \hat{a}_i$ be the Hamiltonian of a system of free fermions where $\varepsilon_N(k)$ is given by Eq. (1) in terms of the interaction strength $h_N(x)$. If there exists an smooth function $\mathcal{E}(p)$ independent of N such that $\varepsilon_N(Np/2\pi) = \mathcal{E}(p) + o(1)$ for all momenta $p_l \equiv 2\pi l/N \mod 2\pi$ in the limit $N \to \infty$, we shall say that $\mathcal{E}(p)$ is the dispersion relation of the system. It follows that the dispersion relation is symmetric about π, and that $\mathcal{E}(0) = 0$ if the interaction strength is translationally invariant (as we shall indeed assume in what follows). We can then extend the dispersion relation to the whole real line as a 2π-periodic function, and assume $\mathcal{E}(-p) = \mathcal{E}(p) = \mathcal{E}(2\pi - p)$. If $\mathcal{E}'(p) > 0$ in $(0, \pi)$ we say that the dispersion relation is monotonic.

Equation (1) shows that if a dispersion relation $\mathcal{E}(p)$ exists then it is also unique and satisfies $\mathcal{E}(-p) = \mathcal{E}(p) = \mathcal{E}(2\pi - p)$. Let us assume in what follows that the system considered has a known dispersion relation according to the definition given above.

5.2. The elliptic chain

There is an important type of interaction satisfying the previous conditions that exhibits a sufficiently rich structure making it possible to examine a number of key properties in an analytic fashion. More precisely, consider the Hamiltonian whose interaction strength is given by the elliptic function

$$h_N(x) = \left(\frac{\alpha}{\pi}\right)^2 \left(\sinh \frac{\pi}{\alpha}\right)^2 \left(\wp_N(i-j) - \frac{2\hat{\eta}_1}{\alpha^2}\right) \tag{9}$$

where $\alpha > 0$ is a real parameter, $\wp_N(x) = \wp(x; \omega_1, \omega_3)$ is the Weierstrass elliptic function with half-periods $\omega_1 = N/2$ and $\omega_3 = \mathrm{i}\alpha/2$ and $\hat{\eta}_1 = \zeta(1/2)$, $\zeta(x) = \zeta(x; \omega_1, \omega_3)$ denoting the Weierstrass zeta function now with half-periods $\omega_1 = 1/2$ and $\omega_3 = \mathrm{i}N/(2\alpha)$ (see, e.g., [95]). We

have actually defined an infinite family of Hamiltonians describing free fermion systems and shall globally refer to them as the elliptic chain.

In Ref. [51] it was shown that the Hamiltonian of the elliptic chain coincides in the limit $\alpha \to 0^+$ with that of the XX model via a Jordan-Wigner transformation (see, e.g., [106]), whereas in the limit $\alpha \to \infty$ it yields the su(1|1) Haldane–Shastry spin chain. In other words, the elliptic chain smoothly interpolates between the XX model and the su(1|1) Haldane–Shastry spin chain. The function $\varepsilon_N(k)$ was computed in closed form in Ref. [51], with the result

$$\mathcal{E}(p) = 2\left(\sinh\frac{\pi}{\alpha}\right)^2\left(\wp(p) - \left(\zeta(p) - \frac{\eta_1 p}{\pi}\right)^2 - \frac{2\eta_1}{\pi}\right)$$

where now $\wp(p) = \wp(p;\pi,\mathrm{i}\pi/\alpha)$, $\zeta(p) = \zeta(p;\pi,\mathrm{i}\pi/\alpha)$ and $\eta_1 = \zeta(\pi)$. Moreover, taking the limits $\alpha \to 0^+$ and $\alpha \to \infty$ in the previous dispersion relation one recovers those of the XX and Haldane–Shastry models, namely, $2(1-\cos p)$ and $p(2\pi-p)/2$ respectively. It can be shown that $\mathcal{E}(p)$ is monotonic in the interval $[0,\pi]$ for all values of α and the set of excited modes of the corresponding ground states have one connected component. Indeed, the ground state is $\rho(v)$ where

$$v = \bar{v} = \{0 \le l \le N-1 : \mathcal{E}(2\pi l/N) < \mu\}$$

is connected[14].

5.3. Examples

There are simple Hamiltonians featuring both short- and long-range interactions whose dispersion relation is not monotonic, thus allowing for critical ground states in different universality classes.

Consider for instance a coupling strength h_N describing nearest- and next-to-nearest-neighbors interactions. Without loss of generality suppose that its values are 1 and J for nearest- and next-to-nearest-neighbors respectively, i.e., $h_N(1) = 1$, $h_N(2) = J$. Since $h_N(x)$ describes the interactions in a closed and translationally invariant system, it follows that $h_N(N-1) = 1$, $h_N(N-2) = J$ and $h_N(x) = 0$ for other values $2 < x < N-2$.

Inserting this interaction strength in Eq. (1) we obtain a well-defined dispersion relation $\mathcal{E}(p)$ with no dependence on N, given by $\mathcal{E}(p) = \mathcal{E}_1(p) +$

[14]Strictly speaking, the previous statement is true only when $\mathcal{E}(2\pi l/N) \neq \mu$ for all l in which case the ground state is unique. In the thermodynamic limit this is generically true and we shall implicitly make this assumption throughout this chapter.

$\mathcal{E}_2(p)$ with $\mathcal{E}_1(p) = 2(1-\cos p)$ and $\mathcal{E}_2(p) = 2J(1-\cos 2p)$, again with no $o(1)$ term. Since $\mathcal{E}'(p) = (2+8J\cos p)\sin p$, the dispersion relation is nonmonotonic if and only if $|J| > 1/4$. If this is the case, the ground state $\rho(v)$ contains a set of modes $v = \bar{v}_1 \uplus \bar{v}_2$ with two connected components for some values of the chemical potential.

Note that the XX model is recovered when $J = 0$, so that if the critical phase of the XX model can be simulated[15], one can hope that adiabatic evolution could in principle run over critical states with different central charges in the same way as the ground state of the XY model is obtained by adiabatic evolution of the ground state of the XX model. In other words, consider Hamiltonians $H(t) = H_0 + t\,H_1$ and write $|\psi_t\rangle = \mathrm{e}^{\mathrm{i}\,H(t)}|\psi_0\rangle$ where $|\psi_0\rangle$ is the ground state of H_0 . In our case H_0 is the nearest-neighbors Hamiltonian of the XX model and H_1 the perturbation described above when $h_N(2) = h_N(N-2) = J$. If $|\psi_0\rangle$ can be simulated and $U(t)$ efficiently implemented, $|\psi_t\rangle$ would simulate the ground state of $H(t)$ if t is slowly increased from $t = 0$. In this way one could obtain the ground state of the Hamiltonian of our example for some J where the set of excited modes in the ground state has two connected components. Since the ground state $|\psi_0\rangle$ of H_0 corresponds to that of the XX model, this would allow to experimentally observe how the integer number c characteristic of the Virasoro algebra of the underlined CFT changes from one critical phase to another.

Consider now long-range interactions of the form $h_N(x) = x^{-\nu}$ for $1 \le x \le N/2$. Using the polylogarithm function $\mathrm{Li}_\nu(z) = \sum_{j=1}^{\infty} z^j/j^\nu$ with $|z| < 1$ we obtain $\mathcal{E}(p) = 2\,\zeta_{\mathrm{R}}(\nu) - 2\,\mathrm{Re}\,\mathrm{Li}_\nu(\mathrm{e}^{\mathrm{i}p})$ for the corresponding dispersion relation. In the previous expression ζ_{R} is the Riemann zeta function; in particular, for $\nu = 2$, the dispersion relation reduces to that of the Haldane–Shastry spin chain (see, e.g., [**95**]). Finally, considering interactions $h_N(x) = x^{-2} - Jx^{-3}$ one has

$$\mathcal{E}(p) = p(2\pi - p)/2 - 2\,J\Big(\zeta_{\mathrm{R}}(3) - \mathrm{Re}\,\mathrm{Li}_3(\mathrm{e}^{\mathrm{i}p})\Big)$$

and thus $\mathcal{E}'(p) = (\pi - p)(1 - J\,\varphi(p))$ with $\varphi(p) = 2\,\mathrm{Im}\,\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}p})/(\pi - p)$. The function $\varphi(p)$ increases monotonically over $(0, \pi)$, with $\varphi(0) = 0$ and

$$\lim_{p\to\pi-} \varphi(p) = 2 \lim_{p\to\pi-} \mathrm{Li}_2'(\mathrm{e}^{\mathrm{i}p}) = 4$$

[15] Which seems to be actually the case, see e.g. [**85**] for the Ising and [**26**] or [**27**] for the XY models. Indeed, the phase transition of the XX chain with N sites could be efficiently simulated with a quantum computer of $\log N$ qubits.

since $\mathrm{Li}_2'(z) = \mathrm{Li}_1(z)/z = -\ln(1-z)/z$. So $\mathcal{E}'(0) = \pi > 0$, and if $J > 1/\ln 4$ there exist points with $\varphi(p) > 1/J$ and $\mathcal{E}'(p) < 0$. Note that $1/\ln 4 = 0.721\cdots$; in particular, if $1/\ln 4 < J < 1$then $h_N(x) > 0$ for all $x \geq 1$ although $\mathcal{E}(p)$ is nonmonotonic.

6. Thermodynamics

The examples of the previous section show that the different critical phases are characterized by the number of solutions of the equation $\mu = \mathcal{E}(p)$. If the dispersion relation has $d+1$ critical points in $[0,\pi]$, the equation $\mu = \mathcal{E}(p)$ has d solutions in $(0,\pi)$ if $\mu_{\min} < \mu < \mu_{\max}$ where $\mu_{\min}$ is the greatest of all minima of the dispersion relation and $\mu_{\max}$ is the smallest of all maxima. Clearly, in this case the set of modes with negative energy, the Fermi sea, is not connected and has exactly d connected components. Our results regarding the entanglement entropy showed that the different critical phases are characterized by the number of solutions of the latter equation which essentially yields the value of the central charge of the effective CFT that describes the low energy spectrum. However, as mentioned in the Introduction it is well-known that both the entanglement entropy of the ground state and the low temperature behaviour of the free energy per unit length $f(T)$ contain information about the central charge of a critical model (see, e.g., Refs. [**P2**]-[**P3**] and references therein). For a $(1+1)$ dimensional CFT the free energy per unit length is approximately given by

$$f = f_0 - \frac{\pi c}{6 v_{\mathrm{s}}} T^2$$

where f_0 is a constant, v_{s} is the effective speed of the conformal fields of the theory [**2**, **21**] (the speed of sound) and c is the central charge associated to the Virasoro algebra.

In order to corroborate our results, we have also computed the free energy per particle $f(T)$ of these models at low temperatures. As we shall see, it only depends on the value of the derivative of the dispersion relation at the points in which it equals the chemical potential. Apart from confirming the results on the value of the central charge derived from the entanglement entropy of the ground state, the computation of $f(T)$ makes it possible to derive all relevant thermodynamic quantities like the internal energy, the entropy, the specific heat, the susceptibility or the mean occupation number. They are presented and plotted in Refs. [**P2**]-[**P3**].

We have shown in Refs. [P2]-[P3] that $f(T)$ coincides with that of a conformal field theory with central charge determined by the Fermi surface; here we shall only sketch its computation. In the first place the free energy per particle is expressed as

$$f(T) = -\frac{T}{\pi}\int_0^\pi \ln\left(1+\mathrm{e}^{-\beta(\mathcal{E}(p)-\mu)}\right)\mathrm{d}p = f_0 - \frac{T}{\pi}\int_0^\pi \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p\,,$$

where $f_0 = f(0) = \int_{\mathcal{E}(p)<\mu}(\mathcal{E}(p)-\mu)\mathrm{d}p$. Suppose now that the equation $\mathcal{E}(p) = \mu$ has $m+1$ simple roots in the interval $(0,\pi)$ and order them increasily as $p_0 < p_1 < \cdots < p_m$. Let P_i be a closed interval of radius Δp centered at p_i and choose Δp small enough such that $P_i \cap P_j = \emptyset$ and $\mathcal{E}'(p) \neq 0$ on $\bigcup P_i$. Note that such a Δp always exists since we are assuming that the roots of the equation $\mathcal{E}(p) = \mu$ are simple. Finally write $Q = [0,\pi] \setminus \bigcup P_i$ for the complement of the union of the intervals P_i in $[0,\pi]$. We obtain

$$I = \int_0^\pi \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p = \int_Q \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p + \sum_{i=0}^m \int_{P_i} \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p\,.$$

The integral over Q is exponentially small in $\beta = T^{-1}$. Indeed let us write $q = \min_{\overline{Q}}|\mathcal{E}(p)-\mu|$ where $\overline{Q}$ denotes the closure of $\overline{Q}$; clearly $q > 0$ since $\mathcal{E}(p) \neq \mu$ on the compact $\overline{Q}$. We conclude that

$$\int_Q \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p < |Q|\mathrm{e}^{-\beta q} < \pi\mathrm{e}^{-\beta q}\,.$$

On the other hand, the low-T behaviour of the remaining integrals was also determined in Refs. [P2]-[P3] in terms of $v_i = |\mathcal{E}'(p_i)|$, namely $I = \sum_i I_i$ with

$$I_i = \int_{P_i} \ln\left(1+\mathrm{e}^{-\beta|\mathcal{E}(p)-\mu|}\right)\mathrm{d}p = \frac{\pi^2 T}{6\,v_i} + O(T^2)\,.$$

We thus have that

$$f = f_0 - \frac{\pi T^2}{6}\sum_{i=0}^m \frac{1}{v_i} + O(T^3)\,.$$

is the expression for the free energy at low temperatures of a CFT with $m+1$ free bosons with Fermi velocities v_i. In particular, the central charge of the model is $c = m+1$ in full agreement with the result obtained through the ground state entanglement entropy.

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A duality principle for the multi-block entanglement entropy of free fermion systems

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The analysis of the entanglement entropy of a subsystem of a one-dimensional quantum system is a powerful tool for unravelling its critical nature. For instance, the scaling behaviour of the entanglement entropy determines the central charge of the associated Virasoro algebra. For a free fermion system, the entanglement entropy depends essentially on two sets, namely the set *A* of sites of the subsystem considered and the set *K* of excited momentum modes. In this work we make use of a general duality principle establishing the invariance of the entanglement entropy under exchange of the sets *A* and *K* to tackle complex problems by studying their dual counterparts. The duality principle is also a key ingredient in the formulation of a novel conjecture for the asymptotic behavior of the entanglement entropy of a free fermion system in the general case in which both sets *A* and *K* consist of an arbitrary number of blocks. We have verified that this conjecture reproduces the numerical results with excellent precision for all the configurations analyzed. We have also applied the conjecture to deduce several asymptotic formulas for the mutual and *r*-partite information generalizing the known ones for the single block case.

One of the distinguishing features of the quantum realm is the existence of entangled states in composite systems, which have no classical analogue and play a fundamental role in quantum information theory and condensed matter physics (see, e.g., refs 1, 2). A widely used quantitative measure of the degree of entanglement between two subsystems A, B of a quantum system $A\cup B$ in a pure state $\rho=|\psi\rangle\langle\psi|$ is the Rényi entanglement entropy[3] $S_\alpha(A)=(1-\alpha)^{-1}\log \mathrm{tr}(\rho_A^\alpha)$, where ρ_A is the reduced density matrix of the subsystem A and $\alpha>0$ is the Rényi parameter (the von Neumann entropy is obtained in the limit $\alpha\to 1$). It is easy to show that $S_\alpha(A)=S_\alpha(B)$, and that the entanglement entropy vanishes when the whole system is in a non-entangled (product) state. Over the last decade, it has become clear that the study of the entanglement between two extended subsystems of a many-body system in one dimension is a powerful tool for uncovering its criticality properties[4–7]. The reason for this is that one-dimensional critical quantum systems are governed by an effective conformal field theory (CFT) in $(1+1)$ dimensions, whose entanglement entropy can be evaluated in closed form in the thermodynamic limit[8–10]. In the simplest case, when the subsystem A consists of a single interval of length L and the whole system is in its ground state, the scaling of $S_\alpha(A)$ for $L\to\infty$ is determined solely by the central charge c. In order to probe the full operator content of the CFT, one needs to analyze more complicated situations in which the set A is the union of a finite number of intervals. In fact, in the last few years there has been a considerable interest in this problem, both for CFTs and one-dimensional lattice models (integrable spin chains or free fermion systems), as witnessed by the number of papers published on this subject (see, e.g., refs 11–18).

In this work we shall extend this analysis to the more general case in which the system's state is also made up of several blocks of consecutive excited momentum modes, which has received comparatively less attention[19–24]. An important motivation for dealing with this type of states is that it makes it possible to treat position and momentum space on a more equal footing, thus revealing certain symmetries that have not been fully exploited so far. This approach naturally leads to a duality principle for the behavior of the entanglement entropy under the exchange of the position and momentum space block configurations, which in fact can be exploited to solve problems that up until now had defied an analytic treatment[25] with standard techniques like the Fisher–Hartwig conjecture[26]. We have applied this duality principle to propose a new conjecture on the composability of the

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entanglement entropy in the multi-block case, which yields a closed asymptotic formula for the Rényi entanglement entropy of a free fermion system in the most general multi-block configuration, both in position and momentum space. This formula, which we have numerically verified for a wide range of configurations both for $0<\alpha<1$ and $\alpha \geq 1$, reduces to the known ones when the configuration in momentum space consists of a single block. It also leads to closed asymptotic formulas for the mutual and the tripartite[12] (or r-partite[18]) information, which again agree with the general CFT predictions.

Results and Methods

Preliminaries and notation. The model considered is a system of N free (spinless) hopping fermions with creation operators $a_j^\dagger$ (where the subindex $j=0, \ldots, N-1$ denotes the site) and Hamiltonian $H=\sum_{i,j=0}^{N-1} g_{\mathcal{N}}(i-j)a_i^\dagger a_j$ preserving the total fermion number. We shall further assume that the hopping amplitude g_N satisfies $g_N(k)=g_N(-k)^*=g_N(k+N)$, so that H is Hermitian and translationally invariant. For this reason, it is convenient to introduce the Fourier-transformed creation operators

$$\hat{a}_j^\dagger = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} jl/N} a_l^\dagger, \quad 0 \leq j \leq N-1. \tag{1}$$

It is straightforward to check that the operators $\hat{a}_j, \hat{a}_j^\dagger$ satisfy the canonical anticommutation relations (CAR), and that they diagonalize H. In fact, we have $H=\sum_{l=0}^{N-1}\varepsilon_N(l)\hat{a}_l^\dagger \hat{a}_l$, with $\varepsilon_N(l)=\sum_{j=0}^{N-1} g_N(j)\mathrm{e}^{2\pi \mathrm{i} jl/N}$. It can be shown that the total momentum operator P is also diagonal in this representation, namely $P=\sum_{l=0}^{N-1} p_l \hat{a}_l^\dagger \hat{a}_l$, with $p_l=2\pi l/N \bmod 2\pi$. Thus the operator $\hat{a}_l^\dagger$ creates a (non-localized) fermion with well-defined energy $\varepsilon_N(l)$ and momentum p_l. Note that $\varepsilon_N(l)$ is obviously real for all modes l, and that the model is critical (gapless) if $\varepsilon_N(l)$ vanishes for some l. We shall suppose in what follows that the system is in a pure energy eigenstate

$$|K\rangle \equiv \hat{a}_{k_1}^\dagger \cdots \hat{a}_{k_M}^\dagger |0\rangle, \quad K=\{k_1, \ldots, k_M\} \subset \{0, \ldots, N-1\}, \tag{2}$$

where $|0\rangle$ is the vacuum, consisting of M fermions with momenta $2\pi k_j/N$. We shall be interested in studying the entanglement properties of a subset of sites $A \equiv \{x_1, \ldots, x_L\} \subset \{0, \ldots, N-1\}$ with respect to the whole system when the latter is in the pure state $|K\rangle$. As is well known, these properties are encoded in the reduced density matrix $\rho_A=\mathrm{tr}_B\rho$, where $\rho \equiv |K\rangle\langle K|$ and $B=\{0, \ldots, N-1\}-A$. As mentioned in the Introduction, the degree of entanglement is usually measured using the Rényi entanglement entropy $S_\alpha(A) \equiv (1-\alpha)^{-1}\log \mathrm{tr}(\rho_A^\alpha)$ (with $\alpha>0$). One of the most efficient ways of computing this entropy is to exploit the connection between the reduced density matrix ρ_A and the correlation matrix C_A, defined by

$$(C_A)_{jk} = \langle K|a_{x_j}^\dagger a_{x_k}|K\rangle, \quad 1 \leq j, k \leq L. \tag{3}$$

This matrix is obviously Hermitian, with eigenvalues $\nu_1, \ldots, \nu_L$ lying in the interval $[0, 1]$. Moreover, since the state $|K\rangle$ is determined by the conditions $\hat{a}_k^\dagger|K\rangle = 0$ for $k \in K$ and $\hat{a}_k|K\rangle = 0$ for $k \notin K$, the expectation value $\langle K|\hat{a}_j^\dagger \hat{a}_k|K\rangle$ vanishes for $k \notin K$ and equals δ_{jk} for $k \in K$. From this fact and Eq. (1) we immediately obtain the following explicit expression for the matrix elements of the correlation matrix C_A:

$$(C_A)_{jk} = \frac{1}{N}\sum_{l\in K} \mathrm{e}^{-2\pi \mathrm{i}(x_j-x_k)l/N}, \quad 1 \leq j, k \leq L. \tag{4}$$

As first shown in refs 4, 27, the reduced density matrix ρ_A factors as the tensor product $\rho_A=\otimes_{l=1}^L \rho_A^{(l)}$, where each $\rho_A^{(l)}$ is a 2×2 density matrix with eigenvalues ν_l and $1-\nu_l$. In particular, the spectrum of ρ_A is the set of numbers

$$\rho_A(\varepsilon_1, \ldots, \varepsilon_L) = \prod_{l=1}^L [\nu_l^{\varepsilon_l}(1-\nu_l)^{1-\varepsilon_l}], \quad \varepsilon_l \in \{0, 1\}. \tag{5}$$

Since the Rényi entropy S_α is additive, it follows that

$$S_\alpha(A) = \sum_{l=1}^L S_\alpha(\rho_A^{(l)}) = (1-\alpha)^{-1}\sum_{l=1}^L \log\bigl(\nu_l{}^\alpha + (1-\nu_l)^\alpha\bigr). \tag{6}$$

Note that the latter method for computing $S_\alpha(A)$ is computationally very advantageous, since it is based on the diagonalization of the $L\times L$ matrix C_A as opposed to direct diagonalization of the $2^L \times 2^L$ matrix ρ_A.

As explained above, it is of great interest to determine the (leading) asymptotic behaviour of the entanglement entropy $S_\alpha(A)$ as the size L of the subsystem A tends to infinity. To this end, note first of all that the matrix C_A is Toeplitz (i.e., $(C_A)_{jk}$ depends only on the difference $j-k$) provided that the subsystem A under consideration is a single block, i.e., a set of consecutive sites. Let us further assume that Eq. (4) has a well-defined limit as $N\to\infty$ with L fixed, in the sense that there exists a piecewise smooth density function $c(p)$ such that $(C_A)_{jk} \to (2\pi)^{-1}\int_0^{2\pi} c(p)\,\mathrm{e}^{-\mathrm{i}(j-k)p}\mathrm{d}p$ in this limit. As first shown by Jin and Korepin[5], it is then possible to apply a particular case of the Fisher–Hartwig conjecture[26] proved by Basor[28] to derive an asymptotic formula for the characteristic polynomial of the correlation matrix C_A, and hence for the entanglement entropy $S_\alpha(A)$ (see also refs 23, 24, 29). However, when the subsystem A is not a single block it is clear from Eq. (4) that C_A is not a Toeplitz matrix, and therefore the method just outlined cannot be used to derive the asymptotic behaviour of

$S_\alpha(A)$ for large L. It should also be stressed that the asymptotic result in ref. 5 is only valid for $N \gg L \gg 1$ (i.e., for an *infinite* chain), since the $N\to\infty$ limit with L fixed is taken before letting $L\to\infty$. In particular, the asymptotic behaviour of $S_\alpha(A)$ when $N\to\infty$ with $L/N\to\gamma_x\in(0,1)$ cannot be directly inferred from the latter result. As we shall explain shortly, these drawbacks can be overcome through the use of a duality principle that we shall introduce below.

The dual correlation matrix. We start by defining the projection of the operator $\hat{a}_j^\dagger$ onto the set $\mathcal{L}(\mathcal{H}_A)$ of linear operators from the Hilbert space $\mathcal{H}_A$ of the subsystem A into itself in the obvious way, namely (cf. Eq. (1))

$$\hat{a}_{A,j}^\dagger = \frac{1}{\sqrt{N}}\sum_{l\in A} \mathrm{e}^{2\pi \mathrm{i} jl/N} a_l^\dagger, \tag{7}$$

and similarly for $\hat{a}_{A,j}$. We shall also denote by $\hat{a}_{B,j}^\dagger$, $\hat{a}_{B,j}$ the corresponding projections onto $\mathcal{L}(\mathcal{H}_B)$, so that $\hat{a}_j = \hat{a}_{A,j} + \hat{a}_{B,j}$, $\hat{a}_j^\dagger = \hat{a}_{A,j}^\dagger + \hat{a}_{B,j}^\dagger$. We then define the *dual correlation matrix* $\widehat{C}_A$ as the $M\times M$ matrix with elements

$$(\widehat{C}_A)_{lm} = \langle 0|\hat{a}_{A,k_l}\hat{a}_{A,k_m}^\dagger|0\rangle, \quad 1\le l,\, m\le M. \tag{8}$$

The dual correlation matrix $\widehat{C}_B$ of the complementary set B is defined similarly. The analogue of the matrix $\widehat{C}_A$ for continuous systems, usually called the overlap matrix, was originally introduced by Klich[30] and has been extensively used in the literature (see, e.g., ref. 31). From the definition (7) of the projected operators $\hat{a}_{A,j}^\dagger$ we immediately obtain the explicit formula

$$(\widehat{C}_A)_{lm} = \frac{1}{N}\sum_{j\in A} \mathrm{e}^{-2\pi \mathrm{i}(k_l-k_m)j/N}, \quad 1\le l,\, m\le M. \tag{9}$$

Comparison with Eq. (4) shows that $\widehat{C}_A$ is obtained from C_A by exchanging the roles played by the sites $x_j\in A$ and the excited modes $k_l\in K$, which justifies the term "dual correlation matrix". We shall show in what follows that this duality can be successfully exploited to obtain the asymptotic behaviour of $S_\alpha(A)$ in situations in which the usual approach based on the correlation matrix C_A is not feasible.

The matrix $\widehat{C}_A$ is clearly Hermitian and positive semidefinite, since for all $z_1, \ldots, z_M \in \mathbb{C}$ we have $\sum_{l,m=1}^M (\widehat{C}_A)_{lm}\, z_l^* z_m = \|(\sum_{l=1}^M z_m \hat{a}_{A,k_m}^\dagger)|0\rangle\|^2$. Thus the eigenvalues $\hat{\nu}_1, \ldots, \hat{\nu}_M$ of $\widehat{C}_A$ are nonnegative. Using the identities $\langle 0|\mathcal{O}_A\mathcal{O}'_B|0\rangle = \langle 0|\mathcal{O}'_B\mathcal{O}_A|0\rangle = 0$, where $\mathcal{O}_A$ and $\mathcal{O}'_B$ are linear operators respectively supported on A and B, it is straightforward to check that $\widehat{C}_B = \mathbb{1}_M - \widehat{C}_A$. Since $\widehat{C}_B$ is also positive semidefinite, from the previous relation it follows that $\hat{\nu}_i \in [0,\, 1]$ for all $i = 1, \ldots, M$. Moreover, the Hermitian character of $\widehat{C}_A$ implies that there exists a unitary $M\times M$ matrix $U \equiv (u_{lm})_{1\le l,m\le M}$ such that $U\widehat{C}_A U^\dagger = \mathrm{diag}(\hat{\nu}_1, \ldots, \hat{\nu}_M)$, and hence $U\widehat{C}_B U^\dagger = \mathbb{1} - U\widehat{C}_A U^\dagger = \mathrm{diag}(1-\hat{\nu}_1, \ldots, 1-\hat{\nu}_M)$. We then define the corresponding rotated operators $\hat{c}_l = \sum_{m=1}^M u_{lm}\hat{a}_{k_m}$ $(1\le l\le M)$, which together with their adjoints satisfy the CAR by the unitarity of U. We shall also need the projections of the latter operators onto the spaces $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$, namely

$$\hat{c}_{A,l} = \sum_{m=1}^M u_{lm}\hat{a}_{A,k_m}, \quad \hat{c}_{B,l} = \sum_{m=1}^M u_{lm}\hat{a}_{B,k_m} = \hat{c}_l - \hat{c}_{A,l}, \tag{10}$$

and similarly for their adjoints. From the above definitions it follows that the vacuum correlators of the operators $\{\hat{c}_{A,l},\, \hat{c}_{A,l}^\dagger\}$ and $\{\hat{c}_{B,l},\, \hat{c}_{B,l}^\dagger\}$ are given by

$$\langle 0|\hat{c}_{A,l}\hat{c}_{A,m}^\dagger|0\rangle = \hat{\nu}_l\delta_{lm}, \quad \langle 0|\hat{c}_{B,l}\hat{c}_{B,m}^\dagger|0\rangle = (1-\hat{\nu}_l)\delta_{lm}, \tag{11}$$

and hence $\|\hat{c}_{A,l}^\dagger|0\rangle\|^2 = \hat{\nu}_l, \|\hat{c}_{B,l}^\dagger|0\rangle\|^2 = 1-\hat{\nu}_l$. Following ref. 30, we note that the state $|\varphi\rangle = \hat{c}_1^\dagger\cdots\hat{c}_M^\dagger|0\rangle$ actually differs from $|K\rangle$ by an irrelevant phase, since by definition of the operators $\hat{c}_l$ we have

$$|\varphi\rangle = \sum_{m_1,\ldots,m_M=1}^M u_{1m_1}^*\cdots u_{M,m_M}^*\hat{a}_{k_{m_1}}^\dagger\cdots\hat{a}_{k_{m_M}}^\dagger|0\rangle = \left(\sum_{\sigma\in S_M}(-1)^\sigma u_{1\sigma_1}^*\cdots u_{M,\sigma_M}^*\right)\hat{a}_{k_1}^\dagger\cdots\hat{a}_{k_M}^\dagger|0\rangle = \det U^*|K\rangle,$$

where $(-1)^\sigma$ denotes the sign of the permutation σ. The latter relation implies that $|K\rangle\langle K| = |\phi\rangle\langle\phi|$, a fact that can be exploited in order to derive an expression for the entanglement entropy $S_\alpha(A)$. To this end, for $\hat{\nu}_l \neq 0, 1$ we define the operators $\hat{d}_{A,l}^\dagger = \hat{c}_{A,l}^\dagger/\sqrt{\hat{\nu}_l}$, $\hat{d}_{B,l}^\dagger = \hat{c}_{B,l}^\dagger/\sqrt{1-\hat{\nu}_l}$, so that by Eq. (11) the states $|1\rangle_{A,l} \equiv \hat{d}_{A,l}^\dagger|0\rangle$, $|1\rangle_{B,l} \equiv \hat{d}_{B,l}^\dagger|0\rangle$ are properly normalized. On the other hand, when $\hat{\nu}_l = 0$ the state $\hat{c}_l^\dagger|0\rangle = \hat{c}_{B,l}^\dagger|0\rangle$ is supported on B by Eq. (11), and is normalized, since the operators $\hat{c}_l$, $\hat{c}_l^\dagger$ obey the CAR. Hence in this case we simply set $\hat{d}_{B,l}^\dagger = \hat{c}_{B,l}^\dagger = \hat{c}_l^\dagger$, $|1\rangle_{B,l} = \hat{d}_{B,l}^\dagger|0\rangle$. Similarly, when $\hat{\nu}_l = 1$ we define $\hat{d}_{A,l}^\dagger = \hat{c}_{A,l}^\dagger = \hat{c}_l^\dagger$, $|1\rangle_{A,l} = \hat{d}_{A,l}^\dagger|0\rangle$, and by the previous definitions we thus have $\hat{c}_l^\dagger = \sqrt{\hat{\nu}_l}\hat{d}_{A,l}^\dagger + \sqrt{1-\hat{\nu}_l}\hat{d}_{B,l}^\dagger$ $(1\le l\le M)$, and therefore $|\varphi\rangle = \otimes_{l=1}^M(\sqrt{\hat{\nu}_l}|1\rangle_{A,l}|0\rangle_{B,l} + \sqrt{1-\hat{\nu}_l}|0\rangle_{A,l}|1\rangle_{B,l})$,where $|0\rangle_{A,b}$, $|0\rangle_{B,l}$ denote the vacuum state in the l-th mode (with respect to the $\hat{c}_m^\dagger$ operators) supported respectively on A or B. Using the identity $|K\rangle\langle K| = |\phi\rangle\langle\phi|$ and tracing over the degrees of freedom of the subsystem B we easily arrive at the fundamental formula

$$\rho_A = \bigotimes_{l=1}^{M} (\hat{\nu}_l |1\rangle_{A,l}\langle 1|_{A,l} + (1-\hat{\nu}_l)|0\rangle_{A,l}\langle 0|_{A,l}) \,. \tag{12}$$

In particular, the spectrum of the matrix ρ_A is the set of numbers

$$\rho_A(\varepsilon_1, \ldots, \varepsilon_M) = \prod_{l=1}^{M} [\hat{\nu}_l^{\varepsilon_l}(1-\hat{\nu}_l)^{1-\varepsilon_l}] \,, \quad \varepsilon_l \in \{0,1\} \,, \tag{13}$$

up to zero eigenvalues. From the additivity of the Rényi entropy and Eqs (12) or (13) it follows that the entanglement entropy $S_\alpha(A)$ is given by

$$S_\alpha(A) = (1-\alpha)^{-1} \sum_{l=1}^{M} \log(\hat{\nu}_l^{\alpha} + (1-\hat{\nu}_l)^{\alpha}), \tag{14}$$

which can be interpreted as the dual of Eq. (6).

The duality principle. As we have seen in the previous subsection, the Rényi entanglement entropy $S_\alpha(A)$ can be computed in two equivalent ways, using the "coordinate" correlation matrix C_A and its "dual" $\widehat{C}_A$ (cf. Eqs (6–14)). This fact strongly suggests the existence of a deeper duality principle that applies to the reduced density matrix ρ_A itself, as evidenced by Eqs (5–13). To formulate this principle, we shall introduce the more precise notation $\rho_A(K)$ to denote the reduced density matrix of the subsystem A when the whole system is in the pure energy eigenstate $|K\rangle$ given by Eq. (2). It should be borne in mind that in this notation both sets A and K are subsets of $\{0, \ldots, N-1\}$, with the subindex always labelling the subsystem sites (in position space) and the argument the set of excited momenta. Let spec$T$ stand for the spectrum of the matrix T, i.e., the set of its eigenvalues, each counted with its respective multiplicity. Likewise, we shall denote by $\mathrm{spec}_0\rho$ the spectrum of a density matrix ρ excluding its zero eigenvalues, i.e., $\mathrm{spec}_0\rho = \mathrm{spec}(\rho|_{(\ker\rho)^\perp})$. We shall then say that two density matrices ρ_i $(i=1,2)$ *are similar up to zero eigenvalues* if $\mathrm{spec}_0\rho_1 = \mathrm{spec}_0\rho_2$, i.e., ρ_1 and ρ_2 have the same nonzero eigenvalues with the same multiplicities. We are now ready to state the following fundamental result:

Theorem 1. *The reduced density matrices $\rho_A(K)$ and $\rho_K(A)$ are similar up to zero eigenvalues.*

Proof. Indeed, by Eqs (5–13) the spectrum of $\rho_A(K)$ excluding the zero eigenvalues can be written in the two equivalent ways

$$\begin{aligned} \mathrm{spec}_0(\rho_A(K)) &= \left\{ \prod_{l=1}^{L} \nu_l^{\varepsilon_l}(1-\nu_l)^{1-\varepsilon_l} | \varepsilon_l \in \{0,1\}, \nu_l \notin \{0,1\} \right\} \\ &= \left\{ \prod_{m=1}^{M} \hat{\nu}_m^{\varepsilon_m}(1-\hat{\nu}_m)^{1-\varepsilon_m} | \varepsilon_m \in \{0,1\}, \hat{\nu}_m \notin \{0,1\} \right\}. \end{aligned} \tag{15}$$

Let us denote by $C_A(K)$ and $\widehat{C}_A(K)$ the correlation matrix (4) and its dual version (9). We then have $\widehat{C}_A(K) = C_K(A), C_A(K) = \widehat{C}_K(A)$, and consequently the sets $\{\nu_l\}_{l=1}^{L}$ and $\{\hat{\nu}_m\}_{m=1}^{M}$ are interchanged by the duality transformation $A \leftrightarrow K$. Applying Eq. (15) to the reduced density matrix $\rho_K(A)$ we conclude that $\mathrm{spec}_0(\rho_A(K)) = \mathrm{spec}_0(\rho_K(A))$, as claimed. □

If S is any entropy functional, from now on we shall use the more precise notation $S(A;K) = S(\rho_A(K))$. Obviously, from the Shannon–Khinchin axioms it follows that two density matrices which are similar up to zero eigenvalues necessarily have the same entropy. From this fact and the previous theorem one can immediately deduce the important *duality principle*

$$S(A;K) = S(K;A) \,, \tag{16}$$

valid for any entropy functional S.

As a first application of this general principle, we shall rigorously derive an asymptotic expression for the Rényi entanglement entropy of a subsystem A consisting of $r>1$ disjoint blocks of consecutive spins when the set K of excited momenta is a single set of M consecutive integers, valid in the limit $N \gg M \gg 1$. More precisely, let $A = \cup_{i=1}^{r} [U_i, V_i)$, $K = [P, Q)$, where $[U_i, V_i)$ denotes the set of all integers l such that $U_i \le l < V_i$ (so that the cardinal of $[U_i, V_i)$ is $V_i - U_i$), and similarly for $[P, Q)$. We first let $N \to \infty$ with M fixed and assume that the following limits exist:

$$\lim_{N\to\infty} \frac{2\pi U_i}{N} \equiv u_i, \quad \lim_{N\to\infty} \frac{2\pi V_i}{N} \equiv v_i,$$

with $u_i, v_i \in [0, 2\pi]$, $u_{i+1} - v_i > 0$, $v_r - u_1 < 2\pi$. We shall be interested in the asymptotic behavior of the Rényi entropy S_α as $M \to \infty$. Thus the problem at hand is precisely the dual of the one solved in refs 20, 24. with the help of the Fisher–Hartwig conjecture. One of the main results of the latter references can be recast in the present context as the asymptotic formula

$$S_\alpha([U,\,V);\,\bigcup_{j=1}^{s}\,[P_j,\,Q_j)) \sim b_\alpha\Big(s\log L + \sum_{j=1}^{s}\log\big(2\sin(\tfrac{q_j-p_j}{2})\big) + \log f(\mathbf{p},\,\mathbf{q})\Big) + sc_\alpha, \tag{17}$$

where $p_j \equiv \lim_{N\to\infty}(2\pi P_j/N)$, $q_j \equiv \lim_{N\to\infty}(2\pi Q_j/N)$,

$$b_\alpha = \frac{1}{6}\Big(1+\frac{1}{\alpha}\Big), \quad c_\alpha = \frac{1}{1-\alpha}\int_0^\infty\left(\alpha\,\mathrm{csch}^2 t - \mathrm{csch}\,t\,\mathrm{csch}(t/\alpha) - \frac{1-\alpha^2}{6\alpha}\,e^{-2t}\right)\frac{dt}{t},$$

$$f(\mathbf{p},\,\mathbf{q}) = \prod_{1\le i<j\le s}\frac{\sin(\frac{q_j-p_i}{2})\sin(\frac{p_j-q_i}{2})}{\sin(\frac{p_j-p_i}{2})\sin(\frac{q_j-q_i}{2})} \tag{18}$$

and the ~ notation means that the difference between the LHS and the RHS tends to 0 as $L\to\infty$. From the duality relation (16) and Eqs (17), (18) it then follows that when $M\to\infty$ we have

$$S_\alpha(\bigcup_{i=1}^{r}\,[U_i,\,V_i);\,[P,\,Q)) = S_\alpha([P,\,Q);\,\bigcup_{i=1}^{r}\,[U_i,\,V_i)) \sim b_\alpha[r\log M + \sum_{i=1}^{r}\log\big(2\sin(\tfrac{v_i-u_i}{2})\big) + \log f(\mathbf{u},\,\mathbf{v})] + rc_\alpha. \tag{19}$$

Taking into account that $f(\mathbf{u},\mathbf{v})=1$ when $r=1$, from the previous formula we obtain the remarkable relation

$$S_\alpha(\bigcup_{i=1}^{r}\,[U_i,\,V_i);\,[P,\,Q)) \sim \sum_{i=1}^{r}S_\alpha([U_i,\,V_i);\,[P,\,Q)) - I_\alpha(\mathbf{u},\,\mathbf{v}),\ \text{with}\ I_\alpha(\mathbf{u},\,\mathbf{v}) \equiv -\,b_\alpha\log f(\mathbf{u},\,\mathbf{v})\,, \tag{20}$$

where the last term can be naturally interpreted as an asymptotic approximation to the *mutual information* shared by the blocks $[U_1, V_1), \ldots, [U_r, V_r)$. We believe that this is the first time that this asymptotic formula, which agrees with well-known CFT results, has been rigorously established using the (proved part of the) Fisher–Hartwig conjecture.

It is important to keep in mind the limiting process leading to Eq. (19) in order to correctly assess its limit of validity. For instance, using the connection between one-dimensional critical systems and $1+1$ dimensional CFTs it follows that the asymptotic behavior of S_α is given (in our notation) by[11,25]

$$S_\alpha(\bigcup_{i=1}^{r}\,[U_i,\,V_i);\,[P,\,Q)) \sim b_\alpha\Big[r\log\Big(\frac{N}{\pi}\sin\Big(\pi\,\frac{M}{N}\Big)\Big) + \sum_{i=1}^{r}\log(v_i-u_i) + \log f^{(\infty)}(\mathbf{u},\,\mathbf{v})\Big] + rc_\alpha, \tag{21}$$

where $f^{(\infty)}(\mathbf{u},\mathbf{v})$ is the product of cross ratios

$$f^{(\infty)}(\mathbf{u},\,\mathbf{v}) = \prod_{1\le i<j\le r}\frac{(v_j-u_i)(u_j-v_i)}{(u_j-u_i)(v_j-v_i)}. \tag{22}$$

The apparent discrepancy between the latter formulas and Eqs (18), (19) is easily explained taking into account that the limiting process in the latter references is the *dual* of the present one, namely $N\to\infty$ with *fixed* U_i, V_i and $2\pi P/N\to p$, $2\pi Q/N\to q$. In other words, Eqs (18), (19) apply when $N\gg M\gg 1$ and arbitrary $L<N$, while Eqs (21), (22) are valid for $N\gg L\gg 1$ and arbitrary $M<N$. It is also obvious that both approaches coincide in the (rather uninteresting) case in which both M/N and L/N tend to zero. On the other hand, it should be apparent that neither Eqs (18), (19) nor (21)-(22) are valid in the general situation in which *both* L/N and M/N tend to a nonzero limit as $N\to\infty$. In fact, it is clear a priori that none of these formulas can hold in the latter range, since they are not consistent with the invariance under complements identity $S(A;K)=S(A^c;K)$ and its dual consequence $S(A;K)=S(A;K^c)$, where A^c and K^c respectively denote the complements of A and K with respect to the set $\{0,\ldots,N-1\}$.

Our next objective is to find an extension of Eqs (19) and (21) valid in the general case in which both L/N and M/N tend to nonzero limits γ_x and γ_p as $N\to\infty$. To this end, consider first the simplest case in which $r=s=1$. By translation invariance and criticality, as $N\to\infty$ we must have $S_\alpha([U,\,V);\,[P,\,Q)) \sim b_\alpha\log N + \sigma_\alpha(\gamma_x,\,\gamma_p)$, where $\gamma_x=(V-U)/N$, $\gamma_p=(Q-P)/N$ and σ_α satisfies: (i) $\sigma_\alpha(\gamma_x,\gamma_p)=\sigma_\alpha(\gamma_p,\gamma_x)$ (on account of the duality principle (16)), (ii) $\sigma_\alpha(\gamma_x,\gamma_p)=\sigma_\alpha(1-\gamma_x,\gamma_p)$ (by the invariance of the entropy under complements), (iii) $\sigma_\alpha(\gamma_x,\gamma_p)=b_\alpha\log(2\gamma_x\cdot\sin(\pi\gamma_p))+c_\alpha+o(1)$, with $\lim_{\gamma_x\to 0}o(1)=0$ (by Eq. (21) with $r=1$). (In fact, combining conditions i) with ii) and iii) it immediately follows that $\sigma_\alpha(\gamma_x,\gamma_p)=\sigma_\alpha(\gamma_x,\,1-\gamma_p)$ and $\sigma_\alpha(\gamma_x,\gamma_p)=b_\alpha\log(2\gamma_p\sin(\pi\gamma_x))+c_\alpha+o(1)$, where $o(1)\to 0$ as $\gamma_p\to 0$.) Obviously, the simplest function satisfying the previous requirements is $\sigma_\alpha(\gamma_x,\,\gamma_p) = b_\alpha\log\big(\frac{2}{\pi}\sin(\pi\gamma_x)\sin(\pi\gamma_p)\big) + c_\alpha$, obtained from Eq. (21) with $r=1$ by the replacement $\pi\gamma_x \mapsto \sin(\pi\gamma_x)$. Numerical calculations show that for all $\alpha>0$ the correct asymptotic formula for $S_\alpha([U;\,V);\,[P,\,Q))$ is indeed the simplest one, namely

$$S_\alpha([U,\,V);\,[P,\,Q)) \sim b_\alpha\log\Big(\frac{2N}{\pi}\sin(\pi\gamma_x)\sin(\pi\gamma_p)\Big) + c_\alpha \tag{23}$$

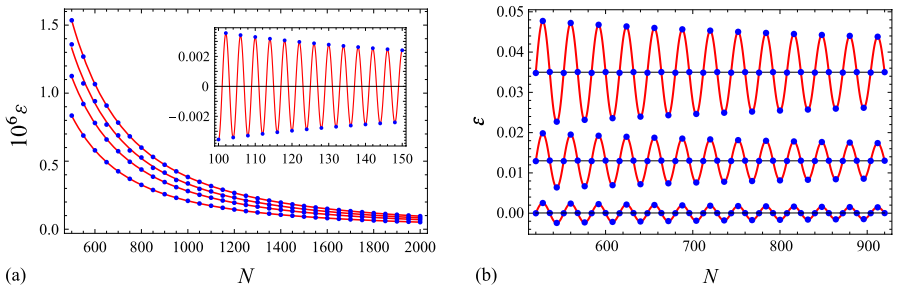

Figure 1. Difference ε between the exact value of the Rényi entropy, computed via Eq. (6) by numerical diagonalization of the correlation matrix (4), and its asymptotic approximation (23) for (**a**) $\gamma_x = \gamma_p = 1/2$ and (**b**) $\gamma_x = 1/8$, $\gamma_p = 1/4$. In panel (a) we have shown the cases (bottom to top) $\alpha = 3/5, 2/3, 3/4, 1$ (von Neumann entropy) and $\alpha = 2$ (inset), while panel (b) depicts the cases $\alpha = 2, 5/2, 3$ (bottom to top, with the horizontal axis displaced respectively by 0.013 and 0.035 in the last two cases to avoid overlap). The solid red lines represent the curves providing the best fits of the data to the laws aN^{-2} (main panel (a)) and $aN^{-2/\alpha}\cos(2\pi\gamma_x\gamma_p N)$ (inset of panel (a) and panel (b)).

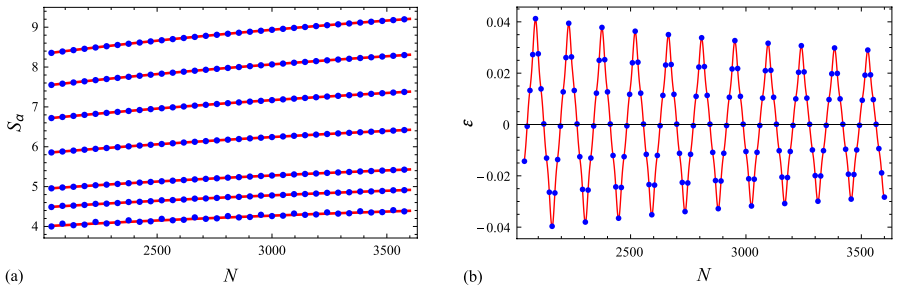

Figure 2. (**a**) Exact Rényi entropy S_α (blue dots) vs. its asymptotic approximation (24) (continuous red line) for a subsystem consisting of three equispaced blocks of equal length $N/12$ when the whole system's state (2) is made up of a sequence of consecutive excited modes of length $N/12$ ($r = 3$, $s = 1$, $\gamma_x = 1/4$, $\gamma_p = 1/12$). The values of the Rényi parameter α considered are (from top to bottom) 1/2, 3/5, 3/4, 1, 3/2, 2 and 3. (**b**) Difference ε between the exact entropy S_3 and its approximation (24) in the previous configuration as a function of the number of fermions N. The continuous red line is the graph of the function $f(N)N^{-2/3}$, with $f(N) = -5.54238\cos(\nu_0 N) - 0.742586\cos(3\nu_0 N) - 0.39794\cos(5\nu_0 N)$ and $\nu_0 = 2\pi\gamma_x\gamma_p/r = \pi/72$.

(see, e.g., Fig. 1 (a) for the most "unfavourable" case $\gamma_x = \gamma_p = 1/2$). This conclusion is also in agreement with the analogous result in ref. 32 for the *XX* model. In fact, we found the leading correction to the approximation (23) to be monotonic in N and $O(N^{-2})$ for $\alpha = 1$, and $O(\cos(2\pi\gamma_x\gamma_p N)N^{-2/\alpha})$ for $\alpha > 1$ (cf. Fig. 1). This behaviour qualitatively agrees with the results of ref. 33 for the error of the Jin–Korepin asymptotic formula for the Rényi entanglement entropy of the ground state of the *infinite XX* chain (Eq. (23) with $\sin(\pi\gamma_x)$ replaced by $\pi\gamma_x$). On the other hand, in the case $0 < \alpha < 1$ (which was not addressed in the latter reference), our numerical calculations suggest that the correction to Eq. (23) is monotonic and $O(N^{-2})$.

At this point, it is very natural to assume that Eq. (20) and its dual are valid for *all* values of the parameters $\gamma_x, \gamma_p \in (0, 1)$, and not just for $\gamma_p \ll 1$ or $\gamma_x \ll 1$, respectively. The latter assumption and Eq. (23) thus lead to the asymptotic formulas

$$S_\alpha\Big(\bigcup_{i=1}^{r}\,[U_i,\,V_i];\,[P,\,Q)\Big) \sim b_\alpha\Big[r\log\Big(\frac{2N}{\pi}\sin(\pi\gamma_p)\Big) + \sum_{i=1}^{r}\log\sin\Big(\tfrac{v_i-u_i}{2}\Big)\Big] - I_\alpha(\mathbf{u},\,\mathbf{v}) + rc_\alpha, \tag{24}$$

$$S_\alpha\Big([U,\,V);\,\bigcup_{i=1}^{s}\,[P_i,\,Q_i)\Big) \sim b_\alpha\Big[s\log\Big(\frac{2N}{\pi}\sin(\pi\gamma_x)\Big) + \sum_{i=1}^{s}\log\sin\Big(\tfrac{q_i-p_i}{2}\Big)\Big] - I_\alpha(\mathbf{p},\,\mathbf{q}) + sc_\alpha. \tag{25}$$

In fact, the validity of the latter equations can be justified by noting that one can go from Eq. (17), which holds for an *infinite* chain, to its analogue for a finite chain by the usual procedure[18,32] of replacing the "arc distance" L by the chord length $(N/\pi)\sin(\pi L/N) = (N/\pi)\sin(\pi\gamma_x)$. In this way Eq. (17) immediately yields Eq. (25), which implies its counterpart (24) by the duality principle (16).

Again, our numerical calculations for several block configurations and a wide range of values of the Rényi parameter α fully corroborate the validity of Eqs (24), (25) (see, e.g., Fig. 2). More precisely, our numerical analysis suggests that for sufficiently large N the error term in the latter equations behaves as $f(N)O(N^{-\min(2,2/\alpha)})$, where $f(N)$ is a periodic function of N. In particular, the error term may not be monotonic in N even for $\alpha \leq 1$, in contrast with what happens in the $r=s=1$ case. The above results are in agreement with those reported in ref. 16 for the (infinite) XY chain and its corresponding free fermion model with $\alpha>1$, $r=2$ and $s=1$.

Multi-block entanglement entropy: conjecture for the general case. We shall address in this section the general problem, in which both sets A and K consist of several blocks of consecutive sites or modes, respectively. To the best of our knowledge, an asymptotic formula for the entanglement entropy in this case has not previously appeared in the literature. As explained above, the main difficulty is now that neither the correlation matrix C_A nor its dual $\widehat{C}_A$ are Toeplitz, so that the standard procedure based on the use of the Fisher–Hartwig conjecture to obtain an asymptotic formula for the characteristic polynomial of the correlation matrix C_A (or of its dual $\widehat{C}_A$) is not applicable. Our approach for deriving a plausible conjecture for the asymptotic behavior of S_α in the general case considered in this subsection relies instead on the general duality principle discussed in the previous section (cf. Theorem 1 and Eq. (16)). In addition, we shall make the natural assumption that when the distance between any two consecutive blocks A_i, A_{i+1} is much larger than the maximum block length (i.e., when $\min_{1\leq i\leq r}(u_{i+1}-v_i) \gg \max_{1\leq i\leq r}(v_i-u_i)$, where $u_{r+1}\equiv u_1+2\pi$) the entanglement entropy is asymptotic to the sum of the single block entropies $S_\alpha(A_i;K)$. The motivation behind this assumption is that when the blocks are far apart their mutual influence should be negligible, and the Rényi entropy is of course additive over independent events.

The simplest asymptotic formula satisfying the above assumption is the trivial one $S_\alpha(A;K) \sim \sum_{i=1}^r S_\alpha(A_i;K)$. However, the latter formula cannot be correct, since it violates the duality principle. The obvious way of fixing this shortcoming would be to add the dual term $\sum_{j=1}^s S_\alpha(A;K_j)$ to the RHS, but the resulting formula violates the above assumption. On the other hand, since by Eq. (20) $\sum_{j=1}^s S_\alpha(A;K_j) \sim \sum_{i=1}^r\sum_{j=1}^s S_\alpha(A_i;K_j) - sI_\alpha(\mathbf{u},\mathbf{v})$, and $I_\alpha(\mathbf{u},\mathbf{v}) \sim 0$ when the blocks in coordinate space are far apart, the heuristic formula

$$S_\alpha(A;K) \sim \sum_{i=1}^r S_\alpha(A_i;K) + \sum_{j=1}^s S_\alpha(A;K_j) - \sum_{i=1}^r\sum_{j=1}^s S_\alpha(A_i;K_j) \tag{26}$$

satisfies the above fundamental assumption. This relation is also clearly consistent with the duality principle (16), since the RHS of Eq. (26) is invariant under the exchange of the sets A and K on account of Theorem 1. We are thus led to conjecture that when $N\to\infty$ the Rényi entropy of a configuration with r blocks A_i in coordinate and s blocks K_j in momentum space satisfies the previous relation. Using Eqs (20), its dual and Eq. (23) we immediately arrive at the closed asymptotic formula

$$\begin{aligned} S_\alpha(A;K) \sim\ & rs\Big(b_\alpha \log\Big(\frac{2N}{\pi}\Big) + c_\alpha\Big) + s\Big(b_\alpha \sum_{i=1}^r \log\sin\Big(\tfrac{v_i-u_i}{2}\Big) - I_\alpha(\mathbf{u},\mathbf{v})\Big) \\ & + r\Big(b_\alpha \sum_{i=1}^s \log\sin\Big(\tfrac{q_i-p_i}{2}\Big) - I_\alpha(\mathbf{p},\mathbf{q})\Big). \end{aligned} \tag{27}$$

The latter equation is manifestly consistent with the duality principle stated in Theorem 1, as expected from the previous remark. It is also apparent that Eq. (27) reduces to Eq. (24) or (25) respectively for $s=1$ or $r=1$, as the asymptotic mutual information I_α vanishes for a single block. Moreover, it is straightforward to explicitly check that when the blocks in coordinate space are far apart the RHS reduces to the sum of the asymptotic approximations (25) to the single-block entropies $S_\alpha(A_i;K)$, since $I_\alpha(\mathbf{u},\mathbf{v}) \sim 0$ in this limit. (By duality, a similar remark applies to the case in which the blocks $[P_j,Q_j)$ in momentum space are far apart from each other.) Finally, it is immediate to check that Eq. (27) satisfies the invariance under complements identity. We have verified through extensive numerical calculations with a wide range of configurations in coordinate and momentum space that when $N \gg 1$ Eq. (27) is correct. In fact, for symmetric configurations (consisting of equally spaced blocks of the same length, both in coordinate and momentum space) the error term in the latter equation behaves as $f(N)N^{-\min(2,2/\alpha)}$, where f is again a periodic function. More precisely (for rational γ_x and γ_p), $f(N)$ is well approximated by a trigonometric polynomial $\sum_{k=0}^{k_{\max}} a_k \cos(k\nu N)$ with small $k_{\max}$ (independent of N), where the main frequency ν is the product of $\nu_0 \equiv 2\pi\gamma_x\gamma_p/rs$ with a simple fraction that can be computed from the configuration parameters r, s, γ_x, γ_p. The behavior of the error is very similar in non-symmetric configurations, except that in some cases it appears to decay faster than N^{-2} for $0<\alpha<1$. As an example, in Fig. 3 we present our results for three different configurations with $(r,s)=(3,2)$, $(7,4)$, $(10,5)$. More precisely, the first and last of these configurations are symmetric, while the middle one is (slightly) asymmetric, as detailed in Fig. 4. As can be seen from Fig. 3(d–f), the error in Eq. (27) behaves in these three cases as described above, where the coefficients a_k of the trigonometric polynomial $f(N)$ and its fundamental frequency ν are listed in Table 1.

It should be noted that the asymptotic formula (27), which we have numerically checked for a *finite* chain, easily yields as a limiting case an analogous formula for an infinite chain. Indeed, if in Eq. (27) we let γ_x tend to 0 we have $\sin((v_i-u_i)/2) \simeq \pi(V_i-U_i)/N$, and similarly for the other arguments of the sine functions appearing in the asymptotic mutual information term $I_\alpha(\mathbf{u},\mathbf{v})$. In this way we easily arrive at the analogue of Eq. (27) for an infinite chain, namely

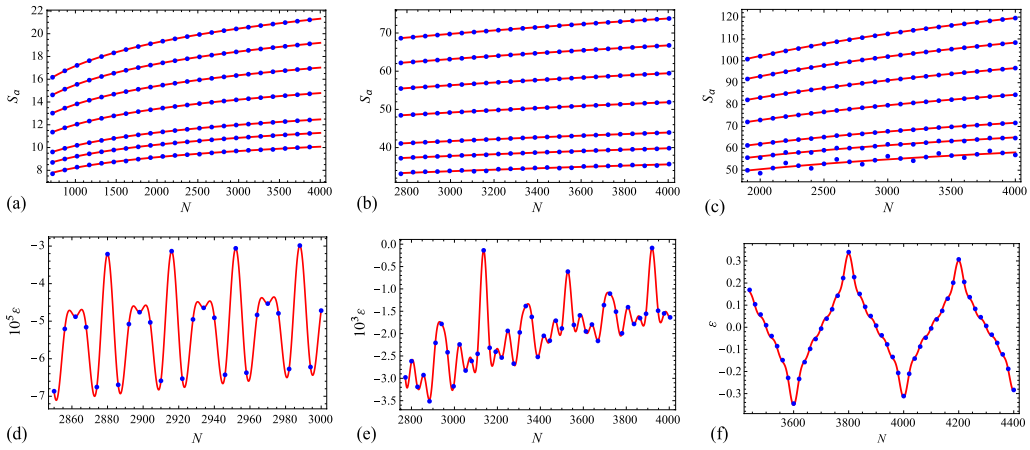

Figure 3. (**a–c**) Exact Rényi entropy S_α (blue dots) and its asymptotic approximation (27) (continuous red line) for $\alpha = 1/2, 3/5, 3/4, 1, 3/2, 2, 3$ (top to bottom) in (**a**) a symmetric configuration with $r = 3$, $s = 2$, $\gamma_x = 1/2$, $\gamma_p = 1/3$, (**b**) an asymmetric configuration with $r = 7$, $s = 4$, $\gamma_x = 1/2$, $\gamma_p = 1/4$ (cf. Fig. 4), and (**c**) a symmetric configuration with $r = 10$, $s = 5$, $\gamma_x = 1/2$, $\gamma_p = 1/4$. (**d–f**) Difference ε between the exact entropy S_α and its approximation (27) for the above configurations and (**d**) $\alpha = 1/2$, (**e**) $\alpha = 1$ (von Neumann entropy), and (**f**) $\alpha = 2$. The red lines represent the corresponding curves $f(N)N^{-\min(2,2/\alpha)}$, with $f(N) = \sum_{k=0}^{k_{\max}} a_k \cos(k\nu N)$ given in Table 1.

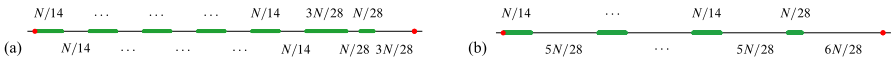

Figure 4. Asymmetric block configuration discussed in Fig. 3(b) in (**a**) coordinate space, (**b**) momentum space (the thick green lines represent the blocks, and the red dots are the two identified endpoints of the chain).

Case	$k_{\max}$	$(a_0, \ldots, a_{k_{\max}})$	ν_0	ν
(d)	2	(−438.485, 105.29, 66.716)	$\pi/18$	ν_0
(e)	14	(−21790.1, 76.0009, 1602.85, 154.097, 5143.99, 397.121, 416.007, 1950.55, 4556.52, 156.444, 756.382, 168.572, 2164.74, 232.817, 2661.63)	$\pi/112$	$2\nu_0/7$
(f)	9	(0, −852.969, 0, −202.359, 0, −99.4396, 0, −57.2755, 0, −55.2294)	$\pi/200$	ν_0

Table 1. Coefficients a_k and fundamental frequency ν of the trigonometric polynomial $f(N) = \sum_{k=0}^{k_{\max}} a_k \cos(k\nu N)$ in the error of Eq. (27) for cases (d)-(f) in Fig. 3.

$$S_\alpha^{(\infty)} \sim sb_\alpha \log\left[\prod_{i=1}^{r}(V_i - U_i)\cdot \prod_{1\le i<j\le r}\frac{(V_j - U_i)(U_j - V_i)}{(U_j - U_i)(V_j - V_i)}\right] + r\left(b_\alpha \sum_{i=1}^{s}\log\sin\left(\tfrac{q_i - p_i}{2}\right) - I_\alpha(\mathbf{p}, \mathbf{q})\right) + rs(b_\alpha \log 2 + c_\alpha). \tag{28}$$

To the best of our knowledge, this general asymptotic formula has not previously appeared in the literature. Note also that for $s = 1$ (i.e., when there is a single block of excited momenta) Eq. (28) implies the asymptotic expression for the mutual information of r blocks in coordinate space conjectured in ref. 25.

From the asymptotic approximation (27) (or its equivalent version Eq. (26)) one can also deduce a remarkable expression for the (asymptotic) mutual information of r blocks $A_i \equiv [U_i, V_i)$ $(1 \le i \le r)$ in position space when the chain is in an energy eigenstate $|K\rangle$ made up of s blocks $K_j \equiv [P_j, Q_j)$ $(1 \le j \le s)$ of excited momentum modes, defined as $\mathcal{I}_\alpha(A_1, \ldots, A_r; K) \equiv \sum_{i=1}^{r} S_\alpha(A_i; K) - S_\alpha(\cup_{i=1}^{r} A_i; K)$. Indeed, using Eqs (20) and (26) we immediately obtain the asymptotic formula

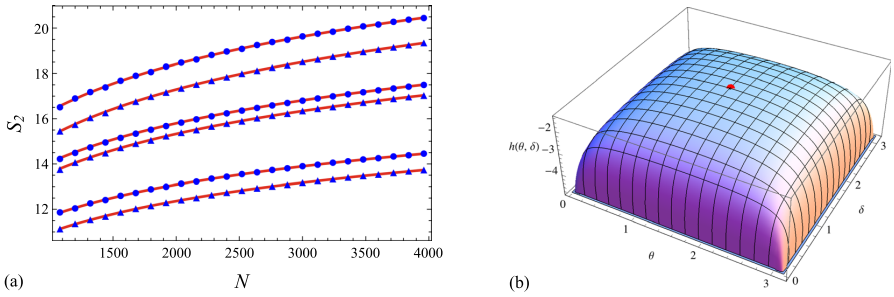

Figure 5. (**a**) Rényi entropy S_2 vs. its asymptotic approximation (24) (red line) in symmetric (blue points) and some non-symmetric (blue triangles) configurations with $\gamma_x = 1/3$, $\gamma_p = 1/2$ and (bottom to top) $4+2$, $5+2$ and $4+3$ blocks. (**b**) 3D plot of the function $h(\theta, \delta)$ in Eq. (31) for $\gamma_x = 1/2$ (the red point corresponds to the symmetric configuration $(\theta, \delta) = (\pi/2, \pi/2)$).

$$\mathcal{I}_\alpha(A_1, \ldots, A_r; K) \sim \sum_{j=1}^{s}\left[\sum_{i=1}^{r} S_\alpha(A_i; K_j) - S_\alpha\Big(\bigcup_{i=1}^{r} A_i; K_j\Big)\right] \sim \sum_{j=1}^{s} I_\alpha(\mathbf{u}, \mathbf{v}) = s I_\alpha(\mathbf{u}, \mathbf{v}). \tag{29}$$

Thus (in the large N limit) the multi-block mutual information $\mathcal{I}_\alpha$ is simply s times the mutual information when the chain's state $|K\rangle$ consists of a single block of consecutive momenta. In particular, we see that $\mathcal{I}_\alpha$ depends only on the *topology* of the state $|K\rangle$ (i.e., the number of blocks of excited momenta), not on its *geometry* (i.e., the particular arrangement and the lengths of these blocks). One could also define the mutual information of s blocks of excited momenta $K_j \equiv [P_j, Q_j]$ $(1 \le j \le s)$ for a fixed configuration $A \equiv \cup_{i=1}^{r} A_i$ in position space. It easily follows from Eq. (29) and the duality principle that this mutual information is asymptotic to $rI_\alpha(\mathbf{p}, \mathbf{q})$. Of course, an analogous formula should hold for the infinite chain replacing the function I_α by its $N \to \infty$ limit $I_\alpha^{(\infty)}(\mathbf{U}, \mathbf{V}) = b_\alpha \log f^{(\infty)}(\mathbf{U}, \mathbf{V})$. In particular, for $s = 1$ the latter expression implies that the model-dependent overall factor appearing in the general formula for the mutual information of a $1+1$ dimensional CFT (see, e.g., refs 11, 13, 18) is equal to 1 for the models under consideration.

An alternative measure of the information shared by the blocks A_i $(1 \le i \le r)$ discussed in ref. 18 is the quantity $\widetilde{\mathcal{I}}_\alpha(A_1, \ldots, A_r) \equiv \sum_{l=1}^{r}(-1)^{l+1}\sum_{1\le i_1<\ldots<i_l\le r} S_\alpha(\cup_{k=1}^{l} A_{i_k})$ (we omit the dependence on the chain's state $|K\rangle$ for conciseness's sake). In particular, for $r = 3$ we obtain the *tripartite information* introduced in ref. 12, whose vanishing characterizes the extensivity of the mutual information $\mathcal{I}_\alpha$. It can be readily checked that the asymptotic relation (27) implies that $\widetilde{\mathcal{I}}_\alpha(A_1, \ldots, A_r)$ vanishes asymptotically for the models under consideration. This follows immediately from Eq. (29) —which is itself a consequence of (27)— and the identities $\sum_{1\le i_1<\ldots<i_l\le r}\sum_{k=1}^{l} S_\alpha(A_{i_k}) = \binom{r-1}{l-1}\sum_{i=1}^{r} S_\alpha(A_i)$, $\sum_{1\le i_1<\ldots<i_l\le r} I_\alpha((u_{i_1}, \ldots, u_{i_l}), (v_{i_1}, \ldots, v_{i_l})) = \binom{r-2}{l-2} I_\alpha(\mathbf{u}, \mathbf{v})$. In particular, this shows that the conjecture (27) implies the asymptotic extensivity of the mutual information $\mathcal{I}_\alpha$ for the models under consideration. (For the infinite chain with $s = 1$, this had already been noted in ref. 25.)

Another noteworthy consequence of the asymptotic formula (27) is the fact that for large N the entanglement entropy can be approximately written as (omitting, for simplicity, its arguments)

$$\begin{aligned} S_\alpha \;\sim\; & rs\Big(b_\alpha \log\Big(\frac{2N}{\pi}\Big) + c_\alpha\Big) + b_\alpha g, \quad \text{with} \quad g \equiv s\Big(\sum_{i=1}^{r} \log\sin\Big(\tfrac{v_i - u_i}{2}\Big) + \log f(u, v)\Big) \\ & + r\Big(\sum_{i=1}^{s} \log\sin\Big(\tfrac{q_i - p_i}{2}\Big) + \log f(\mathbf{p}, \mathbf{q})\Big). \end{aligned} \tag{30}$$

The term in parenthesis in the latter formula, which contains the leading contribution $rsb_\alpha \log N$ to S_α as $N \to \infty$, depends only on the topology of the configuration considered. In particular, from the coefficient of the $\log N$ term we deduce that the models under consideration are critical, behaving as a $1+1$ dimensional CFT with central charge rs. Note also that the fact that the leading asymptotic behavior of the Rényi entanglement entropy S_α depends only on the topology of the configuration in *both* position and momentum space is a generalization of the widespread hypothesis (for the case $r = 1$) that the entanglement properties of critical fermion models are determined by the topology of their Fermi "surface" (see, e.g., ref. 34).

On the other hand, the numerical constant g in the previous equation is independent of N and α, and is solely determined by the geometry of the configuration in both position and momentum space. For instance, for the two symmetric configurations discussed in Fig. 3(a,c) this constant is respectively equal to $-3\log 12$ and $-25\log 1250$.

The asymptotic formula (30) makes it possible to tackle several relevant problems that would otherwise be intractable in practice. For instance, it is natural to conjecture that fixing r, s, γ_x and γ_p the block configuration which maximizes the entropy is the symmetric one (i.e., r equally spaced blocks of equal length in position space, and similarly in momentum space). Our numeric calculations for several configurations suggest that this is indeed the case (see, e.g., Fig. 5(a) for the case $\alpha = 2$). As we see from Eq. (30), this problem reduces to a standard (constrained) maximization problem for the geometric factor g, which in turns splits into two separate problems

for the function $g_1(\mathbf{u}, \mathbf{v}) \equiv \sum_{i=1}^{r} \log \sin\left(\frac{v_i - u_i}{2}\right) + \log f(\mathbf{u}, \mathbf{v})$ and its momentum space counterpart. For instance, when $r = 2$ we can express $g_1(\mathbf{u}, \mathbf{v})$ in terms of the length $L_1 \equiv V_1 - U_1$ of the first block and the interblock distance $d \equiv U_2 - V_1$ as

$$g_1(\mathbf{u}, \mathbf{v}) = \sigma(\theta) + \sigma(2\pi\gamma_x - \theta) + \sigma(2\pi\gamma_x + \delta) + \sigma(\delta) - \sigma(\theta + \delta) - \sigma(2\pi\gamma_x - \theta + \delta) \equiv h(\theta, \delta), \quad (31)$$

where $\sigma(x) \equiv \log \sin(x/2)$, $\theta = 2\pi L_1/N \in (0, 2\pi\gamma_x)$, $\delta = 2\pi d/N \in (0, 2\pi(1-\gamma_x))$. Moreover, from the symmetry of h under $\theta \mapsto 2\pi\gamma_x - \theta$ and $\delta \mapsto 2\pi(1-\gamma_x) - \delta$, it suffices to find the maximum of this function in the rectangle $(0, \pi\gamma_x) \times (0, \pi(1-\gamma_x))$. An elementary calculation shows that h has a local maximum at $\theta = \pi\gamma_x$, $\delta = \pi(1-\gamma_x)$, i.e., at the symmetric configuration, and that ∇h has no other zeros on $(0, \pi\gamma_x] \times (0, \pi(1-\gamma_x)]$. This proves the conjecture in the case $r = 2$ (cf. Fig. 5(b)). For instance, for $r = s = 2$ the maximum value of the entropy is easily found from the latter argument and Eq. (30) to be $4[b_\alpha \log(N \sin(\pi\gamma_x)\sin(\pi\gamma_p)/2\pi) + c_\alpha]$.

Discussion

In this work we have rigorously formulated a general duality principle which posits the invariance of the Rényi entanglement entropy $S(A; K)$ of a chain of free fermions under exchange of the sets of excited momentum modes K and chain sites A of the subsystem under study, where both A and K are the union of an arbitrary (finite) number of blocks of consecutive sites or modes. By means of this principle, we have derived an asymptotic formula for the Rényi entanglement entropy when the set K consists of a single block. From this formula and a natural assumption concerning the additivity of the entropy when the blocks are far apart from each other in either position or momentum space we have conjectured an asymptotic approximation for the entanglement entropy in the general case when both sets A and K consist of an arbitrary number of blocks. We have presented ample numerical evidence of the validity of this formula for different multi-block configurations, and have analyzed its error comparing it with its counterpart for the XX model discussed by Calabrese and Essler[33]. Our conjecture also yields an asymptotic formula for the mutual information of a certain number of blocks in position (or momentum) space valid for arbitrary multi-block configurations, which for $s = 1$ and in the case of an infinite chain is consistent with the general one for $1 + 1$ dimensional CFTs.

The previous results open up several natural research avenues. In the first place, it would be desirable to find a rigorous proof of the fundamental asymptotic relation (26), which leads to the explicit asymptotic formula (27). In particular, it would be of interest to determine the range of models for which this relation holds. Another related problem is to study analytically the precise behavior of the error term in the latter equation. Indeed, our numerical results suggest that this error exhibits a qualitatively similar but considerably more complex behavior than its analogue for an infinite chain with a single block in both position and momentum spaces studied in ref. 33. Finally, an interesting question arising from the discussion after Eq. (30) is the analysis of the configurations *minimizing* the entropy with appropriate constraints, which could be naturally regarded as akin to "semiclassical" states.

Note added in proof. After this article was submitted for review, the authors became aware of the paper by C.H. Lee, P. Ye and X.-L. Qi (J. Stat. Mech.-Theory E. (2014) P10023), in which an alternative proof of Theorem 1 based on previous results of Z. Huang and D.P. Arovas (Phys. Rev. B 86 (2012) 245109) is presented.

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Author Contributions

J.A.C., F.F., A.G.-L. and P.T. contributed equally to this work.

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Critical behavior of su(1|1) supersymmetric spin chains with long-range interactions

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We introduce a general class of su(1|1) supersymmetric spin chains with long-range interactions which includes as particular cases the su(1|1) Inozemtsev (elliptic) and Haldane-Shastry chains, as well as the *XX* model. We show that this class of models can be fermionized with the help of the algebraic properties of the su(1|1) permutation operator and take advantage of this fact to analyze their quantum criticality when a chemical potential term is present in the Hamiltonian. We first study the low-energy excitations and the low-temperature behavior of the free energy, which coincides with that of a $(1+1)$-dimensional conformal field theory (CFT) with central charge $c=1$ when the chemical potential lies in the critical interval $(0,\mathcal{E}(\pi))$, $\mathcal{E}(p)$ being the dispersion relation. We also analyze the von Neumann and Rényi ground state entanglement entropies, showing that they exhibit the logarithmic scaling with the size of the block of spins characteristic of a one-boson $(1+1)$-dimensional CFT. Our results thus show that the models under study are quantum critical when the chemical potential belongs to the critical interval, with central charge $c=1$. From the analysis of the fermion density at zero temperature, we also conclude that there is a quantum phase transition at both ends of the critical interval. This is further confirmed by the behavior of the fermion density at finite temperature, which is studied analytically (at low temperature), as well as numerically for the su(1|1) elliptic chain.

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I. INTRODUCTION

Exactly solvable one-dimensional quantum models are widely used as proving grounds for key ideas in condensed matter physics and the theory of critical phenomena, since their conceptual simplicity often makes it possible to derive exact analytical expressions for the relevant physical quantities. Historically, most of the work in this field has been focused on systems with short-range interactions, like the well-known Heisenberg and (quantum) Ising chains. In the last few years, however, it has become feasible to realize in the laboratory quantum spin chains featuring various types of long-range interactions through different experiments involving, e.g., optical lattices of ultracold Rydberg atoms and trapped ions, or neutral atoms in optical cavities [1–5]. In particular, with the help of hyperfine "clock" states of trapped $^{171}Yb^{+}$ ions it is now possible to simulate quantum spin chains in which the coupling h_{ij} between the ith and jth sites is inversely proportional to a power $\alpha \in (0,3)$ of their distance [1,4]. An important model of this type is the integrable Haldane-Shastry (HS) chain [6,7], whose sites are the equispaced points $z_k = e^{2\pi i k/N}$ $(1 \leqslant k \leqslant N)$ on the unit circle with a coupling proportional to $|z_i - z_j|^{-2}$. In fact, this chain is a limiting case of a more general model due to Inozemtsev, in which the coupling h_{ij} is an elliptic function of the difference $i-j$ with real period N [8].

Although the particles in the original HS chain carried spin $1/2$, the model was shortly generalized to su(m) spin without losing its remarkable integrability properties [9]. As a matter of fact, the su($m|n$) supersymmetric version of the HS chain, originally introduced by Haldane [10], has also been studied in the literature [11,12]. Of particular interest is the su($m|1$) HS chain (with $m>1$), since it is essentially equivalent to an su(m) supersymmetric t-J model [13–15] with exchange and transfer energies proportional to $|z_i - z_j|^{-2}$. This chain, first introduced by Kuramoto and Yokoyama in the su(2) case [16], is an exactly solvable model which provides one of the simplest realizations of spin-charge separation.

In this work we introduce a wide class of su(1|1) supersymmetric spin chains with general translation-invariant couplings $h_{ij}>0$ and a chemical potential term. For zero chemical potential, these models include in particular the supersymmetric elliptic chain studied in Ref. [17] and its two limiting cases, the su(1|1) HS chain and the *XX* model. The class of models under study are technically simpler than their su($m|1$) counterparts, essentially due to the fact that they can be transformed into a system of free spinless fermions in a straightforward way. However, they still exhibit a sufficiently rich structure which makes it possible to examine a number of key properties in the theory of quantum critical systems in an analytic fashion.

More precisely, our main objective is to study whether the models under consideration are quantum critical for suitable values of the chemical potential, and to determine their corresponding central charge. As is well known, a characteristic feature of $(1+1)$-dimensional CFTs is the fact that at low temperature T their free energy per unit length is approximately given (in appropriate units) by $f_0 - \pi c T^2/(6v)$, where f_0 is a constant and v is the effective speed of "sound" [18,19]. Since the low-temperature behavior of f is determined by the low-lying states of the theory, this should also be the case for any one-dimensional quantum system whose low energy spectrum is described by a $(1+1)$-dimensional CFT. In particular, the determination of the low-temperature behavior of the free energy of a one-dimensional critical model provides an efficient way of determining the central charge of its

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underlying CFT. In this way we have been able to show that if the dispersion relation $\mathcal{E}(p)$ is monotonic in the range $[0,\pi]$ the models under study are critical when the chemical potential λ belongs to the open interval $(0,\mathcal{E}(\pi))$, with central charge $c=1$. As further confirmation of this result, we have studied the ground state entanglement entropy, i.e., the entropy of the reduced density matrix of a block of L consecutive spins when the whole chain is in its ground state. Indeed, it is well known that in a $(1+1)$-dimensional CFT the Rényi and von Neumann entanglement entropies scale as $(c/6)(1+1/q)\log L$ and $(c/3)\log L$, respectively, where c is the central charge and q is the Rényi parameter [20–22]. Thus the entanglement entropy of a quantum critical one-dimensional system should be proportional to $\log L$ for $L \gg 1$, where the proportionality constant fixes the central charge of the underlying CFT. Again, we have verified that when the chemical potential belongs to the open critical interval $(0,\mathcal{E}(\pi))$ the entanglement entropy of the models under consideration scales as that of a $(1+1)$-dimensional CFT with $c=1$. We have also examined the behavior of the entanglement entropy and the zero-temperature fermion density as λ approaches the endpoints of the critical interval, showing that it is consistent with a quantum phase transition at both ends. For the su(1|1) chain with elliptic interactions we have studied numerically the fermion density at finite temperature, finding that its behavior is far more complex when the chemical potential lies in the critical interval. More precisely, for suitable values of λ inside this interval the fermion density is not a monotonic function of the temperature, but can rather present up to two extrema.

The paper is organized as follows. In Sec. II we introduce the class of supersymmetric spin chains with which this work is concerned and recall how these models can be fermionized using the algebraic properties of the su(1|1) permutation operator. In Sec. III we analyze the thermodynamics of the general su(1|1) chain (2) when the dispersion relation is monotonic in the range $0 \leqslant p \leqslant \pi$, showing that at low temperature it behaves as a $(1+1)$-dimensional CFT with $c=1$. In Sec. IV we outline the computation of the von Neumann and Rényi ground state entanglement entropies of the latter models in terms of the eigenvalues of the ground state correlation matrix. Section V is devoted to deriving asymptotic formulas for these entropies, both when the size of the block of spins tends to infinity and when the chemical potential approaches the endpoints of the critical interval. In Sec. VI we perform a numerical study of the fermion density of the elliptic su(1|1) chain at finite temperature and determine analytically its low-temperature behavior for arbitrary interactions. Finally, in Sec. VII we summarize our conclusions and discuss some future developments suggested by the present work.

II. THE MODELS

Consider a translation-invariant (closed) spin chain whose N sites are occupied by either a boson or a (spinless) fermion. If we denote by $b_i^\dagger$ and $f_i^\dagger$ the operators that respectively create a boson or a fermion at the ith site, the Hilbert space of the model is the 2^N-dimensional subspace of the infinite-dimensional Fock space determined by the constraints

$$b_i^\dagger b_i + f_i^\dagger f_i = 1, \quad 1 \leqslant i \leqslant N\,. \tag{1}$$

We shall take as the model's Hamiltonian the operator [23]

$$H = \sum_{i<j} h_N(j-i)(1-S_{ij}) - \lambda N_{\mathrm{f}}, \tag{2}$$

where $\lambda \in \mathbb{R}$, $N_{\mathrm{f}} = \sum_i f_i^\dagger f_i$ is the total fermion number operator, h_N is a nonnegative smooth function and S_{ij} is the su(1|1) spin permutation operator [10] defined by

$$S_{ij} = b_i^\dagger b_j^\dagger b_i b_j + f_i^\dagger f_j^\dagger f_i f_j + f_j^\dagger b_i^\dagger f_i b_j + b_j^\dagger f_i^\dagger b_i f_j\,.$$

If we denote by $|0\rangle$ and $|1\rangle$ respectively the states occupied by a boson or a fermion, the action of the operator S_{ij} on the canonical spin basis with elements $|s_1\rangle \otimes \cdots \otimes |s_N\rangle \equiv |s_1,\ldots,s_N\rangle$, with $s_i \in \{0,1\}$, is given by

$$S_{ij}|\ldots,s_i,\ldots,s_j,\ldots\rangle = (-1)^n|\ldots,s_j,\ldots,s_i,\ldots\rangle, \tag{3}$$

where $n = s_i = s_j$ if $s_i = s_j$ while for $s_i \neq s_j$ n equals the number of fermions at the sites $i+1,\ldots,j-1$. Note that S_{ij} is invariant under the supersymmetry transformation $b_i \leftrightarrow f_i$, so that the term $\sum_{i<j} h_N(i-j)(1-S_{ij})$ in H is su(1|1) supersymmetric, while the last term λN_{f}, i.e., the chemical potential of the fermions, transforms into $\lambda(N-N_{\mathrm{f}})$ due to the constraints (1). Furthermore, we shall exclusively be concerned in this paper with *closed* (i.e., periodic) chains, for which $h_N(x) = h_N(N-x)$. It is customary to extend the function h_N to the whole real line as an N-periodic function, so that

$$h_N(x) = h_N(-x) = h_N(x+N) \geqslant 0, \quad \forall x \in \mathbb{R}\,. \tag{4}$$

It was shown in Ref. [17] that any chain of the form (2) can be recast into a model of spinless hopping fermions by identifying the boson state $|0\rangle$ with the fermion vacuum. More precisely, we define a new set of fermion creation operators $a_i^\dagger = f_i^\dagger b_i$, $1 \leqslant i \leqslant N$, which indeed satisfy the canonical anticommutation relations (CAR) on account of (1). For instance, we have

$$\begin{aligned} a_i^\dagger a_i + a_i a_i^\dagger &= f_i^\dagger f_i b_i b_i^\dagger + f_i f_i^\dagger b_i^\dagger b_i = \{f_i^\dagger, f_i\} b_i^\dagger b_i + f_i^\dagger f_i \\ &= b_i^\dagger b_i + f_i^\dagger f_i = 1. \end{aligned}$$

The chain sites can now be either empty (i.e., in the state $|0\rangle$) or occupied by a fermion (in the state $|1\rangle$), and thus the Hilbert space is the whole 2^N-dimensional Fock space built acting on the vacuum $|0,\ldots,0\rangle$ with the operators $a_i^\dagger$. As first shown by Haldane [10], from Eqs. (3) and the constraints (1) it follows that the su(1|1) exchange operator S_{ij} admits the following simple expression in terms of the new fermion operators $a_i, a_i^\dagger$:

$$S_{ij} = 1 - a_i^\dagger a_i - a_j^\dagger a_j + a_i^\dagger a_j + a_j^\dagger a_i\,.$$

Likewise,

$$f_i^\dagger f_i = f_i^\dagger f_i (b_i^\dagger b_i + f_i^\dagger f_i) = f_i^\dagger f_i (b_i b_i^\dagger + f_i^\dagger f_i - 1) = a_i^\dagger a_i$$

(since $f_i^\dagger f_i$ is idempotent), so that $\lambda N_{\mathrm{f}} = \lambda \sum_i a_i^\dagger a_i$ is simply the chemical potential for the new fermions. Taking into account the latter identities, the Hamiltonian (2) can be rewritten as

$$H = -\sum_{i,j} h_N(i-j) a_i^\dagger a_j - \lambda \sum_i a_i^\dagger a_i, \tag{5}$$

where we have set $h_N(0) = -\sum_{j=1}^{N-1} h_N(j)$ (see Ref. [17] for more details). This Hamiltonian describes a system of N hopping (spinless) free fermions on a circle, with hopping amplitude between the ith and jth sites given by $h_N(i-j)$ and chemical potential λ. The translation invariance of this model (encoded in the periodicity of the function h) suggests introducing the Fourier-transformed operators

$$c_l = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \mathrm{e}^{-2\pi \mathrm{i} k l/N} a_k, \quad 0 \leqslant l \leqslant N-1 . \tag{6}$$

It can be readily shown that these operators satisfy the CAR and can therefore be considered as a new set of fermionic operators; in fact, as we shall see below, $c_l^\dagger$ creates a fermion with momentum $p = 2\pi l/N \pmod{2\pi}$. It is shown in Ref. [17] that H is diagonal when written in terms of the new operators c_l and their adjoints. In fact, we have

$$H = \sum_{l=0}^{N-1} [\varepsilon_N(l) - \lambda] c_l^\dagger c_l, \tag{7}$$

where

$$\varepsilon_N(l) = \sum_{j=1}^{N-1} [1 - \cos(2\pi j l/N)] h_N(j) . \tag{8}$$

Likewise, the system's total momentum operator P is given by

$$P = \sum_{l=0}^{N-1} \frac{2\pi l}{N} c_l^\dagger c_l,$$

which shows that the operator $c_l^\dagger$ creates a fermion with momentum $2\pi l/N \pmod{2\pi}$. In this work we shall be concerned with systems for which $\varepsilon_N(l)$ depends on l and N only through the corresponding momentum $2\pi l/N$, i.e.,

$$\varepsilon_N(l) = \mathcal{E}(2\pi l/N), \quad 0 \leqslant l \leqslant N-1,$$

where the *dispersion relation* $\mathcal{E}$ is a smooth function defined in the interval $[0,2\pi]$. It easily follows from Eq. (8) that if such a function $\mathcal{E}$ exists it is necessarily unique and that $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$. An important type of interaction $h_N(x)$ satisfying the above requirement is given by the elliptic function

$$h_N(x) = \left(\frac{\alpha}{\pi}\right)^2 \sinh^2\left(\frac{\pi}{\alpha}\right) \left[\wp_N(x) - \frac{2\hat{\eta}_1}{\alpha^2}\right], \tag{9}$$

where $\alpha > 0$ is a real parameter, $\wp_N(x) \equiv \wp(x; N/2, \mathrm{i}\alpha/2)$, and $\hat{\eta}_1 = \zeta(1/2; 1/2, \mathrm{i}N/(2\alpha))$, $\wp(x;\omega_1,\omega_3)$ and $\zeta(x;\omega_1,\omega_3)$ denoting, respectively, the Weierstrass elliptic and zeta functions with half-periods ω_1 and ω_3 [24,25]. It can be shown [17] that the function (9) satisfies the three conditions in Eq. (4). Moreover, since

$$\lim_{\alpha\to 0+} h_N(x) = \delta_{1,x} + \delta_{N-1,x}, \quad \lim_{\alpha\to\infty} h_N(x) = \frac{(\pi/N)^2}{\sin^2\left(\frac{\pi x}{N}\right)},$$

the model (2) with interaction strength (9) smoothly interpolates between the Heisenberg (for $\alpha = 0$) and Haldane-Shastry (for $\alpha = \infty$) su(1|1) chains (with a chemical potential term added). In fact, the former of these models can be transformed into the spin 1/2 (closed) *XX* Heisenberg Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) + \left(1 - \frac{\lambda}{2}\right) \sum_{i=1}^{N} \left(1 + \sigma_i^z\right),$$

where σ_k^a is the ath Pauli matrix acting on the kth site and $\sigma_{N+1}^a \equiv \sigma_1^a$, with the help of the standard Wigner-Jordan transformation [26]

$$a_k = \sigma_1^z \cdots \sigma_{k-1}^z \cdot \tfrac{1}{2}\left(\sigma_k^x - \mathrm{i}\sigma_k^y\right), \quad 1 \leqslant k \leqslant N .$$

The dispersion relation $\mathcal{E}(p)$ for the elliptic interaction (9) was computed in closed form in Ref. [17]. More precisely, from Eq. (2.21b) in the latter reference and the homogeneity properties of the Weierstrass functions we have

$$\mathcal{E}(p) = 2\sinh^2(\pi/\alpha)\left\{\wp(p) - \left[\zeta(p) - \frac{\eta_1 p}{\pi}\right]^2 - \frac{2\eta_1}{\pi}\right\}, \tag{10}$$

where

$$\wp(p) \equiv \wp(p;\pi,\mathrm{i}\pi/\alpha), \quad \zeta(p) \equiv \zeta(p;\pi,\mathrm{i}\pi/\alpha), \quad \eta_1 = \zeta(\pi) .$$

In particular, we see that in this case the dispersion relation is a pure 2π-periodic [27] function, independent of the number of particles N. Taking the $\alpha \to 0+$ and $\alpha \to \infty$ limits in the above equation for $\mathcal{E}(p)$ one recovers the well-known dispersion relations of the *XX* model [28] and the su(1|1) Haldane-Shastry chain, namely,

$$\mathcal{E}_{XX}(p) = 2(1 - \cos p), \quad \mathcal{E}_{\mathrm{HS}}(p) = \tfrac{1}{2}\, p(2\pi - p) . \tag{11}$$

III. CRITICALITY AND THERMODYNAMICS

In this section we shall exploit the equivalence of the su(1|1) supersymmetric chain (2) to the free fermion model (5) to analyze the critical behavior of this chain as a function of the chemical potential λ. To this end, we first need to determine the ground state of the model (5), which is straightforward from Eq. (7). Indeed, it is obvious from the latter equation that the modes excited in the ground state are precisely those whose momenta $p = 2\pi l/N$ satisfy the condition $\lambda > \mathcal{E}(p)$, so that the ground state is nondegenerate. Strictly speaking, this is true only if we assume that $\mathcal{E}(2\pi l/N) \neq \lambda$ for $l = 0, \ldots, N-1$. Indeed, if $\mathcal{E}(2\pi l/N) = \lambda$ the mode with momentum $2\pi l/N$ [and $2\pi(N-l)/N$, if $l > 0$ and $l \neq N/2$] can be either present or absent in the ground state, which is therefore degenerate. Since we shall be exclusively concerned with the thermodynamic limit $N \to \infty$, from now on we shall implicitly assume without loss of generality that $\mathcal{E}(2\pi l/N) \neq \lambda$ for $0 \leqslant l \leqslant N-1$.

We shall also assume in what follows that the dispersion relation has a positive derivative in the interval $(0,\pi)$, so that it is monotonically increasing in the latter interval and reaches its maximum at $p = \pi$. This is "generically" true, and it certainly holds for the dispersion relation (10) of the elliptic interaction (9) and, in particular, for the *XX* model and the su(1|1) Haldane-Shastry chains. If this is the case, it is straightforward to show that *the model is gapless for* $\lambda \in [0, \mathcal{E}(\pi)]$.

Indeed, first of all, it is clear that the system is gapped for $\lambda < 0$ or $\lambda > \mathcal{E}(\pi)$. For instance, for $\lambda < 0$ the gap

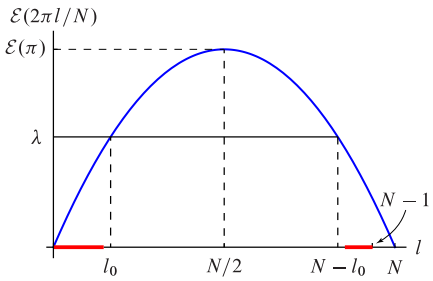

FIG. 1. Dispersion relation $\mathcal{E}(2\pi l/N)$ as a function of the mode number $l = 0, \ldots, N-1$ (the range of modes excited in the ground state for the given λ has been represented by a thick red line).

between the first excited state $c_0^\dagger|0,\ldots,0\rangle$ and the ground state is $\Delta E = |\lambda| > 0$, which remains positive as $N \to \infty$. Similarly, when $\lambda > \mathcal{E}(\pi)$ the gap is approximately equal to $\Delta E = \lambda - \mathcal{E}(\pi) > 0$ independently of N. Suppose, on the other hand, that $0 \leqslant \lambda \leqslant \mathcal{E}(\pi)$, and let l_0 be the root of the equation $\mathcal{E}(2\pi l_0/N) = \lambda$ in the interval $[0, N/2]$, which exists and is unique on account of the monotonicity of $\mathcal{E}$ in the interval $[0,\pi]$. The modes excited in the ground state are now those with $0 \leqslant l \leqslant \lfloor l_0 \rfloor$ and $N - \lfloor l_0 \rfloor \leqslant l \leqslant N-1$, where $\lfloor l_0 \rfloor$ denotes the integer part of l_0 (see Fig. 1). Thus if $0 \leqslant \lambda \leqslant \mathcal{E}(\pi)$ the gap between the first excited state and the ground state, given by

$$\Delta E = \min\big(\lambda - \mathcal{E}\{2\pi\lfloor l_0\rfloor/N\}, \mathcal{E}\{2\pi(\lfloor l_0\rfloor + 1)/N\} - \lambda\big),$$

is $O(1/N)$, since $\lambda = \mathcal{E}(2\pi l_0/N)$. Thus ΔE tends to zero as $N \to \infty$ and the system is gapless, as claimed. (In fact, when l_0 is an integer the modes with $l = l_0$ or $l = N - l_0$ may or may not be present in the ground state, but this does not affect the ground state energy and therefore the foregoing argument.)

We shall next show that when the chemical potential λ belongs to the open interval $(0, \mathcal{E}(\pi))$ the su(1|1) chain (2) is indeed critical, or, more precisely, that at low energies its spectrum is that of a $(1+1)$-dimensional CFT with one free boson. To begin with, we note that when $0 < \lambda < \mathcal{E}(\pi)$ the low-energy excitations of the chain (2) are linear in the excitation momentum. Indeed, let

$$p_0 = 2\pi l_0/N \equiv \mathcal{E}^{-1}(\lambda) \in (0,\pi) \tag{12}$$

denote the Fermi momentum, where $\mathcal{E}^{-1}$ is the inverse function of the restriction of the dispersion relation to the interval $[0,\pi]$. Adding a fermion with momentum $p_0 + \Delta p$ (or, equivalently, $2\pi - p_0 - \Delta p$), with $0 < \Delta p \ll 1$, to the ground state increases the energy by $\mathcal{E}(p_0 + \Delta p) - \lambda = \mathcal{E}(p_0 + \Delta p) - \mathcal{E}(p_0) \simeq \mathcal{E}'(p_0)\Delta p$. The same excess energy is approximately obtained when removing from the ground state a fermion with momentum $p_0 - \Delta p$ (or $2\pi - p_0 + \Delta p$). Thus for low excitation momenta we have $\Delta E \simeq \mathcal{E}'(p_0)\Delta p$, as in a $(1+1)$-dimensional CFT with speed of "sound" $v = \mathcal{E}'(p_0)$.

The simple argument outlined above, based on linearizing the dispersion relation near the Fermi momentum p_0 (or $2\pi - p_0$), the only region in momentum space relevant at low excitation energies, does not provide any information on the central charge of the underlying CFT. A more precise way of establishing the equivalence at low energies of the su(1|1) spin chain (2) with a $(1+1)$-dimensional CFT, and in particular of determining its central charge, is based on the analysis of the chain's free energy. Indeed, as mentioned in the Introduction, at low temperatures the free energy per unit length of a $(1+1)$-dimensional CFT is given (in natural units $\hbar = k_B = 1$) by

$$f(T) \simeq f_0 - \frac{\pi c T^2}{6v}, \tag{13}$$

where $f_0 = f(0)$ is a constant, c is the central charge, and v is the effective speed of sound. On the other hand, by Eq. (7) the free energy of the spin chain (2) is simply given by

$$F(T) = -T \log Z = -T \sum_{l=0}^{N-1} \log Z_l,$$

where $Z_l = 1 + \mathrm{e}^{-\beta(\mathcal{E}(2\pi l/N)-\lambda)}$ (with $\beta \equiv 1/T$) is the partition function of the lth normal mode. Substituting in the previous equation and using the relation $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$ we obtain the closed formula

$$f(T) = \lim_{N\to\infty} \frac{F(T)}{N} = -\frac{T}{\pi}\int_0^\pi \log[1 + \mathrm{e}^{-\beta(\mathcal{E}(p)-\lambda)}]\, dp. \tag{14}$$

In order to determine the low-temperature behavior of $f(T)$, we note that $\mathcal{E}(p) - \lambda$ is negative for $0 < p < p_0$ and positive for $p_0 < p < \pi$, so that $f(T) = f_0 + f_1(T) + f_2(T)$, where

$$f_0 = \frac{1}{\pi}\int_0^{p_0} [\mathcal{E}(p) - \lambda]\, dp = f(0) \tag{15}$$

is constant and

$$f_1(T) = -\frac{T}{\pi}\int_0^{p_0} \log[1 + \mathrm{e}^{-\beta[\lambda - \mathcal{E}(p)]}]\, dp, \tag{16}$$

$$f_2(T) = -\frac{T}{\pi}\int_{p_0}^{\pi} \log[1 + \mathrm{e}^{-\beta[\mathcal{E}(p)-\lambda]}]\, dp \tag{17}$$

vanish at $T = 0$. The low-temperature behavior of $f_1(T)$ can be determined by performing the change of variable $x = [\lambda - \mathcal{E}(p)]/T$, which yields

$$f_1(T) = -\frac{T^2}{\pi}\int_0^{\lambda\beta} \log(1 + \mathrm{e}^{-x})\frac{dx}{\mathcal{E}'(p)}. \tag{18}$$

The condition $\mathcal{E}'(p_0) \neq 0$ implies that $p - p_0 = O(Tx)$ and hence $\mathcal{E}'(p) = v + O(Tx)$, where $v = \mathcal{E}'(p_0)$ is the Fermi velocity. We thus have

$$f_1(T) = -\frac{T^2}{\pi v}\int_0^{\lambda\beta} \log(1 + \mathrm{e}^{-x})\, dx + O(T^3),$$

so that for $T \ll 1$ we obtain [29]

$$f_1(T) = -\frac{T^2}{\pi v}\int_0^{\infty} \log(1 + \mathrm{e}^{-x})\, dx + O(T^3)$$

$$= -\frac{\pi T^2}{12v} + O(T^3).$$

The last term $f_2(T)$ can be similarly dealt with through the change of variable $x = \beta(\mathcal{E}(p) - \lambda)$, with the same result.

Hence at low temperatures we have

$$f(T) = f_0 - \frac{\pi T^2}{6v} + O(T^3),$$

which coincides with Eq. (13) with $c = 1$. This shows that the spin chain (2) is indeed critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = 1$.

The critical behavior of the su(1|1) chain at the endpoints $\lambda = 0, \mathcal{E}(\pi)$ can be similarly investigated. Indeed, suppose to begin with that $\lambda = 0$. In this case $f(T) = f_2(T)$, where f_2 is as in Eq. (17) with $p_0 = 0$, so that performing the change of variable $x = \beta\mathcal{E}(p)$ we obtain

$$f(T) = -\frac{T^2}{\pi} \int_0^{\beta\mathcal{E}(\pi)} \log(1 + \mathrm{e}^{-x}) \frac{dx}{\mathcal{E}'(p)}.$$

The dispersion relation can be expanded around $p = 0$ as $\mathcal{E}(p) = (p/a)^\kappa + O(p^{\kappa+1})$, where $\kappa \geqslant 1$ denotes the order of the lowest nonvanishing derivative of $\mathcal{E}$ at the origin (generically, therefore, $\kappa = 1$) and

$$a \equiv \left[\frac{\kappa!}{\mathcal{E}^{(\kappa)}(0)}\right]^{1/\kappa}. \tag{19}$$

From the latter expansion we have $p/a = (Tx)^{1/\kappa} + O[(Tx)^{2/\kappa}]$, and therefore

$$\mathcal{E}'(p) = \frac{\kappa}{a}\left(\frac{p}{a}\right)^{\kappa-1} + O(p^\kappa) = \frac{\kappa}{a}(Tx)^{1-\frac{1}{\kappa}} + O(Tx). \tag{20}$$

Substituting into the previous equation for $f(T)$ we thus obtain

$$f(T) = -\frac{aI_\kappa}{\kappa\pi} T^{1+\frac{1}{\kappa}} + O(T^{1+\frac{2}{\kappa}}), \quad T \ll 1, \tag{21}$$

with

$$I_\kappa \equiv \int_0^\infty x^{\frac{1}{\kappa}-1} \log(1 + \mathrm{e}^{-x})\, dx.$$

The integral I_κ can actually be evaluated using the technique of Ref. [17], namely:

$$\begin{aligned} I_\kappa &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \int_0^\infty x^{\frac{1}{\kappa}-1} \mathrm{e}^{-nx}\, dx \\ &= \Gamma(\kappa^{-1}) \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^{1+\frac{1}{\kappa}}} \equiv \Gamma(\kappa^{-1})\eta(1+\kappa^{-1}) \\ &= (1 - 2^{-1/\kappa})\Gamma(\kappa^{-1})\zeta_{\mathrm{R}}(1+\kappa^{-1}), \end{aligned}$$

where $\zeta_{\mathrm{R}}(z)$ is Riemann's zeta function, $\eta(z)$ is Dirichlet's eta function, and we have used the identity $\eta(z) = (1 - 2^{1-z})\zeta_{\mathrm{R}}(z)$. Substituting into Eq. (21) we finally obtain

$$f(T) = -\gamma T^{1+\frac{1}{\kappa}} + O\big(T^{1+\frac{2}{\kappa}}\big), \tag{22}$$

with

$$\gamma = \frac{a}{\pi}(1 - 2^{-1/\kappa})\Gamma(1+\kappa^{-1})\zeta_{\mathrm{R}}(1+\kappa^{-1}). \tag{23}$$

We thus see that for $\lambda = 0$ the chain (2) cannot be critical unless $\kappa = 1$, i.e., $\mathcal{E}'(0) \neq 0$. Moreover, for $\kappa = 1$ we have $a = 1/v$, and therefore

$$f(T) = -\frac{\pi T^2}{12v} + O(T^3).$$

This shows that when $\lambda = 0$ and $\mathcal{E}'(0) \neq 0$ the chain (2) is still critical but has central charge $c = 1/2$, and its low-energy behavior is therefore described by a CFT with one free *fermion*. For instance, for the elliptic interaction (9) $\kappa = 2$ for $0 \leqslant \alpha < \infty$, while $\kappa = 1$ for $\alpha = \infty$. In particular, for $0 \leqslant \alpha < \infty$ Eqs. (22)–(23) with $\kappa = 2$ reproduce the result in Ref. [17]. On the other hand, it is well known that the su(1|1) Haldane-Shastry chain (i.e., the $\alpha = \infty$ case) can indeed be described at low energies by a $(1 + 1)$-dimensional CFT with one free fermion [12].

The analysis is totally analogous at the other endpoint $\lambda = \mathcal{E}(\pi)$. Indeed, since now $\mathcal{E}(p) - \mathcal{E}(\pi) < 0$ for $0 \leqslant p < \pi$ we have $f(T) = f_0 + f_1(T)$, where f_0 and f_1 are, respectively, given by Eqs. (15) and (16) with $\lambda = \mathcal{E}(\pi)$. Performing the usual change of variable $x = \beta[\mathcal{E}(\pi) - \mathcal{E}(p)]$ we thus arrive at Eq. (18). Near $p = \pi$ we have $\mathcal{E}(\pi) - \mathcal{E}(p) = [(\pi - p)/b]^\nu + O[(\pi - p)^{\nu+1}]$, where ν denotes the lowest nonvanishing derivative of the dispersion relation at $p = \pi$ and

$$b \equiv \left[-\frac{\nu!}{\mathcal{E}^{(\nu)}(\pi)}\right]^{1/\nu}. \tag{24}$$

Note that, by the symmetry $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$, ν is necessarily *even* and $\mathcal{E}^{(\nu)}(\pi) < 0$. Proceeding as before we readily obtain Eqs. (22)–(23) with a and κ respectively replaced by b and ν. In particular, since in this case $\nu \geqslant 2$ we see that at the endpoint $\lambda = \mathcal{E}(\pi)$ the model (2) is not critical. In summary, our analysis indicates that the latter model is critical for $0 < \lambda < \mathcal{E}(\pi)$, and for $\lambda = 0$ when $\mathcal{E}'(0) \neq 0$.

IV. GROUND STATE ENTANGLEMENT ENTROPY

We shall study in this section the von Neumann entanglement entropy S of the ground state of the su(1|1) supersymmetric model (2), defined as the von Neumann entropy of the reduced density matrix ρ_L of a block of L consecutive sites when the system is in its ground state. In other words, if we denote by $|\psi\rangle$ the ground state of the chain (2), then $\rho_L = \mathrm{tr}_{N-L}\, |\psi\rangle\langle\psi|$, where tr_{N-L} denotes the trace over the Hilbert space of the remaining $N - L$ sites, and the von Neumann entanglement entropy is given by

$$S = -\,\mathrm{tr}(\rho_L \log \rho_L).$$

More generally, we shall also consider the Rényi entropy

$$S_q = \frac{\log \mathrm{tr}\left(\rho_L^q\right)}{1 - q},$$

where $q > 0$ is a real parameter, which reduces to that of von Neumann in the $q \to 1$ limit. As pointed out in the Introduction, the von Neumann and Rényi ground-state entanglement entropies of a $(1 + 1)$-dimensional CFT scale as $r_q \log L$ when $L \to \infty$, where the coefficient r_q is related to the holomorphic and antiholomorphic central charges c and $\bar{c}$ by $r_q = (1 + q^{-1})(c + \bar{c})/12$ (with $q = 1$ for the von Neumann entropy). Since the su(1|1) supersymmetric chain (2) is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = \bar{c} = 1$, it is to be expected that for this model

$$S_q \simeq \frac{1}{6}(1 + q^{-1}) \log L$$

in the limit $L \to \infty$. In fact, we shall rigorously establish this asymptotic formula in the next section [cf. Eq. (32)].

Before addressing the actual computation of the entanglement entropy of the su(1|1) chain (2), we note that the result is the same for its "antiferromagnetic" version $-H$. This is most easily proved by considering the equivalent Hamiltonian (5), whose ground state entanglement entropy is obviously unchanged if we reverse the roles of the occupied and empty sites. In other words, the entanglement entropy is the same for the Hamiltonian (5) as for its image under the replacement $a_i \leftrightarrow a_i^\dagger$. Using the CAR and the even character of the interaction h, it is immediate to show that the latter transformation maps H into $-H - N[\lambda + h_N(0)]$, which establishes our claim.

First of all, it is clear that the ground state is not entangled for λ outside the interval $[0,\mathcal{E}(\pi)]$. Indeed, if (for instance) $\lambda < 0$ the ground state is obviously the vacuum $|0,\ldots,0\rangle$ (i.e., the state with all sites occupied by bosons for the original Hamiltonian (2)), since in this case all the modes have positive energy $\mathcal{E}(2\pi l/N) - \lambda$. In particular, the ground state is a product state ($|0\rangle^{\otimes N}$) and is therefore not entangled. The situation is completely analogous for $\lambda > \mathcal{E}(\pi)$, since in this case $\mathcal{E}(2\pi l/N) - \lambda < 0$ for all l, and therefore all the modes are excited in the ground state. Thus $c_l^\dagger|\psi\rangle = 0$ for all $l = 0,\ldots,N-1$, and therefore

$$a_k^\dagger|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{-2\pi \mathrm{i} kl/N} c_l^\dagger|\psi\rangle = 0, \quad 1 \leqslant k \leqslant N\,.$$

Hence $|\psi\rangle = |1,\ldots,1\rangle = |1\rangle^{\otimes N}$ (i.e., the state with all sites occupied by fermions), which is again a product state and therefore not entangled. (This is also true when $\lambda = \mathcal{E}(\pi)$ if N is *odd*.) From the previous considerations it follows that the ground state entanglement entropy of the model (2) vanishes for λ outside the interval $[0,\mathcal{E}(\pi)]$ (and when $\lambda = \mathcal{E}(\pi)$, if N is odd), since in these cases the ground state is a product state. For this reason, in the rest of this section we shall suppose that λ belongs to the open critical interval $(0,\mathcal{E}(\pi))$.

We shall next find a closed form expression for the entanglement entropy of the su(1|1) chain (2) by applying the method of Ref. [30] to the equivalent fermionic Hamiltonian (5). The first step in our computation is the evaluation of the ground-state correlation matrix A of the latter model, with matrix elements

$$A_{mn} = \langle\psi|a_m^\dagger a_n|\psi\rangle \equiv \langle a_m^\dagger a_n\rangle, \quad 1 \leqslant m,n \leqslant N\,.$$

This matrix can be easily determined (in the thermodynamic limit) from the relations

$$\langle c_j^\dagger c_k\rangle = \begin{cases} 0, & \lfloor l_0\rfloor + 1 \leqslant j \leqslant N - \lfloor l_0\rfloor - 1 \\ \delta_{jk}, & \text{otherwise,} \end{cases}$$

which in turn are a straightforward consequence of the CAR and the conditions

$$\begin{cases} c_j|\psi\rangle = 0, & \lfloor l_0\rfloor + 1 \leqslant j \leqslant N - \lfloor l_0\rfloor - 1 \\ c_j^\dagger|\psi\rangle = 0, & \text{otherwise} \end{cases}$$

characterizing the ground state. Indeed, from the inverse Fourier transform formula

$$a_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} kl/N} c_l$$

and the previous relations it immediately follows that [31]

$$\begin{aligned} A_{mn} &= \frac{1}{N}\left(\sum_{l=0}^{\lfloor l_0\rfloor} + \sum_{l=N-\lfloor l_0\rfloor}^{N-1}\right) \mathrm{e}^{-2\pi \mathrm{i}(m-n)l/N} \\ &= \frac{1}{N} + \frac{2}{N}\sum_{l=1}^{\lfloor l_0\rfloor} \cos[2\pi(m-n)l/N] \\ &\overset{N\gg 1}{\simeq} \frac{1}{\pi}\int_0^{p_0} \cos[p(m-n)]\,dp = \frac{\sin[p_0(m-n)]}{\pi(m-n)}\,. \end{aligned} \tag{25}$$

Let us now consider the analogous correlation matrix A_L for a block of L consecutive sites, which by translation invariance we can take as the first L ones. By the defining property of the reduced density matrix ρ_L [32], for $1 \leqslant m,n \leqslant L$ we have

$$\begin{aligned} (A_L)_{mn} &= \langle a_m^\dagger a_n\rangle_L \equiv \mathrm{tr}_L(a_m^\dagger a_n \rho_L) = \mathrm{tr}(a_m^\dagger a_n|\psi\rangle\langle\psi|) \\ &= \langle\psi|a_m^\dagger a_n|\psi\rangle \equiv A_{mn}, \end{aligned}$$

where tr_L denotes the trace over the Hilbert space of the first L sites. Thus A_L is just the submatrix of A consisting of its first L rows and columns. Following Ref. [30], we now consider an alternative basis of fermionic operators whose correlation matrix is diagonal. More precisely, let $U = (u_{mn})_{1\leqslant m,n\leqslant L}$ be a unitary matrix diagonalizing the Hermitian matrix A_L, i.e., satisfying

$$U A_L U^\dagger = \mathrm{diag}(\mu_1,\ldots,\mu_L) \tag{26}$$

where $\mu_1,\ldots,\mu_L \in [0,1]$ are the eigenvalues of A_L. We then define the operators g_k ($1 \leqslant k \leqslant L$) by $g_k = \sum_{m=1}^L u_{km}^* a_m$; note that g_k, though certainly nonlocal, acts on the Hilbert space of the first L sites. The operators g_k and their adjoints satisfy the CAR by the unitarity of the matrix U, and their correlation matrix is given by

$$\langle g_k^\dagger g_l\rangle_L = \mu_k \delta_{kl}$$

on account of Eq. (26). As shown in Ref. [33], the latter equation and Wick's theorem for Gaussian states imply that the correlation matrix factorizes as $\rho_L = \otimes_{k=1}^L \varrho_k$, with

$$\varrho_k = \mu_k g_k^\dagger g_k + (1-\mu_k) g_k g_k^\dagger\,.$$

The Hilbert space of the system is the tensor product of the two-dimensional spaces spanned by the vectors $|v\rangle_k, g_k^\dagger|v\rangle_k$ ($1 \leqslant k \leqslant N$), where $g_k|v\rangle_k = 0$. Moreover, from the CAR it easily follows that ϱ_k is diagonal in the basis $\{|v\rangle_k, g_k^\dagger|v\rangle_k\}$, with respective eigenvalues $1-\mu_k$ and μ_k. Thus the von Neumann and Rényi entropies of ϱ_k are respectively equal to $s(\mu_k)$ and $s_q(\mu_k)$, where

$$\begin{cases} s(x) = -x\log x - (1-x)\log(1-x), \\ s_q(x) = (1-q)^{-1}\log[x^q + (1-x)^q]. \end{cases} \tag{27}$$

By the additivity property of both of these entropies we then have

$$S = \sum_{k=1}^{L} s(\mu_k), \quad S_q = \sum_{k=1}^{L} s_q(\mu_k). \tag{28}$$

Equations (27)–(28), which are *exact* for any L, make it possible to evaluate numerically the ground state entanglement entropy of *any* supersymmetric su(1|1) chain of the form (2) in *polynomial* time, since they are based on the diagonalization of the $L \times L$ matrix with elements (25). From the latter equations it also follows that the entropy of all of these models is a *universal* function of the Fermi momentum p_0, the difference between two models being manifested only in the dependence of p_0 on the parameter λ through Eq. (12).

V. ASYMPTOTIC FORMULAS FOR THE ENTANGLEMENT ENTROPY

Equations (27)–(28) can be used to obtain approximate expressions for the entanglement entropy of the general su(1|1) supersymmetric chain (2) in several interesting regimes. To begin with, we shall investigate the behavior of the entropy as λ approaches its extreme critical values 0 and $\mathcal{E}(\pi)$. Suppose, in the first place, that λ tends to zero for fixed L, so that the Fermi momentum p_0 is much smaller than $1/L$. In this case all the matrix elements of the correlation matrix A_L in Eq. (25) are approximately equal to p_0/π, so that $A_L = p_0 B_L/\pi$, where B_L is the $L \times L$ matrix with all matrix elements equal to 1. Since the eigenvalues of B_L are 0 (with multiplicity $L-1$) and L, when $Lp_0 \ll 1$ the Rényi entanglement entropy is approximately given by

$$S_q \simeq s_q(Lp_0/\pi) \simeq \begin{cases} \frac{(Lp_0/\pi)^q}{1-q}, & 0 < q < 1; \\ \frac{q}{q-1}\frac{Lp_0}{\pi}, & q > 1. \end{cases}$$

For the same reason, when $Lp_0 \ll 1$ the von Neumann entropy can be approximated by

$$S \simeq s(Lp_0/\pi) \simeq -\frac{Lp_0}{\pi} \log\left(\frac{Lp_0}{\pi}\right).$$

In particular, we see that both S_q and S are continuous [34] at $\lambda = 0$. Similarly, suppose now that p_0 is close to its upper critical value $\mathcal{E}(\pi)$, so that $p_0 = \pi - \varepsilon$ with $\varepsilon \ll 1/L$. In this case we have

$$A_{mn} \simeq -(-1)^{m-n}\frac{\varepsilon}{\pi}, \quad m \neq n,$$

while $A_{nn} = (\pi - \varepsilon)/\pi$. Thus $A_L = \mathbb{I} - (\varepsilon C_L)/\pi$, where C_L is the $L \times L$ matrix with matrix elements $C_{mn} = (-1)^{m-n}$. It is easy to check that the eigenvalues of C_L are again 0 (with multiplicity $L-1$) and L, so that the previous asymptotic expressions for S_q and S still hold with p_0 replaced by $\pi - p_0$. In particular, this shows that the von Neumann and Rényi entanglement entropies are both continuous [35] also at the upper critical value $\lambda = \mathcal{E}(\pi)$. On the other hand, it is clear that these entropies have a discontinuous first derivative (with respect to the chemical potential λ) at both endpoints $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$. For instance, for $0 < \lambda \ll 1$ we have

$$p_0 \simeq a\,\lambda^{1/\kappa}, \tag{29}$$

where κ is the order of the first nonvanishing derivative of $\mathcal{E}$ at $p = 0$ and a is defined in Eq. (19). Thus $dS/d\lambda$ diverges as $\lambda^{1/\kappa-1}|\log\lambda|$ when $\lambda \to 0^+$. Similarly, for $0 < q < 1$ the derivative of the Rényi entropy diverges as $\lambda^{q/\kappa-1}$ in this limit, while for $q > 1$ $dS_q/d\lambda$ diverges as $\lambda^{1/\kappa-1}$ for $\kappa > 1$ and tends to a nonzero finite limit when $\kappa = 1$. The situation is similar at the other endpoint $\lambda = \mathcal{E}(\pi)$, i.e.,

$$\pi - p_0 \simeq b[\mathcal{E}(\pi) - \lambda]^{1/\nu}, \tag{30}$$

with b defined by Eq. (24), except that now ν (the order of the lowest nonvanishing derivative of $\mathcal{E}$ at $p = \pi$) is necessarily even and thus greater than or equal to 2. Hence in all cases the derivatives of S and S_q diverge as $\lambda \to \mathcal{E}(\pi)^-$. The above analysis strongly suggests that there is a quantum phase transition at $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$ between an ordered (nonentangled) and a disordered (entangled) ground state, with the entanglement entropy as the order parameter. This conclusion is confirmed by the behavior of the zero-temperature fermion density n_f, which by translation invariance is simply given by

$$n_f = \langle a_i^\dagger a_i \rangle \equiv A_{ii} = \frac{p_0}{\pi} \tag{31}$$

in the critical interval $0 < \lambda < \mathcal{E}(\pi)$. Indeed, by Eqs. (29)–(30), near the two critical points $\lambda = 0, \mathcal{E}(\pi)$ the fermion density respectively behaves as $(a/\pi)\lambda^{1/\kappa}$ and $1 - (b/\pi)(\mathcal{E}(\pi) - \lambda)^{1/\nu}$. Since $n_f = 0$ for $\lambda < 0$ and $n_f = 1$ for $\lambda > \mathcal{E}(\pi)$, this behavior is typical of a quantum phase transition with exact exponents $1/\kappa$ and $1/\nu$ at the critical points $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$. For instance, for the elliptic interaction (9) it is known [17] that $\nu = 2$ and $\kappa = 2$ for $0 \leqslant \alpha < \infty$, while $\kappa = 1$ for $\alpha = \infty$ (i.e., for the su(1|1) HS chain). The parameters a and b can also be exactly computed in this case, namely,

$$a = \frac{\pi}{\sinh(\pi/\alpha)}\left(\frac{\pi^2}{6}g_2 - 2\eta_1^2\right)^{-1/2},$$

$$b = \frac{\pi}{\sinh(\pi/\alpha)}\left[\pi^2\left(\frac{g_2}{2} - 4e_1^2\right) + 2\eta_1(\eta_1 + 2\pi e_1)\right]^{-1/2},$$

where $e_1 = \wp(\pi)$ and g_2 is the second invariant of the Weierstrass function with half-periods $(\pi, i\pi/\alpha)$ [25].

For the general elliptic su(1|1) model with interactions (9) (with $0 < \alpha < \infty$) and dispersion relation (10), it is of course unfeasible to explicitly invert $\mathcal{E}$ to obtain a closed-form expression for the Fermi momentum $p_0 = \mathcal{E}^{-1}(\lambda)$. Note, however, that the graph of the fermion density n_f admits the simple parametrization $(\mathcal{E}(p), p/\pi)$, with $0 < p < \pi$. In this way we have generated the plot in Fig. 2, where n_f is represented as a function of the normalized parameter $\lambda/\mathcal{E}(\pi)$, where $\mathcal{E}(\pi) = 2\sinh^2(\pi/\alpha)[e_1 - (2\eta_1/\pi)]$, for several values of α in the range [0,50] and for $\alpha = \infty$. The fermion density can be easily computed in closed form for the limiting cases $\alpha = 0$ and $\alpha = \infty$, i.e., for the XX model and the su(1|1) Haldane-Shastry chain, due to the simple form of their dispersion relations. Indeed, from Eq. (11) we immediately obtain

$$n_{f,XX} = \frac{2}{\pi}\arcsin(\sqrt{\lambda}/2), \quad n_{f,HS} = 1 - \sqrt{1 - \frac{2\lambda}{\pi^2}},$$

respectively, for $0 < \lambda < 4$ and $0 < \lambda < \pi^2/2$. As expected, the first of these formulas agrees with the result in Ref. [33], taking into account that our parameter λ is related to the parameter h in the latter reference by $h = 2 - \lambda$. On the other

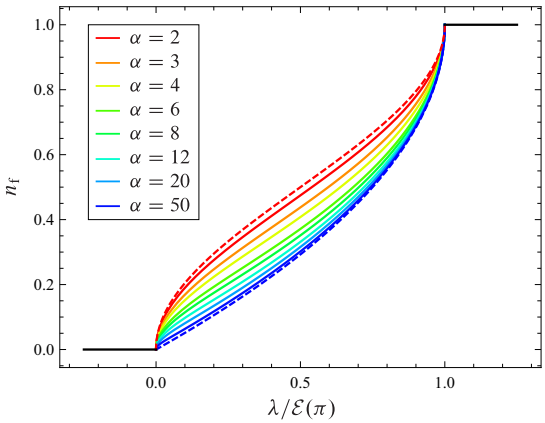

FIG. 2. Zero temperature density of fermions of the su(1|1) chain (2) with elliptic interactions (9) for several values of the parameter α in the range [2,50]. The red (top) and blue (bottom) dashed curves correspond respectively, to the *XX* model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

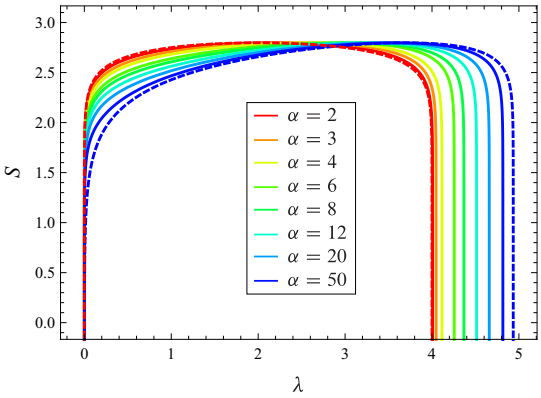

FIG. 3. Approximation (32) to the von Neumann entanglement entropy ($q = 1$) of the elliptic su(1|1) chain (2)–(9) for $L = 1000$ and several values of the parameter α between 2 and 50. The red and blue dashed curves correspond respectively to the *XX* Heisenberg model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

hand, the formula for the su(1|1) HS chain is, to the best of our knowledge, new.

It is also of interest to determine the asymptotic behavior of the von Neumann and Rényi entropies for $0 < \lambda < \mathcal{E}(\pi)$ fixed and $L \gg 1$. To this end, we note that Eq. (25) implies that A_{mn} is a function of $m - n$ only, and hence the correlation matrix A_L is a Toeplitz matrix. This fact can be exploited to find a simple asymptotic formula for the von Neumann and Rényi entanglement entropies in the $L \to \infty$ limit, as shown in Ref. [33] for the *XX* model. The formula in the latter reference, which is based on a particular case of the general Fisher-Hartwig conjecture [36] proved by Basor [37], is also valid for a general model of the form (2) provided only that we express the result in terms of the Fermi momentum p_0. Indeed, this formula relies only on Eq. (25) for the correlation matrix, which, as we have just seen, holds for the model (2) with $p_0 = \mathcal{E}^{-1}(\lambda)$. In this way one obtains the following asymptotic formula for the Rényi entropy in the limit $L \sin p_0 \gg 1$:

$$S_q = \frac{q+1}{6q} \log(L \sin p_0) + \gamma_1^{(q)} + o(1), \qquad (32)$$

while the corresponding formula for the von Neumann entropy is obtained from the above by setting $q = 1$. Here $o(1)$ denotes a function of L and p_0 which tends to 0 as $L \sin p_0 \to \infty$, and $\gamma_1^{(q)}$ is a constant (independent of L and p_0) whose precise value, which can be found in Ref. [33], will not be needed in what follows.

Equation (32) can be easily applied in the case of the *XX* and su(1|1) Haldane-Shastry chains. Indeed, for the former of these models we have $\sin p_0 = \sqrt{\lambda(1 - \frac{\lambda}{4})}$, so that (32) agrees with the result in Ref. [33]. On the other hand, for the su(1|1) HS chain $\sin p_0 = \sin(\sqrt{\pi^2 - 2\lambda}\,)$, and hence Eq. (32) yields the following asymptotic formulas for the von Neumann and Rényi ground state entanglement entropies:

$$S_q = \frac{q+1}{6q} \log[L \sin(\sqrt{\pi^2 - 2\lambda})] + \gamma_1^{(q)} + o(1)\,. \qquad (33)$$

These formulas are valid for λ belonging to the critical interval $(0, \pi^2/2)$, in the asymptotic regime $L \sin\sqrt{\pi^2 - 2\lambda} \gg 1$. For the general elliptic su(1|1) model (9) with $0 < \alpha < \infty$ no such closed formulas are available. However, as for the fermion density, the graph of S_q admits the simple parametrization

$$\left(\mathcal{E}(p), \frac{q+1}{6q} \log[L \sin p] + \gamma_1^{(q)}\right), \quad 0 < p < \mathcal{E}(\pi),$$

where for simplicity we have dropped the $o(1)$ terms.

For instance, in Fig. 3 we present a plot of the approximation (32) to the von Neumann entropy of the elliptic su(1|1) chain (2)–(9) for $L = 1000$ and several values of the parameter α, including the limiting cases $\alpha = 0$ (the Heisenberg *XX* model) and $\alpha = \infty$ (the su(1|1) Haldane-Shastry chain). It is apparent that all of these plots are qualitatively similar, although only in the case of the *XX* model ($\alpha = 0$) is the graph of S symmetric about the midpoint $\lambda = \mathcal{E}(\pi)/2$. More precisely, the maximum of S at $\lambda = \mathcal{E}(\pi/2)$ is increasingly displaced towards the right as α tends to infinity, with $\mathcal{E}(\pi/2)/\mathcal{E}(\pi)$ varying continuously from 1/2 to 3/4 as α ranges from 0 to ∞.

VI. FERMION DENSITY AT FINITE TEMPERATURE

In the previous sections we have seen that the su(1|1) chain (2) is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = 1$. This is confirmed by the asymptotic behavior of the ground state entanglement entropy when the size of the block of spins considered tends to infinity. In this section we shall show that the fermion density at finite temperature, given by

$$n_f = \lim_{N\to\infty} \frac{1}{N} \sum_{l=0}^{N-1} (1 + e^{\beta[\mathcal{E}(2\pi l/N) - \lambda]})^{-1}$$
$$= \frac{1}{\pi} \int_0^\pi \frac{dp}{1 + e^{\beta[\mathcal{E}(p) - \lambda]}}, \qquad (34)$$

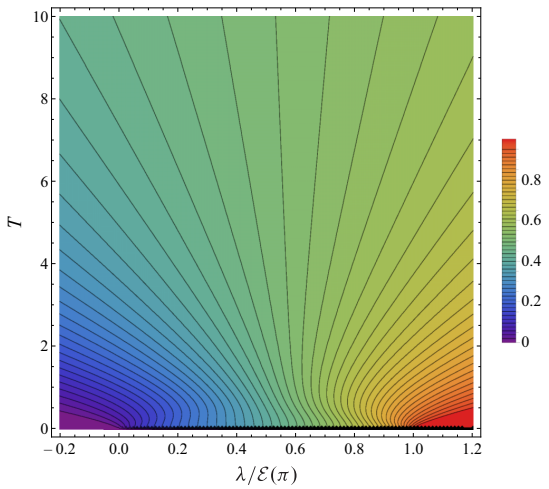

FIG. 4. Contour plot of the fermion density of the elliptic su(1|1) chain (2)–(9) with $\alpha = 5$ for $0 \leqslant T \leqslant 10$ and $-0.2 \leqslant \lambda/\mathcal{E}(\pi) \leqslant 1.2$.

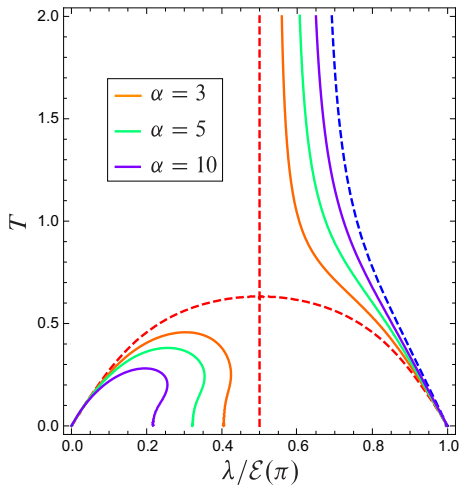

FIG. 5. Plot of the implicit curve $\partial n_f/\partial T = 0$ for the elliptic chain (2)–(9) and several values of the parameter α. The red and blue dashed curves correspond respectively to the *XX* model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

also exhibits a qualitatively richer behavior when λ lies in the critical interval $(0,\mathcal{E}(\pi))$.

As a concrete example, we shall first focus on the su(1|1) elliptic chain (2)–(9). In Fig. 4 we present a contour plot of n_f for this model with $\alpha = 5$ for $T \in [0,10]$ and $\lambda/\mathcal{E}(\pi) \in [-0.2,1.2]$, obtained by numerically evaluating the integral in Eq. (34). For λ outside the critical interval $(0,\mathcal{E}(\pi))$, it is clear that n_f is a monotonic function of T [increasing for $\lambda \leqslant 0$, decreasing for $\lambda \geqslant \mathcal{E}(\pi)$], since

$$\frac{\partial n_f}{\partial T} = \frac{\beta^2}{4\pi}\int_0^\pi \frac{\mathcal{E}(p)-\lambda}{\cosh^2\{\beta[\mathcal{E}(p)-\lambda]/2\}}\,dp\,.$$

On the other hand, it is apparent from Fig. 4 that there is a range of values of λ in the interval $(0,\mathcal{E}(\pi))$ for which the fermion density is not a monotonic function of the temperature. Remarkably, the su(1|1) elliptic chain exhibits this interesting behavior for all values of the parameter α, including the limiting cases $\alpha = 0$ and $\alpha = \infty$. More precisely, for each α there are three critical values λ_i $(i = 1,2,3)$ of the chemical potential λ such that (i) for $0 < \lambda \leqslant \lambda_1$, the fermion density reaches an absolute minimum at some positive temperature and then increases monotonically towards its limiting value $1/2$; (ii) for $\lambda_1 < \lambda < \lambda_2$, n_f first reaches a maximum at some $T > 0$ and then a minimum, after which it tends monotonically to $1/2$; (iii) for $\lambda_2 \leqslant \lambda \leqslant \lambda_3$, n_f is monotonically increasing, and (iv) for $\lambda_3 < \lambda < \mathcal{E}(\pi)$, the fermion density attains an absolute maximum at some $T > 0$ and then decreases monotonically towards $1/2$. This is also true for the limiting values $\alpha = 0$ (*XX* model) and $\alpha = \infty$ (su(1|1) HS chain), for which $\lambda_1 = \lambda_2 = \lambda_3 = \mathcal{E}(\pi)/2 = 2$ and $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2\mathcal{E}(\pi)/3 = \pi^2/3$, respectively. This behavior is qualitatively apparent from Fig. 5, where we have represented the implicit curve $\partial n_f/\partial T = 0$ versus λ and T for $\alpha = 0,3,5,10,\infty$, and is also confirmed by the plots of n_f versus T for these values of α and $\lambda = \mathcal{E}(\pi)/3, 3\mathcal{E}(\pi)/4$ presented in Fig. 6.

Although in general the integral in Eq. (34) cannot be computed in closed form, its low-temperature behavior can be readily determined, as we shall now explain. To begin with, when $\lambda < 0$ the exponent $\beta(\mathcal{E}(p)-\lambda)$ is positive throughout the whole integration range, so that

$$n_f \simeq \frac{1}{\pi}\int_0^\pi \mathrm{e}^{-\beta(\mathcal{E}(p)-\lambda)}\,dp = \frac{T\mathrm{e}^{-|\lambda|\beta}}{\pi}\int_0^{\beta\mathcal{E}(\pi)} \frac{\mathrm{e}^{-x}}{\mathcal{E}'(p)}\,dx,$$

where $x = \beta\mathcal{E}(p)$. Using Eq. (20) and extending the integration range to $+\infty$ we obtain

$$\begin{aligned} n_f &\simeq \frac{a}{\kappa\pi}\,T^{1/\kappa}\mathrm{e}^{-|\lambda|\beta}\int_0^\infty x^{\frac{1}{\kappa}-1}\mathrm{e}^{-x}\,dx \\ &= \frac{a}{\pi}\,\Gamma(1+\kappa^{-1})\,T^{1/\kappa}\mathrm{e}^{-|\lambda|\beta}, \quad \lambda < 0\,. \end{aligned}$$

Proceeding in a similar way we obtain an analogous formula when $\lambda > \mathcal{E}(\pi)$:

$$n_f \simeq 1 - \frac{b}{\pi}\,\Gamma(1+\nu^{-1})\,T^{1/\nu}\mathrm{e}^{-\beta[\lambda-\mathcal{E}(\pi)]}, \quad \lambda > \mathcal{E}(\pi)\,.$$

We thus see that for $\lambda \notin [0,\mathcal{E}(\pi)]$ the fermion density at low temperature is monotonic, approaching exponentially its zero temperature values 0 (for $\lambda < 0$) and 1 (for $\lambda > \mathcal{E}(\pi)$). For $\lambda = 0$, the change of variable $x = \beta\mathcal{E}(p)$ and Eq. (20) easily

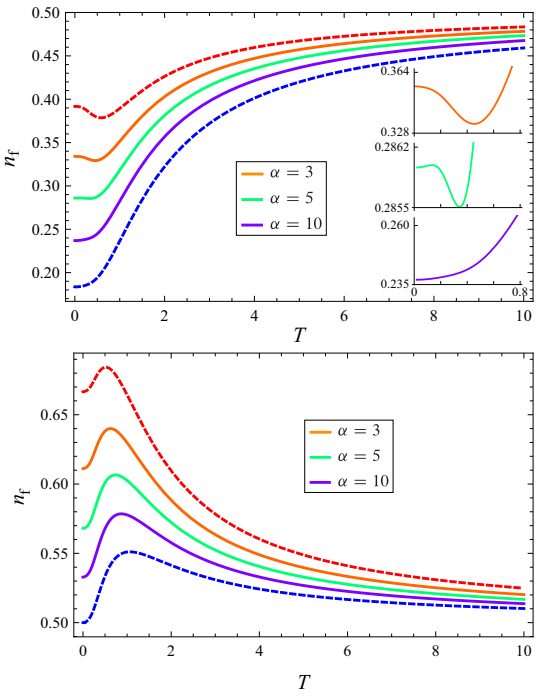

FIG. 6. Top: Plot of the fermion density of the chain (2)–(9) as a function of the temperature T for several values of the parameter α and $\lambda = \mathcal{E}(\pi)/3$ (inset: blowup of the range $0 \leqslant T \leqslant 0.8$). Bottom: Analogous plot for $\lambda = 3\mathcal{E}(\pi)/4$. In both plots, the red and blue dashed lines correspond to the limiting cases $\alpha = 0$ and $\alpha = \infty$.

yield

$$n_f \simeq \frac{aT^{1/\kappa}}{\kappa\pi}\int_0^\infty \frac{x^{\frac{1}{\kappa}-1}\,dx}{1+e^x} = \frac{aT^{1/\kappa}}{\kappa\pi}\int_0^\infty \frac{x^{\frac{1}{\kappa}-1}e^{-x}}{1+e^{-x}}\,dx$$
$$= \frac{aT^{1/\kappa}}{\kappa\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty x^{\frac{1}{\kappa}-1}e^{-nx}\,dx$$
$$= \frac{a}{\pi}\,\Gamma(1+\kappa^{-1})\eta(\kappa^{-1})\,T^{1/\kappa},$$

and therefore

$$n_f = \begin{cases} \frac{\log 2}{\pi v}\,T, & \kappa = 1, \\ \frac{a}{\pi}\,(1-2^{1-1/\kappa})\Gamma(1+\kappa^{-1})\zeta_R(\kappa^{-1})\,T^{1/\kappa}, & \kappa > 1\,. \end{cases}$$

Likewise, at the other endpoint $\lambda = \mathcal{E}(\pi)$ we have

$$n_f = 1 - \frac{b}{\pi}\,(1-2^{1-1/\nu})\Gamma(1+\nu^{-1})\zeta_R(\nu^{-1})\,T^{1/\nu},$$

since now ν is even and hence greater than 1.

Suppose next that λ lies in the critical interval $(0,\mathcal{E}(\pi))$. We start by writing the fermion density as the sum

$$n_f = \frac{p_0}{\pi} - \frac{1}{\pi}\int_0^{p_0}\frac{dp}{1+e^{-\beta[\mathcal{E}(p)-\lambda]}} + \frac{1}{\pi}\int_{p_0}^{\pi}\frac{dp}{1+e^{\beta[\mathcal{E}(p)-\lambda]}}$$
$$\equiv \frac{p_0}{\pi} - n_{f,1} + n_{f,2},$$

where the first term is the value of n_f at $T = 0$ [cf. Eq. (31)]. In this case the leading ($O(T)$) contributions to the two integrals $n_{f,i}$ cancel each other, so that we need to evaluate the $O(T^2)$ corrections. For the first integral, after performing the change of variable $x = \beta[\lambda - \mathcal{E}(p)]$ we have

$$n_{f,1} = \frac{T}{\pi}\int_0^{\lambda\beta}\frac{1}{\mathcal{E}'(p)}\,\frac{dx}{1+e^x}\,.$$

Expanding $\mathcal{E}'(p)$ to first order in $p - p_0$ we obtain

$$\mathcal{E}'(p) = \mathcal{E}'(p_0) + \mathcal{E}''(p_0)(p-p_0) + O[(p-p_0)^2]$$
$$= v - \frac{\mathcal{E}''(p_0)}{v}\,Tx + O[(Tx)^2],$$

where we have used the expansion $Tx = \mathcal{E}(p_0) - \mathcal{E}(p) = -v(p-p_0) + O[(p-p_0)^2]$. We thus have

$$\mathcal{E}'(p)^{-1} = \frac{1}{v}\left\{1 + \frac{\mathcal{E}''(p_0)}{v^2}\,Tx + O[(Tx)^2]\right\}.$$

Substituting in the definition of $n_{f,1}$ and using the estimate

$$\int_{\lambda\beta}^\infty x^j(1+e^x)^{-1}\,dx \leqslant \int_{\lambda\beta}^\infty x^j e^{-x}\,dx = O(\beta^j e^{-\lambda\beta})$$

(with $j = 0,1,\ldots$) we obtain

$$n_{f,1} = \frac{T}{\pi v}\int_0^\infty \frac{dx}{1+e^x} + \frac{\mathcal{E}''(p_0)T^2}{\pi v^3}\int_0^\infty \frac{x\,dx}{1+e^x} + O(T^3)$$
$$= \frac{\log 2}{\pi v}\,T + \frac{\pi\mathcal{E}''(p_0)}{12v^3}\,T^2 + O(T^3)\,. \qquad (35)$$

The term $n_{f,2}$ can be similarly dealt with through the analogous change of variable $x = \beta[\mathcal{E}(p) - \lambda]$, so that $Tx = \mathcal{E}(p) - \mathcal{E}(p_0) = v(p-p_0) + O[(p-p_0)^2]$ and hence

$$\mathcal{E}'(p)^{-1} = \frac{1}{v}\left\{1 - \frac{\mathcal{E}''(p_0)}{v^2}\,Tx + O[(Tx)^2]\right\}.$$

From the definition of $n_{f,2}$ we immediately obtain

$$n_{f,2} = \frac{T}{\pi v}\int_0^\infty \frac{dx}{1+e^x} - \frac{\mathcal{E}''(p_0)T^2}{\pi v^3}\int_0^\infty \frac{x\,dx}{1+e^x} + O(T^3)$$
$$= \frac{\log 2}{\pi v}\,T - \frac{\pi\mathcal{E}''(p_0)}{12v^3}\,T^2 + O(T^3),$$

and combining this result with Eq. (35) we finally have

$$n_f = \frac{p_0}{\pi} - \frac{\pi\mathcal{E}''(p_0)}{6v^3}\,T^2 + O(T^3)\,. \qquad (36)$$

In particular, for the XX and su(1|1) HS chains the low-temperature expansion (36) reads

$$n_{f,XX} = \frac{2}{\pi}\arcsin(\sqrt{\lambda}/2) - \frac{\pi(2-\lambda)}{6\lambda^{3/2}}\,T^2 + O(T^3),$$
$$n_{f,HS} = 1 - \frac{1}{\pi}\sqrt{\pi^2 - 2\lambda} + \frac{\pi T^2}{6(\pi^2-2\lambda)^{3/2}} + O(T^3)\,.$$

The absence of a term linear in T in Eq. (36) is in agreement with the low-temperature behavior of n_f apparent from Fig. 6. It is also interesting to observe that the sign of the leading correction to the $T = 0$ value of n_f is opposite to that of the second derivative of $\mathcal{E}$ at the Fermi momentum p_0. This behavior can be understood by noting that the energy difference between adding a fermion with momentum $p_0 + \Delta p$

(or $2\pi - p_0 - \Delta p$) and removing a fermion with momentum $p_0 - \Delta p$ (or $2\pi - p_0 + \Delta p$), with $0 < \Delta p \ll 1$, is given by $\mathcal{E}(p_0 + \Delta p) + \mathcal{E}(p_0 - \Delta p) - 2\mathcal{E}(p_0) \simeq \mathcal{E}''(p_0)\Delta p^2$. Thus when $\mathcal{E}''(p_0) < 0$ the addition of a fermion is energetically more favorable than its removal for momenta close to the Fermi momentum p_0 (or to $2\pi - p_0$), and consequently the fermion density should increase at sufficiently low temperatures. For instance, for the elliptic interaction (9) with $\alpha \geqslant 0$ finite $\mathcal{E}''(p)$ is positive for p less than a critical momentum (which depends on α) and negative for larger momenta, while for the su(1|1) HS chain (i.e., for $\alpha = \infty$) $\mathcal{E}''(p) = -1$ is always negative. Again, these facts are consistent with the behavior of n_f observed in Fig. 6.

VII. SUMMARY AND OUTLOOK

In this paper we introduce a general class of su(1|1) supersymmetric spin chains with long-range interactions generalizing the su(1|1) Haldane-Shastry and Inozemtsev (elliptic) chains, which can be fermionized using the algebraic properties of the su(1|1) permutation operator. We exploit this fact to study the critical behavior of this class of models (with nonzero chemical potential λ) in terms of their dispersion relation $\mathcal{E}(p)$. More precisely, we show that they are gapless when the chemical potential lies on the interval $[0,\mathcal{E}(\pi)]$, and that their ground state is a product state unless λ belongs to this interval. We prove that the models under study are actually critical when $0 < \lambda < \mathcal{E}(\pi)$ by verifying that their low-energy excitations are linear in the excitation momentum, and that their free energy at low temperature exhibits the characteristic quadratic behavior found in a $(1+1)$-dimensional CFT with $c = 1$ [18,19]. As further confirmation of this critical behavior, we find an exact asymptotic formula for the von Neumann and Rényi entanglement entropies for the ground state, showing that when λ belongs to the open interval $(0,\mathcal{E}(\pi))$ they both scale as $\log L$ when the size L of the block of spins considered tends to infinity. Moreover, in both cases the constant multiplying $\log L$ is the same as for a $(1+1)$-dimensional CFT with central charge $c = 1$ [20–22]. Likewise, we show that the asymptotic behaviors of the entanglement entropy and the zero-temperature fermion density as λ approaches the endpoints of the critical interval $(0,\mathcal{E}(\pi))$ are consistent with a quantum (continuous) phase transition. We also analyze the fermion density at finite temperature for a particular class of models with elliptic interactions, finding that its behavior is nontrivial (for instance, it is not always a monotonic function of the temperature, and it can in fact present up to two extrema at finite temperature) when λ belongs to the critical interval.

The results of this paper suggest several lines for future research. For one thing, they might prove relevant for the su(2) analogs of the models discussed in this paper, and most notably the spin 1/2 Inozemtsev and HS chains in the presence of a magnetic field. Indeed, it has been analytically shown that the su(1|1) HS chain with zero chemical potential λ is equivalent in the thermodynamic limit to its su(2) counterpart with zero magnetic field [17]. More recently, a numerical computation of the free energy of the spin 1/2 elliptic chain with no magnetic field suggests that this model and its su(1|1) version with $\lambda = 0$ studied in this paper are also equivalent in the thermodynamic limit [38]. If this equivalence could be extended to the case of nonzero chemical potential (or magnetic field strength, for the su(2) models), the results of this work could be used, for instance, to evaluate the ground state entanglement entropy of the spin 1/2 elliptic chain and its asymptotic limit when L tends to infinity.

Another line of research suggested by the present work is the study of the entanglement entropy of the low-lying states of the su(1|1) supersymmetric model (2) when the chemical potential lies in the critical interval $(0,\mathcal{E}(\pi))$. Indeed, it has been recently shown [39,40] that in a $(1+1)$-dimensional CFT the quotient between the entanglement entropy of an excited state created by acting on the vacuum with a primary field and that of the ground state is a universal quantity, essentially determined by the conformal weights of the field. Thus the computation of the entanglement entropy of the lowest excited states of the model (2) when $\lambda \in (0,\mathcal{E}(\pi))$, which can be constructed from the equivalent fermionic model (5), could shed some light on its underlying CFT.

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Supersymmetric spin chains with nonmonotonic dispersion relation: Criticality and entanglement entropy

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We study the critical behavior and the ground-state entanglement of a large class of su(1|1) supersymmetric spin chains with a general (not necessarily monotonic) dispersion relation. We show that this class includes several relevant models, with both short- and long-range interactions of a simple form. We determine the low temperature behavior of the free energy per spin, and deduce that the models considered have a critical phase in the same universality class as a $(1+1)$-dimensional conformal field theory (CFT) with central charge equal to the number of connected components of the Fermi sea. We also study the Rényi entanglement entropy of the ground state, deriving its asymptotic behavior as the block size tends to infinity. In particular, we show that this entropy exhibits the logarithmic growth characteristic of $(1+1)$-dimensional CFTs and one-dimensional (fermionic) critical lattice models, with a central charge consistent with the low-temperature behavior of the free energy. Our results confirm the widely believed conjecture that the critical behavior of fermionic lattice models is completely determined by the topology of their Fermi surface.

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I. INTRODUCTION

Integrable spin chains often provide a fertile ground for studying key theoretical concepts in a simple framework that captures the essential features of the problems under consideration. An important example of this assertion is the analysis of the entanglement of a quantum system, which can be considered as one of the fundamental characteristics of the quantum realm [1]. One of the most common ways of measuring the degree of entanglement of a state of a quantum system X is via the bipartite entropy of a subsystem A [2]. This entropy is defined by $S_A = S(\rho_A)$, where $\rho_A = \mathrm{tr}_{X\setminus A}\,\rho$ is the reduced density matrix of the subsystem A, ρ is the density matrix representing the state of the whole system, and S is an appropriate entropy functional (von Neumann, Rényi, etc.). The small class of models for which the entanglement entropy can be evaluated in closed form (at least in the thermodynamic limit) includes certain integrable spin chains, like the Lipkin-Meshkov-Glick model [3], its su(n) generalization [4] and the nearest-neighbors Heisenberg XX and XY models [5–7]. As is well known, the latter two models are critical (gapless) for a certain range of values of the applied magnetic field, their corresponding Virasoro algebras having central charge respectively equal to 1 and 1/2. In both cases, the bipartite Rényi entropy of a block of L consecutive spins when the whole chain is in its ground state scales as $c(1+\alpha^{-1})(\ln L)/6$ in the critical phase, where $\alpha > 0$ is the Rényi parameter ($\alpha = 1$ for the von Neumann entropy) and c is the central charge. This behavior is consistent with the scaling of the Rényi entanglement entropy of a $(1+1)$-dimensional conformal field theory (CFT) [8–10]. In fact, the logarithmic scaling of the ground-state entanglement entropy is a characteristic feature of critical (fermionic) one-dimensional lattice models with short-range interactions (see, e.g., Ref. [11]).

In a previous paper [12], we showed that the above results also apply to a large class of supersymmetric spin chains with general (not necessarily short-range) interactions, which turn out to be equivalent to a suitable free fermion model. The critical character of these chains (for appropriate values of the chemical potential μ) was ascertained via the analysis of the low-temperature behavior of the free energy per spin. Indeed, we proved that when the dispersion relation $\mathcal{E}(p)$ of the corresponding free fermion model is monotonic in the interval $[0,\pi]$, for $0 < \mu < \mathcal{E}(\pi)$ the free energy per spin is approximately given (in natural units $\hbar = k_B = 1$) by

$$f(T) \simeq f_0 - \frac{\pi c T^2}{6v}, \tag{1}$$

where v is the Fermi velocity (or effective speed of "sound") and $c = 1$. This is precisely the expected behavior of the free energy for any critical model (c being the central charge of its Virasoro algebra), since at low temperatures the free energy of a quantum system is determined by its lowest energy levels, and the free energy per spin of a $(1+1)$-dimensional CFT with central charge c satisfies (1) for sufficiently small T [13,14]. We also studied the ground-state Rényi entanglement entropy of the above mentioned supersymmetric spin chains, showing that in the thermodynamic limit $L \to \infty$ it again behaves as that of a $(1+1)$-dimensional CFT with central charge $c = 1$.

The aim of this paper is to extend the results of Ref. [12] by suppressing the requirement that the dispersion relation be monotonic in $[0,\pi]$. As shown in Sec. III, this makes it possible to treat a host of naturally arising models, like supersymmetric spin chains with near and next-to-near interactions, or with long-range rational interactions, whose dispersion relation is not always monotonic. In fact, the entanglement entropy of fairly arbitrary energy eigenstates of one-dimensional free fermionic systems (in particular, of the ground state of such systems with a nonmonotonic dispersion relation) has been previously studied in the literature; see, e.g., Refs. [15,16].

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In general, the entanglement entropy of the ground state of these models grows logarithmically with the size L of the subsystem, with a constant prefactor determined by the number of boundary points of the Fermi "surface" in $[0,2\pi)$. This logarithmic scaling is a manifestation of the so-called "area law," which is believed to hold for critical fermionic systems in an arbitrary number of dimensions [11]. We shall show that the su(1|1) supersymmetric chains studied in this paper do indeed satisfy the area law. More precisely, by analyzing the low-temperature behavior of the free energy we shall first show that the models under consideration are critical for $\mathcal{E}_{\min} < \mu < \mathcal{E}_{\max}$, where $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ respectively denote the minimum and maximum values of the dispersion relation. (As explained in Sec. IV, strictly speaking this is only true if the roots of the equation $\mathcal{E}(p) = \mu$ are all simple.) From the latter analysis it also follows that the central charge of these models is equal to the number $m+1$ of disjoint intervals that make up the Fermi sea. We shall next study the ground-state Rényi entanglement entropy, showing that in the thermodynamic limit $L \to \infty$ it behaves as $k_\alpha \ln L + C_\alpha$. We shall explicitly compute the (nonuniversal) constant C_α, and prove that the prefactor k_α is equal to $(m+1)(1+\alpha^{-1})/6$. This is in agreement with the value of the central charge deduced from the low-temperature analysis of the free energy, and once again confirms the conjecture that the entanglement properties of critical fermion models are entirely determined by the topology of their Fermi surface [11].

We shall end this section with a few words on the paper's organization. In Sec. II we recall the definition of the supersymmetric chains under consideration and review their main properties. Section III is devoted to the analysis of the models' dispersion relation and the construction of simple examples of supersymmetric chains, featuring both short- and long-range interactions, with a nonmonotonic dispersion relation. In Sec. IV we derive the asymptotic behavior of the models' free energy per spin at low temperature, showing that they are critical in an appropriate range of the chemical potential, and determine the central charge of the corresponding Virasoro algebra. The asymptotic behavior of the entanglement entropy of the models' ground state is determined in Sec. V using a particular case of the Fisher-Hartwig conjecture for Toeplitz matrices [17] rigorously proved by Böttcher and Silbermann [18]. We briefly state our conclusions and outline several future developments suggested by the present work in Sec. VI. The paper ends with three appendixes in which we present a review of the application of the Fisher-Hartwig conjecture in the present context, as well as the proofs of several technical results used throughout Sec. V.

II. THE MODELS

The type of models we shall study in this work is the class of su(1|1) supersymmetric spin chains with translationally invariant interactions introduced in Ref. [12]. In the latter models each site is occupied either by a scalar boson or a spinless fermion, whose creation operators we shall respectively denote by $b_i^\dagger$ and $f_i^\dagger$, the subindex $i = 1,\ldots,N$ indicating the site on which these operators act. Thus the Hilbert space is the 2^N-dimensional subspace $\mathfrak{H}$ of the infinite-dimensional Fock space defined by the constraints

$$b_i^\dagger b_i + f_i^\dagger f_i = 1, \quad 1 \leqslant i \leqslant N. \tag{2}$$

The Hamiltonian of the models under consideration is given by [19]

$$H = \sum_{i<j} h_N(j-i)(1-S_{ij}) - \mu N_f, \tag{3}$$

where the operator N_f is the total fermion number

$$N_f = \sum_i f_i^\dagger f_i,$$

so that the real parameter μ has the natural interpretation of the fermions' chemical potential. The real-valued function $h_N(k)$ giving the strength of the interaction between two particles k sites apart is assumed to satisfy the constraint

$$h_N(x) = h_N(N-x), \tag{4}$$

but is otherwise arbitrary [20]. In other words, the chain is *closed*, i.e., translationally invariant. Finally, S_{ij} is the su(1|1) spin permutation operator, defined by [21]

$$S_{ij} = b_i^\dagger b_j^\dagger b_i b_j + f_i^\dagger f_j^\dagger f_i f_j + f_j^\dagger b_i^\dagger f_i b_j + b_j^\dagger f_i^\dagger b_i f_j.$$

Equivalently, let $|s_1,\ldots,s_N\rangle \equiv |s_1\rangle \otimes \cdots \otimes |s_N\rangle$ (with $s_k \in \{0,1\}$) be a state of the canonical spin basis, where $|0\rangle$ and $|1\rangle$ respectively denote the state with one boson or one fermion. The action of S_{ij} on the latter state is then given by

$$S_{ij}|\ldots,s_i,\ldots,s_j,\ldots\rangle = (-1)^n|\ldots,s_j,\ldots,s_i,\ldots\rangle, \tag{5}$$

where $n = s_i = s_j$ if $s_i = s_j$, while for $s_i \neq s_j$, n equals the number of fermions in the state $|s_1,\ldots,s_N\rangle$ occupying the sites $i+1,\ldots,j-1$. The operator S_{ij} is clearly invariant under the supersymmetry transformation $b_i \leftrightarrow f_i$ $(1 \leqslant i \leqslant N)$, and on $\mathfrak{H}$ we have $N_f \mapsto N - N_f = N_b$, where $N_b = \sum_i b_i^\dagger b_i$ is the total boson number. Hence the Hamiltonian (3) is indeed supersymmetric invariant, up to a constant term and the usual relabeling $\mu \mapsto -\mu$.

The fundamental feature of the su(1|1) supersymmetric chain (3), explained in detail in Refs. [12,22], is that it can be mapped into a free-fermion model by interpreting the boson state $|0\rangle$ as the fermion vacuum. More precisely, consider the operators

$$a_i^\dagger = f_i^\dagger b_i, \quad i = 1,\ldots,N,$$

which can be regarded as a new set of fermion creation operators as they obviously satisfy the canonical anticommutation relations (CARs) on $\mathfrak{H}$. It was shown by Haldane [21] that on $\mathfrak{H}$ the su(1|1) permutation operator S_{ij} can be simply expressed as

$$S_{ij} = 1 - a_i^\dagger a_i - a_j^\dagger a_j + a_i^\dagger a_j + a_j^\dagger a_i. \tag{6}$$

Substituting into Eq. (3) we readily obtain

$$H = -\sum_{i\neq j} h_N(|i-j|) a_i^\dagger a_j - (\mu - \mu_0)\sum_i a_i^\dagger a_i, \tag{7}$$

where

$$\mu_0 = \sum_{j=1}^{N-1} h_N(j).$$

We thus see that the spin chain (3) is indeed equivalent to a free-fermion model with hopping amplitude $-h_N(|i-j|)$ and chemical potential $\mu-\mu_0$.

Since the Hamiltonian (7) is translationally invariant on account of Eq. (4), it can be diagonalized by the discrete Fourier transform

$$\hat{a}_l = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-2\pi i k l/N} a_k, \quad 0 \leqslant l \leqslant N-1. \tag{8}$$

Indeed, the operators $\hat{a}_l$ obviously satisfy the CAR, and can therefore be considered as a new set of fermionic operators. Moreover, a straightforward calculation shows that [22]

$$H = \sum_{l=0}^{N-1} [\varepsilon_N(l) - \mu] \hat{a}_l^\dagger \hat{a}_l, \tag{9}$$

where

$$\varepsilon_N(l) = \sum_{j=1}^{N-1} [1 - \cos(2\pi j l/N)] h_N(j). \tag{10}$$

Likewise, the system's total momentum operator $\mathcal{P}$ is given by [23]

$$\mathcal{P} = \sum_{l=0}^{N-1} p_l \, \hat{a}_l^\dagger \hat{a}_l,$$

with

$$p_l = \frac{2\pi l}{N} \pmod{2\pi}.$$

Thus the operator $\hat{a}_l^\dagger$ creates a (nonlocalized) fermion with well-defined energy $\varepsilon_N(l)$ and momentum p_l. It follows from Eq. (9) that the spectrum of H is the set of numbers of the form

$$E_N(\delta_0, \ldots, \delta_{N-1}) = \sum_{l=0}^{N-1} \delta_l \varepsilon_N(l),$$

with $\delta_l \in \{0,1\}$, whose corresponding eigenstates are given by

$$\psi(\delta_0, \ldots, \delta_{N-1}) = (\hat{a}_0^\dagger)^{\delta_0} \cdots (\hat{a}_{N-1}^\dagger)^{\delta_{N-1}} |0, \ldots, 0\rangle.$$

III. DISPERSION RELATION

An essential requirement making it possible to study the chain (3)—or, equivalently, its fermionic counterpart (7)—in the thermodynamic limit is the existence of a smooth function $\mathcal{E}(p)$ *independent of* N such that when $N \to \infty$ we have

$$\varepsilon_N(Np/2\pi) = \mathcal{E}(p) + o(1). \tag{11}$$

When this is the case, we shall refer to $\mathcal{E}(p)$ as the model's *dispersion relation*. From the latter equation and the identity $\varepsilon_N(l) = \varepsilon_N(N-l)$ it follows that the dispersion relation is always symmetric about π, namely

$$\mathcal{E}(p) = \mathcal{E}(2\pi - p). \tag{12}$$

Likewise, $\varepsilon_N(0) = 0$ implies that $\mathcal{E}(0) = 0$. It is also customary to extend $\mathcal{E}(p)$ to the whole real line as a 2π-periodic function, in which case Eq. (12) entails that $\mathcal{E}(p) = \mathcal{E}(-p)$.

For instance, for the su(1|1) Haldane-Shastry chain [21], whose interaction strength is given by

$$h_N(x) = \frac{\pi^2/N^2}{\sin^2(\pi x/N)}, \tag{13}$$

it was shown in Ref. [24] that

$$\varepsilon_N(l) = \frac{2\pi^2}{N^2} l(N-l).$$

Hence in this case (11) holds with

$$\mathcal{E}(p) = \frac{p}{2}(2\pi - p) \tag{14}$$

and no error term. In fact, it can be shown that Eq. (11) also holds (again with no error term) for a suitable dispersion relation $\mathcal{E}$ in the more general chain with elliptic interactions studied in Ref. [22].

We shall next present a few relevant examples of models of the form (3) for which the dispersion relation is guaranteed to exist. To this end, it is convenient to rewrite Eq. (10) to take into account conditions (4), namely

$$\varepsilon_N(l) = 2 \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} [1 - \cos(2\pi j l/N)] h_N(j) + 2[1 - \pi(N)]\, \pi(l)\, h_N(N/2), \tag{15}$$

where $\pi(k) \in \{0,1\}$ denotes the parity of the integer k and $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. Clearly, the values of $h_N(j)$ with $1 \leqslant j \leqslant N/2$ appearing in the latter equation are no longer restricted by Eq. (4). For this reason, from now on we shall implicitly restrict the domain of h_N to the range $1 \leqslant j \leqslant N/2$, since for $N/2 < j \leqslant N-1$ we simply have $h_N(j) = h_N(N-j)$. In this vein, we shall say (with a slight abuse of language) that the interaction is *independent of* N if there is a fixed function $h(x)$ such that $h_N(j) = h(j)$ for $1 \leqslant j \leqslant N/2$. If this is the case we shall simply write $h_N = h$, again implicitly assuming that we are restricting ourselves to the range $1 \leqslant j \leqslant N/2$.

An important class of models of the form (3) for which the dispersion relation $\mathcal{E}(p)$ is guaranteed to exist are those whose interaction strength h_N is short-ranged and independent of N. By this we mean that there is a positive integer r (the range of the interaction) such that $h_N(j) = 0$ for $r < j < N - r$, and

$$h_N(j) = \alpha_j, \quad 1 \leqslant j \leqslant r, \tag{16}$$

with α_j independent of N and $\alpha_r \neq 0$. Obviously, in this case we have

$$\mathcal{E}(p) = 2 \sum_{j=1}^{r} \alpha_j [1 - \cos(jp)]. \tag{17}$$

In fact, the same is true if we drop (16) but assume instead that the limit

$$\lim_{N\to\infty} h_N(j) \equiv \alpha_j$$

exists for all $j = 1, \ldots, r$.

On the other hand, the short range of the interaction h_N is by no means a necessary condition for the existence of the dispersion relation $\mathcal{E}(p)$. Indeed, suppose for simplicity that

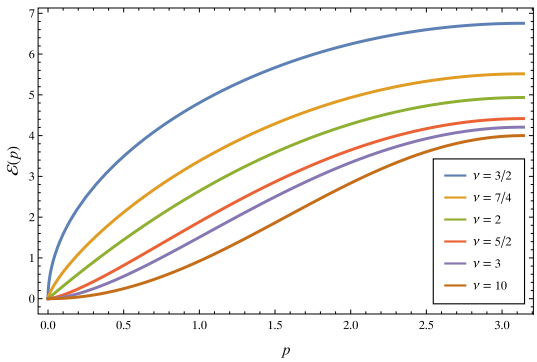

FIG. 1. Dispersion relation of the su(1|1) chain (3) with power-law interaction $h_N(x) = x^{-\nu}$ for several values of the exponent ν between 3/2 and 10.

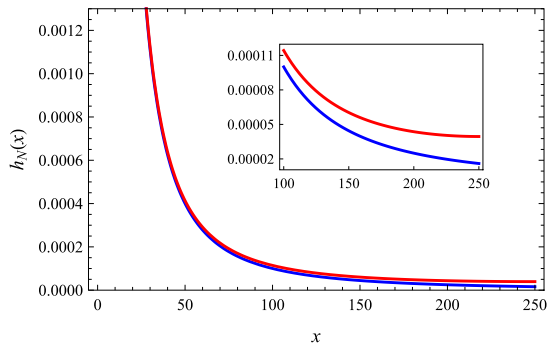

FIG. 2. Comparison of the interaction strength (13) of the su(1|1) HS chain (red, upper) with the simple inverse-square law $h_N(x) = 1/x^2$ (blue, lower) for $N = 500$. Inset: same plot for the range $100 \leqslant x \leqslant 250$.

$h_N = h$ is independent of N, and that the series $\sum_{j=1}^{\infty} h(j)$ is absolutely convergent. Then (11) clearly holds with

$$\mathcal{E}(p) = 2\sum_{j=1}^{\infty} h(j)[1 - \cos(jp)]. \tag{18}$$

For instance, for the power-law interaction $h_N(x) = Cx^{-\nu}$ with $\nu > 1$ the previous series can be summed in closed form in terms of the polylogarithm function [25]

$$\mathrm{Li}_\nu(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\nu}, \quad |z| \leqslant 1,$$

namely (taking, for simplicity, $C = 1$)

$$\begin{aligned}\mathcal{E}(p) &= 2\zeta(\nu) - \mathrm{Li}_\nu(e^{ip}) - \mathrm{Li}_\nu(e^{-ip}) \\ &= 2[\zeta(\nu) - \mathrm{Re}\,\mathrm{Li}_\nu(e^{ip})],\end{aligned} \tag{19}$$

where ζ denotes Riemann's zeta function (cf. Fig. 1).

From the integral representation

$$\mathrm{Li}_\nu(z) = \frac{z}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{e^x - z}\, dx, \tag{20}$$

where Γ is Euler's gamma function, we obtain the equivalent expression

$$\mathcal{E}(p) = 2\zeta(\nu) - \frac{2}{\Gamma(\nu)} \int_0^\infty \frac{(e^x \cos p - 1)x^{\nu-1}}{e^{2x} - 2e^x \cos p + 1}\, dx. \tag{21}$$

Using the latter formula in the identity $\mathcal{E}(p) = \int_0^p \mathcal{E}'(t)dt$ and reversing the order of integration we arrive at the somewhat simpler expression

$$\mathcal{E}(p) = \frac{2^\nu s}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} \coth x}{\sinh^2 x + s}\, dx, \quad s \equiv \sin^2(p/2).$$

Remarkably, for $\nu = 2$ Eq. (19) reduces to Eq. (14) (see, e.g., [25]). Thus the su(1|1) chain with rational interaction $h_N(x) = x^{-2}$ has the same dispersion relation as the Haldane-Shastry chain (13). This is of course not entirely unexpected, since for *fixed* $x \neq 0$ we have $\lim_{N\to\infty} (\pi/N)^2 \sin^{-2}(\pi x/N) = x^{-2}$. Note, however, that for $x \sim N/2$ both interactions, although negligibly small as $N \to \infty$, differ by a factor $\sim (\pi/2)^2$ (cf. Fig. 2).

Of course, although (11) holds for a wide range of interesting interactions, it is not universally true. For instance, it is not satisfied by the N-independent interaction $h_N(x) = C/x$, since

$$\sum_{j=1}^{\infty} \frac{\cos(jp)}{j} = -\ln[2\sin(p/2)]$$

converges for $0 < p < 2\pi$ while the series $\sum_{j=1}^{\infty} j^{-1}$ is divergent.

In a previous paper [12] we analyzed the critical behavior of supersymmetric spin chains of the type (3) whose dispersion relation is monotonic in the interval $[0,\pi]$. These models include the su(1|1) Haldane-Shastry chain [cf. (14)] and, more generally, its elliptic generalization introduced in Ref. [22]. As is apparent from Fig. 1 [and can be analytically checked differentiating Eq. (21)], the chain (3) with power-law interactions also exhibits this property. However, this behavior is not universal, and there are in fact simple examples of supersymmetric chains of the form (3) with a nonmonotonic dispersion relation.

Indeed, consider to begin with the chain (3) with nearest and next-to-nearest interactions, whose Hamiltonian (up to an irrelevant multiplicative constant) is given by

$$H = \sum_i (1 - S_{i,i+1}) + J \sum_i (1 - S_{i,i+2}) - \mu N_f, \tag{22}$$

with $S_{N,N+1} \equiv S_{1N}$, $S_{N-1,N+1} \equiv S_{1,N-1}$, and $S_{N,N+2} \equiv S_{2N}$. Note that when $J = 0$ the fermionic version of the latter model can be mapped to the (closed) Heisenberg XX chain by a Wigner-Jordan transformation [12]. From Eq. (15) with $h_N(1) = 1$ and $h_N(2) = J$ we easily obtain

$$\mathcal{E}(p) = 2(1 - \cos p) + 2J(1 - \cos 2p).$$

Since

$$\mathcal{E}'(p) = 2\sin p(1 + 4J\cos p),$$

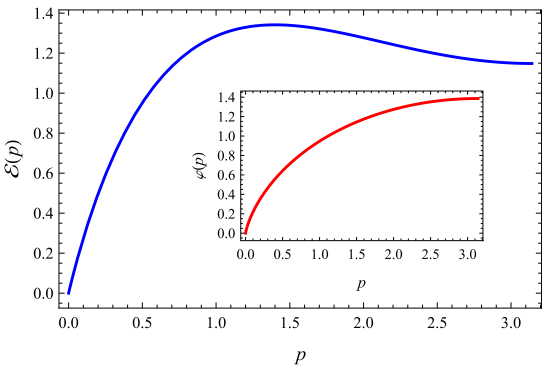

FIG. 3. Dispersion relation of the spin chain (3) with interaction (23) for $J = 0.9$. Inset: plot of the function $\varphi(p)$ in Eq. (24).

the dispersion relation will have a critical point in $(0,\pi)$ if and only if $|J| > 1/4$ (more precisely, a maximum for $J > 1/4$ and a minimum for $J < -1/4$). Thus in this case the dispersion relation is not monotonic in $[0,\pi]$ provided that $|J| > 1/4$. The same is clearly true for chains of the form (3) with interactions of finite range $r > 1$, for suitable values of the interaction strengths.

It is also easy to construct simple examples of chains of the form (3) with long-range interactions with a nonmonotonic dispersion relation. Take, for instance,

$$h_N(x) = \frac{1}{x^2} - \frac{J}{x^3}, \tag{23}$$

whose dispersion relation is given by

$$\mathcal{E}(p) = \frac{p}{2}(2\pi - p) - 2J[\zeta(3) - \mathrm{Re}\,\mathrm{Li}_3(e^{ip})].$$

If $p \in (0,\pi)$, differentiating the latter equation we obtain

$$\mathcal{E}'(p) = (\pi - p)[1 - J\varphi(p)],$$

with

$$\varphi(p) = 2\,\mathrm{Im}\,\mathrm{Li}_2(e^{ip})/(\pi - p). \tag{24}$$

It can be shown (cf. Fig. 3) that the function $\varphi(p)$ increases monotonically over the interval $(0,\pi)$, with $\varphi(0) = 0$ and $\lim_{p\to\pi-0}\varphi(p) = 2\ln 2$ [26], so that $\mathcal{E}'(p)$ changes sign once (from positive to negative) in $(0,\pi)$ if and only if $J > (2\ln 2)^{-1}$. We conclude that the dispersion relation of the chain (3) with interaction (23) is not monotonic on $[0,\pi]$ provided that $J > (2\ln 2)^{-1} \simeq 0.721\,348$. In particular, for $(2\ln 2)^{-1} < J < 1$ the dispersion relation is not monotonic in $[0,\pi]$ even if the interaction strength is positive for $x \geqslant 1$ [see Fig. 3 for a plot of $\mathcal{E}(p)$ when $J = 0.9$].

IV. CRITICAL BEHAVIOR

In this section we shall study the critical properties of the spin chain (3) when its dispersion relation $\mathcal{E}(p)$ is not necessarily monotonic over the interval $[0,\pi]$. To this end, we shall examine the low temperature behavior of the Helmholtz free energy per spin

$$f(T) = -T\lim_{N\to\infty}\frac{\ln Z_N}{N},$$

which for this model is given by (cf. [12])

$$f(T) = -\frac{T}{\pi}\int_0^\pi \ln[1 + e^{-\beta[\mathcal{E}(p)-\mu]}]\,dp. \tag{25}$$

In the previous expressions Z_N denotes the partition function of the chain (3) with N spins, and $\beta = 1/T$ (in natural units $\hbar = k_B = 1$). As remarked in the Introduction, at low temperatures the free energy of a critical model should satisfy Eq. (1). Moreover, it was shown in Ref. [12] that when $\mathcal{E}(p)$ is monotonic and nonnegative in the interval $[0,\pi]$ the model (3) is critical when the chemical potential μ lies in the interval $(0,\mathcal{E}(\pi))$, with central charge $c = 1$, and noncritical for μ outside the closed interval $[0,\mathcal{E}(\pi)]$. We shall next extend this result to the more general case in which $\mathcal{E}(p)$ is not necessarily monotonic (nor nonnegative) in $[0,\pi]$.

To begin with, it is immediate to show that the model (3) is not critical when μ lies outside the interval $[\mathcal{E}_{\min},\mathcal{E}_{\max}]$. Indeed, suppose first that $\mu < \mathcal{E}_{\min}$, so that $\mathcal{E}(p) - \mu > 0$, $f_0 \equiv f(0) = 0$ and

$$|f(T)| < \frac{T}{\pi}\int_0^\pi e^{-\beta[\mathcal{E}(p)-\mu]}\,dp < Te^{-\beta(\mathcal{E}_{\min}-\mu)},$$

in contradiction with the asymptotic behavior (1) characteristic of a critical model. Similarly, when $\mu > \mathcal{E}_{\max}$ we have

$$f_0 \equiv f(0) = \frac{1}{\pi}\int_0^\pi [\mathcal{E}(p) - \mu]dp$$

and

$$|f(T) - f_0| = \frac{T}{\pi}\int_0^\pi \ln[1 + e^{-\beta[\mu-\mathcal{E}(p)]}]\,dp < Te^{-\beta(\mu-\mathcal{E}_{\max})},$$

again in disagreement with Eq. (1). This conclusion is also borne out by the fact that when $\mu > \mathcal{E}_{\max}$ or $\mu < \mathcal{E}_{\min}$ the spectrum is clearly gapped, with energy gap respectively equal to $\mu - \mathcal{E}_{\max}$ or $\mathcal{E}_{\min} - \mu$.

Let us now consider the more interesting case in which $\mu \in (\mathcal{E}_{\min},\mathcal{E}_{\max})$, in which the spectrum is clearly gapless. We shall suppose that the equation $\mathcal{E}(p) = \mu$ has $m + 1 \geqslant 1$ roots $p_0 < p_1 < \cdots < p_m$ in the interval $(0,\pi)$, which we will assume to be *simple*. We start by expressing the free energy as

$$f(T) = f_0 - \frac{T}{\pi}\int_0^\pi \ln(1 + e^{-\beta|\mathcal{E}(p)-\mu|})dp, \tag{26}$$

where

$$f_0 \equiv f(0) = \frac{1}{\pi}\int_{\mathcal{E}(p)<\mu}[\mathcal{E}(p) - \mu]dp$$

and the last integral is extended to the subset of the interval $[0,\pi]$ defined by the inequality $\mathcal{E}(p) < \mu$. Clearly, as $T \to 0+$ the main contribution to the integral in Eq. (26) comes from an increasingly small neighborhood of the "turning points" p_i, near which $|\mathcal{E}(p) - \mu|$ is small. To exploit this fact, we choose $\Delta p > 0$ small enough that $[p_i - \Delta p, p_i + \Delta p] \cap [p_j - \Delta p, p_j + \Delta p] = \emptyset$ for $i \neq j$ and $\mathcal{E}'(p) \neq 0$ on $\cup_{i=0}^m [p_i - \Delta p, p_i + \Delta p]$. This is certainly possible, since by

hypothesis $\mathcal{E}'(p_i) \neq 0$ for all i. Obviously, Δp depends only on the dispersion relation $\mathcal{E}(p)$, and is therefore independent of T. Calling $A = [0,\pi] - \cup_{i=0}^{m}[p_i - \Delta p, p_i + \Delta p]$ we have

$$\int_0^\pi \ln[1 + e^{-\beta|\mathcal{E}(p)-\mu|}]dp = \int_A \ln[1 + e^{-\beta|\mathcal{E}(p)-\mu|}]dp + \sum_{i=0}^{m}\int_{p_i-\Delta p}^{p_i+\Delta p} \ln[1 + e^{-\beta|\mathcal{E}(p)-\mu|}]dp. \quad (27)$$

The first integral can be easily estimated. Indeed, let a be the minimum value of $|\mathcal{E}(p) - \mu|$ on the compact set $\overline{A}$, which is clearly positive since $\mathcal{E}(p) \neq \mu$ on $\overline{A}$, and denote by $|A|$ the length of A. We then have

$$\int_A \ln[1 + e^{-\beta|\mathcal{E}(p)-\mu|}]dp \leqslant e^{-a\beta}|A| \leqslant \pi e^{-a\beta}, \quad (28)$$

with a (and $|A|$) obviously independent of T. Consider next the integral

$$I_i \equiv \int_{p_i-\Delta p}^{p_i+\Delta p} \ln[1 + e^{-\beta|\mathcal{E}(p)-\mu|}]dp. \quad (29)$$

To analyze its low temperature behavior, we perform the change of variable

$$x = \beta|\mathcal{E}(p) - \mu| \quad (30)$$

separately in each of the intervals $[p_i - \Delta p, p_i]$ and $[p_i, p_i + \Delta p]$. Since $\mathcal{E}'(p_i) \neq 0$, this change of variable is one to one and C^∞ in both of the latter intervals, and we have

$$\frac{I_i}{T} = \int_0^{\beta|\mathcal{E}(p_i-\Delta p)-\mu|} \ln(1 + e^{-x})\frac{dx}{|\mathcal{E}'(p)|} + \int_0^{\beta|\mathcal{E}(p_i+\Delta p)-\mu|} \ln(1 + e^{-x})\frac{dx}{|\mathcal{E}'(p)|}. \quad (31)$$

The asymptotic behavior of these integrals as $T \to 0+$ can be easily determined taking into account that by construction $\mathcal{E}'(p)$ does not vanish on both intervals $[p_i - \Delta p, p_i]$ and $[p_i, p_i + \Delta p]$, and therefore

$$\frac{1}{\mathcal{E}'(p)} = \frac{1}{\mathcal{E}'(p_i)} + O(p - p_i) = \frac{1}{\mathcal{E}'(p_i)} + O(Tx),$$

as $x(p_i) = 0$ implies that $p - p_i = O(Tx)$. Since the integral $\int_0^\infty x\ln(1 + e^{-x})dx$ is convergent we have

$$I_i = \frac{T}{v_i}\left(\int_0^{\beta|\mathcal{E}(p_i-\Delta p)-\mu|} \ln(1 + e^{-x})dx + \int_0^{\beta|\mathcal{E}(p_i+\Delta p)-\mu|} \ln(1 + e^{-x})dx\right) + O(T^2), \quad (32)$$

where we have set

$$v_i = |\mathcal{E}'(p_i)|.$$

Moreover, if K is independent of β we have

$$\left|\int_0^{K\beta} \ln(1 + e^{-x})dx - \int_0^\infty \ln(1 + e^{-x})dx\right| \leqslant \int_{K\beta}^\infty e^{-x}dx = e^{-K\beta}.$$

Using this inequality and the integral

$$\int_0^\infty \ln(1 + e^{-x})dx = \frac{\pi^2}{12}$$

in Eq. (32) we thus have

$$I_i = \frac{\pi^2 T}{6v_i} + O(T^2).$$

From Eqs. (26)–(29) we finally obtain the asymptotic estimate

$$f = f_0 - \frac{\pi T^2}{6}\sum_{i=0}^{m}\frac{1}{v_i} + O(T^3), \quad T \to 0+. \quad (33)$$

This is the low temperature behavior of the free energy of a $(1 + 1)$-dimensional CFT with $m + 1$ free bosons with Fermi velocities $v_0, \ldots, v_m$. Thus in this case the model (3) is critical, with central charge $c = m + 1$.

The situation is markedly different if any of the roots of the equation $\mathcal{E}(p) = \mu$ is not simple. Indeed, assume that p_k is a root of order $\nu_k > 1$ of the latter equation, so that we can write

$$\mathcal{E}(p) - \mu = \varepsilon_k\left(\frac{p - p_k}{b_k}\right)^{\nu_k} + O[(p - p_k)^{\nu_k+1}]$$

with

$$b_k = \left(\frac{\nu_k!}{|\mathcal{E}^{(\nu_k)}(p_k)|}\right)^{1/\nu_k}, \quad \varepsilon_k = \operatorname{sgn}\mathcal{E}^{(\nu_k)}(p_k).$$

We now choose $\Delta p > 0$ such that $[p_i - \Delta p, p_i + \Delta p] \cap [p_j - \Delta p, p_j + \Delta p] = \emptyset$ for $i \neq j$ and $\mathcal{E}'(p) \neq 0$ on $[p_i - \Delta p, p_i) \cup (p_i, p_i + \Delta p]$ for all i. Proceeding as before we again arrive at Eqs. (26) and (27) and obtain the estimate (28) for the first integral in Eq. (27). In order to analyze the low temperature behavior of the integral I_k, we again perform the change of variable (30) in each of the intervals $[p_k - \Delta p, p_k]$ and $[p_k, p_k + \Delta p]$, thus obtaining Eq. (31) with $i = k$. In each of the latter intervals we now have

$$|p - p_k| = b_k(Tx)^{1/\nu_k}[1 + O((Tx)^{1/\nu_k})]$$

and

$$|\mathcal{E}'(p)| = \frac{\nu_k}{b_k}\left(\frac{|p - p_k|}{b_k}\right)^{\nu_k-1}[1 + O(p - p_k)],$$

so that

$$|\mathcal{E}'(p)|^{-1} = \frac{b_k}{\nu_k}(Tx)^{1/\nu_k-1}[1 + O((Tx)^{1/\nu_k})].$$

Substituting into Eq. (31) and proceeding as before we easily obtain

$$I_k = \frac{2b_k}{\nu_k}T^{1/\nu_k}\int_0^\infty x^{1/\nu_k-1}\ln(1 + e^{-x})dx + O(T^{2/\nu_k}) = 2b_k(1 - 2^{-1/\nu_k})\Gamma\left(1 + \nu_k^{-1}\right)\zeta(1 + \nu_k^{-1})T^{1/\nu_k} + O(T^{2/\nu_k})$$

(see Ref. [12] for more details on the evaluation of the last integral). Thus at low temperatures the contribution of p_k to the free energy, given by

$$-\frac{TI_k}{\pi} = -\frac{2b_k}{\pi}(1-2^{-1/\nu_k})\Gamma\big(1+\nu_k^{-1}\big)\zeta\big(1+\nu_k^{-1}\big)T^{1+1/\nu_k} + O(T^{1+2/\nu_k}), \tag{34}$$

dominates over the $O(T^2)$ contribution coming from the simple roots p_i. Moreover, since the coefficient of T^{1+1/ν_k} in Eq. (34) is always negative, this term cannot be compensated by similar terms in Eq. (27) coming from other multiple roots. We thus conclude that when $\mu \in (\mathcal{E}_{\min},\mathcal{E}_{\max})$, but the equation $\mathcal{E}(p) = \mu$ has at least one multiple root, the model (3) cannot be critical. A similar analysis shows that this is also the case when $\mu = \mathcal{E}_{\min}$ or $\mu = \mathcal{E}_{\max}$ [27]. This shows that the model (3) is critical if and only if $\mathcal{E}_{\min} < \mu < \mathcal{E}_{\max}$ and all the roots of the equation $\mathcal{E}(p) = \mu$ are simple. When that is the case, the central charge of the model is equal to the number of connected components of its Fermi sea (or, equivalently, half the number of connected components of its Fermi "surface"). Thus, the universality class of the model (3) depends exclusively on the topology of its Fermi sea, which confirms the general assertion in Ref. [11].

V. GROUND-STATE ENTANGLEMENT ENTROPY

As mentioned in the Introduction, one of the hallmarks of a critical fermionic lattice model in one dimension with short-range interactions is the logarithmic growth of its ground-state bipartite entanglement entropy with the length L of the block of spins considered. More precisely, let

$$S_\alpha = (1-\alpha)^{-1}\ln \operatorname{tr}\big(\rho_L^\alpha\big)$$

denote the Rényi entropy of the block when the whole chain is in its ground state $|\psi\rangle$, where $\rho_L = \operatorname{tr}_{N-L}|\psi\rangle\langle\psi|$. The expected behavior of S_α in this type of models is then

$$S_\alpha = \frac{c}{6}(1+\alpha^{-1})\ln L + C_\alpha, \tag{35}$$

where c is the central charge of the corresponding Virasoro algebra and C_α is a nonuniversal constant (independent of L). We showed in a previous paper [12] that the latter formula is also valid for the supersymmetric chains (3) when their dispersion relation is monotonic (and nonnegative) in the interval $[0,\pi]$, even in the case of long-range interactions. In this section we shall extend this result to a general model of the type (3), whose dispersion relation need not be monotonic (or nonnegative) in $[0,\pi]$.

To this end, recall first of all that the ground-state entanglement entropy S_α can be expressed in terms of the eigenvalues of the ground-state correlation matrix A_L, with matrix elements

$$(A_L)_{jk} = \langle\psi|a_j^\dagger a_k|\psi\rangle, \quad 1 \leqslant j,k \leqslant L.$$

Indeed, it was shown in Ref. [5] that

$$S_\alpha = \sum_{i=1}^{L} s_\alpha(\lambda_i), \tag{36}$$

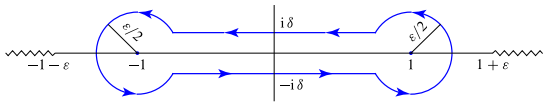

FIG. 4. Integration path $\gamma_{\varepsilon,\delta}$ in Eq. (38).

where

$$s_\alpha(x) = (1-\alpha)^{-1}\ln\left[\left(\frac{1+x}{2}\right)^\alpha + \left(\frac{1-x}{2}\right)^\alpha\right]$$

and $\lambda_1,\ldots,\lambda_L \in [-1,1]$ are the eigenvalues of the matrix $2A_L - \mathbb{1}$. The asymptotic behavior of S_α can be determined following the method developed by Jin and Korepin [6] for the XX model. To this end, for $\varepsilon > 0$ we define the complex-valued function

$$s_\alpha^{(\varepsilon)}(z) = (1-\alpha)^{-1}\ln\left[\left(\frac{1+\varepsilon+z}{2}\right)^\alpha + \left(\frac{1+\varepsilon-z}{2}\right)^\alpha\right], \tag{37}$$

where $\ln z \equiv \ln|z| + i\arg_{(-\pi,\pi]} z$ and $z^a \equiv e^{a\ln z}$. This function has a logarithmic branch cut on the set $|\operatorname{Re} z| \geqslant 1+\varepsilon$ and no other singularities on a sufficiently small open subset (independent of ε) containing the interval $[-1,1]$ [28]. By Cauchy's theorem and Eq. (36), if $\gamma_{\varepsilon,\delta}$ is the path sketched in Fig. 4 we therefore have

$$S_\alpha = \lim_{\varepsilon,\delta\to 0+}\frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\delta}} s_\alpha^{(\varepsilon)}(\lambda)\frac{d}{d\lambda}\ln D_L(\lambda)\,d\lambda, \tag{38}$$

where

$$D_L(\lambda) \equiv \det(\lambda + 1 - 2A_L). \tag{39}$$

As explained in Appendix A, the latter integral can then be approximated using a proved case of the Fisher-Hartwig conjecture to estimate the logarithmic derivative of $D_L(\lambda)$.

A. Asymptotic formula for $D_L(\lambda)$

In order to derive the asymptotic behavior of $D_L(\lambda)$, we first need to determine the symbol of the Toeplitz matrix $T_L \equiv \lambda + 1 - 2A_L$ (see again Appendix A for the definition of the symbol and its calculation in two simple cases). We shall compute this symbol for a general model of the type (3), whose dispersion relation is not assumed to be monotonic over $[0,\pi]$. More precisely, we shall only suppose that the equation $\mathcal{E}(p) = \mu$ has $m+1 \geqslant 1$ *simple* roots $p_0 < p_1 < \cdots < p_m$ in the interval $(0,\pi)$. From the symmetry of $\mathcal{E}$ around π [cf. Eq. (12)] it then follows that the remaining roots of the equation $\mathcal{E}(p) = \mu$ in the interval $(0,2\pi)$ are $2\pi - p_m < \cdots < 2\pi - p_1 < 2\pi - p_0$.

In general, the system's ground state $|\psi\rangle$ is determined by the conditions [29]

$$\begin{cases} \hat{a}_k^\dagger|\psi\rangle = 0, & \varepsilon_N(k) < \mu, \\ \hat{a}_k|\psi\rangle = 0, & \varepsilon_N(k) > \mu, \end{cases}$$

so that

$$\langle\psi|\hat{a}_j^\dagger\hat{a}_k|\psi\rangle = \begin{cases} 0, & \varepsilon_N(k) > \mu, \\ \delta_{jk}, & \varepsilon_N(k) < \mu. \end{cases}$$

It immediately follows from Eq. (8) that the matrix elements of the correlation matrix A_L are given by

$$(A_L)_{jk} = \frac{1}{N}\sum_{l\in I} e^{-2\pi i(j-k)l/N},$$

where the sum ranges over the set I of integers in the range $[0,N-1]$ satisfying the condition $\varepsilon_N(l) < \mu$. In the thermodynamic limit $N \to \infty$ the latter formula becomes

$$(A_L)_{jk} = \frac{1}{2\pi}\int_{\mathcal{E}(p)<\mu} e^{-i(j-k)p}dp, \tag{40}$$

where the integral is extended to the subset of the interval $(0,2\pi)$ defined by the inequality $\mathcal{E}(p) < \mu$. In fact, by the 2π periodicity of the integrand we can replace the interval $(0,2\pi)$ by any interval of length 2π, which we shall take as $[-p_0,2\pi - p_0]$. Let us suppose, for definiteness, that $\mathcal{E}'(p_0) > 0$ [the case $\mathcal{E}'(p_0) < 0$ is dealt with similarly]. From the simple nature of the roots p_j, $2\pi - p_j$, it then follows that the subintervals of the interval $(-p_0,2\pi - p_0)$ on which $\mathcal{E}(p) - \mu$ is negative are

$$(p_{2k-1}, p_{2k}), \quad 0 \leqslant k \leqslant \lfloor m/2 \rfloor,$$

with $p_{-1} \equiv -p_0$, and

$$(2\pi - p_{2k}, 2\pi - p_{2k-1}), \quad 1 \leqslant k \leqslant \lfloor (m+1)/2 \rfloor,$$

with $p_{m+1} \equiv 2\pi - p_m$. By Eq. (40), the symbol of the Toeplitz matrix $T_L = \lambda + 1 - 2A_L$ is given by

$$c(e^{i\theta}) = \begin{cases} \lambda - 1, & -p_0 < \theta < p_0, \\ \lambda + 1, & p_0 < \theta < p_1 \\ \quad\vdots & \quad\vdots \\ \lambda + 1, & 2\pi - p_2 < \theta < 2\pi - p_1. \end{cases} \tag{41}$$

Thus $c(e^{i\theta})$ is piecewise constant and alternates between the two values $\lambda - 1$ and $\lambda + 1$. The discontinuities of this symbol at the points $e^{\pm ip_j}$ (with $0 \leqslant j \leqslant m$) suggest the ansatz

$$c(e^{i\theta}) = b(e^{i\theta})\prod_{j=0}^{m} t_{\beta_j}(e^{i(\theta+p_j)})t_{-\beta_j}(e^{i(\theta-p_j)})$$

for suitable b and β_j. To verify this ansatz, we note that for $p_{j-1} < \theta < p_j$ (with $0 \leqslant j \leqslant m$) we have

$$t_{\beta_k}(e^{i(\theta+p_k)}) = e^{i\beta_k(\theta+p_k-\pi)},$$

$$t_{-\beta_k}(e^{i(\theta-p_k)}) = \begin{cases} e^{-i\beta_k(\theta-p_k-\pi)}, & 0 \leqslant k \leqslant j-1 \\ e^{-i\beta_k(\theta-p_k+\pi)}, & j \leqslant k \leqslant m, \end{cases}$$

whereas for $2\pi - p_j < \theta < 2\pi - p_{j-1}$ (with $1 \leqslant j \leqslant m+1$)

$$t_{\beta_k}(e^{i(\theta+p_k)}) = \begin{cases} e^{i\beta_k(\theta+p_k-\pi)}, & 0 \leqslant k \leqslant j-1 \\ e^{i\beta_k(\theta+p_k-3\pi)}, & j \leqslant k \leqslant m, \end{cases}$$

$$t_{-\beta_k}(e^{i(\theta-p_k)}) = e^{-i\beta_k(\theta-p_k-\pi)},$$

and thus in either case

$$c(e^{i\theta}) = b(e^{i\theta})e^{2i\sum_{k=0}^{m}\beta_k p_k}e^{-2\pi i\sum_{k=j}^{m}\beta_j}.$$

Comparing the latter formula with Eq. (41) we arrive at the system

$$b\, e^{2i\sum_{k=0}^{m}\beta_k p_k}e^{-2\pi i\sum_{k=j}^{m}\beta_j} = \lambda - (-1)^j, \quad 0 \leqslant j \leqslant m+1. \tag{42}$$

These equations easily imply that $\beta_j + \beta_{j+1}$ is an integer multiple of 2π for $j = 0,\ldots,m-1$. We shall take $\beta_j + \beta_{j+1} = 0$, so that calling $\beta_0 = \beta$ we have

$$\beta_j = (-1)^j\beta, \quad 0 \leqslant j \leqslant m. \tag{43}$$

From the equations with $j = m$ and $j = m+1$ we then obtain

$$e^{(-1)^m 2\pi i\beta} = \frac{\lambda + (-1)^m}{\lambda - (-1)^m},$$

so that we can take

$$\beta = \frac{1}{2\pi i}\ln\left(\frac{\lambda+1}{\lambda-1}\right). \tag{44}$$

Finally, from the equation with $j = m+1$ we have

$$\begin{aligned} b &= [\lambda + (-1)^m]e^{-2i\beta\sum_{k=0}^{m}(-1)^k p_k} \\ &= [\lambda + (-1)^m]\left(\frac{\lambda+1}{\lambda-1}\right)^{-\sum_{k=0}^{m}(-1)^k(p_k/\pi)}, \end{aligned} \tag{45}$$

which can also be written as

$$b = (\lambda+1)\left(\frac{\lambda+1}{\lambda-1}\right)^{-P}, \tag{46}$$

where

$$P \equiv \sum_{k=0}^{m}(-1)^k\frac{p_k}{\pi} + \pi(m) \tag{47}$$

and $\pi(m)$ denotes the parity of m. It is easy to check that with this choice of b and β_j Eq. (42) are all satisfied.

Since Eq. (44) coincides with the first Eq. (A9), as explained in Appendix A, the condition $|\mathrm{Re}\,\beta| < 1/2$ is satisfied, so that we can apply the Fisher-Hartwig conjecture to estimate $D_L(\lambda) \equiv \det T_L$. To this end (using the notation in Appendix A), note first of all that $R = 2(m+1)$ and, by Eq. (A4),

$$M = -2(m+1)\beta^2. \tag{48}$$

Moreover, from Eq. (43) it easily follows that

$$\prod_{r=1}^{R} G(1+\beta_r)G(1-\beta_r) = [G(1+\beta)G(1-\beta)]^{2(m+1)}$$

and

$$\begin{aligned} &\prod_{1\leqslant s<r\leqslant R}\left[2\left|\sin\left(\frac{\theta_r-\theta_s}{2}\right)\right|\right]^{2\beta_r\beta_s} \\ &= \prod_{i=0}^{m}(2\sin p_i)^{-2\beta^2}\prod_{0\leqslant j<i\leqslant m}\left[2\sin\left(\frac{p_i-p_j}{2}\right)\right]^{4(-1)^{i+j}\beta^2} \\ &\quad\times\prod_{0\leqslant j<i\leqslant m}\left[2\sin\left(\frac{p_i+p_j}{2}\right)\right]^{-4(-1)^{i+j}\beta^2}. \end{aligned}$$

Equation (A5) and the Fisher-Hartwig conjecture (A3) thus yield the asymptotic formula

$$\begin{aligned} D_L(\lambda) = {} & [f(p_0,\ldots,p_m)L^{m+1}]^{-2\beta^2}(\lambda+1)^L\left(\frac{\lambda+1}{\lambda-1}\right)^{-LP} \\ & \times [G(1+\beta)G(1-\beta)]^{2(m+1)}[1+o(1)], \end{aligned} \tag{49}$$

with

$$f(p_0,\ldots,p_m)=\prod_{i=0}^{m}(2\sin p_i)\times\prod_{0\leqslant j<i\leqslant m}\left[\frac{\sin^2\left(\frac{p_i+p_j}{2}\right)}{\sin^2\left(\frac{p_i-p_j}{2}\right)}\right]^{(-1)^{i+j}} \quad (50)$$

independent of L and λ.

B. Asymptotic behavior of the ground-state entanglement entropy

We shall next use the approximate formula (49) and Eq. (38) to derive an asymptotic formula for the Rényi entanglement entropy of the ground state of a general model of the form (3) in the limit $L\to\infty$. First of all, from Eq. (49) we easily obtain

$$\frac{d}{d\lambda}\ln D_L(\lambda)\simeq L\left(\frac{1-P}{\lambda+1}+\frac{P}{\lambda-1}\right)+\frac{4i\beta}{\pi(1-\lambda^2)}[\ln(L^{m+1}f)+(m+1)\Phi(\lambda)], \quad (51)$$

with

$$\Phi(\lambda)=-\frac{1}{2\beta}\frac{d}{d\beta}\ln[G(1+\beta)G(1-\beta)]=1+\gamma_E+\sum_{n=1}^{\infty}\frac{\beta^2/n}{n^2-\beta^2} \quad (52)$$

[cf. Eq. (A6)]. In fact, the dominant term (proportional to L) in the previous expression does not contribute to Eq. (38), since by Cauchy's residue theorem we have

$$\frac{1}{2\pi i}\int_{\gamma_{\varepsilon,\delta}}s_\alpha^{(\varepsilon)}(\lambda)\frac{d\lambda}{\lambda\mp1}=s_\alpha^{(\varepsilon)}(\pm1)\underset{\varepsilon\to0+}{\longrightarrow}s_\alpha(\pm1)=0.$$

Thus Eqs. (38)–(51) yield

$$S_\alpha\simeq\frac{2}{\pi^2}\lim_{\varepsilon,\delta\to0+}\int_{\gamma_{\varepsilon,\delta}}\frac{s_\alpha^{(\varepsilon)}(\lambda)}{1-\lambda^2}\beta[\ln(fL^{m+1})+(m+1)\Phi(\lambda)]d\lambda.$$

Moreover, it is straightforward to verify that the integral along the circular arcs of $\gamma_{\varepsilon,\delta}$ vanishes identically, since each of these arcs is mapped to the opposite of the other by the transformation $\lambda\mapsto-\lambda$, and the integrand changes sign under the latter mapping [cf. Eqs. (37), (44), and (52)]. We thus obtain

$$S_\alpha\simeq\frac{2}{\pi^2}\left(\int_{-1-i\,0}^{1-i\,0}-\int_{-1+i\,0}^{1+i\,0}\right)\frac{s_\alpha(\lambda)}{1-\lambda^2}\beta[\ln(fL^{m+1})+(m+1)\Phi(\lambda)]d\lambda. \quad (53)$$

In order to evaluate these integrals, we note that along the segments $\lambda=x\pm\mathrm{i}\delta$ with $|x|<1$ we have

$$w\equiv\frac{\lambda+1}{\lambda-1}=\frac{x^2-1+\delta^2\mp2i\delta}{(1-x)^2+\delta^2},$$

so that

$$\lim_{\delta\to0+}|w|=\frac{1+x}{1-x}.$$

On the other hand,

$$\mathrm{Re}\,w=\frac{x^2-1+\delta^2}{(1-x)^2+\delta^2}$$

is negative for sufficiently small δ, while

$$\frac{\mathrm{Im}\,w}{\mathrm{Re}\,w}=\frac{\pm2\delta}{1-x^2-\delta^2}$$

tends to 0 as $\delta\to0+$ and has the same sign as $\pm\delta$, so that

$$\lim_{\delta\to0+}\arg_{(-\pi,\pi]}w=\mp\pi.$$

We thus have

$$\lim_{\delta\to0+}\beta(x\pm i\delta)=\frac{1}{2\pi i}\left[\ln\left(\frac{1+x}{1-x}\right)\mp i\pi\right]\equiv-iB(x)\mp\frac{1}{2},$$

with

$$B(x)=\frac{1}{2\pi}\ln\left(\frac{1+x}{1-x}\right).$$

From Eq. (53) it immediately follows that

$$S_\alpha\simeq[\ln(L^{m+1}f)+(m+1)(1+\gamma_E)]I_1(\alpha)+(m+1)I_2(\alpha), \quad (54)$$

with

$$I_1(\alpha)=\frac{2}{\pi^2}\int_{-1}^{1}\frac{s_\alpha(x)}{1-x^2}\,dx\,, \quad (55)$$

$$I_2(\alpha)=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-1}^{1}\frac{s_\alpha(x)}{1-x^2}\,\mathrm{Re}\left[\frac{\left(\frac12+iB(x)\right)^3}{n^2-\left(\frac12+iB(x)\right)^2}\right]dx. \quad (56)$$

The value of the integral $I_1(\alpha)$ can be deduced from Ref. [6] (cf. also [16]), namely

$$I_1(\alpha)=\frac{1+\alpha}{6\alpha} \quad (57)$$

(see Appendix B for an elementary derivation of the latter formula). We thus obtain

$$S_\alpha\simeq(m+1)\frac{1+\alpha}{6\alpha}\ln(Lf^{1/m+1})+(m+1)\widetilde{C}_\alpha, \quad (58)$$

where

$$\widetilde{C}_\alpha\equiv\frac{1+\alpha}{6\alpha}(1+\gamma_E)+I_2(\alpha)=\frac{1+\alpha}{6\alpha}(1+\gamma_E)+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-1}^{1}\frac{s_\alpha(x)}{1-x^2}\,\mathrm{Re}\left[\frac{\left(\frac12+iB(x)\right)^3}{n^2-\left(\frac12+iB(x)\right)^2}\right]dx. \quad (59)$$

Comparing with Eq. (35), we see that the ground-state Rényi entanglement entropy of the model (3) behaves as that of a critical system with central charge $c=m+1$, as expected. Moreover, the constant term C_α is given in this case by

$$C_\alpha=\frac{1+\alpha}{6\alpha}\ln f(p_0,\ldots,p_m)+(m+1)\widetilde{C}_\alpha, \quad (60)$$

where the first term is model dependent (it depends on μ and $\mathcal{E}(p)$ through the momenta p_i), while $\widetilde{C}_\alpha$ is a universal constant

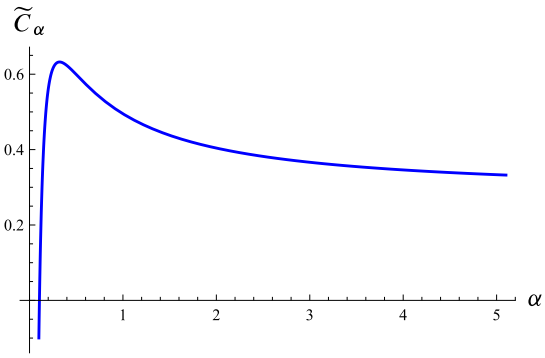

FIG. 5. Plot of the constant term $\widetilde{C}_\alpha$ in Eq. (58).

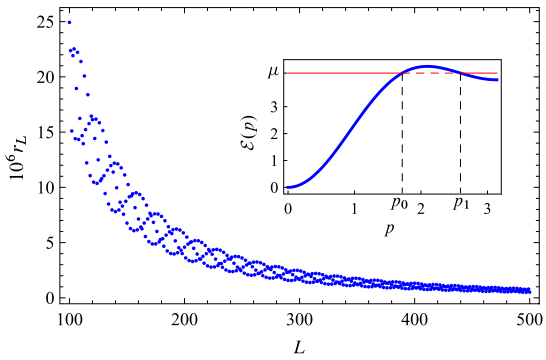

FIG. 6. Relative error $r_L \equiv S_{\rm app}/S - 1$ of the approximation $S_{\rm app}$ in the right-hand side of Eq. (63) to the von Neumann ground-state entanglement entropy of the chain (22) with $J = 1/2$ and $\mu = 17/4$ as a function of the block length L. Inset: dispersion relation of the chain (22) with the latter values of J and μ. [The interval $(\mathcal{E}(\pi),\mathcal{E}_{\rm max})$ is in this case is $(4,9/2)$, $m = 1$, $p_0 \simeq 1.717\,77$, and $p_1 \simeq 2.593\,56$.]

(independent of L and p_i) characteristic of the class of models under consideration. It is shown in Appendix C that $\widetilde{C}_\alpha$ in Eq. (58) can be expressed as

$$\widetilde{C}_\alpha = -\frac{2}{\pi^2}\int_{-1}^{1}\frac{s_\alpha(x)}{1-x^2}\,\mathrm{Re}\left[\psi\left(\frac{1}{2}+iB(x)\right)\right]dx, \qquad (61)$$

where

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z)$$

is the digamma function. In particular, Eq. (61) implies that $\widetilde{C}_\alpha$ coincides with the function $\Upsilon_1^{(\alpha)}$ defined in Eq. (64) of Ref. [6]. Since for $m = 0$ we have $f(p_0) = 2\sin p_0$, Eq. (58) yields the formula derived in Ref. [12] for the Rényi entanglement entropy of the model (3) when its dispersion relation is monotonic over the interval $[0,\pi]$ (which, as explained in the latter reference, includes the XX model studied in Ref. [6]). In fact, using the ideas of Ref. [6] Eq. (61) can be written in the simpler form

$$\widetilde{C}_\alpha = \frac{1}{1-\alpha}\int_0^\infty\left(\alpha\,\mathrm{csch}^2 t - \mathrm{csch}\,t\,\mathrm{csch}(t/\alpha) - \frac{1-\alpha^2}{6\alpha}e^{-2t}\right)\frac{dt}{t} \qquad (62)$$

(see Appendix C for details). From the previous expression it is straightforward to evaluate $\widetilde{C}_\alpha$ numerically for any specific value of the Rényi parameter $\alpha > 0$; cf. Fig. 5. It can be numerically verified that $\widetilde{C}_\alpha$ vanishes for $\alpha \simeq 0.106\,022$, and attains its maximum value $\simeq 0.632\,417$ for $\alpha \simeq 0.321\,699$ (cf. Fig. 5).

It also follows from Eq. (62) that $\widetilde{C}_\alpha \to -\infty$ as $\alpha \to 0+$, and that when $\alpha \to \infty$ $\widetilde{C}_\alpha$ tends to a finite (nonzero) limit, given by

$$\widetilde{C}_\infty = \int_0^\infty\left(\frac{1}{t}\,\mathrm{csch}\,t - \mathrm{csch}^2 t - \frac{e^{-2t}}{6}\right)\frac{dt}{t} \simeq 0.279\,70.$$

Taking the $\alpha \to 1$ limit in the previous formulas we obtain the following asymptotic expression for the von Neumann entropy $S \equiv S_1$:

$$S \simeq \frac{m+1}{3}\ln L + C_1, \qquad (63)$$

where

$$C_1 = \tfrac{1}{3}\ln f(p_0,\ldots,p_m) + (m+1)\widetilde{C}_1$$

and the universal constant $\widetilde{C}_1 \equiv \lim_{\alpha\to 1}\widetilde{C}_\alpha$ is given by

$$\widetilde{C}_1 = \int_0^\infty\left(\frac{\cosh t}{\sinh^3 t} - \frac{1}{t\sinh^2 t} - \frac{e^{-2t}}{3t}\right)dt \simeq 0.495\,018.$$

Note, in particular, that the latter equation agrees with the formula for the analogous constant Υ_1 in Ref. [6].

The formula (58)–(62) [or its counterpart (63) for the von Neumann entropy] provides an excellent approximation to the ground-state Rényi entanglement entropy of the supersymmetric chain (3) for even moderately large values of L. As an example, in Fig. 6 we have represented the relative error $r_L \equiv S_{\rm app}/S - 1$, where $S_{\rm app}$ is the approximation (63) to the von Neumann entropy S, for the finite-range chain (22) in the case $J = 1/2$ and $\mu = 17/4$. The value of S has been numerically computed diagonalizing the correlation matrix A_L and using the exact formula (36) (with $\alpha = 1$). As explained in Sec. III, for the value of J considered the dispersion relation has exactly one maximum in the interval $(0,\pi)$, and hence is not monotonic. In particular, for $\mu \in (\mathcal{E}(\pi),\mathcal{E}_{\rm max}) = (4,9/2)$ the Fermi sea consists of two disjoint intervals, as is also apparent from the inset in Fig. 6. As can be seen from the latter figure, the relative error decreases (though not monotonically) from 2.5×10^{-5} to 10^{-6} when L ranges from 100 to 500.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have analyzed the critical behavior of a large class of supersymmetric spin chains whose dispersion relation $\mathcal{E}(p)$ is not assumed to be monotonic in the interval $[0,\pi]$. We have examined the conditions under which the dispersion relation is well defined (i.e., is a continuous function) in the thermodynamic limit, providing several simple examples of models of this type, with both short- and long-range interactions, whose dispersion relation is not monotonic.

The main conclusion of our work is that the criticality properties of the supersymmetric chains (3) are determined exclusively by the topology (the number of points) of their Fermi "surface." More precisely, through the analysis of the free energy per spin in the critical (gapless) phase, we have shown that these models are equivalent to a system of $m+1$ free bosons with Fermi velocities $v_i = \mathcal{E}'(p_i)$, where $p_0, \ldots, p_m$ are the points of the Fermi surface in the interval $[0,\pi]$. In particular, the central charge is equal to the number $m+1$ of connected components (intervals) of the Fermi sea. This result is corroborated by the asymptotic behavior of the ground-state Rényi entanglement entropy S_α as the block size L tends to infinity, which has been derived applying a proved case of the Fisher-Hartwig conjecture. Indeed, we have shown that $S_\alpha \simeq (m+1)(1+\alpha^{-1})\ln L + C_\alpha$, where C_α is a nonuniversal constant (independent of L) which we have computed in closed form in terms of the momenta $p_0, \ldots, p_m$. In particular, for large L the entanglement entropy exhibits the logarithmic growth characteristic of $(1+1)$-dimensional conformal field theories with central charge $c = m+1$. This behavior, which is typical of critical (fermionic) one-dimensional lattice models with short-range interactions (see, e.g., [11,16]), was recently established by the authors for supersymmetric spin chains of the type considered here under the assumption that the dispersion relation is monotonic in $[0,\pi]$.

The present work opens up several possible lines for future research. In the first place, one could consider a generalization of our results on the ground-state entanglement entropy to more general situations (for instance, considering excited states, as in Ref. [16]), in which the Fermi sea is not necessarily a finite union of disjoint intervals but exhibits a more complicated topological structure. Another interesting generalization of the present work is the analysis of the entanglement of a subset consisting of the union of two or more disjoint blocks. In fact, the entanglement entropy of this type of subsystems has already been discussed in Ref. [30], giving rise to an unproved conjecture on the asymptotic behavior of the determinant of a block Toeplitz matrix.

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APPENDIX A: TOEPLITZ MATRICES AND THE FISHER-HARTWIG CONJECTURE

In this appendix we shall briefly review the Fisher-Hartwig conjecture on the asymptotic behavior of the determinant of a Toeplitz matrix when its order tends to infinity. (Recall that a matrix T is Toeplitz if its matrix elements t_{ij} depend only on $i-j$.)

If $c(z)$ is a (complex-valued) function defined on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, we define its Fourier coefficients c_n $(n \in \mathbb{Z})$ by

$$c_n = \frac{1}{2\pi i}\int_{|z|=1} c(z)\, z^{-n-1}\, dz \equiv \frac{1}{2\pi}\int_0^{2\pi} c(e^{i\theta})e^{-in\theta}\, d\theta.$$

Note that the last integral can in fact be extended to any interval of length 2π, by the 2π periodicity of the integrand. For any $L \in \mathbb{N}$, the function $c : S^1 \to \mathbb{C}$ defines a Toeplitz matrix T_L of order L through the relation

$$(T_L)_{ij} = c_{i-j}, \quad 1 \leqslant i,j \leqslant L.$$

We shall say that the function c is the *symbol* of the Toeplitz matrix T_L. The Fisher-Hartwig conjecture applies to matrices T_L whose symbol satisfies certain requirements that we shall next describe.

More precisely [31], c should be of the form

$$c(z) = b(z)\prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)})[2-2\cos(\theta-\theta_r)]^{\alpha_r}, \qquad \text{(A1)}$$

where $\mathrm{Re}\,\alpha_r > -1/2$, $b : S^1 \to \mathbb{C}$ is a nonvanishing smooth function with zero winding number, and

$$t_\beta(z) = e^{i\beta(\theta-\pi)}, \quad \theta \equiv \arg_{[0,2\pi)} z. \qquad \text{(A2)}$$

Note that $t_\beta(e^{i(\theta-\theta_0)})$ has in general (i.e., unless β is an integer) a single jump discontinuity at $z = e^{i\theta_0}$. If c satisfies Eq. (A1), we denote by l_n $(n \in \mathbb{Z})$ the nth Fourier coefficient of $\ln b$ (which is well defined and smooth, from the smoothness of b and the assumption on its winding number), and define

$$b_\pm(z) = \exp\left(\sum_{n=1}^{\infty} l_{\pm n} z^{\pm n}\right), \quad z \in S_1.$$

It is immediate to show that b_+ (respectively b_-) can be analytically prolonged to the interior (respectively exterior) of the unit circle. It also follows from the definition of $b_\pm$ that on the unit circle we have the *Wiener-Hopf decomposition*

$$b(z) = e^{l_0} b_+(z) b_-(z), \quad z \in S^1.$$

Let us further set

$$E[b] = \exp\left(\sum_{n=1}^{\infty} n\, l_n l_{-n}\right)$$

and

$$E = E[b]\prod_{r=1}^{R} b_+(e^{i\theta_r})^{\beta_r-\alpha_r} b_-(e^{i\theta_r})^{-\alpha_r-\beta_r} \times \prod_{1\leqslant s\neq r\leqslant R}(1-e^{i(\theta_s-\theta_r)})^{(\alpha_r+\beta_r)(\beta_s-\alpha_s)} \times \prod_{r=1}^{R}\frac{G(1+\alpha_r+\beta_r)G(1+\alpha_r-\beta_r)}{G(1+2\alpha_r)},$$

where the Barnes G function is the entire function defined by

$$G(1+z) = (2\pi)^{z/2} e^{-(z+1)(z/2)-\gamma_E(z^2/2)} \times \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^n e^{-z+z^2/2n}\right]$$

and γ_E is the Euler-Mascheroni constant. The Fisher-Hartwig conjecture states [31] that if T_L is the Toeplitz matrix with symbol (A1) then when $L \to \infty$ we have

$$\det T_L = e^{l_0 L} L^M E[1+o(1)],$$

with

$$M = \sum_{r=1}^{R} (\alpha_r^2 - \beta_r^2).$$

The above conjecture was actually proved [18] in the case

$$\alpha_r = 0, \quad |\mathrm{Re}\beta_r| < \tfrac{1}{2}, \quad r = 1, \ldots, R,$$

which, as we shall see below, is the relevant one for our purposes. Furthermore, as explained in Sec. V, we shall only need to consider the case in which b is a constant (i.e., independent of θ). The Fisher-Hartwig conjecture simplifies considerably in this case, since

$$l_n = l_0\, \delta_{0n} \Rightarrow b_\pm = E[b] = 1, \quad e^{l_0} = b,$$

and therefore

$$\det T_L = b^L L^M E[1 + o(1)] \tag{A3}$$

with

$$M = -\sum_{r=1}^{R} \beta_r^2 \tag{A4}$$

and

$$E = \prod_{1 \leqslant s < r \leqslant R} \left[2 \left| \sin\left(\frac{\theta_r - \theta_s}{2} \right) \right| \right]^{2\beta_r \beta_s} \times \prod_{r=1}^{R} G(1 + \beta_r) G(1 - \beta_r). \tag{A5}$$

Note also that the product $G(1+z)G(1-z)$ reduces to

$$G(1+z)G(1-z) = e^{-(1+\gamma_E)z^2} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z^2}{n^2} \right)^n e^{z^2/n} \right]. \tag{A6}$$

We shall be mainly interested in the case in which $T_L = \lambda - (2A_L - \mathbb{1})$, where λ is a spectral parameter and A_L is the correlation matrix of a block of L spins of the su(1|1) supersymmetric spin chain. As a first example, let us express in the form (A1) and (A2) the symbol of the matrix T_L when the chain's dispersion relation is monotonic in the interval $[0,\pi]$. To begin with, in this case we have

$$(A_L)_{jk} = \frac{\sin[p_0(j-k)]}{\pi(j-k)} = \frac{1}{2\pi} \int_{-p_0}^{p_0} e^{-i(j-k)\theta}\, d\theta,$$

where $p_0 \in [0,\pi]$ is the Fermi momentum [12]. Thus the symbol of the Toeplitz matrix A_L is

$$f(e^{i\theta}) = \begin{cases} 1, & -p_0 < \theta < p_0, \\ 0, & p_0 < \theta < 2\pi - p_0, \end{cases}$$

and that of T_L is therefore given by

$$c(e^{i\theta}) = \begin{cases} \lambda - 1, & -p_0 < \theta < p_0, \\ \lambda + 1, & p_0 < \theta < 2\pi - p_0. \end{cases}$$

Note that c has two jump discontinuities on the unit circle at the points $e^{\pm i p_0}$. We shall next show that

$$c(e^{i\theta}) = b(e^{i\theta}) t_\beta(e^{i(\theta+p_0)}) t_{-\beta}(e^{i(\theta-p_0)}) \tag{A7}$$

for suitable β and $b(z)$. Indeed, first of all we have

$$-p_0 < \theta < 2\pi - p_0 \Rightarrow 0 < \theta + p_0 < 2\pi \Rightarrow t_\beta(e^{i(\theta+p_0)}) = e^{i\beta(\theta+p_0-\pi)}.$$

On the other hand, if $-p_0 < \theta < p_0$ then

$$0 \leqslant 2(\pi - p_0) < \theta - p_0 + 2\pi < 2\pi \Rightarrow t_{-\beta}(e^{i(\theta-p_0)}) = e^{-i\beta(\theta-p_0+\pi)},$$

while for $p_0 < \theta < 2\pi - p_0$ we have

$$0 < \theta - p_0 < 2(\pi - p_0) \leqslant 2\pi \Rightarrow t_{-\beta}(e^{i(\theta-p_0)}) = e^{-i\beta(\theta-p_0-\pi)}.$$

Hence

$$t_\beta(e^{i(\theta+p_0)}) t_{-\beta}(e^{i(\theta-p_0)}) = \begin{cases} e^{2i\beta(p_0-\pi)}, & -p_0 < \theta < p_0, \\ e^{2i\beta p_0}, & p_0 < \theta < 2\pi - p_0. \end{cases} \tag{A8}$$

In order for Eq. (A7) to hold we must therefore have

$$b\, e^{2i\beta(p_0-\pi)} = \lambda - 1, \quad b\, e^{2i\beta p_0} = \lambda + 1,$$

from which we easily get

$$e^{2i\beta\pi} = \frac{\lambda+1}{\lambda-1}, \quad b = (\lambda+1) e^{-2i\beta p_0}.$$

Although these equations admit an infinite number of solutions (β, b) provided that $\lambda \neq \pm 1$, it will prove convenient for our purposes to take

$$\beta = \frac{1}{2\pi i} \ln\left(\frac{\lambda+1}{\lambda-1} \right), \quad b = (\lambda+1) \left(\frac{\lambda+1}{\lambda-1} \right)^{-p_0/\pi}, \tag{A9}$$

where $\ln z \equiv \ln|z| + i \arg_{(-\pi,\pi]} z$ and $z^a \equiv e^{a \ln z}$. Note, in particular, that b is a nonvanishing constant. It is also important to observe that

$$\lambda \notin [-1,1] \Rightarrow |\mathrm{Re}\beta| = \frac{1}{2\pi} \arg_{(-\pi,\pi]} \left(\frac{\lambda+1}{\lambda-1} \right) < \frac{1}{2},$$

since by definition $-\pi < \arg_{(-\pi,\pi]} z \leqslant \pi$ and

$$\arg_{(-\pi,\pi]} \left(\frac{\lambda+1}{\lambda-1} \right) = \pi \iff \frac{\lambda+1}{\lambda-1} \in (-\infty, 0) \iff \lambda \in (-1,1).$$

Thus the Fisher-Hartwig conjecture can be applied provided that λ lies outside the closed interval $[-1,1]$, with $R = 1$, $\alpha_1 = 0$ and

$$M = -2\beta^2, \quad E = (2\sin p_0)^{-2\beta^2} G(1+\beta)^2 G(1-\beta)^2.$$

By Eqs. (A3) and (A9), when $L \to \infty$ the characteristic polynomial $D_L(\lambda) \equiv \det(\lambda + 1 - 2A_L)$ is given by

$$D_L(\lambda) = (2L \sin p_0)^{-2\beta^2} (\lambda+1)^L \left(\frac{\lambda+1}{\lambda-1} \right)^{-Lp_0/\pi} \times G(1+\beta)^2 G(1-\beta)^2 [1 + o(1)], \tag{A10}$$

with β defined by Eq. (A9). This is precisely the formula used by Jin and Korepin [6] for the determination of the asymptotic behavior of the ground-state entanglement entropy of the *XX* model.

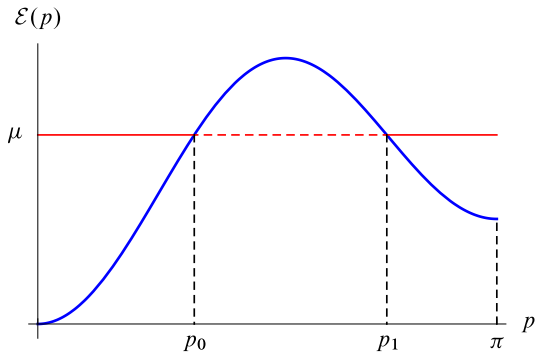

FIG. 7. Dispersion relation $\mathcal{E}(p)$ with a single maximum in $(0,\pi)$.

As a second example, we shall consider a simple case in which the dispersion relation $\mathcal{E}$ is not monotonic in $[0,\pi]$. More precisely, suppose that $\mathcal{E}$ is nonnegative and has a single maximum in the open interval $(0,\pi)$ (cf. Fig. 7).

For $\mu \in (0,\mathcal{E}(\pi))$, the equation $\mathcal{E}(p) = \mu$ has one root p_0 in the interval $(0,\pi)$, and the determinant $D_L(\lambda)$ is approximately given by Eq. (A10). We shall next determine the asymptotic behavior of $D_L(\lambda)$ when $\mu \in (\mathcal{E}(\pi),\mathcal{E}_{\max})$, where $\mathcal{E}_{\max}$ is the maximum value of $\mathcal{E}(p)$, and thus the equation $\mathcal{E}(p) = \mu$ has two roots $p_0 < p_1$ in the interval $(0,\pi)$. To begin with, from Eq. (40) it follows that in this case

$$(A_L)_{jk} = \frac{1}{2\pi}\left(\int_{-p_0}^{p_0} + \int_{p_1}^{2\pi-p_1}\right) e^{-i(j-k)p}\,dp.$$

Thus the symbol of $T_L = \lambda + 1 - 2A_L$ is given by

$$c(e^{i\theta}) = \begin{cases} \lambda - 1, & \theta \in (-p_0,p_0) \cup (p_1, 2\pi - p_1) \\ \lambda + 1, & \theta \in (p_0,p_1) \cup (2\pi - p_1, 2\pi - p_0). \end{cases} \tag{A11}$$

In other words, $c(e^{i\theta})$ alternatively takes on the two values $\lambda - 1$ and $\lambda + 1$ on each of the four intervals $(-p_0,p_0), \ldots, (2\pi - p_1, 2\pi - p_0)$ on which $\mathcal{E}(p) - \mu$ has constant sign, starting with $\lambda - 1$. Since the symbol (A11) has four discontinuities at the points $e^{\pm ip_0}$ and $e^{\pm ip_1}$, we shall try to express it as

$$\begin{aligned} c(e^{i\theta}) = b(e^{i\theta}) & t_{\beta_0}(e^{i(\theta+p_0)}) t_{-\beta_0}(e^{i(\theta-p_0)}) \\ & \times t_{\beta_1}(e^{i(\theta+p_1)}) t_{-\beta_1}(e^{i(\theta-p_1)}) \end{aligned} \tag{A12}$$

for suitably chosen b, β_i. In fact, we only need compute $t_{\pm\beta_1}(e^{i(\theta\pm p_1)})$, which is straightforward:

$$t_{\beta_1}(e^{i(\theta+p_1)}) = \begin{cases} e^{i\beta_1(\theta+p_1-\pi)}, & -p_0 < \theta < 2\pi - p_1 \\ e^{i\beta_1(\theta+p_1-3\pi)}, & 2\pi - p_1 < \theta < 2\pi - p_0 \end{cases}$$

$$t_{-\beta_1}(e^{i(\theta-p_1)}) = \begin{cases} e^{-i\beta_1(\theta-p_1+\pi)}, & -p_0 < \theta < p_1 \\ e^{-i\beta_1(\theta-p_1-\pi)}, & p_1 < \theta < 2\pi - p_0. \end{cases}$$

Combining the previous equations with Eq. (A8) and comparing with Eq. (A11) we immediately arrive at the system

$$\begin{aligned} b\, e^{2i(\beta_0 p_0 + \beta_1 p_1)} e^{-2\pi i(\beta_0+\beta_1)} &= b\, e^{2i(\beta_0 p_0 + \beta_1 p_1)} = \lambda - 1, \\ b\, e^{2i(\beta_0 p_0 + \beta_1 p_1)} e^{-2\pi i \beta_1} &= \lambda + 1. \end{aligned}$$

From the first equation it follows that $\beta_0 + \beta_1$ must be an integer. Choosing the simplest solution $\beta_0 = -\beta_1 \equiv \beta$ and dividing the last equation by the first one we again obtain Eq. (A9) for β. Finally, from the last equation it follows that

$$b = (\lambda+1)e^{2i\beta(p_1-p_0-\pi)} = (\lambda+1)\left(\frac{\lambda+1}{\lambda-1}\right)^{(p_1-p_0-\pi)/\pi}. \tag{A13}$$

Note that β is still given by Eq. (A9), so the condition $|\mathrm{Re}\beta| < 1/2$, necessary for the validity of the Fisher-Hartwig conjecture, also applies in this case if $\lambda \notin [-1,1]$. Since now $M = -4\beta^2$,

$$\prod_{r=1}^{4} G(1+\beta_r)G(1-\beta_r) = G(1+\beta)^4 G(1-\beta)^4$$

and

$$\begin{aligned} \prod_{1\leqslant s<r\leqslant 4} & \left[2\left|\sin\left(\frac{\theta_r-\theta_s}{2}\right)\right|\right]^{2\beta_r\beta_s} \\ &= \left[2\sin\left(\frac{p_1-p_0}{2}\right)\right]^{-4\beta^2}\left[2\sin\left(\frac{p_1+p_0}{2}\right)\right]^{4\beta^2} \\ &\quad \times (4\sin p_0 \sin p_1)^{-2\beta^2}. \end{aligned}$$

By the Fisher-Hartwig conjecture, the determinant $D_L(\lambda)$ is given in this case by

$$\begin{aligned} D_L(\lambda) = & \left[\frac{4L^2 \sin p_0 \sin p_1 \sin^2\left(\frac{p_1-p_0}{2}\right)}{\sin^2\left(\frac{p_1+p_0}{2}\right)}\right]^{-2\beta^2} \\ & \times (\lambda+1)^L \left(\frac{\lambda+1}{\lambda-1}\right)^{(L/\pi)(p_1-p_0-\pi)} \\ & \times G(1+\beta)^4 G(1-\beta)^4 [1+o(1)]. \end{aligned} \tag{A14}$$

APPENDIX B: COMPUTATION OF THE INTEGRAL $I_1(\alpha)$

In this appendix we shall provide an elementary derivation of the integral $I_1(\alpha)$ in Eq. (55), which appears in the asymptotic expression of the Rényi entanglement entropy of the model (3). To begin with, we have

$$I_1(\alpha) = \frac{2}{\pi^2}(1-\alpha)^{-1}\hat{I}_1(\alpha),$$

with

$$\hat{I}_1(\alpha) = \int_{-1}^{1} \ln\left[\left(\frac{1+x}{2}\right)^{\alpha} + \left(\frac{1-x}{2}\right)^{\alpha}\right] \frac{dx}{1-x^2},$$

or equivalently [performing the change of variables $t = (x+1)/2$]

$$\hat{I}_1(\alpha) = \int_0^1 \ln[t^\alpha + (1-t)^\alpha]\frac{dt}{t}. \tag{B1}$$

Integrating by parts we obtain the equivalent expression

$$\hat{I}_1(\alpha) = -\alpha \int_0^1 \frac{t^{\alpha-1} - (1-t)^{\alpha-1}}{t^\alpha + (1-t)^\alpha} \ln t\, dt. \tag{B2}$$

On the other hand, differentiating Eq. (B1) with respect to α we easily get

$$\hat{I}_1'(\alpha) = \int_0^1 \frac{t^{\alpha-1}}{t^\alpha + (1-t)^\alpha} \frac{\ln t}{1-t}\, dt,$$

and hence [25]

$$\hat{I}_1'(\alpha) + \frac{\hat{I}_1(\alpha)}{\alpha} = \int_0^1 \frac{\ln t}{1-t}\, dt = -\frac{\pi^2}{6}.$$

Solving this linear differential equation with the initial condition $\hat{I}_1(1) = 0$ we finally obtain

$$\hat{I}_1(\alpha) = \frac{\pi^2}{12\alpha}(1-\alpha^2), \tag{B3}$$

which immediately yields Eq. (57).

APPENDIX C: SIMPLIFICATION OF THE CONSTANT $\widetilde{C}_\alpha$

In this appendix we derive Eq. (61) for the constant $\widetilde{C}_\alpha$ in Eq. (58), and show that it can be more simply expressed by means of the integral (62). To this end, we use the elementary identity

$$\psi(z) = -\gamma_E + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z-1}\right),$$

which can be immediately derived from the well-known infinite product for the gamma function. Our starting point is the definition (56) of $I_2(\alpha)$, which can be written as

$$I_2(\alpha) = \frac{2}{\pi^2}\int_{-1}^{1} \frac{s_\alpha(x)}{1-x^2}[f(Z(x)) + f(1-Z(x))]dx, \tag{C1}$$

with $Z(x) = 1/2 + i\,B(x)$ and

$$f(z) = \sum_{n=1}^{\infty} \frac{z^3}{n(n^2-z^2)}.$$

From the relation

$$\frac{z^2}{n(z^2-n^2)} = \frac{1}{2}\left(\frac{1}{n+z} - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n-z} - \frac{1}{n}\right)$$

and the functional identity $\psi(z+1) = \psi(z) + 1/z$ satisfied by the digamma function it immediately follows that

$$f(z) = -\frac{z}{2}\,[\psi(z) + \psi(1-z)] - z\gamma_E - \frac{1}{2},$$

and therefore

$$f(z) + f(1-z) = -1 - \gamma_E - \tfrac{1}{2}\,[\psi(z) + \psi(1-z)].$$

Substituting into Eq. (C1) and using Eqs. (57) and (59) we easily obtain Eq. (61).

In order to prove Eq. (62), we first make the change of variable $w = B(x)$ in Eq. (61), which yields

$$(1-\alpha)\widetilde{C}_\alpha = -\frac{2}{\pi}\int_0^\infty \ln\left[\frac{2\cosh(\pi\alpha w)}{[2\cosh(\pi w)]^\alpha}\right] \times \left[\psi\left(\frac{1}{2} + i\,w\right) + \psi\left(\frac{1}{2} - i\,w\right)\right] dw$$

$$= \frac{2}{i\pi}\int_0^\infty \ln\left[\frac{2\cosh(\pi\alpha w)}{[2\cosh(\pi w)]^\alpha}\right] \times \frac{d}{dw}\ln\left[\frac{\Gamma\left(\frac{1}{2} - i\,w\right)}{\Gamma\left(\frac{1}{2} + i\,w\right)}\right] dw.$$

We next integrate by parts, taking into account that by Stirling's formula we have

$$\ln\left[\frac{\Gamma\left(\frac{1}{2} - i\,w\right)}{\Gamma\left(\frac{1}{2} + i\,w\right)}\right] = O(w\ln w)$$

while

$$\ln\left[\frac{2\cosh(\pi\alpha w)}{[2\cosh(\pi w)]^\alpha}\right] = O(e^{-2\pi\min(1,\alpha)w}),$$

so that the boundary term vanishes. We thus obtain

$$(1-\alpha)\widetilde{C}_\alpha = 2i\alpha\int_0^\infty \big[\tanh(\pi\alpha w) - \tanh(\pi w)\big]\ln\left[\frac{\Gamma\left(\frac{1}{2} - i\,w\right)}{\Gamma\left(\frac{1}{2} + i\,w\right)}\right] dw.$$

On the other hand, from Gauss's integral representation of the digamma function [32]

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt$$

it easily follows that

$$\ln\Gamma(z) = \int_0^\infty \left[z - 1 - \frac{1-e^{-(z-1)t}}{1-e^{-t}}\right]\frac{e^{-t}}{t}\, dt$$

and hence

$$\ln\left[\frac{\Gamma\left(\frac{1}{2} - i\,w\right)}{\Gamma\left(\frac{1}{2} + i\,w\right)}\right] = i\int_0^\infty [\mathrm{csch}(t/2)\sin(wt) - 2we^{-t}]\frac{dt}{t}.$$

Substituting into the last formula for $\widetilde{C}_\alpha$ and using the elementary identity

$$\tanh(\pi\alpha w) - \tanh(\pi w) = \frac{2e^{-2\pi w}}{1+e^{-2\pi w}} - \frac{2e^{-2\pi\alpha w}}{1+e^{-2\pi\alpha w}}$$

we obtain

$$(1-\alpha)\widetilde{C}_\alpha = 4\alpha\int_0^\infty \frac{g_1(t) - g_\alpha(t)}{t}\, dt,$$

with

$$g_\alpha(t) = \int_0^\infty \frac{e^{-2\pi\alpha w}}{1+e^{-2\pi\alpha w}}[2we^{-t} - \mathrm{csch}(t/2)\sin(wt)]dw.$$

The latter integral can be evaluated in closed form by elementary means. Indeed,

$$\int_0^\infty \frac{w\,e^{-2\pi\alpha w}}{1+e^{-2\pi\alpha w}}dw = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty w\,e^{-2n\pi\alpha w}dw$$

$$= \frac{1}{4\pi^2\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\zeta(2)}{8\pi^2\alpha^2} = \frac{1}{48\alpha^2},$$

while

$$\int_0^\infty \frac{e^{iwt}\,e^{-2\pi\alpha w}}{1+e^{-2\pi\alpha w}}dw = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty e^{(it-2n\pi\alpha)w}dw$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{i\,t - 2n\pi\alpha},$$

and therefore

$$\int_0^\infty \frac{e^{-2\pi\alpha w}\sin(wt)}{1+e^{-2\pi\alpha w}}dw = \mathrm{Im}\sum_{n=1}^{\infty}\frac{(-1)^n}{i\,t-2n\pi\alpha}$$
$$= t\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{t^2+4\alpha^2\pi^2n^2} = \frac{1}{4\alpha}\left(\frac{2\alpha}{t}-\mathrm{csch}[t/(2\alpha)]\right).$$

We thus obtain

$$g_\alpha(t) = \frac{e^{-t}}{24\alpha^2}+\mathrm{csch}(t/2)\left(\frac{\mathrm{csch}[t/(2\alpha)]}{4\alpha}-\frac{1}{2t}\right),$$

from which Eq. (62) easily follows.

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[26] This limit is easily computed from L'Hôpital's rule and the identity $\mathrm{Li}_2'(z) = \mathrm{Li}_1(z)/z = -\ln(1-z)/z$.

[27] We are assuming here that $\mathcal{E}$ is smooth on $[0,\pi]$, so that $\mathcal{E}'$ vanishes at the points where $\mathcal{E}$ attains its maximum and minimum values over the latter interval.

[28] Indeed, if $g(z) \equiv (1+\varepsilon+z)^\alpha + (1+\varepsilon-z)^\alpha$ then for $x \in [-1,1]$ $g(x)$ is real and satisfies $g(x) \geqslant 2(1+\varepsilon)^\alpha > 2$. On the other hand, if $0 < \varepsilon < 1$ then for $\zeta \in \mathbb{C}$ with $|\zeta| < 1$ and $x \in [-1,1]$ integrating the derivative of $g(z)$ along the segment from x to $x+\zeta$ we obtain

$$\begin{aligned}&|\mathrm{Re}g(x+\zeta) - \mathrm{Re}g(x)|\\ &\leqslant \alpha\int_x^{x+\zeta}|(1+\varepsilon+z)^{\alpha-1} - (1+\varepsilon-z)^{\alpha-1}|dz\\ &\leqslant 2\alpha\int_x^{x+\zeta}(1+\varepsilon+|z|)^{\alpha-1}dz\\ &\leqslant 2\alpha(2+\varepsilon+|\zeta|)^{\alpha-1}|\zeta| \leqslant 2^{2\alpha-1}\alpha|\zeta|.\end{aligned}$$

Thus $\mathrm{Re}g(x+\zeta) > 0$ for $x \in [-1,1]$ provided that $|\zeta| < \min(1, 2^{2(1-\alpha)}/\alpha)$.

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CHAPTER 2

Generalized Lipkin–Meshkov–Glick models

The Lipkin–Meshkov–Glick (LMG) model [56, 89, 92] is one of the few quantum many-body systems that have been solved in the literature. The key to its solvability is the fact that it can be mapped to a system of N spin-$\frac{1}{2}$ particles in a constant magnetic field with long-range uniform interactions of the so-called XY type. In particular, when the model is isotropic (XX interactions), its Hamiltonian is a polynomial in[16] the total spin squared S^2 and its third component S_z. It follows that the isotropic model can be solved exactly for all N. In the general (non-isotropic) case the model can be solved, in principle, via the Bethe ansatz, although in practice numerical diagonalization methods are indeed more efficient.

This chapter is devoted to the so-called generalized Lipkin–Meshkov–Glick (gLMG) models studied in Refs. [P4]-[P5]. We construct generalized models in the sense that, like in the original isotropic LMG model, their ground states are determined by the number of particles of each type (the magnon content) since they are invariant under permutations. In particular, it can be shown [P4] that the original isotropic LMG Hamiltonian is diagonal in a basis of common eigenstates of S^2 and S_z, and the ground state belongs to the maximum spin subspace or sector. It turns out that the maximum spin subspace is always invariant under permutations, independently of the local dimension of the internal space of each particle. Consequently, we define the gLMG models by Hamiltonians acting on the tensor product of N copies of the Hilbert space associated to the fundamental representation of su(m) whose ground state is required to be, in addition, invariant under permutations. We will present two kind of contributions; the study of the ground state phase diagram and its entanglement content (see Ref. [P4]), as well as the exact computation of the partition function (see Ref. [P5]) of a certain subclass of gLMG models of Haldane–Shastry type.

[16] Here $S^2 = S_x^2 + S_y^2 + S_z^2$ with $S_\alpha = \frac{1}{2}\sum_i \sigma_i^\alpha$ (see Fn. 8).

1. Original (isotropic) Lipkin–Meshkov–Glick model

The explicit Hamiltonian of the original isotropic LMG model that we are trying to generalize is [**P4**]

$$
\begin{aligned}
H_{\mathrm{LMG}} &= -\frac{2}{N}\,S^2 + \frac{2}{N}\Big(S_z - \frac{Nh}{2}\Big)^2 - \frac{Nh^2}{2} + 1 \\
&= \frac{2}{N}\sum_{i<j}(1 - P_{ij}) + \frac{2}{N}\Big(S_z - \frac{Nh}{2}\Big)^2 - \frac{N}{2}(1+h^2)\,,
\end{aligned}
\tag{10}
$$

where the sum is over i and j in $\{0,\dots,N-1\}$ with N being the total number of particles, $P_{ij} = \frac{1}{2} + 2\,\vec{S}_i\cdot\vec{S}_j$ is the permutation operator exchanging the spins of the i-th and j-th sites[17], and h with $|h| \le 1$ represents the external magnetic field strength.

Since H_{LMG}, S^2 and S_z are three mutually commuting operators, to solve the latter Hamiltonian one should describe the different irreducible invariant subspaces associated to different eigenvalues of S^2. It is well-known that the eigenvalues of S^2 are of the form $j(j+1)$ with j a non-negative integer or half-integer. We shall also denote by (j) the irreducible representation associated to the eigenvalue $j(j+1)$ of S^2 acting on one copy of the spin-j subspace of dimension $2j+1$. These $2j+1$ different spin-j elements are eigenvectors of S_z with eigenvalues $m_z = -j, -j+1, \dots, j$.

Note that $(\frac{1}{2}) \otimes (j)$ contains the trivial representation (0) if and only if $j = \frac{1}{2}$. Thus, if we define $n_0 = 0$ when N is even and $n_0 = \frac{1}{2}$ when N is odd, the possible values of the spin appearing in the decomposition of $(j)^{\otimes N}$ into irreducible representations are $n_0, n_0+1, \dots, \frac{N}{2}$. All of these irreducible representations have integer spin when the total number of particles N is even and half-(odd-)integer spin when N is odd.

Let us denote by $d\cdot(j) = \bigoplus^d (j)$ the direct sum of d copies of (j). Then the sought-for decomposition of the tensor product of N copies of the spin-$\frac{1}{2}$ irreducible representation is $(\frac{1}{2})^{\otimes N} = \bigoplus_j d_j\cdot(j)$, where the direct sum in the right hand side is over $j = n_0, n_0+1, \dots, \frac{N}{2}$ and

$$
d_j = \binom{N}{j + \frac{N}{2}} - \binom{N}{j + \frac{N}{2} + 1}\,.
$$

We conclude that there exists a basis of common eigenvectors of S^2 and S_z with elements $|\,j\,m_z\,\nu\rangle$ where $\nu = 1, 2, \dots, d_j$ for fixed m_z and j; and

[17] As usual, $\vec{S}_i \cdot \vec{S}_j = \frac{1}{4}(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z)$ (see Fn. 8).

$m_z = -j, -j+1, \ldots, j$ for fixed $j = n_0, n_0+1, \ldots, \frac{N}{2}$. Thus each energy appears with degeneracy d_j at least. The maximum spin sector is the spin-$\frac{N}{2}$ subspace and the ground state belongs to it [**P4**], as we shall now discuss.

Indeed, the dependence on S^2 of the Hamiltonian (10) shows that the minimum energy is obtained when $j(j+1)$ is maximum, i.e., for $j = \frac{N}{2}$. Since the degeneracy of each subspace of eigenvectors of S_z is just one in the maximum spin sector, the ground state is non-degenerate provided the energy attains its minimum for an unique $m_z = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}$. It turns out that this is actually the case since the dependence on S_z of the latter Hamiltonian is quadratic. For instance, one can easily see directly from (10) that this value is the closest integer to $\frac{Nh}{2}$. As we will see, this is a special case of a more general result [**P4**] valid for a larger class of generalized models describing su(m) particles.

2. Generalized Lipkin–Meshkov–Glick models

The Hamiltonian of the generalized model will contain two different terms H_0 and H_1. The contribution of the former to the total energy is minimum if the state is invariant under permutations (like in the maximum spin sector), and H_0 is expressed as a sum of terms proportional to $1 - P_{ij}$ acting on the Hilbert space of N copies of the fundamental representation of su(m), as the in the original Hamiltonian (10) which is recovered for $m = 2$. Permutations P_{ij} being involutions, its eigenvalues are ± 1 and the contribution to the energy of terms of the form $h_{ij}(1 - P_{ij})$ with $h_{ij} > 0$ is minimum for states invariant under permutations that can be characterized by their magnon content, i.e., by numbers $(n_a) = (n_1, \ldots, n_m)$ where n_a is the number of particles of type a. The ground state will be non-degenerate if, for instance, the second term H_1 only depends on the magnon content and its contribution to the energy is minimum for an unique (n_a) (in the same way as $m_z = n_\uparrow - n_\downarrow$ determined above the numbers of spins up and down [**P4**] uniquely). As we shall see, if the dependence of the Hamiltonian is quadratic on the generators of the standard su(m) Cartan subalgebra, such Hamiltonians can be constructed by expressing the permutation operators in terms of the generators of the Lie algebra su(m).

Let us write $\mathcal{H}$ for the Hilbert space of N copies of the fundamental representation of su(m), and H_0 for a Hamiltonian of the general form $H_0 = \sum_{i<j} h_{ij}(1 \pm P_{ij})$, where h_{ij} are real parameters and P_{ij} denotes

the operator permuting the su(m) spins of the i-th and j-th sites. This Hamiltonian describes a gLMG model if we can express[18] the action of permutations in $\mathcal{H}$ in terms of the local generators of the Lie algebra su(m). In fact, P_{ij} can be easily expressed in terms of the Hermitian generators x^α of su(m) in the fundamental representation satisfying the normalization condition $\mathrm{tr}(x^\alpha x^\beta) = \frac{1}{2}\delta_{\alpha\beta}$. The precise relation is $P_{ij} = \frac{1}{m} + 2\vec{x}_i \cdot \vec{x}_j$, where $\vec{x}_i \cdot \vec{x}_j = \sum_\alpha x_i^\alpha x_j^\alpha$ and $x_i^\alpha = (\otimes^{i-1} 1_m) \otimes x^\alpha \otimes (\otimes^{N-i} 1_m)$, the sum being over $\alpha = 1, \ldots, m^2 - 1$. Let us now fix an element $\mathfrak{e} \in \mathrm{GL}(m)$ and consider the operator $\sum_i (\otimes^{i-1} 1_m) \otimes \mathfrak{e} \otimes (\otimes^{N-i} 1_m)$. It is clear that such an operator commutes with all spin exchange operators P_{ij}. If we repeat this construction for each generator of the Cartan subalgebra, we obtain $m-1$ linearly independent and mutually commuting operators each of which also commutes with any permutation P_{ij} and thus with H_0. It is not difficult to note (see [**P4**] or [**P5**]) that the set of operators just constructed can generate, together with the identity, a set of m operators $(\mathfrak{n}_a) = (\mathfrak{n}_1, \ldots, \mathfrak{n}_m)$ where each $\mathfrak{n}_a$ counts the number of particles of type a. These m operators are not linearly independent, since $\mathfrak{n}_1 + \cdots + \mathfrak{n}_m = N \cdot 1$. We shall then say that a system is an su(m) generalized Lipkin–Meshkov–Glick (gLMG) model if it is described by a Hamiltonian of the general form $H = H_0 + H_1$ with H_0 as above and $H_1 = f(\mathfrak{n})$ for some analytic function f. More explicitly,

$$H = H_0 + H_1 = \sum_{i<j} h_{ij}(1 \pm P_{ij}) + \sum_{i_1} \cdots \sum_{i_m} f_{(i_1,\ldots,i_m)}\, \mathfrak{n}_1^{i_1} \ldots \mathfrak{n}_m^{i_m} \tag{11}$$

where h_{ij} and $f_{(i_1,\ldots,i_m)}$ are real numbers, the first sum is over pairs of numbers in $\{0, \ldots, N-1\}$, and the other ones are over non-negative integers. Hamiltonians with the "$-$" sign in Eq. (11) will be called ferromagnetic, and those with the "$+$" sign antiferromagnetic. Note that the original isotropic su(2) LMG model (10) is included in the previous definition as a ferromagnetic gLMG model with $f(t_0, t_1) = \frac{2}{N}(t_0 - t_1 - Nh/2)^2 - \frac{N}{2}(1+h^2)$, where the real number h is interpreted as the external magnetic field strength and $h_{ij} = 2/N$ for all values of i and j in $\{0, \ldots, N-1\}$.

3. Partition function

When H_0 is the Hamiltonian of one of the three families of the su(m) spin chains of Haldane–Shastry type, in [**P5**] we compute the partition function

[18] One can see that this is actually the case. We will follow essentially the approach of [**P5**].

in both the antiferromagnetic and ferromagnetic cases. To this end, we first derived the spectrum of the restrictions $H_0|_{(n_a)}$ of the latter Hamiltonian to subspaces with fixed magnon content $(n_a) = (n_1, \ldots, n_m)$, i.e., subspaces spanned by elements $|s_1 \cdots s_N\rangle$ satisfying $|\{i : s_i = a\}| = n_a$.

More precisely, let us write Z_0 for the partition function of a Hamiltonian $H_0 = \sum_{i<j} h(u_i - u_j)(1 \pm P_{ij})$ and with

$$h(u) = \begin{cases} \frac{1}{2}(\sin u)^{-2}\,, & u_k = \frac{\pi}{N}k \qquad \text{(HS)} \\ u^{-2}\,, & u_k = \zeta_k \qquad \text{(PF)} \\ \frac{1}{2}(\sinh u)^{-2}\,, & u_k = \mathrm{e}^{2\xi_k} \qquad \text{(FI)} \end{cases} \tag{12}$$

where ζ_k and ξ_k respectively denote the k^{th} zero of the Hermite polynomial of degree N and the Laguerre polynomial $L_N^{\beta-2N+1}$ with $\beta > 2N-2$. They are associated to the su(m) versions of the three spin chains of Haldane–Shastry type mentioned in the Introduction, namely the Haldane–Shastry (HS), Polychronakos–Frahm (PF) and Frahm–Inozemtsev (FI) spin chains. They are also usually referred to as trigonometric, rational and hyperbolic HS-type spin chains, respectively. Since $[\mathfrak{n}_a, H_0] = 0$, the Hilbert space decomposes as $\mathcal{H} = \bigoplus_{(n_a)} \mathcal{H}^{(n_a)}$ and consequently

$$Z_0 = \sum_{n_1} \cdots \sum_{n_m} Z_0^{(n_1,\ldots,n_m)} \tag{13}$$

where the sums run over non-negative integers $(n_a) = (n_1, \ldots, n_m)$ such that $n_1 + \cdots + n_m = N$. For any magnon content (n_a), we explicitly obtained in [**P5**] exact expressions for the restricted partition functions $Z_0^{(n_1,\ldots,n_m)}$ of the three su(m) spin chains of HS-type of Eq. (12) for an arbitrary finite number of particles N in both the ferromagnetic and the antiferromagnetic case. It then follows that the partition function Z of an su(m) gLMG model with Hamiltonian $H = H_0 + f(\mathfrak{n})$ is given in terms of the restricted partition functions $Z_0^{(n_1,\ldots,n_m)}$ by

$$Z = \sum_{n_1} \cdots \sum_{n_m} Z_0^{(n_1,\ldots,n_m)}\, q^{f(n_1,\ldots,n_m)}$$

where $q = \mathrm{e}^{-1/T}$. Several thermodynamic quantities can be derived from the exact expressions of [**P5**] for these partition functions as well as a novel conjecture on the distribution of eigenvalues of the three su(m) spin chains of HS-type restricted to subspaces with fixed magnon content. We shall refer to this publication for a further discussion on these results and different graphical illustrations thereof.

4. Ground state

We shall now turn to ferromagnetic su(m) gLMG models with $h_{ij} > 0$. The ground state of these models is completely symmetric under particle permutations, and its degeneracy is given by the number of absolute minima of the function f of m variables characteristic of each model. When f is quadratic, the ground state is unique and the phase diagram can be determined explicitly.

The magnon content characterizes states invariant under permutations. We write $|(n_a)\rangle$ for the unique element of norm one which is invariant under permutations and is an eigenstate of each $\mathfrak{n}_a$ with eigenvalue n_a (with, of course, $\sum_a n_a = N$). These states are sometimes referred to in the literature as su(m) Dicke states. The study of the entanglement content of Dicke states $|(n_a)\rangle$ is motivated by the fact that they are the unique ground state of certain "parent" Hamiltonians. Indeed, it is shown in [**P4**] that the parent Hamiltonian of $|(n_a)\rangle$ is the su(m) ferromagnetic gLMG model of the form

$$H = \sum_{i<j} h_{ij}(1 - P_{ij}) + \sum_{b=1}^{m-1} c_b \left(\mathfrak{n}_b - \mathfrak{n}_m - N h_b\right)^2, \tag{14}$$

where $h_{ij}, c_b > 0$. The energy of such a model is minimal for the Dicke state $|(n_a)\rangle$ with $n_b - n_m = N h_b$. The previous system of $m-1$ equations for the indeterminates (n_a) is easily solved when $(h_1, \ldots, h_{m-1}) \in \mathcal{S}$, where $\mathcal{S} \subset \mathbf{R}^{m-1}$ is the simplex whose m vertices are the weights of the fundamental representation of su(m) for a standard choice of the Cartan subalgebra. We also show in [**P4**] that the ground state is still unique even when the vector $(h_1, \ldots, h_{m-1})$ is outside $\mathcal{S}$, although in this case the corresponding Dicke state does not contain magnons of at least one type. Finally, for the su(3) case we determine in detail the complete ground state phsae diagram in the symmetric case $c_1 = c_2$. We also discuss in [**P4**] how the vertices of this simplex are actually the weights of the fundamental representation of su(m), for the appropriate basis of the Cartan subalgebra.

4.1. The entanglement content

Given $(n_a) = (n_1, \ldots, n_m)$ with $\sum_a n_a = N$, let us denote by $|1^{n_1} \cdots m^{n_m}\rangle$ the unique basis state with magnon content (n_a) and $s_1 \leq \cdots \leq s_N$. Let $\mathfrak{S}_{(n_a)}$ denote the set of *distinct* permutations of the state $|1^{n_1} \cdots m^{n_m}\rangle$

whose total number of elements is obviously given by

$$|\mathfrak{S}_{(n_a)}| = \frac{N!}{n_1! \cdots n_m!} \,. \tag{15}$$

In what follows, N will as usual denote the total number of particles and $L < N$ the number of particles in a proper subsystem. Given two magnon contents (n_a) and (l_a) with $n_1 + \cdots + n_m = N$ and $l_1 + \cdots + l_m = L$, we shall write $(l_a) \prec (n_a)$ if $l_a \leq n_a$ for all $a = 1, \ldots, m$. Since the state $|(n_a)\rangle$ is invariant under permutations, all proper subsystems of length $L < N$ are equivalent to the one labeled by the set $\{1, 2, \ldots, L\}$. We can then decompose the action of each $\sigma \in \mathfrak{S}_{(n_a)}$ as

$$\begin{aligned} \sigma |1^{n_1} \cdots m^{n_m}\rangle &= |s_1 \cdots s_L\rangle \otimes |s'_1 \cdots s'_{N-L}\rangle \\ &= \tau |1^{l_1} \cdots m^{l_m}\rangle \otimes \tau' |1^{n_1 - l_1} \cdots m^{n_m - l_m}\rangle \,, \end{aligned}$$

where $\tau \in \mathfrak{S}_{(l_a)}$, $\tau' \in \mathfrak{S}_{(n_a - l_a)}$ and $(l_a) \prec (n_a)$. It follows that

$$|\mathfrak{S}_{(n_a)}| = \sum_{(l_a) \prec (n_a)} |\mathfrak{S}_{(l_a)}| \cdot |\mathfrak{S}_{(n_a - l_a)}| \,, \tag{16}$$

so that the state $|(n_a)\rangle$ and the reduced density matrix ρ of the subsystem of L sites can be expressed as

$$\begin{aligned} |(n_a)\rangle &= \sum_{(l_a) \prec (n_a)} \sqrt{\lambda(l_a)}\, |(l_a)\rangle \otimes |(n_a - l_a)\rangle \,, \\ \rho &= \sum_{(l_a) \prec (n_a)} \lambda(l_a)\, |(l_a)\rangle\langle(l_a)| \,, \end{aligned} \tag{17}$$

with

$$\lambda(l_a) = |\mathfrak{S}_{(n_a)}|^{-1} \cdot |\mathfrak{S}_{(l_a)}| \cdot |\mathfrak{S}_{(n_a - l_a)}| \,.$$

Note that Eq. (16) implies that the (diagonal) reduced density matrix ρ has unit trace. As it is shown in [**P4**], it is convenient to rewrite the eigenvalues of ρ as

$$\begin{aligned} \lambda(l_1, \ldots, l_m) &= \lambda_1(l_1)\, \lambda_2(l_1, l_2) \cdots \lambda_m(l_1, \ldots, l_m) \\ \lambda_a(l_1, \ldots, l_a) &= \binom{L - \sum_{b<a} l_b}{l_a} \binom{N - L - \sum_{b<a} (n_b - l_b)}{n_a - l_a} \times \\ &\qquad \times \binom{N - \sum_{b<a} n_b}{n_a}^{-1} . \end{aligned}$$

This expression of the eigenvalues as the latter product allows one to take advantage of the Laplace–de Moivre formula to derive an asymptotic expression for the trace (integral) of ρ^q (see [**P4**] for the details). More precisely, we first approximate the behavior of the hypergeometric distribution $\lambda_1(l_1)$ by a suitable Gaussian distribution with the same mean and standard deviation in the thermodynamic limit $N \to \infty$. Since l_1 is sharply picked around its mean value $\langle l_1 \rangle$, $\lambda_2(l_1, l_2)$ can be approximated by $\lambda_2(\langle l_1 \rangle, l_2)$ which is again an hypergeometric distribution on the variable l_2. The repeated application of this procedure leads to a formula for the eigenvalues in terms of a general multivariate normal distribution. In this way we obtained an explicit closed-form expression for the entanglement Hamiltonian H_{ent} of arbitrary Dicke states[19]. These results are, to the best of our knowledge, the first rigorous derivation of the behaviour of the spectrum of the reduced density matrix of a Dicke state for an arbitrary subsystem of length $L < N$ in the su(m) case. In particular, we obtain formulas in terms of $\alpha = L/N$ for all α and not just $\alpha \ll 1$.

Once the covariance matrix of the latter distribution is obtained, any entanglement entropy can be computed straightforwardly; and we refer to [**P4**] for several figures and plots comparing the different behaviors for the Rényi, Tsallis or von Neumann entropies.

[19]The entanglement Hamiltonian is the logarithm of the reduced density matrix: $\rho = \mathrm{e}^{H_{\text{ent}}}$.

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Generalized isotropic Lipkin–Meshkov–Glick models: ground state entanglement and quantum entropies

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Abstract. We introduce a new class of generalized isotropic Lipkin–Meshkov–Glick models with $\mathrm{su}(m+1)$ spin and long-range non-constant interactions, whose non-degenerate ground state is a Dicke state of $\mathrm{su}(m+1)$ type. We evaluate in closed form the reduced density matrix of a block of L spins when the whole system is in its ground state, and study the corresponding von Neumann and Rényi entanglement entropies in the thermodynamic limit. We show that both of these entropies scale as $a \log L$ when L tends to infinity, where the coefficient a is equal to $(m-k)/2$ in the ground state phase with k vanishing $\mathrm{su}(m+1)$ magnon densities. In particular, our results show that none of these generalized Lipkin–Meshkov–Glick models are critical, since when $L \to \infty$ their Rényi entropy R_q becomes independent of the parameter q. We have also computed the Tsallis entanglement entropy of the ground state of these generalized $\mathrm{su}(m+1)$ Lipkin–Meshkov–Glick models, finding that it can be made extensive by an appropriate choice of its parameter only when $m-k \geqslant 3$. Finally, in the $\mathrm{su}(3)$ case we construct in detail the phase diagram of the ground state in parameter space, showing that it is determined in a simple way by the weights of the fundamental representation of $\mathrm{su}(3)$. This is also true in the $\mathrm{su}(m+1)$ case; for instance, we prove that the region for which all the

magnon densities are non-vanishing is an $(m+1)$-simplex in $\mathbb{R}^m$ whose vertices are the weights of the fundamental representation of $\mathrm{su}(m+1)$.

Keywords: solvable lattice models, conformal field theory (theory), entanglement in extended quantum systems (theory)

Contents

1. Introduction

A crucial difference between classical and quantum systems is the fact that in the latter ones the entropy of a subsystem can be positive even when the whole system is in its (pure) ground state at zero temperature. Indeed, classically a positive value of the entropy is simply a reflection of the lack of knowledge of the precise microstate of the system when it is in a certain macrostate. On the other hand, an essential property of quantum systems is the fact that the exact knowledge of the state of the whole system does not give complete information about the state of a subsystem. This is a consequence of the entanglement among different parts of the system, which is perhaps the most paradigmatic quantum phenomenon [1, 2].

One of the most widespread quantitative measures of the degree of entanglement of a quantum system at zero temperature is the bipartite entropy of its ground state $|\psi\rangle$. More precisely, if we divide the system into two subsystems with L and $N-L$ particles the corresponding bipartite entropy is the quantum entropy of the reduced

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density matrix $\rho_L = \mathrm{tr}_{N-L}\rho$, where $\rho \equiv |\psi\rangle\langle\psi|$ and the trace is taken over the degrees of freedom of the latter subsystem. This definition is actually symmetric between both subsystems, since the entropy of ρ_L necessarily coincides with that of ρ_{N-L}. In practice, the von Neumann [1] and Rényi [3, 4] entropies, respectively defined by

$$S = -\mathrm{tr}(\rho_L \log \rho_L), \qquad R_q = \frac{\log \mathrm{tr}(\rho_L^q)}{1-q} \tag{1.1}$$

(where q is a positive real parameter) have been extensively used as quantitative measures of the bipartite entanglement. The exact evaluation of these entropies, or even the determination of their large L limit, is only possible for a handful of mostly one-dimensional models. These models include the XX and XY nearest-neighbors Heisenberg spin chains, for which the asymptotic behavior of both the von Neumann and Rényi entropies was established in the last decade [5–7] with the help of the Fisher–Hartwig conjecture. As is well known, both of these models are critical in a certain region of their parameter space [8]. The large L behavior of the entropy is essentially different in the critical and non-critical regions. More precisely, the entropy scales as $\log L$ in the critical region, while it saturates to a constant in the non-critical one. This is a manifestation of the so-called area law [9], according to which the (von Neumann) bipartite entropy of a critical one-dimensional quantum system with short-range interactions behaves as $\log L$, while for non-critical systems it tends to a constant. This is consistent with the fact that the bipartite entropy of two-dimensional conformal field theories (which describe one-dimensional quantum critical systems in the thermodynamic limit) scales as $(c+\bar{c})(1+q^{-1})(\log L)/12$, where c and $\bar{c}$ denote respectively the holomorphic and anti-holomorphic central charges and $q = 1$ for the von Neumann entropy [10–12].

On the other hand, it is generally believed that the area law need not hold for systems exhibiting long-range interactions, since the stronger correlations in these systems are expected to increase the entropy. One of the few one-dimensional long-range quantum systems that has been exactly solved is the Lipkin–Meshkov–Glick (LMG) model [13–15], originally introduced by the latter authors to describe a system of N fermions in two levels each of which is N times degenerate. This model is equivalent to a system of N spin 1/2 particles with constant XY-type long-range interactions in a transverse magnetic field. In the isotropic (XX) case the model is exactly solvable, and its bipartite von Neumann entropy has also been computed in closed form [16, 17]. The von Neumann entropy vanishes in the gapped (and not entangled) phase, while it grows as $(\log L)/2$ in the gapless one. Thus the LMG model behaves in this respect as a one-dimensional quantum system with *short*-range interactions. The reason of this behavior is the competition between the range of the interactions, which tends to increase the entropy, and the high degree of symmetry of this model, which tends to lower it. Indeed, in this case the ground state is a symmetric Dicke state (i.e. it has maximum total spin $S = N/2$ and a well-defined value of the total spin component S^z) for all values of its parameters, which is easily seen to imply that the entanglement entropy cannot exceed $\log(L+1)$.

A class of one-dimensional models with long-range interactions which has been extensively studied is that of spin chains of Haldane–Shastry (HS) type. The original HS chain [18, 19] is a lattice model of N spin 1/2 particles uniformly arranged on a circle, with pairwise interactions inversely proportional to the square of the chord distance. This

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model was soon afterwards generalized to particles carrying $\mathrm{su}(m+1)$ spin, as well as to rational and hyperbolic interactions [20–23]. The HS chain, which is closely related to the one-dimensional Hubbard model with long-range hopping [24], is exactly solvable via the asymptotic Bethe ansatz, and its partition function can be evaluated in closed form [25]. Among other remarkable properties, the HS chain possesses exact Yangian symmetry for a finite number of sites [26], and provides one of the simplest realizations of anyons in one dimension via Haldane's fractional statistics [26–29].

In this paper we shall introduce a large family of $\mathrm{su}(m+1)$ spin chains which, like the HS-type chains, feature variable long-range interactions, and whose ground state entanglement properties are similar to those of the isotropic LMG model. More precisely, the models we shall construct will admit as non-degenerate ground state a generalized Dicke state of $\mathrm{su}(m+1)$ type, i.e. a state totally symmetric under permutations and with a well defined number of particles in each of the $\mathrm{su}(m+1)$ internal one-particle states ($\mathrm{su}(m+1)$ magnons). This will be achieved by replacing the quadratic term in the total spin operator S^z present in the LMG Hamiltonian by a sum of similar terms in each of the generators of the $\mathrm{su}(m+1)$ Cartan subalgebra. The resulting models can thus be considered a natural generalization of the original (isotropic) LMG model, to which they actually reduce when $m=1$ and all the two-body interactions are constant.

An explicit expression for the reduced density matrix ρ_L of any system whose ground state is an $\mathrm{su}(m+1)$-type Dicke state with arbitrary m first appeared in [30]. The eigenvalues of ρ_L turn out to define a hypergeometric distribution in m variables, thus generalizing the result of [16, 17] for the spin 1/2 ($m=1$) isotropic LMG model. We shall show that in the thermodynamic limit $N\gg 1$ with $L/N\to\alpha$ (finite) the eigenvalues of the reduced density matrix can be well approximated by a Gaussian distribution, whose parameters we evaluate in closed form for arbitrary m and α. With the help of this approximation, we obtain explicit asymptotic expressions for the von Neumann and Rényi entropies in the thermodynamic limit. In particular, our expression for the von Neumann entropy coincides with that of [30], derived by extrapolation from the $\alpha=0$ case. Remarkably, in the region of parameter space for which all the ground state magnon densities are non-vanishing both of these entropies scale as $(m\log L)/2$ as L tends to infinity. Thus the behavior of the von Neumann entropy is that of a critical model with $c=\bar{c}=3m/2$, described by a conformal field theory (CFT) with m fermions and m bosons. This may not seem surprising at first sight, taking into account that many critical one-dimensional spin chains, including the Heisenberg (XXX) and the $\mathrm{su}(m+1)$ HS chains, are effectively described by theories of this type (see, e.g. [26, 31, 32]). Here, however, the situation is more subtle. Indeed, the Rényi entropy R_q, although still proportional to $\log L$ for large L, becomes independent of the parameter q for $L\to\infty$, and as a consequence the family of generalized isotropic LMG models cannot contain any critical instances.

It is well known that a crucial requirement of classical thermodynamics is the extensivity of the (Maxwell–Boltzmann) entropy of a given system, i.e. that $S\propto L^d$, where L is a characteristic length of the system and d is the number of space dimensions. In a quantum context this requirement, at least for the von Neumann entropy, is violated in many cases, as for instance in black hole thermodynamics [33–36]. In fact, the area law mentioned above evidences a non-extensive behavior of von Neumann's entropy in strongly correlated quantum systems. On the other hand, it is very natural to enquire

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whether this feature is shared by all quantum entropies available in the literature. Interestingly enough, this is not the case. For instance, as already noted in [37], the quantum Tsallis entropy [38, 39] can be extensive when von Neumann's is not in several one- and two-dimensional strongly correlated systems, which include the Heisenberg XY model. We have found a similar behavior for the Tsallis entanglement entropy of the ground state of generalized $\mathrm{su}(m+1)$ LMG models with $m-k \geqslant 3$ (where k is the number of vanishing magnon densities), while for $m-k=1,\ 2$ the Tsallis entropy is not extensive for any value of its parameter.

As mentioned above, the ground state of the isotropic LMG model has two quantum phases (entangled and non-entangled), respectively determined by the values of the (suitably normalized) magnetic field strength being less or greater than 1 in absolute value. For the generalized LMG models constructed in this paper, the situation is more subtle. More precisely, we shall show that in this case the ground state can be in exactly $m+1$ quantum phases, each of which is characterized by the vanishing of a certain number of magnon densities. Moreover, in the phase with k vanishing magnon densities both the von Neumann and Rényi entropies scale as $\frac{1}{2}(m-k)\log L$, implying again that none of these phases can contain any critical models. We have performed a detailed analysis of the ground state phases in the $\mathrm{su}(3)$ ($m=2$) case, completely identifying the corresponding regions in parameter space. Remarkably, these regions are entirely determined in a geometric way by the weights of the fundamental representation of $\mathrm{su}(3)$ associated to the choice of the Cartan generators. A similar result holds in the general ($\mathrm{su}(m+1)$) case; for instance, we show that the region for which all the magnon densities are non-vanishing is an $(m+1)$-simplex in $\mathbb{R}^m$ whose vertices are the weights of the fundamental representation of $\mathrm{su}(m+1)$.

We end this section with a few words on the paper's organization. In section 2 we review the main properties of the isotropic LMG model and present the construction of its $\mathrm{su}(m+1)$ generalization with non-constant interactions. In section 3 we evaluate in closed form the ground state entanglement entropies of von Neumann and Rényi, and study their main properties. As a byproduct of our analysis, we obtain a similar expression for the Tsallis entropy and discuss its extensivity. In section 4, a detailed description of the entanglement properties of the ground state as a function of the $\mathrm{su}(m+1)$ magnetic field strength is presented. Special attention is paid to the case $m=2$, for which we obtain a complete phase diagram describing the regions in parameter space with different magnon content. Section 5 is devoted to the presentation of our conclusions and the discussion of some open problems suggested by the previous results. The paper ends with three technical appendices in which we present a detailed derivation of the exact formula for the eigenvalues of the reduced density matrix, compute the first and second moments of a multivariate hypergeometric distribution, and show how a univariate hypergeometric distribution can be approximated by a normal one in a suitable limit.

2. Generalized Lipkin–Meshkov–Glick models

The original Lipkin–Meshkov–Glick model describes a system of N mutually interacting spin 1/2 particles in a constant magnetic field. Its Hamiltonian can be taken as

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$$H = -\frac{\lambda}{N}\sum_{i<j}(\sigma_i^x\sigma_j^x + \gamma\sigma_i^y\sigma_j^y) - h\sum_i \sigma_i^z, \tag{2.1}$$

where $\lambda > 0$, $\gamma \geqslant 0$ and h are real parameters, σ_k^a is the a-th Pauli matrix acting on the k-th spin, and the sums (as hereafter, unless otherwise stated) range from 1 to N. The model (2.1) is thus the analogue of the Heisenberg XY chain with long-range constant interactions. Its Hamiltonian can be expressed in terms of the total spin operators $S^a = \sum_i \sigma_i^a/2$ as

$$H = -\frac{\lambda}{N}(1+\gamma)\left(\mathbf{S}^2 - (S^z)^2 - \frac{N}{2}\right) - 2hS^z - \frac{\lambda}{2N}(1-\gamma)((S^+)^2 + (S^-)^2), \tag{2.2}$$

where $\mathbf{S} = (S^x, S^y, S^z)$ and $S^\pm = S^x \pm \mathrm{i}S^y$.

It is well known [40, 41] that the LMG model undergoes a second-order quantum phase transition with mean field exponents at $h/\lambda = 1$. In the anisotropic case $0 \leqslant \gamma < 1$, the model has been solved only in the thermodynamic limit [42, 43], although its entanglement properties have been extensively studied (see, e.g. [44–46]). In this work we shall focus on the isotropic case $\gamma = 1$, for which H is diagonal in a basis $|S, M, \nu\rangle$ of common eigenstates[3] of $\mathbf{S}^2$ and S^z with respective eigenvalues $S(S+1)$ and M. Here, as usual, $M = -S, -S+1, \ldots, S$ and $S = \pi(N)/2, \pi(N)/2+1, \ldots, N/2$, where $\pi(N)$ denotes the parity of the integer N. We shall further set $\lambda = 1$ (which amounts to a trivial rescaling) and also restrict ourselves, without loss of generality, to nonnegative values of h (since the spectrum of H is clearly independent of the sign of h). With this choice of parameters equation (2.2) implies that $H|S, M, \nu\rangle = E(S, M)|S, M, \nu\rangle$, where the eigenvalue $E(S, M)$ is given by

$$\begin{aligned} E(S,M) &= -\frac{2}{N}(S(S+1) - M^2 - N/2) - 2hM \\ &= -\frac{2}{N}S(S+1) + \frac{2}{N}\left(M - \frac{Nh}{2}\right)^2 - \frac{Nh^2}{2} + 1. \end{aligned} \tag{2.3}$$

The properties of the ground state of the LMG model (2.1) with $\lambda = \gamma = 1$ and $h \geqslant 0$ are essentially different in the two quantum phases $0 \leqslant h < 1$ and $h \geqslant 1$. Indeed, for $h \geqslant 1$ the minimum of equation (2.3) is achieved when $S = M = N/2$, so that the ground state is the product state with all spins up. In particular, for $h \geqslant 1$ the ground state is not entangled. On the other hand, when $0 \leqslant h < 1$ the energy is clearly a minimum when $S = N/2$ and $M = I(hN/2)$, where $I(x)$ denotes the closest integer[4] (for N even) or half-integer (for N odd) to x. In particular, since the total spin S is maximum the ground state must be totally symmetric. It is also non-degenerate, since the number $N_\uparrow$ of 'up' spins is fixed by the condition

[3] The additional quantum number ν, which ranges from 1 to $\binom{N}{N/2+S} - \binom{N}{N/2+S+1}$, takes into account the degeneracy of the eigenspace with a given S and M.

[4] The definition of $I(x)$ is ambiguous when x is an integer (for odd N) or a half-integer (for even N). Since we are ultimately interested in the thermodynamic limit, we shall henceforth implicitly assume that $hN \neq 2p + 1 - \pi(N)$ with $p \in \mathbb{Z}$, so that $I(hN/2)$ is well defined.

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$$M = I(hN/2) = \frac{1}{2}(N_\uparrow - N_\downarrow) = N_\uparrow - \frac{N}{2} \implies N_\uparrow = \frac{N}{2} + I(hN/2), \tag{2.4}$$

where $N_\downarrow = N - N_\uparrow$ denotes the number of 'down' spins. Thus, when $0 \leqslant h < 1$ the ground state of the LMG Hamiltonian (2.1) with $\gamma = \lambda = 1$ is the totally symmetric state[5]

$$|\psi(N_\uparrow, N_\downarrow)\rangle \equiv |N/2, I(hN/2)\rangle, \tag{2.5}$$

where $N_\uparrow$ is given by (2.4). Denoting by $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively the spin up and down one-particle states, the (normalized) ground state (2.5) can be expressed in terms of the elements

$$|s_1, \ldots, s_N\rangle \equiv |s_1\rangle \cdots |s_N\rangle \equiv |s_1\rangle \otimes \cdots \otimes |s_N\rangle, \qquad s_k = \uparrow, \downarrow,$$

of the canonical spin basis as

$$|\psi(N_\uparrow, N_\downarrow)\rangle = \binom{N}{N_\uparrow}^{-\frac{1}{2}} \sum_{p \in S_{N_\uparrow, N_\downarrow}} |p(\underbrace{\uparrow, \ldots, \uparrow}_{N_\uparrow}, \underbrace{\downarrow, \ldots, \downarrow}_{N_\downarrow})\rangle. \tag{2.6}$$

In the latter formula $S_{N_\uparrow, N_\downarrow}$ denotes any set of $\binom{N}{N_\uparrow}$ permutations of N elements inequivalent with respect to the initial state $|\uparrow, \ldots, \uparrow, \downarrow, \ldots, \downarrow\rangle$ (i.e. such that the images of the latter state under any two elements of the set differ).

Our aim is to construct a model generalizing the (isotropic) LMG model (2.1) in two different directions. More precisely, we shall consider an internal space of arbitrary dimension $m+1$, and shall also allow for general (position-dependent) long-range interactions. We shall only require that the ground state of the model (in the thermodynamic limit) be (i) non degenerate, (ii) totally symmetric, and (iii) such that the number N_s of particles in each one-particle state $|s\rangle$ (with $s = 1, \ldots, m+1$) is well-defined, as in the original LMG model. In other words, the ground state of the model should be the *Dicke state*

$$|\psi(N_1, \ldots, N_{m+1})\rangle = \left(\frac{N!}{N_1! \cdots N_{m+1}!}\right)^{-\frac{1}{2}} \sum_{p \in S_{N_1, \ldots, N_{m+1}}} |p(\underbrace{1, \ldots, 1}_{N_1}, \ldots, \underbrace{m+1, \ldots, m+1}_{N_{m+1}})\rangle, \tag{2.7}$$

with $N_1 + \cdots + N_{m+1} = N$. As before, $S_{N_1, \ldots, N_{m+1}}$ denotes any set of $\frac{N!}{N_1! \cdots N_{m+1}!}$ permutations of N elements inequivalent with respect to the state $|1, \ldots, 1, \ldots, m+1, \ldots, m+1\rangle$.

In order to construct these generalized Lipkin–Meshkov–Glick (gLMG) models, we note that the local su(2) spin operators $\mathbf{S}_k \equiv (S_k^x, S_k^y, S_k^z)$ are related to the spin permutation operators S_{ij}, whose action on the canonical spin basis is given by

$$S_{ij}|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N\rangle = |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N\rangle, \tag{2.8}$$

[5] Note that in this case the quantum number ν can be omitted, since the subspace with $S = N/2$ and any M is one-dimensional (see the footnote 3).

by the identity

$$\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2}\left(S_{ij} - \frac{1}{2}\right).$$

From the previous equation it easily follows that

$$\mathbf{S}^2 = \sum_{i<j} S_{ij} - \frac{N}{4}(N-4),$$

so that the Hamiltonian (2.1) with $\lambda = \gamma = 1$ can also be expressed as

$$H = \frac{2}{N}\sum_{i<j}(1 - S_{ij}) + \frac{2}{N}\left(S^z - \frac{Nh}{2}\right)^2 - \frac{N}{2}(1+h^2). \tag{2.9}$$

Motivated by this fact, we consider the general spin permutation operators S_{ij} in equation (2.8) acting on particles with $(m+1)$ internal degrees of freedom. It is well known that these operators can be expressed in terms of the local (Hermitian) generators t_k^a $(a = 1, \ldots, m(m+2))$ of the fundamental representation of the $\mathrm{su}(m+1)$ algebra acting on the k-th site (with the normalization $\mathrm{tr}(t_k^a t_k^b) = \frac{1}{2}\delta_{ab}$) as

$$S_{ij} = \frac{1}{m+1} + 2\sum_{a=1}^{m(m+2)} t_i^a t_j^a \equiv \frac{1}{m+1} + 2\,\mathbf{t}_i \cdot \mathbf{t}_j. \tag{2.10}$$

Furthermore, since S_{ij} is obviously Hermitian and has eigenvalues ± 1, it is clear that the lowest energy eigenspace of the Hamiltonian

$$H_0 = \sum_{i<j} h_{ij}(1 - S_{ij}) \tag{2.11}$$

coincides with the subspace of totally symmetric states provided that $h_{ij} > 0$ for all $i < j$. Note that this Hamiltonian commutes with the total $\mathrm{su}(m+1)$ generators $T^a = \sum_i t_i^a$, since these operators commute with each S_{ij}. In fact, the model (2.11) reduces to the well-known Haldane–Shastry spin chain of $\mathrm{su}(m+1)$ type when the interactions are given by

$$h_{ij} = \frac{\pi^2}{N^2}\,\sin^{-2}\left(\frac{\pi(i-j)}{N}\right).$$

This model, as well as its rational and hyperbolic versions, has been extensively studied in the literature due to its remarkable integrability and solvability properties (see, e.g. [22, 28, 47]).

In order to single out a state of the form (2.7) as the unique ground state we need to add a suitable term to the Hamiltonian H_0, as we shall now explain. By analogy with the $\mathrm{su}(2)$ case, we define the operators J_i^a $(a = 1, \ldots, m)$ acting on the i-th site by

$$J_i^a = E_i^{aa} - E_i^{m+1,m+1}, \tag{2.12}$$

where E^{bc} denotes the $(m+1)\times(m+1)$ matrix whose only nonzero element is a 1 in the b-th row and c-th column[6]. Note that the operators $\mathrm{i}J_k^a$ $(1\leqslant a\leqslant m)$ are a basis of the standard Cartan subalgebra of $\mathrm{su}(m+1)$ (at the k-th site), and that $J_k^1=\sigma_k^z$ in the $\mathrm{su}(2)$ case. The simplest Hamiltonian with ground state satisfying conditions (i)–(iii) above is then given by

$$H=\sum_{i<j}h_{ij}(1-S_{ij})+\sum_{a=1}^{m}c_a(J^a-Nh_a)^2\equiv H_0+H_1, \tag{2.13}$$

where h_{ij}, $c_a>0$, $h_a\in\mathbb{R}$ and

$$J^a=\sum_i J_i^a, \qquad 1\leqslant a\leqslant m.$$

Indeed, note first of all that H_0 commutes with all the operators J^a, and hence with H_1, so that H_0 and H_1 can be simultaneously diagonalized. As mentioned before, the subspace of totally symmetric states is the eigenspace of H_0 with lowest (zero) energy. On the other hand, we have

$$J^a|1,\ldots,1,\ldots,m+1,\ldots,m+1\rangle=(N_a-N_{m+1})|1,\ldots,1,\ldots,m+1,\ldots,m+1\rangle,$$

and therefore

$$J^a|\psi(N_1,\ldots,N_{m+1})\rangle=(N_a-N_{m+1})|\psi(N_1,\ldots,N_{m+1})\rangle,$$

since J^a commutes with all the permutation operators S_{ij}. Thus in the thermodynamic limit $N\to\infty$ with $N_a/N\to n_a$ finite, the energy of the Hamiltonian (2.13) is minimal for the state $|\psi(N_1,\ldots,N_{m+1})\rangle$ whose magnon densities n_a satisfy

$$n_a-n_{m+1}=h_a, \qquad 1\leqslant a\leqslant m. \tag{2.14}$$

The above system is easily solved, with the result

$$n_a=h_a+\frac{1-h}{1+m} \qquad n_{m+1}=\frac{1-h}{1+m}, \tag{2.15}$$

where

$$h\equiv\sum_{a=1}^{m}h_a.$$

The consistency condition for the system (2.14), namely that $0\leqslant n_a\leqslant 1$ for $a=1,\ldots,m+1$, is satisfied provided that

$$-m\leqslant h\leqslant 1, \qquad -m\leqslant h-(m+1)h_a\leqslant 1. \tag{2.16}$$

These are the equations of an m-dimensional (closed) simplex $\mathcal{H}\subset\mathbb{R}^m$ with vertices

$$\boldsymbol{\mu}_a=\mathbf{e}_a \quad (1\leqslant a\leqslant m), \qquad \boldsymbol{\mu}_{m+1}=-\sum_{a=1}^{m}\mathbf{e}_a, \tag{2.17}$$

[6] This particular choice of basis of the standard Cartan subalgebra of $\mathrm{su}(m+1)$ is largely a matter of convenience, in that it results in the simplest form for equations (2.15) and (2.16) below. Note, in particular, that the operators J_i^a are not orthogonal with respect to the usual Killing–Cartan scalar product, i.e. $\mathrm{tr}(J_i^aJ_i^b)\neq 0$ for $a\neq b$.

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where $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is the canonical basis of $\mathbb{R}^m$ (see equation (2.14)). In other words, the vectors $\boldsymbol{\mu}_a$ $(1 \leqslant a \leqslant m+1)$ are the weights of the fundamental representation of $\mathrm{su}(m+1)$ with respect to the basis (2.12) of its Cartan subalgebra.

In summary, we have shown that (in the thermodynamic limit) the model (2.13) with $h_{ij}, c_a > 0$ and parameters h_a satisfying (2.16) has a non-degenerate, totally symmetric ground state $|\psi(N_1, \ldots, N_{m+1})\rangle$ given by equation (2.7), where $N_a = n_a N$ and the magnon densities n_a are determined by $h_1, \ldots, h_m$ through equation (2.15). The Hamiltonian (2.13) is not invariant under the $\mathrm{su}(m+1)$ algebra, due to its second term H_1. However, it can be easily expressed in terms of the (local) $\mathrm{su}(m+1)$ generators using equation (2.10), namely

$$H = -\sum_{i \neq j} h_{ij}\, \mathbf{t}_i \cdot \mathbf{t}_j + \sum_{a=1}^{m} c_a (J^a - N h_a)^2 + E_0,$$

with $E_0 = \frac{m}{m+1} \sum_{i<j} h_{ij}$. We shall thus refer to the model (2.13) with $m+1$ internal degrees of freedom and h_a satisfying equation (2.16) as an $\mathrm{su}(m+1)$ gLMG model. Note that, since the parameters h_a couple to the total $\mathrm{su}(m+1)$ generators J^a, they can be regarded as $\mathrm{su}(m+1)$ magnetic field strengths by analogy with the $\mathrm{su}(2)$ case. In fact, from equation (2.9) and the identity $S^z = J^1/2$ it follows that the isotropic LMG model is an $\mathrm{su}(2)$ gLMG model with $h_{ij} = 2/N$ for all $1 \leqslant i, j \leqslant N$ and $c_1 = 1/(2N)$ (up to an irrelevant constant).

Remark 1. The condition that all the coefficients h_{ij} in equation (2.13) be positive can be considerably relaxed. Indeed, the latter condition is certainly sufficient to guarantee that the ground state of the model be symmetric, but it is by no means necessary. More precisely, it suffices that the transpositions $i \leftrightarrow j$ corresponding to positive h_{ij} generate the full symmetric group. For instance, the consecutive transpositions $i \leftrightarrow i+1$ (with $i = 1, \ldots, N-1$) certainly fulfill this requirement. It follows that the ground state of the Hamiltonian with nearest neighbor interactions

$$H = \sum_i \tilde{h}_i (1 - S_{i,i+1}) + \sum_{a=1}^{m} c_a (J^a - N h_a)^2,$$

where $\tilde{h}_i, c_a > 0$ and $S_{N,N+1} = 1$ or $S_{N,N+1} = S_{1,N}$, is also of the form (2.7). Thus the family of gLMG models can be suitably enlarged to encompass systems with both short-range and long-range interactions.

Remark 2. Although the ground state of the model (2.13) with the restrictions (2.16) is explicitly given by equation (2.7), its full spectrum cannot be computed in closed form for arbitrary values of the parameters h_{ij}, c_a, h_a. Note, however, that the spin permutation operators S_{ij} leave invariant each of the subspaces $\mathcal{S}_{N_1,\ldots,N_{m+1}}$ consisting of states with N_s particles in each internal state $|s\rangle$ (magnons of type s). It trivially follows that H_0 leaves these subspaces invariant, and the same is true of H_1, since each Cartan generator J^a is equal to the constant $N_a - N_{m+1}$ on them. Thus, in order to compute the spectrum of H it suffices to diagonalize the restrictions of H_0 to each of the subspaces $\mathcal{S}_{N_1,\ldots,N_{m+1}}$. More precisely, if $E_i(N_1, \ldots, N_m)$ denotes an arbitrary eigenvalue of $H_0|_{\mathcal{S}_{N_1,\ldots,N_{m+1}}}$ then the eigenvalues of H are given by

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$$E_i(N_1, \ldots, N_{m+1}) + \sum_{a=1}^{m} c_a(N_a - N_{m+1} - Nh_a)^2.$$

3. Quantum entropies

In this section we shall compute the analytic expressions of the (bipartite) entanglement entropies of von Neumann and Rényi for the ground state of the gLMG model (2.13) or, more generally, of any quantum system whose ground state is a Dicke state (2.7). Actually, since the von Neumann entropy is the $q \to 1$ limit of the Rényi entropy, it suffices to compute $\mathrm{tr}(\rho_L^q)$ for arbitrary $q > 0$. Note that, by equation (A.7), the von Neumann entropy is bounded above by $\log d(L, m)$, where $d(L, m)$ is the dimension of the subspace spanned by the Dicke states $|\psi(L_1, \ldots, L_{m+1})\rangle$ with $L_1 + \cdots + L_{m+1} = L$, i.e. the subspace of symmetric states for L particles with an $(m+1)$-dimensional internal space. Since we obviously have

$$d(L, m) = \binom{L+m}{m} \leqslant \frac{(L+m)^m}{m!},$$

the von Neumann entropy satisfies

$$\begin{aligned} S &\leqslant \log\binom{L+m}{m} \leqslant m\log(L+m) - \log(m!) \leqslant m\log(L+m) - \int_1^m \log x \, \mathrm{d}x \\ &= m\log(L+m) - m\log m + m - 1. \end{aligned} \tag{3.1}$$

In other words, the von Neumann (bipartite) entropy of the ground state of the gLMG model (2.13)—or, more generally, of any quantum system whose ground state is a Dicke state (2.7)—cannot grow faster than $\log L$, which is the typical scaling behavior of the entropy observed in many critical spin chains (see, e.g. [9, 17]).

As first shown in [30], the reduced density matrix ρ_L is diagonal in the Dicke basis $|\psi(L_1, \ldots, L_{m+1})\rangle$ (where $0 \leqslant L_i \leqslant N_i$ and $L_1 + \cdots + L_{m+1} = L$), with eigenvalues

$$\lambda(L_1, \ldots, L_m) = \binom{N}{L}^{-1} \prod_{a=1}^{m+1} \binom{N_a}{L_a} \tag{3.2}$$

(see appendix A for a detailed derivation of the latter formula). Our first goal is to analyze the behavior of $\lambda(L_1, \ldots, L_m)$ in the thermodynamic limit $N \to \infty$, with

$$\lim_{N\to\infty} \frac{L}{N} = \alpha, \qquad \lim_{N\to\infty} \frac{N_a}{N} = n_a \tag{3.3}$$

and $0 < \alpha, n_a < 1$ for $a = 1, \ldots, m+1$. Note that, by equation (2.15), the latter condition on the magnon densities n_a will be satisfied provided that the magnetic field strength vector $\mathbf{h} \equiv (h_1, \ldots, h_m)$ lies in the interior of the simplex (2.16). We start by rewriting equation (3.2) as

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$$\lambda(L_1, \ldots, L_m) = \prod_{a=1}^{m} \lambda_a(L_1, \ldots, L_a), \tag{3.4}$$

where each factor

$$\lambda_a(L_1, \ldots, L_a) = \begin{pmatrix} L - \sum\limits_{b<a} L_b \\ L_a \end{pmatrix} \begin{pmatrix} N - L - \sum\limits_{b<a} (N_b - L_b) \\ N_a - L_a \end{pmatrix} \begin{pmatrix} N - \sum\limits_{b<a} N_b \\ N_a \end{pmatrix}^{-1} \tag{3.5}$$

is a hypergeometric distribution in the variable L_a. Note, in particular, that all the binomial coefficients appearing in the latter expression are well defined (i.e. non-vanishing) on account of the inequalities (A.5). The main idea in order to derive the asymptotic behavior of the RHS of equation (3.4) as $N \gg 1$ is to recursively apply the approximation of a hypergeometric distribution by a suitable Gaussian distribution described in appendix C. To begin with, the first factor

$$\lambda_1(L_1) = \begin{pmatrix} L \\ L_1 \end{pmatrix} \begin{pmatrix} N - L \\ N_1 - L_1 \end{pmatrix} \begin{pmatrix} N \\ N_1 \end{pmatrix}^{-1}$$

can be approximated using equations (C.5)–(C.6) with

$$\tilde{L} = L, \quad \tilde{N} = N, \quad l = L_1, \quad n = N_1,$$

and hence $\tilde{\alpha} = \alpha$, $\nu = n_1$. We thus obtain (see equation (C.3))

$$\lambda_1(L_1) \simeq g(L_1; \mu_1, \sigma_1) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \mathrm{e}^{-\frac{x_1^2}{2\sigma_1^2}},$$

where

$$\mu_1 = L n_1, \qquad \sigma_1^2 = L(1-\alpha) n_1 (1 - n_1), \qquad x_1 = L_1 - \mu_1 = L_1 - L n_1.$$

By equations (B.3) and (B.4), the mean and standard deviation of the first magnon number L_1 are respectively equal to μ_1 and $(1 - 1/N)^{-1}\sigma_1^2$, so that $\sigma_1/\mu_1 = \mathrm{O}(L^{-1/2})$. In particular, this implies that in the thermodynamic limit the distribution of L_1 becomes sharply peaked around its mean value $L n_1$. As we shall now see, this fact is crucial for determining the behavior of the second factor

$$\lambda_2(L_1, L_2) = \begin{pmatrix} L - L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} N - L - N_1 + L_1 \\ N_2 - L_2 \end{pmatrix} \begin{pmatrix} N - N_1 \\ N_2 \end{pmatrix}^{-1}.$$

Indeed, we can approximate $\lambda_2(L_1, L_2)$ using equations (C.5)–(C.6) with

$$\tilde{L} = L - L_1, \quad \tilde{N} = N - N_1 \simeq (1 - n_1) N, \quad l = L_2, \quad n = N_2,$$

and hence

$$\tilde{\alpha} = \frac{\alpha - L_1/N}{1 - n_1} \simeq \alpha, \qquad \nu = \lim_{N\to\infty} \frac{N_2}{N - N_1} = \frac{n_2}{1 - n_1} \equiv \nu_2,$$

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where we have used the fact that $L_1 \sim Ln_1$. We thus obtain

$$\lambda_2(L_1, L_2) \simeq g(L_2; \mu_2, \sigma_2),$$

with

$$\mu_2 = \tilde{L}\nu_2 = Ln_2 - \frac{n_2 x_1}{1-n_1},$$

$$\sigma_2^2 = \tilde{L}(1-\tilde{\alpha})\nu_2(1-\nu_2) \simeq L(1-n_1)(1-\alpha)\nu_2(1-\nu_2) = \frac{L(1-\alpha)n_2(1-n_1-n_2)}{1-n_1},$$

where in the second formula we have again taken into account that $L_1 \sim Ln_1$. Setting $x_2 = L_2 - \mu_2$, from the previous formulas it immediately follows that

$$\sigma_1\sigma_2 = L(1-\alpha)n_1n_2(1-n_1-n_2), \qquad L_2 - \mu_2 = x_2 + \frac{n_2x_1}{1-n_1},$$

and hence

$$\lambda_1(L_1)\lambda_2(L_1, L_2) = [2\pi L(1-\alpha)]^{-1}\,[n_1n_2(1-n_1-n_2)]^{-1/2}\exp\left(-\frac{\mathcal{E}(x_1, x_2)}{2L(1-\alpha)}\right),$$

where

$$\mathcal{E}(x_1, x_2) = \frac{x_1^2}{n_1(1-n_1)} + \frac{1-n_1}{n_2(1-n_1-n_2)}\left(x_2 + \frac{n_2x_1}{1-n_1}\right)^2 = \frac{x_1^2}{n_1} + \frac{x_2^2}{n_2} + \frac{(x_1+x_2)^2}{1-n_1-n_2}.$$

The above approximate formula for $\lambda_1\lambda_2$ suggests that in general we have

$$\prod_{b=1}^{a}\lambda_b \simeq [2\pi L(1-\alpha)]^{-m/2}\left(1-\sum_{b=1}^{a}n_b\right)^{-1/2}\prod_{b=1}^{a}n_b^{-1/2}\exp\left(-\frac{\mathcal{E}(x_1, \ldots, x_a)}{2L(1-\alpha)}\right), \tag{3.6}$$

with $x_b \equiv L_b - Ln_b$ and

$$\mathcal{E}(x_1, \ldots, x_a) = \sum_{b=1}^{a}\frac{x_b^2}{n_b} + \frac{\left(\sum_{b=1}^{a}x_b\right)^2}{1-\sum_{b=1}^{a}n_b}. \tag{3.7}$$

This fact can be readily established by induction through a straightforward calculation. In particular, setting $a = m$ in the previous formulas we obtain the following simple asymptotic expression for the eigenvalues of the reduced density matrix ρ_L in the thermodynamic limit:

$$\lambda(L_1, \ldots, L_m) \simeq [2\pi L(1-\alpha)]^{-m/2}\prod_{a=1}^{m+1}n_a^{-1/2}\cdot\exp\left(-\frac{\mathcal{E}(x_1, \ldots, x_m)}{2L(1-\alpha)}\right). \tag{3.8}$$

Remark 3. From the condition $\mathrm{tr}\rho_L = 1$ it follows that the integral over $\mathbb{R}^m$ of the RHS of equation (3.8) is approximately equal to 1. In fact, it is straightforward to check that this integral is exactly equal to 1, which proves that the multivariate hypergeometric distribution $\lambda(L_1, \ldots, L_m)$ can be approximated in the thermodynamic limit (3.3) by a suitable normal distribution. We shall next show that this normal distribution is completely determined by the fact that its first and second moments coincide with

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those of the exact distribution $\lambda(L_1, \ldots, L_m)$ in the thermodynamic limit. In other words (see equations (B.5) and (B.8)), the RHS of equation (3.8) is simply the Gaussian distribution with moments

$$\langle L_i \rangle = Ln_i, \qquad \big\langle L_i L_j \big\rangle - \langle L_i \rangle \big\langle L_j \big\rangle = \big\langle x_i x_j \big\rangle = L(1-\alpha) n_i (\delta_{ij} - n_j),$$

where $x_i \equiv L_i - Ln_i$. Indeed, the covariance matrix of the general normal distribution

$$g(L_1, \ldots, L_m) = \frac{(\det A)^{1/2}}{(2\pi)^{m/2}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^{m} a_{ij} x_i x_j \right) \tag{3.9}$$

with means $\langle L_i \rangle = Ln_i$ is given by

$$\big\langle x_i x_j \big\rangle = \frac{A_{ij}}{\det A} = (A^{-1})_{ij},$$

where A_{ij} is the complementary minor of a_{ij} in the symmetric matrix $A \equiv (a_{ij})_{1 \leqslant i,j \leqslant m}$. Comparing with equation (3.9) we immediately obtain

$$(A^{-1})_{ij} = L(1-\alpha) n_i (\delta_{ij} - n_j),$$

which after an elementary calculation leads to

$$a_{ij} = [L(1-\alpha)]^{-1} \left(\frac{\delta_{ij}}{n_i} + \frac{1}{n_{m+1}} \right).$$

This is exactly the coefficient matrix of the Gaussian distribution (3.8), as claimed. In particular, this observation also shows that the approximation (3.8) coincides with that explicitly derived in [30] for the case $\alpha = 0$ and $m = 2$.

Let us now turn to the computation of the trace

$$\mathrm{tr}(\rho_L^q) = \sum_{L_1, \ldots, L_m} \lambda(L_1, \ldots, L_m)^q,$$

where $\lambda(L_1, \ldots, L_m)$ is given by equation (3.2) and the sum is over all non-negative integers $L_1, \ldots, L_m$ satisfying the inequalities (A.5). In the thermodynamic limit we can approximate $\lambda(L_1, \ldots, L_m)$ by the asymptotic formula (3.8), and the sum over $L_1, \ldots, L_m$ by an integral. Moreover, we can extend the domain of integration to the whole space, since the Gaussian distribution is negligible unless $L_a \sim Ln_a$ for all a. We thus obtain

$$\begin{aligned} \mathrm{tr}(\rho_L^q) &= [2\pi L(1-\alpha)]^{-mq/2} \prod_{a=1}^{m+1} n_a^{-q/2} \cdot \int_{\mathbb{R}^m} \exp\left(-\frac{\mathcal{E}(q^{1/2} x_1, \ldots, q^{1/2} x_m)}{2L(1-\alpha)} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &= q^{-\frac{m}{2}} [2\pi L(1-\alpha)]^{-mq/2} \prod_{a=1}^{m+1} n_a^{-q/2} \cdot \int_{\mathbb{R}^m} \exp\left(-\frac{\mathcal{E}(x_1, \ldots, x_m)}{2L(1-\alpha)} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_m \\ &= q^{-\frac{m}{2}} \left(2\pi(1-\alpha) L \prod_{a=1}^{m+1} n_a^{1/m} \right)^{\frac{m(1-q)}{2}}, \end{aligned} \tag{3.10}$$

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where we have used the fact that the RHS of (3.8) is normalized to 1.

Remark 4. An immediate consequence of the Schmidt decomposition is the fact that $\mathrm{tr}(\rho_L^q) = \mathrm{tr}(\rho_{N-L}^q)$, where ρ_{N-L} is the reduced density matrix of the last $N-L$ spins [2]. It follows from this equality that $\mathrm{tr}(\rho_L^q)$ should be symmetric under $L \to N-L$. This obviously holds for the RHS of equation (3.10), since

$$L(1-\alpha) \simeq \frac{L(N-L)}{N}.$$

Remark 5. The $\alpha \to 0$ limit of equation (3.10) when q is a positive integer can also be obtained using the replica trick and the techniques developed in [48–50] to analyze totally symmetric states. More precisely, the authors of the latter references derive an expression for $\mathrm{tr}(\rho_L^q)$ (with $q \in \mathbb{N}$) in the limit $N \to \infty$ with

$$\lim_{N\to\infty} \frac{N_a}{N} = n_a, \qquad L \text{ fixed}, \tag{3.11}$$

namely

$$\mathrm{tr}(\rho_L^q) = \int_{[0,2\pi]^{mq}} \left(\sum_{a=1}^{m+1} n_a \mathrm{e}^{\mathrm{i}(\theta_a^{k+1} - \theta_a^k)} \right)^L \prod_{k=1}^{q} \prod_{a=1}^{m} \frac{\mathrm{d}\theta_a^k}{2\pi},$$

where $\theta_a^{q+1} \equiv \theta_a^1$ and $\theta_{m+1}^k = 0$. Although this integral cannot be computed in closed form, its asymptotic behavior when $L \gg 1$ can be exactly determined, with the result

$$\mathrm{tr}(\rho_L^q) = q^{-\frac{m}{2}} \left(2\pi L \prod_{a=1}^{m+1} n_a^{1/m} \right)^{\frac{m(1-q)}{2}}. \tag{3.12}$$

This is indeed the $\alpha \to 0$ limit of equation (3.10), as expected, since (3.11) followed by the $L \to \infty$ limit is essentially equivalent to the thermodynamic limit (3.3) followed by the $\alpha \to 0$ limit. In any case, it should be noted that, although equation (3.12) can be obtained from (3.10) taking the $\alpha \to 0$ limit (at least for $q \in \mathbb{N}$), it is of course not possible to derive the latter equation from (3.12).

From equations (1.1) and (3.10) it immediately follows that in the large L limit the Rényi entropy is given by

$$R_q = \frac{m}{2} \frac{\log q}{q-1} + \frac{m}{2} \log \left(2\pi L(1-\alpha) \prod_{a=1}^{m+1} n_a^{1/m} \right). \tag{3.13}$$

Remarkably, R_q depends on q only through the first term, which is irrelevant in the thermodynamic limit $L \to \infty$. This fact, already noted by the authors of [50] in the $\alpha = 0$ case, shall prove important for determining whether the class of generalized LMG models (2.13) contains critical models. Taking the $q \to 1$ limit of equation (3.13) we deduce a similar formula for the von Neumann entropy, namely

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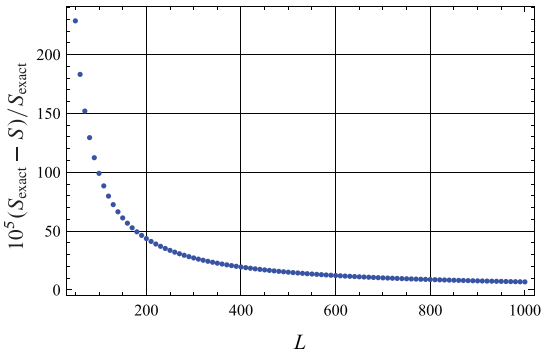

Figure 1. Relative error of the approximation (3.14) to the exact von Neumann entropy S_{exact} for $m = 2$, $h_1 = h_2 = 1/5$, $\alpha = 1/2$ as a function of $L \in [50, 1000]$ (in increments of 10).

$$S = \frac{m}{2} \log\left(2\pi \mathrm{e} L(1-\alpha) \prod_{a=1}^{m+1} n_a^{1/m} \right). \tag{3.14}$$

Equation (3.14) was first derived in [30] by extrapolation from the $\alpha = 0$ case (see also [16]). Note also that this equation is clearly consistent with the a priori bound (3.1) since $L \gg 1$ by hypothesis.

Equations (3.13)–(3.14) provide excellent approximations to the exact values of the Rényi and von Neumann entropies for even moderately large values of L, with a relative error steadily decreasing with L (see, e.g. figure 1 for the case $m = 2$, $h_1 = h_2 = 1/5$ and $\alpha = 1/2$). Remarkably, in this case the Rényi and von Neumann entropies merely differ by a (q-dependent) constant, namely

$$R_q = S + \frac{m}{2}\left(\frac{\log q}{q-1} - 1 \right).$$

Note also that in the su(2) case the von Neumann entropy (3.14) reduces to

$$S = \frac{1}{2} \log(2\pi \mathrm{e} L(1-\alpha) n_1 n_2) = \frac{1}{2} \log\left[\frac{\pi \mathrm{e}}{2} L(1-\alpha) \right] + \frac{1}{2} \log(1-h^2),$$

in agreement with the result of [16] and [17] for the isotropic LMG model.

As expected, the maximum value of the entropies (3.13)–(3.14) is obtained when $n_a = 1/(m+1)$ for all $a = 1, \ldots, m+1$, i.e. for $\mathbf{h} = 0$. On the other hand, the approximations (3.13)–(3.14) to the von Neumann and Rényi entropies tend to $-\infty$ when $\mathbf{h}$ approaches the faces of the simplex $\mathcal{H}$, since each of these faces is determined by the vanishing of one of the magnon densities n_a. This behavior had already been pointed out in [17] for the von Neumann entropy of the isotropic (su(2)) LMG model. It may seem surprising that the approximate von Neumann and Rényi entropies (3.13)–(3.14) become negative in a certain subset of $\mathcal{H}$. In fact, as we shall now discuss in more detail, when $L \gg 1$ the regions in which each of these approximations are negative (and, therefore, break down) are negligibly small.

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In order to substantiate the previous claim, we shall estimate the distance r_0 of a point $\mathbf{h}_0$ lying on one of the $(m-1)$-dimensional faces of the simplex $\mathcal{H}$ to the positive entropy region, or equivalently to the zero entropy hypersurface. For simplicity, we shall deal with the generic situation in which $\mathbf{h}_0$ belongs to the interior of the face. We shall prove that for both entropies under consideration $r_0 \sim L^{-m}$. To this end, suppose to begin with that $\mathbf{h}_0$ lies in the interior of the face where $n_a = 0$, with $1 \leqslant a \leqslant m$. In this case we can approximate the remaining densities n_b by their values $n_b^0 = h_{0b} + (1-h_0)/(m+1)$ at the point $\mathbf{h}_0 \equiv (h_{01}, \ldots, h_{0m})$ (with $h_0 \equiv \sum_{a=1}^{m+1} h_{0a}$), so that (for instance) the von Neumann entropy satisfies

$$S \simeq \frac{m}{2} \log\left[2\pi \mathrm{e} L(1-\alpha)\right] + \frac{1}{2} \sum_{1 \leqslant b \neq a \leqslant m+1} \log n_b^0 + \frac{1}{2} \log n_a.$$

Equating the RHS of this equation to zero and solving for n_a we obtain

$$n_a \equiv \frac{1}{m+1}\left(m h_a - \sum_{1 \leqslant b \neq a \leqslant m+1} h_b + 1\right) = \frac{[2\pi \mathrm{e} L(1-\alpha)]^{-m}}{\prod\limits_{1 \leqslant b \neq a \leqslant m+1} n_b^0}.$$

Thus near $\mathbf{h}_0$ the $S = 0$ hypersurface can be approximated by the above hyperplane, which is parallel to the hyperplane $n_a = 0$ containing the face under consideration. Computing the distance between these two hyperplanes we immediately find the following approximate formula for r_0 in the case of the von Neumann entropy:

$$r_0 \simeq \frac{m+1}{\sqrt{m^2+m-1}} \frac{[2\pi \mathrm{e} L(1-\alpha)]^{-m}}{\prod\limits_{1 \leqslant b \neq a \leqslant m+1} n_b^0} \sim L^{-m}. \tag{3.15}$$

When $\mathbf{h}_0$ lies on the face $n_{m+1} = 0$ a totally analogous calculation leads to the slightly simpler result

$$r_0 \simeq \frac{m+1}{\sqrt{m}} \frac{[2\pi \mathrm{e} L(1-\alpha)]^{-m}}{\prod\limits_{1 \leqslant b \leqslant m} n_b^0} \sim L^{-m}. \tag{3.16}$$

The computation of r_0 for the Rényi entropy proceeds along the same lines, with the result

$$r_0 \simeq q^{\frac{m}{1-q}}\, \mathrm{e}^m\, (r_0)_{\mathrm{S}}, \tag{3.17}$$

where $(r_0)_{\mathrm{S}}$ is the approximate value of r_0 for the von Neumann entropy given by equations (3.15) or (3.16). Interestingly, the RHS of equation (3.17) is less (resp. greater) than $(r_0)_{\mathrm{S}}$ for $0 < q < 1$ (resp. $q > 1$). Note, finally, that the assumption that $\mathbf{h}_0$ belongs to the *interior* of the face is essential for the validity of equations (3.15)–(3.17). More generally, it can be shown that the corresponding value of r_0 for $\mathbf{h}_0$ lying on (the interior of) an $(m-k)$-dimensional face of $\mathcal{H}$ (with $k = 1, \ldots, m$) behaves as $L^{-m/k}$. In other words, even in the most unfavorable case $k = m$ (i.e. when $\mathbf{h}_0$ is one of the vertices of the simplex $\mathcal{H}$) the distance of $\mathbf{h}_0$ to the positive entropy region is $\mathrm{O}(L^{-1})$.

By equations (3.13)–(3.14), when $L \to \infty$ the von Neumann and Rényi ground state entanglement entropies satisfy

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$$S = R_q = \frac{m}{2}\log L + \mathrm{O}(1). \tag{3.18}$$

As remarked above, this behavior of the von Neumann entropy is characteristic of quantum critical (gapless) one-dimensional lattice systems with short-range interactions. More generally, when $d > 1$ the von Neumann entanglement entropy of a d-dimensional critical system featuring short-range interactions is expected to scale as $L^{d-1}\log L$ (for fermionic systems) or L^{d-1} (for bosonic ones) when $L \gg 1$, L being the linear size of the system. On the other hand, for a non-critical (gapped) system with short-range interactions the von Neumann entropy should grow only as L^{d-1}. This so-called *area law* [9] has been verified for a wide range of quantum systems, such as the XX and XY models [6, 7], the Heisenberg (XYZ) spin chain [51, 52], the original (su(2), not necessarily isotropic) LMG model [17], translation-invariant (quadratic) fermionic systems in arbitrary dimension [53], and certain two-dimensional bosonic and fermionic systems [37, 54], to name only a few. On the other hand, for models with long-range interactions it is widely accepted that the area law need not hold, since in general the range of the interaction tends to increase the entropy [9]. This statement should be taken with some caution, since the entanglement entropy is ultimately a property of the state, and two models featuring short-range and long-range interactions may have the same ground state [55]. The logarithmic growth (3.18) of the ground state entanglement entropy of the gLMG models (2.13) does *not* indicate, however, that this class contains critical models. Indeed, the Rényi entropy of these models scales as $a\log L$ with a independent of q, while for a two-dimensional CFT the coefficient a should instead be proportional to $1 + q^{-1}$.

It has been shown in [37] that in some low-dimensional quantum systems whose entanglement (von Neumann) entropy follows the area law the Tsallis entropy, defined by

$$T_q = \frac{\mathrm{tr}(\rho_L^q) - 1}{1 - q}, \tag{3.19}$$

becomes extensive (i.e. scales as L^d, where d is the number of space dimensions and L is a characteristic length) for a suitable value of the positive parameter q. This entropy, which plays an important role in the study of strongly correlated classical systems, has also been extensively applied in other fields ranging from natural and social sciences to linguistics and economics (see the online document http://tsallis.cat.cbpf.br/TEMUCO.pdf for an updated bibliography). Since the Tsallis and Rényi entropies are obviously related by

$$R_q = \frac{\log\left[1 + (1 - q)\,T_q\right]}{1 - q}, \tag{3.20}$$

from equation (3.13) we immediately obtain the following explicit formula for the Tsallis entanglement entropy of the ground state (2.7) of the gLMG model (2.13):

$$T_q = \frac{1}{1 - q}\left(q^{-\frac{m}{2}}\left[2\pi L(1 - \alpha)\prod_{a=1}^{m+1} n_a^{1/m}\right]^{\frac{m(1-q)}{2}} - 1\right). \tag{3.21}$$

Thus for $L \gg 1$ the Tsallis entropy scales as a power law, namely

$$T_q = \frac{L^{\frac{m(1-q)}{2}}}{(1-q)q^{\frac{m}{2}}}\left[2\pi(1-\alpha)\prod_{a=1}^{m+1} n_a^{1/m}\right]^{\frac{m(1-q)}{2}} + \mathrm{O}(1). \tag{3.22}$$

The dominant term of T_q is linear in L if $q = 1 - 2/m$, which requires that $m \geqslant 3$ due to the condition $q > 0$. For this critical value of q the Tsallis entropy is given by

$$T_{1-\frac{2}{m}} = \frac{\pi m L(1-\alpha)}{(1-\frac{2}{m})^{\frac{m}{2}}}\prod_{a=1}^{m+1} n_a^{1/m} - \frac{m}{2}, \tag{3.23}$$

so that the entropy per particle in the thermodynamic limit reads

$$\lim_{L\to\infty}\frac{T_{1-\frac{2}{m}}}{L} = \frac{\pi m(1-\alpha)}{(1-\frac{2}{m})^{\frac{m}{2}}}\prod_{a=1}^{m+1} n_a^{1/m}. \tag{3.24}$$

As we have just seen, in the su(2) and su(3) cases the Tsallis entanglement entropy is not extensive for *any* value of the parameter q. It would therefore be of interest, in this context, to find a generalized entropy (like, e.g. one of the group entropies studied in [56] and [57]) which is extensive for the ground state of the gLMG model (2.13) with $m = 1, 2$.

4. Ground state phase diagram

In the previous sections we have studied the entanglement properties of the ground state of the gLMG model (2.13) when the magnetic field strength $\mathbf{h}$ lies in the interior of the simplex $\mathcal{H}$ given by equation (2.16). The aim of this section is to extend the previous results outside this region, determining how the behavior of the ground state and its entanglement entropy vary with $\mathbf{h}$.

To this end, note first of all that by equation (2.13) the magnon densities n_a in the ground state must minimize the function

$$\varepsilon(n_1, \ldots, n_m) = \sum_{a=1}^{m} c_a(n_a - n_{m+1} - h_a)^2, \tag{4.1}$$

with $n_{m+1} = 1 - \sum_{a=1}^{m} n_a$, in the simplex $\mathcal{N} \subset \mathbb{R}^m$ defined by

$$0 \leqslant n_a \leqslant 1, \qquad \sum_{a=1}^{m} n_a \leqslant 1.$$

Thus the condition for the ground state to have well-defined magnon densities is that ε has a *unique* minimum in $\mathcal{N}$. In fact, from the very definition of the set $\mathcal{H}$ it follows that for $\mathbf{h} \in \mathcal{H}$ the unique minimum of the energy function (4.1) over $\mathbb{R}^m$, given

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by equations (2.14), lies in $\mathcal{N}$. More precisely, when $\mathbf{h}$ belongs to the interior of $\mathcal{H}$ the absolute minimum of ε lies in the interior of $\mathcal{N}$, and therefore $0 < n_a < 1$ for all $a = 1, \ldots, m+1$. Thus in this case the ground state is not only entangled, but contains magnons of each of the $m+1$ types $|a\rangle$. On the other hand, when $\mathbf{h}$ belongs to the boundary of $\mathcal{H}$ the unique global minimum of ε, still given by equation (2.15), now lies on the boundary of $\mathcal{N}$. In particular, at least one of the magnon densities must vanish in this case, so that the ground state is entangled but does not have full magnon content. We shall be mainly interested in this section in the case in which $\mathbf{h}$ lies in the exterior of $\mathcal{H}$, so that the minimum of ε in $\mathcal{N}$ is necessarily attained on its boundary $\partial\mathcal{N}$. This minimum is therefore no longer given by the simple equations (2.14), but must be computed by examining the behavior of ε on $\partial\mathcal{N}$. We shall prove at the end of this section that the minimum of ε on $\partial\mathcal{N}$ is unique for all values of the magnetic field strength $\mathbf{h}$. Thus the ground state of the generalized LMG model (2.13) is always unique, although its magnon content is given by equation (2.15) only for $\mathbf{h} \in \mathcal{H}$.

The boundary of the simplex $\mathcal{N}$ is the union of the $(m-k)$-dimensional sets (or $(m-k)$- *faces*) $\partial\mathcal{N}_{a_1,\ldots,a_k}$ determined by the equations

$$n_{a_1} = \cdots = n_{a_k} = 0,$$

where $k = 1, \ldots, m$ and $1 \leqslant a_1 < \cdots < a_k \leqslant m+1$. Let us suppose, therefore, that the unique minimum of ε in $\partial\mathcal{N}$ is attained on a certain $(m-k)$-face $\partial\mathcal{N}_{a_1,\ldots,a_k}$. If $k = m$ (i.e. if the minimum of ε on $\mathcal{N}$ is attained at one of its vertices) then one of the magnon densities is necessarily equal to 1 and the remaining ones vanish, so that the ground state is not entangled. On the other hand, if $1 \leqslant k < m$ then the ground state is entangled but has not full magnon content, since it only contains magnons of $m-k+1 < m+1$ types. In other words, we expect that in general the ground state can be in exactly one of $m+1$ possible 'phases', characterized by the vanishing of $0 \leqslant k \leqslant m$ magnon densities. Note that we have included the $k = 0$ case, in which $n_a > 0$ for all $a = 1, \ldots, m+1$, which holds when $\mathbf{h}$ lies in the interior of $\mathcal{H}$.

We shall now describe the essential properties of the ground state when $\mathbf{h}$ belongs to the exterior of $\mathcal{H}$. As we have just seen, in this case the minimum of the energy function ε is attained on one of the $(m-k)$-faces $\partial\mathcal{N}_{a_1,\ldots,a_k}$ of $\mathcal{N}$, so that in the ground state $n_{a_1} = \cdots = n_{a_k} = 0$ and $0 < n_{a_{k+1}}, \ldots, n_{a_{m+1}} < 1$, where

$$\{a_{k+1}, \ldots, a_{m+1}\} = \{1, \ldots, m+1\} \setminus \{a_1, \ldots, a_k\}.$$

Thus in the thermodynamic limit (3.3) we have

$$N_{a_1} = \cdots = N_{a_k} = L_{a_1} = \cdots = L_{a_k} = 0,$$

and the eigenvalues of the reduced density matrix ρ_L are given by

$$\begin{aligned}\lambda(L_{a_{k+1}}, \ldots, L_{a_m}) &= \frac{L!}{\prod_{j=k+1}^{m+1} L_{a_j}!} \frac{(N-L)!}{\prod_{j=k+1}^{m+1} (N_{a_j} - L_{a_j})!} \frac{\prod_{j=k+1}^{m+1} N_{a_j}!}{N!} \\ &= \binom{N}{L}^{-1} \prod_{j=k+1}^{m+1} \binom{N_{a_j}}{L_{a_j}},\end{aligned}$$

with $L_{a_{m+1}} = L - \sum_{j=k+1}^{m} L_{a_j}$. The distribution of the $m-k$ independent variables L_{a_j} ($k+1 \leqslant j \leqslant m$) is therefore the analogue of equation (3.2), with m replaced by $m-k$ and $n_1, \ldots, n_{m+1}$ by $n_{a_{k+1}}, \ldots, n_{a_{m+1}}$. Hence equations (3.13), (3.14) and (3.21) for the Rényi, von Neumann and Tsallis entropies still hold in this case, provided that we perform the above replacements; in particular, the von Neumann entropy is given by

$$S = \frac{m-k}{2} \log\left(2\pi \mathrm{e} L(1-\alpha) \prod_{j=k+1}^{m+1} n_{a_j}^{1/(m-k)} \right). \tag{4.2}$$

In the derivation of the latter equation we have tacitly assumed that $k < m$. In fact, when $k = m$ the ground state is not entangled, and hence $S = 0$ in this case.

By equation (4.2), the asymptotic behavior of the von Neumann entropy in the ground state phase with k vanishing magnon densities is given by

$$S = \frac{m-k}{2} \log L + \mathrm{O}(1).$$

As discussed in the previous section, this result does not imply that in this phase there should be critical gLMG models. Indeed, the Rényi entropy R_q also scales as $a \log L$ when $L \gg 1$, where $a = (m-k)/2$ is independent of the parameter q. As we know, this behavior is inconsistent with the characteristic scaling of the Rényi entropy of a one-dimensional CFT, for which $R_q = a(q) \log L$ with $a(q)$ proportional to $1 + q^{-1}$.

Remark 6. By the discussion preceding equation (4.2), the Tsallis entropy in the phase with k vanishing magnon densities scales as $L^{(m-k)(1-q)/2}$, and is therefore extensive for $q = 1 - 2/(m-k)$ provided that $m - k \geqslant 3$. Hence this entropy is not extensive for *any* value of q in the phases with $m-1$ and $m-2$ vanishing magnon densities.

As a concrete example of the previous general statements, we shall next discuss in detail the su(3) case ($m = 2$) with the symmetric choice $c_1 = c_2 = C$. Taking (without loss of generality) $C = 1$, the energy function is simply

$$\varepsilon(\mathbf{n}) = (\mathbf{x}(\mathbf{n}) - \mathbf{h})^2, \tag{4.3}$$

where $\mathbf{x}(\mathbf{n}) \equiv \sum_{s=1}^{3} n_s \boldsymbol{\mu}_s$, $\mathbf{n} = (n_1, n_2) \in \mathcal{N}$ and $n_3 = 1 - n_1 - n_2$. When $\mathbf{n}$ ranges over $\mathcal{N}$, the point $\mathbf{x}(\mathbf{n})$ varies over the triangle $\mathcal{H}$ in a one-to-one fashion, with $\mathbf{n} \in \partial\mathcal{N}$ if and only if $\mathbf{x}(\mathbf{n}) \in \partial\mathcal{H}$. Hence the minimum of $\varepsilon(\mathbf{n})$ is simply the distance of the fixed vector $\mathbf{h}$ to the triangle $\mathcal{H}$. Moreover, since this triangle is convex, the minimum distance of $\mathbf{h}$ to $\mathcal{H}$ is attained at a unique point $\mathbf{x}(\mathbf{n}_0)$ in $\mathcal{H}$.

The problem of minimizing the energy function ε has a very simple geometric solution. Indeed, when $\mathbf{h} \in \mathring{\mathcal{H}}$ the point $\mathbf{x}(\mathbf{n}_0)$ closest to $\mathbf{h}$ is obviously $\mathbf{h}$ itself, so that the magnon density vector $\mathbf{n}_0 = \mathbf{x}^{-1}(\mathbf{h})$ is given by equation (2.15). Suppose, on the other hand, that $\mathbf{h} \notin \mathring{\mathcal{H}}$ (including the limiting case $\mathbf{h} \in \partial\mathcal{H}$). Let us denote by L_{bc} the side of the triangle $\mathcal{H}$ with vertices $\boldsymbol{\mu}_b$ and $\boldsymbol{\mu}_c$. This side is parametrized by a magnon density $\mathbf{n}$ with $n_a = 0$, where $\{a, b, c\} = \{1, 2, 3\}$. Likewise, we shall denote by R_{bc} the half-strip bounded by the side L_{bc} and the two straight lines perpendicular to it through the vertices $\boldsymbol{\mu}_b$ and $\boldsymbol{\mu}_c$ which does not contain the opposite vertex $\boldsymbol{\mu}_a$, with the two limiting half-lines removed (see figure 2). By construction, when $\mathbf{h} \in R_{bc}$ the point in $\mathcal{H}$ closest to

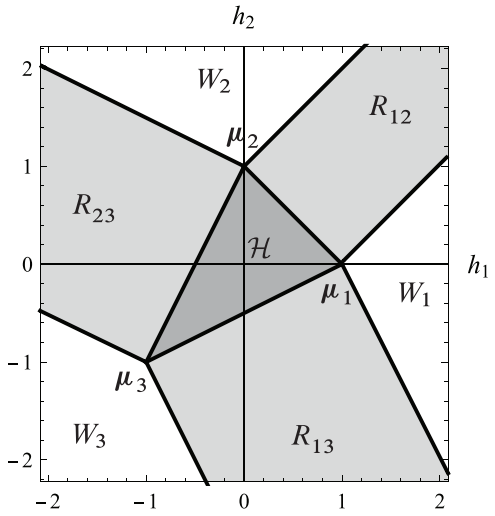

Figure 2. Triangle $\mathcal{H}$, half-strips R_{ab}, wedges W_a and vertices $\boldsymbol{\mu}_a$ in the (h_1, h_2)-plane.

$\mathbf{h}$ lies on the interior of the side $L_{bc} \subset \partial\mathcal{H}$, so that in this case the corresponding density $\mathbf{n}_0$ belongs to the side $n_a = 0$ of $\mathcal{N}$. The value of $\mathbf{n}_0 \equiv (n_{01}, n_{02})$ can be easily computed by minimizing

$$\varepsilon\big|_{n_a=0} = (n_b(\boldsymbol{\mu}_b - \boldsymbol{\mu}_c) + \boldsymbol{\mu}_c - \mathbf{h})^2 \tag{4.4}$$

with respect to n_b (assuming, without loss of generality, that $b \in \{1, 2\}$). In this way we easily obtain

$$n_{0b} = \frac{(\mathbf{h} - \boldsymbol{\mu}_c) \cdot (\boldsymbol{\mu}_b - \boldsymbol{\mu}_c)}{(\boldsymbol{\mu}_b - \boldsymbol{\mu}_c)^2}, \tag{4.5}$$

and of course $n_{0c} = 1 - n_{0b}$. Note that in this case, although $n_{0a} = 0$, the ground state is still entangled, since $0 < n_{0b}, n_{0c} < 1$. On the other hand, let W_a denote the closed wedge with vertex $\boldsymbol{\mu}_a$ limited by the half-strips R_{ab} and R_{ac} (see figure 2). Obviously, if $\mathbf{h}$ lies in W_a the point of the triangle $\mathcal{H}$ closest to $\mathbf{h}$ is the vertex $\boldsymbol{\mu}_a$, whose corresponding magnon densities are $n_a = 1$ and $n_b = n_c = 0$. In summary, we have shown that the 'phase diagram' of the ground state of the su(3) gLMG model (2.13) (with $c_1 = c_2$ for all a) is as follows:

(i) In the interior $\mathring{\mathcal{H}}$ of the triangle $\mathcal{H}$, the ground state is a symmetric state containing all three types of magnons $|a\rangle$ (with $a = 1, 2, 3$).

(ii) In each of the sets R_{bc} the ground state is still entangled, but contains only magnons of the two types $|b\rangle$ and $|c\rangle$.

(iii) In the wedges W_a, the ground state consists of magnons of type $|a\rangle$ only, and is therefore not entangled.

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From the previous remark, the general formula (4.2) and equations (2.15) and (4.5), it follows that when $L \gg 1$ the von Neumann entanglement entropy as a function of the su(3) magnetic field $\mathbf{h}$ is given by

$$S = \begin{cases} S_0 - \dfrac{3}{2}\log 3 + \dfrac{1}{2}\left(\log(1+2h_1-h_2) + \log(1-h_1+2h_2)\right. & \mathbf{h} \in \mathring{\mathcal{H}} \\ \qquad\qquad \left. + \log(1-h_1-h_2)\right), & \\ \dfrac{1}{2}(S_0 + \log\left[1-(h_1-h_2)^2\right] - 2\log 2), & \mathbf{h} \in R_{12} \\ \dfrac{1}{2}(S_0 - 2\log 5 + \log(3+2h_1+h_2) + \log(2-2h_1-h_2)), & \mathbf{h} \in R_{13} \\ \dfrac{1}{2}(S_0 - 2\log 5 + \log(3+h_1+2h_2) + \log(2-h_1-2h_2)), & \mathbf{h} \in R_{23} \\ 0, & \mathbf{h} \in W_1 \cup W_2 \cup W_3, \end{cases}$$

with $S_0 \equiv \log\left[2\pi \mathrm{e} L(1-\alpha)\right]$. In figure 3 we have plotted this entropy as a function of the magnetic field $\mathbf{h}$ in the range $|h_a| \leqslant 2$ for $L(1-\alpha) = 1000$. As mentioned in the previous section, the latter approximation to the von Neumann entropy tends to $-\infty$ when $\mathbf{h}$ approaches a side L_{ab} of the triangle $\mathcal{H}$ from its interior. Note, however, that it has a finite limit when this side is approached from the corresponding half-strip R_{ab}. Similarly, $S \to -\infty$ when $\mathbf{h}$ approaches one of the straight lines limiting a half-strip R_{ab} from its interior, but has a finite limit when this straight line is approached from the corresponding wedge W_a or W_b (except at the vertices $\boldsymbol{\mu}_a$ and $\boldsymbol{\mu}_b$).

Remark 7. A formula similar to the previous equation for the von Neumann entropy can be derived without difficulty for the Rényi and the Tsallis entropies. In fact, comparing equations (3.13)–(3.14) it is clear that the only difference between the von Neumann and the Rényi entropies is the constant term $(m-k)[(\log q)/(q-1)-1]/2$ in the phase with k vanishing magnon densities (i.e. $k = 0$, 1, 2 respectively for $\mathbf{h}$ belonging to $\mathring{\mathcal{H}}$, R_{ab} and W_a). Thus the von Neumann and Rényi entropies are essentially equivalent for the models under consideration.

In the general case (i.e. for $m > 2$ and arbitrary positive values of the parameters c_a), the analysis is very similar. Indeed, in this case the energy function $\varepsilon(\mathbf{n})$ can be written as

$$\varepsilon(\mathbf{n}) = \|\mathbf{x}(\mathbf{n}) - \mathbf{h}\|^2,$$

where

$$\mathbf{x}(\mathbf{n}) = \sum_{s=1}^{m+1} n_s \boldsymbol{\mu}_s, \qquad \mathbf{n} = (n_1, \ldots, n_m), \qquad n_{m+1} = 1 - \sum_{a=1}^{m} n_a,$$

and the norm $\|\cdot\|$ is defined by $\|\mathbf{y}\|^2 \equiv \sum_{a=1}^{m} c_a y_a^2$. As before, when the vector $\mathbf{n}$ varies over the simplex $\mathcal{N}$ the point $\mathbf{x}(\mathbf{n})$ parametrizes the simplex $\mathcal{H}$ in a bijective way, with $\mathbf{n} \in \partial\mathcal{N}$ if and only if $\mathbf{x}(\mathbf{n}) \in \partial\mathcal{H}$. Thus the problem of minimizing $\varepsilon(\mathbf{n})$ is again equivalent to finding the distance (with respect to the norm $\|\cdot\|$) of the fixed vector $\mathbf{h}$ to the simplex $\mathcal{H}$. Since this simplex is a convex polytope, there is a unique point in $\mathcal{H}$ closest to $\mathbf{h}$ for all values of $\mathbf{h} \in \mathbb{R}^m$. This proves that the magnon densities are

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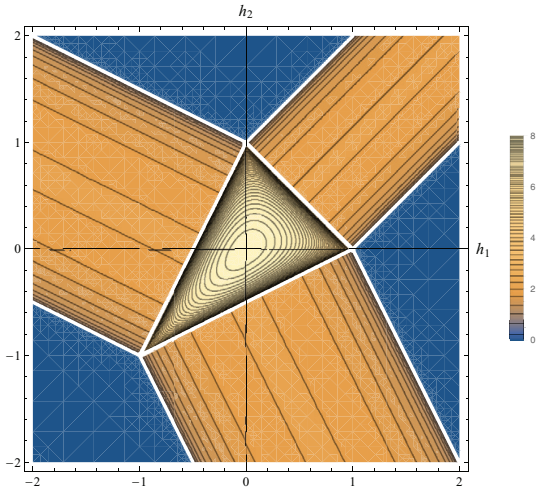

Figure 3. von Neumann entropy of the su(3) gLMG model (2.13) with $c_1 = c_2$ and $L(1-\alpha) = 1000$ as a function of the magnetic field $\mathbf{h} = (h_1, h_2)$.

uniquely determined by the magnetic field strength $\mathbf{h}$, so that the ground state of the gLMG model (2.13) is always unique. More precisely, when $\mathbf{h} \in \mathring{\mathcal{H}}$, the distance of $\mathbf{h}$ to $\mathcal{H}$ is obviously zero and the corresponding density vector $\mathbf{n}_0$ is again determined by the condition $\mathbf{h} = \mathbf{x}(\mathbf{n}_0)$, i.e. by equation (2.15). On the other hand, when $\mathbf{h}$ lies outside $\mathring{\mathcal{H}}$ (in particular, if $\mathbf{h} \in \partial\mathcal{H}$), it is clear from geometric considerations that the minimum distance of $\mathbf{h}$ to the simplex $\mathcal{H}$ is attained at a point $\mathbf{x}(\mathbf{n}_0) \in \partial\mathcal{H}$, so that the corresponding magnon density $\mathbf{n}_0$ lies in $\partial\mathcal{N}$. Hence in the latter case the system is in one of the m phases characterized by the vanishing of $k > 0$ magnon densities.

5. Conclusions and outlook

In this work we have introduced a large class of su$(m+1)$ spin models with long-range interactions possessing a symmetric non-degenerate ground state with well-defined magnon numbers. Our models provide a natural multiparameter generalization of the well-known spin 1/2 isotropic Lipkin–Meshkov–Glick model, featuring non-constant interactions and su$(m+1)$ spin. They are also closely related to Haldane–Shastry type chains, which can be formally obtained from them through specific realizations of the couplings h_{ij} dropping the magnetic field terms.

One of the main results of our work is the detailed derivation of the asymptotic behavior of the eigenvalues of the reduced density matrix of a block of L spins when the system is in its ground state, for arbitrary values of m and $\alpha = \lim_{N\to\infty} L/N$. This makes it possible to compute the von Neumann and Rényi entropies in closed form when $L \gg 1$, and to derive their asymptotic behavior when $L \to \infty$. A notable outcome of our analysis is that both of these entropies scale as $\frac{1}{2}(m-k)\log L$ in the latter limit,

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where k is the number of vanishing magnon densities in the ground state. In particular, from the behavior of the Rényi entropy it follows that the class of generalized su$(m+1)$ LMG models contains no critical models. We have also computed the Tsallis entropy, showing that it can be made extensive when the number of 'effective' internal degrees of freedom $m-k+1$ is greater than 3. Finally, we have completely determined the different phases of the ground state in terms of the su$(m+1)$ magnetic field strength $\mathbf{h}$, and shown that they are related in a simple geometric way to the weights of the fundamental representation of su$(m+1)$.

Our results open up several natural directions for further research and a number of related problems. One such problem is the determination in closed form of the full spectrum of suitable models of the class introduced in this paper, particularly for su$(m+1)$ spin with $m>1$. In fact, the integrability properties of the HS-type chains suggest the possibility of exploring the existence of integrable generalizations thereof with a non-vanishing magnetic field term of the form considered in this work. At the same time, it could also be of interest to extend the analysis of the entanglement entropy performed in this paper to different entropic functionals. Indeed, due to the presence of several parameters in the gLMG Hamiltonian, multiparametric entropies [56, 57] could play an important role in the classification of the possible thermodynamic regimes admitted by the system when one varies the values of its parameters. Finally, the fact that the von Neumann entanglement entropy of generalized LMG models is proportional to $\log L$ in certain regions of parameter space, though as we have seen does not imply the existence of critical models, suggests that these regions may nevertheless contain models with interesting non-generic properties worth investigating. This is certainly true in the $m=1$ case, since (for instance) the isotropic LMG model is gapless precisely in the interval $|h|<1$ for which the von Neumann entropy scales as $\log L$ [40].

Acknowledgments

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Appendix A. Ground-state reduced density matrix for a block of spins

In this appendix we shall compute from first principles the reduced density matrix ρ_L of a block of L spins of the su$(m+1)$ gLMG chain (2.13) when the system is in its ground state (2.7), with magnon densities $n_a=N_a/N$ determined (in the thermodynamic limit) by equation (2.15). This result, first obtained in [30], shall be used in section 3 to evaluate several standard bipartite entanglement entropies in the thermodynamic limit.

More precisely, we need to compute

$$\rho_L=\mathrm{tr}_{N-L}\rho\equiv\mathrm{tr}_{N-L}|\psi(N_1,\ldots,N_{m+1})\rangle\langle\psi(N_1,\ldots,N_{m+1})|, \tag{A.1}$$

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where tr_{N-L} is the trace over the degrees of freedom of the remaining $N-L$ spins. Since the ground state $|\psi(N_1,\ldots,N_{m+1})\rangle$ is invariant under permutations, the result is obviously independent of the specific positions of the L spins considered. We shall therefore assume in what follows that the two blocks under consideration consist of the first L and the last $N-L$ spins. In order to evaluate the RHS of equation (A.1), it is convenient to label the states of the canonical spin basis $|s_1,\ldots,s_N\rangle$ by the positions of its magnons. More explicitly, we shall use the notation $|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle$ to denote a state whose type a magnons (with $a=1,\ldots,m$) are located in the positions specified by the N_a components of the ordered multi-index $\mathbf{i}_a$, while those of type $m+1$ lie in the remaining positions. Note, in particular, that no two multi-indices $\mathbf{i}_a$ and $\mathbf{i}_b$ with $a\neq b$ can have any common components, which we shall denote by $\mathbf{i}_a\cap\mathbf{i}_b=\varnothing$. For instance, with this notation the su(4) basis state $|3,1,1,4,1,3,4\rangle$ will be denoted by $|(2,3,5),(),(1,6)\rangle$.

Using the above notation, the ground state (2.7) can be written as

$$|\psi(N_1,\ldots,N_{m+1})\rangle=\left(\frac{N!}{\prod_{a=1}^{m+1}N_a!}\right)^{-\frac{1}{2}}\sum_{\mathbf{i}_1,\ldots,\mathbf{i}_m}|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle, \tag{A.2}$$

where the sum is over all ordered multi-indices $\mathbf{i}_a\in\{1,\ldots,N\}^{N_a}$ such that $\mathbf{i}_a\cap\mathbf{i}_b=\varnothing$ for $a\neq b$. We thus have

$$\rho=\frac{\prod_{a=1}^{m+1}N_a!}{N!}\sum_{\substack{\mathbf{i}_1,\ldots,\mathbf{i}_m\\ \mathbf{j}_1,\ldots,\mathbf{j}_m}}|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle\langle\mathbf{j}_1,\ldots,\mathbf{j}_m|,$$

where the sum is again over all ordered multi-indices $\mathbf{i}_a$, $\mathbf{j}_a$ satisfying the above condition. Thus, in order to evaluate ρ_L we need only compute $\mathrm{tr}_{N-L}|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle\langle\mathbf{j}_1,\ldots,\mathbf{j}_m|$. To this end, we decompose each multi-index $\mathbf{i}_a$ as

$$\mathbf{i}_a=(\mathbf{i}_a^L,\mathbf{i}_a^{N-L}),$$

where the components of each $\mathbf{i}_a^L$ range from 1 to L and those of $\mathbf{i}_a^{N-L}$ from $L+1$ to N, and similarly for $\mathbf{j}_a$. It is then straightforward to show that

$$\mathrm{tr}_{N-L}|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle\langle\mathbf{j}_1,\ldots,\mathbf{j}_m|=|\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L\rangle\langle\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L|\prod_{a=1}^{m}\delta_{\mathbf{i}_a^{N-L},\mathbf{j}_a^{N-L}}. \tag{A.3}$$

Indeed,

$$\langle s_{L+1},\ldots,s_N|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle\langle\mathbf{j}_1,\ldots,\mathbf{j}_m|s_{L+1},\ldots,s_N\rangle=0$$

unless the last $N-L$ spin components of the basis states represented by $|\mathbf{i}_1,\ldots,\mathbf{i}_m\rangle$ and $|\mathbf{j}_1,\ldots,\mathbf{j}_m\rangle$ are both equal to $s_{L+1},\ldots,s_N$, which accounts for the product of Kronecker deltas in equation (A.3). Moreover, when $\mathbf{i}_a^{N-L}=\mathbf{j}_a^{N-L}$ for all $a=1,\ldots,m$ the only state of the canonical basis of the Hilbert space of the last $N-L$ spins that can contribute to the trace is $|\mathbf{i}_1^{N-L},\ldots,\mathbf{i}_m^{N-L}\rangle=|\mathbf{j}_1^{N-L},\ldots,\mathbf{j}_m^{N-L}\rangle$, which immediately yields equation (A.3).

Using equation (A.3) it is straightforward to obtain the expression

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$$\rho_L = \frac{\prod_{a=1}^{m+1} N_a!}{N!} \sum_{\substack{\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L,\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L,\mathbf{i}_1^{N-L},\ldots,\mathbf{i}_m^{N-L} \\ |\mathbf{i}_a^L|=|\mathbf{j}_a^L|=N_a-|\mathbf{i}_a^{N-L}|}} |\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L\rangle\langle\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L|, \tag{A.4}$$

where $|\mathbf{i}|$ denotes the number of components of the multi-index $\mathbf{i}$. In order to evaluate the latter sum, we introduce the notation $L_a = |\mathbf{i}_a^L|$, $a = 1,\ldots,m$, where the magnon numbers $L_1,\ldots,L_m$ satisfy the obvious inequalities

$$0 \leqslant L_a \leqslant N_a, \qquad \sum_{a=1}^{m} L_a \leqslant L, \qquad \sum_{a=1}^{m} (N_a - L_a) \leqslant N - L. \tag{A.5}$$

Equation (A.4) can then be written as

$$\rho_L = \frac{\prod_{a=1}^{m+1} N_a!}{N!} \sum_{L_1,\ldots,L_m} \sum_{\substack{\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L,\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L \\ |\mathbf{i}_a^L|=|\mathbf{j}_a^L|=L_a}} \sum_{\substack{\mathbf{i}_1^{N-L},\ldots,\mathbf{i}_m^{N-L} \\ |\mathbf{i}_a^{N-L}|=N_a-L_a}} |\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L\rangle\langle\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L|, \tag{A.6}$$

where the outermost sum is over the range specified by equation (A.5). The sum over the multi-indices $\mathbf{i}_a^{N-L}$ is clearly the number of different $(N-L)$-particles states of the form $|\mathbf{i}_1^{N-L},\ldots,\mathbf{i}_m^{N-L}\rangle$ with a fixed number $N_a - L_a$ of type a magnons ($1 \leqslant a \leqslant m+1$), namely the combinatorial number

$$\frac{(N-L)!}{\prod_{a=1}^{m+1}(N_a - L_a)!}$$

with $L_{m+1} \equiv L - \sum_{a=1}^{m} L_a$. Thus equation (A.6) reduces to

$$\rho_L = \frac{\prod_{a=1}^{m+1} N_a!}{N!} \sum_{L_1,\ldots,L_m} \frac{(N-L)!}{\prod_{a=1}^{m+1}(N_a - L_a)!} \sum_{\substack{\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L \\ |\mathbf{i}_a^L|=L_a}} |\mathbf{i}_1^L,\ldots,\mathbf{i}_m^L\rangle \cdot \sum_{\substack{\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L \\ |\mathbf{j}_a^L|=L_a}} \langle\mathbf{j}_1^L,\ldots,\mathbf{j}_m^L|.$$

Using equation (A.2) with N and N_a respectively replaced by L and L_a we finally arrive at the following explicit formula for the reduced density matrix ρ_L:

$$\rho_L = \sum_{L_1,\ldots,L_m} \lambda(L_1,\ldots,L_m)|\psi(L_1,\ldots,L_{m+1})\rangle\langle\psi(L_1,\ldots,L_{m+1})|, \tag{A.7}$$

where the summation range is again given by equation (A.5) and

$$\lambda(L_1,\ldots,L_m) = \frac{L!}{\prod_{a=1}^{m+1} L_a!} \frac{(N-L)!}{\prod_{a=1}^{m+1}(N_a - L_a)!} \frac{\prod_{a=1}^{m+1} N_a!}{N!} = \binom{N}{L}^{-1} \prod_{a=1}^{m+1} \binom{N_a}{L_a}. \tag{A.8}$$

Thus ρ_L is diagonal in the basis $|\psi(L_1,\ldots,L_{m+1})\rangle$ with $L_1,\ldots,L_m$ satisfying (A.5) (and $L_{m+1} = L - \sum_{a=1}^{m} L_a$), and its eigenvalues $\lambda(L_1,\ldots,L_m)$ are given by equation (A.8) (see

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[30]). In the particular case $m = 1$ (A.8) reduces to a hypergeometric distribution [58], as shown in [17] for the isotropic LMG model (see also [16]).

According to the Schmidt decomposition theorem (see, e.g. [2]), the ground state $|\psi(N_1, \ldots, N_{m+1})\rangle$ can be expressed as

$$|\psi(N_1, \ldots, N_{m+1})\rangle = \sum_i b_i |\psi_i\rangle \otimes |\varphi_i\rangle, \tag{A.9}$$

where $\{|\psi_j\rangle\}$ and $\{|\varphi_k\rangle\}$ are appropriate orthonormal bases of the Hilbert spaces of the first L and last $N - L$ particles, and the Schmidt coefficients b_i are non-negative real numbers. From this formula it immediately follows that

$$\rho_L = \sum_i b_i^2 \, |\psi_i\rangle\langle\psi_i|,$$

and comparing with equation (A.7) we obtain

$$|\psi_i\rangle = |\psi(L_1, \ldots, L_{m+1})\rangle, \qquad b_i = \sqrt{\lambda(L_1, \ldots, L_{m+1})}\,.$$

In fact, in this case it is straightforward to derive the Schmidt decomposition (A.9) directly. Indeed, using the previous notation for the multi-indices $\mathbf{i}_a$ and equation (A.2) we have

$$\begin{aligned}
|\psi(N_1, \ldots, N_{m+1})\rangle &= \left(\frac{N!}{\prod_{a=1}^m N_a!}\right)^{-\frac{1}{2}} \sum_{\substack{\mathbf{i}_1^L, \ldots, \mathbf{i}_m^L \\ \mathbf{i}_1^{N-L}, \ldots, \mathbf{i}_m^{N-L}}} |\mathbf{i}_1^L, \ldots, \mathbf{i}_m^L\rangle \otimes |\mathbf{i}_1^{N-L}, \ldots, \mathbf{i}_m^{N-L}\rangle \\
&= \left(\frac{N!}{\prod_{a=1}^{m+1} N_a!}\right)^{-\frac{1}{2}} \sum_{L_1, \ldots, L_m} \left(\sum_{\substack{\mathbf{i}_1^L, \ldots, \mathbf{i}_m^L \\ |\mathbf{i}_a^L| = L_a}} |\mathbf{i}_1^L, \ldots, \mathbf{i}_m^L\rangle \right) \otimes \left(\sum_{\substack{\mathbf{i}_1^{N-L}, \ldots, \mathbf{i}_m^{N-L} \\ |\mathbf{i}_a^{N-L}| = N_a - L_a}} |\mathbf{i}_1^{N-L}, \ldots, \mathbf{i}_m^{N-L}\rangle \right) \\
&= \sum_{L_1, \ldots, L_m} \sqrt{\lambda(L_1, \ldots, L_m)}\, |\psi(L_1, \ldots, L_{m+1})\rangle \otimes |\psi(N_1 - L_1, \ldots, N_{m+1} - L_{m+1})\rangle.
\end{aligned}$$

Again, in the su(2) case the latter equation reduces to the analogous formula in [17].

It is also important to observe that all the results in this section are based exclusively on the fact that the ground state of the gLMG model (2.13) is of the form (2.7), i.e. it is symmetric and has well-defined magnon densities. Thus the above results hold, in general, for *any* quantum system whose ground state is of the latter form.

Appendix B. Moments of the multivariate hypergeometric distribution

In this appendix we shall compute in closed form the first and second moments of the multivariate hypergeometric distribution $\lambda(L_1, \ldots, L_m)$ given by equation (3.2), with support (A.5). Our starting point is the elementary identity

$$\prod_{i=1}^{m+1} (1 + t_i)^{N_i} = \sum_{L_1=0}^{N_1} \cdots \sum_{L_{m+1}=0}^{N_{m+1}} \prod_{i=1}^{m+1} \binom{N_i}{L_i} \cdot t_1^{L_1} \cdots t_{m+1}^{L_{m+1}},$$

from which we deduce that

$$\sum_{L_1=0}^{N_1}\cdots\sum_{L_{m+1}=0}^{N_{m+1}}\prod_{k=1}^{n}(L_i-k+1)\cdot\prod_{j=1}^{m+1}\binom{N_j}{L_j}\cdot t_1^{L_1}\cdots t_{m+1}^{L_{m+1}}=t_i^n\frac{\partial^n}{\partial t_i^n}\prod_{j=1}^{m+1}(1+t_j)^{N_j}. \tag{B.1}$$

The previous formula can be applied to compute the moments of the distribution $\lambda(L_1,\ldots,L_m)$ in a straightforward way. Indeed, let

$$\langle f(L_1,\ldots,L_m)\rangle\equiv\sum_{L_1,\ldots,L_m}f(L_1,\ldots,L_m)\lambda(L_1,\ldots,L_m),$$

where the sum ranges over the set determined by the inequalities (A.5), denote the average of the function $f(L_1,\ldots,L_m)$ with respect to the distribution (3.2). Equation (B.1) with $N_1+\cdots+N_{m+1}=N$ implies that

$$\left\langle\prod_{k=1}^{n}(L_i-k+1)\right\rangle=\binom{N}{L}^{-1}\left\{t_i^n\frac{\partial^n}{\partial t_i^n}\prod_{j=1}^{m+1}(1+t_j)^{N_j}\Big|_{t_1=\cdots=t_{m+1}=t}\right\}_L, \tag{B.2}$$

where $\{\phi(t)\}_L$ denotes the coefficient of t^L in the polynomial $\phi(t)$. From the latter formula with $n=1$ we obtain

$$\langle L_i\rangle=\binom{N}{L}^{-1}\{N_it(1+t)^{N-1}\}_L=N_i\frac{\binom{N-1}{L-1}}{\binom{N}{L}}=\frac{LN_i}{N}=Ln_i. \tag{B.3}$$

Similarly, equation (B.2) with $n=2$ yields

$$\begin{aligned}\langle L_i(L_i-1)\rangle&=\binom{N}{L}^{-1}\{N_i(N_i-1)t^2(1+t)^{N-2}\}_L=N_i(N_i-1)\frac{\binom{N-2}{L-2}}{\binom{N}{L}}\\&=\frac{L(L-1)N_i(N_i-1)}{N(N-1)},\end{aligned}$$

so that

$$\left\langle x_i^2\right\rangle\equiv\langle(L_i-\langle L_i\rangle)^2\rangle=\langle L_i(L_i-1)\rangle-\langle L_i\rangle(\langle L_i\rangle-1)=Ln_i(1-n_i)\frac{N-L}{N-1}. \tag{B.4}$$

In particular, in the thermodynamic limit $N\to\infty$ with $\lim_{N\to\infty}(L/N)=\alpha$ finite we obtain the asymptotic formula

$$\left\langle x_i^2\right\rangle\simeq L(1-\alpha)n_i(1-n_i). \tag{B.5}$$

The latter equation generalizes the analogous formula derived in [30] by approximating $\lambda(L_1,\ldots,L_m)$ by a multinomial distribution, valid only for $\alpha=0$. Finally, the covariances $\langle x_ix_j\rangle=\langle L_iL_j\rangle-\langle L_i\rangle\langle L_j\rangle$ with $i\neq j$ can also be easily evaluated from the identity

$$\left\langle L_iL_j\right\rangle=\binom{N}{L}^{-1}\left\{t_it_j\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}\prod_{k=1}^{m+1}(1+t_k)^{N_k}\Big|_{t_1=\cdots=t_{m+1}=t}\right\}_L, \tag{B.6}$$

whose proof is similar to that of equation (B.2). From equation (B.6) we easily obtain

$$\left\langle L_i L_j \right\rangle = \binom{N}{L}^{-1} \{N_i N_j t^2 (1+t)^{N-2}\}_L = N_i N_j \frac{\binom{N-2}{L-2}}{\binom{N}{L}} = \frac{L(L-1)N_i N_j}{N(N-1)}, \qquad i \neq j,$$

and hence, by equation (B.3),

$$\left\langle x_i x_j \right\rangle = \frac{L(L-1)N_i N_j}{N(N-1)} - \frac{L^2 N_i N_j}{N^2} = -L n_i n_j \frac{N-L}{N-1}, \qquad i \neq j. \tag{B.7}$$

Again, in the thermodynamic limit we obtain the asymptotic formula

$$\left\langle x_i x_j \right\rangle \simeq -L(1-\alpha) n_i n_j, \qquad i \neq j, \tag{B.8}$$

which for $\alpha = 0$ yields the analogous formula in [30].

Appendix C. Limit of the hypergeometric probability distribution

In this appendix we shall provide a brief self-contained proof of the approximation of a hypergeometric probability distribution by a suitable Gaussian distribution, used in section 3 to derive the behavior of the eigenvalues of the reduced density matrix ρ_L in the thermodynamic limit.

Consider the hypergeometric probability distribution

$$p_l = \frac{\binom{\tilde{L}}{l}\binom{\tilde{N}-\tilde{L}}{n-l}}{\binom{\tilde{N}}{n}}, \qquad l = 0, 1, \ldots, \tilde{L}, \tag{C.1}$$

where $0 \leqslant n, \tilde{L} \leqslant \tilde{N}$ are fixed. We are interested in approximating p_l when $\tilde{N} \to \infty$, assuming that $\lim_{\tilde{N}\to\infty}(\tilde{L}/\tilde{N}) \equiv \tilde{\alpha}$ and $\lim_{\tilde{N}\to\infty}(n/\tilde{N}) \equiv \nu$ with $0 < \tilde{\alpha}, \nu < 1$. To this end, note that we can write

$$p_l = \frac{\binom{\tilde{L}}{l} x^l (1-x)^{\tilde{L}-l} \cdot \binom{\tilde{N}-\tilde{L}}{n-l} x^{n-l} (1-x)^{\tilde{N}-\tilde{L}-n+l}}{\binom{\tilde{N}}{n} x^n (1-x)^{\tilde{N}-n}} \tag{C.2}$$

for arbitrary $x \in (0,1)$. According to the Laplace–de Moivre theorem, for $K \gg 1$ a binomial distribution

$$\binom{K}{k} x^k (1-x)^{K-k}$$

can be approximated by the continuous Gaussian distribution

$$g(k;\mu,\sigma) \equiv \frac{\mathrm{e}^{-\frac{(k-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \tag{C.3}$$

where

$$\mu = xK, \qquad \sigma^2 = Kx(1-x).$$

By equations (B.3) and (B.5) with $m = 1$, in the limit considered the mean and variance of the random variable l are given by

$$\langle l\rangle = \nu\tilde{L}, \qquad \langle l^2\rangle - \langle l\rangle^2 \simeq \tilde{L}(1-\tilde{\alpha})\nu(1-\nu). \tag{C.4}$$

Hence $(\langle l^2\rangle - \langle l\rangle^2)^{1/2}/\langle l\rangle = \mathrm{O}(\tilde{L}^{-1/2})$, so that the hypergeometric distribution (C.1) is sharply peaked around its average $\nu\tilde{L}$. It is immediate to check that when l is near $\nu\tilde{L}$ we can simultaneously approximate the three binomial distributions in equation (C.2) using the Laplace–de Moivre formula with $x = \nu$ and suitable choices of k and K. We thus obtain

$$p_l \simeq \sqrt{2\pi\tilde{N}\nu(1-\nu)}\cdot\frac{\mathrm{e}^{-\frac{(l-\tilde{N}\tilde{\alpha}\nu)^2}{2\tilde{N}\tilde{\alpha}\nu(1-\nu)}}}{\sqrt{2\pi\tilde{N}\tilde{\alpha}\nu(1-\nu)}}\cdot\frac{\mathrm{e}^{-\frac{(\tilde{N}\nu-l-\tilde{N}(1-\tilde{\alpha})\nu)^2}{2\tilde{N}(1-\tilde{\alpha})\nu(1-\nu)}}}{\sqrt{2\pi\tilde{N}(1-\tilde{\alpha})\nu(1-\nu)}} = g(l;\mu,\sigma), \tag{C.5}$$

with

$$\mu = \tilde{N}\tilde{\alpha}\nu, \qquad \sigma^2 = \tilde{N}\tilde{\alpha}(1-\tilde{\alpha})\nu(1-\nu). \tag{C.6}$$

Note, finally, that these values of μ and σ^2 respectively coincide with the asymptotic values of the mean and variance of the random variable l (see equation (C.4)).

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PAPER: Quantum statistical physics, condensed matter, integrable systems

Generalized Lipkin–Meshkov–Glick models of Haldane–Shastry type

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Abstract. We introduce a class of generalized Lipkin–Meshkov–Glick (gLMG) models with su(m) interactions of Haldane–Shastry type. We compute the partition function of these models in closed form by exactly evaluating the partition function of the restriction of a spin chain Hamiltonian of Haldane–Shastry type to subspaces with well-defined magnon numbers. As a byproduct of our analysis, we obtain strong numerical evidence of the Gaussian character of the level density of the latter restricted Hamiltonians, and study the distribution of the spacings of consecutive unfolded levels. We also discuss the thermodynamic behavior of a large family of su(2) and su(3) gLMG models, showing that it is qualitatively similar to that of a two-level system.

Keywords: integrable spin chains and vertex models, solvable lattice models

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Contents

1. Introduction

One of the first, and still one of the few, quantum mechanical many-body models that has been solved in the literature is the Lipkin–Meshkov–Glick (LMG) model [1–3], which describes a system of N fermions with two N-fold degenerate one-particle levels. The original motivation for introducing this model was testing the validity of different approximation schemes from solid state physics or field theory in the context of nuclear physics. Over the years, the LMG model has appeared in connection with a wide range of problems of physical interest, including shape transitions in nuclei [4], trapped ion and optical cavity experiments [5, 6], two-modes Bose–Einstein condensates [7–9], and quantum information theory [10–14]. In particular, it has been shown that the von Neumann entanglement entropy of its ground state grows logarithmically with the size of the subsystem, as is the case for one-dimensional critical systems [15–18] (although this model is actually not critical [19]).

As already noted in the original papers, the key to the solvability of the LMG model is the fact that it can be mapped to a system of N spin-1/2 particles with constant long-range interactions of XY type in an external transverse magnetic field. In the isotropic (XX) case the Hamiltonian of this effective model is a polynomial in $\mathbf{J}^2$ and J_z, where $\mathbf{J}$ is the the total spin operator, and can thus be exactly solved for arbitrary N. The general (non-isotropic) LMG model can be solved in principle via the Bethe ansatz [20, 21], though in practice this is less efficient than brute-force numerical diagonalization.

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In the thermodynamic limit, however, the density of states of the latter model in the highest spin sector ($J = N/2$) has been derived by means of a spin-coherent-state formalism [22, 23].

A wide family of models with long-range interactions of su(m) type generalizing the *isotropic* LMG model was recently introduced in [19]. In analogy with the latter model, the non-degenerate ground state of these novel models is given by a Dicke state whose reduced density matrix for a subsystem of $L < N$ spins can be computed in closed form, which in turn yields the entanglement entropy in the thermodynamic limit $N \to \infty$ with $L/N = \alpha$ finite. Although both the von Neumann and the Rényi entanglement entropies grow logarithmically with the size L of the subsystem, the corresponding prefactor is independent of the Rényi parameter, which implies that none of these models can be critical. Interestingly, for $m > 3$ there is at least one quantum phase whose Tsallis entanglement entropy [24, 25] becomes extensive for a suitable value of the Tsallis parameter. However, the full spectrum of these models in general cannot be evaluated in closed form.

In this paper we introduce a family of generalized Lipkin–Meshkov–Glick (gLMG) models, with interactions governed by an su(m) integrable spin chain of Haldane–Shastry type. The latter chains are the celebrated Haldane–Shastry (HS) su(m) spin chain [26–28], which describes a circular array of equispaced spins with two-body long-range interactions inversely proportional to the square of the (chord) distance, and its rational [29, 30] and hyperbolic [31] analogues. Although the HS chain was originally introduced as a model whose exact ground state coincides with Gutzwiller's variational wave function for the Hubbard model in the limit of large on-site interaction [32, 33], it soon proved of interest per se in condensed matter and theoretical physics. Indeed, as pointed out by Haldane [34], the spinon excitations of this chain provide one of the simplest examples of a quantum system featuring fractional statistics (see also [28, 35, 36]). The HS chain is closely connected to important conformal field theories like the $k = 1$ Wess–Zumino–Novikov–Witten model [34, 37], and has recently been related to infinite matrix product states [38]. Integrable extensions of the Haldane–Shastry chain with long-range interactions involving more than two spins also play a key role for describing non-perturbatively the spectrum of planar $\mathcal{N} = 4$ gauge theory in the context of the AdS-CFT correspondence [39, 40]. The interest in spin chains of HS type has been further reinforced by recent developments in quantum simulation, as witnessed by the proposal of an experimental realization of the HS chain using two internal atomic states of atoms trapped in a photonic crystal waveguide [41].

One of the key features of spin chains of Haldane–Shastry type is the fact that their partition functions can be exactly computed for any number of spins [42–44] by exploiting their connection with a corresponding spin dynamical model of Calogero–Sutherland type [45–48] through a mechanism known as the Polychronakos 'freezing trick' [42]. This has made it possible to check the validity of several fundamental conjectures on the characterization of quantum chaos versus integrability [49, 50]. In particular, it has been shown that spin chains of HS type do not behave as expected for a 'generic' integrable system, in the sense that the distribution of the spacings between consecutive levels is not Poissonnian [43, 44, 51].

The gLMG models that we introduce in this paper can also be regarded as a deformation of the su(m) spin chains of HS type. More precisely, we add to the HS-type

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Hamiltonian a term depending on the generators of the standard su(m) Cartan subalgebra, which commutes with the former Hamiltonian. In particular, when this extra term is linear in the Cartan generators it can be interpreted as an su(m) external magnetic field, and the corresponding models are the ones studied in reference [52]. Likewise, when the extra term is a suitable quadratic combination of the Cartan generators we recover the models introduced in reference [19], which include the isotropic LMG model. We shall see that the Hilbert space of a general gLMG model decomposes as a direct sum of subspaces with fixed magnon numbers, which are separately invariant under the action of both the original HS-type Hamiltonian and the new term. By suitably adapting the freezing trick, we shall be able to compute the partition function of the restriction of the Hamiltonians of the three spin chains of HS type to the latter invariant subspaces. This in turn yields the partition function of the full gLMG Hamiltonian, since the Cartan generators are proportional to the identity on these subspaces. The knowledge of the partition function of the gLMG models of HS type, as well as the restricted partition functions of the corresponding spin chains, enables one to study several statistical properties of the spectrum of the latter models. In particular, we have obtained strong numerical evidence that the level density of the restriction of the HS-type chain Hamiltonians to subspaces with fixed magnon numbers follows a Gaussian distribution in the large N limit, as is known to be the case for the full spectrum of these models [53, 54]. We have also studied the distribution of the spacings between consecutive levels of the restrictions of these models to the invariant subspaces, showing that it follows the characteristic law for an approximately equispaced spectrum with normally distributed energy levels [44, 51]. Finally, we have numerically computed the thermodynamic functions of gLMG models of HS type whose extra term is quadratic in the Cartan generators, comparing them with the exact results for the original (HS-type) chains in the thermodynamic limit derived in reference [52].

2. The models

The models we shall study in this paper are deformations of su(m) spin chains with Hamiltonians of the form

$$H_0 = \sum_{1\leqslant i<j\leqslant N} h_{ij}(1-\epsilon S_{ij}), \quad \epsilon = +,-, \tag{2.1}$$

with $h_{ij}\in\mathbb{R}$. In the latter equation S_{ij} is the operator permuting the su(m) spins of the ith and jth particles, whose action on the canonical su(m) spin basis

$$\mathcal{S} = \big\{|s_1\rangle\otimes\cdots\otimes|s_N\rangle \equiv |s_1,\ldots,s_N\rangle \mid s_i = 1,\ldots,m,\ 1\leqslant i\leqslant N\big\}, \tag{2.2}$$

is given by

$$S_{ij}|s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N\rangle = |s_1,\ldots,s_j,\ldots,s_i,\ldots,s_N\rangle.$$

These operators can be expressed in terms of the local (Hermitian) generators t_k^a ($a=1,\ldots,m^2-1$) of the fundamental representation of the su(m) algebra acting on the kth site (with the normalization $\mathrm{tr}(t_k^a t_k^b) = \frac{1}{2}\,\delta_{ab}$) as

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$$S_{ij} = \frac{1}{m} + 2\sum_{a=1}^{m^2-1} t_i^a t_j^a \equiv \frac{1}{m} + 2\,\mathbf{t}_i \cdot \mathbf{t}_j. \tag{2.3}$$

We can thus write[1]

$$H_0 = -\epsilon \sum_{i\neq j} h_{ij}\,\mathbf{t}_i \cdot \mathbf{t}_j + E_0,$$

with $E_0 = (1 - \frac{\epsilon}{m})\sum_{i<j} h_{ij}$. In particular, for $m = 2$ we have $\mathbf{t}_k = \frac{1}{2}\boldsymbol{\sigma}_k$, where $\boldsymbol{\sigma}_k = (\sigma_k^1, \sigma_k^2, \sigma_k^3)$ are the three Pauli matrices acting on the kth site.

Let $\mathcal{N}_a$ denote the ath magnon number operator defined by

$$\mathcal{N}_a|s_1, \ldots, s_N\rangle = N_a|s_1, \ldots, s_N\rangle, \qquad 1 \leqslant a \leqslant m, \tag{2.4}$$

where[2]

$$N_a = \big|\big\{k = 1, \ldots, N \mid s_k = a\big\}\big|. \tag{2.5}$$

The latter operators are related to the Hermitian generators of the standard Cartan subalgebra of the Lie algebra $\mathrm{su}(m)$, as we shall now explain. Indeed, let J_k^a denote the operator whose action on the Hilbert space of the kth particle is given by

$$J_k^a|s_k\rangle = (\delta_{a,s_k} - \delta_{m,s_k})|s_k\rangle, \qquad 1 \leqslant a \leqslant m-1. \tag{2.6}$$

The $m-1$ commuting operators $\mathrm{i}J_k^a$ generate the standard Cartan subalgebra[3] of $\mathrm{su}(m)$ at each site k. We then define the global (Hermitian) Cartan generators

$$J^a \equiv \sum_{k=1}^{N} J_k^a, \qquad 1 \leqslant a \leqslant m-1.$$

From equation (2.6) it then follows that

$$J^a = \mathcal{N}_a - \mathcal{N}_m, \qquad 1 \leqslant a \leqslant m-1.$$

Summing over a and taking into account that $\sum_{a=1}^{m}\mathcal{N}_a = N$ we obtain

$$\sum_{a=1}^{m-1} J^a = N - m\mathcal{N}_m, \qquad 1 \leqslant a \leqslant m-1.$$

Using the last two equations we can express the magnon number operators in terms of the Cartan subalgebra generators as

$$\mathcal{N}_a = J^a(1 - \delta_{am}) - \overline{J} + \frac{N}{m}, \qquad 1 \leqslant a \leqslant m, \tag{2.7}$$

where

[1] Here and throughout the paper, all sums and products run from 1 to N unless otherwise specified.
[2] We shall denote in what follows by $|A|$ the cardinal of the set A.
[3] This choice of the generators of the standard Cartan subalgebra of $\mathrm{su}(m)$ is simply a matter of convenience. Note, however, that these generators are not orthogonal with respect to the usual Killing–Cartan scalar product, i.e. $\mathrm{tr}(J_k^a J_k^b) \neq 0$ for $a \neq b$.

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$$\overline{J} \equiv \frac{1}{m}\sum_{a=1}^{m-1} J^a.$$

We shall consider in what follows deformations $H = H_0 + H_1$ of (2.1) in which

$$H_1 = h(\mathcal{N}_1, \ldots, \mathcal{N}_m) \tag{2.8}$$

is an analytic function of the magnon number operators $\mathcal{N}_a$. Note, first of all, that the previous expression for H_1 is not ambiguous, since $[\mathcal{N}_a, \mathcal{N}_b] = 0$ for $1 \leqslant a, b \leqslant m$. It is also clear that $\mathrm{i}H_1$ lies in the enveloping algebra of the $\mathrm{su}(m)$ Cartan subalgebra on account of equation (2.7). For this reason, we shall say that

$$\begin{aligned} H = H_0 + H_1 &= \sum_{i<j} h_{ij}(1 - \epsilon S_{ij}) + h(\mathcal{N}_1, \ldots, \mathcal{N}_m) \\ &= -\varepsilon \sum_{i \neq j} h_{ij}\, \mathbf{t}_i \cdot \mathbf{t}_j + h(\mathcal{N}_1, \ldots, \mathcal{N}_m) + E_0. \end{aligned} \tag{2.9}$$

is an $\mathrm{su}(m)$ *generalized Lipkin–Meshkov–Glick* (gLMG) model. In particular, when $h_{ij} > 0$ for all $i < j$, $\epsilon = +$ and h is the quadratic polynomial

$$h(x_1, \ldots, x_m) = \sum_{a=1}^{m-1} c_a (x_a - x_m - N h_a)^2, \qquad \text{with} \quad h_a \in \mathbb{R},\ c_a > 0,$$

we obtain the models whose ground state entanglement entropy was computed in closed form in reference [19]. The latter models include the original (su(2), isotropic) LMG model when $h_{ij} = 2/N$ for all $i < j$ and $c_1 = 1/(2N)$, up to a constant energy.

One of the fundamental properties of the Hamiltonian (2.9) is that it preserves the subspaces of the Hilbert space $\mathcal{H} \equiv (\mathbb{C}^m)^{\otimes N}$ with a fixed magnon configuration. Indeed, let us denote by $\mathcal{H}(\mathbf{N})$, where $\mathbf{N} = (N_1, \ldots, N_m)$ and $|\mathbf{N}| \equiv N_1 + \cdots + N_m = N$, the subspace of $\mathcal{H}$ whose elements are linear combinations of basis states $|s_1, \ldots, s_N\rangle \equiv |\mathbf{s}\rangle$ with magnon numbers N_a (see equation (2.5)). Clearly H_0 leaves $\mathcal{H}(\mathbf{N})$ invariant, since each permutation operator S_{ij} does. On the other hand, $\mathcal{N}_a|s\rangle = N_a|s\rangle$ on $\mathcal{H}(\mathbf{N})$ by construction, and therefore

$$H_1 = h(\mathbf{N}) \quad \text{on } \mathcal{H}(\mathbf{N}).$$

Thus $H = H_0 + H_1$ preserves $\mathcal{H}(\mathbf{N})$, as stated. It is also clear from the above discussion that $[H_0, H_1] = 0$, and that the eigenvalues of $H^{\mathbf{N}} \equiv H|_{\mathcal{H}(\mathbf{N})}$ can be expressed as

$$E_i^0(\mathbf{N}) + h(\mathbf{N}), \qquad 1 \leqslant i \leqslant \dim \mathcal{H}(\mathbf{N}),$$

where $\{E_i^0(\mathbf{N})\}_{1 \leqslant i \leqslant \dim \mathcal{H}(\mathbf{N})}$ is the spectrum of $H_0^{\mathbf{N}} \equiv H_0|_{\mathcal{H}(\mathbf{N})}$. Hence the partition function $Z^{\mathbf{N}}(T)$ of $H^{\mathbf{N}}$ is given by

$$Z^{\mathbf{N}}(T) = q^{h(\mathbf{N})} \sum_{i=1}^{\dim \mathcal{H}(\mathbf{N})} q^{E_i^0(\mathbf{N})} \equiv q^{h(\mathbf{N})} Z_0^{\mathbf{N}}(T), \qquad q \equiv \mathrm{e}^{-1/k_{\mathrm{B}}T},$$

where $Z_0^{\mathbf{N}}(T)$ is the partition function of $H_0^{\mathbf{N}}$. Since

$$\mathcal{H} = \bigoplus_{|\mathbf{N}|=N} \mathcal{H}(\mathbf{N}),$$

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the partition function of H is given by

$$Z(T) = \sum_{|\mathbf{N}|=N} Z^{\mathbf{N}}(T) = \sum_{|\mathbf{N}|=N} q^{h(\mathbf{N})} Z_0^{\mathbf{N}}(T). \tag{2.10}$$

Thus the partition function of the model (2.9) is completely determined by the partition functions $Z_0^{\mathbf{N}}(T)$ of the restrictions of the spin chain Hamiltonian H_0 to each of the subspaces $\mathcal{H}(\mathbf{N})$. We shall see in the following sections that the latter partition functions can be computed in closed form when H_0 is the Hamiltonian of one of the three spin chains of HS type, namely the Haldane–Shastry [26, 27], Polychronakos–Frahm (PF) [29, 30] and Frahm–Inozemtsev (FI) [31] chains. The chain sites of these integrable spin chains can be expressed as

$$z_k = \begin{cases} k\pi/N, & \text{for the HS chain} \\ \zeta_k, & \text{for the PF chain} \\ \mathrm{e}^{2\xi_k}, & \text{for the FI chain,} \end{cases} \tag{2.11}$$

where ζ_k and ξ_k respectively denote the kth zero of the Hermite polynomial of degree N and the generalized Laguerre polynomial $L_N^{\beta-2N+1}$ with $\beta > 2(N-1)$. In all three cases, the interaction strength is a function $h_{ij} = h(z_i - z_j)$ of the difference $z_i - z_j$, namely

$$h(x) = \begin{cases} \frac{1}{2}\sin^{-2} x, & \text{for the HS chain} \\ x^{-2}, & \text{for the PF chain} \\ \frac{1}{2}\sinh^{-2} x, & \text{for the FI chain.} \end{cases} \tag{2.12}$$

Remarkably, the (total) partition function $Z_0(T) = \sum_{|\mathbf{N}|=N} Z_0^{\mathbf{N}}(T)$ of all of these models can be computed in closed form by exploiting their close connection with their associated spin Calogero–Sutherland models (see, e.g. [42–44]). In the following sections we shall adapt this technique, known in the literature as Polychronakos's freezing trick [42], to evaluate the restricted partition functions $Z_0^{\mathbf{N}}(T)$.

3. The freezing trick

In this section we shall outline the computation of the restricted partition function $Z_0^{\mathbf{N}}$ for the Haldane–Shastry spin chain, which is the best known of these models and presents certain technical subtleties stemming from its translation invariance. To this end, we first recall that in this case H_0 is related to the strong interaction limit of the spin Sutherland model

$$H_{\mathrm{sp}} = -\Delta + a\sum_{i\neq j} \sin^{-2}(x_i - x_j)(a - \epsilon S_{ij}), \qquad a > 0,$$

where $\Delta \equiv \sum_i \partial_{x_i}^2$. Indeed, we can write

$$H_{\mathrm{sp}} = H_{\mathrm{sc}} + 4a\hat{H}_0(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_N)$,

$$H_{\rm sc} = -\Delta + a(a-1)\sum_{i\neq j}\sin^{-2}(x_i - x_j)$$

is the scalar Sutherland model and

$$\hat{H}_0(\mathbf{x}) = \frac{1}{2}\sum_{i<j}\sin^{-2}(x_i - x_j)(1 - \epsilon S_{ij})$$

is obtained from H_0 replacing the chain sites z_i by the dynamical variables x_i. Since $H_{\rm sp}$ and $H_{\rm sc}$ are translation invariant, the total momentum is conserved and can be set to zero by working in the center of mass frame. In the strong interaction limit $a \to \infty$ the eigenfunctions of $H_{\rm sp}$ become sharply peaked at the coordinates of the minimum of the scalar potential

$$U(\mathbf{x}) = \sum_{i\neq j}\sin^{-2}(x_i - x_j)$$

in the configuration space (A_{N-1} Weyl chamber)

$$A = \{\mathbf{x} \in \mathbb{R}^N \mid x_1 < \cdots < x_N\},$$

which (up to an overall translation) coincide with the chain sites $z_k = k\pi/N$. Thus, when $a \gg 1$ the eigenvalues of $H_{\rm sp}$ are approximately given by

$$E_{ij} \simeq E_i^{\rm sc} + 4aE_j^0, \qquad a \gg 1,$$

where $E_i^{\rm sc}$ and E_j^0 respectively denote two arbitrary eigenvalues of $H_{\rm sc}$ and H_0. From the latter equation it immediately follows that the partition function $Z_0(T)$ of the Haldane–Shastry chain is given by the freezing trick formula

$$Z_0(T) = \lim_{a\to\infty}\frac{Z_{\rm sp}(4aT)}{Z_{\rm sc}(4aT)}. \tag{3.1}$$

This is the basis for the computation of $Z_0(T)$ in reference [43]. We shall now show that the same procedure can be carried out to compute the restricted partition functions $Z_0^{\mathbf{N}}(T)$. Essentially, this is due to the fact that the spin Hamiltonian $H_{\rm sp}$ preserves the subspaces $L^2(A)\otimes\mathcal{H}(\mathbf{N})$ of its Hilbert space $L^2(A)\otimes\mathcal{H}$. Thus, $Z_0^{\mathbf{N}}$ can be obtained from the analogue of equation (3.1), namely

$$Z_0^{\mathbf{N}}(T) = \lim_{a\to\infty}\frac{Z_{\rm sp}^{\mathbf{N}}(4aT)}{Z_{\rm sc}(4aT)}, \tag{3.2}$$

where $Z_{\rm sp}^{\mathbf{N}}$ is the partition function of $H_{\rm sp}^{\mathbf{N}} = H_{\rm sp}|_{L^2(A)\otimes\mathcal{H}(\mathbf{N})}$.

To begin with, note that the Hamiltonian $H_{\rm sp}$ is equivalent to its symmetric/antisymmetric extension to the Hilbert space $\Lambda_\pm\big(L^2(\mathbb{R}^N)\otimes\mathcal{H}\big)$, where Λ_+ (resp. Λ_-) is the symmetrizer (resp. antisymmetrizer) with respect to permutations of the particles' coordinates and su(m) spin variables. This is basically due to the fact that any point $\mathbf{x} \in \mathbb{R}^N$ not lying on the singular hyperplanes $x_i - x_j = 0$ can be mapped in a unique way to a point in A by a suitable permutation. As we shall see below, it shall be convenient for what follows to identify $H_{\rm sp}$ with its symmetric (resp. antisymmetric) extension when $\epsilon = 1$ (resp. $\epsilon = -1$). With this identification, it can be shown [43] that

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$H_{\rm sp}$ is represented by an upper triangular matrix in the appropriately ordered (non-orthonormal) basis with elements

$$|\mathbf{p},\mathbf{s}\rangle = \Lambda_\epsilon\Big(\mathrm{e}^{2\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\prod_{i<j}\sin(x_i-x_j)^a\,|\mathbf{s}\rangle\Big), \tag{3.3}$$

where $|\mathbf{s}\rangle \in \mathcal{S}$ and $\mathbf{p} \equiv (p_1,\dots,p_N) \in \mathbb{R}^N$ satisfy the following conditions:

(i) The differences $n_i \equiv p_i - p_{i+1}$ $(1 \leqslant i \leqslant N-1)$ are nonnegative integers.

(ii) If $p_i = p_{i+1}$ then $s_i \prec s_{i+1}$.

(iii) The total momentum of the state $|\mathbf{p},\mathbf{s}\rangle$ vanishes, i.e. $\sum_i p_i = 0$.

In the second condition, the notation $s_i \prec s_j$ stands for $s_i < s_j$ when $\epsilon = -1$ and $s_i \leqslant s_j$ when $\epsilon = 1$. The first condition is justified in [43], the second one can be arranged due to the symmetric/antisymmetric nature of the states (3.3), while the last one simply reflects that we are working in the center of mass frame. As shown in the latter reference, the states $|\mathbf{p},\mathbf{s}\rangle$ should be ordered in such a way that $|\mathbf{p},\mathbf{s}\rangle$ precedes $|\mathbf{p}',\mathbf{s}'\rangle$ whenever $\mathbf{p} < \mathbf{p}'$, where the last notation means that $\mathbf{p}$ precedes $\mathbf{p}'$ in the lexicographic order. With this partial order, the action of $H_{\rm sp}$ on the basis (3.3) is upper triangular. More precisely [43],

$$H_{\rm sp}|\mathbf{p},\mathbf{s}\rangle = E(\mathbf{p})|\mathbf{p},\mathbf{s}\rangle + \sum_{\mathbf{p}'<\mathbf{p};\,\mathbf{s}'} c(\mathbf{p}',\mathbf{s}')\,|\mathbf{p}',\mathbf{s}'\rangle, \tag{3.4}$$

with $c(\mathbf{p}',\mathbf{s}') \in \mathbb{C}$ and

$$E(\mathbf{p}) = \sum_i \big[2p_i + a(N+1-2i)\big]^2. \tag{3.5}$$

Since $H_{\rm sp}$ preserves $\mathcal{H}(\mathbf{N})$, if the vector $\mathbf{s}$ in equation (3.4) is such that $|\mathbf{s}\rangle \in \mathcal{H}(\mathbf{N})$ then $|\mathbf{s}'\rangle \in \mathcal{H}(\mathbf{N})$ for all vectors $\mathbf{s}'$ appearing in the RHS of the latter equation. In other words, $H^{\mathbf{N}}_{\rm sp}$ is also upper triangular with respect to the basis (3.3), where $|\mathbf{s}\rangle \in \mathcal{S}\cap\mathcal{H}(\mathbf{N})$ and the quantum numbers $(\mathbf{p},\mathbf{s})$ satisfy conditions (i)–(iii) above, ordered as previously explained. Moreover, by equation (3.4) the eigenvalues of $H^{\mathbf{N}}_{\rm sp}$ are given by equation (3.5). Expanding the latter equation in powers of a we obtain

$$E(\mathbf{p}) = E_{\rm GS} + 4a\sum_i p_i(N+1-2i) + \mathrm{O}(1),$$

where

$$E_{\rm GS} = a^2\sum_i (N+1-2i)^2 = \frac{a^2}{3}\,N(N^2-1)$$

is the ground state energy of the ferromagnetic model ($\epsilon = 1$). Thus in the limit $a \to \infty$ we have

$$\lim_{a\to\infty} q^{-E_{\rm GS}/4a} Z^{\mathbf{N}}_{\rm sp}(4aT) = \sum_{\mathbf{p},\mathbf{s}} q^{\sum_i p_i(N+1-2i)},$$

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where the sum is extended to all $(\mathbf{p},\mathbf{s})$ satisfying conditions (i)–(iii) above with $|\mathbf{s}\rangle \in \mathcal{S}\cap\mathcal{H}(\mathbf{N})$. Since the exponent is independent of the spin variables $\mathbf{s}$, the sum over $\mathbf{s}$ can be immediately carried out, namely

$$\lim_{a\to\infty} q^{-E_{\mathrm{GS}}/4a} Z^{\mathbf{N}}_{\mathrm{sp}}(4aT) = \sum_{\mathbf{p}} d(\mathbf{p},\mathbf{N},\epsilon)\, q^{\sum_i p_i(N+1-2i)}, \tag{3.6}$$

where the spin degeneracy factor $d(\mathbf{p},\mathbf{N},\epsilon)$ is the number of multiindices $\mathbf{s}$ satisfying condition (ii) above for a given $\mathbf{p}$ such that $|\mathbf{s}\rangle \in \mathcal{S}\cap\mathcal{H}(\mathbf{N})$. In other words,

$$d(\mathbf{p},\mathbf{N},\epsilon) = \left|S(\mathbf{p},\epsilon)\cap\mathcal{H}(\mathbf{N})\right| \tag{3.7}$$

where

$$S(\mathbf{p},\epsilon) \equiv \left\{|\mathbf{s}\rangle\in\mathcal{S} \mid p_i = p_{i+1} \Rightarrow s_i \prec s_{i+1},\ 1\leqslant i\leqslant N-1\right\}. \tag{3.8}$$

In order to evaluate the sum in equation (3.6), we note that by conditions (i) and (iii) above we can write the multiindex $\mathbf{p}$ as

$$\mathbf{p} = (\overbrace{\rho_1,\ldots,\rho_1}^{k_1},\ldots,\overbrace{\rho_r,\ldots,\rho_r}^{k_r}), \tag{3.9}$$

with

$$k_1+\cdots+k_r = N, \quad k_1\rho_1+\cdots+k_r\rho_r = 0, \quad \rho_i>\rho_{i+1}, \quad \rho_i-\rho_{i+1}\in\mathbb{N}. \tag{3.10}$$

Thus the multiindex $\mathbf{p}$ consists of r *blocks* of lengths $k_1,\ldots,k_r$. Calling

$$K_i = \sum_{j=1}^{i} k_j, \qquad 0\leqslant i\leqslant r, \tag{3.11}$$

we have

$$\sum_i p_i(N+1-2i) = \sum_{i=1}^{r}\rho_i \sum_{j=K_{i-1}+1}^{K_i}(N+1-2j) = \sum_{i=1}^{r}\rho_i k_i(N-2K_i+k_i).$$

Since $d(\mathbf{p},\mathbf{N},\epsilon)$ obviously depends on $\mathbf{p}$ only through $\mathbf{k}\equiv(k_1,\ldots,k_r)$, we can rewrite equation (3.6) as

$$\lim_{a\to\infty} q^{-E_{\mathrm{GS}}/4a} Z^{\mathbf{N}}_{\mathrm{sp}}(4aT) = \sum_{r=1}^{N}\sum_{\mathbf{k}\in\mathcal{P}^r_N} d(\mathbf{p},\mathbf{N},\epsilon) \sum_{\substack{\rho_1>\cdots>\rho_r,\ \rho_i-\rho_{i+1}\in\mathbb{N}\\ k_1\rho_1+\cdots+k_r\rho_r=0}} q^{\sum_{i=1}^r \rho_i k_i(N-2K_i+k_i)}, \tag{3.12}$$

where $\mathcal{P}^r_N$ denotes the set of all partitions of the integer N in r parts with order taken into account. The inner sum in equation (3.12) was evaluated in [43], with the result

$$\sum_{\substack{\rho_1>\cdots>\rho_r,\ \rho_i-\rho_{i+1}\in\mathbb{N}\\ k_1\rho_1+\cdots+k_r\rho_r=0}} q^{\sum_{i=1}^r \rho_i k_i(N-2K_i+k_i)} = \prod_{i=1}^{r-1}\frac{q^{\mathcal{E}(K_i)}}{1-q^{\mathcal{E}(K_i)}}, \tag{3.13}$$

where

$$\mathcal{E}(k) = k(N-k). \tag{3.14}$$

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Substituting equations (3.13) into (3.12) we obtain

$$\lim_{a\to\infty} q^{-E_{\rm GS}/4a} Z^{\mathbf{N}}_{\rm sp}(4aT) = \sum_{r=1}^{N} \sum_{\mathbf{k}\in\mathcal{P}^r_N} d(\mathbf{p},\mathbf{N},\epsilon) \prod_{i=1}^{r-1} \frac{q^{\mathcal{E}(K_i)}}{1-q^{\mathcal{E}(K_i)}}, \tag{3.15}$$

where $\mathbf{p}$ is any multiindex of the form (3.9). The partition function for the scalar Hamiltonian was also evaluated in [43] in the large a limit, namely

$$\lim_{a\to\infty} q^{-E_{\rm GS}/4a} Z_{\rm sc}(4aT) = \prod_{i=1}^{N} (1-q^{\mathcal{E}(i)})^{-1}. \tag{3.16}$$

Combining equations (3.15) and (3.16) with (3.2) we finally obtain the following explicit formula for the restricted partition function $Z^{\mathbf{N}}_0(T)$:

$$Z^{\mathbf{N}}_0(T) = \sum_{r=1}^{N} \sum_{\mathbf{k}\in\mathcal{P}^r_N} d(\mathbf{p},\mathbf{N},\epsilon) \prod_{i=1}^{r-1} q^{\mathcal{E}(K_i)} \cdot \prod_{j=1}^{N-r} (1-q^{\mathcal{E}(K'_j)}), \tag{3.17}$$

where

$$\{K'_1,\dots,K'_{N-r}\} = \{1,\dots,N-1\} - \{K_1,\dots,K_{r-1}\}$$

and $\mathbf{p}$ is determined by $\mathbf{k}$ through equation (3.9). Following a similar procedure for the PF and FI chains we again obtain equation (3.17), but with $\mathcal{E}(k)$ in equation (3.14) respectively given by k and $k(\beta-2N+k+1)$ (see [44, 51] for more details). In summary, the restricted partition function $Z^{\mathbf{N}}_0(T)$ for the three chains of HS type is given by equation (3.17), with *dispersion relation*

$$\mathcal{E}(k) = \begin{cases} k(N-k), & \text{for the HS chain} \\ k, & \text{for the PF chain} \\ k(\beta-2N+k+1), & \text{for the FI chain.} \end{cases} \tag{3.18}$$

Equations (2.10)–(3.17) yield an explicit formula for the partition function of the su(m) gLMG model (2.9) with interactions $h_{ij} = h(z_i - z_j)$ given by equations (2.11) and (2.12), once the degeneracy factor $d(\mathbf{p},\mathbf{N},\epsilon)$ is known.

4. Degeneracy factor

As we have seen in the previous section, in order to evaluate the partition function $Z(T)$ of an su(m) gLMG model of HS type through equations (2.10)–(3.17), we only need to determine the degeneracy factor $d(\mathbf{p},\mathbf{N},\epsilon)$ defined in equation (3.7). To this end, let us fix $\mathbf{p}$ in equation (3.9) (with $\mathbf{k}\in\mathcal{P}^r_N$) and take $\mathbf{N} = (N_1,\dots,N_m)$ such that $N_a \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}$ and $|\mathbf{N}| = N$. The degeneracy factor $d(\mathbf{p},\mathbf{N},\epsilon)$ is obviously much easier to compute in the antiferromagnetic case ($\epsilon=-1$), since by Pauli's principle the su(m) spins in each block of length $k_1,\dots,k_r$ in which the components of $\mathbf{p}$ are equal must all be different (in fact, arranged in a strictly increasing sequence according to condition (ii) in the previous section).

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4.1. Anti-ferromagnetic case

Let us define the vector $\mathbf{r} = (r_1, \ldots, r_m)$ by

$$r_i \equiv \big|\{\, j = 1, \ldots, r \mid k_j = i\}\big| \in \mathbb{N}_0, \quad 1 \leqslant i \leqslant m,$$

so that

$$r_1 + \cdots + r_m = r, \qquad r_1 + 2r_2 + \cdots + mr_m = N. \tag{4.1}$$

In other words, r_i is the number of blocks of length i in the expression (3.9) for $\mathbf{p}$. Obviously $d(\mathbf{p}, \mathbf{N}, -) \equiv D_m(\mathbf{r}, \mathbf{N})$, where $D_m(\mathbf{r}, \mathbf{N})$ denotes the number of ways one can distribute N_1 spins $|1\rangle$, N_2 spins $|2\rangle$, , N_m spins $|m\rangle$ in r_1 blocks of one site, r_2 blocks of two sites,..., r_m blocks of m sites, with all spins different in each block.

For $1 \leqslant i, j \leqslant m$, let us denote by $N_{i,j} \in \mathbb{N}_0$ the number of spins $|i\rangle$ in the r_j blocks of j sites, and define $\mathbf{N}_i = (N_{i,1}, \ldots, N_{i,m})$ such that $|\mathbf{N}_i| = N_i$ for $1 \leqslant i \leqslant m$. We can find an expression for the degeneracy factor by counting the number of ways one can fill the pattern of blocks so that all the spins in each block are different. To this end, we start with an empty pattern and fill it as follows:

(i) *Fill all the r_m blocks of m sites.*
In the r_m blocks of m sites there must be r_m spins of each type. We are left with $N_1 - r_m$ spins $|1\rangle$, $N_2 - r_m$ spins $|2\rangle$,..., $N_m - r_m$ spins $|m\rangle$ and a pattern of r_1 blocks of one site, r_2 blocks of two sites,..., r_{m-1} blocks of $m-1$ sites.

(ii) *Distribute the remaining $N_m - r_m$ spins of type $|m\rangle$ in the $r - r_m$ empty blocks left.*
As in the previous step, we next fix a vector

$$\mathbf{x} = (x_1, \ldots, x_{m-1})$$

with $x_i \equiv N_{m,i} \in \mathbb{N}_0$ and $|\mathbf{x}| = N_m - r_m$. Clearly, the number of ways of distributing the $N_m - r_m$ spins $|m\rangle$ in the available $r - r_m$ blocks is given by the product of binomial coefficients $\prod_{i=1}^{m-1} \binom{r_i}{x_i}$.

(iii) *For each $\mathbf{x}$ in step (ii), we are left with a new pattern $\hat{\mathbf{r}} \in \mathbb{N}_0^{m-1}$ and new spins of types $1, \ldots, m-1$ with magnon numbers $(\hat{N}_1, \ldots, \hat{N}_{m-1}) \equiv \hat{\mathbf{N}}$.*
Remarkably, the new pattern $\hat{\mathbf{r}}$ has no blocks of m sites and the new vector $\hat{\mathbf{N}}$ has no spins $|m\rangle$. More precisely, for $i = 1, \ldots, m-1$ there are now $\hat{r}_i = r_i - x_i + x_{i+1}$ blocks of i sites, i.e. the previous r_i minus the occupied blocks of i sites plus the occupied blocks of $i+1$ sites (note that we must take $x_m = 0$, since all the blocks of m sites were filled up in the first step). Thus, the new pattern $\hat{\mathbf{r}} \equiv \hat{\mathbf{r}}(\mathbf{x})$ and magnon vector $\hat{\mathbf{N}}$ are given by

$$\hat{r}_i = r_i - x_i + x_{i+1}, \;\; 1 \leqslant i \leqslant m-2; \quad \hat{r}_{m-1} = r_{m-1} - x_{m-1}, \tag{4.2}$$

$$\hat{N}_i = N_i - r_m, \;\; 1 \leqslant i \leqslant m-1, \tag{4.3}$$

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and therefore

$$D_m(\mathbf{r},\mathbf{N}) = \sum_{|\mathbf{x}|=N_m-r_m} \prod_{i=1}^{m-1} \binom{r_i}{x_i} \cdot D_{m-1}\big(\hat{\mathbf{r}}(\mathbf{x}),\hat{\mathbf{N}}\big). \tag{4.4}$$

Note that the new vectors $\hat{\mathbf{N}}$ and $\hat{\mathbf{r}}$ satisfy a relation analogous to the last equation (4.1), namely (by equations (4.2) and (4.3))

$$|\hat{\mathbf{N}}| = N - N_m - (m-1)r_m = \hat{r}_1 + 2\,\hat{r}_2 + \cdots + (m-1)\,\hat{r}_{m-1}. \tag{4.5}$$

(iv) *Iterate the process described above.*
By equation (4.4), we can express the degeneracy factor

$$D_m(\mathbf{r},\mathbf{N}) \equiv D_m(\mathbf{r}^{(m)},\mathbf{N}^{(m)})$$

as a linear combination of degeneracy factors

$$D_{m-1}(\hat{\mathbf{r}},\hat{\mathbf{N}}) \equiv D_{m-1}(\mathbf{r}^{(m-1)},\mathbf{N}^{(m-1)}).$$

This process can be iterated, by expressing each term $D_{m-1}\left(\mathbf{r}^{(m-1)},\mathbf{N}^{(m-1)}\right)$ in equation (4.4) in terms of degeneracy factors

$$D_{m-2}\left((\mathbf{r}^{(m-1)})\hat{},(\mathbf{N}^{(m-1)})\hat{}\right) \equiv D_{m-2}(\mathbf{r}^{(m-2)},\mathbf{N}^{(m-2)}),$$

and so on. We thus obtain the recursion relation

$$D_k(\mathbf{r}^{(k)},\mathbf{N}^{(k)}) = \sum_{|\mathbf{x}|=N_k^{(k)}-r_k^{(k)}} \prod_{i=1}^{k-1} \binom{r_i^{(k)}}{x_i} \cdot D_{k-1}\big(\mathbf{r}^{(k-1)}(\mathbf{x}),\mathbf{N}^{(k-1)}\big), \tag{4.6}$$

where

$$\mathbf{r}^{(k)},\mathbf{N}^{(k)} \in \mathbb{N}_0^k; \quad \mathbf{x},\mathbf{r}^{(k-1)}(\mathbf{x}),\mathbf{N}^{(k-1)} \in \mathbb{N}_0^{k-1},$$

with

$$r_i^{(k-1)}(\mathbf{x}) = r_i^{(k)} - x_i + x_{i+1} \ \ (x_k \equiv 0), \quad N_i^{(k-1)} = N_i^{(k)} - r_k^{(k)}$$

and $\mathbf{r}^{(m)} \equiv \mathbf{r}$, $\mathbf{N}^{(m)} \equiv \mathbf{N}$. The above recursion relation, together with the obvious initial condition $D_1 = 1$, fully determines $D_m(\mathbf{r},\mathbf{N})$.

In section 5 we shall illustrate the above procedure for computing the degeneracy factor $d(\mathbf{p},\mathbf{N},-) \equiv D_m(\mathbf{r},\mathbf{N})$ with several examples. Once D_m is determined, the restricted partition $Z_0^{\mathbf{N},(-)}(T)$ in the antiferromagnetic case is obtained from equation (3.17), namely

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$$Z_0^{\mathbf{N},(-)}(T) = \sum_{r=\lceil N/m\rceil}^{N} \sum_{\mathbf{k}\in\mathcal{P}_N^r} D_m(\mathbf{r},\mathbf{N}) \prod_{i=1}^{r-1} q^{\mathcal{E}(K_i)} \cdot \prod_{j=1}^{N-r} (1-q^{\mathcal{E}(K'_j)}), \tag{4.7}$$

where the range of the last sum comes from condition (ii) above, since in the antiferromagnetic case the lengths k_i of the blocks in equation (3.9) are all at most equal to m. The partition function of the corresponding gLMG model of HS type (2.9) can then be computed from equation (2.10), with the result

$$Z^{(-)}(T) = \sum_{|\mathbf{N}|=N} q^{h(\mathbf{N})} \sum_{r=\lceil N/m\rceil}^{N} \sum_{\mathbf{k}\in\mathcal{P}_N^r} D_m(\mathbf{r},\mathbf{N}) \prod_{i=1}^{r-1} q^{\mathcal{E}(K_i)} \cdot \prod_{j=1}^{N-r} (1-q^{\mathcal{E}(K'_j)}). \tag{4.8}$$

4.2. Ferromagnetic case

A similar procedure could be followed in principle to compute the degeneracy factor $d(\mathbf{p},\mathbf{N},+)$ in the ferromagnetic case $\epsilon = 1$. The main difference is that now each value of the su(m) spin can be used more than once to fill the blocks of length $k_1,\dots,k_r$ determined by the multiindex $\mathbf{p}$ in equation (3.9), which considerably complicates matters.

In practice, it is much easier to derive the ferromagnetic partition function $Z^{(+)}$ from the antiferromagnetic one $Z^{(-)}$ computed in the previous subsection by means of the identity

$$H_0^{(+)} + H_0^{(-)} = \sum_{i\neq j} h_{ij} \equiv E_{\max}(N), \tag{4.9}$$

where $H_0^{(\pm)}$ denotes the Hamiltonian (2.1) with $\epsilon = \pm$. The constant $E_{\max}(N)$, which is the maximum energy of $H_0^{(-)}$, can be easily computed in closed form for each of the interactions (2.12) and (2.11) taking into account the identity [55]

$$E_{\max}(N) = \sum_{i=1}^{N-1} \mathcal{E}(i), \tag{4.10}$$

namely

$$E_{\max} = \begin{cases} \frac{N}{6}(N^2-1), & \text{for the HS chain} \\ \frac{N}{2}(N-1), & \text{for the PF chain} \\ \frac{N}{6}(N-1)(3\beta-4N+2), & \text{for the FI chain.} \end{cases}$$

From equation (4.9) it immediately follows that the restricted partition functions $Z_0^{\mathbf{N},(\pm)}$ of $H_0^{(\pm)}$ are related by

$$Z_0^{\mathbf{N},(+)}(q) = q^{E_{\max}(N)} Z_0^{\mathbf{N},(-)}(q^{-1}).$$

Using equations (3.17) and (4.10) we easily obtain

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$$Z_0^{\mathbf{N},(+)}(T) = q^{E_{\max}(N)} \sum_{r=\lceil N/m\rceil}^{N} \sum_{\mathbf{k}\in\mathcal{P}_N^r} D_m(\mathbf{r},\mathbf{N}) \prod_{i=1}^{r-1} q^{-\mathcal{E}(K_i)} \cdot \prod_{j=1}^{N-r} (1-q^{-\mathcal{E}(K'_j)})$$
$$= \sum_{r=\lceil N/m\rceil}^{N} (-1)^{N-r} \sum_{\mathbf{k}\in\mathcal{P}_N^r} D_m(\mathbf{r},\mathbf{N}) \prod_{i=1}^{N-r} (1-q^{\mathcal{E}(K'_i)}), \tag{4.11}$$

where $D(\mathbf{r},\mathbf{N})$ is the antiferromagnetic degeneracy factor computed in the previous subsection. By equation (2.10), the partition function of the ferromagnetic Hamiltonian $H^{(+)} \equiv H_0^{(+)} + H_1$ is given by

$$Z^{(+)}(T) = \sum_{|\mathbf{N}|=N} q^{h(\mathbf{N})} \sum_{r=\lceil N/m\rceil}^{N} (-1)^{N-r} \sum_{\mathbf{k}\in\mathcal{P}_N^r} D_m(\mathbf{r},\mathbf{N}) \prod_{i=1}^{N-r} (1-q^{\mathcal{E}(K'_i)}). \tag{4.12}$$

5. Examples

5.1. su(2)

In this case $\mathbf{r} = (r_1, r_2)$, $\mathbf{N} = (N_1, N_2)$, and the recursion relation (4.6) with $D_1 = 1$ immediately yields

$$D_2(\mathbf{r},\mathbf{N}) = \binom{r_1}{N_2 - r_2}.$$

Expressing r_1, r_2 in terms of r and N by means of the relations $r = r_1 + r_2$, $N = N_1 + N_2 = r_1 + 2\,r_2$ we finally obtain

$$D_2(\mathbf{r},\mathbf{N}) = \binom{2r-N}{r-N_1} = \binom{2r-N}{r-N_2}.$$

Thus the restricted partition function of the su(2) chains (2.1) of HS type is given by

$$Z_0^{\mathbf{N}}(T) = \sum_{r=1}^{N} (-1)^{\frac{1+\epsilon}{2}(N-r)} \sum_{\mathbf{k}\in\mathcal{P}_N^r} \binom{2r-N}{r-N_1} q^{\frac{1-\epsilon}{2}\sum_{i=1}^{r-1}\mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1-q^{\mathcal{E}(K'_i)}).$$

By equation (2.10), the partition function of the corresponding su(2) gLMG model reads

$$Z(T) = \sum_{N_1=0}^{N} q^{h(N_1,N-N_1)} \sum_{r=1}^{N} (-1)^{\frac{1+\epsilon}{2}(N-r)} \sum_{\mathbf{k}\in\mathcal{P}_N^r} \binom{2r-N}{r-N_1} q^{\frac{1-\epsilon}{2}\sum_{i=1}^{r-1}\mathcal{E}(K_i)} \prod_{i=1}^{N-r} (1-q^{\mathcal{E}(K'_i)}).$$

5.2. su(3)

Let $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{N} = (N_1, N_2, N_3)$ such that $r = r_1 + r_2 + r_3$, $N = N_1 + N_2 + N_3$ and $r_1 + 2\,r_2 + 3\,r_3 = N$. We then have

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$$D_3(\mathbf{r},\mathbf{N}) = \sum_{x_1+x_2=N_3-r_3} \binom{r_1}{x_1}\binom{r_2}{x_2} D_2(\hat{\mathbf{r}},\hat{\mathbf{N}}) = \sum_{x_1+x_2=N_3-r_3} \binom{r_1}{x_1}\binom{r_2}{x_2}\binom{2\,\hat{r}-\hat{N}}{\hat{r}-\hat{N}_2}$$
$$= \sum_{x_1+x_2=N_3-r_3} \binom{r_1}{x_1}\binom{r_2}{x_2}\binom{2r-2x_1-N+N_3}{r-x_1-N_2}, \tag{5.1}$$

where we have used the identities $\hat{r} = r - r_3 - x_1$, $\hat{N} = N - N_3 - 2\,r_3$ and $\hat{N}_2 = N_2 - r_3$.

5.3. su(4)

Let $\mathbf{r} = (r_1, r_2, r_3, r_4)$ and $\mathbf{N} = (N_1, N_2, N_3, N_4)$ such that $r = r_1 + r_2 + r_3 + r_4$, $N = N_1 + N_2 + N_3 + N_4 = r_1 + 2\,r_2 + 3\,r_3 + 4\,r_4$. Using equation (5.1) with $(\hat{\mathbf{r}}, \hat{N}) \equiv (\mathbf{r}^{(3)}, \mathbf{N}^{(3)})$ in place of $(\mathbf{r}, \mathbf{N})$ we easily obtain

$$D_4(\mathbf{r},\mathbf{N}) = \sum_{x_1+x_2+x_3=N_4-r_4} \binom{r_1}{x_1}\binom{r_2}{x_2}\binom{r_3}{x_3} D_3(\hat{\mathbf{r}},\hat{\mathbf{N}})$$
$$= \sum_{x_1+x_2+x_3=N_4-r_4} \tbinom{r_1}{x_1}\tbinom{r_2}{x_2}\tbinom{r_3}{x_3} \sum_{y_1+y_2=N_3-r_4-r_3+x_3} \tbinom{r_1-x_1+x_2}{y_1}\tbinom{r_2-x_2+x_3}{y_2}\tbinom{2\,r-2\,x_1-2\,y_1-N_1-N_2}{r-x_1-y_1-N_2}. \tag{5.2}$$

6. The LMG-PF model

When H_0 is the Hamiltonian of the PF chain the restricted partition function $Z_0^{\mathbf{N}}(T)$, and hence the partition function $Z(T)$ of the corresponding LMG-PF model (2.9), can be considerably simplified. Indeed, in this case

$$H_{\rm sp} = -\Delta + ar^2 + \sum_{i\neq j} \frac{a(a-\epsilon S_{ij})}{(x_i-x_j)^2} = H_{\rm sc} + 2a\hat{H}_0(\mathbf{x}), \tag{6.1}$$

where $r^2 \equiv \sum_i x_i^2$,

$$H_{\rm sc} = -\Delta + ar^2 + \sum_{i\neq j} \frac{a(a-1)}{(x_i-x_j)^2}$$

is the scalar Calogero model and

$$\hat{H}_0(\mathbf{x}) = \sum_{i<j} \frac{1-\epsilon S_{ij}}{(x_i-x_j)^2}$$

is obtained from H_0 by the formal substitution $z_i \mapsto x_i$. Proceeding as in section 3 we obtain the analogue of equation (3.2), namely

$$Z_0^{\mathbf{N}}(T) = \lim_{a\to\infty} \frac{Z_{\rm sp}^{\mathbf{N}}(2aT)}{Z_{\rm sc}(2aT)}, \tag{6.2}$$

where the partition function $Z_{\rm sc}(2aT)$ of the scalar Calogero model is given by [51]

$$Z_{\rm sc}(2aT) = q^{E_{\rm GS}/2a} \prod_i (1-q^i)^{-1}, \quad E_{\rm GS} \equiv a^2 N(N-1) + aN. \tag{6.3}$$

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In order to compute $Z^{\mathbf{N}}_{\mathrm{sp}}(2aT)$, we note [51] that the Hamiltonian (6.1) of the spin Calogero model is upper triangular in the basis with elements

$$|\mathbf{p},\mathbf{s}\rangle = \mathrm{e}^{-ar^2/2}\prod_{i<j}|x_i-x_j|^a\Lambda_\epsilon\Big(\prod_i x_i^{p_i}|\mathbf{s}\rangle\Big) \tag{6.4}$$

partially ordered by the total degree $|\mathbf{p}|$, with corresponding eigenvalues

$$E(\mathbf{p}) = 2a(\,p_1+\cdots+p_N) + E_{\mathrm{GS}}. \tag{6.5}$$

Of course, we must choose the quantum numbers $(\mathbf{p},\mathbf{s})$ in such a way that the states (6.4) are actually a basis. The main difference with the HS and FI models is that only in this case $E(\mathbf{p})$ and the admissible partial order of the basis states (6.4) do not depend on the ordering of the components of $\mathbf{p}$ [43, 44, 51]. As a consequence, we can choose the quantum numbers $(\mathbf{p},\mathbf{s})$ in each subspace $\Lambda_\epsilon\big[L^2(\mathbb{R}^N)\otimes\mathcal{H}(\mathbf{N})\big]$ as follows:

(i) We first order the components of the spin quantum number $\mathbf{s}$ increasingly, so that

$$\mathbf{s} = \big(\overbrace{1,\ldots,1}^{N_1},\ldots,\overbrace{m,\ldots,m}^{N_m}\big)$$

is now fixed.

(ii) In each block of $\mathbf{s}$ with fixed magnon number $|a\rangle$ we order the corresponding components of the vector $\mathbf{p}$ also increasingly, so that $\mathbf{p}=(\boldsymbol{\rho}^1,\ldots,\boldsymbol{\rho}^m)$ with

$$\boldsymbol{\rho}^j \equiv (\rho^j_1,\ldots,\rho^j_{N_j})$$

and $\rho^j_i \prec \rho^j_{i+1}$.

We thus have

$$E(\mathbf{p}) = E_{\mathrm{GS}} + 2a\sum_{i=1}^N p_i = E_{\mathrm{GS}} + 2a\sum_{j=1}^m\sum_{i=1}^{N_j}\rho^j_i,$$

and therefore

$$q^{-E_{\mathrm{GS}}/2a}Z^{\mathbf{N}}_{\mathrm{sp}}(2aT) = \sum_{\rho^k_i\prec\rho^k_{i+1}}\prod_{j=1}^m q^{\rho^j_1+\cdots+\rho^j_{N_j}} = \prod_{j=1}^m\sum_{0\leqslant\rho^j_1\prec\cdots\prec\rho^j_{N_j}} q^{\rho^j_1+\cdots+\rho^j_{N_j}}.$$

The inner sum in the latter formula can be computed in closed form, with the result

$$\sum_{0\leqslant\rho^j_1\prec\cdots\prec\rho^j_{N_j}} q^{\rho^j_1+\cdots+\rho^j_{N_j}} = q^{\frac{1-\epsilon}{4}N_j(N_j-1)}\prod_{i=1}^{N_j}(1-q^i)^{-1} \equiv q^{\frac{1-\epsilon}{4}N_j(N_j-1)}(q)^{-1}_{N_j},$$

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and thus

$$Z_{\rm sp}^{\mathbf{N}}(2aT) = \prod_{j=1}^{m} q^{\frac{1-\epsilon}{4}N_j(N_j-1)}(q)_{N_j}^{-1} = q^{\frac{1-\epsilon}{4}\sum_{j=1}^{m} N_j(N_j-1)} \prod_{j=1}^{m}(q)_{N_j}^{-1}.$$

From this equation and equations (6.2) and (6.3) we obtain the following closed-form expression for the restricted partition function of the PF chain:

$$Z_0^{\mathbf{N}}(T) = q^{\frac{1-\epsilon}{4}\sum_{j=1}^{m} N_j(N_j-1)} \frac{(q)_N}{\prod_{j=1}^{m}(q)_{N_j}}.$$

Finally, by equation (2.10) the partition function of the LMG-PF model is given by

$$\begin{aligned} Z(T) &= \sum_{N_1+\cdots+N_m=N} q^{h(\mathbf{N})+\frac{1-\epsilon}{4}\sum_{j=1}^{m} N_j(N_j-1)} \frac{(q)_N}{\prod_{j=1}^{m}(q)_{N_j}} \\ &\equiv \sum_{N_1+\cdots+N_m=N} q^{h(\mathbf{N})+\frac{1-\epsilon}{4}\sum_{j=1}^{m} N_j(N_j-1)} \begin{bmatrix} N \\ N_1,\ldots,N_m \end{bmatrix}_q. \end{aligned} \tag{6.6}$$

In particular, for $h = 0$ we recover the well-known formula for the partition function of the PF chain in reference [42].

7. Analysis of the spectrum and thermodynamics

In this section we shall take advantage of the knowledge of the restricted partition function of the gLMG models (2.9) to study several statistical properties of their spectrum and analyze the behavior of their thermodynamic functions for large N. To begin with, we have examined the level density of the restriction of the Hamiltonian to subspaces with a fixed magnum content. Since H_1 is constant on these subspaces, this is of course equivalent to studying the level density of the corresponding spin chains of HS type. It is well-known in this respect [53, 54] that the level density of the *complete* spectrum of the latter models becomes normally distributed in the $N \to \infty$ limit, essentially due to the existence of a description of the spectrum in terms of Haldane's motifs [28, 55]. We have computed the spectrum of the HS chain for up to $N = 26$ for su(2) and $N = 24$ for su(3) in the largest subspace $\mathcal{H}(\mathbf{N})$ (with $N_i = N/m$ for all i). Our results clearly indicate that the spectrum of the restriction of H_0 to this subspace is also normally distributed (see figure 1, left), with parameters μ and σ given by the mean and standard deviation of the restricted spectrum. For the FI and PF chains we have obtained similar results. This fact suggests [54] that in all three cases there might be a formula for the energies in each sector of the spectrum with fixed magnon numbers in terms of motifs.

Since the continuous part of the cumulative level density in each sector can be well approximated by a Gaussian distribution, the energies of the 'unfolded' spectrum [56] can be taken as

$$\eta_i = \int_{-\infty}^{E_i} g(t)\mathrm{d}t, \qquad g(E) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{(E-\mu)^2}{2\sigma^2}}, \quad i = 1,\ldots,n.$$

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According to a long-standing conjecture due to Berry and Tabor [49], the distribution of the (normalized) spacings between consecutive levels of the unfolded spectrum, defined as

$$s_i = (n-1)\frac{\eta_{i+1}-\eta_i}{\eta_n-\eta_1}, \quad i = 1, \ldots, n-1,$$

is expected to be Poissonian for 'generic' integrable systems. On the other hand, for a chaotic system the well-known Bohigas–Giannoni–Schmit conjecture posits that this distribution should be given by the Wigner distribution corresponding to the appropriate ensemble of random matrices [50]. In references [43, 44, 51] it was observed that the distribution $p(s)$ of the spacings between consecutive levels of the *whole* spectrum of all three chains of HS type follows none of the above distributions, but is typically given by the 'square root of a logarithm' law

$$P(s) = 1 - \frac{2}{\sqrt{\pi}\, s_{\max}}\sqrt{\log\left(\frac{s_{\max}}{s}\right)}, \tag{7.1}$$

where $P(s) = \int_0^s p(s')\mathrm{d}s'$ is the cumulative distribution and $s_{\max}$ is the maximum spacing. As shown in [57, 58], this is due to the fact that the raw spectrum of the latter chains is approximately equispaced and normally distributed. We have computed the distribution of consecutive (normalized) spacings in the subspaces mentioned above for the HS, PF and FI chains for $m = 2$ and 3. In all cases, the cumulative spacings distribution $P(s)$ fits equation (7.1) with remarkable accuracy (see the insets figure 1, right, for the HS chain). This clearly suggests that the (raw) spectrum of the restriction of the three HS-type chains to subspaces with fixed magnon content is also approximately equispaced. We have also verified that this conclusion is indeed correct for all three chains of HS type. For instance, for the su(2) HS chain with $N_1 = N_2 = 13$ (see figure 1, top right) 93.8% of the spacings between consecutive levels of the raw spectrum are equal to 1, while for the su(3) HS chain with $N_1 = N_2 = N_3 = 8$ (see figure 1, bottom right) the predominant spacing is again 1 and occurs 95.7% of the times.

We shall next analyze the thermodynamics of a class of LMG models of HS type whose deformation Hamiltonian (2.8) is given by

$$h(x_1, \ldots, x_m) = \frac{1}{N}\sum_{a=1}^{m}(x_a - n_a N)^2, \tag{7.2}$$

where the parameters n_a $(1 \leqslant a \leqslant m)$ are assumed to lie in the interval $(0, 1)$ and $n_1 + \cdots + n_m = 1$. These parameters thus represent the magnon densities of the ground state in the ferromagnetic case ($\epsilon = 1$). The motivation for considering a quadratic deformation Hamiltonian is, first of all, that in the original, isotropic LMG model the external term H_1 is precisely of this form. More recently, generalized LMG models with a quadratic external term have proved of interest in the context of quantum information theory, since they are some of the few systems for which the bipartite entanglement entropy of the ground state can be computed in closed form [11, 13, 19]. Using the exact formulas (4.8)–(4.12) and (6.6), we have evaluated the partition function of this class of models for a relatively large number of spins, of the order of 100 (resp. 50) for the su(2) (resp. su(3)) ferromagnetic LMG-PF models. From the resulting

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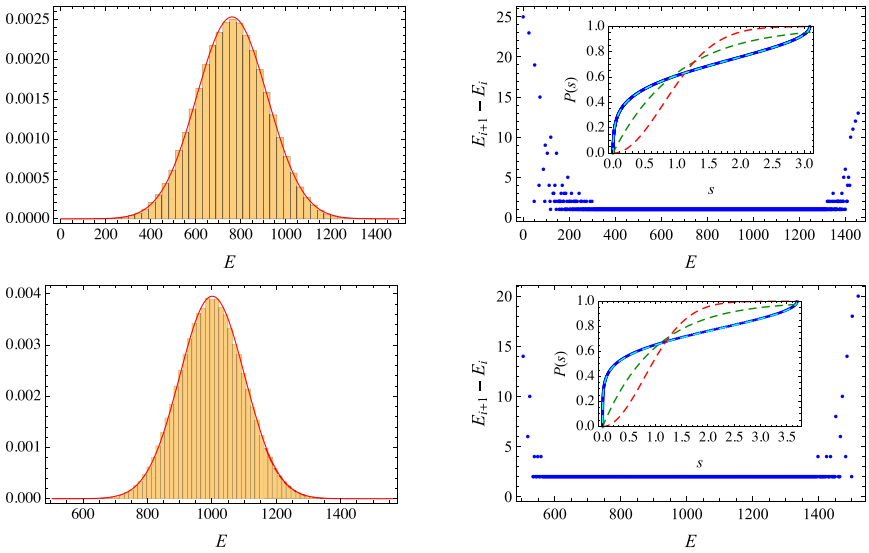

Figure 1. Left: level density histogram (normalized to unity) for the HS chain in a subspace with fixed magnon numbers $N_1 = N_2 = 13$ (su(2), top) and $N_1 = N_2 = N_3 = 8$ (su(3), bottom) compared to a Gaussian distribution (continuous red line). Right: differences between consecutive levels of the raw spectrum (main plots) and cumulative spacings distribution of the unfolded spectrum (insets) in the latter configurations. The dashed red, green and light blue curves in the insets are respectively the Wigner (GOE) and Poisson cumulative distributions and the law (7.1). (In all cases, we have used natural units $\hbar = 2M = 1$.).

expression, we have computed the free energy f, the internal energy u, the entropy s and the specific heat c (per spin, in all cases) via the formulas

$$f(T) = -\frac{T}{N}\log Z(T), \qquad u(T) = \frac{T^2}{N}\frac{\partial \log Z(T)}{\partial T}, \tag{7.3}$$

$$s(T) = \frac{\partial}{\partial T}\left(\frac{T}{N}\log Z(T)\right), \qquad c(T) = \frac{2T}{N}\frac{\partial \log Z(T)}{\partial T} + \frac{T^2}{N}\frac{\partial^2 \log Z(T)}{\partial T^2}, \tag{7.4}$$

where we have taken Boltzmann's constant $k_B = 1$. We have first verified that the thermodynamic functions are practically independent of N for $N \lesssim 100$ (in the su(2) case) and $N \lesssim 50$ (in the su(3) case). Thus the thermodynamic functions for $N = 100$ (in the su(2) case) and $N = 50$ (in the su(3) case) can be regarded as a reasonable approximation of their $N \to \infty$ counterparts. As an additional check, we have compared the results for the su(2) PF chain with no deformation Hamiltonian and $N = 100$ spins with the exact $N \to \infty$ formulas derived in [52], finding them in excellent agreement (see figure 2). In particular, the extensive behavior of the *thermodynamic* entropy contrasts with the logarithmic growth of the ground-state *entanglement* entropy of the ferromagnetic 'quadratic' gLMG models studied in reference [19].

In figures 2 and 3 we present the plots of the free and internal energies, the entropy and the specific heat (per spin) respectively of the su(2) and su(3) models (2.9)–(7.2)

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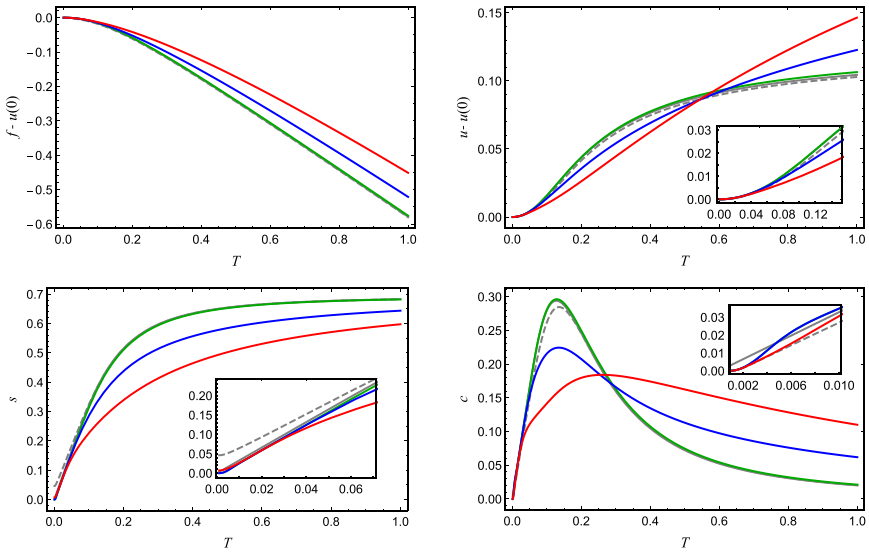

Figure 2. Thermodynamic functions for the su(2) ferromagnetic LMG-PF model with $h(N_1, N_2) = [(N_1 - n_1 N)^2 + (N_2 - n_2 N)^2]/N$ for $N = 100$ spins. The red, blue and green lines correspond respectively to the magnon densities $(n_1, n_2) = (1/8, 7/8)$, $(1/4, 3/4)$ and $(1/2, 1/2)$, while the continuous gray line represents the $h = 0$, $N = \infty$ exact result. (In all cases, we have used natural units $\hbar = 2M = k_B = 1$.).

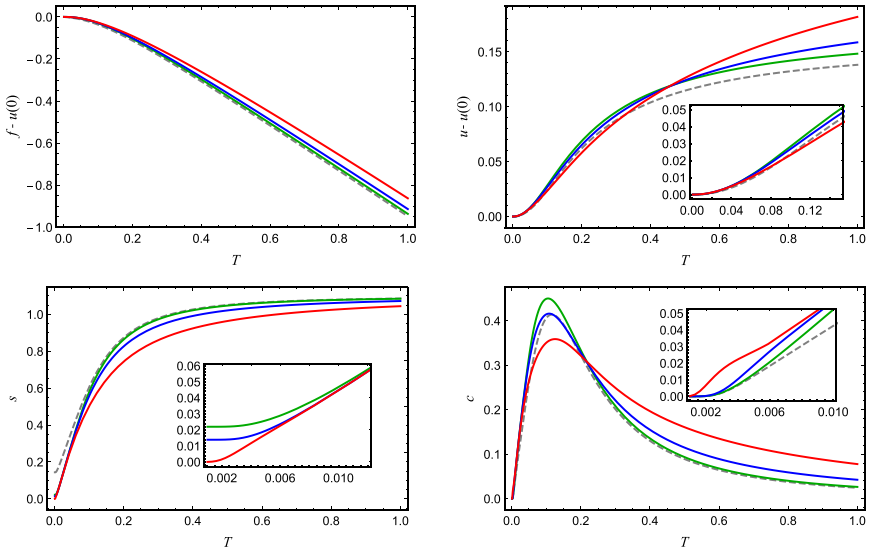

Figure 3. Thermodynamic functions for the su(3) ferromagnetic LMG-PF model with $h(N_1, N_2, N_3) = \sum_{i=1}^{3} (N_i - n_i N)^2/N$ for $N = 50$ spins. The red, blue and green lines correspond respectively to the magnon densities $(n_1, n_2, n_3) = (1/8, 1/4, 5/8)$, $(1/4, 1/4, 1/2)$ and $(1/3, 1/3, 1/3)$. (In all cases, we have used natural units $\hbar = 2M = k_B = 1$.).

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in the PF case. It is apparent from these figures that both the su(2) and the su(3) thermodynamic functions qualitatively behave like those of a two-level system, as for instance the one-dimensional Ising model at zero magnetic field or a paramagnetic spin 1/2 ion [59]. In particular, from figures 2 and 3 we see that the specific heat exhibits the Schottky peak characteristic of the latter systems. Finally, it may seem surprising that the entropy per spin does not appear to vanish at $T = 0$ in some cases, especially when $h = 0$ (see, e.g. figure 3). Of course, the explanation for this behavior is that the number of spins N is finite (though large), so that $s(0) = (\log d(m, N))/N$, where $d(m, N)$ is the ground state degeneracy. In the ferromagnetic case under consideration, it follows from equation (2.9) that when $h = 0$ the ground states are the symmetric states, so that

$$d(m, N) = \binom{N + m - 1}{m - 1} \simeq \frac{N^{m-1}}{(m-1)!},$$

and thus $s(0) \simeq (m - 1)(\log N)/N$ is small but nonzero. On the other hand, when h does not vanish identically the H_1 term in equation (2.9) breaks the ground state degeneracy almost completely (the more so in the less symmetric cases, in which the densities n_a are all different), so that $s(0)$ is significantly smaller than its $h = 0$ counterpart.

8. Conclusions

We shall finish this paper with a brief summary of its main results. We have introduced a family of generalized su(m) Lipkin–Meshkov–Glick models whose interacting term is a spin chain of Haldane–Shastry type, which can be equivalently regarded as the deformation of a spin chain of HS type H_0 by the addition of a term H_1 in the enveloping algebra of the Cartan subalgebra of su(m). The Hilbert space of the system is a direct sum of subspaces $\mathcal{H}(\mathbf{N})$ with fixed magnon numbers, in which the action of the deformation term is diagonal, so that the model's partition function decomposes as in equation (2.10). By a suitable adaptation of Polychronakos's freezing trick, we have been able to compute in a closed form the partition functions of the restrictions of the spin chain Hamiltonian H_0 to the subspaces $\mathcal{H}(\mathbf{N})$. In view of the previous remarks, this immediately yields the partition function of the associated gLMG model. In particular, when H_0 is the Hamiltonian of the Polychronakos–Frahm spin chain we have obtained an alternative, simpler expression for the partition function akin to Polychronakos's formula [42] for the case $H_1 = 0$. This closed-form expression for the partition function of the restriction of H_0 to the subspaces $\mathcal{H}(\mathbf{N})$ has been used in numerical calculations to provide strong evidence that the level density of the latter restriction is Gaussian when the number of spins tends to infinity. In view of the results of [54], this suggests that there exists a description of the spectrum of $H_0\big|_{\mathcal{H}(\mathbf{N})}$ in terms of motifs, a fact that deserves further investigation. We have also numerically studied the distribution of the spacings of consecutive unfolded levels of $H_0\big|_{\mathcal{H}(\mathbf{N})}$, showing that it follows the same characteristic law previously found for the complete spectrum. As a final application, we have computed the free and internal energies, the entropy and the specific heat per spin of a class of su(2) and su(3) gLMG models with quadratic H_1. We have checked

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that these functions are virtually independent of the number of spins N when this number is sufficiently large, which indicates that they yield reasonable approximations to their respective thermodynamic limits. Our analysis shows that the thermodynamic functions of these models are qualitatively similar to those of a two-level system, as already observed in [52] for the su(2) chains of HS type. In the latter chains, this similarity is ultimately due to the existence of a description of the spectrum in terms of motifs, which leads to simple closed formulas for the thermodynamic functions in terms of the dispersion relation. This again suggests that such a description should also exist for the more general models studied in this paper.

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CHAPTER 3

The long-range t-J model of Kuramoto and Yokoyama

This chapter is devoted to the study of the so-called long-range t-J model introduced by Kuramoto and Yokoyama [86,87]. Its Hamiltonian is shown to be equivalent to a suitable modification of a spin chain of Haldane–Shastry type [62,112] in terms of supersymmetric su(1|2) exchange operators. Its spectrum can be generated, with the correct degeneracy for each energy level, from supersymmetric Young tableaux and their corresponding motifs. All these results remain true for a larger class of systems; indeed, we construct generalized su(m) generalizations of the Kuramoto-Yokoyam model that can be solved in a similar as the original model.

1. The model

The so-called t-J model describing strongly correlated electrons in a lattice is well-known as one of the simplest toy models of high-temperature superconductivity featuring spin-charge separation. Each lattice site in this model can be either empty or occupied by one fermion that interacts with its nearest-neighbors through spin exchange and charge repulsion, and can also hop to one of its contiguous sites if it is empty.

We are interested in the long-range version of the t-J model introduced by Kuramoto and Yokoyama [86, 87] in the nineties (the KY model in what follows). In contrast to its original (short-range) counterpart, this new model exhibits strong spin-charge separation[20] and is solvable using asymptotic the Bethe ansatz. However, althoug in principle its energies can be obtained as the solution of a system of transcendental equations, this approach does not provide the degeneracy of each energy level. An unproved conjecture proposed by Saiga and Kuramoto [107] states that

[20]Spin and charge velocities (and not just their corresponding susceptibilities) are independent of the charge density and the magnetization, respectively, at low temperatures.

the spectrum (i.e., the set of different eigenvalues each counted with its corresponding multiplicity) for the KY model can be described in terms of a modified (supersymmetric) version of the celebrated Haldane motifs [**64**] and and their corresponding Young tableaux.

In [**P6**] we completely solved the problem of finding a full description of the spectrum of the KY model for a finite number of sites N, including the determination of the degeneracy of each energy level and the spin content characterizing the quantum numbers of the corresponding eigenfunctions. Our results provide a rigorous proof of the conjecture of Saiga and Kuramoto and, indeed, can be used to fully characterize the spectrum of a generalized system of su(m) fermions for all $m \geq 1$ in a lattice of N sites[21]. Indeed, we introduce a larger class of long-range t-J models describing systems in which each site is either empty or occupied by one fermion with flavour $a = 1, 2, \ldots, m$ thus generalizing the (long-range) su(2) KY model and being consequently called (long-range) su(m) KY models.

It is well-known that the su(2) KY model can be mapped to a supersymmetric su(1|2) HS spin chain perturbed with a suitable chemical potential term. This is done by reinterpreting the vacuum at each site of the su(2) KY model as the unique bosonic state at each position of the su(1|2) HS spin chain. Suppose that we know the spectrum of the Hamiltonian of the su(1|2) HS spin chain in subspaces spanned by eigenstates of the number operators, i.e., subspaces of fixed magnon content (n_a). Since the Hilbert space decomposes as the direct sum of subspaces with fixed magnon content, the full spectrum of the (unrestricted) perturbed Hamiltonian follows from the knowledge of its spectra when restricted to subspaces of fixed magnon content.

Summarizing, to solve the su(2) KY model is to obtain the spectrum of the su(1|2) HS spin chain when its Hamiltonian is restricted to subspaces with fixed magnon content. We study this model in [**P6**] and generalize the previous approach introducing (and solving) su(m) KY models related to supersymmetric su(1|m) HS spin chains perturbed by the addition of suitable chemical potential terms.

[21] In particular, for $m = 1$ we obtain a model that can be studied with the techniques of the first chapter and for $m = 2$ we obtain the system of spin-$\frac{1}{2}$ fermions first studied by Kuramoto and Yokoyama.

In [P7] we use the transfer matrix method of Ref. [52] to efficiently compute the partition function of the supersymmetric su($m|p$) Haldane–Shastry spin chain perturbed with the addition of a function linear in the corresponding $m+p$ number operators. This approach is particularly efficient in the thermodynamic limit. As explained in [52], in that case the trace of the product of (positive) site-dependent transfer matrices is replaced by the integral of their largest eigenvalue, whose existence and uniqueness is guaranteed by the classical theorem of Perron–Froebenius. From the latter partition function we derive the partition functions of each of the restrictions[22] of the perturbed Hamiltonian to subspaces of fixed magnon content.

1.1. The Hamiltonian

The su(2) KY model introduced by Kuramoto and Yokoyama in the nineties is a long-range version of the original (nearest-neighbors) supersymmetric t-J model. More generally, we shall consider here a class of su(m) KY models, consisting of a one-dimensional lattice with N sites each of which can be either empty or occupied by a single fermion with m internal degrees of freedom. We shall be mainly interested

Here our focus will be concentrated mainly in long-ranged models of this kind, in which both the interaction strength and the hopping amplitude are non-null between any pair of fermions. More precisely, we shall take as Hamiltonian of the su(m) KY model

$$H_{\mathrm{KY}} = J \sum_{i<j} \frac{1}{2} (\sin u_{ij})^{-2} \mathcal{P} \left(O_{ij}^{(1)} + O_{ij}^{(2)} \right) \mathcal{P} , \tag{18}$$

where $t \in \mathbf{R}$, $u_{ij} = u_i - u_j$ with $u_k = k\frac{\pi}{N}$ and the sum is over pairs of numbers in $\{0, \dots, N-1\}$. All the operators in the Hamiltonian can be written in terms of the m fermionic operators $c_{1i}, c_{2i}, \dots, c_{mi}$ and their adjoints where $c_{ai}^\dagger$ the operator that creates a fermion of flavor a at position i. Finally, $\mathcal{P}$ projects onto single-occupancy states in which each site is

[22] Write H_0 for any supersymmetric su($m|p$) Hamiltonian of Haldane–Shastry type and $H_1 = \mu_1 \mathfrak{n}_1 + \cdots + \mu_{m+p} \mathfrak{n}_{m+p}$ for the perturbation linear in the number operators. If H_0 commutes with each magnon operator $\mathfrak{n}_a$, the partition function of H_0 restricted to the subspace of magnon content (n_a) is given by the coefficient of $q^{n_1\mu_1} \cdots q^{n_{m+p}\mu_{m+p}}$ in the partition function of $H_0 + H_1$.

occupied by at most one fermion and

$$O_{ij}^{(1)} = -\sum_{a=1}^{m}\left(c_{ai}^{\dagger}c_{aj} + c_{aj}^{\dagger}c_{ai}\right) - \left(1-\frac{1}{m}\right)\sum_{a=1}^{m}\sum_{b=1}^{m} c_{ai}^{\dagger}c_{ai}c_{bj}^{\dagger}c_{bj}$$
$$O_{ij}^{(2)} = 2\left(x_i^1 x_j^1 + \cdots + x_i^{m^2-1}x_j^{m^2-1}\right),$$

with

$$x_i^{\alpha} = \sum_{a=1}^{m}\sum_{b=1}^{m}(x^{\alpha})_{ab}\, c_{ai}^{\dagger}c_{bi}$$

where $(x^{\alpha})_{ab}$ are the matrix elements of x^{α}, the Hermitian generators of su(m) in the fundamental representation satisfying the normalization condition $\mathrm{tr}(x^{\alpha}x^{\beta}) = \frac{1}{2}\delta_{\alpha\beta}$.

Let us write $\mathcal{F}$ for the Fock space and $\mathcal{PF}$ for the span of all single-occupancy states, the relevant subspace in which the previous Hamiltonian is not identically zero. Note that the canonical basis of $\mathcal{PF}$ contains elements $O_1O_2\cdots O_N|0\rangle$ with $|0\rangle$ the vacuum and each O_i being equal to one of the creation operators $c_{1i}^{\dagger},\ldots,c_{mi}^{\dagger}$ or the identity.

Consider the Hilbert space $\mathcal{H} = \otimes^N \mathbf{C}^{m+1}$ and the standard basis with elements $|s_1\cdots s_N\rangle$ where $s_i = 0,1,2,\ldots,m$. A unitary transformation $U : \mathcal{PF} \to \mathcal{H}$ can be defined such that the partition function of $H' = UH_{\mathrm{KY}}U^{\dagger}$, that coincides with that of H_{KY}, can be derived. Indeed, let us denote by $c_{0i}^{\dagger} = 1$ the identity operator in the Fock space and consider the linear map U defined by its action over canonical basis elements as $O_1\cdots O_N|0\rangle \mapsto |\varphi_1(O_1)\cdots\varphi_N(O_N)\rangle$ where $\varphi_i(c_{ai}^{\dagger}) = a$ for all values $a = 0,1,2,\ldots,m$. In this way, the canonical basis of $\mathcal{PF}$ is mapped into the standard basis of $\mathcal{H}$. If one writes $\varphi_i^{-1}(a) = c_{ai}^{\dagger}$ then it is clear that the linear map U is indeed a unitary transformation whose inverse $U^{-1} = U^{\dagger}$ can be defined through its action over the basis elements of $\mathcal{H}$ as $|s_1\cdots s_N\rangle \mapsto \varphi_1^{-1}(s_1)\cdots\varphi_N^{-1}(s_N)|0\rangle$.

Consider now a splitting $\mathbf{C}^{m+1} = \mathbf{C}\oplus\mathbf{C}^m$ associated to a system with one bosonic degree of freedom and m fermionic ones, and assume for definiteness that label 0 is bosonic and labels $1,2,\ldots,m$ are fermionic. The action of su($1|m$) graded permutations $P_{ij}^{(1|m)}$ is defined exactly as in Chapter 1 (Sec. 5.1) by the relation

$$P_i^{(m|p)}|\cdots s_i s_{i+1}\cdots\rangle = \begin{cases} -1|\cdots s_{i+1}s_i\cdots\rangle, & s_i, s_{i+1} > 0 \\ +1|\cdots s_{i+1}s_i\cdots\rangle, & \text{otherwise} \end{cases} \tag{19}$$

It was shown in [P6] that under the previous mapping

$$H' = UH_{\rm KY}U^\dagger = J\sum\nolimits_{i<j}\frac{1}{2}(\sin u_{ij})^{-2}\big(1-P_{ij}^{(1|m)}\big) - t_0\,\mathfrak{q} \tag{20}$$

$$J \equiv \frac{2\pi^2 t}{N^2} \in \mathbf{R} \qquad t_0 \equiv \frac{J}{6}(N^2-1)$$

where we have denoted by $\mathfrak{q} = \mathfrak{n}_1 + \mathfrak{n}_2 + \cdots + \mathfrak{n}_m = N\cdot 1 - \mathfrak{n}_0$ for the total fermion number (charge) operator. In view of the above mapping, we shall equivalently regard the bosons as uncharged spinless particles or holes. In the $m = 2$ case, the sum of the two species of fermions is then proportional to the total charge and its difference to the total magnetization. If we write H_0 for the Hamiltonian of the supersymmetric su$(1|m)$ HS spin chain, then it is apparent from the previous expression that $H' = JH_0 + H_1'$ with $H_1' = -t_0(N-\mathfrak{n}_0)$.

2. Spectrum

Obviously eigenvectors of $H_{\rm KY}$ and $H' = UH_{\rm KY}U^\dagger$ are related by U in a one-to-one fashion and have the same eigenvalue. As shown in Ref. [P6], the energy levels of H', and thus of $H_{\rm KY}$, can be parametrized by a supersymmetric motif $\delta = (\delta_1,\ldots,\delta_{N-1})$ and a magnon content $(n_a) = (n_0, n_1, \ldots, n_m)$ with $n_1 + \cdots + n_m = N - n_0$.

In fact, in Refs. [P6]-[P7] we actually study Hamiltonians H of the general form $H = H' + H_1$ with

$$H = H' - \frac{1}{2}\sum_{b=1}^{m-1} h_b(\mathfrak{n}_b - \mathfrak{n}_m) - \mu'\mathfrak{q} = JH_0 - \sum_{a=1}^{m}\mu_a\mathfrak{n}_a$$

where $(h_1,\ldots,h_{m-1})$ are the real components of a generalized magnetic field, $\mu' \in \mathbf{R}$ is the chemical potential of the fermions, and we have introduced the convenient quantities $\mu_a = \mu' + t_0 + \frac{1}{2}h_a$ for $a = 1,\ldots,m$ with $h_m = -(h_1 + \cdots + h_{m-1})$. It is shown in Ref. [P7] that the spectrum of H_0 in the subspace with magnon content (n_a) consists of the energies $E = J\mathcal{E}_\delta - \sum_{a=1}^m \mu_a n_a$ where

$$\mathcal{E}_\delta = \sum_{i=1}^{N-1}\delta_i(N-i)$$

and $\delta = (\delta_1,\ldots,\delta_{N-1})$ with $\delta_i \in \{0,1\}$ is a motif. Moreover, the degeneracy $S_\delta(n_a)$ (possibly equal to zero) of this level is given by the number

of sequences $(s_1, \dots, s_N) \in \{0, \dots, m\}^N$ (equivalent to supersymmetric Young tableaux) such that

(1) $|\{i : s_i = a\}| = n_a$,
(2) $\delta_i = 1$ if and only if $s_i > s_{i+1}$ or $s_i = s_{i+1} > 0$.

To illustrate this way of proceed, we shall focus in what follows on the su(2) case, and write $\mathfrak{n}_\uparrow$ and $\mathfrak{n}_\downarrow$ for the number operators associated to the two different fermionic degrees of freedom. We introduce the total magnetization operator $\mathfrak{m} = \frac{1}{2}(\mathfrak{n}_\uparrow - \mathfrak{n}_\downarrow)$ with eigenvalues $n_\mathfrak{m} = \frac{1}{2}(n_\uparrow - n_\downarrow)$ and write $n_\mathfrak{q} = n_\uparrow + n_\downarrow$ for the eigenvalues of $\mathfrak{q} = \mathfrak{n}_\uparrow + \mathfrak{n}_\downarrow$. Consider the Hamiltonian $H = JH_0 - h\mathfrak{m} - \mu\mathfrak{q}$, where $h \in \mathbf{R}$ represents the applied magnetic field and $\mu - t_0 = \mu' \in \mathbf{R}$ is the chemical potential of the fermions. In this case we can characterize each energy level and its degeneracy with a motif δ and a magnon content $(n_\uparrow, n_\downarrow)$ with generic energy levels

$$E = J\mathcal{E}_\delta - \Big(\mu + \frac{h}{2}\Big) n_\uparrow - \Big(\mu - \frac{h}{2}\Big) n_\downarrow = J\mathcal{E}_\delta - \mu n_\mathfrak{q} - h n_\mathfrak{m} \tag{21}$$

appearing with degeneracy given by $S_\delta(n_0, n_\uparrow, n_\downarrow)$ with $n_0 = N - n_\uparrow - n_\downarrow$. Of course, since $\mathfrak{n}_\uparrow = \frac{1}{2}\mathfrak{q} + \mathfrak{m}$ and $\mathfrak{n}_\downarrow = \frac{1}{2}\mathfrak{q} - \mathfrak{m}$, we can also describe the spectrum in terms of a motif δ and a magnon content of the form $(n_\mathfrak{q}, n_\mathfrak{m})$ with the replacements $n_\uparrow \mapsto \frac{1}{2}n_\mathfrak{q} + n_\mathfrak{m}$ and $n_\downarrow \mapsto \frac{1}{2}n_\mathfrak{q} - n_\mathfrak{m}$. Let us write $\varphi(\delta, n_\uparrow, n_\downarrow)$ for a given eigenfunction of the operators $\mathfrak{n}_\uparrow$ and $\mathfrak{n}_\downarrow$, with energy given by Eq. (21). The motif $\delta = (\delta_1, \dots, \delta_{N-1})$ in the latter equation satisfies satisfy $\delta_i = \delta(s_i, s_{i+1})$, with

$$\delta(s, s') = \begin{cases} 0\,, & (s, s') = (0,0), (0,\uparrow), (0,\downarrow), (\uparrow,\downarrow) \\ 1\,, & (s, s') = (\uparrow,0), (\downarrow,0), (\uparrow,\uparrow), (\downarrow,\downarrow), (\downarrow,\uparrow) \end{cases}$$

in accordance to the function previously defined if we take the ordering $\uparrow < \downarrow$ in the fermionic labels. Moreover, $n_\uparrow = n_\uparrow(s)$ and $n_\downarrow = n_\downarrow(s)$ with $n_\uparrow(s) = \{i : s_i = \uparrow\}$ and $n_\downarrow(s) = \{i : s_i = \downarrow\}$ respectively. Then the spectrum of H, with the correct degeneracy of each energy level, is given by the evaluation over all $(s_1, \dots, s_N)$ with $s_i \in \{0, \uparrow, \downarrow\}$ of the energy function

$$E(s_1, \dots, s_N) = J \sum_i \delta(s_i, s_{i+1}) \cdot i(N - i) - \mu \left(n_\uparrow(s) + n_\downarrow(s)\right) - \frac{h}{2}\left(n_\uparrow(s) - n_\downarrow(s)\right). \tag{22}$$

3. Ground state energy

Let us now choose[23] $t = 1/(2\pi^2)$ and without loss of generality, $h \geq 0$ (since the ground state associated to $h < 0$ is obtained from that associated to $-h$ by flipping the state of each spin). Within each subspace of magnon content specified by $n_\uparrow$ and $n_\downarrow$ (with $n_\uparrow \geq n_\downarrow$ since $h \geq 0$), the minimum energy is given by the minimum of the function $J\mathcal{E}_\delta = \sum_i \delta_i X_i(1 - X_i)$, with $X_k = k/N$, restricted to motifs δ associated to basis states $|s\rangle$ with magnon content $n_\uparrow(s) = n_\uparrow$ and $n_\downarrow(s) = n_\downarrow$. One can see [**P6**] that the basis state $|\bar{s}\rangle$ minimizing the energy within this subspace must be of the form

$$|\bar{s}_1 \cdots \bar{s}_N\rangle = |\cdots\uparrow\cdots\rangle\otimes|\cdots\uparrow\downarrow\cdots\rangle\otimes|\cdots 0\cdots\rangle\otimes|\cdots\uparrow\downarrow\cdots\rangle\otimes|\cdots\uparrow\cdots\rangle\,.$$

The previous expression for the ground-state candidate is invariant under the mapping $\bar{s}_i \mapsto \bar{s}_{N-i+1}$ reversing its entries, and the number of one-particle tensor factors in $|\cdots\uparrow\cdots\rangle$ and $|\cdots\uparrow\downarrow\cdots\rangle$ is respectively $\frac{1}{2}(n_\uparrow - n_\downarrow) = n_{\mathfrak{m}}$ and $n_\downarrow$ if we assume that both $n_\uparrow$ and $n_\downarrow$ are even numbers. (The variations when these two integers are not both even are minimal and the final expression for the ground state energy obtained in the thermodynamic limit is not affected.) Consequently, the number of one-particle tensor factors in $|\cdots 0\cdots\rangle$ is necessarily given by $n_0 = N - n_\uparrow - n_\downarrow$.

The motif $\bar{\delta} = \delta(\bar{s})$ associated to the previous basis state $|\bar{s}\rangle$ has entries $\bar{\delta}_i = \delta_i(\bar{s}_i, \bar{s}_{i+1})$ in terms of the function $\delta(s, s')$ introduced at the beginning of this section and which can be written conveniently as the following symmetric concatenation of five terms

$$\bar{\delta} = (1,\ldots,1)\,(0,1,\ldots,0,1)\,(0,\ldots,0)\,(0,1,\ldots,0,1)\,(1,\ldots,1)$$

where $\bar{\delta}_i = \bar{\delta}_{N-i}$. The first $\frac{1}{2}(n_\uparrow - n_\downarrow) = n_{\mathfrak{m}}$ entries of $\bar{\delta}$ are equal to one, the next $n_\downarrow$ entries are alternatively 0 and 1 and the central sector of entries equal to 0 has length n_0. According to Eq. (22), if we introduce $\bar{n}_{\mathfrak{q}} = n_{\mathfrak{q}}/N$ and $\bar{n}_{\mathfrak{m}} = n_{\mathfrak{m}}/N$ as respectively the density of charge and magnetization per particle, then in terms of the variables $u = \frac{1}{2}\bar{n}_{\mathfrak{q}}$ and $v = \bar{n}_{\mathfrak{m}}$ the energy per particle $E(\bar{s})/N$ is given by

$$\frac{E(\bar{s})}{N} = \frac{2}{N}\sum_{k=1}^{Nv} X_k(1 - X_k) + \frac{2}{N}\sum_{k=1}^{N(u-v)/2} X_{Nv+2k}\,(1 - X_{Nv+2k}) - 2\mu u - h v$$

[23] With this choice, $J = 1/N^2$ according to Eq. (20).

where we have used the fact that the energy is symmetric around $N/2$. Finally, introducing the function $G_\lambda(w) = -\lambda\, w + \int_0^w x\,(1-x)\,\mathrm{d}x$ we are left [**P6**] with the following expression for the energy per spin in the thermodynamic limit

$$e(u,v) = \lim_{N\to\infty} \frac{E(\bar{s})}{N} = G_{2\mu}(u) + G_h(v)\,.$$

The problem of finding the ground state has been reduced to that of minimizing $H(u,v)$ in the triangle $0 \le v \le u \le \frac{1}{2}$. We did so in [**P6**], obtaining a phase diagram composed of five different regions depending on wether the minimum of $e(u,v)$ is obtained in the interior of the previous triangle or on its sides. In particular, we find that the function e and its first derivatives are continuous in the latter triangle while the second derivatives being discontinuous at the boundaries of the previously referred different five ground state phases. Each of these phases is characterized by their magnon content: a phase with no vanishing magnon numbers ($n_a \neq 0$ for all $a = 0, \uparrow, \downarrow$), a phase with no bosons ($n_0 = 0$), a phase with bosons and only one type of fermions ($n_\downarrow = 0$), a phase without bosons and only one type of fermions ($n_0 = n_\downarrow = 0$), and a phase with only bosons ($n_\downarrow = n_\uparrow = 0$).

4. The su(m) elliptic t-J model

In Ref. [**P6**], we also introduce a novel su(m) t-J model that smoothly interpolates between the su(m) KY model discussed in the previous sections and its short-range counterpart. We shall refer to these models as su(m) Inozemtsev t-J, since because the relation between them and the su($1|m$) Inozemtsev's elliptic spin chain [**73**] is the same as that between su(m) KY models and su($1|m$) HS spin chains. Indeed, they are mapped to a supersymmetric su($1|m$) Inozemtsev spin chain perturbed with suitable chemical potential terms under the isomorphism U previously discussed.

Here we shall content ourselves with presenting the general expressions H_I and $UH_IU^\dagger$ analogous to Eqs. (18) and (20), for the su(m) Inozemtsev t-J models and its supersymmetric spin chain version. In general, we can consider Hamiltonians akin to H_{KY}, but with interaction and hopping amplitude of the form $t_{ij} = t(i-j)$ for some $t(k)$ satisfying $t(k) = t(-k) = t(N-k)$. Writing $t_0 = \sum_{k=1}^{N-1} t(k)$, we have [**P6**]

$$U \cdot \sum_{i<j} t_{ij}\, \mathcal{P}\left(O_{ij}^{(1)} + O_{ij}^{(2)}\right)\mathcal{P} \cdot U^\dagger = \sum_{i<j} t_{ij}\big(1 - P_{ij}^{(1|m)}\big) - t_0\, \mathfrak{q}\,,$$

which reduces to Eq. (20) for $t(k) = 2\pi^2 t/N^2 \sin^{-2}(k\pi/N)$. In fact, the previous Hamiltonian reduces to the su(m) version of the original (nearest-neighbor) t-J model for $t(k)$ proportional to an N periodic extension of $\delta_{k,1} + \delta_{k,N-1}$, in which case $t_{ij} \neq 0$ only when $|i-j|$ equals 1 or $N-1$. The Inozemtsev su(m) t-J model is described by a Hamiltonian $H_{\rm I}$ of the previous form with interaction strength $t(x) = t\, h_N(x)$ where $t \in \mathbf{R}$ and $h_N(x)$ is given by Eq. (9) in terms of Weierstrass elliptic and zeta functions. More precisely

$$UH_{\rm I}U^\dagger = t\left(\frac{\alpha}{\pi}\right)^2 \left(\sinh\frac{\pi}{\alpha}\right)^2 \sum_{i<j}\Big(\wp_N(i-j) - \frac{2\hat\eta_1}{\alpha^2}\Big)(1 - P_{ij}^{(1|m)}) - t_0\,\mathfrak{q}$$

$$t_0 = \frac{2t}{\pi^2}\left(\sinh\frac{\pi}{\alpha}\right)^2 (\hat\eta_1 - \eta_1)\,,$$

where $\eta_1 = \zeta(1/2; 1/2, \mathrm{i}(2\alpha))$. Note that we have omitted the explicit dependence on parameter α. It can be shown that[24]

$$UH_{t\text{-}J}U^\dagger = \lim_{\alpha\to 0+} UH_{\rm I}U^\dagger = t\sum_i (1 - P_{i,i+1}^{(1|m)}) - 2t\,\mathfrak{q}\,,$$

$$UH_{\rm KY}U^\dagger = \lim_{\alpha\to\infty} UH_{\rm I}U^\dagger\,,$$

where $H_{t\text{-}J}$ is the Hamiltonian of the su(m) version of the original nearest-neighbor t-J model.

Thus the Inozemtsev su(m) t-J model with Hamiltonian $H_{\rm I}$ interpolates between the su(m) version of the original t-J model and the su(m) KY model. Let us finally note that the previous expressions are valid for all m; in particular, setting $m = 1$ we recover the the su(1|1 supersymmetric spin chains of Chapter 1.

5. Thermodynamics

Due to the fact that the energy function $E(s)$ is linear in $n_a(s)$, we can write it as a sum of a function of just two variables evaluated at every nearest neighbor pair s_i, s_{i+1}. Let us then set $\mu_\uparrow = \mu + \frac{h}{2}$ and $\mu_\downarrow = \mu - \frac{h}{2}$ and rewrite Eq. (22) as

$$E(s_1, \ldots, s_N) = K\sum_{i=1}^N \delta(s_i, s_{i+1})\frac{i}{N}\left(1 - \frac{i}{N}\right) - \sum_i \mu_{s_i}$$

[24] See [51] for a detailed derivation.

where $K = 2\pi^2 t$ and note that as we have explained, the spectrum is generated (with the correct degeneracy) by evaluating $E(s)$ over all vectors $s = (s_1, \ldots, s_N)$ with $s_i = 0, 1, 2, \ldots, m$. The partition function can then be written as

$$Z = \operatorname{tr} A(X_0)A(X_1)\cdots A(X_{N-1})$$
$$A_{rs}(X) = q^{K\delta(r,s)\,\varepsilon(X)-\frac{1}{2}(\mu_r+\mu_s)}$$

where $X_k = k/N$ and $\varepsilon(x) = x(1-x)$. Note that $A(X)$ is a positive matrix, and thus has a simple eigenvalue of largest absolute magnitude $\lambda_1(X)$ according to the classical Perron–Froebenius theorem. It was shown in [**52**] that in the thermodynamic limit the free energy per particle $f = F/N = -(T/N)\log Z$ is given by

$$f = -2T\int_0^{1/2} \log \lambda_1(x)\mathrm{d}x\,.$$

In [**P7**] we are able to rigorously derive the thermodynamics from the previous approach by explicitly obtaining the complete asymptotic series for the free energy per particle. This is possible since the matrix A has order three and is degenerate, so that closed-form expressions for $\lambda_1(x)$ can be obtained.

It is worth mentioning the equivalence of our approach in Ref. [**P7**] with the so-called Kuramoto–Kato (KK) equation from which the first and second coefficients of this asymptotic series expansion can be derived ([**81**]). More precisely, these authors postulate that the spectrum of the su(2) KY model is effectively given by a set of free particles with generalized statistics and obtain what we call the KK equation as a self-consistency requirement. We show in [**P7**] how a solution of the KK equation and the Perron–Froebenius eigenvalue are related. Indeed, we construct a generalized KK equation with solution equivalent to the Perron–Froebenius eigenvalue of matrix A for arbitrary $\varepsilon(x)$ and conjecture an analogous relation for the general su($m|p$) supersymmetric models with Hamiltonian

$$H = \sum_{i<j} h(i-j)(1 - P_{ij}^{(p|m)})\,.$$

We refer to [**P7**] for further discussions regarding this topic and also for figures and plots presenting the results obtained for the relevant thermodynamic functions. In particular, for zero temperature we reproduce the

ground state results obtained previously with a completely different approach in [**P6**], and also find that the characteristic spin-charge separation of the model holds to all orders in the asymptotic expansion.

Supersymmetric t-J models with long-range interactions: partition function and spectrum

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Abstract.

We study the spectrum of the long-range supersymmetric su(m) t-J model of Kuramoto and Yokoyama in the presence of an external magnetic field and a charge chemical potential. To this end, we first establish the precise equivalence of a large class of models of this type to a family of su($1|m$) spin chains with long-range exchange interactions and a suitable chemical potential term. We exploit this equivalence to compute in closed form the partition function of the long-range t-J model, which we then relate to that of an inhomogeneous vertex model with simple interactions. From the structure of this partition function we are able to deduce an exact formula for the restricted partition function of the long-range t-J model in subspaces with well-defined magnon content in terms of its analogue for the equivalent vertex model. This yields a complete analytical description of the spectrum in the latter subspaces, including the precise degeneracy of each level, by means of the supersymmetric version of Haldane's motifs and their related skew Young tableaux. As an application, we determine the structure of the motifs associated with the ground state of the spin 1/2 model in the thermodynamic limit in terms of the magnetic field strength and the charge chemical potential. This leads to a complete characterization of the distinct ground state phases, determined by their spin content, in terms of the magnetic field strength and the charge chemical potential.

Keywords: integrable spin chains and vertex models, solvable lattice models

1. Introduction

Among lattice models of strongly correlated fermions, the t-J model holds a prominent position due to its role in the theoretical description of high-temperature superconductivity and as an example of a simple model featuring spin-charge separation [1–3]. In this model each lattice site can be either empty or occupied by one fermion, which interacts with its nearest neighbors through spin exchange and charge repulsion and can also hop between contiguous lattice sites.. In the one-dimensional case the t-J model is of particular interest, as it is both supersymmetric and exactly solvable through the nested Bethe ansatz when its parameters are suitably related [1, 4–8]. In the early nineties, Kuramoto and Yokoyama [9, 10] introduced a long-range version of the supersymmetric t-J model featuring $1/r^2$ interactions, which reduces to the su(2) Haldane–Shastry (HS) chain [11, 12] in the high density limit (i.e., when all the sites are occupied). Among other interesting features, the Kuramoto–Yokoyama (KY) model exhibits strong spin-charge separation, in the sense that at low temperatures the spin and charge velocities are respectively independent of the charge density and the magnetization. At low energies the KY model is known to be a Luttinger liquid [13], with spin and charge excitations independently described by a $c = 1$ conformal field theory (CFT).

The supersymmetric KY model is exactly solvable through the asymptotic Bethe ansatz pioneered by Sutherland and Shastry [14], as its energies can be found in principle by solving a system of transcendental equations in the asymptotic momenta [15, 16]. However, this method does not completely determine the spectrum, since it does not provide complete information on the degeneracy of each level. Based on numerical calculations, Wang et al. [17] proposed an empirical description of the degeneracies of the spectrum of the su(2) KY model reminiscent of the rule for filling the border strips associated to Haldane's motifs [18–20]. This description, however, is known to be incorrect in certain situations, although the needed corrections vanish in the thermodynamic limit [17]. Inspired by the equivalence between the su(2) KY model and the su(1|2) supersymmetric HS chain (up to a term proportional to the total electric charge), Saiga and Kuramoto [21] conjectured a description of the former model's spectrum essentially in terms of su(1|2)-supersymmetric Haldane motifs, which accounted for the numerical results for $N \leqslant 16$ spins. To the best of our knowledge, this conjecture has remained unproved in the literature.

In this paper we address and completely solve the problem of finding a full description of the spectrum of the supersymmetric su(m) KY model with a general chemical potential term for a finite number of sites, including the determination of the levels' degeneracies and spin content. In particular, our results provide a rigorous proof of the Saiga–Kuramoto conjecture for arbitrary m and N. Our approach, which we shall now briefly summarize, is new and bypasses the usual machinery of the asymptotic Bethe ansatz, transfer matrix, Yangian highest-weight states, etc. Indeed, we start by establishing the precise connection between the su(m) KY model and the su(1|m)

supersymmetric HS spin chain with a suitable chemical potential term for arbitrary m, thus generalizing the well-known result for $m = 2$. Since the partition function of the latter chain was recently evaluated in Ref. [22] by using Polychronakos's freezing trick, this immediately yields a novel closed formula for the partition function of the su(m) KY model. A remarkable property of this partition function is that it can be recast as the partition function of an equivalent (inhomogeneous) vertex model with simple interactions [23]. We show that both partition functions are polynomials in appropriate variables, whose coefficients are nothing but the corresponding restricted partition functions on subspaces with well-defined magnon content. This crucial observation provides a closed formula for the restricted partition function of the su(m) KY model (in the presence of an external magnetic field and a chemical potential term) on each of these subspaces. Finally, by analyzing the restricted partition function of the equivalent vertex model we are able to express the spectrum of the su(m) KY model in each subspace with well-defined magnon content in terms of suitably restricted supersymmetric Haldane motifs and their corresponding Young tableaux. This yields a complete and rigorous description of the spectrum in the latter subspaces, including a systematic way for determining the degeneracy of each level, which implies the Saiga–Kuramoto conjecture as a particular case.

It should be noted that, while the traditional freezing trick allows one to compute the partition function (and, in principle, the spectrum) of certain integrable systems, it does not provide any information about the corresponding wave functions. On the other hand, the analysis of the spectrum of the su(m) KY model by the method described above, extends the freezing trick and, more importantly, the equivalence to a vertex model, to subspaces of the Hilbert space with well-defined magnon content. It is in fact this connection with a restricted vertex model what makes it possible to identify all energy eigenvalues within any subspace of the Hilbert space with well-defined magnon content. In other words, our approach not only yields the complete spectrum of the su(m) KY model but also the magnon numbers or spin content of the corresponding wave functions.

Our approach yields several additional results that we shall now briefly discuss. In the first place, as a consequence of the general discussion of the equivalence of the su(m) KY model to an su($1|m$) supersymmetric HS chain with a suitable chemical potential term, we construct a new model of KY type with general elliptic interactions which can be mapped to a corresponding su($1|m$) generalization of Inozemtsev's chain [24]. This model certainly deserves further study, since it smoothly interpolates between the standard (nearest-neighbors) t-J model and the (long-range) KY model. Secondly, as an application of the description of the spectrum of the KY model in terms of supersymmetric Young tableaux, we determine the ground state of the spin 1/2 model in an external magnetic field in the thermodynamic limit. In particular, we give a complete description of the different ground state phases, characterized by their spin content — i.e., su($1|2$), su($1|1$) and su($0|2$), apart from the trivial phases consisting of only holes or fermions of one type— in terms of the magnetic field strength and the charge chemical

potential. This description goes beyond previously known results, which are restricted to the genuinely su(1|2) phase. In particular, we show that the strong spin-charge separation characteristic of the long-range t-J model at low temperatures [25] occurs in all nontrivial phases.

The paper is organized as follows. In Section 2 we introduce the model and show its equivalence to a supersymmetric HS chain with suitable chemical potential terms. We also introduce the model's elliptic version and discuss its connection with the supersymmetric extension of Inozmetsev's elliptic chain. The model's partition function is then computed in Section 3 by exploiting its equivalence to a supersymmetric HS chain. In Section 4 we recast the partition function as that of a suitable inhomogeneous vertex model and derive the model's restricted partition function on subspaces with well-defined magnon content. As explained above, this yields an explicit and complete description of the spectrum on the latter subspaces, including each level's degeneracy, in terms of suitable supersymmetric Young tableaux. Section 5 is devoted to a complete analysis of the ground state phases of the spin 1/2 model in an external magnetic field in the thermodynamic limit. In Section 6 we present our conclusions and outline several future developments. The paper ends with a short technical appendix in which we present the proof of a new result regarding the degeneracy of reverse motifs.

2. The models

We shall deal in this paper with a class of su(m) t-J type models, consisting of a one-dimensional lattice with N sites each of which can be either empty or occupied by a single fermion with m internal degrees of freedom. We shall be mainly interested in long-ranged models of the latter type, in which the spin and charge interactions among the fermions and their hopping amplitude involve all possible pairs of lattice sites. More precisely, we shall take as the model's Hamiltonian

$$H_0 = \sum_{1\leqslant i<j\leqslant N} \mathcal{P}\Big\{ -t_{ij} \sum_{\sigma=1}^{m} (c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma}) + 2J_{ij}\mathbf{T}_i\cdot\mathbf{T}_j - 2V_{ij}n_i n_j \Big\}\mathcal{P} \equiv \sum_{1\leqslant i<j\leqslant N} H_{ij}\,, \tag{2.1}$$

or equivalently‡

$$H_0 = \sum_{i\neq j} \mathcal{P}\Big\{ -t_{ij} \sum_{\sigma} c_{i\sigma}^\dagger c_{j\sigma} + J_{ij}\mathbf{T}_i\cdot\mathbf{T}_j - V_{ij}n_i n_j \Big\}\mathcal{P}\,, \tag{2.2}$$

where $t_{ij} = t_{ji}$, $J_{ij} = J_{ji}$, $V_{ij} = V_{ji}$ are real constants. We shall also assume that the model (2.1)-(2.2) is *translation-invariant*, i.e., that

$$t_{ij} = t(i-j)\,, \quad J_{ij} = J(i-j)\,, \quad V_{ij} = V(i-j) \tag{2.3}$$

with

$$t(x) = t(-x) = t(N-x)\,, \tag{2.4}$$

‡ Here and in what follows, unless otherwise stated sums and products over Latin indexes run over the set $1,\dots,N$, while Greek indices range from 1 to m.

and similarly for $J(x)$, $V(x)$. In the latter equations $c^\dagger_{i\sigma}$ (respectively $c_{i\sigma}$) denotes the operator creating (resp. destroying) a fermion of type σ at site i and $n_i = \sum_\sigma n_{i\sigma}$, where $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$, is the total number of fermions at site i. The operator $\mathcal{P}$ is the projector onto single-occupancy states, in which each site is occupied by at most one fermion. Finally, $\mathbf{T}_i \equiv (T^1_i, \dots, T^{m^2-1}_i)$, where T^r_i is the r-th su(m) Hermitian generator in the fundamental representation acting on the i-th site (with a suitable normalization that we shall specify below). Thus the first term (proportional to t_{ij}) in Eqs. (2.1)-(2.2) accounts for the hopping of fermions between sites i and j, while the last two terms respectively model the spin (exchange) and charge interaction between the latter sites.

The Hamiltonian (2.1)-(2.2) encompasses several well-known models, which we shall briefly review. To begin with, note that a nearest-neighbors version of the Hamiltonian (2.1)-(2.2) is obtained by taking $t(x)$, $J(x)$, $V(x)$ proportional to the N-periodic extension of

$$\delta_{1,x} + \delta_{N-1,x}\,, \qquad 1 \leqslant x \leqslant N-1\,. \tag{2.5}$$

When $m = 2$, this is the original t-J model introduced in Ref. [1]. On the other hand, the long-ranged supersymmetric Kuramoto–Yokoyama model [9, 10] follows from Eqs. (2.2)-(2.3) when

$$t(x) = J(x) = 4V(x) = \frac{t\pi^2}{N^2} \sin^{-2}(\pi x/N)\,,$$

where t is a positive real parameter. More generally, when $m = 2$ the model (2.2)-(2.3) is supersymmetric provided that $t(x) = J(x) = 4V(x)$. In fact, one of our aims in this section is to generalize the latter result to the su(m) case with arbitrary $m > 2$. As we shall next see, the key idea in this respect is to map the original Hamiltonian (2.1) to that of a suitable supersymmetric spin chain in which the holes are regarded as bosons.

More precisely, consider a one-dimensional lattice (*spin chain*) each of whose sites are occupied either by a boson or an su(m) fermion. Thus the model's Hilbert space is $\hat{\mathcal{H}} = \otimes_{i=1}^N \hat{\mathcal{H}}_i$, where $\hat{\mathcal{H}}_i$ is the linear span of the one-particle states $b^\dagger_i|\hat{\Omega}\rangle_i$, $f^\dagger_{i\sigma}|\hat{\Omega}\rangle_i$ with $\sigma = 1, \dots, m$, $b^\dagger_i$, $f^\dagger_{i\sigma}$ are the operators creating respectively a boson and a fermion of type σ at the i-th site, and $|\hat{\Omega}\rangle_i$ is the vacuum. Similarly, denote by $\mathcal{H} = \otimes_{i=1}^N \mathcal{H}_i$ the Hilbert space of the original model (2.1), where $\mathcal{H}_i$ is spanned by the vacuum $|\Omega\rangle_i$ and the one-particle states $c^\dagger_{i\sigma}|\Omega\rangle_i$. We now introduce the unitary mapping $\varphi : \mathcal{H} \to \hat{\mathcal{H}}$ defined by

$$\varphi|\Omega\rangle_i = b^\dagger_i|\hat{\Omega}\rangle_i\,, \quad \varphi(c^\dagger_{i\sigma}|\Omega\rangle_i) = f^\dagger_{i\sigma}|\hat{\Omega}\rangle_i\,.$$

This mapping induces a natural way of associating to each linear operator A acting on $\mathcal{H}$ a corresponding linear operator $\hat{A} = \varphi A \varphi^{-1} = \varphi A \varphi^\dagger$ acting on $\hat{\mathcal{H}}$. Note, in particular, that $\widehat{(A^\dagger)} = \varphi A^\dagger \varphi^\dagger \equiv \hat{A}^\dagger$. It is straightforward to check that under this correspondence $\hat{c}_{i\sigma} = b^\dagger_i f_{i\sigma}$, since both operators agree on the canonical basis of $\hat{\mathcal{H}}_i$. Note than on $\mathcal{H}_i$ we have $c_{i\sigma} = \mathcal{P} c_{i\sigma} \mathcal{P}$, so that we can also write

$$\widehat{(\mathcal{P} c_{i\sigma} \mathcal{P})} = b^\dagger_i f_{i\sigma} \equiv X^{0\sigma}_i, \tag{2.6}$$

and therefore (since $\mathcal{P}$ is Hermitian, being a projector)

$$(\mathcal{P}c_{i\sigma}^\dagger\mathcal{P})\hat{} = \hat{c}_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i \equiv X_i^{\sigma 0}\,. \tag{2.7}$$

We shall also need in the sequel the relations

$$(\mathcal{P}c_{i\sigma}^\dagger c_{i\sigma'}\mathcal{P})\hat{} = f_{i\sigma}^\dagger f_{i\sigma'} \equiv X_i^{\sigma\sigma'}\,, \tag{2.8}$$

$$[\mathcal{P}(1-n_i)\mathcal{P}]\hat{} = b_i^\dagger b_i \equiv X_i^{00}\,, \tag{2.9}$$

which easily follow from the previous ones. For instance, taking into account that $c_{i\sigma'}\mathcal{P} = \mathcal{P}c_{i\sigma'}\mathcal{P}$ and $\mathcal{P}^2 = \mathcal{P}$ we have

$$\begin{aligned}(\mathcal{P}c_{i\sigma}^\dagger c_{i\sigma'}\mathcal{P})\hat{} &= (\mathcal{P}c_{i\sigma}^\dagger\mathcal{P}\cdot\mathcal{P}c_{i\sigma'}\mathcal{P})\hat{}\\ &= f_{i\sigma}^\dagger b_i b_i^\dagger f_{i\sigma'} = X_i^{\sigma\sigma'} + f_{i\sigma}^\dagger b_i^\dagger f_{i\sigma} b_i = X_i^{\sigma\sigma'},\end{aligned}$$

since $f_{i\sigma}b_i = 0$ on $\hat{\mathcal{H}}_i$.

Consider next the su$(1|m)$ supersymmetric permutation operators $P_{ij}^{(1|m)} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ (with $i<j$), whose action on the canonical basis

$$|\sigma_1\cdots\sigma_N\rangle \equiv a_{1\sigma_1}^\dagger\cdots a_{N\sigma_N}^\dagger|\hat{\Omega}\rangle\,,\quad \sigma_1,\ldots,\sigma_N\in\{0,\ldots,m\},$$

where $a_{i0}=b_i$ and $a_{i\sigma}=f_{i\sigma}$ for $\sigma\geqslant 1$ and $|\hat{\Omega}\rangle = \otimes_i|\hat{\Omega}\rangle_i$ is the global vacuum, is given by [26]

$$P_{ij}^{(1|m)}|\cdots\sigma_i\cdots\sigma_j\cdots\rangle = \epsilon(\boldsymbol{\sigma})|\cdots\sigma_j\cdots\sigma_i\cdots\rangle\,.$$

The sign $\epsilon(\boldsymbol{\sigma})$ is 1 (respectively -1) if $\sigma_i=\sigma_j=0$ (resp. $\sigma_i,\sigma_j\geqslant 1$), while for $\sigma_i\sigma_j=0$ and $\sigma_i\neq\sigma_j$ it is equal to the number of fermionic spins σ_k with $i+1\leqslant k\leqslant j-1$. It is well known [19,26,27] that $P_{ij}^{(1|m)}$ can be expressed in terms of creation and annihilation operators as follows:

$$P_{ij}^{(1|m)} = \sum_{\alpha,\beta=0}^{m} a_{i\alpha}^\dagger a_{j\beta}^\dagger a_{i\beta}a_{j\alpha} = \sum_{\alpha,\beta=0}^{m}(-1)^{p(\beta)}X_i^{\alpha\beta}X_j^{\beta\alpha}\,,$$

where $p(0)=0$ and $p(\sigma)=1$ for $\sigma\geqslant 1$. We thus have

$$P_{ij}^{(1|m)} = X_i^{00}X_j^{00} + \sum_\sigma (X_i^{\sigma 0}X_j^{0\sigma} - X_i^{0\sigma}X_j^{\sigma 0}) - P_{ij}\,, \tag{2.10}$$

where

$$P_{ij} = \sum_{\sigma,\sigma'} X_i^{\sigma\sigma'}X_j^{\sigma'\sigma}$$

is the ordinary permutation operator when acting on purely fermionic states.

Our next goal is to relate the product $\mathbf{T}_i\cdot\mathbf{T}_j$ appearing in the Hamiltonian (2.1) with the supersymmetric permutation operator $P_{ij}^{(1|m)}$. To this end, note first of all that the components T_i^r of $\mathbf{T}_i$ are defined in the usual way as

$$T_i^r = \sum_{\sigma,\sigma'} T_{\sigma\sigma'}^r c_{i\sigma}^\dagger c_{i\sigma'}\,, \tag{2.11}$$

where the complex numbers $T_{\sigma\sigma'}^r$ are the matrix elements of the r-th (Hermitian) generator of su(m) in the fundamental representation. We shall normalize the $m\times m$ matrices $T^r \equiv (T_{\sigma\sigma'}^r)_{1\leqslant\sigma,\sigma'\leqslant m}$ so that

$$\mathrm{tr}(T^rT^s) = \frac{1}{2}\,\delta_{rs}\,.$$

In particular, when $m = 2$ the operator T^r can be taken as the usual spin 1/2 operator $S^r = \sigma^r/2$, where σ^r is the r-th Pauli matrix. In order to relate $\mathbf{T}_i \cdot \mathbf{T}_j$ with $P_{ij}^{(1|m)}$ we shall make use of the identity

$$2\sum_{r=1}^{m^2-1}(T^r)_{\sigma\sigma'}(T^r)_{\mu\mu'} = \delta_{\sigma\mu'}\delta_{\sigma'\mu} - \frac{1}{m}\,\delta_{\sigma\sigma'}\delta_{\mu\mu'}\,, \tag{2.12}$$

which is a direct consequence of the completeness of the generators T^r together with the identity matrix. From Eqs. (2.8), (2.11) and (2.12) we obtain

$$\begin{aligned}2(\mathcal{P}\mathbf{T}_i \cdot \mathbf{T}_j\mathcal{P})\hat{} &= 2\sum_{r=1}^{m^2-1}\sum_{\substack{\sigma,\sigma'\\ \mu,\mu'}} T^r_{\sigma\sigma'}T^r_{\mu\mu'}X_i^{\sigma\sigma'}X_j^{\mu\mu'} = \sum_{\sigma,\sigma'} X_i^{\sigma\sigma'}X_j^{\sigma'\sigma} - \frac{1}{m}\sum_{\sigma,\mu}X_i^{\sigma\sigma}X_i^{\mu\mu}\\ &= P_{ij} - \frac{1}{m}\,\hat{n}_i\hat{n}_j\,,\end{aligned} \tag{2.13}$$

where $\hat{n}_k = \sum_\sigma f^\dagger_{k\sigma}f_{k\sigma}$ denotes the total number of fermions (created by the operators $f^\dagger_{k\sigma}$) at the k-th site. From Eq. (2.10) for the supersymmetric permutation operator $P_{ij}^{(1|m)}$ we obtain, after some algebra,

$$P_{ij}^{(1|m)} + \hat{n}_i + \hat{n}_j - 1 = \hat{A}_{ij}\,, \tag{2.14}$$

where

$$A_{ij} = \mathcal{P}\Big[\sum_\sigma (c^\dagger_{i\sigma}c_{j\sigma} + c^\dagger_{j\sigma}c_{i\sigma}) - 2\mathbf{T}_i\cdot\mathbf{T}_j + \Big(1-\frac{1}{m}\Big)n_in_j\Big]\mathcal{P}.$$

Comparing with Eq. (2.1) we deduce that H_{ij} will be proportional to A_{ij} provided that

$$t(x) = J(x) = 2\Big(1-\frac{1}{m}\Big)^{-1}V(x)\,, \tag{2.15}$$

and in that case

$$\hat{H}_{ij} = t_{ij}(1 - P_{ij}^{(1|m)} - \hat{n}_i - \hat{n}_j)\,.$$

In other words, when condition (2.15) is satisfied, i.e., when H_0 is of the form

$$H_0 = \sum_{i<j} t_{ij}\mathcal{P}\Big[-\sum_\sigma (c^\dagger_{i\sigma}c_{j\sigma} + c^\dagger_{j\sigma}c_{i\sigma}) + 2\mathbf{T}_i\cdot\mathbf{T}_j - \big(1-\tfrac{1}{m}\big)\,n_in_j\Big]\mathcal{P}\,, \tag{2.16}$$

the corresponding Hamiltonian $\hat{H}_0$ is given by

$$\hat{H}_0 = \sum_{i<j} t_{ij}(1 - P_{ij}^{(1|m)} - \hat{n}_i - \hat{n}_j) = \sum_{i<j} t_{ij}(1 - P_{ij}^{(1|m)}) - \sum_{i\neq j} t_{ij}\hat{n}_j\,.$$

Note that so far we have not used the translation-invariance conditions (2.3)-(2.4), so that the previous result is valid in full generality. On the other hand, when the model (2.1) is translation-invariant we can use Eqs. (2.3)-(2.4) to further simplify the last term in the previous equation. Indeed, in this case

$$\sum_{i\neq j} t_{ij}\hat{n}_j = \sum_j \Big(\sum_{i;i\neq j} t_{ij}\Big)\hat{n}_j\,,$$

with

$$\sum_{i;i\neq j} t_{ij} = \sum_{k=1-j}^{-1} t(k) + \sum_{k=1}^{N-j} t(k) = \sum_{k=N-j+1}^{N-1} t(k-N) + \sum_{k=1}^{N-j} t(k) = \sum_{k=1}^{N-1} t(k) \equiv t_0\,,$$

so that

$$\hat{H}_0 = \sum_{i<j} t_{ij}(1 - P_{ij}^{(1|m)}) - t_0 \mathcal{F}, \tag{2.17}$$

where

$$\mathcal{F} \equiv \sum_i \hat{n}_i$$

is the total number of fermions.

Summarizing, we have shown that the translation-invariant su(m) model (2.1)-(2.3) is supersymmetric if its coefficients are related by Eq. (2.15). When this is the case this general model reduces to (2.16), which is equivalent to the su($1|m$) supersymmetric spin chain (2.17). It should be stressed that the coefficient of the charge interaction term $n_i n_j$ in the supersymmetric t-J Hamiltonian (2.16) must depend on m as specified in the latter equation, a fact that does not seem to have been previously noted in the literature.

Note that Eqs. (2.16)–(2.17) are also valid for $m = 1$. In this case the Hamiltonian (2.16) is that of a free fermion system (the terms proportional to $\mathbf{T}_i \cdot \mathbf{T}_j$ and $n_i n_j$ vanish identically), as first noted by Göhmann and Wadati [28]. This fact was recently exploited in Refs. [29, 30] to evaluate the entanglement entropy of the ground state of (translation-invariant) su($1|1$) spin chains of HS type.

In the rest of this work we shall be mainly concerned with the supersymmetric t-J model (2.16) with

$$t(x) = t(\pi/N)^2 \sin^{-2}(\pi x/N)\,, \tag{2.18}$$

which is the su(m) version of the original KY model. In this case [31, 32]

$$\sum_{i\neq j} \sin^{-2}(\pi(i-j)/N) = \frac{N}{3}\,(N^2-1)\,, \tag{2.19}$$

so that

$$t_0 = \frac{t\pi^2}{3N^2}(N^2-1)\,. \tag{2.20}$$

Hence the Hamiltonian $\hat{H}_0$ of the equivalent su($1|m$) supersymmetric spin chain can be written as

$$\hat{H}_0 = \frac{2t\pi^2}{N^2}\Big[H_{\rm HS}^{(1|m)} - \frac{1}{6}(N^2-1)\mathcal{F}\Big]\,, \tag{2.21}$$

where

$$H_{\rm HS}^{(1|m)} = \frac{1}{2}\sum_{i<j} \frac{1 - P_{ij}^{(1|m)}}{\sin^2(\pi(i-j)/N)} \tag{2.22}$$

is the Hamiltonian of the su($1|m$) Haldane–Shastry spin chain [11, 12, 19, 33]. In fact, the model (2.21) was introduced by Kawakami in the early 90's [15]. On the other hand, when $t(x)$ is proportional to Eq. (2.5) the Hamiltonian (2.16) reduces to the su(m) version of the original (nearest-neighbor) t-J model

$$H_0 = t\sum_i \mathcal{P}\Big[-\sum_\sigma (c_{i\sigma}^\dagger c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger c_{i\sigma}) + 2\mathbf{T}_i\cdot\mathbf{T}_{i+1} - \Big(1-\tfrac{1}{m}\Big)\,n_i n_{i+1}\Big]\mathcal{P}, \tag{2.23}$$

where $N+1 \equiv 1$. The equivalent su$(1|m)$ supersymmetric chain Hamiltonian is given by

$$\hat{H}_0 = t \sum_i (1 - P_{i,i+1}^{(1|m)}) - 2t\mathcal{F}\,, \tag{2.24}$$

where $P_{N,N+1}^{(1|m)} \equiv P_{1N}^{(1|m)}$, which is essentially the su$(1|m)$ Uimin–Lai–Sutherland model [4, 34, 35].

Next, inspired by Inozemtsev's elliptic spin chain [24], we introduce a one-parameter family of supersymmetric su(m) t-J models (2.16) which smoothly interpolate between the su(m) KY model (2.16)-(2.18) and the (periodic) nearest-neighbors su(m) t-J model (2.23). More precisely, let

$$t(x) = t\Big(\frac{\alpha}{\pi}\Big)^2 \sinh^2(\pi/\alpha)\Big(\wp_N(x) - \frac{2\tilde{\eta}_1}{\alpha^2}\Big)\,, \tag{2.25}$$

where

$$\wp_N(x) \equiv \wp(x; N/2, \mathrm{i}\alpha/2), \quad \tilde{\eta}_1 \equiv \zeta(1/2; 1/2, \mathrm{i}N/(2\alpha))$$

and $\alpha > 0$. In the latter formulas $\wp(x;\omega_1,\omega_3)$ and $\zeta(x;\omega_1,\omega_3)$ denote respectively the Weierstrass elliptic function with half-periods ω_1, ω_3 and its corresponding zeta function, defined by

$$\begin{aligned}
\zeta(z;\omega_1,\omega_3) &= \frac{1}{z} + \sum_{l,n\in\mathbb{Z}^2-\{(0,0)\}} \Big[\frac{1}{z - 2l\omega_1 - 2n\omega_3} + \frac{z}{(2l\omega_1 + 2n\omega_3)^2}\Big]\,,\\
\wp(z;\omega_1,\omega_3) &= -\zeta'(z;\omega_1,\omega_3)\,.
\end{aligned}$$

It can be shown [36, 37] that when $1 \leqslant x \leqslant N-1$

$$\lim_{\alpha\to 0+} t(x) = t(\delta_{1,x} + \delta_{-1,x}),$$

while the $\alpha \to +\infty$ limit of Eq. (2.25) is Eq. (2.18). Note that the constant t_0 for the function (2.25) is given by

$$t_0 = \frac{2t}{\pi^2} \sinh^2(\pi/\alpha)(\tilde{\eta}_1 - \eta_1)\,, \quad \eta_1 \equiv \zeta(1/2; 1/2, \mathrm{i}/(2\alpha))$$

(see, e.g., Ref. [38]). Thus the Hamiltonian (2.17) of the su$(1|m)$ supersymmetric spin chain equivalent to the su(m) t-J model (2.16) with elliptic interactions (2.25) is given by

$$\hat{H}_0 = tH_I^{(1|m)} - \frac{2t}{\pi^2} \sinh^2(\pi/\alpha)(\tilde{\eta}_1 - \eta_1)\mathcal{F},$$

where

$$H_I^{(1|m)} = \Big(\frac{\alpha}{\pi}\Big)^2 \sinh^2(\pi/\alpha) \sum_{i<j} \Big(\wp_N(i-j) - \frac{2\tilde{\eta}_1}{\alpha^2}\Big)(1 - P_{ij}^{(1|m)})$$

is the su$(1|m)$ version of Inozemtsev's elliptic spin chain [24]. In fact, for $m = 1$ the partition function and thermodynamics of the latter chain were derived in Ref. [37], and the entanglement entropy of its ground state was analyzed in Ref. [29].

3. Partition function

In this section we shall compute in closed form the partition function of the su(m) KY model (2.16)-(2.18) by exploiting its equivalence with the su($1|m$) spin chain Hamiltonian of Haldane–Shastry type (2.21)-(2.22). As a matter of fact, we shall consider the more general Hamiltonian

$$H = H_0 - \frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (n^\sigma - n^m) - \mu_c \sum_\sigma n^\sigma \equiv H_0 + H_1, \tag{3.1}$$

where H_0 is given by Eqs. (2.16)-(2.18) and

$$n^\sigma \equiv \sum_i n_{i\sigma}$$

denotes the total number of fermions of type σ. The last term in H_1 is the chemical potential of the fermions (or, equivalently, of the total electric charge), while the first one has a natural interpretation as arising from the interaction with an external su(m) magnetic field with strengths $h_1, \ldots, h_{m-1}$. Indeed, for $m = 2$ the term $-(h_1/2)(n^1 - n^2)$ equals $-h_1 S^z$, where S^z is the z component of the total spin operator. This is indeed the contribution to the energy arising from the interaction with the magnetic field $h_1 \mathbf{e}_z$ of a charged fermion (with gyromagnetic ratio $g = 2$, and unit mass and electric charge in natural units). More generally, for arbitrary $m \geqslant 2$ we have

$$n^\sigma - n^m = \sum_i (n_{i\sigma} - n_{im}),$$

where the operators $\mathrm{i}(n_{k\sigma} - n_{km})$ generate the standard su(m) Cartan subalgebra at the k-th site. By Eq. (2.8), the su($1|m$) spin chain Hamiltonian $\hat{H}$ equivalent to H is $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_1 = -\frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (\mathcal{N}_\sigma - \mathcal{N}_m) - \mu_c \mathcal{F}$$

and $\mathcal{N}_\sigma \equiv \sum_i \hat{n}^\sigma$ is the total numbers of fermions (created by $f^\dagger_{i\sigma} f_{i\sigma}$) of type σ. More explicitly, we have

$$\hat{H} = J H^{(1|m)}_{\mathrm{HS}} - \frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (\mathcal{N}_\sigma - \mathcal{N}_m) - (t_0 + \mu_c)\mathcal{F}\,, \tag{3.2}$$

with t_0 given by Eq. (2.20) and $J = 2t\pi^2/N^2$. The latter equation can be more concisely rewritten as

$$\hat{H} = J H^{(1|m)}_{\mathrm{HS}} - \sum_\sigma \mu_\sigma \mathcal{N}_\sigma, \tag{3.3}$$

where μ_σ is the chemical potential of the fermion of type σ, given by

$$\mu_\sigma = \frac{1}{2} h_\sigma + \mu_c + t_0\,, \quad 1 \leqslant \sigma \leqslant m-1\,; \tag{3.4}$$

$$\mu_m = -\frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma + \mu_c + t_0\,. \tag{3.5}$$

From the above remarks, it follows that the partition function $\mathcal{Z}$ of the supersymmetric su(m) KY model (3.1) coincides with that of the su($1|m$) supersymmetric Haldane–Shastry chain with a chemical potential term given in Eq. (3.3). The partition function of the latter model has been recently evaluated in Ref. [22] by taking advantage of its connection with the supersymmetric spin Sutherland model via Polychronakos's freezing trick [39, 40], with the result

$$\mathcal{Z}(q;\boldsymbol{\mu}) = \sum_{\mathbf{k}\in\mathcal{P}_N} d(\mathbf{k}) q^{\sum_{i=1}^{r-1} JK_i(N-K_i)} \prod_{i=1}^{N-r} (1-q^{JK'_i(N-K'_i)})\,. \tag{3.6}$$

Here $q \equiv \mathrm{e}^{-1/T}$, $\mathbf{k} = (k_1,\dots,k_r) \in (\mathbb{N}\cup\{0\})^r$, $\mathcal{P}_N$ is the set of partitions (with order taken into account) of the integer N, $K_i \equiv \sum_{j=1}^{i} k_j$, and the $N-r$ positive integers K'_i are defined by

$$\{K'_1,\dots,K'_{N-r}\} = \{1,\dots,N-1\} - \{K_1,\dots,K_{r-1}\}\,.$$

For each multiindex $\mathbf{k} \in (\mathbb{N}\cup\{0\})^r$, the coefficient $d(\mathbf{k})$ is defined by

$$d(\mathbf{k}) = \prod_{i=1}^{r} \sum_{j=0}^{\min(k_i,m)} e_j(q^{-\boldsymbol{\mu}})\,, \quad q^{-\boldsymbol{\mu}} \equiv (q^{-\mu_1},\dots,q^{-\mu_m}), \tag{3.7}$$

where

$$e_j(y_1,\dots,y_m) \equiv \sum_{1\leqslant i_1<\cdots<i_j\leqslant m} y_{i_1}\cdots y_{i_j} \tag{3.8}$$

denotes the elementary symmetric polynomial of degree $j \leqslant m$ in m variables $y_1,\dots,y_m$ (with $e_0 \equiv 1$). By the above remarks, Eq. (3.6) yields also the partition function of the supersymmetric su(m) KY model (3.1). This is one of the main results of the present work.

Equation (3.7) for $d(\mathbf{k})$ can be considerably simplified introducing the numbers $\nu_i(\mathbf{k})$ defined by

$$\begin{aligned} \nu_l(\mathbf{k}) &= |\{i\in\{1,\dots,r\} : k_i = l\}|\,, \quad l = 1,\dots,m-1\,,\\ \nu_m(\mathbf{k}) &= |\{i\in\{1,\dots,r\} : k_i \geqslant m\}|\,, \end{aligned}$$

where $|A|$ denotes the cardinal of the set A. We then have

$$d(\mathbf{k}) = \prod_{l=1}^{m-1} \left[\sum_{j=0}^{l} e_j(q^{-\boldsymbol{\mu}})\right]^{\nu_l(\mathbf{k})} \cdot \prod_{j=1}^{m} (1+q^{-\mu_j})^{\nu_m(\mathbf{k})}. \tag{3.9}$$

For instance, in the su(2) and su(3) cases we respectively have

$$d(\mathbf{k}) = (1+q^{-\mu_1}+q^{-\mu_2})^{\nu_1(\mathbf{k})}[(1+q^{-\mu_1})(1+q^{-\mu_2})]^{\nu_2(\mathbf{k})}$$

and

$$d(\mathbf{k}) = \Big(1+\sum_\sigma q^{-\mu_\sigma}\Big)^{\nu_1(\mathbf{k})}\Big(1+\sum_\sigma q^{-\mu_\sigma}+\sum_{\sigma<\sigma'} q^{-(\mu_\sigma+\mu_{\sigma'})}\Big)^{\nu_2(\mathbf{k})}\prod_\sigma (1+q^{-\mu_\sigma})^{\nu_3(\mathbf{k})}\,.$$

The latter expressions becomes even simpler when applied to the "pure" supersymmetric KY model (i.e., without magnetic field or chemical potential terms), for which $\mu_\sigma = t_0$ for all $\sigma = 1, \dots, m$ according to Eqs. (3.4)-(3.5). Indeed, in this case we have

$$e_j(q^{-\boldsymbol{\mu}}) = e_j(q^{-t_0}, \dots, q^{-t_0}) = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant m} q^{-jt_0} = \binom{m}{j} q^{-jt_0} \,,$$

and hence

$$d(\mathbf{k}) = (1+q^{-t_0})^{m\nu_m(\mathbf{k})} \prod_{l=1}^{m-1} \left[\sum_{j=0}^{l} \binom{m}{j} q^{-jt_0} \right]^{\nu_l(\mathbf{k})} .$$

For instance, the partition function of the original (su(2)) supersymmetric KY model is given by

$$\mathcal{Z}_N(q) = \sum_{\mathbf{k} \in \mathcal{P}_N} (1+2q^{-t_0})^{\nu_1(\mathbf{k})} (1+q^{-t_0})^{2\nu_2(\mathbf{k})} q^{\sum_{i=1}^{r-1} JK_i(N-K_i)} \prod_{i=1}^{N-r} (1-q^{JK_i'(N-K_i')}) \,.$$

4. Spectrum and motifs

In this section we shall give a complete description of the spectrum of the supersymmetric su(m) KY model (2.16)-(2.18) —or, more generally, the Hamiltonian (3.1)— in each subspace with well-defined spin content in terms of the supersymmetric version of Haldane's motifs [19] and their associated skew Young tableaux [20, 23, 41]. In particular, this description implies the validity of the Saiga–Kuramoto conjecture, which is one of the main results of this paper.

We start by recalling that the partition function (3.6) of the su($1|m$) spin chain (3.2) —and, hence, of the supersymmetric su(m) KY model (3.1)— exactly coincides with the partition function of the inhomogeneous vertex model with energies [22, 23]

$$E(\mathbf{s}) = J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i(N-i) - \sum_i \mu_{s_i} \,, \tag{4.1}$$

where $\mu_0 \equiv 0$, $\mathbf{s} \in \{0, \dots, m\}^N$ and $\delta(s, s')$ is defined by

$$\delta(s,s') = \begin{cases} 1, & s > s' \text{ or } s = s' > 0 \\ 0, & s < s' \text{ or } s = s' = 0 \,. \end{cases}$$

The first sum in Eq. (4.1) can be interpreted as the energy of a one-dimensional vertex model with $N+1$ vertices $0, \dots, N$ joined by N bonds with values $s_1, \dots, s_N \in \{0, \dots, m\}$, the energy associated to the i-th vertex being equal to $\delta(s_i, s_{i+1}) i(N-i)$. For this reason, we shall henceforth refer to the vector $\mathbf{s}$ as the *bond vector*. Likewise, the vectors $\boldsymbol{\delta}(\mathbf{s})$ with components $\delta(s_i, s_{i+1})$ $(1 \leqslant i \leqslant N-1)$ in Eq. (4.1) can be identified with su($1|m$) motifs [19, 23, 27]. Thus the spectrum of the su(m) KY model (with the correct degeneracy for each level) can be computed from Eq. (4.1) by letting $\mathbf{s}$ run over all possible $(m+1)^N$ bond vectors. It is important to note that the energies (4.1) depend not only on the motif $\boldsymbol{\delta}$ but also on the chemical potentials μ_α through the last term. This term will in general break the huge degeneracy associated to the motifs $\boldsymbol{\delta}$,

which is in part due to the invariance of the model (3.3) with $\mu_\alpha = 0$ (i.e., the su$(1|m)$ supersymmetric HS chain) under the Yangian $Y(\mathrm{gl}(1|m))$ [41]. In other words, the general model (3.3) should be far less degenerate than the su$(1|m)$ supersymmetric HS spin chain.

Let us denote by $\psi_{\mathbf{s}}$ the unique eigenfunction of the supersymmetric su(m) KY model (3.1) corresponding to the eigenvalue $E(\mathbf{s})$ (4.1) associated with the bond vector $\mathbf{s}$. Our aim is to determine the magnon numbers or spin content of the eigenfunction $\psi_{\mathbf{s}}$ directly from the structure of its bond vector $\mathbf{s}$. To this end, we shall now show how the spectrum of the restriction of the Hamiltonian (3.1) to each subspace with well-defined spin content can be fully generated from the motif formula (4.1) by suitably restricting the components of the bond vector $\mathbf{s}$.

More precisely, let us denote by

$$n(N_0, \ldots, N_m) \equiv n(\mathbf{N}), \qquad N_0 + \cdots + N_m \equiv |\mathbf{N}| = N,$$

the subspace of the Hilbert space of the su(m) KY model in which each number operator n^α has a well-defined value N_α. We shall henceforth refer to the vector $\mathbf{N}$ as the *magnon content* of the subspace $n(\mathbf{N})$. Note next that, by Eqs. (3.1) and (3.2), the KY Hamiltonian H in Eq. (3.1) can be written as

$$H = H_0' - \sum_\sigma \mu_\sigma n^\sigma, \qquad H_0' \equiv H_0 + t_0 \sum_\sigma n^\sigma, \tag{4.2}$$

where H_0 and t_0 are respectively given by Eqs. (2.16)-(2.18) and (2.20). Clearly, H_0' does not depend on the chemical potentials μ_σ (since H_0 is also independent of μ_σ); in fact, $\widehat{H_0'} = J H_{\mathrm{HS}}^{(1|m)}$. It should be noted at this point that the subspaces $n(\mathbf{N})$ are invariant under both H and H_0', since the number operators n^σ obviously commute with the operators $c_{k\sigma}^\dagger c_{l\sigma}$, n_k, $\mathbf{T}_k \cdot \mathbf{T}_l$. Consequently, by Eq. (4.2), the partition function of the Hamiltonian H can be written as

$$\mathcal{Z}(q; \boldsymbol{\mu}) = \sum_{\mathbf{N};|\mathbf{N}|=N} q^{-\sum_\sigma \mu_\sigma N_\sigma} \mathcal{Z}_0^{\mathbf{N}}(q), \tag{4.3}$$

where $\mathcal{Z}_0^{\mathbf{N}}(q)$ denotes the partition function of the restriction of H_0' to the subspace $n(\mathbf{N})$. We also know that

$$\mathcal{Z}(q; \boldsymbol{\mu}) = \mathcal{Z}_V(q; \boldsymbol{\mu}), \tag{4.4}$$

where $\mathcal{Z}_V(q; \boldsymbol{\mu})$ denotes the partition function of the vertex model (4.1). Let us now rewrite the latter equation as

$$E(\mathbf{s}) = J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i(N-i) - \sum_\sigma \mu_\sigma N_\sigma(\mathbf{s}),$$

where $N_\sigma(\mathbf{s})$ denotes the number of components of the bond vector $\mathbf{s}$ equal to σ. We then have

$$\mathcal{Z}_V(q; \boldsymbol{\mu}) = \sum_{\mathbf{s}} q^{E(\mathbf{s})} = \sum_{\mathbf{N};|\mathbf{N}|=N} q^{-\sum_\sigma \mu_\sigma N_\sigma} \sum_{\mathbf{s};N_\alpha(\mathbf{s})=N_\alpha} q^{J\sum_{i=1}^{N-1} \delta(s_i,s_{i+1}) i(N-i)}, \tag{4.5}$$

where the last sum is restricted to all bond vectors $\mathbf{s}$ such that $N_\alpha(\mathbf{s}) = N_\alpha$ for all $\alpha = 0, \ldots, m$. Note that the right-hand sides of Eqs. (4.3) and (4.5) are both

polynomials in the variables $x_\alpha \equiv \mathrm{e}^{\beta\mu_\alpha}$ $(\alpha = 0, \ldots, m)$. Equating the coefficient of $x_0^{N_0} \cdots x_m^{N_m}$ in both equivalent expressions for $\mathcal{Z}(q; \boldsymbol{\mu})$ we immediately deduce that the partition function of the restriction of H_0' to the subspace $n(\mathbf{N})$ is given by

$$\mathcal{Z}_0^{\mathbf{N}}(q) = \sum_{\mathbf{s}: N_\alpha(\mathbf{s}) = N_\alpha} q^{J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i(N-i)} . \tag{4.6}$$

It follows from the latter equation that the spectrum of H_0' in the subspace $n(\mathbf{N})$ can be generated from the motif formula

$$E_0(\mathbf{s}; \mathbf{N}) = J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i(N-i), \qquad \text{with } N_\alpha(\mathbf{s}) = N_\alpha \text{ for all } \alpha = 0, \ldots, m\,.$$

In view of Eq. (4.2), the spectrum of the full Hamiltonian H restricted to the subspace $n(\mathbf{N})$ (with the correct degeneracies for all levels) is generated by the analogous formula

$$E(\mathbf{s}; \mathbf{N}) = E_0(\mathbf{s}; \mathbf{N}) - \sum_\sigma \mu_\sigma N_\sigma, \qquad \text{with } N_\alpha(\mathbf{s}) = N_\alpha \text{ for all } \alpha = 0, \ldots, m\,. \tag{4.7}$$

We thus conclude that the spectrum of H on $n(\mathbf{N})$ is obtained from Eq. (4.1) by imposing the natural conditions $N_\alpha(\mathbf{s}) = N_\alpha$ on the bond vector $\mathbf{s}$. It follows from this assertion that we can label the eigenfunctions of H in such a way that each $\psi_{\mathbf{s}}$ belongs to the subspace $n(\mathbf{N}(\mathbf{s}))$ containing exactly $N_\alpha(\mathbf{s})$ magnons of type α. This is in fact one of the main results of this section, whose consequences we shall explore next.

To begin with, we shall use the method of Ref. [23] to express the spectrum (4.7) of the supersymmetric KY model (3.1) on the invariant subspace $n(\mathbf{N})$ in an alternative way. To this end, we first note that the numerical values of the energies can be computed from the formula

$$E_{\boldsymbol{\delta},\mathbf{N}} = J \sum_{i=1}^{N-1} \delta_i \cdot i(N-i) - \sum_\sigma \mu_\sigma N_\sigma\,, \tag{4.8}$$

where now $\boldsymbol{\delta} \equiv (\delta_1, \ldots, \delta_{N-1})$ is a supersymmetric motif, i.e., a sequence of $N-1$ zeros and ones. Secondly, the degeneracy of each energy $E_{\boldsymbol{\delta},\mathbf{N}}$ (which could be zero) is evaluated by counting the number of ways of filling the border strip associated to the motif $\boldsymbol{\delta}$ according to the usual su$(1|m)$ rules [41,42], with the additional restriction that each number $\alpha \in \{0, \ldots, m\}$ must appear exactly N_α times. More precisely, given the motif $(\delta_1, \ldots, \delta_{N-1})$ its associated border strip is constructed by starting with one box, and then reading the motif from left to right and adding a box below (resp. to the left of) the i-th box provided that δ_i is equal to 0 (resp. 1); see, e.g., Fig. 1. This border strip should then be filled with the numbers $0, 1, \ldots, m$ according to the following rules:

i) The numbers in each row form a nondecreasing sequence, allowing only the repetition of positive numbers.

ii) The numbers in each column (read from top to bottom) form a nondecreasing sequence, allowing only the repetition of 0.

iii) Each number $\alpha \in \{0, \ldots, m\}$ must appear exactly N_α times.

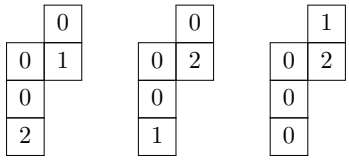

Figure 1. Allowed Young tableaux for the motif $(0,1,0,0)$ in the subspace $n(3,1,1)$.

Each filling of a border strip according to the previous rules is called a (skew) Young tableau. Given such a tableau, it is straightforward to check that its associated motif can be obtained from the bond vector $(s_1,\ldots,s_N)$ whose components are the numbers in the tableau read from top to bottom by setting $\delta_i = \delta(s_i, s_{i+1})$. The equivalence between the description of the spectrum through Eq. (4.8) (where the degeneracy of a motif $\boldsymbol{\delta}$ is evaluated counting all the fillings of its associated border strip allowed by rules i)–iii)) and Eq. (4.7) essentially follows from this observation. Note also that, by Eq. (4.8), all the Young tableaux associated to a given motif have the same energy within each invariant subspace $n(\mathbf{N})$.

Consider, for instance, the motif $\boldsymbol{\delta} = (0,1,0,0)$ for $N = 5$ particles in the su(1|2) case. As explained above, the degeneracy associated to this motif in each invariant subspace $n(\mathbf{N})$ is given by all possible Young tableaux generated from it according to the above three rules. For instance, for the subspace $n(3,1,1)$ it is easy to check that there are exactly three allowed Young tableaux for the above motif (cf. Fig. 1), with energy $6J - \mu_1 - \mu_2$. In fact, it is a straightforward matter to verify that there are exactly 8 invariant subspaces $n(\mathbf{N})$ with fixed spin content compatible with the motif $(0,1,0,0)$ (i.e., with nonzero degeneracy), whose respective degeneracies and energies are listed in Table 1.

N	Degeneracy	Energy
(1,2,2)	1	$6J - 2(\mu_1 + \mu_2)$
(3,0,2)	1	$6J - 2\mu_2$
(3,2,0)	1	$6J - 2\mu_1$
(4,0,1)	1	$6J - \mu_2$
(4,1,0)	1	$6J - \mu_1$
(2,1,2)	2	$6J - (\mu_1 + 2\mu_2)$
(2,2,1)	2	$6J - (2\mu_1 + \mu_2)$
(3,1,1)	3	$6J - (\mu_1 + \mu_2)$

Table 1. Invariant subspaces $n(\mathbf{N})$ compatible with the motif $(0,1,0,0)$ in the su(1|2) case.

The above general results considerably simplify for the supersymmetric su(m) KY model without magnetic field or chemical potential terms, given by Eq. (2.16)-(2.18).

Motif	0	1	2	3	4	5	Energy
$(0,0,0,0)$	0	0	0	1	2	1	$-4J(5-N_0)$
$(0,0,0,1)$, $(1,0,0,0)$	0	0	2	4	2	0	$-4J(4-N_0)$
$(0,0,1,0)$, $(0,1,0,0)$	0	1	4	5	2	0	$-2J(7-2N_0)$
$(1,0,0,1)$	0	4	8	4	0	0	$-4J(3-N_0)$
$(0,0,1,1)$, $(1,1,0,0)$	0	3	6	3	0	0	$-2J(5-2N_0)$
$(1,0,1,0)$, $(0,1,0,1)$	2	8	10	4	0	0	$-2J(5-2N_0)$
$(0,1,1,0)$	2	7	8	3	0	0	$-4J(2-N_0)$
$(1,0,1,1)$, $(1,1,0,1)$	6	12	6	0	0	0	$-2J(3-2N_0)$
$(0,1,1,1)$, $(1,1,1,0)$	4	8	4	0	0	0	$-4J(1-N_0)$
$(1,1,1,1)$	6	5	0	0	0	0	$4JN_0$

Table 2. Spectrum of the su(1|2) KY model (2.16)-(2.18) with $N = 5$ sites. The integers appearing in the columns labeled $0,\dots,5$ are the degeneracies of the motif(s) in each row for the subspaces with $N_0 = 0,\dots,5$ holes.

Indeed, in this case $h_\sigma = \mu_c = 0$, so that from Eq. (3.5) we obtain

$$\mu_\sigma = t_0 = \frac{t\pi^2}{3N^2}(N^2-1), \qquad 1 \leqslant \sigma \leqslant m\,.$$

Thus Eq. (4.8) becomes

$$E_{\boldsymbol{\delta},\mathbf{N}} = J\sum_{i=1}^{N-1}\delta_i{\cdot}i(N{-}i){-}t_0(N{-}N_0) = J\left[\sum_{i=1}^{N-1}\delta_i\cdot i(N-i) - \frac{1}{6}\,(N^2-1)(N-N_0)\right], \quad (4.9)$$

which depends on the spin content $\mathbf{N}$ only through N_0. Thus for a given motif $\boldsymbol{\delta}$ all of its compatible invariant subspaces $n(\mathbf{N})$ with the same number of holes N_0 will now have the same energy. For instance, for the case $N = 5$, $m = 2$ and the motif $(0,1,0,0)$ considered above there are only four sectors with different energies, corresponding to $N_0 = 1$ (singlet), 2 (four times degenerate), 3 (five times degenerate) and 4 (twice degenerate), with respective energies $-2J(7-2N_0)$. In general, in order to compute the degeneracy associated to each motif $\boldsymbol{\delta}$ in a sector with a given number of holes N_0 we just have to count the number of allowed Young tableaux according to rules i) and ii) above, replacing rule three by the following:

iii') The number 0 must appear exactly N_0 times.

The rules i), ii), iii') provide a complete description of the spectrum of the supersymmetric su(m) KY model (2.16)-(2.18), with the correct degeneracy for each energy (4.9), for arbitrary values of m and the number of sites N. In particular, when applied to the simplest su(2) case these rules provide the first rigorous proof known to the authors to the long-standing conjecture of Saiga and Kuramoto mentioned in the Introduction.

As a simple illustration of the above assertion, we present in Table 2 the detailed spectrum of the (original) su(2) KY model (2.16)-(2.18) with $N = 5$ sites. More precisely, in each of the columns of this table labeled with the integers $N_0 = 0,\dots,5$

we list the degeneracy associated to the motif(s) in each row for the subspace with N_0 holes. This degeneracy is computed by first generating all the skew Young tableaux compatible with each of the $2^{N-1} = 16$ possible motifs according to rules i), ii) and iii') above, which can be easily accomplished using a simple *Mathematica* program. We then sort the resulting tableaux according to the number of holes (zeros) in each of them. By Eq. (4.9), a motif $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{N-1})$ and its reverse $\boldsymbol{\delta}' = (\delta_{N-1}, \ldots, \delta_1)$ clearly have the same energy. In fact, it can be shown that two such motifs give rise to the same numbers of compatible Young tableaux in each subspace $n(\mathbf{N})$ (see the appendix for details). For this reason, we have grouped together in Table 2 two motifs that are the reverse of each other. More generally, if we exchange any two components δ_k and δ_{N-k} of a motif $\boldsymbol{\delta}$ we obtain a motif with the same energy as $\boldsymbol{\delta}$ in each subspace with N_0 holes. However, these two motifs may not necessarily yield the same number of compatible Young tableaux in such a subspace. For instance, the motifs $(0,0,1,1)$ and $(1,0,1,0)$ have the same energy $-2J(5-2N_0)$ in a subspace with N_0 holes, but it is clear from Table 2 that their degeneracies differ in each of these subspaces for $N_0 = 0, \ldots, 3$. Finally, it is apparent from Table 2 that the ground state is obtained from the motifs $(0,0,1,0)$, $(0,1,0,0)$ (in the sector with one hole) and $(1,0,1,0)$, $(0,1,0,1)$ (in the sector with no holes). It is thus six times degenerate, with energy $-10J$.

5. Ground state phases for the supersymmetric spin 1/2 KY model

The complete description of the spectrum of the supersymmetric KY model in terms of motifs, bond vectors and their associated skew Young tableaux developed in the previous section is particularly suited to studying its ground state. As an example, we shall compute next the ground state energy per site of the spin 1/2 KY model in the thermodynamic limit for all possible values of the magnetic field strength $h \equiv h_1$ and chemical potential $\mu \equiv \mu_c + t_0$. To this end, let us first choose the unit of energy so that $t = 1/(2\pi^2)$, so that Eq. (4.8) reads

$$E_{\boldsymbol{\delta},\mathbf{N}} = \frac{1}{N^2} \sum_{i=1}^{N-1} \delta_i \cdot i(N-i) - \frac{h}{2}(N_1 - N_2) - \mu(N_1 + N_2)\,. \tag{5.1}$$

We shall also assume without loss of generality that $h \geqslant 0$, since taking $h < 0$ simply reverses the role of the "up" ($\sigma = 1$) and "down" ($\sigma = 2$) fermions. It is then clear from the term proportional to h in Eq. (5.1) that the ground state(s) must belong to an invariant subspace $n(\mathbf{N})$ with $N_1 \geqslant N_2$. Since the dispersion function $i(N-i)$ is symmetric about $i = N/2$ and has an absolute maximum at this point, by Eq. (4.1) for N large enough the bond vector minimizing the energy in the subspace $n(\mathbf{N})$ must be

of the form§

$$\mathbf{s}_0 = (\underbrace{1\cdots1}_{(N_1-N_2)/2}\,\underbrace{12\cdots12}_{N_2}\,\underbrace{0\cdots0}_{N_0}\,\underbrace{12\cdots12}_{N_2}\,\underbrace{1\cdots1}_{(N_1-N_2)/2})\,. \tag{5.2}$$

By Eq. (5.1), the energy per site of the corresponding motif

$$\boldsymbol{\delta}_0 = (\underbrace{1,\ldots,1}_{(N_1-N_2)/2},\,\underbrace{0,1,\ldots,0,1}_{N_2},\,\underbrace{0,\ldots,0}_{N_0},\,\underbrace{0,1,\ldots,0,1}_{N_2},\,\underbrace{1,\ldots,1}_{(N_1-N_2)/2-1})$$

is given by

$$\frac{E_{\boldsymbol{\delta}_0,\mathbf{N}}}{N} = \frac{2}{N}\sum_{k=1}^{Nt}\varepsilon(x_k) + \frac{2}{N}\sum_{k=1}^{N(s-t)/2}\varepsilon(x_{Nt+2k}) - ht - 2\mu s \equiv u_{\boldsymbol{\delta}_0,\mathbf{N}}\,,$$

where the factor of 2 before the sums comes from the obvious symmetry of $\boldsymbol{\delta}_0$ around $N/2$ and we have set

$$s = \frac{N_1+N_2}{2N}, \qquad t = \frac{N_1-N_2}{2N}, \qquad x_k = \frac{k}{N}\,, \qquad \varepsilon(x) = x(1-x)\,.$$

Hence in the thermodynamic limit the minimum energy per site in a subspace $n(\mathbf{N})$ with $N_1 \geqslant N_2$ is given by

$$\begin{aligned} u(s,t) \equiv \lim_{N\to\infty} u_{\boldsymbol{\delta}_0,\mathbf{N}} &= 2\int_0^t \varepsilon(x)\,\mathrm{d}x + \int_t^s \varepsilon(x)\,\mathrm{d}x - ht - 2\mu s \\ &= \int_0^s \varepsilon(x)\,\mathrm{d}x + \int_0^t \varepsilon(x)\,\mathrm{d}x - ht - 2\mu s \equiv f_{2\mu}(s) + f_h(t)\,, \end{aligned} \tag{5.3}$$

where

$$f_\lambda(s) = \int_0^s \varepsilon(x)\,\mathrm{d}x - \lambda s \equiv f(s) - \lambda s\,.$$

The ground state energy of the spin 1/2 KY model in the thermodynamic limit is thus the minimum value of the function $u(s,t)$ in the triangle

$$D = \{(s,t) \in \mathbb{R}^2 : 0 \leqslant t \leqslant s \leqslant 1/2\}\,.$$

In order to compute this minimum value, note first of all that $f'_\lambda(x) = \varepsilon(x) - \lambda$, with $\varepsilon(x)$ monotonically increasing from 0 to 1/4 in the interval $[0,1/2]$. Hence f_λ is monotonically increasing‖ over the interval $[0,1/2]$ for $\lambda \leqslant 0$, monotonically decreasing for $\lambda \geqslant 1/4$, and has a unique global minimum at the point

$$x_0(\lambda) \equiv \varepsilon^{-1}(\lambda) = \frac{1}{2}\,(1 - \sqrt{1-4\lambda}\,) \in (0,1/2)$$

for $0 < \lambda < 1/4$. We thus have the following possibilities (recall that we are assuming throughout that $h \geqslant 0$):

§ We are actually assuming here that both N_1 and N_2 are even. In other cases the form of the minimizing bond vector differs slightly from Eq. (5.2), but the formula for its energy coincides in the thermodynamic limit with the one given below. Note also that unless N_1 and N_2 are both odd there is actually an additional bond vector with the same energy as (5.2) (or its variants, for N_1 and N_2 of opposite parity).

‖ For the sake of conciseness, we shall implicitly assume in what follows that the functions f_λ and ε are restricted to the interval of interest $[0,1/2]$.

i) $\dfrac{h}{2} < \mu < 1/8$

In this case $f_h(t)$ and $f_{2\mu}(s)$ both have a unique minimum over the interval $[0,1/2]$ respectively at the points $t_0 = x_0(h) \in [0,1/2)$ and $s_0 = x_0(2\mu) \in (0,1/2)$, with $t_0 < s_0$ since $\varepsilon(t_0) = h < 2\mu = \varepsilon(s_0)$. Hence u attains its global minimum on D at the point (s_0,t_0) (which is an interior point if $h>0$), and the ground state energy density u is consequently given by

$$\begin{aligned} u = u(s_0,t_0) &= \int_0^{x_0(h)} \varepsilon(x)\,\mathrm{d}x + \int_0^{x_0(2\mu)} \varepsilon(x)\,\mathrm{d}x - hx_0(h) - 2\mu x_0(2\mu) \\ &= \frac{1}{6}\left[1 - 3(h+2\mu) - \frac{1}{2}(1-4h)^{3/2} - \frac{1}{2}(1-8\mu)^{3/2}\right]. \end{aligned} \tag{5.4}$$

Note also that $t = (N_1 - N_2)/(2N)$ and $2s = (N_1+N_2)/N$ are respectively equal to the magnetization and the charge density per site (assuming that the fermions have unit charge and gyromagnetic ratio equal to 2). Hence the zero-temperature magnetization and charge densities are given by

$$m_s = x_0(h) = \frac{1}{2}\,(1 - \sqrt{1-4h}\,)\,, \qquad n_c = 2x_0(2\mu) = 1 - \sqrt{1-8\mu}\,.$$

The corresponding magnetic and charge susceptibilities are obtained by differentiation, namely,

$$\chi_s = \frac{\partial m_s}{\partial h} = (1-4h)^{-1/2} = (1-2m_s)^{-1}\,, \quad \chi_c = \frac{\partial n_c}{\partial \mu} = 4(1-8\mu)^{-1/2} = 4(1-n_c)^{-1}\,,$$

in agreement with known results (see, e.g., [25]).

If (h,μ) lies outside the region $\{(h,\mu) \in \mathbb{R}^2 : 0 \leqslant h/2 < \mu < 1/8\}$, the system

$$\varepsilon(s) = 2\mu\,, \qquad \varepsilon(t) = h$$

determining the critical points of u has no solutions within the interior of D. Thus the function $u(s,t)$ must necessarily attain its minimum value in the triangle D on its sides. It is worth noting in this respect that this minimum cannot be reached at an interior point of the horizontal side $t=0$ unless $h=0$, since for $h>0$ we have $f_h'(0) = -h < 0$ and thus

$$u(s,0) = f_{2\mu}(s) > f_{2\mu}(s) + f_h(t) = u(s,t)$$

for sufficiently small $t>0$. We are left with the following possibilities:

ii) $h \geqslant 1/4$, $\mu + \dfrac{h}{2} \geqslant 1/4$

In this case $u(s,s) = f_h(s) + f_{2\mu}(s) = 2f_{h/2+\mu}(s)$ and $u(1/2,t) = f_h(t) + f_{2\mu}(1/2)$ are both decreasing, so that these functions have a unique global minimum on the interval $[0,1/2]$ at the right endpoint $1/2$, with common value

$$u(1/2,1/2) = 2f_{h/2+\mu}(1/2) = \frac{1}{6}\,[1 - 3(h+2\mu)]\,. \tag{5.5}$$

This must be the unique minimum of u on D, since as $h>0$ this minimum cannot be attained on the interior of the side $t=0$. Hence in this case the ground state energy per

site is given by Eq. (5.5), while the zero-temperature magnetization and charge densities are simply

$$n_c = 2m_s = 1\,.$$

Note that $s = t = 1/2$ is equivalent to $N_1 = N$, so that this phase consists only of up fermions.

iii) $\mu + \dfrac{h}{2} \leqslant 0$

Since $\mu \leqslant -h/2 \leqslant 0$, the minimum of $u(s,0) = f_{2\mu}(s)$ and $u(s,s) = 2f_{h/2+\mu}(s)$ on $[0,1/2]$ is located at $s = 0$, while

$$u(1/2,t) = f(t) + f(1/2) - ht - \mu > -\frac{h}{2} - \mu \geqslant 0 = u(0,0)\,.$$

Thus in this case the ground state energy, zero-temperature magnetization and charge all vanish. In fact, since $s = t = 0$ is equivalent to $N_0 = N$, this is the trivial phase consisting only of holes.

iv) $h < 1/4$, $\mu \geqslant 1/8$

In this case $u(1/2,t)$ attains its minimum value in the interval $[0,1/2]$ at $t_0 = x_0(h) \in [0,1/2)$. On the other hand,

$$u(s,s) = f_h(s) + f_{2\mu}(s) > f_h(t_0) + f_{2\mu}(1/2) = u(1/2,t_0),$$

since $f_{2\mu}$ is decreasing in the interval $[0,1/2]$ on account of the condition $\mu \geqslant 1/8$. Similarly,

$$u(s,0) = f_{2\mu}(s) \geqslant f_{2\mu}(1/2) = u(1/2,0) \geqslant u(1/2,t_0),$$

with equality only if $h = 0$ and $s = 1/2$. Thus the global minimum of u on D is attained at the point $(1/2,t_0)$. In particular, the ground state energy per site is given by

$$\begin{aligned} u = u(1/2,t_0) &= \int_0^{x_0(h)} \varepsilon(x)\,\mathrm{d}x - hx_0(h) + \int_0^{1/2} \varepsilon(x)\,\mathrm{d}x - \mu \\ &= \frac{1}{6}\left[1 - 3(h+2\mu) - \frac{1}{2}(1-4h)^{3/2}\right], \end{aligned} \tag{5.6}$$

while the magnetization and charge per site at zero temperature read

$$m_s = x_0(h)\,, \qquad n_c = 1\,.$$

The $T = 0$ magnetic susceptibility is again

$$\chi_s = (1-4h)^{-1/2} = (1-2m_s)^{-1}\,,$$

while the charge susceptibility vanishes. Since $s = 1/2$ is equivalent to $N_1 + N_2 = N$, this is an su(2) phase characterized by the absence of holes.

v) $0 < h/2 + \mu < 1/4$, $\mu \leqslant h/2$

The above inequalities imply that $u(s,s) = 2f_{h/2+\mu}(s)$ attains its unique global minimum over the interval $[0,1/2]$ at the point $s_0 = x_0(h/2+\mu) \in (0,1/2)$. It is straightforward to check that $u(s,t)$ achieves its minimum over the domain D at the point (s_0,s_0). Indeed,

Region	Ground state energy per site	Spin content
$\frac{h}{2} < \mu < \frac{1}{8}$	$\frac{1}{6}[1-3(h+2\mu)-\frac{1}{2}(1-4h)^{3/2}-\frac{1}{2}(1-8\mu)^{3/2}]$	B, F_1, F_2
$h \geqslant \frac{1}{4}, \quad \mu+\frac{h}{2} \geqslant \frac{1}{4}$	$\frac{1}{6}\,[1-3(h+2\mu)]$	F_1
$\mu+\frac{h}{2} \leqslant 0$	0	B
$h < \frac{1}{4}, \quad \mu \geqslant \frac{1}{8}$	$\frac{1}{6}[1-3(h+2\mu)-\frac{1}{2}(1-4h)^{3/2}]$	F_1, F_2
$0 < \mu+\frac{h}{2} < \frac{1}{4}, \quad \mu \leqslant \frac{h}{2}$	$\frac{1}{6}\big[1-3(h+2\mu)-(1-2h-4\mu)^{3/2}\big]$	B, F_1

Table 3. Ground state energy per site and spin content ($B \equiv$ hole (boson), $F_\sigma \equiv$ fermion of type σ) for the spin 1/2 KY model as a function of its parameters $h \geqslant 0$ and μ (the unit of energy has been taken as $2\pi^2 t$).

as $h > 0$ in this case this minimum cannot be reached on the interior of the side $t = 0$. Consider next the side $s = 1/2$. If $h \geqslant 1/4$ the function $f_h(t)$ is decreasing, so that

$$u(1/2,t) = f_h(t) + f_{2\mu}(1/2) \geqslant f_h(1/2) + f_{2\mu}(1/2) = u(1/2,1/2) > u(s_0,s_0).$$

On the other hand, if $0 \leqslant h < 1/4$ then f_h has a global minimum on $[0,1/2]$ at $t_0 = x_0(h) \geqslant x_0(h/2+\mu) = s_0$, with $t_0 \in [0,1/2)$. Since $\varepsilon(x) - 2\mu > 0$ for $x > t_0$ (this is obvious for $\mu \leqslant 0$, while for $0 < \mu \leqslant 1/8$ it is a consequence of the inequality $x_0(2\mu) \leqslant x_0(h)$, which in turn follows from $2\mu \leqslant h$) we have

$$f_{2\mu}(1/2) = \int_0^{1/2} (\varepsilon(x) - 2\mu)\,\mathrm{d}x > \int_0^{t_0} (\varepsilon(x) - 2\mu)\,\mathrm{d}x = f_{2\mu}(t_0)\,.$$

Hence

$$\begin{aligned} u(s_0,s_0) - u(1/2,t_0) &= 2f_{h/2+\mu}(s_0) - f_h(t_0) - f_{2\mu}(1/2) \\ &< 2f_{h/2+\mu}(s_0) - f_h(t_0) - f_{2\mu}(t_0) = 2f_{h/2+\mu}(s_0) - 2f_{h/2+\mu}(t_0) \leqslant 0\,, \end{aligned}$$

which completes the proof of our assertion. In summary, in this case the ground state energy per site is given by

$$\begin{aligned} u = u(s_0,s_0) &= 2\int_0^{x_0(h/2+\mu)} \varepsilon(x)\,\mathrm{d}x - (h+2\mu)x_0(\tfrac{h}{2}+\mu) \\ &= \frac{1}{6}\big[1-3(h+2\mu)-(1-2h-4\mu)^{3/2}\big], \end{aligned} \tag{5.7}$$

while the magnetization, the charge density and their susceptibilities (per site) read

$$\begin{aligned} n_c &= 2m_s = 2x_0(h/2+\mu) = 1-\sqrt{1-2h-4\mu}\,, \\ \chi_c &= 4\chi_s = 2(1-2h-4\mu)^{-1/2} = \frac{2}{1-n_c} = \frac{2}{1-2m_s}\,. \end{aligned}$$

Note, finally, that the equality $s = t$ is equivalent to $N_2 = 0$. This is thus an su(1|1) phase, consisting only of holes and up fermions.

Our results are summarized in Table 3. Note, in particular, that u is continuous (indeed, of class C^1) over its domain, although its second derivatives are discontinuous along the boundaries of the regions listed in Table 3 (cf. Fig. 2).

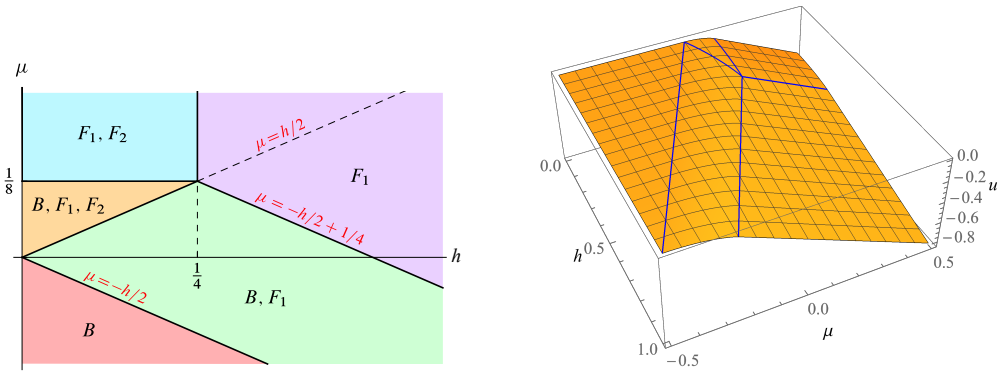

Figure 2. Left: phase diagram of the ground state of the spin 1/2 KY model ($B \equiv$ hole (boson), $F_\sigma \equiv$ fermion of type σ). Right: ground state energy per site of the latter model as a function of its parameters $h \geqslant 0$ and μ (the blue lines represent the boundaries of the regions in the phase diagram). In both plots, the unit of energy has been taken as $2\pi^2 t$.

6. Conclusions

Although the supersymmetric su(m) t-J model with long-range interactions, also known as the su(m) KY model, has been studied extensively during the last few decades, an analytical derivation of its spectrum and exact partition function has been missing so far. With the purpose of filling up this gap, in this paper we first establish the precise equivalence of the su(m) KY model to a suitable modification of the su($1|m$) HS spin chain with chemical potential terms. This equivalence allows us to obtain the partition function of the former model from that of the latter, which was recently computed in Ref. [22]. A remarkable property of this partition function is the fact that it can be rewritten as the partition function of a suitable inhomogeneous vertex model. Analyzing the structure of these two equivalent partition functions, we not only obtain the complete spectrum of the su(m) KY model in the presence of an arbitrary magnetic field and charge chemical potential, but also develop a novel method for determining the magnon numbers or spin content of the corresponding wave functions. This yields an exhaustive description of the spectrum in the subspaces with well-defined magnon content in terms of suitably restricted bond configurations of the equivalent vertex model, which are closely connected with supersymmetric versions of Haldane's motifs and their related skew Young tableaux. For the particular case $m = 2$, this description provides a rigorous proof of a long-standing conjecture by Saiga and Kuramoto [21] based on numerical evidence.

As a concrete application of our results, we study various thermodynamic properties of the su(2) KY model in the zero temperature limit. To this end, we determine the structure of the motifs and bond configurations yielding the ground state of the latter model in the thermodynamic limit for different values of the external parameters. The structure of such bond configurations leads to a complete description of the different

ground state phases in terms of the magnetic field strength and the charge chemical potential. These phases are characterized by the spin content (magnon numbers) of the corresponding wave functions, namely an su(1|2) phase where holes and fermions of both types co-exist, an su(1|1) phase with holes and fermions of only one type, and an su(0|2) phase with fermions of both types, apart from the trivial phases consisting of only holes or fermions of one type. We also compute the zero-temperature values of the energy, magnetization and charge density, along with the magnetic and charge susceptibilities, for each ground state phase. This description of the thermodynamic properties at zero temperature goes beyond previously known results, which were derived by different methods and restricted to the su(1|2) phase. In particular, our analysis confirms that the strong spin-charge separation characteristic of the long-range *t-J* model at low temperatures [25] occurs in all nontrivial phases.

Note, finally, that the description of the spectrum of the su(m) KY model in terms of supersymmetric motifs and their associated Young tableaux derived in this paper makes it possible to compute in closed form the model's thermodynamic functions at finite temperature, by means of the transfer matrix method developed in Refs. [22,43]. In fact, work on this problem is currently in progress and shall be presented in a forthcoming publication.

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Appendix A. Degeneracy of a reverse motif

In this appendix we shall show that the degeneracy $d(\boldsymbol{\delta}, \mathbf{N})$ of the motif $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{N-1})$ in the subspace $n(\mathbf{N})$, i.e., the number of allowed Young tableaux for this motif containing N_α instances of each of the integer $\alpha \in \{0, \ldots, m\}$, is equal to that of its reverse $\boldsymbol{\delta}' \equiv (\delta_{N-1}, \ldots, \delta_1)$. To this end, define the super-Schur polynomial associated to the motif $\boldsymbol{\delta}$ by

$$S_{\boldsymbol{\delta}}(x, \mathbf{y}) = \sum_{T \in \boldsymbol{\delta}} x^{N_0(T)} y_1^{N_1(T)} \cdots y_m^{N_m(T)} ,$$

where the sum is extended to all allowed Young tableaux T associated to $\boldsymbol{\delta}$ according to rules i)–iii) in Section 4, and $N_\alpha(T)$ denotes the number of times the integer α appears in T. It is well known (see, e.g., [20,27,42]) that this polynomial can be computed from

the determinantal formula

$$S_{\boldsymbol{\delta}} = \begin{vmatrix} E_{k_r} & E_{k_{r-1}+k_r} & E_{k_{r-2}+k_{r-1}+k_r} & \cdots & E_{k_1+\cdots+k_r} \\ 1 & E_{k_{r-1}} & E_{k_{r-2}+k_{r-1}} & \cdots & E_{k_1+\cdots+k_{r-1}} \\ 0 & 1 & E_{k_{r-2}} & \cdots & E_{k_1+\cdots+k_{r-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & E_{k_2} & E_{k_1+k_2} \\ 0 & \cdots & 0 & 1 & E_{k_1} \end{vmatrix} \equiv S\langle k_1, \ldots, k_r \rangle \,,$$

where $\sum_{j=1}^{i} k_j$ (with $1 \leqslant i \leqslant r-1$) denotes the position of the i-th 1 in the motif $\boldsymbol{\delta}$,

$$k_r = N - \sum_{i=1}^{r-1} k_i \,,$$

and the polynomials $E_k(x, \mathbf{y})$ are defined in terms of the elementary symmetric polynomials (3.8) by

$$E_k(x, \mathbf{y}) = \sum_{l=0}^{k} x^{k-l} e_l(\mathbf{y}) \,.$$

Clearly, for the reverse motif $\boldsymbol{\delta}'$ we have $k'_i = k_{r+1-i}$, so that

$$S_{\boldsymbol{\delta}'} = S\langle k_r, \ldots, k_1 \rangle \,.$$

Since, by definition of $S_{\boldsymbol{\delta}}$, $d(\boldsymbol{\delta}, \mathbf{N})$ is the coefficient of $x^{N_0} y_1^{N_1} \cdots y_m^{N_m}$ in $S_{\boldsymbol{\delta}}$, to prove the equality of $d(\boldsymbol{\delta}, \mathbf{N})$ and $d(\boldsymbol{\delta}', \mathbf{N})$ it suffices to show that

$$S\langle k_1, \ldots, k_r \rangle = S\langle k_r, \ldots, k_1 \rangle \,. \tag{A.1}$$

We shall establish the latter equality by induction on r. To begin with, note that (A.1) is trivially obvious for $r = 1$. Suppose now that the latter equation holds for determinants $S\langle\cdot\rangle$ of order up to $r-1$. Expanding $S\langle k_1, \ldots, k_r \rangle$ by the first column we obtain:

$$S\langle k_1, \ldots, k_r \rangle = E_{k_r} S\langle k_1, \ldots, k_{r-1} \rangle - S\langle k_1, \ldots, k_{r-2}, k_{r-1} + k_r \rangle \,. \tag{A.2}$$

Similarly, expanding $S\langle k_r, \ldots, k_1 \rangle$ by the last row we have

$$S\langle k_r, \ldots, k_1 \rangle = E_{k_r} S\langle k_{r-1}, \ldots, k_1 \rangle - S\langle k_{r-1} + k_r, k_{r-2}, \ldots, k_1 \rangle \,. \tag{A.3}$$

Equation (A.1) follows immediately from Eq. (A.2) using the induction hypothesis and Eq. (A.3). Note, finally, that the latter proof can be easily adapted to the su($n|m$) case with arbitrary n.

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Supersymmetric t-J models with long-range interactions: thermodynamics and criticality

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Abstract. We analyze the thermodynamics and the critical behavior of the supersymmetric su(m) t-J model with long-range interactions. Using the transfer matrix formalism, we obtain a closed-form expression for the free energy per site both for a finite number of sites and in the thermodynamic limit. Our approach, which is different from the usual ones based on the asymptotic Bethe ansatz and generalized exclusion statistics, can in fact be applied to a large class of models whose spectrum is described in terms of supersymmetric Young tableaux and their associated Haldane motifs. In the simplest and most interesting su(2) case, we identify the five ground state phases of the model and derive the complete low-temperature asymptotic series of the free energy per site, the magnetization and charge densities, and their susceptibilities. We verify the model's characteristic spin-charge separation at low temperatures, and show that it holds to all orders in the asymptotic expansion. Using the low-temperature asymptotic expansions of the free energy, we also analyze the critical behavior of the model in each of its ground state phases. While the standard su(1|2) phase is described by two independent CFTs with central charge $c = 1$ in correspondence with the spin and charge sectors, we find that the low-energy behavior of the su(2) and su(1|1) phases is that of a single $c = 1$ CFT. We show that the model exhibits an even richer behavior on the boundary between zero-temperature phases, where it can be non-critical but gapless, critical in the spin sector but not in the charge one, or critical with central charge $c = 3/2$.

1. Introduction

The su(m) t-J model is one of the most intensively studied lattice models of strongly correlated fermions, due to its relevance for the theoretical understanding of high-temperature superconductivity and as one of the simplest quantum systems exhibiting spin-charge separation [1–4]. The sites of this model can be occupied by at most one charged fermion with m internal degrees of freedom, which can hop between contiguous lattice sites and interacts with its nearest neighbors through spin exchange and charge repulsion. The one-dimensional t-J model is of particular interest, since it is supersymmetric and exactly solvable through the nested Bethe ansatz when its two parameters are suitably related [1, 5–9].

In a recent paper [10] we have computed in closed form the partition function of the supersymmetric su(m) t-J model with long-range interactions introduced by Kuramoto and Yokoyama [11, 12]. The lattice sites of the latter model are equispaced on a circle, and each fermion can now interact with any other and hop among any two sites. Moreover, both the interaction strength and the hopping amplitude are inversely proportional to the square of the chord distance between the corresponding sites. The supersymmetric character of the su(m) Kuramoto–Yokoyama (KY) model can be established by mapping it to a suitable modification of the su($1|m$) Haldane–Shastry spin chain [13]. This connection can in fact be exploited to fully determine the spectrum of the former model in terms of supersymmetric motifs and their corresponding Young tableaux [10, 14].

The thermodynamics of the supersymmetric KY model has been actively investigated ever since its introduction. In fact, in the original reference [11] the low-temperature asymptotic behavior of the magnetic and charge susceptibilities was determined by means of the asymptotic Bethe ansatz (see, e.g., [15]). A few years later, the thermodynamics of the su(m) KY model at arbitrary temperature in the $N \to \infty$ limit was studied by Kato and Kuramoto [16] applying Polychronakos's freezing trick [17] to the su($1|m$) supersymmetric spin Sutherland model [18, 19]. This method, which is rather involved, requires first establishing the equivalence of the latter model to a system of non-interacting su($1|m$) particles and then modding out the contribution of the dynamical degrees of freedom. Moreover, it essentially relies on specific properties of the HS chain such as its equivalence to a model of free particles with generalized momenta obeying fractional statistics. On a more practical level, the formula for the grand potential obtained by Kato and Kuramoto depends on a function which must be determined by solving an implicit equation with an appropriate choice of branch.

In this paper we propose a novel direct method for analyzing the thermodynamics of the supersymmetric KY model, which can be applied to a wide range of models with (complete or broken) Yangian symmetry. We shall show how a formula for the grand potential of these models, akin to Kato and Kuramoto's for the long-range supersymmetric t-J model, emerges in a transparent way from their partition function without requiring that they be described by generalized pseudo-momenta or fractional

statistics. In the simplest and most interesting case $m = 2$, the corresponding implicit equation is quadratic and can therefore be explicitly solved, which leads to a new closed-form expression for the grand potential of the spin 1/2 KY model.

The starting point in our method is the explicit formula for the partition function of the KY model with an arbitrary (finite) number of sites N obtained in Ref. [10], which can be recast into that of a related inhomogeneous vertex model. This key observation makes it feasible to apply the transfer matrix method in Refs. [20,21] to derive a closed-form expression for the grand potential of the su(m) KY model in the thermodynamic limit in terms of the largest eigenvalue (in modulus) of a site-dependent transfer matrix. The characteristic equation of this matrix, when expressed in an appropriate variable, is precisely the implicit equation deduced by Kato and Kuramoto (henceforth referred to as the KK equation). In fact, our method can be applied to any model described by an effective inhomogeneous vertex model, whose energy function is expressible in terms of a dispersion relation and the supersymmetric Young tableaux associated to finite-dimensional representations of the Yangian acting on tame modules [22–24]. By varying the dispersion relation we can derive the thermodynamics of a large class of (partially or totally) Yangian-invariant systems, which includes not only the KY model (or, equivalently, the su($1|m$) supersymmetric HS chain) but other well-known lattice models like the Polychronakos–Frahm (PF) [17,25] or the Frahm–Inozemtsev (FI) [26,27] spin chains. The grand potential of all of these models can again be expressed in terms of the largest eigenvalue of a suitable transfer matrix. In the su($1|m$) case, we explicitly show that the characteristic equation of this transfer matrix is equivalent to a generalized KK equation for a system of one boson and m fermions. This strongly suggests that the models in this class can be reformulated as systems of "free" su($1|m$) particles (holons and spinons) interacting via appropriate fractional statistics. So far, this conjecture has only been proved by an ad hoc method for the su($1|2$) case in Ref. [16].

A further aim of this paper is to take advantage of the explicit formula for the grand potential of the spin 1/2 supersymmetric KY model in order to analyze in detail the low-temperature behavior of its main thermodynamic functions, going beyond the first-order calculations in Refs. [16,28]. To this end, we first determine the zero-temperature limit of the magnetization and charge densities for all values of the magnetic field h and the charge chemical potential μ. In this way we identify the model's five ground state phases in (h, μ) space, characterized by their content of holes and fermions of both species. We then compute the asymptotic expansion of the grand potential to all orders in T, which turns out to be different in each of these ground state phases. From the asymptotic series of the grand potential we derive analogous infinite asymptotic expansions for the main thermodynamic functions of interest, namely the magnetization and charge densities and their respective susceptibilities. Apart from recovering the lowest-order results of Ref. [28], we show that the strong spin-charge separation characteristic of the model under consideration is a non-perturbative property, in the sense that it persists at all orders in the low-temperature asymptotic expansion of both susceptibilities. We also use our low-temperature asymptotic expansions to briefly analyze the critical behavior

of the spin 1/2 KY model for arbitrary values of the magnetic field and the charge chemical potential. In the genuinely su(1|2) phase we confirm the well-known result that the model is described by two independent $c = 1$ conformal field theories (CFT), one for each of the spin and charge sectors [29]. On the other hand, in the su(2) and su(1|1) phases we interestingly find that the model, while still critical, is instead described by a single $c = 1$ CFT. The situation is even more complex on the boundary between ground-state phases, where the model can be non-critical but gapless, critical in the spin sector but not in the charge one, or have fractional central charge $c = 3/2$.

The paper is organized as follows. In Section 2 we introduce the model and recall from Ref. [10] its precise equivalence to (a modification of) the su(1|m) supersymmetric Haldane–Shastry chain. We then exploit this equivalence to obtain explicit formulas for the free energy per site and the main thermodynamic functions in the thermodynamic limit. In Section 3 we discuss the derivation of the Kato–Kuramoto equations and their generalizations by means of the transfer matrix formalism. In the remaining sections we focus on the simplest and most interesting case, namely the spin 1/2 supersymmetric KY model. More precisely, in Section 4 we obtain exact expressions for the zero-temperature magnetization and charge densities, and apply them to identify the different ground state phases in terms of the magnetic field strength and the charge chemical potential. Section 5 is devoted to the derivation of the complete asymptotic series of the free energy per site in each of the ground state phases, which we then use to analyze in detail the model's critical behavior. In Section 6 we compute the corresponding series for the magnetization per site, the charge density and their susceptibilities, and discuss the spin-charge separation characteristic of the model under study. We present our conclusions and outline some future developments in Section 7. The paper ends with three appendices in which we deal with several technical questions arising in the derivation of the asymptotic series in Section 5.

2. Free energy of the su(m) Kuramoto–Yokoyama model

2.1. The model

As shown in our previous paper [10], the Hamiltonian of the supersymmetric su(m) KY model can be written as‡

$$H_0 = \frac{t\pi^2}{N^2} \sum_{i<j} \sin^{-2}\left(\tfrac{\pi}{N}(i-j)\right) \mathcal{P}\Big[-\sum_{\sigma} (c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma}) + 2\mathbf{T}_i \cdot \mathbf{T}_j - \left(1 - \tfrac{1}{m}\right) n_i n_j \Big] \mathcal{P}. \quad (2.1)$$

In the latter equations $c^\dagger_{i\sigma}$ (respectively $c_{i\sigma}$) denotes the operator creating (resp. destroying) a fermion of type $\sigma \in \{1, \dots, m\}$ at site i and $n_i = \sum_\sigma n_{i\sigma}$, where $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$, is the total number of fermions at site i. The operator $\mathcal{P}$ is the projector onto single-occupancy states, in which each site is occupied by at most one fermion.

‡ Here and in what follows, unless otherwise stated, sums and products over Latin indexes run over the set $1, \dots, N$ while Greek indices range from 1 to m.

Finally, $\mathbf{T}_i \equiv (T_i^1, \ldots, T_i^{m^2-1})$, where T_i^r is the r-th su(m) Hermitian generator in the fundamental representation acting on the i-th site. More precisely,

$$T_i^r = \sum_{\sigma,\sigma'} T^r_{\sigma\sigma'} c^\dagger_{i\sigma} c_{i\sigma'} \,, \tag{2.2}$$

where the complex numbers $T^r_{\sigma\sigma'}$ are the matrix elements of the r-th (Hermitian) generator of su(m) in the fundamental representation with the normalization

$$\mathrm{tr}(T^r T^s) = \frac{1}{2}\,\delta_{rs} \,.$$

Thus the first term between braces in Eq. (2.1) accounts for the hopping of fermions between sites i and j, while the last two terms respectively model the spin (exchange) and charge interaction between the latter sites.

As shown in Ref. [10], the KY Hamiltonian (2.1) can be mapped to (a suitable modification of) the su($1|m$) Haldane–Shastry spin chain Hamiltonian by identifying the holes of the KY model with the bosons of the HS spin chain. Indeed, the Hilbert space of the latter chain is $\hat{\mathcal{H}} = \otimes_{i=1}^N \hat{\mathcal{H}}_i$, where $\hat{\mathcal{H}}_i$ is the linear span of the one-particle states $b_i^\dagger|\hat\Omega\rangle_i$, $f_{i\sigma}^\dagger|\hat\Omega\rangle_i$ ($\sigma = 1, \ldots, m$), $b_i^\dagger$, $f_{i\sigma}^\dagger$ are the operators creating respectively a boson and a fermion of type σ at the i-th site and $|\hat\Omega\rangle_i$ is the vacuum in $\hat{\mathcal{H}}_i$. Similarly, let $\mathcal{H} = \otimes_{i=1}^N \mathcal{H}_i$ denote the Hilbert space of the original model (2.1), where $\mathcal{H}_i$ is the space spanned by its vacuum $|\Omega\rangle_i$ and the one-particle states $c_{i\sigma}^\dagger|\Omega\rangle_i$. The unitary mapping $\varphi : \mathcal{H} \to \hat{\mathcal{H}}$ defined by

$$\varphi|\Omega\rangle_i = b_i^\dagger|\hat\Omega\rangle_i \,, \quad \varphi(c_{i\sigma}^\dagger|\Omega\rangle_i) = f_{i\sigma}^\dagger|\hat\Omega\rangle_i \tag{2.3}$$

induces a natural way of associating to each linear operator $A : \mathcal{H} \to \mathcal{H}$ a corresponding linear operator $\hat A = \varphi A \varphi^{-1} = \varphi A \varphi^\dagger$ acting on $\hat{\mathcal{H}}$. It is shown in Ref. [10] that under this correspondence the Hamiltonian H_0 in Eq. (2.1) is transformed into

$$\hat H_0 = \frac{t\pi^2}{N^2} \Big\{ \sum_{i<j} \sin^{-2}\!\big(\tfrac{\pi}{N}\,(i-j)\big)\big(1 - P_{ij}^{(1|m)}\big) - \frac{1}{3}\,(N^2-1)\,\mathcal{F} \Big\} \,, \tag{2.4}$$

where $\mathcal{F} \equiv \sum_i \hat n_i$ is the total number of su($1|m$) fermions and $P_{ij}^{(1|m)}$ denotes the su($1|m$) supersymmetric permutation operator. Recall that the action of $P_{ij}^{(1|m)}$ on the canonical basis of $\hat{\mathcal{H}}$ is given by

$$P_{ij}^{(1|m)} |\cdots \sigma_i \cdots \sigma_j \cdots\rangle = \epsilon(\boldsymbol{\sigma}) |\cdots \sigma_j \cdots \sigma_i \cdots\rangle \,,$$

where $\epsilon(\boldsymbol{\sigma})$ is 1 (respectively -1) if $\sigma_i = \sigma_j = 0$ (resp. $\sigma_i, \sigma_j \geqslant 1$), while for $\sigma_i \sigma_j = 0$ and $\sigma_i \neq \sigma_j$ it is equal to the number of fermionic spins σ_k with $i+1 \leqslant k \leqslant j-1$. Thus the first term in $\hat H_0$ coincides with the Hamiltonian of the su($1|m$) supersymmetric HS chain [13, 30–33], as we had anticipated.

2.2. Free energy

We shall next explain how to compute in closed form the grand potential of the su(m) KY model (2.1) by exploiting its equivalence with the su($1|m$) HS spin chain

Hamiltonian (2.4). In the thermodynamic limit, this is equivalent to computing the free energy of the Hamiltonian

$$H = H_0 - \frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (n^\sigma - n^m) - \mu_c \sum_\sigma n^\sigma \equiv H_0 + H_1, \tag{2.5}$$

where $n^\sigma \equiv \sum_i n_{i\sigma}$ denotes the total number of fermions of type σ. The last term in H_1 is the chemical potential of the fermions (or, equivalently, of the electric charge), while the first one can be interpreted as arising from the interaction with an external su(m) magnetic field with strengths $h_1, \ldots, h_{m-1}$ along each (Hermitian) generator of the standard Cartan subalgebra of su(m) (see Ref. [10] for more details). In particular, for $m = 2$ the term $-(h_1/2)(n^1 - n^2)$ equals $-h_1 S^z$, where S^z is the z component of the total spin operator. The su($1|m$) spin chain Hamiltonian $\hat{H}$ equivalent to H under the mapping (A.2) is $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_1 = -\frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (\mathcal{N}_\sigma - \mathcal{N}_m) - \mu_c \mathcal{F}$$

and $\mathcal{N}_\sigma \equiv \hat{n}^\sigma$ is the total numbers of su($1|m$) fermions of type σ. We can thus write

$$\hat{H} = J H_{\rm HS}^{(1|m)} - \frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma (\mathcal{N}_\sigma - \mathcal{N}_m) - (t_0 + \mu_c)\mathcal{F}\,, \tag{2.6}$$

where

$$H_{\rm HS}^{(1|m)} = \frac{1}{2}\sum_{i<j} \sin^{-2}\!\left(\tfrac{\pi}{N}(i-j)\right)\left(1 - P_{ij}^{(1|m)}\right)$$

and

$$J = \frac{2t\pi^2}{N^2}\,, \qquad t_0 = \frac{t\pi^2}{3N^2}\,(N^2 - 1)\,. \tag{2.7}$$

The Hamiltonian $\hat{H}$ can be more concisely expressed as

$$\hat{H} = J H_{\rm HS}^{(1|m)} - \sum_\sigma \mu_\sigma \mathcal{N}_\sigma, \tag{2.8}$$

where μ_σ is the chemical potential of the fermion of type σ, given by

$$\mu_\sigma = \frac{1}{2}\,h_\sigma + \mu_c + t_0\,, \quad 1 \leqslant \sigma \leqslant m-1\,; \tag{2.9}$$

$$\mu_m = -\frac{1}{2}\sum_{\sigma=1}^{m-1} h_\sigma + \mu_c + t_0\,. \tag{2.10}$$

As shown in Refs. [10] and [21], the spectrum of the su($1|m$) spin chain (2.8), and hence of the equivalent Hamiltonian (2.5), can be generated from the formula

$$E(\mathbf{s}) = J\sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i(N-i) - \sum_i \mu_{s_i}\,, \tag{2.11}$$

where $\mu_0 \equiv 0$, $\mathbf{s} \in \{0, \ldots, m\}^N$ and $\delta(s, s')$ is defined by

$$\delta(s,s') = \begin{cases} 1, & s > s' \ \text{ or } \ s = s' > 0 \\ 0, & s < s' \ \text{ or } \ s = s' = 0\,. \end{cases} \tag{2.12}$$

The vectors $\boldsymbol{\delta}(\mathbf{s})$ with components $\delta(s_i, s_{i+1})$ $(1 \leqslant i \leqslant N-1)$ in Eq. (2.11) are su(1|m) motifs [13, 14, 34]. In fact, the first sum in Eq. (2.11) can be interpreted as the energy of a one-dimensional vertex model with $N+1$ vertices $0, \dots, N$ joined by N bonds with values $s_1, \dots, s_N \in \{0, \dots, m\}$, the energy associated to the ith vertex being equal to $\delta(s_i, s_{i+1}) i(N-i)$ [14].

Equation (2.11) is the key ingredient in the exact computation of the free energy in the thermodynamic limit through the (site-dependent) transfer matrix method developed in Ref. [21]. Indeed, from Eq. (2.11) it follows that the partition function can be expressed as

$$\mathcal{Z} = \operatorname{tr}(A(x_0) \cdots A(x_{N-1})), \tag{2.13}$$

where $x_k \equiv k/N$ and the $(m+1) \times (m+1)$ transfer matrix $A(x)$ has matrix elements

$$A_{\alpha\beta}(x) = q^{K\varepsilon(x)\delta(\alpha,\beta) - \frac{1}{2}(\mu_\alpha + \mu_\beta)}, \quad 0 \leqslant \alpha, \beta \leqslant m\,. \tag{2.14}$$

In the latter equation we have defined

$$q = \mathrm{e}^{-1/T}\,, \qquad \varepsilon(x) = x(1-x)\,, \qquad K = 2t\pi^2 > 0\,, \tag{2.15}$$

and as above we have taken (without loss of generality) $\mu_0 \equiv 0$. It can then be shown that in the thermodynamic limit $N \to \infty$ Eq. (2.13) yields the following closed-form expression for the free energy per site of the Hamiltonian (2.5):

$$f = -2T \int_0^{1/2} \log \lambda_1(x)\, \mathrm{d}x\,, \tag{2.16}$$

where $\lambda_1(x)$ is the largest eigenvalue in modulus of the matrix $A(x)$ (simple and positive, by the Perron–Frobenius theorem). In fact, in the next section we shall explain how the latter formula leads to the expression for the grand potential derived by a more laborious method in Ref. [16].

From now on we shall restrict ourselves to the su(2) case, for which the eigenvalue $\lambda_1(x)$ can be computed in closed form. Indeed, in this case the matrix $A(x)$ is given by

$$A(x) = \begin{pmatrix} 1 & q^{-\mu_1/2} & q^{-\mu_2/2} \\ q^{K\varepsilon(x) - \mu_1/2} & q^{K\varepsilon(x) - \mu_1} & q^{-(\mu_1+\mu_2)/2} \\ q^{K\varepsilon(x) - \mu_2/2} & q^{K\varepsilon(x) - (\mu_1+\mu_2)/2} & q^{K\varepsilon(x) - \mu_2} \end{pmatrix}.$$

By Eqs. (2.7) and (2.9)-(2.10), the chemical potentials μ_σ are given (in the thermodynamic limit) by

$$\mu_\sigma = (-1)^{\sigma+1} \frac{h}{2} + \mu\,,$$

where $h \equiv h_1$ and $\mu \equiv \mu_c + K/6$. Taking these relations into account, the Perron–Frobenius eigenvalue of the matrix $A(x)$ reads

$$\lambda_1(x) = \mathrm{e}^{\beta(\mu - K\varepsilon(x))} \Big[b(x) + \sqrt{b^2(x) + \mathrm{e}^{K\beta\varepsilon(x)} - 1} \Big],$$

where $\beta \equiv 1/T$ and

$$b(x) = \frac{1}{2}\, \mathrm{e}^{\beta(K\varepsilon(x) - \mu)} + \cosh(\tfrac{\beta h}{2}). \tag{2.17}$$

The previous equations yield the following *exact* explicit formula for the free energy per site of the su(2) KY model in the presence of a magnetic field h and charge chemical potential μ_c:

$$f = -\mu_c - 2T \int_0^{1/2} \log\Big[b(x) + \sqrt{b^2(x) + \mathrm{e}^{K\beta\varepsilon(x)} - 1}\,\Big]\mathrm{d}x\,. \tag{2.18}$$

The magnetization (per site) $m_s = \langle n^1 - n^2\rangle/(2N)$ and the charge (fermion) density $n_c = \langle n^1 + n^2\rangle/N$ (where $\langle\cdot\rangle$ denotes thermal average) are easily computed in closed form differentiating the latter equation, namely

$$m_s = -\frac{\partial f}{\partial h} = \sinh(\tfrac{\beta h}{2}) \int_0^{1/2} D(x)^{-1/2}\mathrm{d}x, \tag{2.19}$$

$$n_c = -\frac{\partial f}{\partial \mu_c} = 1 - \int_0^{1/2} D(x)^{-1/2}\mathrm{e}^{\beta(K\varepsilon(x)-\mu)}\mathrm{d}x, \tag{2.20}$$

with

$$D(x) \equiv b^2(x) + \mathrm{e}^{K\beta\varepsilon(x)} - 1.$$

The corresponding susceptibilities χ_s and χ_m are then given by

$$\chi_s = \frac{\partial m_s}{\partial h} = \frac{\beta}{2} \int_0^{1/2} \mathrm{e}^{\beta(K\varepsilon(x)-\mu)} D(x)^{-3/2}\Big[1 + \tfrac{1}{2}\sinh^2(\tfrac{\beta h}{2}) + \cosh(\tfrac{\beta h}{2})\Big(\mathrm{e}^{\beta\mu} + \tfrac{1}{4}\,\mathrm{e}^{\beta(K\varepsilon(x)-\mu)}\Big)\Big]\mathrm{d}x, \tag{2.21}$$

$$\chi_c = \frac{\partial n_c}{\partial \mu_c} = \beta \int_0^{1/2} \mathrm{e}^{\beta(K\varepsilon(x)-\mu)} D(x)^{-3/2}\Big[\sinh^2(\tfrac{\beta h}{2}) + \tfrac{1}{2}\,\mathrm{e}^{\beta(K\varepsilon(x)-\mu)}\cosh(\tfrac{\beta h}{2}) + \mathrm{e}^{K\beta\varepsilon(x)}\Big]\mathrm{d}x. \tag{2.22}$$

Other thermodynamic functions, like the energy u, entropy $s = \beta(u - f)$ and specific heat (per site) $c_V = (\partial u)/(\partial T)$, are easily computed from Eq. (2.18). For instance,

$$u = \frac{\partial}{\partial\beta}(\beta f) = -\mu_c - \int_0^{1/2} D(x)^{-1/2}\Big[h\sinh(\tfrac{\beta h}{2}) + (K\varepsilon(x) - \mu)\mathrm{e}^{\beta(K\varepsilon(x)-\mu)}\Big]\,\mathrm{d}x\,. \tag{2.23}$$

3. Derivation of a generalized Kato–Kuramoto equation through the transfer matrix formalism

As explained in the Introduction, Kato and Kuramoto [16] obtained an expression for the grand potential per spin ω of the KY Hamiltonian H_0 in Eq. (2.1) in terms of a function implicitly determined by an algebraic equation that we have named the Kato–Kuramoto equation. It should be noted that the deduction of this equation in Ref. [16] requires that (in our notation) the dispersion relation $\varepsilon(x)$ of the model be given by Eq. (2.15)§. In this Section we shall first of all show how the KK equation emerges in a transparent way from Eq. (2.16) for the free energy per spin of the su(m) KY model with the general chemical potential term H_1 in Eq. (2.5). More importantly, we shall

§ In point of fact, in Ref. [16] there is a more general derivation of the KK equation based on the equivalence of the KY model to a system of g-ons with an appropriately chosen statistical matrix, but this derivation is valid only for the su(1|2) case.

outline how this equation can be generalized to a large class of solvable lattice models with (complete or partial) Yangian invariance, including the supersymmetric PF and FI spin chains.

To this end, let us denote by $P_m(\lambda) = \det(\lambda - A(x))$ the characteristic polynomial of the transfer matrix $A(x)$ in Eq. (2.14), where we have suppressed the dependence of P_m on x for the sake of conciseness. As remarked in Ref. [21] $P_m(\lambda)$ is divisible by λ, so that

$$Q_m(\lambda) = \frac{P_m(\lambda)}{\lambda}$$

is an m-th degree monic polynomial. We shall next show that Q_m satisfies the recursion relation

$$Q_m(\lambda) = \lambda Q_{m-1}(\lambda) - \eta\, a_m^2 \prod_{\sigma=1}^{m-1} \left(\lambda + (1-\eta)a_\sigma^2\right), \qquad m \geqslant 2\,, \tag{3.1}$$

where we have set

$$\eta = \mathrm{e}^{-K\beta\varepsilon(x)}\,, \qquad a_\sigma = \mathrm{e}^{\beta\mu_\sigma/2}\,, \quad 1 \leqslant \sigma \leqslant m\,.$$

To see this, note that we can write

$$(-1)^{m+1} P_m(\lambda) = \begin{vmatrix} 1-\lambda & a_1 & a_2 & \cdots & a_{m-1} & a_m \\ \eta a_1 & \eta a_1^2 - \lambda & a_1 a_2 & \cdots & a_1 a_{m-1} & a_1 a_m \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta a_{m-1} & \eta a_{m-1} a_1 & \eta a_{m-1} a_2 & \cdots & \eta a_{m-1}^2 - \lambda & a_{m-1} a_m \\ \eta a_m & \eta a_m a_1 & \eta a_m a_2 & \cdots & \eta a_m a_{m-1} & \eta a_m^2 - \lambda \end{vmatrix}.$$

Multiplying the first row of the latter determinant by ηa_m and subtracting it from the last one, after a straightforward calculation we obtain

$$Q_m(\lambda) = \lambda Q_{m-1}(\lambda) - \eta a_m^2 \begin{vmatrix} a_1 & a_2 & \cdots & a_{m-1} & 1 \\ \eta a_1^2 - \lambda & a_1 a_2 & \cdots & a_1 a_{m-1} & a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ \eta a_{m-1} a_1 & \eta a_{m-1} a_2 & \cdots & \eta a_{m-1}^2 - \lambda & a_{m-1} \end{vmatrix}.$$

The determinant in the previous formula can be easily evaluated by subtracting the last column multiplied by a_i from the i-th column for $i = 1, \ldots, m-1$, which yields Eq. (3.1). From the latter recursion relation and the initial condition

$$Q_1(\lambda) = \lambda - \eta a_1^2 - 1$$

we readily obtain the following explicit formula for $Q_m(\lambda)$:

$$Q_m(\lambda) = \lambda^m - \lambda^{m-1} - \eta \sum_{k=0}^{m-1} \lambda^k a_{m-k}^2 \prod_{\sigma=1}^{m-k-1} \left(\lambda + (1-\eta)a_\sigma^2\right).$$

This expression can be somewhat simplified with the help of the identities

$$\prod_{\sigma=1}^{p} \left(\lambda + (1-\eta)a_\sigma^2\right) = \sum_{l=0}^{p} \lambda^l (1-\eta)^{p-l} e_{p-l}(a_1^2, \ldots, a_p^2)$$

and

$$e_k(x_1,\dots,x_{k+q}) = \sum_{l=0}^{q} x_{k+l} e_{k-1}(x_1,\dots,x_{k+l-1})\,,$$

where

$$e_k(x_1,\dots,x_r) \equiv \sum_{1\leqslant\sigma_1<\cdots<\sigma_k\leqslant r} x_{\sigma_1}\cdots x_{\sigma_k}$$

denotes the elementary symmetric polynomial of degree k in $r \geqslant k$ variables. Indeed, after a lengthy but straightforward calculation we obtain

$$(1-\eta)Q_m(\lambda) = \lambda^m - (1-\eta)\lambda^{m-1} - \eta\prod_{\sigma=1}^{m}\Big(\lambda + a_\sigma^2(1-\eta)\Big). \tag{3.2}$$

Performing the change of variable

$$X = \frac{1-\eta}{\lambda}, \qquad \lambda \neq 0\,, \tag{3.3}$$

we thus arrive at the fundamental identity

$$\lambda^{-m}(1-\eta)Q_m(\lambda) = 1 - X - \eta\prod_{\sigma=1}^{m}\left(1 + a_\sigma^2 X\right), \qquad \lambda \neq 0\,. \tag{3.4}$$

In view of the previous discussion, we next rewrite Eq. (2.16) for the free energy per site of the supersymmetric su(m) KY model with a chemical potential term as

$$f = 2T\int_0^{1/2} \log\left(\frac{X_1(x)}{1-\mathrm{e}^{-K\beta\varepsilon(x)}}\right)\mathrm{d}x\,, \tag{3.5}$$

where

$$X_1(x) \equiv \frac{1-\mathrm{e}^{-K\beta\varepsilon(x)}}{\lambda_1(x)}\,.$$

Since $\lambda_1(x)$ is a nonzero root of the characteristic equation of the matrix $A(x)$, from Eqs. (3.3)-(3.4) and the definition of η we deduce that the function $X_1(x)$ satisfies

$$1 - X_1 = \eta\prod_{\sigma=1}^{m}\left(1 + a_\sigma^2 X_1\right),$$

or equivalently

$$K\beta\varepsilon(x) = -\log(1-X_1(x)) + \sum_{\sigma}\log(1+\mathrm{e}^{\beta\mu_\sigma}X_1(x))\,. \tag{3.6}$$

Note that, since λ_1 does not vanish and $\varepsilon(0) = 0$, we must have

$$X_1(0) = 0\,. \tag{3.7}$$

Equations (3.5)-(3.7) are equivalent to the expression for the grand potential per site ω of the su(m) KY model in Ref. [16]. To see this in more detail, note that in the thermodynamic limit the Hamiltonian $\hat{H}_0$ in Eq. (2.4) (with $t = 1$) is related to the analogous Hamiltonian

$$\mathcal{H}_{t\text{-}J} = -\frac{\pi^2}{N^2}\sum_{i<j}\sin^{-2}(\tfrac{\pi}{N}(i-j))P_{ij}^{(1|m)} \tag{3.8}$$

in Ref. [16] by

$$\mathcal{H}_{t\text{-}J} = \hat{H}_0 + \frac{\pi^2}{3}\,\mathcal{F} - \frac{N\pi^2}{6},$$

where we have used the identity

$$\sum_{i\neq j} \sin^{-2}(\pi(i-j)/N) = \frac{N}{3}\,(N^2-1)\,. \tag{3.9}$$

Thus in the thermodynamic limit the grand potential of $\mathcal{H}_{t\text{-}J}$ should be equal to the free energy of the right-hand side of Eq. (3.8) with the addition of a chemical potential term. Since we have absorbed the term proportional to $\mathcal{F}$ in the definition of the fermion chemical potentials μ_σ (cf. Eqs. (2.9)-(2.10)), we must then show that

$$\omega = f - \frac{\pi^2}{6} = 2T\int_0^{1/2} \log\Big(\frac{X_1(x)}{1-\mathrm{e}^{-K\beta\varepsilon(x)}}\Big)\,\mathrm{d}x - \frac{\pi^2}{6}\,. \tag{3.10}$$

Performing the change of variable $x = (\pi - p)/(2\pi)$ in the integral, under which

$$K\varepsilon(x) = 2\pi^2 x(1-x) = \frac{1}{2}\,(\pi^2 - p^2) \equiv \varepsilon_0(p)\,,$$

we see that it suffices to show that

$$\omega = \frac{T}{\pi}\int_0^{\pi} \log\Big(\frac{\tilde{X}_1(p)}{1-\mathrm{e}^{-\beta\varepsilon_0(p)}}\Big)\,\mathrm{d}p - \frac{\pi^2}{6}\,, \tag{3.11}$$

where by Eqs. (3.6)-(3.7) $\tilde{X}_1(p) \equiv X_1(x(p))$ satisfies

$$\beta\varepsilon_0(p) = -\log(1-\tilde{X}_1(p)) + \sum_\sigma \log(1+\mathrm{e}^{\beta\mu_\sigma}\tilde{X}_1(p)) \tag{3.12}$$

and

$$\tilde{X}_1(\pi) = X_1(0) = 0\,. \tag{3.13}$$

Our claim now follows from the fact that Eqs. (3.11) and (3.12) are nothing but Eqs. (2.33) and (2.22) in Ref. [16] (taking into account that we have set, without loss of generality, $\mu_0 = 0$), while Eq. (3.13) is the condition that according to the latter reference determines the appropriate branch $\tilde{X}_1(p)$ of the algebraic function defined by Eq. (3.12).

A few remarks on the equivalence of Eq. (2.16) to (3.11)-(3.13)— or, more generally, (3.5)-(3.7)— are now in order. The approach of Ref. [16] is based on the derivation of the thermodynamics of the su(1|m) spin Sutherland model in the $N\to\infty$ limit, which yields the thermodynamics of the su(1|m) HS chain in the strong coupling limit through Polychronakos's freezing trick. An essential ingredient in this approach is the equivalence of the su(1|m) spin Sutherland model to a system of non-interacting su(1|m) particles whose spectrum can be effectively described in terms of generalized momenta obeying an appropriate exclusion statistics (see also [35]). From this description follows an integral relation satisfied by the one-particle energy $\tilde{\varepsilon}(p)$ (defined by $\tilde{X}_1(p) = \mathrm{e}^{-\beta\tilde{\varepsilon}(p)}$), which in turn yields Eq. (3.12). Equation (3.11) is then derived through a fairly elaborate argument, by first expressing the grand potential of the spin Sutherland model in terms of the one-particle energy and then subtracting the phonon contribution. By

contrast, in our approach Eq. (2.16) follows in a straightforward way from Eqs. (2.11)-(2.12) applying the transfer matrix method, as summarized in the previous section (see Ref. [21] for full details). That the spectrum of the su($1|m$) HS chain is completely described by equations analogous to (2.11)-(2.12) is in fact a fundamental property of all su($n|m$) spin chains of HS type, stemming directly from the structure of their partition function [14]. This description does not rely at all on the properties of the associated spin Sutherland model, such as the existence of generalized quasimomenta satisfying an appropriate exclusion statistics, and may thus apply even to other types of Yangian-invariant models not necessarily derived from a dynamical spin model. In view of the above argument, the free energy of all these models, including the three families of su($n|m$) spin chains of HS type, should also be described by equations analogous to (3.5)–(3.7). This is indeed remarkable, and it underscores the fact that the range of applicability of the latter equations is much wider than could naively be expected from their original derivation in Ref. [16].

As a simple example of the last assertion, consider the su($n|m$) Polychronakos–Frahm and Frahm–Inozemtsev chains with a chemical potential term $-\sum_{\alpha=1}^{m+n-1}\mu_\alpha \mathcal{N}_\alpha$. When $n = 1$, the spectra of these models can be obtained replacing the Haldane–Shastry dispersion relation $i(N-i)$ in Eq. (2.11) respectively by i and $i(i+N\gamma)$ (with $N\gamma > -1$) [20, 21]. It is then straightforward to show that if we set

$$K = \begin{cases} NJ, & \text{for the PF chain} \\ N^2 J, & \text{for the FI chain,} \end{cases}$$

the partition function of these models is still given by Eqs. (2.13)-(2.14), but with $\varepsilon(x)$ replaced by

$$\varepsilon(x) = \begin{cases} x\,, & \text{for the PF chain} \\ x(x+\gamma)\,, & \text{for the FI chain} \end{cases} \tag{3.14}$$

(γ being now a nonnegative parameter). Consequently, in the thermodynamic limit their free energy can be expressed in the form‖

$$f = -T\int_0^1 \log\lambda_1(x)\,\mathrm{d}x\,, \tag{3.15}$$

where $\lambda_1(x)$ is the Perron–Frobenius eigenvalue of the transfer matrix (2.14) with $\varepsilon(x)$ as in Eq. (3.14). In the $n = 1$ case, from the latter equation we deduce reasoning as above that the grand potential of the PF and FI chains can be written as

$$\omega = T\int_0^1 \log\Big(\frac{X_1(x)}{1-\mathrm{e}^{-K\beta\varepsilon(x)}}\Big)\,\mathrm{d}x\,, \tag{3.16}$$

where $X_1(x)$ satisfies the generalized KK equation (3.6)-(3.7) with $\varepsilon(x)$ given by (3.14). By the same token, in the general su($n|m$) case with $n > 1$ we conjecture that (3.15) is equivalent to (3.16), where now $X_1(x)$ should satisfy the generalized KK equation

$$K\beta\varepsilon(x) = -\sum_{\alpha=0}^{n-1}\log(1-\mathrm{e}^{\beta\mu_\alpha}X_1(x)) + \sum_{\alpha=n}^{m+n-1}\log(1+\mathrm{e}^{\beta\mu_\alpha}X_1(x)) \quad (\text{with } \mu_0 \equiv 0) \tag{3.17}$$

‖ The missing factor of 2 and the different range of integration in Eq. (3.15) compared to the analogous Eq. (2.16) are due to the lack of symmetry about $x = 1/2$ of $\varepsilon(x)$ in Eq. (3.14).

with the condition $X_1(0) = 0$. More generally, this should be true for *any* model whose spectrum be given by an equation of the form

$$E(\mathbf{s}) = \frac{K}{N^\alpha} \sum_{i=1}^{N-1} \mathcal{E}_N(i)\delta(s_i, s_{i+1}) - \sum_i \mu_{s_i}\,, \tag{3.18}$$

together with the su$(n|m)$ analogue of Eq. (2.12):

$$\delta(s, s') = \begin{cases} 1, & s > s' \ \text{ or } \ s = s' \geqslant n \\ 0, & s < s' \ \text{ or } \ s = s' < n\,, \end{cases} \tag{3.19}$$

provided that $\lim_{N\to\infty} \mathcal{E}_N(Nx) \equiv \varepsilon(x)$ exists for all $x \in [0, 1]$. In fact, an equation akin to (3.17) has been proposed in Ref. [35] for the supersymmetric Sutherland model.

On a more technical level, our approach also helps clarify an important issue concerning the definition of the function $\tilde{X}_1(p)$ through Eqs. (3.12)-(3.13), or equivalently $X_1(x)$ through Eq. (3.6) and the condition $X_1(0) = 0$. Indeed, from the previous discussion it follows that the latter equation is equivalent to the algebraic equation (with coefficients depending on the parameter $x \in [0, 1/2]$)

$$\widehat{Q}(X) \equiv 1 - X - \mathrm{e}^{-K\beta\varepsilon(x)} \prod_{\sigma=1}^{m} \left(1 + \mathrm{e}^{\beta\mu_\sigma} X\right) = 0 \tag{3.20}$$

for the variable X. When $x = 0$ it is clear that $X = 0$ is a simple root of this equation, since $\widehat{Q}$ vanishes at the origin and the coefficient of X in the latter polynomial is $-1 - \sum_{\sigma=1}^{m} \mathrm{e}^{\beta\mu_\sigma} \neq 0$. By the implicit function theorem, the condition $X_1(0) = 0$ uniquely defines a branch of the algebraic function (3.20) near $x = 0$. However, it is not clear whether this is still the case —i.e., whether there is no branch crossing— as x increases. From a practical standpoint, the actual computation of $X_1(x)$ through Eqs. (3.20) and (3.7) at a point $x > 0$ is arduous at best, since it requires following the appropriate branch of the algebraic function (3.20) all the way from $x = 0$. Both problems are solved by our approach, since it is now clear from Eq. (3.3) that $X_1(x)$ is simply given by $(1 - \mathrm{e}^{-K\beta\varepsilon(x)})/\lambda_1(x)$, where $\lambda_1(x)$ is the eigenvalue of the matrix $A(x)$ with the largest modulus (whose uniqueness for arbitrary x is guaranteed by the Perron–Frobenius theorem).

4. Ground state phases

In this section we shall use Eqs. (2.19)-(2.20) to derive exact expressions for the zero-temperature magnetization and charge densities of the spin 1/2 KY model for arbitrary values of the magnetic field strength h and charge chemical potential μ. In this way we shall identify the model's five ground state phases, which in turn determine the form of the low-temperature asymptotic series that we shall compute in the next section.

In order to simplify the calculations, in the rest of the paper we shall take K as the unit of energy and hence of temperature (since $k_\mathrm{B} = 1$ from the outset). We can also suppose without loss of generality that $h \geqslant 0$, since changing the sign of h is equivalent to exchanging n^1 with n^2, or equivalently replacing (m_s, n_c) by $(-m_s, n_c)$.

Region	Equation	Species content
$\mathcal{B}_0$	$0 < \mu + \frac{h}{2} < \frac{1}{4}, \quad \mu < \frac{h}{2}$	Bosons and "up" fermions
$\mathcal{B}_1$	$h < \frac{1}{4}, \quad \mu > \frac{1}{8}$	Fermions
$\mathcal{T}$	$\frac{h}{2} < \mu < \frac{1}{8}$	Bosons and fermions
$\mathcal{W}_0$	$\mu < -\frac{h}{2}$	Bosons
$\mathcal{W}_1$	$h > \frac{1}{4}, \quad \mu > -\frac{h}{2} + \frac{1}{4}$	"Up" fermions

Table 1. Definitions of the regions $\mathcal{B}_i$, $\mathcal{T}$ and $\mathcal{W}_i$ in the half-plane $h \geqslant 0$ (cf. Fig. 1 left) and their species content.

The magnetization per site m_s clearly vanishes for $h = 0$. On the other hand, for $h > 0$ and $T \to 0$ we can replace $\sinh(\beta h/2)$ and $\cosh(\beta h/2)$ by $\mathrm{e}^{\beta h/2}/2$ up to an exponentially small term $\mathrm{O}(\mathrm{e}^{-\beta h/2})$. Hence at low temperatures we can write

$$m_s \simeq \int_0^{1/2} [4\mathrm{e}^{-\beta h} D(x)]^{-1/2} \mathrm{d}x, \tag{4.1}$$

with

$$4\mathrm{e}^{-\beta h} D(x) \simeq 1 + 4\mathrm{e}^{\beta(\varepsilon(x)-h)} + 2\mathrm{e}^{\beta(\varepsilon(x)-\mu-\frac{h}{2})} + \mathrm{e}^{2\beta(\varepsilon(x)-\mu-\frac{h}{2})},$$

where we have dropped several exponentially small terms $\mathrm{O}(\mathrm{e}^{-\beta h})$. From the previous equations it immediately follows that the zero-temperature magnetization is given by

$$m_s = \left| \{ x \in [0, \tfrac{1}{2}] : \varepsilon(x) < h, \ \varepsilon(x) < \mu + \tfrac{h}{2} \} \right|,$$

where $|A|$ denotes the measure of the set A. Thus, at $T = 0$ we have

$$m_s = \begin{cases} x_0(h), & (h,\mu) \in \mathcal{B}_1 \cup \mathcal{T} \\ \dfrac{1}{2}, & (h,\mu) \in \mathcal{W}_1 \\ x_0(\mu + \frac{h}{2}), & (h,\mu) \in \mathcal{B}_0 \\ 0, & (h,\mu) \in \mathcal{W}_0 \,, \end{cases} \tag{4.2}$$

where

$$x_0(t) = \frac{1}{2}\left(1 - \sqrt{1 - 4t}\,\right) \tag{4.3}$$

is the unique root of the equation $\varepsilon(x) = t$ in the interval $[0, 1/2]$, and the regions $\mathcal{B}_i$, $\mathcal{T}$ and $\mathcal{W}_i$ are defined in Table 1 (cf. Fig. 1 left). It is also straightforward to check that m_s is continuous on the boundaries of the latter sets, and hence everywhere.

Similarly,

$$1 - n_c = 2\int_0^{1/2} [4\mathrm{e}^{-2\beta(\varepsilon(x)-\mu)} D(x)]^{-1/2} \mathrm{d}x,$$

where the term in brackets can be approximated at low temperatures by

$$\begin{aligned} &4\mathrm{e}^{-2\beta(\varepsilon(x)-\mu)}(\mathrm{e}^{\beta\varepsilon(x)} - 1) + (1 + \mathrm{e}^{-\beta(\varepsilon(x)-\mu-\frac{h}{2})})^2 \\ &\qquad = 1 + \mathrm{e}^{-2\beta(\varepsilon(x)-\mu-\frac{h}{2})} + 2\mathrm{e}^{-\beta(\varepsilon(x)-\mu-\frac{h}{2})} + 4\mathrm{e}^{-\beta(\varepsilon(x)-2\mu)} - 4\mathrm{e}^{-2\beta(\varepsilon(x)-\mu)} \end{aligned}$$

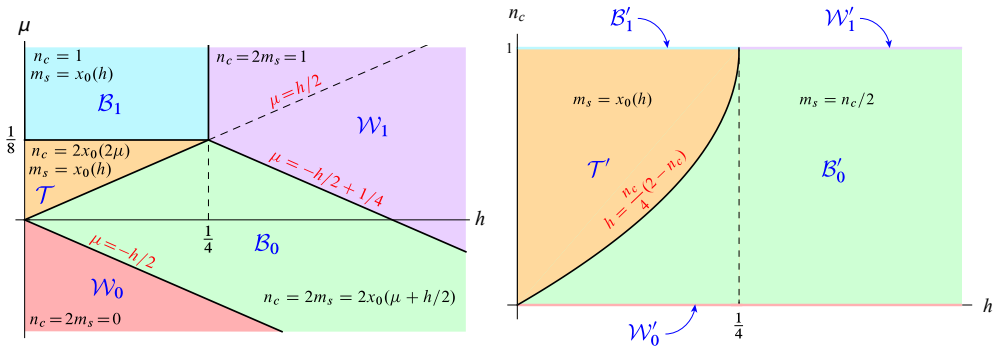

Figure 1. Left: zero temperature magnetization and charge densities as functions in each of the regions $\mathcal{B}_i$, $\mathcal{T}$, $\mathcal{W}_i$ defined in Table 1. Right: Images $\mathcal{B}'_i$, $\mathcal{T}'$, $\mathcal{W}'_i$ of the regions $\mathcal{B}_i$, $\mathcal{T}$, $\mathcal{W}_i$ under the mapping $(h, \mu) \mapsto (h, n_c)$.

Proceeding as before we obtain the following expression for the zero-temperature charge density:

$$\frac{1}{2}(1 - n_c) = \Big|\{x \in [0, \tfrac{1}{2}] : \varepsilon(x) > 2\mu,\ \varepsilon(x) > \mu + \tfrac{h}{2}\}\Big|,$$

or equivalently

$$n_c = 2\,\Big|\{x \in [0, \tfrac{1}{2}] : \varepsilon(x) < 2\mu \text{ or } \varepsilon(x) < \mu + \tfrac{h}{2}\}\Big|.$$

Thus the zero-temperature charge density is given by

$$n_c = \begin{cases} 1, & (h, \mu) \in \mathcal{B}_1 \cup \mathcal{W}_1 \\ 2x_0(2\mu), & (h, \mu) \in \mathcal{T} \\ 2x_0(\mu + \frac{h}{2}), & (h, \mu) \in \mathcal{B}_0 \\ 0, & (h, \mu) \in \mathcal{W}_0\,. \end{cases} \tag{4.4}$$

As before, it is easily verified that n_c is continuous across the boundaries of the regions $\mathcal{B}_i$, $\mathcal{W}_i$, $\mathcal{T}$. It should also be noted that Eqs. (4.2)-(4.4) were derived in Ref. [10] by a more laborious method, based on determining the magnon content of the ground state for arbitrary values of the parameters h and μ using Eqs. (2.11)-(2.12) for the energies of the equivalent vertex model.

As remarked in Ref. [10], the regions $\mathcal{B}_i$, $\mathcal{T}$ and $\mathcal{W}_i$ defined above have a clear interpretation as different zero-temperature phases of the model. Indeed, taking into account that

$$m_s = \frac{1}{2N}\,\langle n^1 - n^2 \rangle\,, \qquad n_c = \frac{1}{N}\,\langle n^1 + n^2 \rangle\,,$$

it is clear the latter regions are characterized by their different species content as listed in Table 1. Thus in the bands $\mathcal{B}_0$ and $\mathcal{B}_1$ the spin 1/2 KY model is respectively equivalent (at zero temperature) to an su(1|1) and an (antiferromagnetic) su(2) Haldane–Shastry chain, while in the wedges $\mathcal{W}_i$ it is trivial. In fact, the value of the zero-temperature magnetization in the bands $\mathcal{B}_{0,1}$ coincides with the corresponding one for the su(1|1)

and su(2) HS spin chains computed in Refs. [36] and [20]. On the other hand, in the triangle $\mathcal{T}$ the model is genuinely of su(1|2) type. We shall see in the next sections that in each of the zero-temperature phases found above the thermodynamic quantities have a different low-temperature asymptotic series.

Equation (4.2) for m_s can be expressed in terms of the independent variables $h \geqslant 0$ and $0 \leqslant n_c \leqslant 1$ by taking into account how the regions $\mathcal{B}_i$, $\mathcal{T}$ and $\mathcal{W}_i$ transform under the (non-invertible) mapping $(h, \mu) \mapsto (h, n_c)$ determined by Eq. (4.4) (cf. Fig. 1 right). To begin with, it is obvious that the wedge $\mathcal{W}_0$ collapses into the line $n_c = 0$. Similarly, the wedge $\mathcal{W}_1$ is transformed into the horizontal half-line $n_c = 1$, $h > 1/4$, while the vertical band $\mathcal{B}_1$ goes into the segment $n_c = 1$, $h < 1/4$. On the other hand, the triangle $\mathcal{T}$ is mapped into the bounded region to the left of the parabola $h = h_0(n_c)$, where

$$h_0(n_c) = \varepsilon(n_c/2) = \frac{n_c}{4}\,(2 - n_c)\,.$$

Indeed, in the triangle $\mathcal{T}$ we have

$$n_c = 2x_0(2\mu) \iff \varepsilon(n_c/2) = 2\mu > h\,.$$

Likewise, the oblique band $\mathcal{B}_0$ is transformed into the unbounded region to the right of the parabola $h = h_0(n_c)$, since in this band

$$n_c = 2x_0(\mu + \tfrac{h}{2}) \iff \varepsilon(n_c/2) = \mu + \frac{h}{2} < h\,.$$

From these considerations it readily follows that the zero-temperature magnetization is expressed in terms of the variables (h, n_c) by

$$m_s = \begin{cases} x_0(h), & 0 \leqslant h \leqslant h_0(n_c) \\ \dfrac{1}{2}\,n_c, & h \geqslant h_0(n_c) \end{cases} \tag{4.5}$$

(cf. Ref. [29]). Note that m_s should be continuous everywhere in (h, n_c) space, since it is continuous when expressed in terms of the variables (h, μ). In particular, the previous expression for the critical magnetic field h_0 is recovered by imposing the continuity of m_s across the parabola $h = h_0(n_c)$.

From Eqs. (4.2)-(4.4) it is straightforward to compute the zero-temperature magnetic and charge susceptibilities by differentiation. To begin with, the magnetic susceptibility vanishes in the wedges $\mathcal{W}_0$ and $\mathcal{W}_1$. On the other hand, in the region $\mathcal{B}_1 \cup \mathcal{T}$ (including the segment $\mu = 1/8$, $h < 1/4$) we have

$$\chi_s = \frac{\partial m_s}{\partial h} = x_0'(h) = \frac{1}{1 - 2x_0(h)} = \frac{1}{1 - 2m_s}\,, \qquad (h, \mu) \in \mathcal{B}_1 \cup \mathcal{T}\,.$$

Likewise, in the band $\mathcal{B}_0$ the magnetic susceptibility is given by

$$\chi_s = \frac{1}{2}\,x_0'(\mu + \tfrac{h}{2}) = \frac{1}{2(1 - 2m_s)} = \frac{1}{2(1 - n_c)}\,, \qquad (h, \mu) \in \mathcal{B}_0\,.$$

A similar analysis for the charge susceptibility $\chi_c \equiv \partial n_c/\partial\mu_c = \partial n_c/\partial\mu$ yields the result

$$\chi_c = \begin{cases} 0, & (h,\mu) \in \mathcal{B}_1 \cup \mathcal{W}_0 \cup \mathcal{W}_1 \\ \dfrac{4}{1-n_c}\,, & (h,\mu) \in \mathcal{T} \\ \dfrac{2}{1-n_c}\,, & (h,\mu) \in B_0\,. \end{cases}$$

(In fact, χ_c vanishes also on the segment $h = 1/4$, $\mu > 1/8$.) Comparing with the previous expressions for m_s and n_c at zero temperature, we deduce that χ_s diverges on the half-lines $\{h = 1/4, \mu \geqslant 1/8\}$ and $\{\mu + h/2 = 1/4, \mu \leqslant 1/8\}$, while χ_c is divergent on the half-lines $\{\mu = 1/8, h < 1/4\}$ and $\{\mu + h/2 = 1/4, \mu \leqslant 1/8\}$. This is a well-known fact (see, e.g., Ref. [28]). More surprising is the behavior of the magnetic and charge susceptibilities at the boundaries of the su(1|1) phase with the su(1|2) phase and the vacuum. Indeed, these functions present jump discontinuities on the segment $\{\mu = h/2, 0 \leqslant h \leqslant 1/4\}$ and the half-line $\{\mu + h/2 = 0, h \geqslant 0\}$. The discontinuity on the latter segment (which, to the best of our knowledge, had not been previously pointed out in the literature) is particularly interesting, since it is due to the fact that $(1-2m_s)\chi_s$ and $(1-n_c)\chi_c$ are different constants on each side of this segment. It is also worth mentioning that $\chi_c = 4\chi_s$ on the union of the half-planes $\mu < h/2$ and $h > 1/4$. Note, finally, that our formulas for χ_s and χ_m agree with those of Ref. [28] in the triangle $\mathcal{T}$, which is the only region considered in the latter reference¶. In any case, the dependence of χ_s (resp. χ_c) exclusively on m_s (resp. on n_c) at zero temperature is a manifestation of the spin-charge separation characteristic of the t-J model.

It is also straightforward to express the zero-temperature susceptibilities as functions of the variables (h, n_c). The key fact in this respect is that the segment $\mu = h/2$, $0 \leqslant h \leqslant 1/4$ is mapped to the arc of the parabola $n_c = 2x_0(h) \equiv 1 - \sqrt{1-4h}$ (or, equivalently, $h = h_0(n_c)$) with $0 \leqslant h \leqslant 1/4$. The susceptibilities are then given by

$$\chi_s = \begin{cases} 0, & (h,n_c) \in \mathcal{W}_0' \cup \mathcal{W}_1' \\ \dfrac{1}{1-2m_s} \equiv \dfrac{1}{\sqrt{1-4h}}, & (h,n_c) \in \mathcal{B}_1' \cup \mathcal{T}' \\ \dfrac{1}{2(1-2m_s)} \equiv \dfrac{1}{2(1-n_c)}, & (h,n_c) \in \mathcal{B}_0' \end{cases}$$

and

$$\chi_c = \begin{cases} 0\,, & (h,n_c) \in \mathcal{B}_1' \cup \mathcal{W}_0' \cup \mathcal{W}_1' \\ \dfrac{4}{1-n_c}\,, & (h,n_c) \in \mathcal{T}' \\ \dfrac{2}{1-n_c}\,, & (h,n_c) \in\ \mathcal{B}_0'\,. \end{cases}$$

In particular, we see that $\chi_c = 4\chi_s$ in the infinite region $\sqrt{1-4h} < 1 - n_c$.

¶ Note that the magnetic field and the magnetic moment in the Ref. [28] are respectively $h/2$ and $2m_s$ in our notation. Indeed, in the latter reference the magnetic field interaction in the Hamiltonian is taken as $h(n^1 - n^2)$, while the magnetization is defined as $\langle n^1 - n^2\rangle$.

5. Asymptotic series for the free energy and criticality

Starting from the exact formula (2.18), in this section we shall derive the complete asymptotic series of the free energy of the su(2) KY model[+] at $T = 0$ for arbitrary values of the parameters h and μ. We shall also use this asymptotic series to analyze the model's criticality properties and the low-temperature behavior of its main thermodynamic functions.

5.1. Wedges $\mathcal{W}_0$ and $\mathcal{W}_1$

To begin with, it is straightforward to show that in the wedges $\mathcal{W}_{0,1}$ the free energy is exponentially small in β as $T \to 0$. Indeed, we can rewrite Eq. (2.18) as

$$f(T) = -\mu_c - \frac{h}{2} - 2T \int_0^{1/2} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\, \mathrm{d}x\,, \tag{5.1}$$

with

$$\widetilde{b}(x) \equiv \mathrm{e}^{-\frac{\beta h}{2}} b(x) = \frac{1}{2}[1 + \mathrm{e}^{-\beta(\mu + h/2 - \varepsilon(x))} + \mathrm{e}^{-\beta h}]. \tag{5.2}$$

Clearly all the exponents in the previous formula for f are strictly negative in the region $\mathcal{W}_1$, so that $f(0) = -\mu_c - h/2$. Taking in to account that $\varepsilon(x) \leqslant \varepsilon(1/2) = 1/4$ we easily obtain

$$|f(T) - f(0)| \leqslant T \log[a + \sqrt{a^2 + \mathrm{e}^{-\beta(h-1/4)}}\,],$$

with $a \equiv \widetilde{b}(1/2) > 1/2$. From the elementary inequality $\sqrt{a^2 + x} \leqslant a + x/(2a)$ (where $x > 0$) it then follows that

$$a + \sqrt{a^2 + \mathrm{e}^{-\beta(h-1/4)}} \leqslant 2a + \mathrm{e}^{-\beta(h-1/4)}\,,$$

which easily yields the estimate

$$|f(T) - f(0)| = \mathrm{O}\Big(T\mathrm{e}^{-\beta \min\left(h - \frac{1}{4}, \mu + \frac{h}{2} - \frac{1}{4}\right)}\Big), \quad (h, \mu) \in \mathcal{W}_1\,.$$

A similar analysis in the region $\mathcal{W}_0$ shows that

$$|f(T) - f(0)| = \mathrm{O}\Big(T\mathrm{e}^{-\beta|\mu + \frac{h}{2}|}\Big), \quad (h, \mu) \in \mathcal{W}_0\,.$$

Recall that at low temperatures the free energy per unit length of a $(1+1)$-dimensional CFT (in natural units $\hbar = k_{\mathrm{B}} = 1$) behaves as [37, 38]

$$f(T) \simeq f(0) - \frac{\pi c T^2}{6v}\,, \tag{5.3}$$

where c is the central charge and v is the Fermi velocity (effective speed of light). From the previous estimates for $f(T) - f(0)$ at low temperatures it then follows that the su(2) KY model is *not* critical when (h, μ) lies on the wedges $\mathcal{W}_0$ and $\mathcal{W}_1$. In fact, the exponentially small bounds in β for $|f(T) - f(0)|$ found above show that the spin 1/2 KY model is gapped on the wedges $\mathcal{W}_{0,1}$, with energy gap given by $|\mu + h/2|$ in $\mathcal{W}_0$ and $\min(h - 1/4, \mu + h/2 - 1/4)$ in $\mathcal{W}_1$.

[+] From now on, by "su(m) KY model" we shall understand the full Hamiltonian (2.5), whose free energy (in the thermodynamic limit) is therefore the grand potential of the original KY Hamiltonian H_0 in Eq. (2.1).

5.2. Vertical band $\mathcal{B}_1$

Splitting the integration interval in Eq. (5.1) into $[0, x_0(h)]$ and $[x_0(h), 1/2]$, and setting

$$\widehat{b}(x) \equiv \mathrm{e}^{-\frac{\beta}{2}\varepsilon(x)} b(x) = \frac{1}{2}\Big[\mathrm{e}^{-\frac{\beta}{2}(\varepsilon-h)} + \mathrm{e}^{-\frac{\beta}{2}(\varepsilon+h)} + \mathrm{e}^{-\beta(\mu-\frac{\varepsilon}{2})}\Big], \tag{5.4}$$

we can write

$$\begin{aligned} f(T) = f_0 &- 2T \int_0^{x_0(h)} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\,\mathrm{d}x \\ &- 2T \int_{x_0(h)}^{1/2} \log\Big[\widehat{b} + \sqrt{\widehat{b}^2 + 1 - \mathrm{e}^{-\beta\varepsilon}}\,\Big]\,\mathrm{d}x\,, \end{aligned} \tag{5.5}$$

where

$$f_0 = -\mu_c - h x_0(h) - \int_{x_0(h)}^{1/2} \varepsilon(x)\mathrm{d}x. \tag{5.6}$$

When $(h, \mu) \in \mathcal{B}_1$ all the exponents in the previous formulas for $\widetilde{b}$ and $\widehat{b}$ are negative for $0 \leqslant x < x_0(h)$ and $x_0(h) < x \leqslant 1/2$, respectively. Thus both integrals in Eq. (5.5) vanish at $T = 0$, and hence $f_0 = f(0)$. Furthermore, we have

$$\int_0^{x_0(h)} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\,\mathrm{d}x = I_1 + \mathrm{O}(\mathrm{e}^{-\beta \min(h, \mu - \frac{h}{2})}), \tag{5.7}$$

$$\int_{x_0(h)}^{1/2} \log\Big[\widehat{b} + \sqrt{\widehat{b}^2 + 1 - \mathrm{e}^{-\beta\varepsilon}}\,\Big]\,\mathrm{d}x = I_2 + \mathrm{O}(\mathrm{e}^{-\beta \min(h, \mu - \frac{1}{8})}), \tag{5.8}$$

where

$$I_1 \equiv \int_0^{x_0(h)} \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-\beta(h-\varepsilon)}}\,\Big]\mathrm{d}x\,, \tag{5.9}$$

$$I_2 \equiv \int_{x_0(h)}^{1/2} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(\varepsilon-h)} + \sqrt{1 + \tfrac{1}{4}\,\mathrm{e}^{-\beta(\varepsilon-h)}}\,\Big]\,\mathrm{d}x \tag{5.10}$$

(cf. Appendix A). We shall next derive the full asymptotic series of the integrals I_k in powers of T. To this end, let us perform in the integral I_1 the change of variable $y = \beta(h - \varepsilon(x))$, or equivalently

$$x = x_0(h - Ty)\,, \tag{5.11}$$

obtaining

$$I_1 = T \int_0^{\beta h} \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\,\Big] x_0'(h - Ty)\,\mathrm{d}y\,.$$

We next expand the last term in the previous equation around $T = 0$ taking into account the identity $x_0' = (1 - 2x_0)^{-1}$, with the result

$$x_0'(h - Ty) = \sum_{l=0}^{\infty} a_l(h)(-Ty)^l\,,$$

where

$$a_l(s) = \frac{2^l(2l-1)!!}{l!\,[1 - 2x_0(s)]^{2l+1}} = \frac{2^l(2l-1)!!}{l!\,(1-4s)^{l+\frac{1}{2}}} \tag{5.12}$$

and $(-1)!! \equiv 1$. As shown in Appendix B, the asymptotic series of I_1 is then given by

$$I_1 \sim \sum_{l=0}^{\infty} (-1)^l \, a_l(h) \, T^{l+1} \int_0^\infty y^l \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\Big] \mathrm{d}y \,. \tag{5.13}$$

Consider next the integral I_2. Since the natural change of variable

$$x = x_0(h + Ty) \tag{5.14}$$

is singular at the endpoint $y = \beta(1/4 - h)$, we first subdivide the integration range into the intervals $x_0(h) \leqslant x \leqslant x_0(\frac{1}{8} + \frac{h}{2})$ and $x_0(\frac{1}{8} + \frac{h}{2}) \leqslant x \leqslant 1/2$. The integral over the second interval is clearly $\mathrm{O}(\mathrm{e}^{-\frac{\beta}{4}(\frac{1}{4}-h)})$, so that

$$I_2 = \int_{x_0(h)}^{x_0(\frac{1}{8}+\frac{h}{2})} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(\varepsilon-h)} + \sqrt{1 + \tfrac{1}{4}\,\mathrm{e}^{-\beta(\varepsilon-h)}}\,\Big] \mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\frac{\beta}{4}(\frac{1}{4}-h)}) \,. \tag{5.15}$$

Performing the change of variable (5.14) in the integral in Eq. (5.15) and proceeding as before we obtain

$$I_2 \sim \sum_{l=0}^{\infty} a_l(h) \, T^{l+1} \int_0^\infty y^l \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-y/2} + \sqrt{1 + \tfrac{1}{4}\,\mathrm{e}^{-y}}\Big] \mathrm{d}y. \tag{5.16}$$

From Eqs. (5.5)–(5.8) and (5.13)-(5.16) we finally obtain the asymptotic series of the free energy per site in the open band $\mathcal{B}_1$:

$$f(T) - f(0) \sim -2 \sum_{l=0}^{\infty} a_l(h) \, \mathcal{I}_l \, T^{l+2}, \quad (h, \mu) \in \mathcal{B}_1 \,, \tag{5.17}$$

where

$$\mathcal{I}_l = \int_0^\infty y^l \Big\{ \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-y/2} + \sqrt{1 + \tfrac{1}{4}\,\mathrm{e}^{-y}}\Big] + (-1)^l \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\Big] \Big\} \mathrm{d}y. \tag{5.18}$$

As explained in Appendix C, the latter integrals can be expressed in several alternative ways, to wit

$$\mathcal{I}_l = \frac{1}{2(l+1)} \int_{-\infty}^{\infty} x^{l+1} \left[\frac{1}{\sqrt{1 + 4\mathrm{e}^x}} - \theta(-x) \right] \mathrm{d}x = \frac{1}{(l+1)(l+2)} \int_{-\infty}^{\infty} \frac{x^{l+2}\,\mathrm{e}^x}{(1 + 4\mathrm{e}^x)^{3/2}} \, \mathrm{d}x, \tag{5.19}$$

where $\theta(t) = (1 + \operatorname{sgn} t)/2$ is Heaviside's step function. The integrals $\mathcal{I}_l$ can actually be computed in closed form for low values of l, namely

$$\mathcal{I}_0 = \frac{\pi^2}{6} \,, \quad \mathcal{I}_1 = \zeta(3) \,, \quad \mathcal{I}_2 = \frac{\pi^4}{10} \,, \quad \mathcal{I}_3 = 2[\pi^2 \zeta(3) + 9\zeta(5)],$$

where $\zeta(z)$ denotes Riemann's zeta function. We thus obtain the low temperature expansion

$$f(T) - f(0) = -\frac{\pi^2 T^2}{3(1-4h)^{1/2}} - \frac{4\zeta(3) T^3}{(1-4h)^{3/2}} - \frac{6\pi^4 T^4}{5(1-4h)^{5/2}} + \mathrm{O}(T^5), \quad (h, \mu) \in \mathcal{B}_1. \tag{5.20}$$

Equation (5.20) strongly suggests that the su(2) KY model is critical in the vertical band $\mathcal{B}_1$. To ascertain this fact and compute the central charge, however, we first need to determine the Fermi velocity v of the low-energy excitations above the ground state. To this end, recall first of all that in the limit $N \to \infty$ the ground state energy of the spin 1/2 KY model is approximately given by

$$E_0 \simeq \sum_{k=1}^{Nm_s} (\varepsilon(x_k) - h) + \sum_{k=1}^{Nn_c/2} (\varepsilon(x_k) - 2\mu) \,, \qquad x_k \equiv k/N \,, \tag{5.21}$$

while its momentum (mod. 2π) can be written as

$$P \simeq 2\pi \sum_{k=1}^{Nm_s} x_k + 2\pi \sum_{k=1}^{Nn_c/2} x_k \tag{5.22}$$

(see [10] and, e.g., [39, 40]). As shown in Section 4, in the vertical band $\mathcal{B}_1$ we have

$$m_s = x_0(h)\,, \qquad n_c = 1\,,$$

and hence

$$E_0 \simeq \sum_{k=1}^{Nx_0(h)} (\varepsilon(x_k) - h) + \sum_{k=1}^{N/2} (\varepsilon(x_k) - 2\mu)\,, \qquad P \simeq 2\pi \sum_{k=1}^{Nx_0(h)} x_k + 2\pi \sum_{k=1}^{N/2} x_k\,.$$

Low-energy excitations above the ground state are obtained by adding a "mode" with $k = Nx_0(h) + 1$ or removing one with $k = Nx_0(h) - 1$. The energy of these excitations is thus $E_0 + \Delta E$, with

$$\Delta E = \pm[\varepsilon(x_0(h) \pm \tfrac{1}{N}) - h] \simeq \pm\Big[\varepsilon(x_0(h)) \pm \frac{\varepsilon'(x_0(h))}{N} - h\Big] = \frac{\varepsilon'(x_0(h))}{N}\,,$$

On the other hand, the momentum carried by the mode added (respectively removed) is

$$p = 2\pi x_k \equiv p_0 \pm \Delta p\,,$$

where $p_0 = 2\pi x_0(h)$ is the Fermi momentum and $\Delta p = 2\pi/N$. The Fermi velocity of the low-energy excitations is therefore given by

$$v = \frac{\Delta E}{\Delta p} = \frac{\varepsilon'(x_0(h))}{2\pi} = \frac{1 - 2x_0(h)}{2\pi} = \frac{\sqrt{1 - 4h}}{2\pi}\,. \tag{5.23}$$

From Eqs. (5.20) and (5.23) it then follows that in this case the model is critical* with central charge $c = 1$. It should also be noted that in the limit $T \to 0$ the only contribution to the integrals $I_{1,2}$ in Eqs. (5.7)-(5.8), in terms of which

$$f - f(0) \sim -2T(I_1 + I_2)\,,$$

comes from an arbitrarily small neighborhood of the point $x_0(h) = p_0/(2\pi)$ up to exponentially small terms in β. We shall express this relationship by saying that these integrals are critical at $x = x_0(h)$. Thus the Fermi velocity (5.23) is proportional to the derivative of the dispersion relation $\varepsilon(x)$ at the unique critical point of the integrals $I_{1,2}$.

5.3. Oblique band $\mathcal{B}_0$

In this case Eq. (5.5) becomes

$$\begin{aligned} f(T) = f_0 &- 2T \int_0^{x_0(\frac{h}{2}+\mu)} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\mathrm{d}x \\ &- 2T \int_{x_0(\frac{h}{2}+\mu)}^{1/2} \log\Big[\overline{b} + \sqrt{\overline{b}^2 + \mathrm{e}^{-\beta(\varepsilon-2\mu)} - \mathrm{e}^{-2\beta(\varepsilon-\mu)}}\,\Big]\mathrm{d}x\,, \end{aligned} \tag{5.24}$$

* It is also important to mention in this respect that the ground state of the spin 1/2 KY model has finite degeneracy (at most 4) [40].

where now

$$f_0 = -\mu_c - h\, x_0(\tfrac{h}{2}+\mu) - 2\int_{x_0(\frac{h}{2}+\mu)}^{1/2} (\varepsilon(x)-\mu)\,\mathrm{d}x \tag{5.25}$$

and

$$\overline{b}(x) \equiv \mathrm{e}^{\beta(\mu-\varepsilon(x))}\, b(x) = \frac{1}{2}\Big[1 + \mathrm{e}^{-\beta(\varepsilon-\frac{h}{2}-\mu)} + \mathrm{e}^{-\beta(\varepsilon+\frac{h}{2}-\mu)}\Big]. \tag{5.26}$$

Again, all the exponents appearing in Eq. (5.24) are nonpositive, so that $f_0 = f(0)$. Moreover, proceeding as above we obtain the estimates

$$\int_0^{x_0(\frac{h}{2}+\mu)} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\mathrm{d}x = I_3 + \mathrm{O}(\mathrm{e}^{-\beta\min(h,\frac{h}{2}-\mu)}), \tag{5.27}$$

$$\int_{x_0(\frac{h}{2}+\mu)}^{1/2} \log\Big[\overline{b} + \sqrt{\overline{b}^2 + \mathrm{e}^{-\beta(\varepsilon-2\mu)} - \mathrm{e}^{-2\beta(\varepsilon-\mu)}}\,\Big]\mathrm{d}x = I_4 + \mathrm{O}(\mathrm{e}^{-\beta\min(h,\frac{h}{2}-\mu)}), \tag{5.28}$$

where

$$I_3 \equiv \int_0^{x_0(\frac{h}{2}+\mu)} \log\Big[1 + \mathrm{e}^{-\beta(\frac{h}{2}+\mu-\varepsilon)}\Big]\mathrm{d}x\,, \tag{5.29}$$

$$I_4 \equiv \int_{x_0(\frac{h}{2}+\mu)}^{1/2} \log\Big[1 + \mathrm{e}^{-\beta(\varepsilon-\frac{h}{2}-\mu)}\Big]\,\mathrm{d}x = \int_{x_0(\frac{h}{2}+\mu)}^{x_0(\frac{1}{8}+\frac{h}{4}+\frac{\mu}{2})} \log\Big[1 + \mathrm{e}^{-\beta(\varepsilon-\frac{h}{2}-\mu)}\Big]\,\mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\frac{\beta}{2}(\frac{1}{4}-\frac{h}{2}-\mu)})\,. \tag{5.30}$$

The asymptotic series of the integral I_3 is obtained as above through the change of variable $y = \beta(\frac{h}{2}+\mu-\varepsilon(x))$, namely

$$\begin{aligned} I_3 &\sim \sum_{l=0}^{\infty} (-1)^l a_l(\tfrac{h}{2}+\mu)\, T^{l+1} \int_0^{\infty} y^l \log(1+\mathrm{e}^{-y})\mathrm{d}y \\ &= \sum_{l=0}^{\infty} (-1)^l a_l(\tfrac{h}{2}+\mu)\, l!(1-2^{-l-1})\zeta(l+2)\, T^{l+1}. \end{aligned} \tag{5.31}$$

Likewise, performing the analogous change of variable $y = \beta(\varepsilon(x) - \frac{h}{2} - \mu)$ in the RHS of Eq. (5.30) we obtain the asymptotic series

$$I_4 \sim \sum_{l=0}^{\infty} a_l(\tfrac{h}{2}+\mu)\, l!(1-2^{-l-1})\zeta(l+2)\, T^{l+1}. \tag{5.32}$$

Combining Eqs. (5.31)-(5.32) we finally arrive at the following asymptotic series for the free energy per site in the oblique band $\mathcal{B}_0$:

$$f(T) - f(0) \sim -2\sum_{l=0}^{\infty} \frac{(2^{2l+1}-1)(4l-1)!!}{[1-2(h+2\mu)]^{2l+\frac{1}{2}}}\, \zeta(2l+2)\, T^{2l+2}. \tag{5.33}$$

In particular, the first few terms in the latter series are explicitly given by♯

$$f(T) - f(0) = -\frac{\pi^2 T^2}{3[1-2(h+2\mu)]^{1/2}} - \frac{7\pi^4 T^4}{15[1-2(h+2\mu)]^{5/2}} + \mathrm{O}(T^6), \quad (h,\mu) \in \mathcal{B}_0.$$

♯ The coefficients $\zeta(2l+2)$ can be expressed as

$$\zeta(2l+2) = \frac{(2\pi)^{2l+2}}{2(2l+2)!}\,|B_{2l+2}|\,,$$

where B_k is the k-th Bernoulli number defined by $x/(\mathrm{e}^x - 1) = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$.

As before, the previous asymptotic expansion indicates that the model is critical when (h,μ) lie in the oblique band $\mathcal{B}_0$. In this case we have

$$m_s = \frac{n_c}{2} = x_0(\mu + \tfrac{h}{2})\,,$$

so that the ground state energy, momentum and Fermi momentum are given by

$$E_0 \simeq 2\sum_{k=1}^{Nx_0(\mu+\frac{h}{2})} (\varepsilon(x_k) - \mu - \tfrac{h}{2})\,, \qquad P \simeq 2\cdot 2\pi \sum_{k=1}^{Nx_0(\mu+\frac{h}{2})} x_k\,, \qquad p_0 = 4\pi x_0(\mu+\tfrac{h}{2})\,,$$

and therefore the low-energy excitations satisfy

$$\Delta E = \frac{2}{N}\,\varepsilon'(x_0(\mu+\tfrac{h}{2}))\,, \qquad \Delta p = \frac{4\pi}{N}\,.$$

We conclude that the Fermi velocity is given in this case by

$$v = \frac{\varepsilon'(x_0(\mu+\frac{h}{2}))}{2\pi} = \frac{\sqrt{1-2(h+2\mu)}}{2\pi}\,,$$

and thus the central charge is again $c=1$. Note that, as in the previous case, $x_0(\mu+\frac{h}{2})$ is the unique critical point of the integrals $I_{3,4}$ determining the asymptotic expansion of $f(T)-f(0)$ in the oblique band $\mathcal{B}_0$.

5.4. Triangle $\mathcal{T}$

We now have

$$\begin{aligned} f(T)-f(0) = &-2T\int_0^{x_0(h)} \log\Big[\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\Big]\mathrm{d}x \\ &-2T\int_{x_0(h)}^{x_0(2\mu)} \log\Big[\widehat{b} + \sqrt{\widehat{b}^2 + 1 - \mathrm{e}^{-\beta\varepsilon}}\,\Big]\mathrm{d}x \\ &-2T\int_{x_0(2\mu)}^{1/2} \log\Big[\overline{b} + \sqrt{\overline{b}^2 + \mathrm{e}^{-\beta(\varepsilon-2\mu)} - \mathrm{e}^{-2\beta(\varepsilon-\mu)}}\,\Big]\mathrm{d}x \end{aligned} \tag{5.34}$$

with

$$f(0) = -\mu_c - hx_0(h) - \int_{x_0(h)}^{x_0(2\mu)} \varepsilon(x)\,\mathrm{d}x - 2\int_{x_0(2\mu)}^{1/2} (\varepsilon(x)-\mu)\,\mathrm{d}x. \tag{5.35}$$

The first integral in Eq. (5.34) coincides with the LHS of Eq. (5.7), so that its asymptotic series is given by Eq. (5.13). The last integral in Eq. (5.34) differs from

$$\int_{x_0(2\mu)}^{x_0(\frac{1}{8}+\mu)} \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-\beta(\varepsilon-2\mu)}}\,\Big]\mathrm{d}x$$

by terms $\mathrm{O}(\mathrm{e}^{-\beta(\frac{1}{8}-\mu)})$, and thus its asymptotic series can be computed performing the change of variable $y = \beta(\varepsilon(x)-2\mu)$ in the latter integral, with the result

$$\sum_{l=0}^{\infty} a_l(2\mu)\,T^{l+1}\int_0^{\infty} y^l \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\,\Big]\mathrm{d}y\,. \tag{5.36}$$

Finally, the second integral in Eq. (5.34) is dominated by the term $\frac{1}{2}\mathrm{e}^{-\beta(\varepsilon-h)}$ in $\widehat{b}$ for $x_0(h) \leqslant x \leqslant x_0(\frac{h}{2}+\mu)$, while in the interval $[x_0(\frac{h}{2}+\mu), x_0(2\mu)]$ the dominant term is instead $\frac{1}{2}\mathrm{e}^{-\beta(\mu-\frac{\varepsilon}{2})}$. More precisely, we have

$$\int_{x_0(h)}^{x_0(\frac{h}{2}+\mu)} \log\Big[\widehat{b}+\sqrt{\widehat{b}^2+1-\mathrm{e}^{-\beta\varepsilon}}\,\Big]\mathrm{d}x$$
$$= \int_{x_0(h)}^{x_0(\frac{h}{2}+\mu)} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(\varepsilon-h)}+\sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-\beta(\varepsilon-h)}}\,\Big]\mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\beta\min(\frac{1}{2}(\mu-\frac{h}{2}),h)}) \tag{5.37}$$

and

$$\int_{x_0(\frac{h}{2}+\mu)}^{x_0(2\mu)} \log\Big[\widehat{b}+\sqrt{\widehat{b}^2+1-\mathrm{e}^{-\beta\varepsilon}}\,\Big]\mathrm{d}x$$
$$= \int_{x_0(\frac{h}{2}+\mu)}^{x_0(2\mu)} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(2\mu-\varepsilon)}+\sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-\beta(2\mu-\varepsilon)}}\,\Big]\mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\frac{\beta}{2}(\mu-\frac{h}{2})}). \tag{5.38}$$

Comparing with Eq. (5.15) we conclude that the asymptotic series of the LHS of Eq. (5.37) is given by Eq. (5.16). On the other hand, the asymptotic series of the LHS of Eq. (5.38) is easily derived performing the change of variable $y = \beta(2\mu - \varepsilon(x))$ in the RHS, with the result

$$\sum_{l=0}^{\infty}(-1)^l a_l(2\mu)\,T^{l+1}\int_0^{\infty} y^l \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-y/2}+\sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-y}}\,\Big]\mathrm{d}y\,. \tag{5.39}$$

Putting all of the above together we obtain the following asymptotic series for the free energy per site in the triangle $\mathcal{T}$:

$$f(T) - f(0) \sim -2\sum_{l=0}^{\infty}\Big[a_l(h) + (-1)^l a_l(2\mu)\Big]\,\mathcal{I}_l\,T^{l+2}, \tag{5.40}$$

where the integral $\mathcal{I}_l$ is given by Eq. (5.18). In particular, in this case

$$f(T) - f(0) \sim \psi(T,h) + \psi(-T,2\mu)\,, \tag{5.41}$$

where

$$\psi(T,h) = -2\sum_{l=0}^{\infty} a_l(h)\,\mathcal{I}_l\,T^{l+2} \tag{5.42}$$

is the asymptotic series for $f(T) - f(0)$ in the region $\mathcal{B}_1$ (cf. Eq. (5.17)).

As in the previous two cases, the asymptotic behavior of the free energy near $T = 0$ spelled out in Eq. (5.40) is a strong indication that the model is critical when (h,μ) belongs to the triangle $\mathcal{T}$. To confirm this indication and compute the central charge, note first that in this case

$$m_s = x_0(h)\,, \qquad n_c = 2x_0(2\mu)\,,$$

and therefore the ground state energy and momentum are given by

$$E_0 \simeq \sum_{k=1}^{Nx_0(h)}(\varepsilon(x_k)-h) + \sum_{k=1}^{Nx_0(2\mu)}(\varepsilon(x_k)-2\mu)\,, \qquad P \simeq 2\pi\sum_{k=1}^{Nx_0(h)} x_k + 2\pi\sum_{k=1}^{Nx_0(2\mu)} x_k\,.$$

We thus have two types of low-energy excitations associated to spin and charge, with $\Delta E = \varepsilon'(x_0(h))/N$ and $\Delta E = \varepsilon'(x_0(2\mu))/N$ respectively for the spin and charge

excitations. Since in both cases $\Delta p = 2\pi/N$, the Fermi velocities of the spin and charge excitations are respectively given by

$$v_s = \frac{\varepsilon'(x_0(h))}{2\pi} = \frac{\sqrt{1-4h}}{2\pi}\,, \qquad v_c = \frac{\varepsilon'(x_0(2\mu))}{2\pi} = \frac{\sqrt{1-8\mu}}{2\pi}\,. \tag{5.43}$$

The leading term in the expansion (5.40) can therefore be expressed as

$$\begin{aligned} f(T) - f(0) &\simeq -2[a_0(h) + a_0(2\mu)]\mathcal{I}_0 T^2 = -\frac{\pi^2 T^2}{3}\left(\frac{1}{\sqrt{1-4h}} + \frac{1}{\sqrt{1-8\mu}}\right) \\ &= -\frac{\pi T^2}{6}\left(\frac{1}{v_s} + \frac{1}{v_c}\right). \end{aligned} \tag{5.44}$$

Hence both the charge and the spin sectors of the model are described at low energies by a CFT with central charge $c = 1$. This is indeed known to be the case, as first shown in Ref. [29] using the asymptotic Bethe ansatz. Note finally that, as in the previous cases, the points $x_0(h)$ and $x_0(\mu)$ appearing in Eq. (5.43) for the Fermi velocities are nothing but the critical points of the integrals in the RHS of Eq. (5.34) for $f(T) - f(0)$.

5.5. Critical behavior on the boundaries

The asymptotic behavior of the free energy on the boundaries of the five ground-state phases $\mathcal{W}_i$, $\mathcal{B}_i$, $\mathcal{T}$ can be analyzed in much the same way as above. In particular, it is not difficult, and is certainly of interest, to examine the criticality properties of the model on these boundaries. Consider, as an example, the segment $\mu = 1/8$, $0 < h < 1/4$ separating the triangle $\mathcal{T}$ (su(1|2) phase) from the vertical band $\mathcal{B}_1$ (su(2) phase). From Eqs. (5.34) and (5.38) with $\mu = 1/8$ we readily obtain

$$f(T) - f(0) \sim -\frac{\pi T^2}{6v_s} - 2T\int_{x_0(\frac{h}{2}+\frac{1}{8})}^{1/2} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(\frac{1}{4}-\varepsilon)} + \sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-\beta(\frac{1}{4}-\varepsilon)}}\Big]\mathrm{d}x\,,$$

where the Fermi (spin) velocity v_s is given by Eq. (5.43). Although the last integral is critical at $x = 1/2$, it is not asymptotic to $-\pi T^2/(6v_c)$ as the Fermi velocity v_c vanishes at $\mu = 1/8$. In fact, performing the usual change of variable

$$y = \beta(1/4 - \varepsilon(x)) = \left(\frac{1}{2} - x\right)^2$$

we have

$$\begin{aligned} 2T\int_{x_0(\frac{h}{2}+\frac{1}{8})}^{1/2} &\log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{\beta}{2}(\frac{1}{4}-\varepsilon)} + \sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-\beta(\frac{1}{4}-\varepsilon)}}\Big]\mathrm{d}x \\ &= T^{3/2}\int_0^{\frac{\beta}{2}(\frac{1}{4}-h)} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{y}{2}} + \sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-y}}\Big]\frac{\mathrm{d}y}{\sqrt{y}} \sim \kappa_c T^{3/2}, \end{aligned}$$

with

$$\kappa_c \equiv \int_0^\infty \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-\frac{y}{2}} + \sqrt{1+\tfrac{1}{4}\,\mathrm{e}^{-y}}\Big]\frac{\mathrm{d}y}{\sqrt{y}} \simeq 1.2255036\,.$$

Thus the first two nonvanishing terms in the asymptotic expansion of $f(T) - f(0)$ on the segment $\mu = 1/8$, $0 < h < 1/4$ are††

$$f(T) - f(0) \sim -\kappa_c T^{3/2} - \frac{\pi T^2}{6 v_s}\,.$$

We conclude that the model is not critical on this segment, due to the term in the expansion proportional to $T^{3/2}$. However, the second term (proportional to T^2) can be interpreted as signaling that on the spin sector the model is critical with central charge $c = 1$. This conclusion is borne out by the behavior of the ground state energy and momentum, which in this case are given by

$$E_0 \simeq \sum_{k=1}^{Nx_0(h)} \left(\varepsilon(x_k) - h\right) + \sum_{k=1}^{N/2} \left(\varepsilon(x_k) - \tfrac{1}{4}\right), \qquad P \simeq 2\pi \sum_{k=1}^{Nx_0(h)} x_k + 2\pi \sum_{k=1}^{N/2} x_k\,.$$

Since now $\varepsilon'(1/2) = 0$, the low energy excitations obtained by removing the mode with $k = N/2 - 1$ (or adding the one with $k = N/2 + 1$) carry an energy

$$\Delta E = \frac{1}{4} - \varepsilon(\tfrac{1}{2} - \tfrac{1}{N}) = -\frac{\varepsilon''(\frac{1}{2})}{2N^2} = \frac{1}{N^2}\,,$$

which is *quadratic* in $\Delta p = 2\pi/N$. These are in fact the excitations responsible for the $T^{3/2}$ asymptotic behavior of $f(T) - f(0)$ at low temperatures. On the other hand, exciting the mode with $k = Nx_0(h) + 1$ (or suppressing the one with $k = Nx_0(h) - 1$) increases the energy by

$$\Delta E = \frac{\varepsilon'(x_0(h))}{N} = v_s \Delta p\,.$$

These excitations are therefore described by a CFT with Fermi velocity v_s.

Proceeding in an analogous way we can determine the critical behavior (including, where appropriate, the value of the central charge) in the remaining parts of the boundary. In general, the vanishing of the spin or charge Fermi velocity implies that the model is not critical in the corresponding sector (and, hence, as a whole), although it can still be critical in the other sector provided that its Fermi velocity is nonzero. Since v_s (respectively v_c) vanishes only for $h = 1/4$ (respectively $\mu = 1/8$), we conclude that the su(2) KY model should be critical at the vertical segment $h = 0$, $0 \leqslant \mu < 1/8$, non-critical (but critical in the spin sector) at the point $(0, 1/8)$ and non-critical (in both sectors) at the other vertex $(1/4, 1/8)$. This is indeed confirmed by a detailed calculation (see Table 2 for a summary of the results). This calculation also shows that in all the non-critical parts of the boundary the model is gapless, with $f(T) - f(0)$ growing as $T^{3/2}$ at low temperatures.

An interesting situation presents itself on the oblique segment $\mu = h/2$, $0 < h < 1/4$. Indeed, by Eq. (5.34) —or (5.24)— with $\mu = h/2$ we have

$$f(T) - f(0) = -2T \int_0^{x_0(h)} \log\left[\tilde{b} + \sqrt{\tilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}\,\right] \mathrm{d}x$$

†† In fact, it is straightforward to obtain the full asymptotic series

$$f(T) - f(0) \sim -\kappa_c T^{3/2} - 2 \sum_{l=0}^{\infty} a_l(h) \mathcal{I}_l\, T^{l+2}\,.$$

Region	Central charge
$h=0,\ \ 0<\mu<\frac{1}{8}$	$1+1$
$h=0,\ \ \mu>\frac{1}{8}$	1
$\mu=\frac{1}{8},\ \ 0\leqslant h<\frac{1}{4}$	1 (only spin sector)
$\mu=\frac{h}{2},\ \ 0<h<\frac{1}{4}$	$\frac{3}{2}$
$h=\frac{1}{4},\ \ \mu>\frac{1}{8}$	Non-critical
$\mu=\frac{1}{4}-\frac{h}{2},\ \ h>\frac{1}{4}$	Non-critical
$(0,0)$	1
$(\frac{1}{4},\frac{1}{8})$	Non-critical

Table 2. Critical behavior of the spin 1/2 KY model on the boundaries of the ground-state phases $\mathcal{W}_i$, $\mathcal{B}_i$, $\mathcal{T}$. In all the non-critical regions (including the segment $\mu=1/8$, $0\leqslant h<1/4$) the leading term in the low-temperature asymptotic expansion of $f(T)-f(0)$ is proportional to $T^{3/2}$ and the model is gapless.

$$-2T\int_{x_0(h)}^{1/2}\log\Big[\bar{b}+\sqrt{\bar{b}^2+\mathrm{e}^{-\beta(\varepsilon-h)}-\mathrm{e}^{-\beta(2\varepsilon-h)}}\,\Big]\mathrm{d}x\equiv-2T(J_1+J_2)\,,$$

where

$$\widetilde{b}=\frac{1}{2}\Big(1+\mathrm{e}^{-\beta(h-\varepsilon(x))}+\mathrm{e}^{-\beta h}\Big)\,,\qquad \bar{b}=\frac{1}{2}\Big(1+\mathrm{e}^{-\beta(\varepsilon(x)-h)}+\mathrm{e}^{-\beta\varepsilon(x)}\Big)\,.$$

Since both integrals are critical at $x_0(h)\equiv x_0(2\mu)$, the Fermi velocity is expected to be

$$v=\frac{\varepsilon'(x_0(h))}{2\pi}=\frac{\sqrt{1-4h}}{2\pi}>0\,.$$

This is indeed the case, since the continuity of the ground state energy, momentum, magnetization and charge density implies that when $h=2\mu$ we have

$$m_s=\frac{n_c}{2}=x_0(h)\,,\quad E_0\simeq 2\sum_{k=1}^{Nx_0(h)}(\varepsilon(x_k)-h)\,,\quad P\simeq 2\cdot 2\pi\sum_{k=1}^{Nx_0(h)}x_k\,,$$

and therefore the low-energy excitations satisfy

$$\Delta E=\frac{2}{N}\,\varepsilon'(x_0(h))\,,\qquad \Delta p=\frac{4\pi}{N}\,.$$

Performing the changes of variable $y=\beta(h-\varepsilon(x))$ and $y=\beta(\varepsilon(x)-h)$ respectively in the integrals J_1 and J_2, and proceeding as above, we easily obtain

$$J_{1,2}\sim-\frac{T}{2\pi v}\int_0^\infty\log\Big[\tfrac{1}{2}(1+\mathrm{e}^{-y})+\sqrt{\tfrac{1}{4}(1+\mathrm{e}^{-y})^2+\mathrm{e}^{-y}}\,\Big]\mathrm{d}y=-\frac{\pi T}{16v}\,,$$

and therefore

$$f(T)-f(0)\sim-\frac{\pi T^2}{4v}\,.$$

Thus in the segment $\mu=h/2$, $0<h<1/4$ the low-temperature behavior of the model is described by a single CFT with $c=3/2$. It is also interesting to note that, although both Fermi velocities are nonzero in this case, this result cannot be obtained setting $v_s=v_c$ in Eq. (5.44) for the triangle $\mathcal{T}$. The reason is of course that the terms $\mathrm{e}^{-\beta(\mu+h/2-\varepsilon(x))}$ and

$\mathrm{e}^{-\beta(\varepsilon(x)-\mu-h/2)}$ appearing respectively in $\widetilde{b}$ and $\overline{b}$ are exponentially small throughout their whole integration ranges $[0, x_0(h)]$ and $[x_0(2\mu), 1/2]$ (and can therefore be discarded) only if μ is strictly greater than $h/2$.

5.6. Discussion

It should be noted that the expansions (5.17), (5.33) and (5.40) are all true asymptotic series, i.e., their radius of convergence vanishes. Indeed, for Eqs. (5.17) and (5.40) this stems from the following bound on the integral $\mathcal{I}_l$ in Eq. (5.18):

$$\mathcal{I}_l \geqslant \int_0^\infty [y^l (c\,\mathrm{e}^{-y/2} - \mathrm{e}^{-y})]\mathrm{d}y = (2^{l+1}c - 1)l!,$$

where $c \equiv \log(\frac{1+\sqrt{5}}{2})$. To derive the latter bound simply observe that the function

$$\phi(y) \equiv \mathrm{e}^{y/2} \log\Big[\tfrac{1}{2}\,\mathrm{e}^{-y/2} + \sqrt{1 + \tfrac{1}{4}\,\mathrm{e}^{-y}}\Big]$$

is monotonically increasing on $[0, \infty]$, so that $\phi(y) \geqslant \phi(0) = c$. As to Eq. (5.33), from the elementary identity

$$\log(1 + \mathrm{e}^{-y}) \geqslant \log 2 \cdot \mathrm{e}^{-y}, \qquad y \geqslant 0,$$

it follows that

$$\int_0^\infty y^k \log(1 + \mathrm{e}^{-y})\mathrm{d}y \geqslant \log 2 \int_0^\infty y^k \mathrm{e}^{-y}\mathrm{d}y = k! \log 2\,.$$

Our claim then follows from the fact that the coefficient of T^{2l+2} in Eq. (5.33) is proportional to

$$a_{2l}(\tfrac{h}{2} + \mu) \int_0^\infty y^{2l} \log(1 + \mathrm{e}^{-y})\mathrm{d}y$$

(cf. (5.31) and its analog

$$I_4 \sim \sum_{l=0}^\infty a_l(\tfrac{h}{2} + \mu)\, T^{l+1} \int_0^\infty y^l \log(1 + \mathrm{e}^{-y})\mathrm{d}y$$

for I_4).

In Fig. 2 we compare the free energy per site numerically computed through Eq. (2.18) with its asymptotic expansions up to four terms at the points $h = \mu = 1/12$ in the triangle $\mathcal{T}$ and $h = 1/4, \mu = 0$ in the oblique band $\mathcal{B}_0$ (cf. Eqs. (5.40) and (5.33)). We see from this figure that the agreement between the exact value of f and its expansions is quite good at sufficiently low temperature. To be more precise, from the estimates for each of the integrals in Eq. (5.34) it can be checked that the exponentially small terms discarded to obtain the asymptotic series (5.40) for f in the triangle $\mathcal{T}$ are $\mathrm{O}(T\mathrm{e}^{-\beta/48})$ at the point $h = \mu = 1/12$. Thus the latter series should not be expected to provide a good approximation for f unless $T \lesssim 0.02$. This is clearly in agreement with the numerical results represented in Fig. 2 (left). In particular, from the inset in the latter figure it is apparent that the absolute value of the error of the asymptotic expansions up to four terms does not exceed 1.5×10^{-5} for $T \leqslant 0.01$. A similar remark can be made for the band $\mathcal{B}_0$ (see Fig. 2, right). Note also that, although the absolute value of

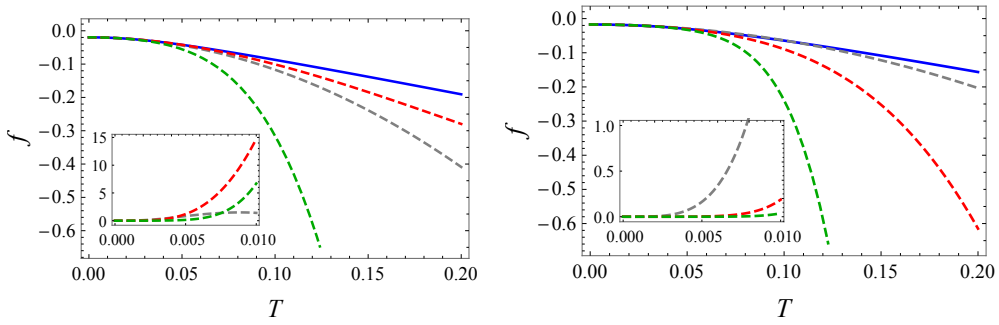

Figure 2. Left: comparison between f (solid blue line) and its asymptotic expansions (5.40) with two, three and four terms (gray, red and green dashed lines, respectively) at the point $h = \mu = 1/12$ in the triangle $\mathcal{T}$. Right: analogous plot for the asymptotic expansion (5.33) at the point $h = 1/4$, $\mu = 0$ in the oblique band $\mathcal{B}_0$. Insets: absolute value of the errors of the latter expansions (in units of 10^{-6}) in the smaller range $0 \leqslant T \leqslant 0.01$. In all plots, the unit of temperature and energy is $K = 2\pi^2 t$.

the error of the three asymptotic expansions considered diminishes with their order at sufficiently low temperature (cf. the insets in Fig. 2), this need not be the case at higher temperatures. In fact, it is a well-known feature of divergent asymptotic series that the optimum order varies with the range of the independent variable considered.

5.7. Comparison with the su(1|1) *and* su(2) *HS chains*

It should be clear from the above results that the asymptotic series of the free energy per site exhibits a different qualitative behavior in each of the regions $\mathcal{B}_i$, $\mathcal{W}_i$ and $\mathcal{T}$ identified in the previous section (cf. Table 1 and Fig. 1, left). Moreover, we shall next show that in the bands $\mathcal{B}_0$ and $\mathcal{B}_1$, corresponding respectively to the su(1|1) and su(2) zero-temperature phases, the asymptotic series for f coincides term by term with those for the free energy of the su(1|1) and su(2) HS chains (with a chemical potential and a magnetic field term, respectively). This does not mean that the supersymmetric KY model is equivalent to the su(1|1) and su(2) HS chains in these regions, since their respective free energies differ by exponentially small terms in β which become significant as T increases (cf. Fig. 3).

Consider, to begin with, the su(1|1) HS chain with a chemical potential term, whose Hamiltonian shall be taken in accordance with Ref. [36] as

$$\hat{H}^{(1|1)} = \frac{\pi^2}{N^2} \sum_{i<j} \frac{1 - P_{ij}^{(1|1)}}{\sin^2(\pi(i-j)/N)} - \lambda \mathcal{F} \,. \tag{5.45}$$

In the oblique band $\mathcal{B}_0$, the number of "down" fermions at $T = 0$ is $n^2 = 0$. We should then compare (5.45) with the (spin chain version of) the Hamiltonian of the

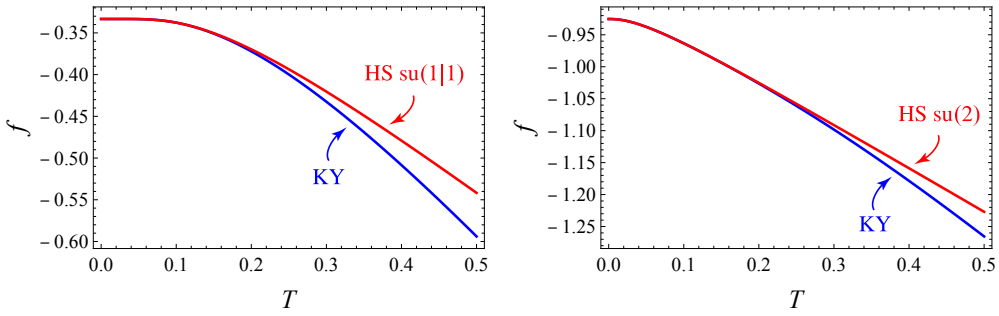

Figure 3. Left: comparison between the free energy of the supersymmetric spin 1/2 Kuramoto–Yokoyama model at the point $h = 1$, $\mu = 0$ in the band $\mathcal{B}_0$ and that of the (appropriately rescaled) su(1|1) HS chain (5.45) (cf. Eq. (5.47)). Right: analogous comparison at the point $h = 1/8$, $\mu = 1$ in the band $\mathcal{B}_1$ with the (antiferromagnetic) rescaled su(2) HS chain. The unit of energy and temperature in both plots is $K = 2\pi^2 t$.

supersymmetric KY model in the sector $n^2 = 0$, given by

$$\hat{H}|_{\mathcal{N}_2=0} = \frac{1}{2N^2} \sum_{i<j} \frac{1 - P_{ij}^{(1|1)}}{\sin^2(\pi(i-j)/N)} - (\mu + \tfrac{h}{2})\mathcal{N}_1 \tag{5.46}$$

(cf. Eq. (2.6)). We thus see that $\hat{H}^{(1|1)} = 2\pi^2 \hat{H}|_{\mathcal{N}_2=0}$ provided that $\lambda = 2\pi^2(\mu + h/2)$. The free energy of the Hamiltonian $\hat{H}^{(1|1)}$ was computed in Ref. [36], namely

$$f^{(1|1)}(T) = -\frac{T}{\pi} \int_0^\pi \log\left[1 + \mathrm{e}^{-2\pi^2\beta(\varepsilon(\frac{p}{2\pi}) - \frac{h}{2} - \mu)}\right] \mathrm{d}p\,.$$

Taking into account the connection between $\hat{H}^{(1|1)}$ and $\hat{H}|_{\mathcal{N}_2=0}$, we should then expect that in the oblique band $\mathcal{B}_0$, and at sufficiently low temperature,

$$f(T) \simeq \frac{1}{2\pi^2} f^{(1|1)}(2\pi^2 T) = -2T \int_0^{1/2} \log\left[1 + \mathrm{e}^{-\beta(\varepsilon(x) - \frac{h}{2} - \mu)}\right] \mathrm{d}x\,. \tag{5.47}$$

In fact, from Eqs. (5.25)-(5.28) it readily follows that

$$\begin{aligned} f(T) &= f_0 - 2T(I_3 + I_4) + \mathrm{O}(\mathrm{e}^{-\beta \min(h, \frac{h}{2} - \mu)}) \\ &= -2T \int_0^{1/2} \log\left[1 + \mathrm{e}^{-\beta(\varepsilon(x) - \frac{h}{2} - \mu)}\right] \mathrm{d}x + \mathrm{O}(\mathrm{e}^{-\beta \min(h, \frac{h}{2} - \mu)})\,. \end{aligned}$$

Thus the low-temperature asymptotic series of $f^{(1|1)}(2\pi^2 T)/(2\pi^2)$ coincides term by term with that of f in the band $\mathcal{B}_0$, as claimed.

Similarly, in the vertical band $\mathcal{B}_1$ the number of bosons at $T = 0$ is $n^0 = 0$. The (spin chain version of the) Hamiltonian of the supersymmetric KY model in the subspace $n^0 = 0$ is given by

$$\hat{H}|_{\mathcal{N}_0=0} = \frac{1}{2N^2} \sum_{i<j} \frac{1 + P_{ij}}{\sin^2(\pi(i-j)/N)} - \frac{h}{2}(\mathcal{N}_1 - \mathcal{N}_2) - \mu N\,, \tag{5.48}$$

where P_{ij} is the ordinary permutation operator. This should be compared with the Hamiltonian of the (antiferromagnetic) su(2) HS chain in an external magnetic field

from Ref. [20] (with $K=-1$ and $2B=h$), namely

$$\hat{H}^{(2)}=\frac{1}{2N^2}\sum_{i<j}\frac{P_{ij}-1}{\sin^2(\pi(i-j)/N)}-\frac{h}{2}(\mathcal{N}_1-\mathcal{N}_2)\,. \tag{5.49}$$

From Eq. (3.9) it follows that in the thermodynamic limit $\hat{H}|_{\mathcal{N}_0=0}=\hat{H}^{(2)}-\mu_c N$. It should therefore be expected that in the vertical band $\mathcal{B}_1$, and at sufficiently low temperatures,

$$f(T)\simeq f^{(2)}(T)-\mu_c=-\mu_c-2T\int_0^{1/2}\log\Big[\cosh(\tfrac{\beta h}{2})+\sqrt{\sinh^2(\tfrac{\beta h}{2})+\mathrm{e}^{\beta\varepsilon(x)}}\,\Big]\mathrm{d}x\equiv g(T),$$

where we have used the exact formula for $f^{(2)}$ from Ref. [20]. The asymptotic series of g around $T=0$ can be obtained following the above procedure. More precisely, we first write

$$\begin{aligned}g(T)=g(0)&-2T\int_0^{x_0(h)}\log\Big[\tfrac{1}{2}(1+\mathrm{e}^{-\beta h})+\sqrt{\tfrac{1}{4}(1-\mathrm{e}^{-\beta h})^2+\mathrm{e}^{-\beta(h-\varepsilon(x))}}\,\Big]\mathrm{d}x\\&-2T\int_{x_0(h)}^{1/2}\log\Big[\tfrac{1}{2}(1+\mathrm{e}^{-\beta h})\mathrm{e}^{-\frac{\beta}{2}(\varepsilon(x)-h)}+\sqrt{1+\tfrac{1}{4}(1-\mathrm{e}^{-\beta h})^2\mathrm{e}^{-\beta(\varepsilon(x)-h)}}\,\Big]\mathrm{d}x\,,\end{aligned}$$

where

$$g(0)=-\mu_c-hx_0(h)-\int_{x_0(h)}^{1/2}\varepsilon(x)\,\mathrm{d}x=f(0)$$

(cf. Eq. (5.6)). Discarding the exponentially small term $\mathrm{e}^{-\beta h}$, and comparing with Eqs. (5.9)-(5.10), we thus see that

$$g(T)\sim f(0)-2T(I_1+I_2)\,.$$

From Eqs. (5.5) and (5.7)-(5.8) we conclude that g has the same asymptotic series as f in the vertical band $\mathcal{B}_1$, as stated.

6. Low-temperature asymptotic expansions of densities and susceptibilities

Differentiating the asymptotic series for the free energy per site obtained in the previous section with respect to the parameters h and μ, we shall next to derive analogous series for the magnetization per site, the charge density and their corresponding susceptibilities. Since the asymptotic series of all these quantities are trivially equal to their zero temperature values on the wedges $\mathcal{W}_0$ and $\mathcal{W}_1$, we shall concentrate in what follows on the remaining regions $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{T}$.

6.1. $\mathcal{B}_1\cup\mathcal{T}$

In the region $\mathcal{B}_1\cup\mathcal{T}$, by Eq. (5.41) the asymptotic series for the magnetization is given by

$$m_s\sim-\frac{\partial f(0)}{\partial h}-\frac{\partial\psi(T,h)}{\partial h}.$$

From Eqs. (5.6) and (5.35) it follows that

$$-\frac{\partial f(0)}{\partial h}=x_0(h)\,,$$

which coincides with the value of $m_s(0)$ computed in Section 4. Using Eq. (5.42) we thus obtain

$$m_s \sim x_0(h) + \sum_{l=0}^{\infty} \frac{2^{l+2}(2l+1)!!}{l!\,(1-4h)^{l+\frac{3}{2}}}\,\mathcal{I}_l\,T^{l+2}, \tag{6.1}$$

where $\mathcal{I}_l$ is given by Eq. (5.18). The first few terms in this series are thus

$$m_s = x_0(h) + \frac{2\pi^2 T^2}{3(1-4h)^{3/2}} + \frac{24\zeta(3)T^3}{(1-4h)^{5/2}} + \frac{12\pi^4 T^4}{(1-4h)^{7/2}} + \mathrm{O}(T^5),$$

Differentiating Eq. (6.1) with respect to h we obtain the corresponding asymptotic series for the magnetic susceptibility χ_s:

$$\chi_s \sim (1-4h)^{-1/2} + \sum_{l=0}^{\infty} \frac{2^{l+3}(2l+3)!!}{l!\,(1-4h)^{l+\frac{5}{2}}}\,\mathcal{I}_l\,T^{l+2}. \tag{6.2}$$

To the best of our knowledge, for (h,μ) lying on the vertical band $\mathcal{B}_1$ the asymptotic expansions (6.1)-(6.2) are new.

Consider next the charge density n_c and its susceptibility χ_c. To begin with, it is obvious that $n_c \sim 1$ and $\chi_c \sim 0$ in the vertical band $\mathcal{B}_1$, since in this region $f(0)+\mu$ and the asymptotic series for $f(T)-f(0)$ depend only on h (cf. Eqs. (5.6) and (5.17)). On the other hand, in the triangle $\mathcal{T}$ the asymptotic series of n_c and χ_c follow immediately from those of m_s and χ_s taking into account Eq. (5.41), namely

$$n_c \sim 2x_0(2\mu) + \sum_{l=0}^{\infty} \frac{2^{l+3}(2l+1)!!}{l!\,(1-8\mu)^{l+\frac{3}{2}}}\,\mathcal{I}_l\,(-T)^{l+2}, \tag{6.3}$$

$$\chi_c \sim 4(1-8\mu)^{-1/2} + \sum_{l=0}^{\infty} \frac{2^{l+5}(2l+3)!!}{l!\,(1-8\mu)^{l+\frac{5}{2}}}\,\mathcal{I}_l\,(-T)^{l+2}, \quad (h,\mu)\in\mathcal{T}. \tag{6.4}$$

In fact, Eqs. (6.2) and (6.4) agree to all orders in T with the asymptotic series which can be obtained from the low-temperature approximations (up to exponentially small terms) to m_s and n_c in Ref. [28]. To see this, note first that in our notation the latter approximations read

$$1-n_c \sim 2\int_{-\infty}^{\infty} \frac{\sqrt{1-8\mu+4Tx}}{(1+4\mathrm{e}^x)^{3/2}}\,\mathrm{e}^x \mathrm{d}x\,, \qquad 1-2m_s \sim 2\int_{-\infty}^{\infty} \frac{\sqrt{1-4h-4Tx}}{(1+4\mathrm{e}^x)^{3/2}}\,\mathrm{e}^x \mathrm{d}x\,.$$

Expanding the square root in either integral in powers of T and using the alternative expression (5.19) for the integrals $\mathcal{I}_l$ we readily obtain Eqs. (6.1) and (6.3), from which the corresponding asymptotic series for χ_s and χ_c follow by term-by-term differentiation.

From the previous formulas it follows that in the triangle $\mathcal{T}$ the magnetization m_s and its susceptibility χ_s depend only on h, while n_c and χ_c depend only on μ, up to terms $\mathrm{O}(T^k \mathrm{e}^{-c\beta})$ (with $c>0$). This is an indication that spin-charge separation is valid at low temperatures up to terms exponentially small in β, as we shall more explicitly show in what follows.

Indeed, inverting the asymptotic expansion (6.1) of m_s to obtain an expansion of $1-2m_s(0) \equiv (1-4h)^{1/2}$ to a given order in T and substituting into Eq. (6.2) we can

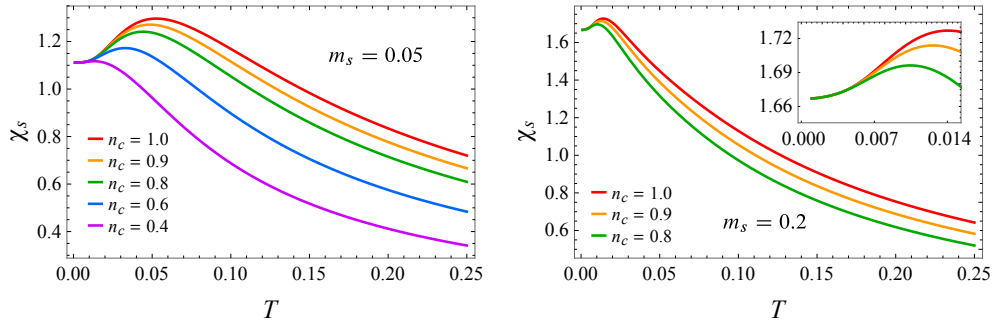

Figure 4. Magnetic susceptibility per site χ_s vs. temperature T at constant charge density n_c and magnetization $m_s = 0.05$ (left) and $m_s = 0.2$ (right). Inset: detail of the plot in the smaller range $0 \leqslant T \leqslant 0.015$. In all plots, T and $1/\chi_s$ are measured in units of $K = 2\pi^2 t$.

derive a corresponding asymptotic expansion of χ_s in terms of m_s. For instance, up to third order in T we have

$$\chi_s = \frac{1}{1-2m_s}\left[1 + \frac{8\pi^2 T^2}{3(1-2m_s)^4} + \frac{192\zeta(3)T^3}{(1-2m_s)^6}\right] + \mathrm{O}(T^4). \tag{6.5}$$

Comparing the asymptotic series (6.3)-(6.4) for n_c and χ_c with the corresponding series (6.1)-(6.2) for m_s and χ_s we conclude that the asymptotic series of χ_c in the triangle $\mathcal{T}$ is given by $\chi_c \sim 4X(\frac{n_c}{2}, -T)$, where $X(m_s, T)$ is the asymptotic series for χ_s in terms of m_s and T. From Eq. (6.5) we thus obtain

$$\chi_c = \frac{4}{1-n_c}\left[1 + \frac{8\pi^2 T^2}{3(1-n_c)^4} - \frac{192\zeta(3)T^3}{(1-n_c)^6}\right] + \mathrm{O}(T^4), \qquad (h,\mu) \in \mathcal{T}. \tag{6.6}$$

From the asymptotic expansions (6.5)-(6.6) it is clear that strong spin-charge separation holds at sufficiently low temperatures if we discard exponentially small terms. Previously, this result had been numerically checked only for $h = 0$ (i.e., $m_s = 0$)

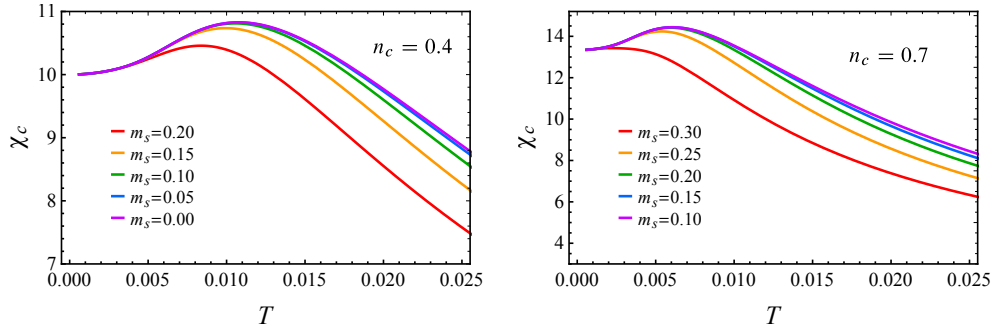

Figure 5. Charge susceptibility per site vs. temperature T at constant magnetization density m_s and charge density $n_c = 0.4$ (left) and 0.7 (right). In all plots, T and $1/\chi_c$ are measured in units of $K = 2\pi^2 t$.

and within the triangle $\mathcal{T}$ [16, 28]. With the help of the exact formulas (2.21)-(2.22) for the susceptibilities derived in Section 2, we have been able to numerically verify the strong spin-charge separation for non-zero magnetic fields and several charge densities (see Figs. 4-5). As is customary, we have taken (m_s, n_c) instead of (h, μ) as independent variables, which requires solving for the latter variables in terms of the former by means of Eqs. (2.19)-(2.20). For this to be possible, the mapping $(h, \mu) \mapsto (m_s, n_c)$ should be invertible in the temperature range considered. At zero temperature, this is the case provided that (h, μ) lies in the triangle $\mathcal{T}$, which is mapped to the triangle $0 < 2m_s < n_c < 1$. For this reason, in the previous plots we have taken magnetizations not greater than $n_c/2$.

Again, when $(h, \mu) \in \mathcal{B}_1$ the asymptotic expansion (6.5) appears to be new, while in the triangle $\mathcal{T}$ Eqs. (6.5)-(6.6) coincide with those derived in Ref. [28] to order T^2. (To verify this assertion one should first replace the dimensionless quantities T, χ_s and χ_c in the latter equations by their true values T/K, $K\chi_s$ and $K\chi_c$, with $K = 2\pi^2 t$, and take into account that in the latter reference t has been set to 1.) Note also that the asymptotic expansions of χ_s and χ_c do *not* have the same functional form, due to the different sign of the coefficients of the odd powers of T. This fact had not been previously noted, since it can only be detected at order T^3 or higher.

6.2. $\mathcal{B}_0$

Consider, finally, the oblique band $\mathcal{B}_0$. The asymptotic series for m_s and χ_s are easily obtained differentiating Eqs. (5.25) and (5.33), i.e.,

$$m_s \sim x_0(\tfrac{h}{2} + \mu) + 2\sum_{l=0}^{\infty} \frac{(2^{2l+1}-1)(4l+1)!!}{[1-2(h+2\mu)]^{2l+\frac{3}{2}}}\, \zeta(2l+2)\, T^{2l+2}, \tag{6.7}$$

$$\chi_s \sim \frac{1}{2}\,[1-2(h+2\mu)]^{-1/2} + 2\sum_{l=0}^{\infty} \frac{(2^{2l+1}-1)(4l+3)!!}{[1-2(h+2\mu)]^{2l+\frac{5}{2}}}\, \zeta(2l+2)\, T^{2l+2}. \tag{6.8}$$

Proceeding as above, it is straightforward to derive from the previous series an asymptotic expansion of χ_s in terms of the variables (m_s, T) in the oblique band B_0 to any desired order. The first few terms in this expansion are

$$\chi_s \sim \frac{1}{2(1-2m_s)}\left[1 + \frac{4\pi^2 T^2}{3(1-2m_s)^4} + \frac{28\pi^4 T^4}{9(1-2m_s)^8} + \frac{3352\pi^6 T^6}{27(1-2m_s)^{12}} + \mathrm{O}(T^8)\right]. \tag{6.9}$$

From Eqs. (5.25) and (5.33) it follows that the asymptotic series of the free energy is a function of $h + 2\mu$. Thus the asymptotic series of n_c and χ_c are proportional to those of m_s and χ_s, namely,

$$n_c \sim 2m_s, \qquad \chi_c \sim 4\chi_s\,.$$

Thus in the band $\mathcal{B}_0$ the magnetic and the charge quantities are proportional, up to exponentially small terms in β. In particular, from Eq. (6.9) we obtain the following asymptotic expansion of χ_c in terms of $1 - n_c$ in the band $\mathcal{B}_0$:

$$\chi_c \sim \frac{2}{1-n_c}\left[1 + \frac{4\pi^2 T^2}{3(1-n_c)^4} + \frac{28\pi^4 T^4}{9(1-n_c)^8} + \frac{3352\pi^6 T^6}{27(1-n_c)^{12}} + \mathrm{O}(T^8)\right]. \tag{6.10}$$

7. Conclusions

In this paper we analyze the thermodynamics of the supersymmetric su(m) t-J model with long-range interactions through a novel approach based on the transfer matrix method. This method exploits the equivalence of the latter model to a modification of the su($1|m$) supersymmetric Haldane–Shastry spin chain, whose spectrum coincides with that of an inhomogeneous vertex model with a simple dispersion function. The energy function of this vertex model is related to suitable representations of the Yangian associated to supersymmetric Young tableaux and their corresponding Haldane motifs. This makes it possible to express the partition function by means of an appropriate site-dependent transfer matrix, which in the thermodynamic limit yields a simple closed-form expression for the free energy per site in terms of the largest eigenvalue (in modulus) of the latter matrix. One of the main advantages of our method is the fact that it can be applied to a wide range of models with (broken or unbroken) Yangian symmetry and arbitrary dispersion relations, including the supersymmetric Polychronakos–Frahm and Frahm–Inozemtsev spin chains. In the su($1|m$) case analyzed in the paper, we explicitly show that the free energy per site of all of these models can be expressed in terms of a function of one variable obeying an algebraic equation which generalizes the one derived by Kato and Kuramoto for multi-component boson-fermion systems [16,35]. We conjecture that this is still the case for more general su($n|m$) models with $n > 1$.

In the spin 1/2 case, we apply the explicit expression for the free energy to analyze in detail the thermodynamic and criticality properties of the model. To this end, we first determine all the ground state phases by computing the zero-temperature values of the magnetization and charge densities for arbitrary values of the magnetic field strength and the charge chemical potential. In particular, we show that the magnetic and charge susceptibilities present hitherto unnoticed jump discontinuities along the common boundary of the su($1|2$) and su($1|1$) phases. We then derive the complete asymptotic series of the free energy per site, showing that it takes different forms on each of the ground state phases. From the lowest-order term in the asymptotic series we determine the regions in parameter space in which the model is described at low energies by an effective CFT, and compute its corresponding central charge. Our results confirm that in the su($1|2$) phase the model is described by a CFT with conformal charge $c = 1$ in both the spin and the charge sectors. However, in the su(2) and su($1|1$) phases we find that the model is equivalent to a single CFT with $c = 1$. We also analyze in detail the critical behavior on the boundary between zero-temperature phases, finding that the system can be critical, gapless but not critical or even critical in the spin sector but not in the charge one. Using the asymptotic series for the free energy, we also derive the complete asymptotic series of the magnetization and charge densities and their corresponding susceptibilities. We numerically verify the strong spin-charge separation characteristic of the model for different (nonzero) values of the magnetization and the charge density, and show that it persists at all orders in the asymptotic expansion. This can be regarded as an analytic confirmation of spin-charge separation in a sufficiently

small range of temperatures near $T = 0$, where the asymptotic expansions provide an excellent approximation for the thermodynamic functions.

Although in this paper we have concentrated on the su(2) case, it would be of interest to apply the transfer matrix method to investigate the ground-state phases, thermodynamics and criticality properties of the general su(m) KY model with $m > 2$. In fact, as explained above, this method could in principle be extended to more general models with partial or total Yangian symmetry provided that their spectrum coincides with that of the inhomogeneous vertex model (3.18)-(3.19) for a suitable dispersion relation $\mathcal{E}_N$.

Acknowledgments

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Appendix A. Derivation of Eqs. (5.7)-(5.8)

In this appendix we shall establish the validity of the estimates (5.7)-(5.8) for the integrals appearing in Eq. (5.5). To begin with, the difference between the first of these integrals and the integral I_1 in Eq. (5.7) is given by

$$\Delta_1 \equiv \int_0^{x_0(h)} \log\left[\frac{\widetilde{b} + \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}}}{\frac{1}{2} + \sqrt{\frac{1}{4} + \mathrm{e}^{-\beta(h-\varepsilon)}}}\right] \mathrm{d}x = \int_0^{x_0(h)} \log(1 + \phi_1(x))\mathrm{d}x\,, \tag{A.1}$$

where

$$\phi_1(x) \equiv \frac{\frac{1}{2}(\mathrm{e}^{-\beta(\mu+\frac{h}{2}-\varepsilon)} + \mathrm{e}^{-\beta h}) + \Delta R_1}{\frac{1}{2} + \sqrt{\frac{1}{4} + \mathrm{e}^{-\beta(h-\varepsilon)}}} \tag{A.2}$$

and

$$\Delta R_1 \equiv \sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}} - \sqrt{\tfrac{1}{4} + \mathrm{e}^{-\beta(h-\varepsilon)}} = \frac{\widetilde{b}^2 - \frac{1}{4} - \mathrm{e}^{-\beta h}}{\sqrt{\widetilde{b}^2 + \mathrm{e}^{-\beta(h-\varepsilon)} - \mathrm{e}^{-\beta h}} + \sqrt{\frac{1}{4} + \mathrm{e}^{-\beta(h-\varepsilon)}}}\,.$$

Since $\widetilde{b} > 1/2$ the denominator in ΔR_1 is $\geqslant 1$, and hence

$$\begin{aligned}|\Delta R_1| &\leqslant \mathrm{e}^{-\beta h} + \widetilde{b}^2 - \tfrac{1}{4} = \mathrm{e}^{-\beta h} + \frac{1}{4}\Big(\mathrm{e}^{-\beta(\mu+\frac{h}{2}-\varepsilon)} + \mathrm{e}^{-\beta h}\Big)\Big(2 + \mathrm{e}^{-\beta(\mu+\frac{h}{2}-\varepsilon)} + \mathrm{e}^{-\beta h}\Big) \\ &\leqslant \mathrm{e}^{-\beta(\mu-\frac{h}{2})} + 2\mathrm{e}^{-\beta h},\end{aligned}$$

where we have used the inequality $\varepsilon(x) \leqslant h$ valid in the interval $[0, x_0(h)]$. From Eq. (A.2) we thus obtain

$$|\phi_1(x)| \leqslant \tfrac{3}{2}\,\mathrm{e}^{-\beta(\mu-\frac{h}{2})} + \tfrac{5}{2}\,\mathrm{e}^{-\beta h} = \mathrm{O}(\mathrm{e}^{-\beta\min(\mu-\frac{h}{2},h)})\,,$$

which in particular shows that when $(h,\mu) \in \mathcal{B}_1$ the function $\phi_1(x)$ tends to zero as $T \to 0$ uniformly in $x \in [0, x_0(h)]$. Since $|\log(1+\phi_1(x))| = \mathrm{O}(\phi_1(x))$ when $\phi_1(x) \to 0$, from Eq. (A.1) it immediately follows that

$$\Delta_1 = \mathrm{O}(\mathrm{e}^{-\beta \min(\mu - \frac{h}{2}, h)}).$$

Similarly, the difference between the LHS of Eq. (5.8) and the integral (5.8) can be written as

$$\Delta_2 = \int_{x_0(h)}^{1/2} \log(1+\phi_2(x))\mathrm{d}x\,, \tag{A.3}$$

where

$$\phi_2(x) \equiv \frac{\frac{1}{2}(\mathrm{e}^{-\frac{\beta}{2}(\varepsilon+h)} + \mathrm{e}^{-\beta(\mu-\frac{\varepsilon}{2})}) + \Delta R_2}{\frac{1}{2}\mathrm{e}^{-\frac{\beta}{2}(\varepsilon-h)} + \sqrt{1+\frac{1}{4}\mathrm{e}^{-\beta(\varepsilon-h)}}}$$

$$\Delta R_2 \equiv \sqrt{\widehat{b}^2 + 1 - \mathrm{e}^{-\beta\varepsilon}} - \sqrt{1+\tfrac{1}{4}\mathrm{e}^{-\beta(\varepsilon-h)}} = \frac{\widehat{b}^2 - \frac{1}{4}\mathrm{e}^{-\beta(\varepsilon-h)} - \mathrm{e}^{-\beta\varepsilon}}{\sqrt{\widehat{b}^2 + 1 - \mathrm{e}^{-\beta\varepsilon}} + \sqrt{1+\frac{1}{4}\mathrm{e}^{-\beta(\varepsilon-h)}}}\,.$$

Proceeding as before, and taking into account that in the interval $[x_0(h), 1/2]$ we have

$$\varepsilon(x) \geqslant h\,, \qquad \mu - \frac{\varepsilon(x)}{2} \geqslant \mu - \frac{1}{8}\,,$$

after a straightforward calculation we obtain the estimate

$$\left|\widehat{b}^2 - \tfrac{1}{4}\mathrm{e}^{-\beta(\varepsilon-h)} - \mathrm{e}^{-\beta\varepsilon}\right| \leqslant \mathrm{e}^{-\frac{\beta}{2}(\varepsilon+h)} + \mathrm{e}^{-\beta(\mu-\frac{\varepsilon}{2})} + \mathrm{e}^{-\beta\varepsilon} \leqslant 2\mathrm{e}^{-\beta h} + \mathrm{e}^{-\beta(\mu-\frac{1}{8})}\,.$$

From the definition of $\phi_2(x)$ it immediately follows that

$$|\phi_2(x)| \leqslant \tfrac{3}{2}\,\mathrm{e}^{-\beta(\mu-\frac{1}{8})} + \tfrac{5}{2}\,\mathrm{e}^{-\beta h} = \mathrm{O}(\mathrm{e}^{-\beta\min(\mu-\frac{1}{8}, h)}),$$

which easily yields (5.8) on account of Eq. (A.3).

Appendix B. Asymptotic series for the integral I_1

In this appendix we derive the asymptotic series (5.13) for the integral I_1. Calling for simplicity $a_l \equiv a_l(h)$ and setting

$$g(y) \equiv \log\left[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\,\right], \qquad \phi(z) \equiv \sum_{l=0}^{\infty} (-1)^l a_l z^l,$$

we need to show that

$$\int_0^{\beta h} g(y)\phi(Ty)\mathrm{d}y \sim \sum_{l=0}^{\infty} (-1)^l a_l\, T^l \int_0^{\infty} y^l g(y)\mathrm{d}y\,.$$

Note first of all that the power series $\phi(z)$ converges for $|z| < 1/4$. Since $h < 1/4$ when $(h,\mu) \in \mathcal{B}_1$, it follows that Ty lies inside the convergence disc of $\phi(z)$ for fixed h and all $y \in [0, \beta h]$. We must check that for all $n \in \mathbb{N}$

$$\sum_{l=0}^{n} (-1)^l a_l\, T^l \int_0^{\infty} y^l g(y)\mathrm{d}y - \int_0^{\beta h} g(y)\phi(Ty)\mathrm{d}y = \mathrm{o}(T^n)\,,$$

i.e.,

$$\sum_{l=0}^{n}(-1)^l a_l\, T^l \int_{\beta h}^{\infty} y^l g(y)\mathrm{d}y - \int_0^{\beta h} \mathrm{d}y\, g(y) \sum_{l=n+1}^{\infty} (-1)^l a_l\, (Ty)^l = \mathrm{o}(T^n). \tag{B.1}$$

Since

$$0 \leqslant \int_{\beta h}^{\infty} y^l g(y)\mathrm{d}y \leqslant \int_{\beta h}^{\infty} y^l \mathrm{e}^{-y}\mathrm{d}y = \mathrm{O}(\beta^l \mathrm{e}^{-\beta h}),$$

the first sum in Eq. (B.1) is $\mathrm{O}(\mathrm{e}^{-\beta h})$. As to the second term, note that

$$\sum_{l=n+1}^{\infty} (-1)^l a_l\, (Ty)^l = (Ty)^{n+1}\, \widetilde{\phi}(Ty),$$

where $\widetilde{\phi}(z)$ is a convergent power series and hence analytic for $|z| < 1/4$. Since $Ty \in [0, h] \subset [0, 1/4)$ when $y \in [0, \beta h]$, it follows that $|\widetilde{\phi}(Ty)| < M(h)$ independently of β. Hence

$$\begin{aligned}\Big| \int_0^{\beta h} \mathrm{d}y\, g(y) \sum_{l=n+1}^{\infty} (-1)^l a_l\, (Ty)^l \Big| &\leqslant M(h)\, T^{n+1} \int_0^{\beta h} y^{n+1} g(y)\mathrm{d}y \\ &\leqslant \Big(M(h) \int_0^{\infty} y^{n+1} e^{-y}\mathrm{d}y \Big)\, T^{n+1} = (n+1)!M(h)\, T^{n+1},\end{aligned}$$

so that both terms in the LHS of Eq. (B.1) are indeed $\mathrm{o}(T^n)$.

Appendix C. Alternative expression for the integrals $\mathcal{I}_l$

In this appendix we will derive the alternative expressions (5.19) for the integrals $\mathcal{I}_l$ appearing in the asymptotic series for the free energy in the triangle $\mathcal{T}$ and the vertical band $\mathcal{B}_1$ (cf. Eq. (5.18)). To begin with, consider the integral

$$\mathcal{I}_{l,1} \equiv \int_0^{\infty} y^l \Big\{ \log\Big[\tfrac{1}{2}\, \mathrm{e}^{-y/2} + \sqrt{1 + \tfrac{1}{4}\, \mathrm{e}^{-y}}\, \Big] \Big\} \mathrm{d}y = \int_0^{\infty} y^l \operatorname{arcsinh}(\tfrac{1}{2}\, \mathrm{e}^{-y/2})\, \mathrm{d}y\,.$$

Writing $\operatorname{arcsinh}(\frac{1}{2}\, \mathrm{e}^{-y/2})$ as the integral of its derivative, namely

$$\operatorname{arcsinh}(\tfrac{1}{2}\, \mathrm{e}^{-y/2}) = \frac{1}{2} \int_y^{\infty} \frac{\mathrm{d}x}{\sqrt{1 + 4\mathrm{e}^{x}}}\,,$$

we can express $\mathcal{I}_{l,1}$ as a double integral as

$$\mathcal{I}_{l,1} = \frac{1}{2} \int_0^{\infty} \mathrm{d}y \int_y^{\infty} \mathrm{d}x\, \frac{y^l}{\sqrt{1 + 4\mathrm{e}^{x}}}\,.$$

Reversing the order of integration we obtain

$$\mathcal{I}_{l,1} = \frac{1}{2} \int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{1 + 4\mathrm{e}^{x}}} \int_0^{x} \mathrm{d}y\, y^l = \frac{1}{2(l+1)} \int_0^{\infty} \frac{x^{l+1}}{\sqrt{1 + 4\mathrm{e}^{x}}}\, \mathrm{d}x\,. \tag{C.1}$$

Consider next the second integral in Eq. (5.18), namely

$$\mathcal{I}_{l,2} = \int_0^{\infty} y^l \Big\{ \log\Big[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \mathrm{e}^{-y}}\, \Big] \Big\} \mathrm{d}y = \int_0^{\infty} y^l \Big[-\tfrac{y}{2} + \operatorname{arcsinh}(\tfrac{1}{2}\, \mathrm{e}^{y/2}) \Big] \mathrm{d}y\,.$$

Proceeding as before we write

$$-\frac{y}{2} + \operatorname{arcsinh}(\tfrac{1}{2}\, \mathrm{e}^{y/2}) = \frac{1}{2} \int_y^{\infty} \left(1 - \frac{1}{\sqrt{1 + 4\mathrm{e}^{-x}}} \right) \mathrm{d}x$$

and therefore

$$\mathcal{I}_{l,2} = \frac{1}{2}\int_0^\infty \mathrm{d}y \int_y^\infty \mathrm{d}x\, y^l \left(1 - \frac{1}{\sqrt{1+4\mathrm{e}^{-x}}}\right) = \frac{1}{2}\int_0^\infty \mathrm{d}x \left(1 - \frac{1}{\sqrt{1+4\mathrm{e}^{-x}}}\right) \int_0^x \mathrm{d}y\, y^l$$
$$= \frac{(-1)^l}{2(l+1)} \int_{-\infty}^0 x^{l+1} \left(\frac{1}{\sqrt{1+4\mathrm{e}^x}} - 1\right) \mathrm{d}x\,. \tag{C.2}$$

Combining Eqs. (C.1) and (C.2) we obtain the first equality in Eq. (5.19). The second equality in the latter equation easily follows integrating by parts in Eqs. (C.1) and (C.2).

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CHAPTER 4

Short-range many-body systems in one dimension

Since their introduction aroundin the early the seventies, quantum integrable many-body models of Calogero–Sutherland [32, 33, 113, 114] (CS) type have been extensively studied in the literature due to their simplicity and their rich mathematical structure. A remarkable property of this kind of models is that their ground state ψ factorizes over the A_{N-1} root system and can be written as

$$\psi(x_1, \ldots, x_N) \propto \prod_i \rho(x_i) \prod_{j<k} \chi(x_j - x_k)\,.$$

Moreover, ψ is square integrable in some configuration space $C \subset \mathbf{R}^N$, with $L^2(C)$ being the Hilbert space of the system. For certain choices of the functions of one variable ρ and χ, $|\psi|^2$ coincides with the joint probability density of eigenvalues of the three types of both Gaussian or Dyson's circular ensembles. Exploiting the role of $|\psi|^2$ as the joint probability density of eigenvalues of random matrices in an ensemble, certain correlation functions can be computed in closed form.

It is natural to seek for the more general CS Hamiltonian whose ground state is of the previous form some pair ρ, χ of functions of one variable. A weaker formulation of this problem with $\rho = 1$ already appeared in Sutherland's early work [113], who later found a solution for this weak version of the problem with two-body interactions given by a potential in terms of elliptic functions [115, 117]. Shortly afterwards, Calogero [33] showed that this is in fact the most general solution of the problem with $\rho = 1$. The general problem (with ρ not necessarily equal to 1) was tackled by Inozemtsev and Meshcheryakov [75], who claimed to have found all possible solutions. A few years later, however, Forrester [54] found a CS model whose ground state is of the required factorized form and yet did not appear in the set of solutions of Ref. [75]. The classification of all possible solutions to the general unrestricted problem was finally completed several

years later by Koprucki and Wagner [84], which included Forrester's model as a particular case.

The probability distribution $p_\beta(s)$ of the (normalized) spacing s between two consecutive eigenvalues of the Gaussian β-ensembles is approximately given by Wigner's surmise $p_\beta(s) = A_\beta s^\beta \mathrm{e}^{-c_\beta s^2}$, where the positive parameters A_β, c_β are fixed by normalization and the condition that the mean spacing be equal to 1 (see, e.g., Refs. [55, 91]). By contrast, it has been conjectured [20] that the spacings distribution of a "generic" quantum integrable model is Poissonian, i.e., $p(s) = \mathrm{e}^{-s}$. The latter distributions are actually obeyed by the spectra of a wide range of either fully chaotic or completely integrable systems [91]. However, for certain so-called pseudo-integrable systems (like, for instance, the Aharonov–Bohm billiard [19] and the three-dimensional Anderson model at the metal-insulator transition point [3]) the spectrum statistics (in particular, the spacings distribution) was found to be quite different from those of either chaotic or generic integrable systems (see, e.g., Refs. [39, 59]).

In the late nineties, Bogomolny, Gerland and Schmit [23, 24] tried to account for this discrepancy by assuming that for the latter systems the probability density $p(\lambda_1, \ldots, \lambda_N)$ of obtaining N eigenvalues within small intervals around $\lambda_1, \ldots, \lambda_N$ respectively (in a finite range of the spectrum) is given by a nearest-neighbors version of the joint probability distribution of the eigenvalues of the Gaussian β-ensembles, namely[25]

$$p(\lambda_1, \ldots, \lambda_N) \propto \prod_{i=1}^{N} \exp\Big(-\frac{\beta}{2}\lambda_i^2\Big) \cdot \prod_{i=1}^{N-1} |\lambda_i - \lambda_{i+1}|^\beta \,.$$

If we identify the eigenvalue λ_k with the coordinate x_k of a quantum particle, the above distribution is the probability density of the ground state of the N-body Hamiltonian

$$H = -\sum_{i=1}^{N} \partial_{x_i}^2 + \omega \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{N-1} \frac{2\alpha(\alpha-1)}{(x_i - x_{i+1})^2} - \sum_{i=1}^{N-1} \frac{2\alpha^2}{(x_i - x_{i+1})(x_{i+1} - x_{i+2})}$$

[25] More precisely, the latter authors considered a periodic version of this density obtained by discarding the first factor, whose contribution is negligible in the limit $N \to \infty$, and adding an interaction term between the first and last particles.

with $2\alpha = 2\omega = \beta$ [**76**]. By contrast to the Calogero model, the latter Hamiltonian features only nearest- (two-body) and next-to-nearest- (three-body) neighbors interactions among the particles. Proceeding in a similar way with the joint probability density of the eigenvalues of Dyson's circular ensembles one obtains a nearest-neighbors version of the ground state of the Sutherland model, which is the ground state of a quantum many-body Hamiltonian with trigonometric two- and three-body near-neighbors interactions. This connection between random matrix theory and quantum many-body models with near-neighbors interactions of Calogero–Sutherland type has in fact spurred the construction of further such models (including particles with spin and interactions of arbitrary finite range) and the study of their properties (see, e.g., Refs. [**15**, **46**, **47**, **96**, **123**]).

The purpose of [**P8**] is to classify all quantum many-body models in one dimension with nearest- and next-to-nearest-neighbors (translation invariant) interactions whose ground state factorizes as

$$\psi(x_1, \ldots, x_N) = \prod \rho(x_i) \prod \chi(x_i - x_{i+1})\,. \tag{23}$$

These models include the versions of the Calogero and Sutherland Hamiltonians with near-neighbors interactions introduced in Ref. [**76**]. In other words, our goal is to perform the near-neighbors analog of the well-known classification of long-range CS models, started by Sutherland and Calogero and ultimately completed by Koprucki and Wagner.

1. Short-range Calogero–Sutherland models

In Ref. [**P8**] we have shown that if we assume certain homogeneity conditions, all short-range ground states in one dimension for non-relativistic spinless quantum many-body systems can be classified. More precisely, we have classified all potentials with at most three-body near-neighbors translation invariant interactions of the form

$$V = \sum V_1(x_i) + \sum V_2(x_i - x_{i+1}) + \sum V_3(x_i - x_{i+1}, x_{i+1} - x_{i+2})$$

whose ground state is of the *Jastrow-like* form (23). When this paper was published, we were not aware that this classification provided the first examples of parent Hamiltonians with ground states exactly given by the family of states introduced in [**126**], the so-called continuous matrix product states (cMPS) since they represent the natural continuum counterpart of the celebrated matrix product states (MPS). We shall take

advantage of this connection here to relate the results of [**P8**] to cMPS for the first time.

In particular, we have proved a negative result stating that the presence of a three-body interaction term V_3 is unavoidable as was particularly the case with all previously known examples. We proceed by finding a solution in terms of a function which is essentially arbitrary, containing as particular instances, all previously known examples. The classification follows from the uniqueness of the general form of this solution. We refer to [**P8**] for examples and further discussion on these results as, for instance, a new hyperbolic potential or the introduction of a model with elliptic interactions.

The solutions to our classification problem are given by four functions of one variable V_1, V_2, ρ and χ, and one function of two variables V_3.

2. A negative result

Observe that if both V_2 and V_3 are identically zero then the system is non-interacting and it trivially has a Jastrow-like ground state ψ given by the product $\psi = \rho(x_1)\cdots\rho(x_N)$ where $\rho(x)$ is the ground state of the one-particle Hamiltonian $H_1 = -\partial_x^2 + V_1(x)$. We shall say that a Jastrow-like wavefunction ψ of the form (23) is non-trivial if $\psi \neq \rho(x_1)\cdots\rho(x_N)$ for all functions $\rho(x)$. Equivalently, if ψ is a non-trivial Jastrow-like ground state of a short-range CS Hamiltonian H as above, then its potential $V \neq \sum V_1(x_i)$ is that of an interacting system. Our first result can be stated as follows:

If $V_3 = 0$ then there is no non-trivial short-range CS model with non-trivial Jastrow-like ground state.

The previous assertion is proved in [**P8**] and states that no short-range CS model described by a Hamiltonian H of the form

$$H = -\sum \partial_{x_i}^2 + \sum V_1(x_i) + \sum V_2(x_i - x_{i+1}) \tag{24}$$

can exist with a Jastrow-like ground state unless V_2 is identically zero (in which case χ is identically one). At this point, one can try to investigate what kind of wavefunctions could be good candidates to become the ground state of such a Hamiltonian. For the case of motion in the circle, it is natural to try with a wavefunction given by the trace of a product of

matrices. For instance, we can take

$$\begin{aligned} \psi &= \operatorname{tr} W \\ W &= R(x_1)T(x_1-x_2)R(x_2)\cdots T(x_{N-1}-x_N)R(x_N)T(x_N-x_1) \end{aligned} \tag{25}$$

where $R(x)$ and $T(u)$ are suitable matrices whose coefficients $R_{kl}(x)$ and $Q_{kl}(u)$ are functions of one variable. In the case of motion in the real line an ansatz is given by a similar expression modified by the inclusion of two new matrix-valued functions of appropriate dimensions playing the role of the points at infinity.

The ansatz (25) is certainly invariant under cyclic permutations and is the wavefunction of a translation invariant system of N particles. It could well be called a continuous matrix product state (cMPS) and Hamiltonians of the form (24) represent interesting candidates to be their parent Hamiltonians.

3. The classification

In [**P8**] we have shown that the classification problem posed admits the following solution:

For essentially any arbitrary function of one variable φ and $\omega \in \mathbf{R}$ the following expressions

$$\begin{aligned} V_1(x) &= \omega^2 x^2 \\ V_2(u) &= 2\varphi'(u) + 2\varphi^2(u) - 2\omega u\varphi(u) \\ V_3(u,v) &= -2\varphi(u)\varphi(v) \end{aligned} \tag{26a}$$

defines a potential V and thus a short-range CS model whose ground state, with energy $N\omega$, is determined by taking

$$\begin{aligned} \rho(x) &= \exp\left(-\frac{1}{2}\omega x^2\right) \\ \chi(u) &= \exp\left(\int^u \varphi(s)\mathrm{d}s\right) \end{aligned} \tag{26b}$$

in the expression (23), and is thus Jastrow-like.

Taking into account our convention for sums and products [**P8**], the previous expressions provide a solution both on the circle and the real line as it can be explicitly checked. Note that we can assume that φ' is not identically zero since in that case we obtain a trivial non-interacting solution. Furthermore, we have also assumed in [**P8**] for simplicity's sake

that φ is meromorphic. Apart from that, φ is arbitrary except for the requirement that the Jastrow-like wavefunction (23) with the previous ρ and χ be square-integrable. We discuss certain sufficient conditions ensuring square-integrability, providing several explicit examples. On the other hand, apart from these examples we have also proved the following statement:

Every non-trivial solution (given by four functions of one variable V_1, V_2, ρ and χ, and one function of two variables V_3) is of the form (26) for some function φ and $\omega \in \mathbf{R}$.

It is left to show that the corresponding function ψ is the ground state of $H = -\sum \partial_{x_i}^2 + V$ for that potential V, and that $\psi \in L^2(C)$ for some configuration space $C \in \mathbf{R}^N$. This was done in [**P8**], obtaining the known trigonometric, hyperbolic and rational models as particular instances and a new model featuring elliptic interactions.

4. Parent Hamiltonians for a subclass of continuous matrix product states

Two main facts relate Jastrow-like ground states to continuous matrix product states (cMPS): from the one side, proposals for ground states as the trace of a product of matrix-valued functions can be readily connected to the family of states introduced in [**126**] in a natural way; on the other hand, our classification is a classification of certain subclass of the family of states considered in the latter reference, the so-called rank-1 cMPS. Following [**126**], we shall introduce this subclass and show that it belongs to the set of Jastrow-like wavefunctions (23).

I have tried to obtain the ground state of an interacting Hamiltonian with V_3 identically zero as in Eq. (24) in terms of cMPS of greater rank, without success. The general matrix equation we obtained behaved differently when the cMPS is of full-rank, and in most cases it turns out that in this case the solution can only be trivial. Since the case of rank one is strictly contained within the class of Jastrow-like ground states considered in [**P8**], we conjecture that the problem of finding a Hamiltonian of the form (24) with ground state as in (25) has no solution for matrix-valued functions W, R, T of order two.

The role of cMPS in quantum field theories is analogous to that of MPS in the description of spin chains. As a matter of fact, in the seminar paper [**126**], it is written at this respect

> "Just as MPS capture the entanglement structure of low-energy states of quantum spin systems, the entanglement structure of cMPS is tailored to describe the low-energy states of quantum field theories."

Let us denote by $|\Omega\rangle$ the Fock vacuum and let $|\Phi\rangle$ be some cMPS that will depend, in general, on several matrix-valued functions of one variable. Let us say, for the sake of simplicity, that it depends on two square matrices $R(\lambda)$ and $Q(\lambda)$ of order d where $\lambda \in \mathbf{R}$ is a parameter[26]. This cMPS describes a system of fermions or bosons in a one-dimensional ring and does not have a well defined number of particles, as is usual in the framework of second quantization. It can be written as $|\Phi\rangle = \sum_{n=0}^{\infty} |\Phi_n\rangle$ with

$$|\Phi_n\rangle = \int_{C_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \psi_n(x_1, \ldots, x_n) \, \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_n) |\Omega\rangle$$

for some configuration spaces $C_n \subset \mathbf{R}^n$ and functions of n variables ψ_n, where Ψ is the field operator satisfying the canonical commutation $(+)$ or anticommutation $(-)$ relations $\{\Psi(x), \Psi^\dagger(y)\}_\pm = \delta(x-y)$ respectively for bosons $(+)$ or fermions $(-)$. As usual, the Fock vacuum is characterized by the relations $\Psi(x)|\Omega\rangle = 0$ and the configuration spaces C_n are given by $0 < x_1 < \cdots < x_n < 2\pi$, in the case of a ring of length 2π. Finally, the amplitude ψ_n can be written as

$$\psi_n = \mathrm{tr}\left(u_Q(0, x_1) R(x_1) u_Q(x_1, x_2) \cdots u_Q(x_{n-1}, x_n) R(x_n) u_Q(x_n, 0)\right)$$

where $u_Q(x, y)$ is the path-ordered exponential of $Q(\lambda)$ when integrated between the points x and y. If $\mathcal{T}$ denotes the string-ordering operator mapping string-like products of the form[27] $O_1(x_1) \cdots O_n(x_n)$ to string-like

[26] This can be interpreted as a fictitious time, as the spectral parameter in the inverse scattering method literature, as a segment in energy-, frequency-, spatial- or momentum-space ...

[27] In other words, field-like variables $O_i(x)$ are expressed in terms of $\Psi(x)$ or $\Psi^\dagger(x)$.

products $O_{i_1}(x_{i_1})\cdots O_{i_n}(x_{i_n})$ when $x_{i_1}<\cdots<x_{i_n}$, then

$$\begin{aligned} u_Q(x,y) &= \mathcal{P}\exp\left(\int_x^y Q(z)\mathrm{d}z\right) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\int_x^y \mathrm{d}z_1\cdots\int_x^y \mathrm{d}z_m \mathcal{T}Q(z_1)\cdots Q(z_m) \\ &= \sum_{m=0}^{\infty}\int_{C_m(x,y)} Q(z_1)\cdots Q(z_m)\,\mathrm{d}z_1\cdots\mathrm{d}z_m \end{aligned}$$

where $\mathcal{P}\exp$ denotes the path-ordered exponential and $C_m(x,y)$ is the configuration space C_m for a segment $[x,y]$ given by $x\le z_1<\cdots<z_m\le y$.

If both R and Q are constant matrices the previous function simplifies to $u_Q(x,y)=\mathrm{e}^{(y-x)Q}$. Furthermore, if the rank of R is exactly one, the amplitudes ψ_n are Jastrow-like wavefunctions describing systems of n particles. They can be written in the form of Eq. (23), with ρ identically one and $\chi(u)=\mathrm{tr}(R\cdot\mathrm{e}^{uQ})$. We call cMPS of rank one the set of cMPS with order-d constant matrices R of rank one and matrices Q of arbitrary rank.

The classification discussed above applies and one can interpret the cMPS of rank one as follows. In principle, the cMPS $|\Phi\rangle$ is a superposition of states of different number of particles, as usual in the second quantization formalism. However, for each positive integer n its contribution to the sector of n particles is given by the ground state solution ψ_n of the Schrödinger equation for an effective short-range CS Hamiltonian.

More precisely, this Hamiltonian is of the form $H=-\sum_i\partial_{x_i}^2+V$. The potential V is given in terms of the three functions V_1, V_2 and V_3 appearing in Eq. (26a) for a suitable function φ. Finally, for any pair of order-d constant matrices R of rank one and Q of arbitrary rank, $\varphi(u)=\partial\chi(u)/\chi(u)$ is the logarithmic derivative of the function $\chi(u)=\mathrm{tr}(R\cdot\mathrm{e}^{uQ})$.

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Jastrow-like ground states for quantum many-body potentials with near-neighbors interactions

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HIGHLIGHTS

- Complete classification of near-neighbors CS models with a Jastrow-like ground state.
- Examples of such models with rational, trigonometric and hyperbolic interactions.
- Construction of a new elliptic model which yields these examples in suitable limits.

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ABSTRACT

We completely solve the problem of classifying all one-dimensional quantum potentials with nearest- and next-to-nearest-neighbors interactions whose ground state is Jastrow-like, i.e., of Jastrow type but depending only on differences of consecutive particles. In particular, we show that these models must necessarily contain a three-body interaction term, as was the case with all previously known examples. We discuss several particular instances of the general solution, including a new hyperbolic potential and a model with elliptic interactions which reduces to the known rational and trigonometric ones in appropriate limits.

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1. Introduction

Since their introduction in the early 70's, the quantum integrable many-body models of Calogero [1] and Sutherland [2,3] have been extensively studied due to their conceptual simplicity and their outstanding properties. In fact, the fundamental character of these models is attested by their appearance in such diverse areas as soliton theory [4,5], orthogonal polynomials [6–8], random matrix theory [9–11], fractional statistics and anyons [12,13], quantum Hall effect [14,15], conformal field theory [16–18], general relativity [19,20], hydrodynamics of cold atomic gases [21], and quantum quenching [22].

A remarkable feature of the Calogero and Sutherland models is that their ground-state wave function ψ is factorized over the A_{N-1} root system, i.e., is of the form

$$\psi(\mathbf{x}) \propto \prod_{i=1}^{N} \rho(x_i) \cdot \prod_{1\leqslant i<j\leqslant N} \chi(x_i - x_j). \tag{1.1}$$

Here $\mathbf{x} \equiv (x_1, \ldots, x_N)$ and

$$\rho(x) = \mathrm{e}^{-\frac{1}{2}\omega x^2}, \qquad \chi(x) = |x|^a$$

for the (harmonic) Calogero model, while

$$\rho(x) = 1\,, \qquad \chi(x) = |\sin x|^a$$

for the Sutherland model (with $\omega > 0$ and $a > -1/2$). As first noted by Sutherland [2,23], this property makes it possible to compute in closed form certain correlation functions of the latter models by exploiting their connection with random matrix theory. Indeed, for Calogero's model ψ^2 coincides with the joint probability density of the eigenvalues of the Gaussian orthogonal, unitary and symplectic ensembles respectively for $2\omega = 2a = 1, 2, 4$, while the same relation holds for the ground state of the Sutherland model and Dyson's circular unitary ensembles (with eigenvalues parametrized as $\mathrm{e}^{2\mathrm{i}x_k}$) [24]. Later on, Dyson [25] showed how to construct analogs of the Gaussian ensembles with eigenvalues distributed according to ψ^2 in Eq. (1.1), with essentially arbitrary ρ and $\chi(x) = |x|^{\beta/2}$ (as usual, β will be assumed to take the values 1, 2, 4 for the orthogonal, unitary and symplectic ensembles, respectively).

In view of these results, it is natural to look for the most general quantum Hamiltonian of Calogero–Sutherland (CS) type (i.e, with one- and two-body long-range interactions) whose ground state is of the form (1.1). A restricted version of this problem (with $\rho = 1$) was already formulated by Sutherland himself [2], who later found a solution thereof with an elliptic two-body interaction potential [26,27]. Shortly afterwards, Calogero [28] showed that this is in fact the most general solution of this restricted problem. The general problem (with ρ not necessarily equal to 1) was tackled by Inozemtsev and Meshcheryakov [29], who claimed to have found a complete solution. A decade later, however, Forrester [30] found a model of CS type whose ground state, which exhibits long-range crystalline order in the thermodynamic limit, is of the factorized form (1.1) and yet did not appear in the classification of Ref. [29]. The latter classification was finally completed several years later by Koprucki and Wagner [31], who obtained Forrester's model as a particular case.

The probability distribution $p_\beta(s)$ of the (normalized) spacing s between two consecutive eigenvalues of the Gaussian β-ensembles is approximately given by Wigner's surmise $p_\beta(s) = A_\beta s^\beta \mathrm{e}^{-c_\beta s^2}$, where the positive parameters A_β, c_β are fixed by normalization and the condition that the mean spacing be equal to 1 (see, e.g., Refs. [32,33]). By contrast, it has been conjectured [34] that the spacings distribution of a "generic" quantum integrable model is Poissonian, i.e., $p(s) = \mathrm{e}^{-s}$. The latter distributions are actually obeyed by the spectra of a wide range of either fully chaotic or completely integrable systems [32]. However, for certain so-called pseudo-integrable systems (like, for instance, the Aharonov–Bohm billiard [35] and the three-dimensional Anderson model at the metal–insulator transition point [36]) the spectrum statistics (in particular, the spacings distribution) was found to be quite different from those of either chaotic or generic integrable systems (see, e.g., Refs. [37,38]). In the late 90's, Bogomolny, Gerland and Schmit [39,40] tried to account for this discrepancy by assuming that for the latter systems the probability density $p(\lambda_1, \ldots, \lambda_N)$ of the eigenvalues λ_k (in a finite range

of the spectrum) is given by a nearest-neighbors version of the joint probability distribution of the eigenvalues of the Gaussian β-ensembles, namely [1]

$$p(\lambda_1, \ldots, \lambda_N) \propto \prod_{i=1}^{N} \mathrm{e}^{-\frac{\beta}{2}\lambda_i^2} \cdot \prod_{i=1}^{N-1} |\lambda_i - \lambda_{i+1}|^{\beta} .$$

If we identify the eigenvalue λ_k with the coordinate x_k of a quantum particle, the above distribution is the probability density of the ground state of the N-body Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \omega \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{N-1} \frac{2\alpha(\alpha-1)}{(x_i - x_{i+1})^2} - \sum_{i=1}^{N-1} \frac{2\alpha^2}{(x_i - x_{i+1})(x_{i+1} - x_{i+2})},$$

with $2\alpha = 2\omega = \beta$ [41]. Note that, by contrast to the Calogero model, the latter Hamiltonian features only nearest-neighbors (two-body) and next-to-nearest-neighbors (three-body) interactions among the particles. Proceeding in a similar way with the joint probability density of the eigenvalues of Dyson's circular ensembles one obtains a nearest-neighbors version of the ground state of the Sutherland model, which is the ground state of a quantum many-body Hamiltonian with trigonometric two- and three-body near-neighbors interactions. This connection between random matrix theory and quantum many-body models with near-neighbors interactions of Calogero–Sutherland type has in fact spurred the construction of further such models (including particles with spin and interactions of arbitrary finite range) and the study of their properties (see, e.g., Refs. [42–46]).

The purpose of this paper is to classify all quantum many-body models in one dimension with nearest- and next-to-nearest-neighbors (translation invariant) interactions whose ground state factorizes as in Eq. (1.1), but with the differences $x_i - x_j$ replaced by the nearest-neighbors differences $x_i - x_{i+1}$. As we have just remarked, these models include the versions of the Calogero and Sutherland Hamiltonians with near-neighbors interactions introduced in Ref. [41]. In other words, our goal is to perform the near-neighbors analog of the well-known classification of CS-type models with a factorized ground state of the form (1.1), started by Sutherland and Calogero and ultimately completed by Koprucki and Wagner. We shall present the general solution of this classification problem, both for motion on the real line and on a circle. By contrast with the corresponding problem featuring long-range interactions, this general solution depends on an arbitrary function of one variable and (for motion in the real line) an arbitrary positive parameter. Moreover, we shall show that the three-body term appearing in all previously known examples is unavoidable. In other words, this term must necessarily be present in any potential whose ground state is of the sought-for form. We shall also see that the general solution contains an elliptic potential which yields in a suitable limit the rational and trigonometric models introduced in Ref. [41], as well as a new hyperbolic model akin to the long-range one discussed by Forrester [30].

We shall finish this Introduction with a brief outline of the paper's organization. In Section 2 we obtain a solution of the classification problem depending on an arbitrary function, and prove that the three-body term that it contains cannot be expressed as an external potential plus a two-body term. We show in Section 3 that there is no other solution, thus completing the proposed classification. Section 4 is devoted to verifying that the factorized eigenfunction associated with this solution is actually the (square-integrable) ground state of the corresponding Hamiltonian, provided that the arbitrary function which appears in the solution satisfies some natural physical requirements. In Section 5 we discuss some particular models included in the general solution, recovering the rational and trigonometric potentials of Ref. [41] and introducing the new hyperbolic and elliptic models mentioned above. The paper ends with a concluding section where we summarize our results and indicate possible future developments.

[1] More precisely, the latter authors considered a periodic version of this density obtained by discarding the first factor, whose contribution is negligible in the limit $N \to \infty$, and adding an interaction term between the first and last particles.

2. General solution

We shall consider quantum many-body Hamiltonians of the form

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_N)\,, \tag{2.1}$$

where the potential

$$V(\mathbf{x}) = \sum_i V_1(x_i) + \sum_i V_2(x_i - x_{i+1}) + \sum_i V_3(x_i - x_{i+1}, x_{i+1} - x_{i+2}) \tag{2.2}$$

features at most three-body near-neighbors translation invariant interactions. We shall assume that the particles move either on a circle or on the real line. In the first case the coordinates x_i are typically angular variables, and the particles 1 and N are considered to be nearest neighbors. In particular, in this case all sums and products will be assumed to run from 1 to N, with the identifications $x_{N+k} \equiv x_k$ for all $k \in \mathbb{Z}$. On the other hand, when the particles move on the real line the coordinates x_i are unbounded, and we shall not make the latter identifications. Thus in this case all sums and products will be taken to run over the largest meaningful range between 1 and N. For instance, in this case

$$\sum_i V_1(x_i) \equiv \sum_{i=1}^{N} V_1(x_i)\,, \quad \sum_i V_2(x_i - x_{i+1}) \equiv \sum_{i=1}^{N-1} V_2(x_i - x_{i+1})\,,$$

and, in general,

$$\sum_i f(x_i, x_{i+1}, \ldots, x_{i+k}) \equiv \sum_{i=1}^{N-k} f(x_i, x_{i+1}, \ldots, x_{i+k})\,. \tag{2.3}$$

With this convention, we will be able to present the results for the cases of motion on the circle or on the real line in a unified way.

Our goal is to classify the many-body potentials of the form (2.2), both on a circle and on the real line, for which H admits a ground state $\psi(\mathbf{x})$ of the form

$$\psi(\mathbf{x}) = \prod_i \rho(x_i) \cdot \prod_i \chi(x_i - x_{i+1})\,. \tag{2.4}$$

Here ρ and χ are two functions of one variable such that ψ is square-integrable. In other words (cf. Eq. (1.1)), the wave function (2.4) factorizes over the (positive) *simple* roots of the A_{N-1} root system. We shall say that such a wave function is *Jastrow-like*, by analogy with the usual Jastrow-type form (1.1). We will show that there is no solution featuring only two-body interactions, while the general solution with three-body interactions depends on a constant and an arbitrary function of one variable. The latter solution includes the well-known (rational and trigonometric) potentials of Refs. [41,43,44,47,48] and their hyperbolic counterpart, as well as a new elliptic potential which encompasses the previously known ones.

Imposing that $\psi(\mathbf{x})$ in Eq. (2.4) be an eigenfunction of the Hamiltonian (2.1) with energy E we readily obtain

$$\begin{aligned} V(\mathbf{x}) - E \;&= \sum_i \big(\tau^2(x_i) + \tau'(x_i)\big) + 2\sum_i \big(\varphi^2(x_i - x_{i+1}) + \varphi'(x_i - x_{i+1})\big) \\ &\quad + 2\sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big) - 2\sum_i \varphi(x_i - x_{i+1})\varphi(x_{i+1} - x_{i+2}), \end{aligned} \tag{2.5}$$

where the one-variable functions τ and φ are by definition the logarithmic derivatives of ρ and χ, i.e.,

$$\tau(x) \equiv \frac{\rho'(x)}{\rho(x)}\,, \qquad \varphi(x) \equiv \frac{\chi'(x)}{\chi(x)}\,. \tag{2.6}$$

Here and in what follows we shall assume that φ' is not identically zero, since otherwise $\psi(\mathbf{x})$ is a product of one-particle states and there is no interaction between the particles. We shall also assume, for simplicity's sake, that τ and φ are meromorphic functions.

It is readily apparent from Eq. (2.5) that when $\tau'' \equiv 0$ the potential V is already of the sought-for form (2.2), both for the circle and the real line. Without loss of generality (modulo a trivial overall translation of the coordinates), we can take $\tau(x) = -\omega x$. We thus obtain the following formulas for the potential V and its Jastrow-like eigenfunction $\psi(\mathbf{x})$:

$$V(\mathbf{x}) = \omega^2 r^2 + 2\sum_i \big(\varphi'(x_i - x_{i+1}) + \varphi^2(x_i - x_{i+1})\big) - 2\omega \sum_i (x_i - x_{i+1})\varphi(x_i - x_{i+1}) - 2\sum_i \varphi(x_i - x_{i+1})\varphi(x_{i+1} - x_{i+2}), \tag{2.7}$$

$$\psi(\mathbf{x}) \propto \mathrm{e}^{-\omega r^2/2} \prod_i \chi(x_i - x_{i+1}), \quad \chi(x) \equiv \exp\Big(\int^x \varphi(s)\,\mathrm{d}s\Big), \tag{2.8}$$

with $r^2 \equiv \sum_i x_i^2$ and energy $E = N\omega$. Note that φ is an arbitrary function, except for the requirement that the Jastrow-like eigenfunction (2.8) be square-integrable and other restrictions that we shall discuss below. We emphasize that the previous result is valid both for motion on the circle and on the line, with the convention for the range of the indices in sums and products explained above. However, in the former case one should set $\omega = 0$, since the external potential $\omega^2 r^2$ is not periodic in the angular variables x_i. In addition, in both cases there are several conditions that the function φ should satisfy stemming from natural physical requirements. In the first place, the two- and three-body potentials

$$V_2(x) = 2\big(\varphi'(x) + \varphi^2(x) - \omega x\varphi(x)\big), \qquad V_3(x, y) = -2\varphi(x)\varphi(y) \tag{2.9}$$

should be even functions of their arguments, i.e., $V_2(-x) = V_2(x)$ and $V_3(-x, -y) = V_3(x, y)$. This immediately implies that $\varphi(x)$ should be an odd function of x. Furthermore, in the case of motion in the circle the boundary terms in the second and third sums in Eq. (2.2) should be consistent with the geometry of the system. For instance, the last term in the second sum, given by

$$V_2(x_N - x_{N+1}) \equiv V_2(x_N - x_1),$$

should be equal to $V_2(l + x_N - x_1)$, since $l - (x_1 - x_N)$ is the (arc) distance between the consecutive particles N and 1 on a circle of circumference l (cf. Fig. 1). This implies that V_2 should be an l-periodic function, so that

$$V_2(x) = V_2(l + x) = V_2(l - x), \tag{2.10}$$

where the last equality is a consequence of the even character of V_2. Similarly, from the last two terms in the third sum of Eq. (2.2) we obtain the relations

$$V_3(x_{N-1} - x_N, x_N - x_{N+1}) \equiv V_3(x_{N-1} - x_N, x_N - x_1) = V_3(x_{N-1} - x_N, l + x_N - x_1),$$
$$V_3(x_N - x_{N+1}, x_{N+1} - x_{N+2}) \equiv V_3(x_N - x_1, x_1 - x_2) = V_3(l + x_N - x_1, x_1 - x_2),$$

which lead to the periodicity conditions

$$V_3(x + l, y) = V_3(x, y + l) = V_3(x, y). \tag{2.11}$$

From Eq. (2.9) we easily see that conditions (2.10)–(2.11) above are equivalent to the relation

$$\varphi(x + l) = \varphi(x). \tag{2.12}$$

In summary, φ should be an odd function of its argument and, in the case of motion on a circle, l-periodic. Thus in the latter case we have

$$\varphi(l - x) = \varphi(-x) = -\varphi(x), \tag{2.13}$$

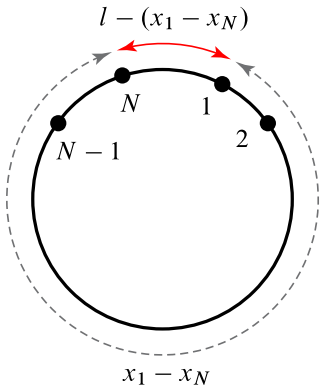

Fig. 1. Arc distance $l-(x_1-x_N)$ between the particles 1 and N on a circle of circumference l.

so that φ is odd about $l/2$. Noting that[2]

$$\chi(x) = c\exp\left(\int_{l/2}^{x}\varphi(t)\mathrm{d}t\right), \qquad 0<x<l,$$

(where c is a constant) we deduce that χ is symmetric about $l/2$ in the interval $(0, l)$, i.e.,

$$\chi(x) = \chi(l-x), \qquad 0<x<l.$$

The latter formula and the periodicity conditions (2.13) imply that

$$\chi(x) = \chi(l-x) = \chi(l+x)$$

holds everywhere (except, at most, at integers multiples of l). Thus in the case of motion on the circle χ is an even, l-periodic function. In particular, it follows from Eq. (2.8) that in this case the Jastrow-like wave function ψ is l-periodic in each variable.

To end this section, we shall next show that the solution (2.7) does not include potentials with purely two-body interactions. In other words, we must prove that the three-body term in Eq. (2.7) cannot be expressed as a sum of an external potential and a two-body term, i.e., that the equation

$$\sum_i \varphi(x_i-x_{i+1})\varphi(x_{i+1}-x_{i+2}) = \sum_i \lambda(x_i) + \sum_i F(x_i-x_{i+1}), \tag{2.14}$$

where λ, F are functions of one variable, cannot be satisfied unless φ is constant. To this end, consider first the case of motion on a circle, which is technically simpler due to the symmetry under the cyclic group. In this case, using the elementary identities

$$\sum_i \lambda(x_i) = \frac{1}{3}\sum_i\big(\lambda(x_i)+\lambda(x_{i+1})+\lambda(x_{i+2})\big), \tag{2.15}$$

$$\sum_i F(x_i-x_{i+1}) = \frac{1}{2}\sum_i\big(F(x_i-x_{i+1})+F(x_{i+1}-x_{i+2})\big), \tag{2.16}$$

and calling $(x_i, x_{i+1}, x_{i+2}) \equiv (x, y, z)$ we deduce that Eq. (2.14) is equivalent to the functional equation

$$\frac{1}{3}\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + \frac{1}{2}\big(F(x-y)+F(y-z)\big) = \varphi(x-y)\varphi(y-z). \tag{2.17}$$

[2] We cannot represent χ by the following formula outside the open interval $(0, l)$, since φ is typically singular at $x=0$, and hence at integer multiples of the period l. We shall assume in the following discussion that φ has no other singularities.

In terms of the independent variables x, $u \equiv x - y$, $v \equiv y - z$, the latter equation can be written as

$$\frac{1}{3}\big(\lambda(x) + \lambda(x-u) + \lambda(x-u-v)\big) + \frac{1}{2}\big(F(u) + F(v)\big) = \varphi(u)\varphi(v)\,. \tag{2.18}$$

Differentiating with respect to x and setting $u = v = 0$ we deduce that $\lambda = \lambda_0$ is constant, so that

$$\varphi(u)\varphi(v) = \frac{1}{2}\big(F(u) + F(v)\big) + \lambda_0,$$

and hence

$$\varphi(u)^2 = F(u) + \lambda_0\,.$$

Substituting back in the previous equation we conclude that $\varphi(u) - \varphi(v)$ must be constant. This implies that φ itself is constant, which is excluded. In the case of motion on the line, Eq. (2.14) still holds (with the convention (2.3) for the summation range), but the identities (2.15)–(2.16) should be replaced by

$$\sum_i \lambda(x_i) = \frac{1}{3}\sum_i \big(\lambda(x_i) + \lambda(x_{i+1}) + \lambda(x_{i+2})\big) + \frac{1}{3}\big(2\lambda(x_1) + \lambda(x_2) + \lambda(x_{N-1}) + 2\lambda(x_N)\big), \tag{2.19}$$

$$\sum_i F(x_i - x_{i+1}) = \frac{1}{2}\sum_i \big(F(x_i - x_{i+1}) + F(x_{i+1} - x_{i+2})\big) + \frac{1}{2}\big(F(x_1 - x_2) + F(x_{N-1} - x_N)\big). \tag{2.20}$$

Consequently, in this case Eq. (2.14) is equivalent to Eq. (2.17) or Eq. (2.18), together with the following two relations coming from the boundary terms in Eqs. (2.19)–(2.20):

$$\frac{1}{3}\big(2\lambda(x) + \lambda(y)\big) + \frac{1}{2}F(x-y) = -\frac{1}{3}\big(\lambda(x) + 2\lambda(y)\big) - \frac{1}{2}F(x-y) = c\,,$$

where c is a constant. Since we have just seen that Eq. (2.18) cannot be satisfied unless φ is a constant, we conclude that there are no potentials of the form (2.7) with only two-body interactions also in the case of motion on the line.

3. Uniqueness

We shall show in this section that (2.7) is the most general potential of the form (2.2) admitting a Jastrow-like eigenfunction (2.4). Together with the result at the end of the previous section, this implies that no potential of the form (2.2) with only two-body interactions admits a Jastrow-like eigenfunction (2.4). For clarity's sake, we shall deal separately with the case of motion on a circle and on the real line.

3.1. Motion on a circle

To begin with, note that in this case we can express the sum of a two- and a three-body term as a pure three-body term, namely

$$\sum_i F_2(x_i - x_{i+1}) + \sum_i F_3(x_i - x_{i+1}, x_{i+1} - x_{i+2}) = \sum_i F(x_i - x_{i+1}, x_{i+1} - x_{i+2})\,, \tag{3.1}$$

with $F(x,y) = F_3(x,y) + (F_2(x) + F_2(y))/2$. Hence the RHS of Eq. (2.5) will be of the form (2.2) provided that there exist a function λ of one variable and a function F of two variables such that

$$\sum_i \lambda(x_i) + \sum_i F(x_i - x_{i+1}, x_{i+1} - x_{i+2}) = 2\sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big). \tag{3.2}$$

Taking into account Eq. (2.15) and the analogous identity

$$2\sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big)$$
$$= \sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big) + \sum_i \varphi(x_{i+1} - x_{i+2})\big(\tau(x_{i+1}) - \tau(x_{i+2})\big),$$

and calling again $(x_i, x_{i+1}, x_{i+2}) \equiv (x, y, z)$, we arrive at the functional equation

$$\frac{1}{3}\big(\lambda(x) + \lambda(y) + \lambda(z)\big) + F(x - y, y - z)$$
$$= \varphi(x - y)\big(\tau(x) - \tau(y)\big) + \varphi(y - z)\big(\tau(y) - \tau(z)\big). \tag{3.3}$$

Equivalently, setting $u \equiv x - y$ and $v \equiv y - z$ we can rewrite the latter equation as

$$L(x, u, v) = F(u, v)\,, \tag{3.4}$$

where the function $L(x, u, v)$ is defined as

$$L(x, u, v) \equiv \varphi(u)\big(\tau(x) - \tau(x - u)\big) + \varphi(v)\big(\tau(x - u) - \tau(x - u - v)\big)$$
$$- \frac{1}{3}\big(\lambda(x) + \lambda(x - u) + \lambda(x - u - v)\big). \tag{3.5}$$

In particular, from the last two equations it easily follows that λ and F are meromorphic functions of their arguments.

We shall now show that Eq. (3.4) implies that $\tau'' \equiv 0$, which, as explained in the previous section, yields the potential (2.7). The key idea in our proof is to note that by Eq. (3.4) the partial derivative of $L(x, u, v)$ with respect to x must vanish identically, i.e.,

$$\frac{1}{3}\big(\lambda'(x) + \lambda'(x - u) + \lambda'(x - u - v)\big)$$
$$= \varphi(u)\big(\tau'(x) - \tau'(x - u)\big) + \varphi(v)\big(\tau'(x - u) - \tau'(x - u - v)\big). \tag{3.6}$$

Letting $v \to -u$ in the latter equation and taking into account the odd character of φ we obtain

$$2\lambda'(x) + \lambda'(x - u) = 6\,\varphi(u)\big(\tau'(x) - \tau'(x - u)\big).$$

If $\tau''(x) \not\equiv 0$ we can solve for $\varphi(u)$ in the latter equation, with the result

$$6\,\varphi(u) = \frac{2\lambda'(x) + \lambda'(x - u)}{\tau'(x) - \tau'(x - u)}\,. \tag{3.7}$$

Expanding the RHS of this equality in a Laurent series around $u = 0$ we readily obtain

$$6\,\varphi(u) = \frac{3\lambda'(x)}{\tau''(x)u} + \frac{3\lambda'(x)\tau'''(x)}{2\tau''(x)^2} - \frac{\lambda''(x)}{\tau''(x)} + \mathrm{O}(u)\,.$$

The coefficient of $1/u$ in the latter equation must be a constant 6α, while that of u^0 must vanish on account of the odd character of φ. We thus deduce that

$$\lambda'(x) = 2\alpha\tau''(x)\,, \qquad \alpha\tau'''(x) = 0\,.$$

From the latter equations it also follows that $\alpha \neq 0$, since otherwise φ would vanish identically on account of Eq. (3.7). Hence $\tau''' = 0$, so that can write

$$\tau(x) = \tau_0 + \tau_1 x + \tau_2 x^2\,, \qquad \lambda(x) = \lambda_0 + 4\alpha\tau_2 x\,,$$

with λ_0, τ_i constant and $\tau_2 \neq 0$. Substituting into Eq. (3.7) we easily obtain

$$\varphi(u) = \frac{\alpha}{u}\,.$$

However, this solution is not acceptable, since it does not satisfy the periodicity condition (2.12) that should hold in this case. We thus conclude that in the case of motion on the circle there is no solution of the problem posed with $\tau'' \not\equiv 0$, as claimed.

3.2. Motion on the real line

We shall next discuss the case of motion on the real line, in which the coordinates x_i are unbounded. To begin with, in this case the identity (3.1) should be replaced by

$$\sum_i F_2(x_i - x_{i+1}) + \sum_i F_3(x_i - x_{i+1}, x_{i+1} - x_{i+2})$$
$$= \sum_i F(x_i - x_{i+1}, x_{i+1} - x_{i+2}) + \frac{1}{2}\big(F_2(x_1 - x_2) + F_2(x_{N-1} - x_N)\big), \tag{3.8}$$

where as before $F(x, y) = F_3(x, y) + (F_2(x) + F_2(y))/2$, and we are using the convention (2.3) on the range of summation indices. Consequently, Eq. (3.2) now reads

$$\sum_i \lambda(x_i) + \sum_i F(x_i - x_{i+1}, x_{i+1} - x_{i+2}) + G(x_1 - x_2) + G(x_{N-1} - x_N)$$
$$= 2\sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big), \tag{3.9}$$

where λ, G are functions of one variable and F is a function of two variables. Using Eq. (2.19) and the identity

$$2\sum_i \varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big)$$
$$= \varphi(x_1 - x_2)\big(\tau(x_1) - \tau(x_2)\big) + \varphi(x_{N-1} - x_N)\big(\tau(x_{N-1}) - \tau(x_N)\big)$$
$$+ \sum_i \Big[\varphi(x_i - x_{i+1})\big(\tau(x_i) - \tau(x_{i+1})\big) + \varphi(x_{i+1} - x_{i+2})\big(\tau(x_{i+1}) - \tau(x_{i+2})\big)\Big]$$

in Eq. (3.9) we readily obtain Eq. (3.3), or equivalently (3.4)–(3.5), plus the additional constraints

$$\big(\tau(x) - \tau(y)\big)\varphi(x - y) = \frac{1}{3}\big(2\lambda(x) + \lambda(y)\big) + G(x - y) + c$$
$$= \frac{1}{3}\big(\lambda(x) + 2\lambda(y)\big) + G(x - y) - c\,, \tag{3.10}$$

where c is a constant. From the latter equations we immediately deduce that $\lambda = \lambda_0$ must be a constant. This in turn implies that $\tau(x) - \tau(y)$ is a function of $x - y$, so that τ must be linear in x. Thus also in this case there is no solution with $\tau'' \not\equiv 0$, as claimed.

4. Ground state conditions

We shall show in this section that the Jastrow-like eigenfunction (2.8) is actually the ground state of the potential (2.7), provided that the function φ satisfy very general assumptions that we shall now discuss.

We shall start our discussion with the case of motion on the line. First of all, it is natural on physical grounds to require that φ be analytic everywhere except at the origin, so that the only singularities of the potential (2.7) are located on the hyperplanes $x_i = x_{i+1}$. For simplicity, we shall further assume that φ has a simple pole at the origin, i.e.,

$$\varphi(x) = \frac{\alpha}{x} + \varphi_0(x)\,,$$

with $\alpha \neq 0$ and φ_0 analytic on the real line. The wave function (2.8) can thus be written as

$$\psi(\mathbf{x}) \propto \mathrm{e}^{-\omega r^2/2} \prod_i |x_i - x_{i+1}|^\alpha \cdot \prod_i \mathrm{e}^{\Phi(x_i - x_{i+1})}\,, \tag{4.1}$$

with $\Phi(x) = \int^x \varphi_0(s)\mathrm{d}s$ analytic on the real line. Note that we must have $\alpha > 1/2$, to ensure that the expected value of the kinetic energy of the eigenstate ψ be finite. This automatically guarantees the

square integrability of ψ near the singular hyperplanes $x_i = x_{i+1}$. Moreover, since the potential (2.7) diverges near these hyperplanes as $\alpha(\alpha-1)(x_i - x_{i+1})^{-2}$, if $\alpha \neq 1$ the particles cannot overtake each other [1,49]. Thus we can fix the ordering of the particles as, e.g., $x_1 > \cdots > x_N$, which amounts to taking the configuration space of the system as the open set

$$A = \{\mathbf{x} \in \mathbb{R}^N \mid x_1 > \cdots > x_N\}.$$

It is then clear that ψ does not vanish on A by Eq. (4.1). Hence to show that ψ is indeed the ground state of H it suffices to verify that it is square-integrable at infinity. To this end, we need only impose that[3] $\omega > 0$ and $\Phi(x) \leqslant cx^2$ with $c < \omega/8$. Indeed, if this is the case we have

$$\mathrm{e}^{-\omega r^2/2} \prod_i \mathrm{e}^{\Phi(x_i - x_{i+1})} \leqslant \exp\Big(-\tfrac{1}{2}\,\omega r^2 + c \sum_i (x_i - x_{i+1})^2\Big) \equiv \mathrm{e}^{-\frac{1}{2}\sum_{i,j} b_{ij} x_i x_j}\,, \tag{4.2}$$

where $B \equiv (b_{ij})_{1 \leqslant i,j \leqslant N}$ is the circulant matrix with first row $(\omega - 4c, 2c, \ldots, 2c)$ (the dots standing for zeros) [50]. Since the eigenvalues of B, given by[4]

$$\lambda_j = \omega - 8c \sin^2(j\pi/N)\,, \qquad j = 0, \ldots, N-1\,,$$

are all positive on account of the condition $\omega > 8c$, the associated quadratic form is positive definite. By the inequality (4.2), this implies that the eigenfunction (4.1) is square-integrable at infinity.

In the case of motion on a circle, we shall again require that φ have a simple pole at the origin with residue $\alpha \neq 0$. By the periodicity condition (2.12), φ must have simple poles with residue α at integer multiples of the circle's circumference l, so that

$$\varphi(x) = \alpha\left(\frac{1}{x} + \frac{1}{x-l}\right) + \varphi_0(x),$$

with φ_0 analytic on the interval $[0, l]$. The Jastrow-like eigenfunction (2.4) (with $\omega = 0$) is then given by

$$\psi(\mathbf{x}) \propto \prod_i \big|(x_i - x_{i+1})(l - x_i + x_{i+1})\big|^{\alpha} \cdot \prod_i \mathrm{e}^{\Phi(x_i - x_{i+1})}, \tag{4.3}$$

where $\Phi(x) = \int^x \varphi_0(s)\mathrm{d}s$ is analytic on the interval $[0, l]$. As before, the square integrability of the eigenfunction ψ at $x_i - x_{i+1} = kl$ (with $k \in \mathbb{Z}$) and the finiteness of the average kinetic energy require that $\alpha > 1/2$. Furthermore, the potential (2.7) diverges near the singular hyperplanes $x_i - x_{i+1} = kl$ as $\alpha(\alpha-1)(x_i - x_{i+1} - kl)^{-2}$. Hence if $\alpha \neq 1$ the particles cannot overtake each other, and the system's configuration space can thus be taken as the open set

$$A = \{\mathbf{x} \in \mathbb{R}^N \mid x_1 > \cdots > x_N > x_1 - l\}\,. \tag{4.4}$$

The Jastrow-like eigenfunction ψ in Eq. (4.3) is square-integrable on A, since the potential is translation-invariant and we can therefore regard the differences $x_i - x_{i+1}$, which range over the bounded interval $(0, l)$, as independent variables after separating the center of mass motion. Moreover, ψ does not vanish on the configuration space (4.4), and is thus again the system's ground state.

5. Examples

As we have seen in the previous sections, the most general potential of the form (2.2) admitting a Jastrow-like eigenfunction (2.8) depends on an essentially arbitrary function φ of one variable and, in the case of motion on the line, an additional constant ω. In particular, choosing φ appropriately one should be able to recover all the potentials of the form (2.2) previously proposed in the literature, as

[3] If $\omega = 0$, the potential (2.7) is translation-invariant, so that the total momentum is conserved. In this case we can separate the center of mass motion and regard the differences $x_i - x_{i+1}$, $1 \leqslant i \leqslant N-1$, as independent variables. Consequently, the eigenfunction (2.4) with $\omega = 0$ will be square-integrable at infinity provided that $\Phi(x) \leqslant -c_0 \log|x| + c_1$ with $c_0 > 1$.

[4] For $N = 2$, the eigenvalues of B are $\omega - 2c$ and $\omega - 6c$.

well as several interesting generalizations thereof. Thus, if $\varphi(x) = \alpha/x$, from Eqs. (2.7) and (2.8) we obtain the rational potential introduced in Ref. [41], namely

$$V(\mathbf{x}) = \omega^2 r^2 + \sum_i \frac{2\alpha(\alpha-1)}{(x_i - x_{i+1})^2} - \sum_i \frac{2\alpha^2}{(x_i - x_{i+1})(x_{i+1} - x_{i+2})},$$
$$\psi(\mathbf{x}) \propto \mathrm{e}^{-\omega r^2/2} \prod_i |x_i - x_{i+1}|^\alpha, \qquad E = N\omega + 2(N-1)\alpha\omega.$$

Although the previous formulas for V and ψ are *formally* valid both for the circle and the real line, as discussed in Section 2 the above potential has a natural physical interpretation only in the latter case.[5]

Similarly, the choice $\varphi(x) = (\pi\alpha/l)\cot(\pi x/l)$ with $\omega = 0$ leads to the trigonometric potential [41]

$$V(\mathbf{x}) = \left(\tfrac{\pi}{l}\right)^2 \sum_i \frac{2\alpha(\alpha-1)}{\sin^2\left(\frac{\pi}{l}(x_i - x_{i+1})\right)} - 2\left(\tfrac{\pi\alpha}{l}\right)^2 \sum_i \cot\left(\tfrac{\pi}{l}(x_i - x_{i+1})\right)\cot\left(\tfrac{\pi}{l}(x_{i+1} - x_{i+2})\right),$$
$$\psi(\mathbf{x}) \propto \prod_i \left|\sin\left(\tfrac{\pi}{l}(x_i - x_{i+1})\right)\right|^\alpha, \qquad E = 2N\left(\frac{\pi\alpha}{l}\right)^2.$$

The natural interpretation of this model is on a circle of radius $l/(2\pi)$.

A hyperbolic version of the previous potential is easily obtained by taking $l = \mathrm{i}\pi/\beta$ (with $\beta > 0$) in the previous formula for $\varphi(x)$. We thus obtain $\varphi(x) = \alpha\beta\coth(\beta x)$ and

$$V(\mathbf{x}) = \omega^2 r^2 - 2\alpha\beta\omega \sum_i (x_i - x_{i+1})\coth\left(\beta(x_i - x_{i+1})\right) + \sum_i \frac{2\alpha(\alpha-1)\beta^2}{\sinh^2\left(\beta(x_i - x_{i+1})\right)}$$
$$- 2\alpha^2\beta^2 \sum_i \coth\left(\beta(x_i - x_{i+1})\right)\coth\left(\beta(x_{i+1} - x_{i+2})\right), \tag{5.1}$$
$$\psi(\mathbf{x}) \propto \mathrm{e}^{-\omega r^2/2} \prod_i \left|\sinh\left(\beta(x_i - x_{i+1})\right)\right|^\alpha, \qquad E = N\omega - 2(N-1)\alpha^2\beta^2 \tag{5.2}$$

(cf. Fig. 2). Note that we have taken $\omega > 0$, since the latter potential has a natural physical interpretation only on the line, and the term $\mathrm{e}^{-\omega r^2/2}$ in the expression for ψ is therefore needed to guarantee its square integrability. The hyperbolic model (5.1) can be regarded as the near-neighbors analog of the long-range model of CS type introduced by Forrester [30]. The ground state of the latter model, which is similar to (5.2) but is factorized over the whole A_{N-1} root system, was shown by Forrester to describe a Wigner solid in the thermodynamic limit.

In the classification of long-range interaction potentials with two-body interactions and Jastrow-type ground state performed in Refs. [28,29,31], the rational, trigonometric and hyperbolic solutions are obtained precisely from the three choices of the function φ used in the previous examples. In fact, in this case there is an additional solution given by

$$\varphi(x) = \alpha\zeta(x) + \gamma x, \tag{5.3}$$

where $\gamma \in \mathbb{R}$ and $\zeta(x) \equiv \zeta(x; g_2, g_3)$ is the Weierstrass zeta function with invariants g_2 and g_3 [54], which yields the above three choices of φ as particular cases on account of the identities

$$\begin{cases} \zeta(x; \frac{4}{3}\beta^4, \frac{8}{27}\beta^6) - \dfrac{\beta^2 x}{3} = \beta\cot(\beta x), \\ \zeta(x; 0, 0) = \dfrac{1}{x}, \\ \zeta(x; \frac{4}{3}\beta^4, -\frac{8}{27}\beta^6) + \dfrac{\beta^2 x}{3} = \beta\coth(\beta x). \end{cases}$$

[5] Note, however, that the variant of the latter potential with cyclic symmetry and its spin version have been used in Ref. [51] to construct an analog of the Polychronakos–Frahm spin chain [52,53] with nearest-neighbors interactions, whose first few eigenvalues can be computed in closed form.

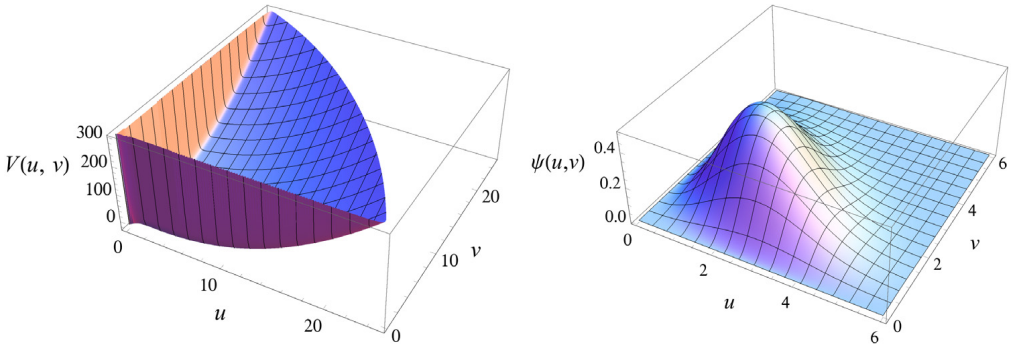

Fig. 2. Hyperbolic potential (5.1) (left) and its normalized Jastrow-like eigenfunction (5.2) (right) for $N = 3, \alpha = 2, \omega = \beta = 1$ in the center of mass frame $x_1 + x_2 + x_3 = 0$ as a function of the relative coordinates $u \equiv x_1 - x_2$, $v \equiv x_2 - x_3$.

It is therefore natural to consider the potential (2.7) generated by the function φ in Eq. (5.3). We shall assume that the invariants $g_{2,3}$ are real and satisfy the condition $g_2^3 > 27g_3^2$, so that ζ is real for real values of its argument and the corresponding Weierstrass function $\wp(z) \equiv -\zeta'(z)$ has a real fundamental period $l < \infty$ and a purely imaginary one ω_3 (with $\operatorname{Im}\omega_3 > 0$). Since the ζ function has simple poles at integer multiples of these periods, the corresponding potential (2.7) is naturally defined on a circle of circumference l. As explained in Section 2, this requires that $\omega = 0$ and that φ be an l-periodic function. In view of the identity

$$\zeta(z + l) = \zeta(z) + 2\eta_1,$$

where $\eta_1 \equiv \zeta(l/2)$, the latter condition will be satisfied if and only if $\gamma = -2\alpha\eta_1/l$. We are thus led to consider the choice

$$\varphi(x) = \alpha\left(\zeta(x) - \frac{2\eta_1}{l}x\right), \tag{5.4}$$

whose associated potential is given by

$$\begin{aligned} V(\mathbf{x}) \;&= -2\alpha\sum_i \wp(x_i - x_{i+1}) + 2\sum_i \varphi(x_i - x_{i+1})^2 \\ &\quad - 2\sum_i \varphi(x_i - x_{i+1})\varphi(x_{i+1} - x_{i+2}) \end{aligned} \tag{5.5}$$

with $\alpha > 1/2$. The corresponding Jastrow-like eigenfunction and energy read

$$\psi(\mathbf{x}) \propto \exp\Big(-\frac{\alpha\eta_1}{l}\sum_i (x_i - x_{i+1})^2\Big)\prod_i |\sigma(x_i - x_{i+1})|^\alpha\,, \qquad E = \frac{4N\eta_1\alpha}{l}, \tag{5.6}$$

where the Weierstrass σ function is defined by $\sigma'/\sigma = \zeta$ and $\lim_{z\to 0}\sigma(z)/z = 1$ (see Fig. 3 for a plot of the potential (5.5) and its Jastrow-like eigenfunction (5.6) for $N = 3$ particles when $\alpha = 2, l = 1$ and $\operatorname{Im}\omega_3 = 1/2$). Recall that σ is entire and odd, and it vanishes only at the periods of $\wp$, so that in particular $\sigma(kl) = 0$ for all $k \in \mathbb{Z}$. Hence ψ does not vanish on the configuration space (4.4), and is therefore the system's ground state. We also know from the general discussion of Section 2 (and is also obvious from the l-periodicity of $\wp$ and φ) that the two-body potential

$$V_2(x) = 2\big(-\alpha\wp(x) + \varphi(x)^2\big) \tag{5.7}$$

is l-periodic and symmetric about $l/2$ (cf. Eq. (2.10)), and that the Jastrow-like eigenfunction (5.6) is also l-periodic in each of its variables. The latter fact can also be checked directly with the help of the identity

$$\sigma(z + l) = -\mathrm{e}^{2\eta_1(z+\frac{l}{2})}\sigma(z)\,.$$

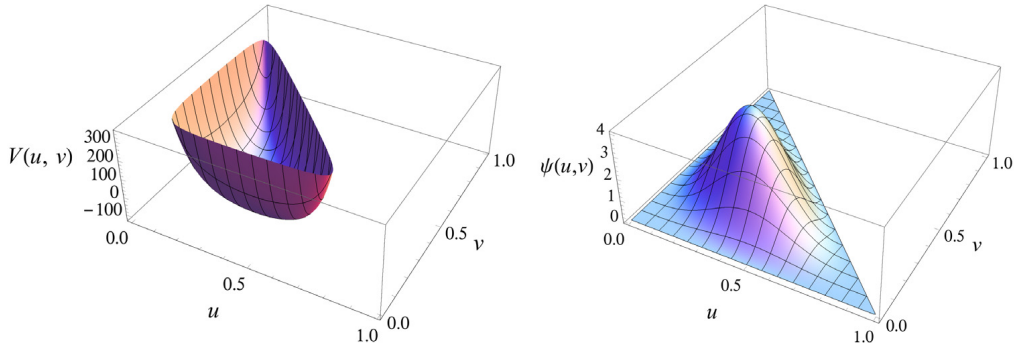

Fig. 3. Elliptic potential (5.5) (left) and its normalized Jastrow-like eigenfunction (5.6) (right) for $N = 3$, $\alpha = 2$, $l = 1$ and $\mathrm{Im}\,\omega_3 = 1/2$ as a function of the relative coordinates $u \equiv x_1 - x_2$, $v \equiv x_2 - x_3$.

In Fig. 4 we present a plot of V_2 and χ for $\alpha = 2$, $l = 1$ and several values of $\mathrm{Im}\,\omega_3$ in the half-period $0 < x < 1/2$.

The potential (5.5) depends on three real parameters, namely $\alpha > 1/2$, $\mathrm{Im}\,\omega_3 > 0$, and $l > 0$. Note, however, that from the well known identities

$$\wp(\mu x|\mu l/2, \mu\omega_3) = \frac{1}{\mu^2}\wp(x), \quad \zeta(\mu x|\mu l/2, \mu\omega_3) = \frac{1}{\mu}\zeta(x)$$

(where $f(z|\mu l/2, \mu\omega_3)$ denotes the corresponding Weierstrass function f with periods μl and $2\mu\omega_3$) it easily follows that either $\mathrm{Im}\,\omega_3$ or l can be rescaled to (say) 1 by an appropriate overall dilation of the coordinates. Note also that when $\mathrm{Im}\,\omega_3 \to \infty$ we have

$$\wp(x) \to \frac{\pi^2}{l^2}\left(\sin^{-2}\left(\tfrac{\pi x}{l}\right) - \frac{1}{3}\right), \quad \zeta(x) \to \frac{\pi^2 x}{3l^2} + \frac{\pi}{l}\cot\left(\tfrac{\pi x}{l}\right), \quad \eta_1 \to \frac{\pi^2}{6l}$$

(see, e.g., Refs. [55,56]), and consequently

$$\varphi(x) \to \frac{\alpha\pi}{l}\cot\left(\tfrac{\pi x}{l}\right), \qquad V_2(x) \to \frac{2\alpha(\alpha-1)}{\sin^2\left(\frac{\pi x}{l}\right)} - 2\left(\frac{\pi}{l}\right)^2\alpha\left(\alpha - \frac{1}{3}\right)$$

(cf. Fig. 4). From these equations it readily follows that as $\mathrm{Im}\,\omega_3 \to \infty$ the potential (5.5) tends to

$$V_{\mathrm{trig}}(\mathbf{x}) - 2\left(\frac{\pi}{l}\right)^2 N\alpha\left(\alpha - \frac{1}{3}\right),$$

where $V_{\mathrm{trig}}(\mathbf{x})$ is the trigonometric potential in Ref. [41] discussed above.

6. Summary and outlook

In this paper we completely solve the problem of classifying all one-dimensional quantum Hamiltonians with nearest- and next-to-nearest-neighbors (translation invariant) interactions admitting a Jastrow-like ground state, both for motion on the real line and on a circle. This is the simplest near-neighbors analog of the well-known problem for Calogero–Sutherland models with long-range interactions proposed shortly after their introduction and completely solved in Ref. [31]. Our solution differs in two fundamental ways with its long-range counterpart. In the first place, we show that the potential must necessarily contain a three-body interaction term, which by construction is absent in the long-range solution. Secondly, the near-neighbors solution depends on an essentially arbitrary function of one variable (and, for motion on the line, on an additional positive parameter). The general solution contains a potential featuring elliptic interactions, which yields the (rational and trigonometric) particular solutions considered so far [41] as limiting cases.

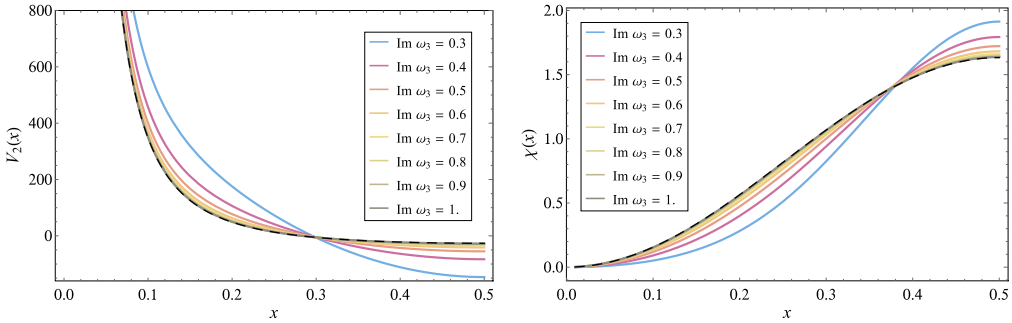

Fig. 4. Left: elliptic two-body potential (5.7) with $l = 1$, $\alpha = 2$ and several values of $\mathrm{Im}\,\omega_3$, compared to its limiting trigonometric potential $4\sin^{-2}(\pi x) - 20\pi^2/3$ (dashed black line). Right: analogous plot for the corresponding functions $\chi(x)$ determining the Jastrow-like eigenfunction (2.8) (with the normalization $\int_0^1 |\chi(x)|^2 dx = 1$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Our results suggest several lines of work for further research. To begin with, it would certainly be of interest to study in detail the potentials contained in the general solution, and in particular determine whether one can exactly compute other eigenfunctions besides the ground state. This is known to be true for the previously known rational and trigonometric models, and it would therefore be very natural to verify if it is also the case for the more general elliptic potential introduced in Section 5 or its hyperbolic limit. Another possible line for future research is the construction and analysis of the spin versions of the near-neighbors models considered (see Refs. [43,44] for the rational and trigonometric models), and their associated short-range spin chains (as was done in Ref. [51] for the rational model). Similarly, it would be of interest to study the extension of our results to more general Jastrow-like ground states depending on differences $x_i - x_{i+k}$ with k less than a fixed range $r > 1$ (see, e.g., Refs. [45,46]), as well as to ground states factorized over other root systems like BC_N [47,48]. Finally, another topic worth investigating is the explicit computation of the correlation functions of the eigenvalue probability densities given by the Jastrow-like ground states considered in this paper, like, e.g., the elliptic wave function in Eq. (5.6). This can be done in principle with the techniques of Refs. [57,58], although the evaluation of the resulting integrals could be far from trivial in this case. In fact, the analogous problem for the density (2.8) with $\omega = 0$ and $\chi(x) = |x|^{\beta/2}$ has already been solved in Ref. [39]. The corresponding distribution of the spacings of consecutive eigenvalues has been shown in the latter reference to be a good approximation to this statistic for certain pseudo-integrable billiards and for the Anderson model at the transition point.

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CHAPTER 5

New algebraic structures for entanglement detection: formal rings and entanglement monotones

Based on formal group theory, new quantum information measures can be constructed that generalize the notion of logarithmic negativity [**P9**]. It can be shown that the family of generalized negativity functions we present possess certain algebraic properties that make them suitable for studying entanglement in many-body systems. In particular, our measures are quantum monotones in a precise sense, as we shall explain in the forthcoming sections.

1. Motivation

Given any entropy functional and a composed system, one can define a measure of the degree of entanglement for pure states between a subsystem and its complement, called the entanglement entropy. It is defined as the entropy functional applied to the distribution of eigenvalues of the reduced density matrix in one of the subsystems.

It can be shown, and it was discussed in detail in [**P4**], that the low-energy behaviour of quantum Hamiltonians (14) can not be described by a CFT even though they present the typical asymptotic behaviour of the entanglement entropy, which is proportional to $\log L$ where $L \gg 1$ is the size of the subsystem considered. From an operational point of view, this can be thought of as a consequence of the fact that for CFTs one is able to compute the trace $\operatorname{tr} \rho^n$ of any power $n = 1, 2, \ldots$ of the reduced density matrix ρ of a subsystem of length L when the whole system of length $N > L$ is in a pure state. Indeed, the characteristic behaviour for a CFT is of the form $\operatorname{tr} \rho^n = L^{f(n)}$ for a suitable function $f(n)$. Note that Rényi and Tsallis entanglement entropy are respectively $R_n = c(1-n)^{-1} f(n) \log L$ and $T_n = (1-n)^{-1}(\mathrm{e}^{nR_{1-n}} - 1)$ for CFTs. Consequently, von Neumann–Shannon entanglement entropy $S = -\operatorname{tr} \rho \log \rho$ can be obtained only when

$\lim_{n\to 1}(1-n)^{-1}f(n)$ exists, which is actually the case since for CFTs $f(n) = c(1-n^2)/(6n)$. Note that if one computes these quantities via the so-called replica trick for a (positive) integer n, additional hypotheses are needed like the existence of the analytic continuation of the functional considered. Equations (17) show that once having diagonalized the reduced density matrix, in the case of models (14) the trace of any power (not necessarily neither an integer nor even a real number) of the reduced density matrix can be explicitly computed. The asymptotic behaviour of the von Neumann entanglement entropy of models (14) with reduced density matrix given in Eqs. (17) is exactly that of a CFT. However, for a CFT the Rényi (equivalently Tsallis) entanglement entropy behaves like $\propto c(1+n)/(6n)\log L$ in contrast to the models considered in which one finds also a behaviour proportional to $\log L$ but with a prefactor independent of n.

Both the Rényi and Tsallis entropy functionals can detect criticality in situations where the von Neumann–Shannon can not. The reason for this is that not only $\operatorname{tr}\rho\log\rho$ but also $\operatorname{tr}\rho^n$ can be computed for a CFT where ρ represents the reduced density matrix in a subsystem when the whole system is in a pure state. In a sense, the Rényi and Tsallis entropy functionals are uni-parametric generalizations of the von Neumann–Shannon one: the latter can be recovered in a certain limit. Suppose one finds a multi-parametric quantity which is a function of the reduced density matrix that can be computed for a CFT. Then, using this quantity one could try to seek for a multi-parametric entropy functional which can be used as a more refined criterion to detect criticality since the extra parameters could help discriminating further situations where even Rényi entropy fails to be a good entanglement detector.

Keeping on mind the previous discussion as a motivation, we concentrate on mixed states due to the fact that, in this case, multi-parametric measures of entanglement are much less common than for pure states.

General mixed states

We shall focus on mixed states and try to obtain entanglement information measures generalizing the logarithmic negativity since, as von Neumann–Shannon entanglement entropy for pure states, it does not depend on any free parameter. It vanishes over states with positive partial transpose (PPT) and does not increase on average under probabilistic completely positive (CP) PPT operations, a set that includes LOCC. This function,

according to Peres criterion, provides a necessary and sufficient condition for separability of mixed states in low-dimensional systems. Also, it represents an upper bound for quantum distillability. In contrast to the case of pure states, where many entanglement entropies generalize that of von Neumann–Shannon, to the best of our knowledge, there is no proper generalization of the fundamental notion of logarithmic negativity.

There are information measures that apply to general mixed states and do not come from an entropy functional evaluated at the reduced density matrix. Our main result is that there is an infinite "tower" of multi-parametric quantities satisfying the following three requirements: (a) each of them does not increase on average under CP-PPT operations; (b) for each of them, the original logarithmic negativity can be recovered for a precise choice of the parameters involved; and (c) each of them is composable.

The second requirement above can be reformulated by saying that the knowledge of the ordinary logarithmic negativity is not sufficient to reconstruct a generalized negativity, so that there could exist situations in which the generalized negativities could describe some class of universal behaviour when the ordinary logarithmic negativity cannot, in the same way as the Rényi entropy is a thinner functional with respect to the von Neumann one.

We introduce these generalized negativities with the hope that they can describe some class of universal behaviour in situations where the ordinary logarithmic negativity can not and serve as a finer discriminant in the same way as it is the Rényi entropy functional with respect to the von Neumann–Shannon one.

2. Statement of the problem

A fundamental requirement of a measure E is its non-increasing beahaviour on-average under LOCC or PPT operations [98], [70], [71]. Precisely, we assume that

$$E(\rho) \geq \sum_i p_i E(\rho_i)$$

where the stats ρ_i is obtained with probability p_i under a protocol involving LOCC (or PPT operations). Another important requirement is that E can discriminate the set of states that can be created using LOCC or PPT protocols. We shall say that an entanglement monotone is a quantity satisfying both properties.

The construction of quantum measures of entanglement analytically determined and satisfying these requirements is an interesting general problem. Important examples are provided by the negativity and generalize logarithmic negativity, introduced in the seminal paper [127]. In particular, in [97] it is shown that the logarithmic negativity is an entanglement monotone. In recent years, entanglement and logarithmic negativity have been largely investigated, due to their prominent role as measures of entanglement [97, 99] also in quantum field theory and conformal field theory [30, 31, 42, 105].

Our aim is to establish a general mathematical framework that allows us to generate a new, large class of quantum information measures that will play the role of entanglement monotones. More precisely, we will show that a wide class of generalized negativities can be defined by means of formal group theory.

3. Main proposal: an outline

In Ref. [P9] we introduce new entropic measures of entanglement based on formal group theory. Hereafter we shall briefly resume our main proposal in this direction.

3.1. Formal groups

The following results are well-known (see, e.g., Refs. [68, 111]). Let R be a commutative associative ring with identity, and $R[\![x_1, x_2, \ldots]\!]$ be the ring of formal power series in the variables $x_1, x_2, \ldots$ with coefficients in R. We shall assume that R is torsion-free. A commutative one-dimensional formal group law over R is a formal power series $\Phi(x, y) \in R[\![x, y]\!]$ such that

(1) $\Phi(x, 0) = \Phi(0, x) = x$
(2) $\Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z))$

and the formal group law is said to be commutative if $\Phi(x, y) = \Phi(y, x)$.

Observe that the existence of an inverse formal series $\varphi(x) \in R[\![x]\!]$ such that $\Phi(x, \varphi(x)) = 0$ is a consequence of the previous definition. Let $B = \mathbf{Z}[\![b_1, b_1, \ldots]\!]$ and consider the following series in $B[\![s]\!]$

$$F(s) = s + \sum_{i=1}^{\infty} b_i \frac{s^{i+1}}{i+1}\,. \tag{27a}$$

Writing $G(t)$ for its compositional inverse satisfying $F(G(t)) = t$ and $G(F(s)) = s$, one has

$$G(t) = t + \sum_{k=1}^{\infty} a_k \frac{t^{k+1}}{k+1} \tag{27b}$$

with $a_1 = -b_1$, $a_2 = \frac{3}{2}b_1^2 - b_2$ and so on. For any commutative one-dimensional formal group law $\Phi(x, y)$ over R, there exists a formal series $\phi(x) \in R[\![x]\!] \otimes \mathbf{Q}$ such that $\phi(x) = x + O(x^2)$ and

$$\Phi(x, y) = \phi^{-1}\left(\phi(x) + \phi(y)\right) \in R[\![x, y]\!] \otimes \mathbf{Q}.$$

The Lazard formal group law is defined by the formal power series

$$\Phi_{\mathrm{L}}(s_1, s_2) = G^{-1}(G(s_1) + G(s_2)).$$

The coefficients of the power series $G(G^{-1}(s_1) + G^{-1}(s_2))$ lie in the ring $B \otimes \mathbf{Q}$ and generate over $\mathbf{Z}$ a subring $L \subset B \otimes \mathbf{Q}$, called the Lazard ring. For any commutative one-dimensional formal group law over any ring R, there exists a unique homomorphism $L \to R$ under which the Lazard group law is mapped into the given group law (the universal property of the Lazard group).

3.2. Entanglement monotones and formal groups

From a technical point of view, group negativities

$$L_G^{(p)}(\rho) = \log_G \|\rho^{\Gamma}\|_p$$

are multi-parametric concave functions, depending on the p-norm $\|\cdot\|_p$ of the partial transposition ρ^{Γ} of a quantum state ρ where we have written $\log_G x = G^{-1}(\log x)$ for the generalized logarithm determined by the formal power series G of the form of Eq. (27b) above. For any matrix M its (Scatten) p-norm is defined in terms of its singular values $s_i(M)$ as $\|M\|_p = (\sum_i s_i(M)^p)^{1/p}$ and the case of standard trace norm is recovered when $p = 1$. These group negativities correspond to quantized versions of the group entropies proposed in [**119**], computed over partially transposed states. They satisfy the following three main properties:

1. For $p = 1$ and all G they are computable measures of entanglement and provide tests of separability for mixed bipartite states.
2. For $p = 1$ and all G, they are entanglement monotones.
3. They are composable: the negativity of a quantum system which is obtained by merging two independent subsystems can be computed in terms of the group negativities of its constituents only.

The previous quantities for $p > 1$ represent auxiliary measures, that also possess interesting properties and appear to be useful in the study of bounds for entanglement distillation processes where composability seems to be important. In fact, we can compute directly the behaviour of negativities on n copies of a quantum system from the knowledge of one of them.

One of our results is that, under mild conditions, the quantities introduced are entanglement monotones for $p = 1$ and quasi-monotones for $p > 1$. This means that the increasing of p-norm negativities under LOCC is bounded on average. For all $\epsilon > 0$, changing the number of particles of the system and the index p of the norm considered, one can obtain an generalized negativity such that ϵ is an upper bound for its increasing under LOCC. The example of the p-norm q-negativity is proposed. This generalized negativity depends on a real parameter q and, for large values of q, it follows that the latter upper bound goes to zero, which restores monotonocity. Explicitly, it reads

$$L^{(p)}_{G_q}(\rho) = \log_{G_q} \|\rho^\Gamma\|_p = \frac{\|\rho^\Gamma\|_p^{1-q} - 1}{1-q} \tag{28a}$$

where we have written $\log_{G_q} x = G_q^{-1}(\log x)$ and

$$G_q^{-1}(t) = \frac{\mathrm{e}^{(1-q)t} - 1}{1-q}$$

according to Eq. (27b). Since G_q defines a formal group law, one has

$$G_q \circ L^{(p)}_{G_q}(\rho \otimes \sigma) = G \circ L^{(p)}_{G_q}(\rho) + G \circ L^{(p)}_{G_q}(\sigma)$$

Accordingly, the last relation implies the following composition law:

$$L^{(p)}_{G_q}(\rho \otimes \sigma) = L^{(p)}_{G_q}(\rho) + L^{(p)}_{G_q}(\sigma) + (1-q)\, L^{(p)}_{G_q}(\rho) \cdot L^{(p)}_{G_q}(\sigma). \tag{28b}$$

From a mathematical point of view, the construction of group negativities relies on the theory of formal groups [**22**], [**68**], which represents an important branch of algebraic topology, with many applications in combinatorics and number theory (see e.g. [**118**], [**120**]). In particular, the property of composability, implies the existence of an underlying formal group law, which controls the composition process of independent subsystems.

4. Some future perspectives

As we have shown, group theory could offer a natural way to generalize the notion of negativity, and to define a new class of easily computable entanglement tests.

Several aspects of the theory deserve further analysis. We think that composability is important to study distillability of entanglement. Therefore, an interesting open problem is to ascertain whether some generalized negativity can provide upper or lower bounds to the asymptotic distillation rate, when we consider a large number of copies of the state $\rho^{\otimes n_\alpha}$.

At the same time, it would be very interesting to compute the generalized negativities suggested in this work, and in Ref. [**P9**], for some critical one-dimensional quantum systems and compare them with available results for conformal field theories at finite temperature [**31**]. From this point of view, generalized negativites (depending on an extra free parameter) could play a role similar to that played by Rényi's entropy in the case of the entanglement detecion of the ground state of one-dimensional many body systems, and in the study of their criticality properties [**34**].

NEW COMPUTABLE ENTANGLEMENT MONOTONES AND WITNESSES FROM FORMAL GROUP THEORY

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Abstract. We present a construction of new quantum information measures that generalize the notion of logarithmic negativity. Our approach is based on formal group theory. We shall prove that the family of generalized negativity functions we present are suitable for studying entanglement in many-body systems due their interesting algebraic properties.

Indeed, under mild hypotheses, the new measures are computable entanglement monotones, non-increasing under LOCC. Also, they are composable: their evaluation over tensor products can be computed in terms of the evaluations over each factor, by means of a certain group law. In principle, being multi-parametric witnesses of entanglement, they could be useful to study separability and (in perspective) criticality of mixed states, playing a role similar to that of Rényi's entanglement entropy in the discrimination of criticality and conformal sectors for pure states.

Contents

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1. Introduction

The study of entanglement of many-body systems represents one of the most relevant challenges of modern research in quantum physics, due to its intrinsic theoretical interest and the very many possible applications. In this context, the determination of suitable information measures, allowing one to detect the entanglement properties of complex quantum systems is of outmost relevance [3].

It is very common and natural, when analyzing compound systems made up of spatially separated parties that can communicate with each other, to focus on protocols that consist on local operations assisted by classical communication (LOCC); they map the set of separable states into itself. Operations preserving the positivity of the density matrix after partial transposition (PPT) are also of special relevance since all LOCC protocols are in particular PPT, which in turn map the set of states with positive partial transpose into themselves.

When designing an entanglement or information measure E, certain conditions should be satisfied regarding LOCC or PPT operations. A fundamental requirement is the *non-increase on average* of E under LOCC or PPT operations [24, 17, 19]. Precisely, we assume that

$$E(\rho) \geqslant \sum_i p_i E(\rho_i) \tag{1}$$

where each of the states ρ_i is obtained with probability p_i after some LOCC or PPT operation is applied to ρ. Another desirable requirement would be that E is able to discriminate whether a state is separable or not. Often this strong requirement is replaced by the weaker condition that E could discriminate whether a state belongs or not to the set of PPT states. Indeed, this set contains the set of separable states, but there are PPT states that are not separable (they are said to contain bound entanglement).

We shall say that an *entanglement monotone* is a quantity satisfying both the two properties above, namely it is non-increasing on average and can discriminate the set of PPT or separable states.

The construction of quantum measures of entanglement analytically determined and satisfying these requirements is an interesting general problem. Important examples are provided by the negativity and the logarithmic negativity, introduced in the seminal paper [38]. In particular, in [23] it is shown that the logarithmic negativity is an entanglement monotone. Also, the logarithmic negativity can recognize PPT states.

In recent years, both the negativity and logarithmic negativity have been largely investigated, due to their prominent role as entanglement measures for mixed states [23, 25] , as well as in the context of quantum field theory and in particular in conformal field theory [11, 26, 6, 7].

As is well known, both Rényi's and Tsallis's entropies can detect criticality in some specific situations when the von Neumann–Shannon entropy can not. The reason for this is that for a Conformal Field Theory (CFT) it turns out

that $\operatorname{tr}\rho\log\rho$ as well as $\operatorname{tr}\rho^n$ can represent useful quantities, where ρ is the reduced density matrix in a subsystem when the whole system is in a pure state. Suppose one finds a multi-parametric quantity which is not just a function of the trace of different powers of the reduced density matrix and assume that this new quantity could be computed for a CFT. Then a stronger criterion would have been found to decide whether the universality class of a one-dimensional quantum critical system corresponds to that of a CFT.

There are one-dimensional quantum systems whose von Neumann–Shannon entanglement entropy coincides with that of a CFT but their Rényi entanglement entropies do not for all n. Consequently, instead of comparing their spectra (which would be the only definitive way of asserting that the quantum critical system is effectively described by a CFT) one can compute their Rényi entanglement entropies as a thinner criterion than the von Neumann–Shannon one (see also [8] for the relevance of Rényi's entropy in the study of multi-block entanglement entropy of free fermion systems). Since the standard (non-parametric) negativity has been recently computed for CFTs our ultimate goal is to find a meaningful and computable parametric generalization of the negativity in order to provide eventually thinner and more specific criteria to classify universality classes of one-dimensional quantum critical systems.

Precisely, the aim of this article is to establish a general mathematical framework that allows us to generate a new, large class of parametric quantum information measures playing the role of entanglement monotones for mixed states. Precisely, we will show that a wide class of generalized entropic information functions can be defined by means of formal group theory. Due to the fact that these new functions widely generalize the notion of logarithmic negativity, we shall call them *group negativities*.

As we will show, from a technical point of view, group negativities are multi-parametric concave functions (generalized logarithms), depending on the p-norm of the partial transposition of a quantum state. The trace-norm subclass is recovered when $p = 1$ and corresponds to quantized versions of the group entropies proposed in [34], computed over partially transposed states. If, in addition, the generalized logarithm is chosen to be the standard one, one recovers the original logarithmic negativity introduced in [38].

More specifically, the trace-norm subclass is made up of multi-parametric *computable measures of entanglement*. Indeed, as shown in Proposition 2 as a consequence of Peres criterion [22], any trace-norm group negativity allows one to detect entanglement in mixed bipartite states: the *strict positivity* of the functional is sufficient to ensure that the state is entangled. Peres criterion (positivity of the partial transpose of a state) is a necessary and sufficient condition for the separability of 2×2 and 2×3 systems, and is still necessary in higher dimensions [18]. In particular, for the trace-norm subclass of group negativities, our main results are the following:

(1) They are *computable* measures of entanglement and provide separability tests for bipartite mixed states.
(2) They are entanglement monotones.
(3) They are *composable*: every element of the whole class of group negativities can be computed for a pure separable state in terms of the group negativities of each of its (non necessarily pure) reductions.

The composability property is guaranteed by the specific functional form of group negativities and could be important in the context of distillability. Indeed, if we have multiple copies of a bipartite state ρ, its distillation rate is the best ratio between the number of maximally entangled pairs which can be obtained from it (distilled) by means of some LOCC protocol and the number of copies of the original state needed. The group negativity of the n copies of ρ can be expressed through the group negativity of ρ.

However, as we will see, generalized negativities with the standard logarithm and p-norms with $p > 1$ are useful as well and represent intrinsically new objects that shall be called *p-norm group negativities*; they represent auxiliary measures and could be useful to determine bounds to distillability rates in distillation processes under different scenarios. Precisely, the simplest p-norm group negativity (see Eq. (13) below) is associated to the additive formal group law (equivalently, the generalized logarithm is the standard one) for all $p \geqslant 1$ as in Definition 10; we shall prove in Theorem 3 that they provide upper bounds for the entropy of distillation not only for $p = 1$.

Concerning genuine p-norm group negativities, our main result is that, under mild conditions, they are *quasi-monotones* in the sense that their increasing after an LOCC (or PPT) operation is bounded on average by a function $k(p)$ independent of the state considered. This function can be made arbitrarily small by using the free parameters typically allowed by group negativities. The explicit example of the p-norm q-negativity is proposed: it corresponds to the genuine p-norm group negativity associated to the multiplicative formal group law from which the real parameter q is inherit. We show that $k(p) \to 0$ for large values of q restoring monotonicity in that regime.

From a mathematical point of view, the construction of group negativities relies on the theory of formal groups [4, 16], which represents an important branch of algebraic topology, with many applications in combinatorics and number theory (see e.g. [30, 32]). According to the arguments exposed above, we consider the composability property essential in order to discuss distillability, since the underlying formal group law controls how the available information is redistributed when independent subsystems are combined into a new one. In Section 2, we shall review briefly the basic facts of formal group theory, especially its role in the theory of generalized entropies from which group negativities are inspired. In Section 3, the main definitions of the class of group negativities and the trace-norm and p-norm subclasses

are introduced. Their main properties are discussed in Section 4 where it is proved that trace-norm negativities are entanglement monotones. Some open problems and future perspectives are discussed in the final Section 6.

2. Groups and entropies: a general approach

We start by recalling some aspects of the group-theoretical classification of generalized entropies and describing how this approach can be used in our formulation of generalized negativities. We first review some definitions of formal group theory (see also [16] for a thorough exposition, and [27] for a shorter introduction to the topic).

2.1. **The Composability Axiom.**

The notion of composability, introduced in [36], has been put in axiomatic form in [33], [34] and related to formal group theory. We shall briefly discuss the concepts of composability and formal group laws as in [33] in order to illustrate the potential relevance of the group-theoretical machinery described above in the study of composite quantum systems.

Definition 1. *An entropy S is strictly (or strongly) composable if there exists a continuous function of two real variables $\Phi(x, y)$ such that the following properties are satisfied.*

(C1) Composability: $S(A \cup B) = \Phi(S(A), S(B))$*, where A and B are two arbitrary statistically independent systems with probability distributions $\{p_i\}_{i=1}^{P}$ and $\{q_j\}_{j=1}^{Q}$, respectively.*

(C2) Symmetry: $\Phi(x, y) = \Phi(y, x)$.

(C3) Associativity: $\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z)$

(C4) Null-composability: $\Phi(x, 0) = x$

Observe that the mere existence of a function $\Phi(x, y)$ taking care of the composition process as in (C1) is necessary, but not sufficient to ensure that a given entropy may be suitable for thermodynamic purposes: this function must satisfy all the requirements above to be admissible. Indeed, in general the entropy of the system composed by subsystems A and B should not vary if we exchange labels A and B, thus justifying condition (C2). In the same vein, condition (C3) guarantees the composability of more than two systems in an associative way, this property being crucial to define a zeroth law. Finally, condition (C4) is also necessary since if we compose two systems A and B and the latter has zero entropy, then the total entropy must coincide with that of the former.

The set of requirements (C2)–(C4) altogether represent the *composability axiom*, which replaces the additivity axiom in the set of the four Shannon-Khinchin axioms. These axioms, introduced by Shannon and Khinchin as

conditions for an uniqueness theorem for the Boltzmann entropy, represent fundamental, non-negotiable requirements that an entropy $S[p]$ should satisfy to be physically meaningful: continuity with respect to all variables $p_1, \ldots, p_W$, maximization over the uniform distribution, expansibility (adding an event of zero probability does not affect the value of $S[p]$).

A *Group entropy* is a function satisfying the first three SK axioms and the composability axiom. Our construction of group negativities is inspired by this notion.

We shall see now that a function $\Phi(x, y)$ satisfying the properties (C2)–(C4) is a formal group law. This is the origin of the connection between entropic measures and formal group theory, as we shall illustrate in the subsequent considerations.

2.2. **Formal groups and formal rings.** Let R be a commutative associative ring with identity, and $R[\![x_1, x_2, \ldots]\!]$ be the ring of formal power series in the variables $x_1, x_2, \ldots$ with coefficients in R. We shall assume that R is torsion-free.

Definition 2. [4] *A commutative one-dimensional formal group law over R is a formal power series $\Phi \in R[\![x, y]\!]$ such that*

(1) $\Phi(x, 0) = \Phi(0, x) = x$

(2) $\Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z))$.

The formal group law is said to be commutative if $\Phi(x, y) = \Phi(y, x)$.

Observe that the existence of an inverse formal series $\varphi \in R[\![x]\!]$ such that $\Phi(x, \varphi(x)) = 0$ is a direct consequence of the previous definition. This justifies the "group" terminology for these algebraic structures.

Let $B = \mathbf{Z}[\![b_1, b_1, \ldots]\!]$ and consider the following series in $B[\![s]\!]$

$$F(s) = s + \sum_{i=1}^{\infty} b_i \frac{s^{i+1}}{i+1}. \tag{2a}$$

If $G \in B[\![t]\!]$ is its compositional inverse (satisfying $F(G(t)) = t$ and $G(F(s)) = s$), one has

$$G(t) = t + \sum_{k=1}^{\infty} a_k \frac{t^{k+1}}{k+1} \tag{2b}$$

with $a_1 = -b_1$, $a_2 = \frac{3}{2}b_1^2 - b_2, \ldots$. Given the formal power series F and G as in Eqs. (2), the Lazard formal group law [16] is defined by the formal power series

$$\Phi_{\mathrm{L}}(s_1, s_2) = G(G^{-1}(s_1) + G^{-1}(s_2))$$

whose coefficients, generate over $\mathbf{Z}$ a subring $L \subset B \otimes \mathbf{Q}$. In other words, the Lazard ring is defined over a subring of the original ring $B \otimes \mathbf{Q}$, called the Lazard ring.

One of the most important property that shall be used in the rest of this work can be recast into the following statement. For any commutative

one-dimensional formal group law over any ring R, there exists a unique homomorphism $L \to R$ under which the Lazard group law is mapped into the given group law. This is called the universal property of the Lazard group. Also, it is important to notice that for any commutative one-dimensional formal group law $\Phi(x, y)$ over R, there exists a series $\phi(x) \in R[[x]] \otimes \mathbb{Q}$ such that

$$\phi(x) = x + O(x^2), \quad \text{and} \quad \Phi(x, y) = \phi^{-1}\left(\phi(x) + \phi(y)\right) \in R[[x, y]] \otimes \mathbb{Q}.$$

Finally, let us also define the notion of formal ring recently introduced in [9].

Definition 3. *Let $(R, +, \cdot)$ be a unital ring. A formal ring is a triple (R, Φ, Ψ) where $\Phi, \Psi \in R[\![x, y]\!]$ are formal power series such that*

(1) Φ is a commutative formal group law according to Def. 2.

(2) Ψ satisfies the relations

$$\Psi(\Psi(x, y), z) = \Psi(x, \Psi(y, z))$$
$$\Psi(x, \Phi(y, z)) = \Phi(\Psi(x, y), \Psi(x, z))$$
$$\Psi(\Phi(x, y), z) = \Phi(\Psi(x, z), \Psi(y, z)).$$

and the formal ring will be said to be commutative if $\Psi(x, y) = \Psi(y, x)$.

3. Trace-norm group negativities

We will show that the notion of logarithmic negativity can be generalized by means of a mathematical formalism based on formal group theory. Our main result is the following: there exists a "tower" of new information measures, each of them reducing to the logarithmic negativity in a certain regime, a priori depending on a set of free parameters. Our construction relies on the notion of group logarithm associated to every formal group law. The standard logarithm is associated to the additive formal group law.

3.1. **Group logarithms.**

Definition 4. *A group logarithm is a strictly increasing and strictly concave function $\log_G : (0, \infty) \to \mathbb{R}$, with $\log_G(1) = 0$ (possibly depending on a set of real parameters); a functional equation of the form*

$$\log_G(xy) = \chi(\log_G(x), \log_G(y)) \tag{3}$$

will be called the group law associated with $\log_G(\cdot)$.

The inverse of a group logarithm will be called the associated group exponential; it is defined by

$$\exp_G(x) = e^{G^{-1}(x)}. \tag{4}$$

We can realize the group law (3) associated with a group logarithm by means of the simple formula

$$\chi(x, y) := G(G^{-1}(x) + G^{-1}(y)), \tag{5}$$

being $G(x) := \log_G(\mathrm{e}^x)$ a strictly increasing continuous function, vanishing at zero. An useful result is the following simple proposition, which allows us to construct easily infinitely many group logarithms.

Proposition 1. *Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function vanishing at zero. The function $\Lambda_G(x)$ defined by*

$$\Lambda_G(x) := G\left(\ln x^\gamma\right), \qquad x > 0, \qquad \gamma > 0 \tag{6}$$

is a group logarithm.

Proof. The function (6) satisfies the functional equation (3), where $\chi(x, y)$ is the group law (5):

$$\begin{aligned} \Lambda_G(xy) &= G\left(\ln x^\gamma + \ln y^\gamma\right) = G\left(G^{-1}(\Lambda_G(x)) + G^{-1}(\Lambda_G(y))\right) \\ &= \chi(\Lambda_G(x), \Lambda_G(y)). \end{aligned}$$

Besides, since the function $G\left(\ln x^\gamma\right)$ is the composition of a strictly increasing function with a strictly concave one, we deduce that it is a group logarithm. □

Remark 1. From now on, we shall focus on group logarithms of the form

$$\log_G(x) = G(\ln x) \tag{7}$$

where $G(\cdot)$ is a strictly increasing function of the form (2b).

Remark 2. Let G be a strictly increasing (real analytic) function of the form (2b). For $\log_G(x) = G(\ln x)$, the requirement of concavity is guaranteed, for instance, by the simple condition

$$a_k > (k+1)a_{k+1} \qquad \forall k \in \mathbb{N}, \qquad \text{with } \{a_k\}_{k\in\mathbb{N}} > 0, \tag{8}$$

which is also sufficient to ensure that the series $G(t)$ is convergent absolutely and uniformly over the compacts with a radius $r = \infty$. Many other choices are allowed.

A first, relevant example of nontrivial group logarithm is given by the so called q-logarithm. We have

$$G(t) = \frac{\mathrm{e}^{(1-q)t} - 1}{1-q}, \qquad \log_q(x) = G(\ln x) = \frac{x^{1-q} - 1}{1-q}, \qquad q > 0\,. \tag{9}$$

This logarithm has been largely investigated in connection with nonextensive statistics [35], [36]. Concerning group exponentials, notice that when $G(t) = t$, we have returned to the standard exponential; when as before $G(t) = \frac{e^{(1-q)t}-1}{1-q}$, we recover the q-exponential $e_q(x) = [1 + (1-q)t]^{\frac{1}{1-q}}$, and so on.

Infinitely many other examples of group logarithms and exponentials are provided, for instance, in [31].

3.2. **Z-entropies.** The group entropies already mentioned use generalized logarithms associated to formal group laws in a similar way. The so-called Z-entropies introduced in [34] are strongly composable and generalize both the Boltzmann and the Rényi entropies. Their general form, for $\alpha > 0$, is

$$Z_{G,\alpha}(p_1, \ldots, p_W) := \frac{\log_G \left(\sum_{i=1}^{W} p_i^{\alpha} \right)}{1 - \alpha} . \tag{10}$$

3.3. **Negativity and PPT operations.** Let us denote by $\mathcal{B}_1$ and $\mathcal{B}_2$ the space of bounded linear operators of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. For a bipartite mixed state $\rho \in \mathcal{B}_1 \otimes \mathcal{B}_2$, let us denote by ρ^{Γ} its partial transposition with respect to $\mathcal{H}_2$ (the final result will not change if we choose $\mathcal{H}_1$ in this definition). The action of partial transposing is defined in the space $\mathcal{B}$ of bounded linear operators of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ extending by linearity the following action over pure separable states $\sigma \otimes \tau \in \mathcal{B}$ with $\sigma \in \mathcal{B}_1$ and $\tau \in \mathcal{B}_2$. Precisely,

$$(\sigma \otimes \tau)^{\Gamma} = \sigma \otimes \tau^{\mathrm{T}}$$

where $\tau^{\mathrm{T}} \in \mathcal{B}_2$ is the transpose of τ.

Definition 5. *Given an element $A \in \mathcal{B}$, write $|A| = \{A\}_+ - \{A\}_-$ where $\{\cdot\}_+$ and $\{\cdot\}_-$ are its positive and negative parts, i.e., its restrictions to the eigenspaces of positive and negative eigenvalues respectively. The trace-norm $\|\cdot\|_1$ of an operator A is defined as $\|A\|_1 = \mathrm{tr}|A|$.*

Note that $A = \{A\}_+ + \{A\}_-$ and if A is Hermitian $\{A\}_+ - \{A\}_- = \sqrt{AA^{\dagger}}$ where $\sqrt{B}$ represents any operator $C \in \mathcal{B}$ such that $C^2 = B \in \mathcal{B}$.

Definition 6. *Given a bipartite mixed state ρ, its negativity is defined to be the function $N(\rho) := \frac{1}{2}(\|\rho^{\Gamma}\|_1 - 1)$ whereas its logarithmic negativity is the function $L(\rho) := \ln\|\rho^{\Gamma}\|_1$.*

The monotonicity of $L(\rho)$ was proved in [23]. Precisely, the following inequality holds

$$L(\rho) \geqslant \sum p_i L(\rho_i) ,$$

where $\rho_i \propto \mathcal{A}_i(\rho)$ is the normalized state associated to outcome i after applying the trace-preserving completely positive operation $\mathcal{A} = \sum_i \mathcal{A}_i$. Note that $\mathcal{A}$ maps the set of PPT states into itself and also that the result of applying $\mathcal{A}$ to ρ can be seen as an ensemble with elements ρ_i appearing with probabilities $p_i = \mathrm{tr}\,\mathcal{A}_i(\rho)$. The logarithmic negativity L is also an upper bound to distillable entanglement, as was shown in [38].

In the same way as the logarithmic negativity is associated to the additive formal group law, one can define analogous entanglement measures associated to different composition laws by means of formal group theory.

3.4. **Group negativities and strict quantum composability.** We propose here one of our main definitions.

Definition 7. *A trace-norm group negativity $L_G : \mathcal{B} \to \mathbf{R}$, where $\mathcal{B}$ is the space of bounded linear operators of a Hilbert space $\mathcal{H}$, is the function*

$$L_G(\rho) := \log_G \| \rho^{T_A} \|_1, \qquad \rho \in \mathcal{B}, \tag{11}$$

where $\log_G(x)$ is a group logarithm of the form (7).

Remark 3. From the discussion above, we conclude that any strictly continuous and invertible function $G(t)$ of the form (2b) generates a group logarithm $\log_G(x) = G(\ln x)$ and, in turn, a group negativity.

The trace-norm group negativities can be regarded as a new quantum version of the Z-entropies introduced in [34]. The main novelty of the present construction is that the functional (11) is the trace-norm of the partial transposition of a quantum state whose spectrum need not be, in general, a probability distribution. Indeed, according to Peres criterion [22], the spectrum of the partial transposition of all separable density matrices are probability distributions: when the partial transpose contains a negative eigenvalue (so that since partial transposition preserves the trace it follows that the absolute values of the eigenvalues do not represent a probability distribution) then the state is entangled.

Proposition 2. *Any trace-norm group negativity is positive semi-definite over the space $\mathcal{B}$ of bounded linear operators of a Hilbert space $\mathcal{H}$ and vanish for states with positive partial transpose. Furthermore, it is strictly composable: if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ then*

$$L_G(\sigma \otimes \tau) = \Phi(L_G(\sigma), L_G(\tau))$$

for any pair of states $\sigma \in \mathcal{B}_1$ and $\tau \in \mathcal{B}_2$ where $\Phi(x, y) = G^{-1}(G(x)+G(y))$.

Proof. Since $\operatorname{tr}\rho = \operatorname{tr}\rho^\Gamma$, it follows that $\{\rho^\Gamma\}_- = 0$ only when $\|\rho^\Gamma\|_1 = 1$. Definition 7 implies that $L_G(\rho) = 0$ in this case, whereas $L_G(\rho) > 0$ in the other cases, neing $\log_G$ strictly increasing. Composability is ensured by the functional equation associated with the group logarithm $\log_G(x) = G(\ln x)$ underlying the definition of trace norms. □

In our framework, the original logarithmic negativity [38] corresponds to the choice $G(t) = t$ which leads to the additive group $\Phi(x, y) = x + y$. A new non-trivial example is provided by the use of the q-logarithm of Eq. (9).

Definition 8. *The trace-norm q-negativity of a state $\rho \in \mathcal{B}$ is the function*

$$L^{(q)}(\rho) := \frac{\|\rho^\Gamma\|_1^{1-q} - 1}{1-q}, \qquad q > 0\,.$$

For $q \to 1$, it reduces to the logarithmic negativity $\lim_{q\to 1} L^{(q)}(\rho) = L(\rho)$. Furthermore, for any pair $\sigma \in \mathcal{B}_1$ and $\tau \in \mathcal{B}_2$ it follows

$$L^{(q)}(\sigma \otimes \tau) = L^{(q)}(\sigma) + L^{(q)}(\tau) + (1-q)L^{(q)}(\sigma)L^{(q)}(\tau)\,.$$

The trace-norm q-negativity is thus associated to the same composition law (called in algebraic topology the multiplicative formal group law [16]) $\Phi(x,y) = x + y + (1-q)x\,y$ as both the classical and the quantum versions of the Tsallis entropy [35, 36]. It is worth mentioning that the entanglement entropy associated to the Tsallis entropy (its evaluation over the reduced density matrix of a bipartite pure state) has been used in [37] to characterize the separability of a family of quantum states, correctly recovering Peres criterion for a concrete family of states.

However, Definition 8 is naturally adapted to the Peres criterion, which can be applied to *any* state. Whenever Peres criterion is sufficient and necessary, then $L_q > 0$ for entangled states only. Clearly, the trace-norm q-negativity suggests further generalization in terms of general entanglement witnesses [25], namely quantities that separate an entangled state from the set of separable ones in more general scenarios, whereas partial transpose separates in a necessary and sufficient way quantum systems associated to Hilbert spaces of dimension strictly lower than eight, namely when $\min_{i=1,2} \dim \mathcal{H}_i = 2$ and $\max_{i=1,2} \dim \mathcal{H}_i = 2, 3$.

The fact that trace-form group negativities are strictly composable (they are composable irregardless of the tensor factors considered) is a non-trivial property, essentially due to their non-trace functional form. Precisely, when dealing with standard entropies over a probability space, classical strict composability prevents the use of infinitely many trace-form entropies, namely functions of probability distributions $(p_1, \ldots, p_W)$ of the form

$$\sum_{i=1}^{W} f(p_i), \qquad f(0) = f(1) = 0 \ .$$

Precisely, a theorem proved in [12] states, under very general hypotheses, that the most general trace-form entropy strictly composable is Tsallis's entropy (recovering Boltzmann's entropy when $q \to 1$). Thus, using the more commonly adopted trace-form functionals one is lead to weakly composable negativities [33], namely composable only over the product of uniform distributions. Instead, strictly composable entropies are possible in the non-trace-form class. Indeed, each Z-entropy in Eq. (10) is strictly composable, with a specific composition law, to which one can associate a trace-norm group negativity. The first such pair is made up of the original Tsallis's q-entropy and the trace-norm q-negativity of Definition 8.

When $\dim \mathcal{H} \geqslant 8$, positivity of partial transposition is only necessary for separability and thus there exist entangled states with positive partial transpose. For them all $\{\rho^\Gamma\}_- = 0$ and $L_G(\rho) = 0$, in particular, when $G(t) = t$ one recovers the well-known fact that the logarithmic negativity vanishes $L(\rho) = 0$ for PPT entangled states. We shall address this issue in the next section where monotonicity of trace-norm group negativities is shown. We introduce a new class of p-norm group negativities which,

although are not necessarily zero over PPT states, they are *entanglement ε-monotones*, i.e., they are non-increasing on average up to a positive quantity $\varepsilon > 0$. Precisely, there exists a family of functionals that we shall call p-norm group negativities, where $p \geqslant 1$, each satisfying

$$\varepsilon + E(\rho) \geqslant \sum_i p_i E(\rho_i)$$

which is to be compared with Eq. (1).

4. p-NORM GROUP NEGATIVITIES

The previous construction of trace-norm group negativities can be further extended by considering different p-norms, apart the standard trace norm corresponding to $p = 1$ in the following treatment. As before, we shall be concerned with a composite quantum systems with associated Hilbert space $\mathcal{H}$ of dimension N and write $\mathcal{B}(\mathcal{H})$ for the linear space of bounded linear operators in $\mathcal{H}$.

Consider the Scatten p-norms

$$\|A\|_p = \left(s_1(A)^p + \cdots + s_N(A)^p\right)^{1/p}, \qquad p \geqslant 1$$

for any $A \in \mathcal{B}$ with singular values $s_i(B)$; the limit $p \to \infty$ will be denoted by $\|\cdot\|_\infty$. We introduce now the main objects of our analysis.

Definition 9. *The function $\mathcal{L}_{G,p} : \mathcal{B} \to \mathbf{R}$, for any state $\rho \in \mathcal{B}$ and $p \geqslant 1$*

$$\mathcal{L}_{G,p}(\rho) = \log_G \|\rho^\Gamma\|_p \tag{12}$$

is said to be the p-norm group negativity of the state ρ. Here $\log_G(x)$ is a group logarithm of the form (7).

Clearly, trace-norm group negativities are obtained when $p = 1$. Also, any function G as in Definition 7, under mild hypotheses induces a p-norm group negativity via a group logarithm. A simple but interesting, new and non-trivial case is the additive one, obtained when $G(t) = t$ for $p > 1$.

Definition 10. *The function*

$$\mathcal{L}_p(\rho) = \ln \|\rho^\Gamma\|_p, \qquad p \geqslant 1 \tag{13}$$

will be called the logarithmic p-norm negativity of a mixed state ρ.

Remark 4. Obviously, in information-theoretical applications one could replace $\ln(x)$ with $\log_2(x)$ in Eq. (13) (as in the standard definition for $p = 1$, by Vidal and Werner in [38]), without altering the main properties of the function.

We will show that the quantity $\mathcal{L}_{G,p}(\cdot)$ is bounded on average under LOCC (CP-PPT) operations. This bound can be arbitrarily close to zero (and in particular is exactly zero in the limit $p = 1$).

We shall first deal with a deterministic trace-preserving CP-PPT operation $\mathcal{A}$ and later we consider a general not necessarily deterministic operation

mapping a state ρ to an ensemble of states $\rho_i = \mathcal{A}_i(\rho)$ each appearing with probability $p_i = \operatorname{tr} \mathcal{A}_i(\rho)$ where each operation $\mathcal{A}_i$ is CP-PPT and $\sum_i \mathcal{A}_i$ is trace-preserving. Let us consider the partial transposition as an operator $\Gamma : \mathcal{B} \to \mathcal{B}$; we write $\Gamma(\rho) = \rho^\Gamma$. This operation is clearly involutive. We can define a linear map $\mathcal{A}^\Gamma : \mathcal{B} \to \mathcal{B}$ as $\mathcal{A}^\Gamma(\sigma) := \Gamma \circ \mathcal{A} \circ \Gamma(\sigma)$; equivalently, $\mathcal{A}^\Gamma \circ \Gamma = \Gamma \circ \mathcal{A}$.

We propose a simple Lemma, useful in the forthcoming discussion.

Lemma 1. *Let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be any PPT quantum operation. If $\mathcal{A}$ is (completely) positive and preserves the positivity of the partial transpose, then its partial transpose $\mathcal{A}^\Gamma$ is (completely) positive.*

Proof. Since $\mathcal{A}$ is a PPT operation, $\Gamma \circ \mathcal{A}(\rho)$ is positive if $\Gamma(\rho)$ is positive. Since Γ is an involution, writing $\sigma = \Gamma(\rho)$ and $\rho = \Gamma(\sigma)$ we conclude that $\Gamma \circ \mathcal{A} \circ \Gamma(\sigma)$ is positive if $\Gamma \circ \Gamma(\sigma) = \sigma$ is positive. The proof is completed because $\mathcal{A}^\Gamma = \Gamma \circ \mathcal{A} \circ \Gamma$. □

The following result is due to Plenio [23].

Lemma 2. *Let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be a positive operation. Then* $\operatorname{tr}|\mathcal{A}(\rho)| \leqslant \operatorname{tr}|\rho|$.

Proof. Note that

$$\{\mathcal{A}(\cdot)\}_+ = \{\mathcal{A}(\{\cdot\}_+) + \mathcal{A}(\{\cdot\}_-)\}_+ \leqslant \mathcal{A}(\{\cdot\}_+) \,, \tag{14}$$

due to the fact that $\mathcal{A}(\{\cdot\}_-) = -\mathcal{A}(-\{\cdot\}_-)$ and $-\{\cdot\}_-$ is positive or zero so that by linearity $\mathcal{A}(\{\cdot\}_-)$ is negative or zero. Finally, observe that

$$\begin{aligned} |\mathcal{A}(\rho)| = \{\mathcal{A}(\rho)\}_+ - \{\mathcal{A}(\rho)\}_- &= \{\mathcal{A}(\rho)\}_+ + \{-\mathcal{A}(\rho)\}_+ \\ &= \{\mathcal{A}(\rho)\}_+ + \{\mathcal{A}(-\rho)\}_+ \leqslant \mathcal{A}(\rho_+) + \mathcal{A}(\{-\rho\}_+) \end{aligned}$$

were we have used inequality (14) twice after using linearity in the second term. The desired result follows by noting that $\rho_- = -\{-\rho\}_+$ and using linearity in the RHS of the last inequality to obtain $|\rho| = \rho_+ - \rho_-$ as the argument of $\mathcal{A}$. □

Definition 11. *A group logarithm* $\log_G(x)$ *such that*

$$\log_G(xy) \leqslant \log_G x + \log_G y \tag{15}$$

will be said to be subadditive.

In order to prove the main results of this section, we first propose the following

Lemma 3. *The inequality*

$$\sum_i p_i \|\rho_i^\Gamma\|_p \leqslant \operatorname{tr}|\rho^\Gamma| \,. \tag{16}$$

holds for $p \geqslant 1$ under CP-PPT operations.

Proof. Observe that

$$\sum_i p_i \|\rho_i^\Gamma\|_p = \sum_i \|\mathcal{A}_i^\Gamma(\rho^\Gamma)\|_p = \sum_i (\mathrm{tr}|\mathcal{A}_i^\Gamma(\rho^\Gamma)|^p)^{1/p} \ ,$$

where $|\mathcal{A}_i^\Gamma(\rho^\Gamma)|$ is of course a positive operator. Also, for any positive operator A we have that

$$\mathrm{tr}\, A^p \leqslant (\mathrm{tr}\, A)^p, \qquad p \geqslant 1 \ .$$

Thus, we obtain

$$\left(\mathrm{tr}|\mathcal{A}_i^\Gamma(\rho^\Gamma)|^p\right)^{1/p} \leqslant \mathrm{tr}|\mathcal{A}_i^\Gamma(\rho^\Gamma)| \leqslant \mathrm{tr}\, \mathcal{A}_i^\Gamma(|\rho^\Gamma|) \ , \tag{17}$$

where the last inequality follows from Lemma 2. Therefore we have shown that

$$\sum_i p_i \|\rho_i^\Gamma\|_p \leqslant \sum_i \mathrm{tr}\, \mathcal{A}_i^\Gamma(|\rho^\Gamma|) = \mathrm{tr}|\rho^\Gamma| \ . \tag{18}$$

□

A relevant result is the following

Theorem 1. *The group p-norm negativity* $\mathcal{L}_{G,p}(\rho) = \log_G \|\rho^\Gamma\|_p$ *associated with a subadditive group logarithm, for any* $p \geqslant 1$ *is* bounded on average *under CP-PPT operations, that is, there exists a constant* $k(p)$ *such that*

$$\sum_i p_i \mathcal{L}_{G,p}(\rho_i) - \mathcal{L}_{G,p}(\rho) \leqslant k(p) \ . \tag{19}$$

Proof. Since by definition a group logarithm is a concave function, then

$$\sum_i p_i \, \mathcal{L}_{G,p}(\rho_i) = \sum_i p_i \log_G \|\rho_i^\Gamma\|_p \leqslant \log_G \left(\sum_i p_i \|\rho_i^\Gamma\|_p \right) . \tag{20}$$

Now, since $\log_G$ is strictly increasing, using Lemma 3, we have

$$\sum_i p_i \mathcal{L}_{G,p}(\rho_i) \leqslant \log_G \mathrm{tr}|\rho^\Gamma| \, .$$

Let us denote by N the dimension of the ambient Hilbert space $\mathcal{H}$. We can observe that

$$\mathrm{tr}|\rho^\Gamma| = \|\rho^\Gamma\|_1 \leqslant c(p) \|\rho^\Gamma\|_p \ ,$$

where $c(p) = N^{1-1/p}$, and $\rho \in \mathcal{B}(\mathcal{H})$.
Finally, due to subadditivity of $\log_G$, we conclude that

$$\sum_i p_i \mathcal{L}_{G,p}(\rho_i) \leqslant \log_G \|\rho^\Gamma\|_1 \leqslant \log_G c(p) + \log_G \|\rho^\Gamma\|_p \, .$$

To conclude, we define

$$k(p) := \log_G c(p) \tag{21}$$

and the previous inequality reduces to relation (19). □

Remark 5. The hypotheses of Theorem 1 are actually satisfied by an infinite family of group logarithms. For instance, the group exponential and logarithm considered in Eq. (3) satisfy all required properties: Indeed, $\log_q(x)$ is strictly concave, monotonically increasing and subadditive for $q > 1$, since $\log_q(\rho^X \otimes \rho^Y) = \log_q(\rho^X) + \log_q(\rho^Y) + (1-q)\log_q(\rho^X)\log_q(\rho^Y)$. We introduce now the corresponding negativity measure.

Definition 12. *The p-norm q-negativity for any $\rho \in \mathcal{B}$ is the function*

$$\mathcal{L}_p^{(q)}(\rho) := \frac{(\| \rho^{T_A} \|_p)^{1-q} - 1}{1-q}, \qquad q > 0 \ . \tag{22}$$

Remark 6. An interesting aspect of inequality (19) is that $k(p)$ can be made arbitrarily small in two ways. The first one is to consider norms with $p = 1 + \delta$, with $\delta > 0$ arbitrarily small (namely, small deformations of the trace norm). A second, more specific possibility is to select properly the group logarithm in Eq. (21) and to consider suitable intervals of values of its parameters. For instance, for the q-logarithm (9), we have

$$k(p) = \log_q c(p) = \frac{(N^{1-1/p})^{1-q} - 1}{1-q} \ .$$

Thus, for any $\epsilon > 0$ there exists a value q^* such that for $q > q^*$, $k(p) < \epsilon$. Then

$$\sum_i p_i \mathcal{L}_p^{(q)}(\rho_i) \leqslant \mathcal{L}_p^{(q)}(\rho) + \epsilon \ . \tag{23}$$

Due to the latter property, we shall say that the p-norm q-negativity $\mathcal{L}_q^{(p)}(\rho)$ is an *ϵ-monotone* (or *quasi-monotone*).

As an immediate consequence of Theorem 1, we have the following result.

Theorem 2. *A trace-norm group negativity $L_G(\rho) = \log_G \|\rho^\Gamma\|$ associated with a subadditive group logarithm is an entanglement monotone:*

$$\sum_i p_i L_G(\rho_i) \leqslant L_G(\rho) \ . \tag{24}$$

Proof. It suffices to assume $p = 1$ in the previous discussion. In particular, we have identically $c(1) = 1$ and $k(1) = 0$ into Eq. (19). □

5. p-norm negativity as an upper bound for distillability

A crucial property of the additive p-norm (13) introduced in the present work is the fact that it represents an upper bound for distillability.

In our analysis, we shall closely follow the notation and the discussion of Ref. [38].

We define $\mathcal{L}_p(\Omega) := \ln\|\Omega\|_p = \frac{1-p}{p} \ln N$, where Ω is the (diagonal) density matrix of the maximally mixed (separable) state. Then, we can introduce the *normalized p-norm negativity*

$$\tilde{\mathcal{L}}_p(\rho) := \mathcal{L}_p(\rho) - \mathcal{L}_p(\Omega) \ . \tag{25}$$

Note that the standard (trace-norm) logarithmic negativity is already normalized:

$$\tilde{\mathcal{L}}_1(\rho) = L(\rho) \ .$$

In a completely analogous way, one can normalize any p-norm group negativity

Assume that we have a bipartite state ρ and multiple copies of it by means of LOCC. We recall that its distillation rate is the best rate at which we can extract near-perfect singlet states from its copies.

In particular, given a large number of copies of the state, its *asymptotic* distillation rate is called its entanglement of distillation $E_D(\rho)$.

Let us consider n_α copies of ρ and let Y be a maximally entangled state of two qubits. Then, we are interested in the best approximation to m_α copies of Y that can be obtained from $\rho^{\otimes n_\alpha}$ by means of LOCC.

We introduce [38]

$$\Delta(Y^{\otimes m_\alpha, \rho^{\otimes n_\alpha}}) = \inf_P \parallel Y^{\otimes m_\alpha} - P(\rho^{\otimes n_\alpha}) \parallel_1 \ . \tag{26}$$

Here P runs over all deterministic protocols obtained from LOCC.

We say that c is an achievable distillation rate for ρ , if for any sequences n_α, $m_\alpha \to \infty$ of integers such that $\limsup_\alpha (n_\alpha/m_\alpha) \leqslant c$ we have

$$\lim_\alpha \ \Delta(Y^{\otimes m_\alpha, \rho^{\otimes n_\alpha}}) = 0 \ . \tag{27}$$

Thus, the distillable entanglement is the supremum of all achievable distillation rates. If we allow a small error level, we can introduce the distillable entanglement at error level ϵ, denoted by $E_D^\epsilon(\rho)$, which is characterized by the weaker condition

$$\lim_\alpha \ \Delta(Y^{\otimes m_\alpha, \rho^{\otimes n_\alpha}}) \leqslant \epsilon \ . \tag{28}$$

In this context, our main result is the following

Theorem 3. *Let $\tilde{\mathcal{L}}_p(\rho)$ be the normalized logarithmic p-norm negativity. Then, for any $p \geqslant 1$ we have*

$$\tilde{\mathcal{L}}_p(\rho) \geqslant E_D^\epsilon \ . \tag{29}$$

Proof. As is well known [38], the standard logarithmic negativity satisfies the upper bound

$$L(\rho) \geqslant E_D^\epsilon. \tag{30}$$

We also remind the inequalities ($p > 1$)

$$\| \cdot \|_p \leqslant \| \cdot \|_1 \leqslant N^{1-1/p} \| \cdot \|_p \ . \tag{31}$$

Consequently, from the first inequality (31), we get

$$\mathcal{L}_p(\rho) \leqslant L(\rho). \tag{32}$$

From the second one, we have

$$L(\rho) \leqslant -\mathcal{L}_p(\Omega) + \mathcal{L}_p(\rho) \ .$$

Consequently, due to inequality (30), we get

$$\mathcal{L}_p(\rho) - \mathcal{L}_p(\Omega) \geqslant E_D^{\epsilon} \ . \tag{33}$$

Using Definition 25, we conclude that

$$\tilde{\mathcal{L}}_p(\rho) \geqslant E_D^{\epsilon} \ . \tag{34}$$

□

6. Future Perspectives

As we have shown, group theory offers a natural way to generalize the notion of negativity. This work represents a first exploration of a new, infinite class of easily computable entropic-type measures of entanglement.

Several aspects of the theory deserve further analysis. It is clear that composability is crucial in order to compute entanglement entropy of bipartite or multipartite systems in a natural way, starting from the knowledge of the entropy of its constituents. As we suggested, such a property is fundamental to study distillable entanglement. Therefore, an interesting open problem is to ascertain if all of the group-theoretical negativities introduced here, apart the logarithmic p-norm negativity, can provide upper or lower bounds to the asymptotic distillation rate by means of LOCC, when we consider a large number of copies of the state $\rho^{\otimes n_\alpha}$.

At the same time, it would be very interesting to apply the large family of entropic functionals introduced in this work in the study of finite temperature systems in conformal field theories [7]. From this point of view, one-parametric (or multi-parametric) entanglement monotones could play a role similar to that played by Rényi's entropy in the case of the entanglement detection of the ground state of one-dimensional many body systems, and in the study of their criticality properties [5].

We wish to point out that the language of formal group theory can be directly related to the study of alternative formulations of both classical and quantum mechanics. Indeed, as shown in [13], the linear structure of the theory can be replaced by a non-additive structure generated by means of a suitable diffeomorphism, which would play the same role as the group logarithm of the present theory. In particular, this perspective opens the possibility of performing non-equivalent Weyl quantizations of physical systems, circumventing the von Neumann uniqueness theorem. The generalized negativities introduced in the present work could play a significant role in these alternative formulations. We shall discuss these aspects in detail elsewhere.

Another interesting problem is to give an interpretation of group negativities within the context of quantum information geometry, especially in connection with the problem of tomographic reconstruction of quantum metrics [1], [2], [21], [10].

Finally, we also plan to apply generalized negativities to the study of entanglement properties of some concrete examples of quantum systems, in

particular integrable spin chains of Haldane-Shastry type [14] [15]. Work is in progress along these lines.

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Conclusions and outlook

In Chapter 1 (Ref. [**P1**]) we have rigorously formulated a general duality principle which posits the invariance of the entanglement entropy $S(u, v) = S(v, u)$ of a system of free fermions under exchange of the sets of excited momentum modes v and sites u of the subsystem under study, where both u and v are the union of an arbitrary (finite) number of blocks of consecutive sites or modes. By means of this principle, we have derived an asymptotic formula for the Rényi entanglement entropy when the set v consists of a single block. From this formula and a natural assumption concerning the additivity of the entropy when the blocks are far apart from each other in either position or momentum space, we have conjectured an asymptotic approximation for the entanglement entropy in the general case when both sets u and v consist of an arbitrary number of blocks.

A large class of su(1|1) supersymmetric spin chains with long-range interactions generalizing the su(1|1) Haldane-Shastry and Inozemtsev (elliptic) chains, which can be fermionized using the algebraic properties of the su(1|1) permutation operator has been introduced in this thesis. We exploit this fact to study the critical behavior of this class of models (with nonzero chemical potential μ) in terms of their dispersion relation $\mathcal{E}(p)$, which can be either monotonic (Ref. [**P2**]) or nonmonotonic (Ref. [**P3**]) in $[0, \pi]$, providing simple examples of models of this type with both short- and long-range interactions. The main conclusion of our work is that the criticality properties of the supersymmetric chains (8) are determined exclusively by the number of points of the boundary of their Fermi sea. By analyzing both the entanglement entropy of their ground states and their free energy per particle at low temperatures, we have shown that they are equivalent to $m + 1$ free bosons with Fermi velocities $v_i = \mathcal{E}'(p_i)$, where $p_0, \ldots, p_m$ are the solutions of the equation $\mathcal{E}(p) = \mu$ in $[0, \pi]$ and μ is the chemical potential of the fermions. In particular, the central charge is equal to the number $m + 1$ of connected components (intervals) of the Fermi sea.

These results open up some natural research lines. In the first place, it would be desirable to find a rigorous proof of the conjectured behaviour of the entanglement entropy in the general case, i.e, when both subsets u, v consist of several connected components. Another interesting question is the analysis of the configurations *minimizing* the entropy with appropriate constraints, which could be naturally regarded as akin to "semi-classical" states. One could also consider a generalization of our results on the ground-state entanglement entropy to more general situations (for instance, considering excited states), in which the Fermi sea exhibits a more complicated topological structure.

In Chapter 2 (Ref. [**P4**]) we have introduced a family of generalized su(m) Lipkin–Meshkov–Glick models, and computed the partition function of a subclass whose interacting term is a spin chain of Haldane–Shastry type (Ref. [**P5**]), which can be equivalently regarded as the modification $H = H_0 + H_1$ of a spin chain of HS type H_0 by the addition of a term H_1 in the enveloping algebra of the Cartan subalgebra of su(m). The Hilbert space of the system is a direct sum of subspaces with fixed magnon numbers, in which the action of the deformation term is diagonal, so that the model's partition function decomposes as in equation (13) above. By a suitable adaptation of Polychronakos's freezing trick, we have been able to compute in a closed form the partition functions of the restrictions $H_0|_{(n_a)}$ of the spin chain Hamiltonian H_0 to the subspaces spanned by elements with magnon numbers (n_a). The analysis performed in [**P5**] shows that the thermodynamic functions of these models are qualitatively similar to those of a two-level system, as previously already observed in [**45**] for the su(2) chains of HS type. In the latter chains, this similarity is ultimately due to the existence of a description of the spectrum in terms of motifs, which leads to simple closed formulas for the thermodynamic functions in terms of the dispersion relation. This suggests that such a description should also exist for the more general class of gLMG models of HS type.

In point of fact, one can see that this description in terms of motifs for the spin chains of HS type restricted to subspaces of fixed magnon content can be used to study other physical systems of interest, as is done in Refs. [**P6**]-[**P7**] for the long-range t-J introduced by Kuramoto and Yokoyama in the nineties. In Chapter 3 we show how the partition function yields a complete analytical description of the spectrum in the latter subspaces, including the precise degeneracy of each level. This leads to a complete characterization of the distinct ground state phases, determined by their spin content, as we have obtained in [**P6**]. The latter

approach, together with the results of [52], allowed us to complete the previous study of the su(2) t-J model deriving the complete low-temperature asymptotic expansion of the free energy in [**P7**]. We have also analyzed the critical behavior of the model in each of its ground state phases. The main conclusions are that, while the standard su(1|2) phase is described by two independent CFTs with central charge $c = 1$ in correspondence with the spin and charge sectors, the low-energy behavior of the su(2) and su(1|1) phases is that of a single $c = 1$ CFT. We have shown that the model exhibits an even richer behavior on the boundary between zero-temperature phases, where it can be non-critical but gapless, critical in the spin sector but not in the charge one, or critical with central charge $c = \frac{3}{2}$.

In Chapter 4 we completely solve the problem of classifying all one-dimensional quantum potentials with nearest- and next-to-nearest-neighbors interactions whose ground state is Jastrow-like. It follows from our analysis [**P8**] that all these models must necessarily contain a three-body interaction term, as was the case with all previously known examples. In addition, the family of Jastrow-like states considered can be related to certain subclass of continuous matrix product states, as we outlined in Chapter 4.

On the other hand, another main result included in Chapter 2 regarding a class of gLMG models whose ground state is a Dicke state of type m, is the detailed derivation of the asymptotic behavior of their entanglement spectrum, for arbitrary values of m and $\alpha = \lim_{N\to\infty} L/N$ where L is the size of the subsytem considered. This makes it possible to compute the von Neumann and Rényi entropies in closed form, and to derive their asymptotic behavior. A notable outcome of our analysis is that both of these entropies scale identically in the latter limit. In particular, this behaviour of the Rényi entropy implies that the models are not critical thus showing an explicit example of one-dimensional quantum many-body system whose von Neuman entanglement entropy scales as $\log L$ which is not effectively described by a CFT. Indeed, this fact motivated the study presented in Chapter 5 where generalized entanglement measures for mixed states are proposed. Since the logarithmic negativity does not contain any free parameter, generalized entanglement measures for mixed states could be useful to study criticality experimentally, where quantum states are never guaranteed to be completely pure. Regarding this topic, the main conclusion at this moment is that there is always at least some

value of the parameters for which the new generalized negativities become entanglement monotones [**P9**].

List of publications

[**P1**] José A. Carrasco, F. Finkel, A. González-López, and P. Tempesta, *A duality principle for the multi-block entanglement entropy of free fermion systems*, Sci. Rep.-UK **7** (2017), 11206.

[**P2**] José A. Carrasco, F. Finkel, A. González-López, M. A. Rodríguez, and P. Tempesta, *Critical behavior of* su(1|1) *supersymmetric spin chains with long-range interactions*, Phys. Rev. E **93** (2016), 062103(12).

[**P3**] José A. Carrasco, F. Finkel, A. González-López, and M. A. Rodríguez, *Supersymmetric spin chains with nonmonotonic dispersion relation: Criticality and entanglement entropy*, Phys. Rev. E **95** (2017), 012129(15).

[**P4**] José A. Carrasco, F. Finkel, A. González-López, M. A. Rodríguez, and P. Tempesta, *Generalized isotropic Lipkin—Meshkov—Glick models: ground state entanglement and quantum entropies*, J. Stat. Mech.-Theory E. **2016** (2016), 033114(33).

[**P5**] José A. Carrasco, F. Finkel, and A. González-López, *Generalized Lipkin–Meshkov–Glick models of Haldane–Shastry type*, J. Stat. Mech.-Theory E. **2017** (2017), 103102.

[**P6**] B. Basu-Mallick, N. Bondyopadhaya, J. A. Carrasco, F. Finkel, and A. González-López, *Supersymmetric t-J models with long-range interactions: partition function and spectrum*, J. Stat. Mech.-Theory E. (in press). `arXiv:1811.10297`

[**P7**] B. Basu-Mallick, N. Bondyopadhaya, J. A. Carrasco, F. Finkel, and A. González-López, *Supersymmetric t-J models with long-range interactions: thermodynamics and criticality*. `arXiv:1903.12541`

[**P8**] M. Baradaran, José A. Carrasco, F. Finkel, and A. González-López, *Jastrow-like ground states for quantum many-body potentials with near-neighbors interactions*, Ann. Phys.-New York **388** (2018), 147–161.

[**P9**] J. Carrasco, B. Marmo and P. Tempesta, *New computable entanglement monotones and witness from formal group theory*, `arXiv:1904.10691`

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