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ORIGINAL RESEARCH

Superdifferential Analysis of the Takagi-Van Der Waerden Functions

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Abstract
In this work we completely describe the superdifferential of the Takagi-Van der Waerden functions and, as a consequence, the local maxima of these functions are characterized. Regarding the set of points where the superdifferential is not empty, we calculate its Hausdorff dimension as well as its corresponding Hausdorff measure. To do so, for any even integer greater than or equal to two we determine the 1/2-dimensional Hausdorff measure of the set of points where Takagi-Van der Waerden functions attain their global maximum.

Keywords Takagi-Van der Waerden functions · Superdifferential · Local maxima set · Global maxima set · Hausdorff dimension · Hausdorff measure

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1 Introduction

Continuous nowhere differentiable functions have caught the eye of a large number of mathematicians throughout the course of history. Some years after the emergence of

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Weierstrass's function, T. Takagi provided a simple example of a continuous nowhere differentiable function, known today as the Takagi function (see [16]), defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}\phi(2^n x), \quad x \in [0, 1]$$

where $\phi(x)$ denotes the distance from the point x to the nearest integer. Throughout the 20th century, the Takagi function has been rediscovered on a frequent basis, and it continues to pop up within different mathematical contexts such as mathematical analysis, probability theory and number theory. The surveys [2] and [14] contain a lot of information about the Takagi function, and they are highly recommended for readers who want to acquaint themselves with this function.

In 1930 a variant of the Takagi function was rediscovered by B. L. Van der Waerden (see [17]), where this author used base ten instead of base two, that is the reason why the family of functions we consider in this work receives such name.

For every integer $r \geq 2$, the Takagi-Van der Waerden function $f_r : [0, 1] \to \mathbb{R}$ is defined as follows

$$f_r(x) = \sum_{n=0}^{\infty} \frac{1}{r^n}\phi(r^n x),$$

and the first proof that we know of its nowhere differentiability can be found in the work developed by F. A. Behrend (see [4]). These functions have been studied by many authors such as H. Whitney (see [18]), J. B. Brown and G. Kozlowski (see [5]), A. Shidfar and K. Sabetfakhri (see [15]), J.P. Kahane (see [13]), or the authors we will mention hereunder. Recently, P. C. Allaart has examined the level sets of these functions (see [1]) and the set of points where their lateral derivatives are infinite has been characterized by the authors (see [11]).

Now, we rewrite the Takagi-Van der Waerden function with the sole objective of acquiring a more user-friendly notation. Let us consider the set $D = \{kr^{-n} \in [0, 1] : k, n \in \mathbb{Z}^+\}$ and we decompose it as an increasing sequence of finite subsets of D given by

$$D_n = \left\{ \frac{k}{r^{n-1}} \in [0, 1] : k \in \mathbb{Z} \right\}.$$

Then, we have

$$f_r(x) = \sum_{n=1}^{\infty} g_n(x) = \lim_n G_n(x),$$

where $g_n(x) = \operatorname{dist}(x, D_n)$ denotes the distance from x to the set D_n and $G_n = g_1 + \cdots + g_n$. Furthermore, the set of middle points of consecutive points of D_n is denoted by $\widetilde{D}_n$ and we define $\widetilde{D} = \cup_n \widetilde{D}_n$.

These functions are also very interesting from the subdifferential viewpoint; the following result was obtained by P. Gora and R. J. Stern (see [12]) for the Takagi function and it was proved in a more general setting, which includes the Takagi-Van der Waerden functions, by the first two authors later on (see [9]).

The Fréchet subdifferential of a function at a point will be defined in Section 2 below.

Theorem 1.1 *Let $r \geq 2$ be an integer. If $x \in D$ then the Fréchet subdifferential of the Takagi-Van der Waerden function at x is the whole $\mathbb{R}$, otherwise the Fréchet subdifferential is empty.*

The first two authors have previously calculated the superdifferential of the Takagi function (see [10]). Among other results, in this paper we completely describe the superdifferential of the Takagi-Van der Waerden functions, which we denote by $\partial^+ f_r(\cdot)$.

The different nature that these functions have whether r is odd or even can be seen in the main results of this work:

Theorem 1.2 *Let $r \geq 3$ be an odd integer. If $x \in \widetilde{D}$ then the function $f_r(z) - z\xi$ attains a local maximum at x for every $\xi \in \mathbb{R}$, and in particular we have $\partial^+ f_r(x) = \mathbb{R}$, otherwise $\partial^+ f_r(x) = \emptyset$.*

Theorem 1.3 *Let $r \geq 2$ be an even integer and $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$ with $\varepsilon_n \in \{0, 1, \ldots, r-1\}$. Then, $\partial^+ f_r(x) \neq \emptyset$ if and only if there exists $m_0 \geq 1$ such that for all $i \geq 0$ one of the following situations arises:*

$$\begin{aligned} &(1) \quad \varepsilon_{m_0+2i} = \frac{r}{2} - 1 \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \geq \frac{r}{2}, \ \textit{or} \\ &(2) \quad \varepsilon_{m_0+2i} = \frac{r}{2} \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \leq \frac{r}{2} - 1. \end{aligned} \tag{$\mathbf{S_{m_0}}$}$$

Theorem 1.4 *Let $r \geq 2$ be an even integer and $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$ with $\varepsilon_n \in \{0, 1, \ldots, r-1\}$ such that $\partial^+ f_r(x) \neq \emptyset$. If there exists $n_0 \geq 1$ such that for all $i \geq 0$ we have $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{n_0+2i+1} = \frac{r}{2}$, then*

$$\partial^+ f_r(x) = G'_{n_0-1}(x) + [0, 1],$$

otherwise we have

$$\partial^+ f_r(x) = \{G'_{m_0-1}(x)\},$$

where m_0 is any positive integer satisfying ($\mathbf{S_{m_0}}$).

In Theorem 1.4 observe that the value of $G'_{m_0-1}(x)$ is independent of the choice of m_0.

As a consequence of the previous theorems, we characterize the set of points where Takagi-Van der Waerden functions have a local maximum. This completes the study of their maxima initiated by Y. Baba (see [3]), which proved that these functions attain their global maximum only at $1/2$ whenever r is odd, and concerning the even case, the following result is obtained:

Theorem 1.5 (Y. Baba, 1984) *Let $r \geq 2$ be an even integer and $x = \sum_{n=1}^{\infty} \omega_n r^{-2n}$ with $\omega_n \in \{0, 1, \ldots, r^2 - 1\}$. Then, f_r attains its global maximum at x if and only if*

$$\frac{r^2 - r}{2} \leq \omega_n \leq \frac{r^2 + r - 2}{2}$$

for every n. Furthermore, the Hausdorff dimension of the global maxima set is $1/2$.

However, the $1/2$-dimensional Hausdorff measure of the global maxima set was not calculated and up until now, it even was not known for the Takagi function. We prove the following result which answers this open question.

Theorem 1.6 *Let $r \geq 2$ be an even integer. Then, the $1/2$-dimensional Hausdorff measure of the global maxima set of f_r is $1/\sqrt{r+1}$.*

Joining the previous theorem and Y. Baba's result we conclude the following:

Theorem 1.7 *Let $r \geq 2$ be an even integer. Then, the set of points where the superdifferential of f_r is not empty has Hausdorff dimension $1/2$ and its $1/2$-dimensional Hausdorff measure is plus infinity.*

Finally, we would like to highlight a special property that these functions have, which is obtained as a consequence of our results; Takagi-Van der Waerden functions have empty subdifferential and superdifferential almost everywhere.

2 Preliminary and Tools

The aim of this section is twofold. On the one hand, we introduce some more notation and nonsmooth definitions in order to state and explain our results more precisely. For any unexplained terms of facts in Nonsmooth Analysis we refer to the books [8] and [6].

Recall that, given an upper semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ and a point $x \in \mathbb{R}$, the Fréchet superdifferential of f at x, denoted by $\partial^+ f(x)$, is defined as the set of $\xi \in \mathbb{R}$ such that

$$\limsup_{h \to 0} \frac{f(x+h) - f(x) - \xi h}{|h|} \leq 0.$$

The function f is said to be superdifferentiable at x if $\partial^+ f(x) \neq \emptyset$.

Similarly, for a lower semicontinuous function $f : \mathbb{R} \to \mathbb{R}$, the Fréchet subdifferential of f at x, denoted by $\partial f(x)$, may be defined as

$$\partial f(x) = -\partial^+(-f)(x)$$

and the function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$. Futhermore, a continuous function f is derivable at x if and only if $\partial^+ f(x) \neq \emptyset \neq \partial f(x)$. In such case, we have that $\partial^+ f(x) = \partial f(x) = \{f'(x)\}$.

In the one dimensional case, the superdifferential may be characterized in terms of the Dini derivatives:

$$\begin{aligned} d_- f(x) &= \liminf_{h \uparrow 0} \frac{f(x+h) - f(x)}{h} \\ D^+ f(x) &= \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}. \end{aligned}$$

Proposition 2.1 *An upper semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ is superdifferentiable at $x \in \mathbb{R}$ if and only if*

$$D^+ f(x) \leq d_- f(x) \quad \textit{and} \quad [D^+ f(x), d_- f(x)] \cap \mathbb{R} \neq \emptyset.$$

In such case, $\partial^+ f(x) = [D^+ f(x), d_- f(x)] \cap \mathbb{R}$.

On the other hand, we devote the second part of this section to set some notation and elementary facts that will be used throughout the work.

For a real number $x \in [0, 1]$ we consider its base r expansion given by

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{r^n}, \qquad \varepsilon_n \in \{0, 1, \ldots, r-1\}.$$

It is clear that $x \in D$ if and only if there exists n_0 such that either $\varepsilon_n = 0$ for every $n \geq n_0$ or $\varepsilon_n = r - 1$ for every $n \geq n_0$. In this case, unless expressly stated otherwise, we will choose the representation ending in all zeros.

Regarding the set of middle points, we have to distinguish two cases: if r is even then $\widetilde{D} \subset D$ and we are done, but for r odd, we have that $D \cap \widetilde{D} = \emptyset$ and $x \in \widetilde{D}$ if and only if there exists n_1 such that $\varepsilon_n = \frac{r-1}{2}$ for every $n \geq n_1$.

It is immediate to see that when $x \in D \cup \widetilde{D}$, $g_n'(x)$ does not exist for n big enough, meanwhile if $x \notin D \cup \widetilde{D}$ then the derivative $g_n'(x)$ exists for all n. In order to state the following fact, which completely describes the behavior of such derivatives, we denote

$$k_n(x) = \min \left\{ j \geq n : \varepsilon_j \neq \frac{r-1}{2} \right\}$$

for every $x \notin D \cup \widetilde{D}$, and if no confusion arises we simply write k_n instead of $k_n(x)$. It is important to observe that the third case of this fact can only occur when r is odd.

Fact 1 *Let $r \geq 2$ be an integer. If $x \notin D \cup \widetilde{D}$, then the derivatives $g_n'(x) \in \{-1, 1\}$ are determined as follows:*

(1) *If $\varepsilon_n < \frac{r-1}{2}$, then $g_n'(x) = 1$.*
(2) *If $\varepsilon_n > \frac{r-1}{2}$, then $g_n'(x) = -1$.*
(3) *If $\varepsilon_n = \frac{r-1}{2}$, then $g_n'(x) = g_{k_n}'(x)$.*

For the sake of simplicity, we will take advantage of the fact that these functions can be extended periodically to the whole real line, which implies that f_r is an even function. Hence for every $x \notin D \cup \widetilde{D}$ we get $g_n'(x) = -g_n'(-x)$ for every n, and

$$\begin{aligned} D^+ f_r(x) &= \limsup_{h \downarrow 0} \frac{f_r(x+h) - f_r(x)}{h} = \limsup_{h \uparrow 0} \frac{f_r(-x+h) - f_r(-x)}{-h} \\ &= -\liminf_{h \uparrow 0} \frac{f_r(-x+h) - f_r(-x)}{h} = -d_- f_r(-x). \end{aligned} \tag{2.1}$$

Finally, as a consequence of Theorem 1.1, we have that $\partial^+ f_r(x) = \emptyset$ for every $x \in D$. In the sequel, for every $x \notin D \cup \widetilde{D}$ we will denote

$$a_n = \max\{y \in D_n : y < x\}, \quad b_n = \min\{y \in D_n : x < y\}$$

and c_n the midpoint of (a_n, b_n).

3 The Case *r* Odd

First of all, we have to bear in mind that $\widetilde{D} \cap D = \emptyset$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n whenever r is odd.

Proof of Theorem 1.2 On the one hand, if $x \in \widetilde{D}$ then there exists an index $n_0 \geq 1$ such that $x \in \widetilde{D}_{n_0}$ and $x \notin \widetilde{D}_k$ provided that $k < n_0$. For every $\xi \in \mathbb{R}$ we consider n satisfying that $2n_0 + |\xi| \leq n$. Now, if $2|h| < r^{1-n}$ then $g_k(x+h) - g_k(x) = -|h|$ for every $n_0 \leq k \leq n$, and $g_k(x+h) - g_k(x) \leq 0$ for every $k > n$. Therefore, we obtain

$$\begin{aligned} f_r(x+h) - f_r(x) - \xi h &\leq \sum_{k=1}^{n_0-1} (g_k(x+h) - g_k(x)) - (n-n_0)|h| + |\xi||h| \\ &\leq |h|(2n_0 + |\xi| - n - 1) \leq 0 \end{aligned}$$

which proves that $f_r(z) - z\xi$ attains a local maximum at x.

On the other hand, if $x \notin D \cup \widetilde{D}$ we may consider the function

$$\varphi(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2} \frac{1}{r^{n-1}} - g_n(z) \right) = \frac{r}{2(r-1)} - f_r(z),$$

which is a Generalized Takagi-Van der Waerden function associated to the set $\widetilde{D}$ in the sense defined in [9], and by Theorem 1.2. of [9] we have $\partial\varphi(x) = \emptyset$. Thus the result follows immediately since $\partial\varphi(x) = -\partial^+ f_r(x)$. However, for the sake of self-containedness, we give below another demonstration. For every n, we consider

$$x_n^* := \begin{cases} 2c_{k_n+1} - x & \text{if } g'_{k_n+1}(x) = -1 \\ 2a_{k_n+1} - x & \text{if } g'_{k_n+1}(x) = 1 \end{cases}$$

and observe that $x_n^* < x$. Firstly, we have $g_j(2a_{k_n+1} - x) = g_j(-x) = g_j(x)$ for all $j \geq k_n + 1$. Secondly, since $2c_{k_n+1} = a_{k_n+1} + b_{k_n+1}$ we obtain $g_j(2c_{k_n+1} - x) = g_j(-x) = g_j(x)$ for all $j \geq k_n + 1$. Therefore, we get

$$f_r(x_n^*) - f_r(x) = \sum_{j=1}^{k_n} (g_j(x_n^*) - g_j(x)).$$

If $g'_{k_n+1}(x) = -1$, then $x_n^* = 2c_{k_n+1} - x \in (a_{k_n+1}, b_{k_n+1})$ and $(a_{k_n+1}, b_{k_n+1}) \cap \widetilde{D}_j = \emptyset$ for every $j \leq k_n$, so we get

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G'_{k_n}(x).$$

Otherwise, we have that $a_m < a_{m+1} = \cdots = a_{k_n+1}$ for some $m \leq k_n$, which implies $\varepsilon_{m+1} = \cdots = \varepsilon_{k_n} = 0$. Since $x_n^* = 2a_{m+1} - x \in (a_{m+1} - 2^{-1}r^{-k_n}, a_{m+1})$ and $g_j(x_n^*) = g_j(x)$ for all $j \geq m+1$ we get

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G'_m(x) \leq G'_{k_n}(x).$$

Therefore,

$$d_- f_r(x) \leq \liminf_n \frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} \leq \liminf_n G'_{k_n}(x).$$

As we have mentioned above, since f_r is an even function we have $G'_{k_n}(-x) = -G'_{k_n}(x)$, and consequently

$$D^+ f_r(x) = -d_- f_r(-x) \geq -\liminf_n G'_{k_n}(-x) = \limsup_n G'_{k_n}(x).$$

Therefore, we obtain

$$d_- f_r(x) \leq \liminf_n G'_{k_n}(x) \leq \limsup_n G'_{k_n}(x) \leq D^+ f_r(x)$$

and if $\partial^+ f_r(x) \neq \emptyset$, then by Proposition 2.1, we deduce that $\lim_n G'_{k_n}(x)$ exists. However, this limit can not exist since $G'_{k_{n+1}}(x) - G'_{k_n}(x) = g'_{k_{n+1}}(x)(k_{n+1} - k_n)$ and $\lim_n k_n = +\infty$. This proves the result. □

As a consequence of Theorem 1.2 we have that the function f_r has a local maximum at x if and only if $x \in \widetilde{D}$, and as this set is countable, the Hausdorff dimension of the local maxima set is zero.

4 The Case *r* even: Superdifferential Analysis

This section is devoted to proving Theorems 1.3 and 1.4. In the sequel, we assume that $r \geq 2$ is even, and consequently we have $\widetilde{D}_n \subset D_{n+1}$ for every n. Hence, by Theorem 1.1, it only remains to study the case when $x \notin D$.

Lemma 4.1 *Let $r \geq 2$ be an even integer and $x \notin D$. Then*

$$\begin{aligned} d_- f_r(x) &\leq \liminf_n G_n'(x) + 1, \\ D^+ f_r(x) &\geq \limsup_n G_n'(x) - 1. \end{aligned}$$

Proof For every n we consider $x_n^* = 2a_{n+1} - x$ and we have that $g_j(x_n^*) = g_j(-x) = g_j(x)$ for all $j \geq n+1$, so we get

$$f_r(x_n^*) - f_r(x) = \sum_{j=1}^{n} (g_j(x_n^*) - g_j(x)).$$

If $a_{n+1} = c_n$ then $2a_{n+1} = 2c_n = a_n + b_n \in D_n$, $g_n'(x) = -1$, $g_n(x_n^*) = g_n(x)$ and hence

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_{n-1}'(x) = G_n'(x) + 1.$$

If $a_n < a_{n+1} \neq c_n$ then

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_n'(x).$$

Finally, if $a_m < a_{m+1} = \cdots = a_n = a_{n+1}$ for some $m < n$ then $g_j'(x) = 1$ for every $m + 1 \leq j \leq n$. Thus, we get

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_{m-1}'(x) = G_m'(x) + 1 < G_n'(x) + 1$$

provided that $a_{m+1} = c_m$, and

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_m'(x) < G_n'(x)$$

otherwise. From these facts we deduce

$$d_- f_r(x) \leq \liminf_n \frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} \leq \liminf_n G_n'(x) + 1.$$

As we have mentioned in the introduction to this section, since f_r is an even function we obtain $g_j'(x) = -g_j'(-x)$ for all $j \geq 1$, and consequently

$$D^+ f_r(x) = -d_- f_r(-x) \geq -\liminf_n G_n'(-x) - 1 = \limsup_n G_n'(x) - 1.$$

□

Lemma 4.2 *Let $r \geq 2$ be an even integer and $x \notin D$. Then*

$$g_n(x) + g_{n+1}(x) \leq \frac{r}{2r^n}$$

for every n. Furthermore, the equality holds if and only if one of the following situations arises:

(1) $\varepsilon_n = \frac{r}{2} - 1$ *and* $\varepsilon_{n+1} \geq \frac{r}{2}$, *or*
(2) $\varepsilon_n = \frac{r}{2}$ *and* $\varepsilon_{n+1} \leq \frac{r}{2} - 1$.

Proof Firstly, we prove

$$g_1(x) + g_2(x) \leq \frac{1}{2}. \tag{4.1}$$

Indeed, as $g_1 + g_2$ is differentiable with null derivative on $(\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r})$ it is constant on $[\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}]$. Moreover from the periodicity of g_2 it is clear that $[\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}]$ is the set of maxima of $g_1 + g_2$. This proves Eq. 4.1. Furthermore, the equality in Eq. 4.1 holds if and only if $\varepsilon_1 = \frac{r}{2} - 1$ and $\varepsilon_2 \geq \frac{r}{2}$, or $\varepsilon_1 = \frac{r}{2}$ and $\varepsilon_2 \leq \frac{r}{2} - 1$.

Using $g_{n+1}(x) = r^{-n} g_1(r^n x)$ and by Eq. 4.1 we get

$$\begin{aligned} g_n(x) + g_{n+1}(x) &= \frac{1}{r^{n-1}} g_1(r^{n-1}x) + \frac{1}{r^n} g_1(r^n x) \\ &= \frac{1}{r^{n-1}} g_1(r^{n-1}x) + \frac{1}{r^{n-1}} g_2(r^{n-1}x) \leq \frac{1}{2r^{n-1}}. \end{aligned}$$

As we mentioned at the end of Section 2, the functions g_1 and g_2 may be extended periodically to the whole real line with period one, so if we observe that $r^{n-1}x = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{n-1}, \varepsilon_n \varepsilon_{n+1}$ then the result follows immediately (see Fig. 1).

□

We need the following notation. For every $x \notin D$ we denote

$$\begin{aligned} S(x) &:= S = \limsup_n G'_n(x), \text{ and} \\ I(x) &:= I = \liminf_n G'_n(x). \end{aligned}$$

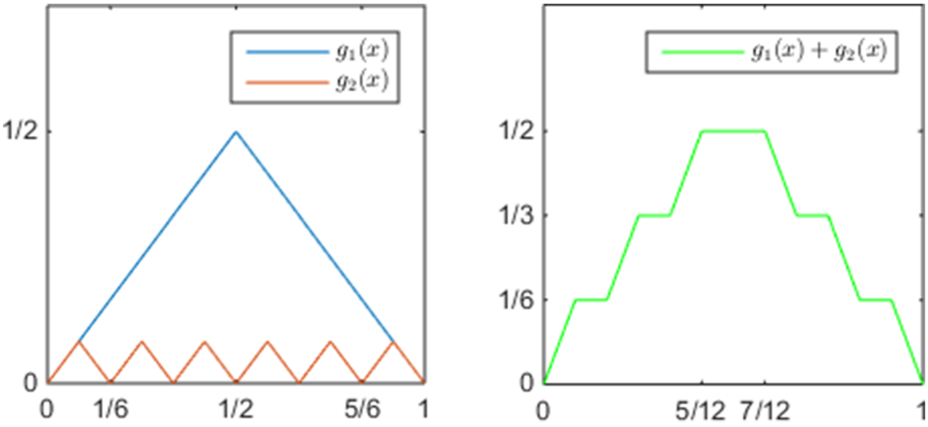

Fig. 1 Functions $g_1(x)$ and $g_2(x)$ for $r = 6$

Theorem 4.3 *Let $r \geq 2$ be an even integer and $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$ with $\varepsilon_n \in \{0, 1, \dots, r-1\}$. If $\partial^+ f_r(x) \neq \emptyset$ then there exists $m_0 \geq 1$ such that for all $i \geq 0$ one of the following situations arises:*

$$\begin{aligned} &(1) \quad \varepsilon_{m_0+2i} = \frac{r}{2} - 1 \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \geq \frac{r}{2}, \ \textit{or} \\ &(2) \quad \varepsilon_{m_0+2i} = \frac{r}{2} \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \leq \frac{r}{2} - 1. \end{aligned} \qquad (\mathbf{S_{m_0}})$$

Proof By Proposition 2.1 and Lemma 4.1 we have that if $S = +\infty$ or $I = -\infty$ then $\partial^+ f_r(x) = \emptyset$. Furthermore, if both limits are finite and $S - I > 2$, then

$$D^+ f_r(x) - d_- f_r(x) \geq S - I - 2 > 0,$$

and by Proposition 2.1 again, we get $\partial^+ f_r(x) = \emptyset$. Therefore, S and I are finite and $S - I \in \{1, 2\}$.

There exists an integer $m \geq 1$ such that $I \leq G'_n(x) \leq S$ for all $n \geq m - 1$. Without loss of generality we may assume that $G'_{m-1}(x) = I + 1$ and we have $|G'_n(x) - G'_{m-1}(x)| \leq 1$ for all $n \geq m$. In particular, we have $G'_{m+2i+1}(x) = I + 1$ and necessarily for all $i \geq 0$ either

(a) $\varepsilon_{m+2i} \leq \frac{r}{2} - 1$ and $\varepsilon_{m+2i+1} \geq \frac{r}{2}$, or
(b) $\varepsilon_{m+2i} \geq \frac{r}{2}$ and $\varepsilon_{m+2i+1} \leq \frac{r}{2} - 1$.

If there are infinitely many indices $(i_j)_{j \geq 1}$ such that $\varepsilon_{m+2i_j} < \frac{r}{2} - 1$, then $\varepsilon_{m+2i_j+1} \geq \frac{r}{2}$ and we consider $y_j^* = 2b_{m+2i_j+1} - x > x$. Hence we get

$$\frac{f_r(y_j^*) - f_r(x)}{y_j^* - x} = G'_{m+2i_j}(x) = I + 2,$$

and then $D^+ f_r(x) \geq I + 2$. By Lemma 4.1 we obtain $D^+ f_r(x) \geq I + 2 > I + 1 \geq d_- f_r(x)$ which implies $\partial^+ f_r(x) = \emptyset$.

Otherwise, suppose there are infinitely many indices $(i_j)_{j \geq 1}$ such that $\varepsilon_{m+2i_j} > \frac{r}{2}$. Then $\varepsilon_{m+2i_j+1} \leq \frac{r}{2} - 1$ and we consider $x_j^* = 2a_{m+2i_j+1} - x < x$, so we get

$$\frac{f_r(x_j^*) - f_r(x)}{x_j^* - x} = G'_{m+2i_j}(x) = I.$$

Therefore, $d_- f_r(x) \leq I$ and by Eq. 2.1 we get $D^+ f_r(x) \geq S$. We have $\partial^+ f_r(x) = \emptyset$ since $d_- f_r(x) \leq I < S \leq D^+ f_r(x)$.

If we are not under the previous situations then there exists $m_0 m$ such that $(\mathbf{S_{m_0}})$ is satisfied. □

Once we have obtained what a point with non-empty superdifferential looks like, we determine such superdifferential.

Lemma 4.4 *Let $r \geq 2$ be an even integer and $x \notin D$. If there exists $m_0 \geq 1$ such that for all $i \geq 0$ one of the following situations arises:*

$$\begin{aligned} &(1) \ \varepsilon_{m_0+2i} = \tfrac{r}{2} - 1 \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \geq \tfrac{r}{2}, \ \textit{or} \\ &(2) \ \varepsilon_{m_0+2i} = \tfrac{r}{2} \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \leq \tfrac{r}{2} - 1, \end{aligned} \qquad (\mathbf{S_{m_0}})$$

then

$$\begin{aligned} d_- f_r(x) &= \liminf_n G_n'(x) + 1, \text{ and} \\ D^+ f_r(x) &= \limsup_n G_n'(x) - 1. \end{aligned}$$

Proof If $x^* \in (a_{m_0}, x)$, then there exists $n > m_0$ such that $a_{n-1} < x^* \le c_{n-1} < x$, which implies $g_{n-1}'(x) = -1$ and $G_{n-2}'(x) = G_{n-1}'(x) + 1 \ge I + 1$ where the inequality holds since $G_j'(x) \ge I$ for all $j \ge m_0$. Hence we have

$$\begin{aligned} \frac{f_r(x^*) - f_r(x)}{x^* - x} &= G_{n-2}'(x) + \frac{1}{x^* - x} \sum_{j=n-1}^{\infty} (g_j(x^*) - g_j(x)) \\ &\ge I + 1 + \frac{1}{x - x^*} \sum_{j=n-1}^{\infty} (g_j(x) - g_j(x^*)). \end{aligned}$$

Furthermore, by Lemma 4.2 we get

$$g_{m_0+2i}(x) + g_{m_0+2i+1}(x) - (g_{m_0+2i}(x^*) + g_{m_0+2i+1}(x^*)) \ge 0$$

for all $i \ge 0$, and consequently we obtain

$$\begin{aligned} &\sum_{j=m_0+2p}^{\infty} (g_j(x) - g_j(x^*)) \\ &= \sum_{i=p}^{\infty} \left[g_{m_0+2i}(x) + g_{m_0+2i+1}(x) - (g_{m_0+2i}(x^*) + g_{m_0+2i+1}(x^*)) \right] \ge 0. \end{aligned}$$

With the previous fact in mind, we distinguish two cases: if $n = m_0 + 2p + 1$ for some $p \ge 0$, then

$$\frac{f_r(x^*) - f_r(x)}{x^* - x} \ge I + 1.$$

If $n = m_0 + 2p$ for some $p \ge 0$, then $G_{n-1}'(x) = I + 1$. As $g_{n-1}'(x) = -1$ we necessarily have $g_{n-2}'(x) = 1$, which implies $G_{n-2}'(x) = I + 2$. Hence we get

$$\begin{aligned} \frac{f_r(x^*) - f_r(x)}{x^* - x} &\ge G_{n-2}'(x) + \frac{g_{n-1}(x^*) - g_{n-1}(x)}{x^* - x} \\ &\quad + \sum_{j=m_0+2p}^{\infty} \frac{1}{x^* - x} (g_j(x^*) - g_j(x)) \ge G_{n-2}'(x) + \frac{g_{n-1}(x^*) - g_{n-1}(x)}{x^* - x} \end{aligned}$$

which implies $d_- f_r(x) = I + 1$ by Lemma 4.1. The other equality follows from the fact $D^+ f_r(x) = -d_- f_r(-x)$. □

Theorem 4.5 *Let $r \ge 2$ be an even integer and $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$, where $\varepsilon_n \in \{0, 1, \dots, r-1\}$, satisfying* ($\mathbf{S_{m_0}}$).

a) *If there exists $n_0 \ge m_0$ such that for all $i \ge 0$ we have $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{n_0+2i+1} = \frac{r}{2}$, then*

$$\partial^+ f_r(x) = G_{n_0-1}'(x) + [0, 1].$$

b) *Otherwise we have*

$$\partial^+ f_r(x) = \{G_{m_0-1}'(x)\}.$$

Proof If there exists $n_0 \geq m_0$ such that for all $i \geq 0$ we have $\varepsilon_{n_0+2i} = \frac{r}{2}-1$ and $\varepsilon_{n_0+2i+1} = \frac{r}{2}$, then $g'_n(x) + g'_{n+1}(x) = 0$ for all $n \geq n_0$. Therefore we have

$$\begin{aligned} G'_{n_0+2i}(x) &= G'_{n_0}(x) + \sum_{j=1}^{i}(g'_{n_0+2j-1}(x) + g'_{n_0+2j}(x)) = G'_{n_0}(x) \\ &= G'_{n_0-1}(x) + g'_{n_0}(x), \end{aligned}$$

and

$$G'_{n_0+2i+1}(x) = G'_{n_0-1}(x) + \textstyle\sum_{j=0}^{i}(g'_{n_0+2j}(x) + g'_{n_0+2j+1}(x)) = G'_{n_0-1}(x)$$

for all $i \geq 0$. Since $\varepsilon_{n_0} = \frac{r}{2} - 1$ we have $g'_{n_0}(x) = 1$, which implies $I = G'_{n_0-1}(x)$ and $S = G'_{n_0-1}(x)+1$. By Lemma 4.4 we have $D^+ f_r(x) = G'_{n_0-1}(x)$ and $d_- f_r(x) = G'_{n_0-1}(x)+1$, so we get

$$\partial^+ f_r(x) = G'_{n_0-1}(x) + [0, 1].$$

Otherwise, we have $g'_{m_0+2i}(x) + g'_{m_0+2i+1}(x) = 0$ for all $i \geq 0$. Therefore we get

$$G'_{m_0+2i+1}(x) = G'_{m_0-1}(x) + \sum_{j=0}^{i}(g'_{m_0+2j}(x) + g'_{m_0+2j+1}(x)) = G'_{m_0-1}(x),$$

and

$$G'_{m_0+2i}(x) = G'_{m_0+2i-1}(x) + g'_{m_0+2i}(x) = G'_{m_0-1}(x) + g'_{m_0+2i}(x)$$

for all $i \geq 0$. Furthermore, we have $\liminf_i g'_{m_0+2i}(x) = -1$ and $\limsup_i g'_{m_0+2i}(x) = 1$ since x does not fulfill condition a). Hence we get $I = G'_{m_0-1}(x)-1$ and $S = G'_{m_0-1}(x)+1$. By Lemma 4.4 we have $D^+ f_r(x) = G'_{m_0-1}(x)$ and $d_- f_r(x) = G'_{m_0-1}(x)$, which implies $\partial^+ f_r(x) = \{G'_{m_0-1}(x)\}$, and this completes the proof of the result. □

Theorems 4.3 and 4.5 give us the proof of Theorem 1.3, meanwhile the proof of Theorem 1.4 is obtained by Theorems 1.3 and 4.5.

On the other hand, Theorem 1.3, Theorem 1.4 and Lemma 4.2 allow us to characterize the local maxima of the Takagi-Van der Waerden functions.

Corollary 4.6 *Let $r \geq 2$ be an even integer and $x \in [0, 1]$. Then, f_r has a local maximum at x if and only if $\partial^+ f_r(x) \neq \emptyset$ and $G'_{m_0-1}(x) = 0$, where $m_0 \geq 1$ is any positive integer satisfying ($\boldsymbol{S_{m_0}}$).*

5 The Case *r* even: Hausdorff Measure of the Set of Points with Non-empty Superdifferential

For any even integer $r \geq 2$, we denote by E_r the set of points where the Takagi-Van Waerden functions attain their global maximum, which is given by $\frac{r^2}{2r^2-2}$.

Taking advantage of Y. Baba's result (see Theorem 1.5 above), we obtain the 1/2-dimensional Hausdorff measure of the set E_r, which is denoted by $\mathcal{H}^{1/2}(E_r)$. To do so, we need to have a better understanding of how this set is constructed. It is important to point out that we follow the mathematical notation used in [7], which is standard, and of course we refer to this book as a general reference for concepts related to the Hausdorff measure of a set.

For every $n \geq 1$ we consider the set

$$A_n = \left\{ \sum_{k=1}^{n} \frac{\omega_k}{r^{2k}} : \frac{r^2-r}{2} \leq \omega_k \leq \frac{r^2+r-2}{2} \right\}$$

and we arrange this set as $A_n = \{x_0, \dots, x_{r^n-1}\}$ satisfying that $x_0 < x_1 < \cdots < x_{r^n-1}$. Therefore,

$$x_0 = \frac{r^2-r}{2} \sum_{k=1}^{n} \frac{1}{r^{2k}}$$

and for $0 \leq i = \sum_{k=0}^{n-1} \varepsilon_k r^k < r^n$ with $0 \leq \varepsilon_k < r$ we have

$$x_i = x_0 + \sum_{k=0}^{n-1} \frac{\varepsilon_k}{r^{2(n-k)}}.$$

For every $0 \leq i \leq j < r^n$ we define the closed interval

$$I_{i,j}^n = \left[x_i + \frac{r^2-r}{2} \sum_{k=n+1}^{\infty} \frac{1}{r^{2k}}, x_j + \frac{r^2+r-2}{2} \sum_{k=n+1}^{\infty} \frac{1}{r^{2k}} \right],$$

and, by Theorem 1.5 it is immediate to see

$$E_r = \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{r^n-1} I_{i,i}^n, \tag{5.1}$$

which implies that E_r is a compact set.

Lemma 5.1 *Let $m \geq 0$ be an integer. The function $F : [1-r, r-1]^m \times [1, r-1] \to \mathbb{R}$ defined by*

$$F(x_0, \dots, x_m) = (r+1) \sum_{k=0}^{m} x_k r^{2k} - \left(\sum_{k=0}^{m} x_k r^k + 1 \right)^2 + 1 \tag{5.2}$$

is nonnegative.

Proof The case $m = 0$ is immediate, so we suppose $m \geq 1$. Since F is a concave function, it necessarily attains its minimum at some vertex of the simplex $[1-r, r-1]^m \times [1, r-1]$. Let $V = \{1-r, r-1\}^m \times \{1, r-1\}$ be the set of vertices of such simplex.

If $\mathbf{x} = (x_0, \dots, x_m) \in V$ and $x_m = 1$ then

$$F(\mathbf{x}) \geq (r+1)r^{2m} - r^{2m} + 1 - \left[r^m + r^m - 1 + 1 \right]^2 + 1 = r \cdot r^{2m} - 4r^{2m} + 2,$$

which is greater than or equal to zero provided that $r \geq 4$.

If $\mathbf{x} = (x_0, \dots, x_m) \in V$, $x_m = r-1$ and there exists $p \in \{1, \dots, m\}$ such that $x_m = \cdots = x_p = r-1$ and $x_{p-1} = 1-r$, then

$$\begin{aligned} F(\mathbf{x}) &\geq r^{2m+2} - r^{2p} - r^{2p} + 1 - \left[r^{m+1} - 1 - 2(r-1)r^{p-1} + 1 \right]^2 + 1 \\ &= \left[2r^{m+1} - 2(r-1)r^{p-1} \right] 2(r-1)r^{p-1} - 2r^{2p} + 2 \geq 0. \end{aligned}$$

Finally,

$$F(r-1, \dots, r-1) = r^{2m+2} - 1 + 1 - (r^{m+1})^2 = 0.$$

□

Lemma 5.2 *For every* $0 \leq i \leq j < r^n$ *we have*

$$|I^n_{i,j}| \geq \frac{(j-i+1)^2}{r^{2n}(r+1)}.$$

Proof Firstly, we have

$$|I^n_{i,i}| = (r-1)\sum_{k=n+1}^{\infty}\frac{1}{r^{2k}} = \frac{1}{r^{2n}(r+1)} \tag{5.3}$$

and consequently, it only remains to prove the result for $i < j$. Let

$$i = \sum_{k=0}^{n-1}\varepsilon_k r^k, \quad j = \sum_{k=0}^{n-1}\varepsilon'_k r^k, \quad \text{with} \quad 0 \leq \varepsilon_k, \varepsilon'_k < r.$$

Then

$$|I^n_{i,j}| = x_j - x_i + (r-1)\sum_{k=n+1}^{\infty}\frac{1}{r^{2k}} = \sum_{k=0}^{n-1}\frac{\varepsilon'_k - \varepsilon_k}{r^{2(n-k)}} + \frac{1}{r^{2n}(r+1)}.$$

Now, we consider $m = \max\{k : \varepsilon'_k - \varepsilon_k \neq 0\}$ and let F be the function defined by Eq. 5.2. We have

$$r^{2n}(r+1)|I^n_{i,j}| - (j-i+1)^2 = F(\varepsilon'_0 - \varepsilon_0, \ldots, \varepsilon'_m - \varepsilon_m) \geq 0$$

by Lemma 5.1. □

Proof of Theorem 1.6 In view of Eqs. 5.1 and 5.3, if we consider $\delta_n = |I^n_{i,i}|$ then

$$\mathcal{H}^{1/2}_{\delta_n}(E_r) \leq r^n\sqrt{\delta_n} = \frac{1}{\sqrt{r+1}}$$

and as we make δ_n as small as necessary by taking n sufficiently large, we have that $\mathcal{H}^{1/2}(E_r) \leq 1/\sqrt{r+1}$.

Now, it only remains to prove the other inequality. Let $\{U_k\}_{k=1}^{\infty}$ be a covering of E_r consisting of open intervals. The compactness of E_r tells us that it is actually covered by a finite number of them, so suppose $E_r \subset \cup_{k=1}^{m} U_k$.

We claim that if we take a n sufficiently large, then for every $k = 1, \ldots, m$ there are a couple of indexes $0 \leq i_k \leq j_k < r^n$ such that

(1) $I^n_{i_k,j_k} \subset U_k$.
(2) The family of intervals $\{I^n_{i_k,j_k}\}_{k=1}^{m}$ is pairwise disjoint and it covers E_r.

Indeed, without loss of generality we may suppose that $U_i \cap U_j \cap U_k = \emptyset$ for any three different indices $1 \leq i, j, k \leq m$ and we have $U_i \not\subset U_j$ and $U_j \not\subset U_i$ for $i \neq j$. Let $I = (a, b)$ and $J = (c, d)$ be two intervals of $U_1, \ldots, U_m$ such that $\overline{I} \cap \overline{J} \neq \emptyset$. We may assume that $I \cap J = [c, b]$. If $c \neq b$ then we may pick $x \in [c, b] \setminus E_r$ since E_r does not contain any interval, and othwerwise we take $x = c = b$. Furthermore, as E_r is a closed set there is $\alpha > 0$ such that $(x - \alpha, x + \alpha) \subset \overline{I} \cup \overline{J} \setminus E_r$. Then, we have

$$((a, x-\alpha) \cup (x+\alpha, d)) \cap E_r = (I \cup J) \cap E_r.$$

Repeating this process for the $m - 1$ possible intersections we obtain a new covering of E_r, named $I_1, \ldots I_m$, formed by open intervals such that $\overline{I}_j \cap \overline{I}_i = \emptyset$ for $i \neq j$ and $I_k \subset U_k$ for every $k = 1, \ldots, m$. Now, let $\delta > 0$ be the minimum of the distance between the intervals

$I_1, \dots I_m$. If we take n sufficiently large such that $\frac{1}{r^{2n}(r+1)} < \delta$ then every interval of the form $I^n_{i,i}$ is contained in only one interval of $I_1, \dots I_m$. Hence, for every $k = 1, \dots, m$ if we denote $i_k = \min\{i : I^n_{i,i} \subset I_k\}$ and $j_k = \max\{j : I^n_{j,j} \subset I_k\}$ then we have $i_k \leq j_k$ and $I^n_{i_k,j_k} \subset I_k$. Since $I_1, \dots I_m$ are pairwise disjoint intervals and they cover E_r, we obtain that the family of intervals $\{I^n_{i_k,j_k}\}_{k=1}^m$ is pairwise disjoint and it covers E_r. This proves the claim.

Therefore, if we conveniently reorder the family $\{I^n_{i_k,j_k}\}_{k=1}^m$, then it can be written as

$$I^n_{0,p_1}, I^n_{p_1+1,p_2}, \dots, I^n_{p_{m-1}+1,r^n-1}.$$

and consequently, by Lemma 5.2 we get

$$\begin{aligned}\sum_{k=1}^m \sqrt{|U_k|} &\geq \sum_{k=1}^m \sqrt{|I^n_{i_k,j_k}|} \\ &\geq \frac{1}{r^n\sqrt{r+1}}(p_1 + 1 + p_2 - p_1 + \dots + r^n - p_{m-1} - 1) \\ &= \frac{1}{\sqrt{r+1}}\end{aligned}$$

which gives us the result. □

Finally, we are now in the position to prove the last main result of this work.

Proof of Theorem 1.7 From Lemma 4.2 it is immediate to see that $x \in E_r$ if and only if it is satisfied that either

(1) $\varepsilon_{2n-1} = \frac{r}{2} - 1$ and $\varepsilon_{2n} \geq \frac{r}{2}$, or
(2) $\varepsilon_{2n-1} = \frac{r}{2}$ and $\varepsilon_{2n} \leq \frac{r}{2} - 1$

for every n. If we denote $\mathcal{A}_r = \{x \in [0,1] : \partial^+ f_r(x) \neq \emptyset\}$, then by Theorem 1.3 we may write

$$\mathcal{A}_r = \bigcup_{m=1}^{\infty} \left(D_m + \frac{1}{r^{m-1}} E_r\right),$$

and, as a consequence of Theorem 1.5, we conclude that $\mathcal{A}_r$ has Hausdorff dimension $1/2$. Furthermore, using Theorem 1.6 we obtain

$$\begin{aligned}\mathcal{H}^{1/2}\left(D_m + \frac{1}{r^{m-1}}E_r\right) &= \#D_m \mathcal{H}^{1/2}\left(\frac{1}{r^{m-1}}E_r\right) \\ &= \frac{r^{m-1}+1}{\sqrt{r^{m-1}}\sqrt{r+1}} \geq \frac{\sqrt{r^{m-1}}}{\sqrt{r+1}}\end{aligned}$$

where the first equality follows since $D_m + \frac{1}{r^{m-1}}E_r$ is a disjoint union of $\#D_m$ smaller similar copies of E_r. Hence we have

$$\frac{\sqrt{r^{m-1}}}{\sqrt{r+1}} \leq \mathcal{H}^{1/2}(\mathcal{A}_r)$$

for every m, and the result follows immediately. □

The proof of Theorem 1.7 does not need the precise value of $\mathcal{H}^{1/2}(E_r)$, it is enough to note that this value is strictly positive, and this follows from the standard theory of self-similar sets with open set condition.

In a similar way as we have done in the proof of Theorem 1.7, the following result is obtained:

Proposition 5.3 *Let* $r \geq 2$ *be an even integer. Then, the set of points where* f_r *attains a local maximum has Hausdorff dimension* $1/2$ *and its* $1/2$*-Hausdorff measure is plus infinity.*

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