ЖУРНАЛ
ВЫЧИСЛИТЕЛЬНОЙ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОЙ ФИЗИКИ

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ON A GENERAL APPROACH TO EXTINCTION AND BLOW-UP FOR QUASI-LINEAR HEAT EQUATIONS

Исследуется асимптотическое поведение неотрицательных решений $u = u(|x|, t)$ уравнения теплопроводности общего вида с источником или стоком тепла $u_t = \Delta\varphi(u) \pm Q(u)$, где φ', Q — заданные неотрицательные функции. Показано, что тщательное использование метода, предложенного Фридманом и Маклаодом, позволяет получить асимптотические оценки решений вблизи момента обострения или полного остывания, которые, как было установлено, являются точными для некоторых частных видов функций φ и Q.

§ 1. Introduction

In recent years, equations of the type

$$(1.1)_\pm \qquad u_t = \Delta\,\varphi(u) \pm Q\,(u)\,,$$

have deserved considerable attention, both because of their relevance as physical models in Continuum Mechanics as well as for their intrinsic mathematical interest. Local (in time) existence of classical solutions for the various initial and boundary value problems associated to $(1.1)_\pm$ follows at once from standard parabolic theory when $\varphi(s) = s$ and, say, $Q(s)$ is continuous (cf. for instance [1], [2]). However solutions need to be defined in a generalized way if, for instance, $\varphi(u) = u^m$ with $m > 1$ (see [3], [4]). We shall assume throughout this paper that

$$(1.2) \qquad \varphi \in C^2((0, \infty)) \cap C^1([0, \infty))\,, \quad Q \in C^1((0, \infty)) \cap C([0, \infty))\,,$$
$$\varphi'(s) > 0, \quad Q(s) > 0 \text{ for } s > 0.$$

Moreover, we shall assume that a suitable theory of (possibly generalized) solutions is available in any of the cases to be considered. These will be referred to henceforth as the solutions without any further specification.

It is well known that, for a wide choice of functions φ and Q, solutions of $(1.1)_\pm$ may develop critical behaviours in a finite time. To simplify this exposition, we shall reduce ourselves henceforth to the situation where $(1.1)_\pm$ holds for $x \in \mathbb{R}^N$, $N \geq 1$, in some time interval $(0, T)$ with $T \leq +\infty$, and

$$(1.3) \qquad u(x, 0) = u_0(x) \text{ for } x \in \mathbb{R}^N, \text{ where } u_0(x) \text{ is continuous, nonnegative and bounded.}$$

The foregoing arguments can be extended, however, to cover initial boundary value problems on bounded domains. Let us recall now some notation. A nonnegative solution $u(x, t)$ of $(1.1)_+$, (1.3) is said to *blow-up* in a finite *blow-up time* $T > 0$ if $u(x, t)$ solves $(1.1)_+$ in any strip $S_\tau = R^N \times (0,\tau)$ with $\tau < T$, and

$$\lim_{t\to T} \left(\sup_{x\in R^N} u(x, t)\right) = \infty$$

If this happens, we shall say that x_0 is *a blow-up point* if there exist sequences $\{x_n\}$, $\{t_n\}$ such that

$$\lim_{n\to\infty} x_n = x_0 , \quad \lim_{n\to\infty} t_n = T , \quad \lim_{n\to\infty} u(x_n, t_n) = +\infty .$$

As suggested by the first order PDE obtained by dropping the diffusion term in $(1.1)_+$, a necessary condition for blow-up to happen is

$$\int^{\infty} \frac{d\eta}{Q(\eta)} < \infty \tag{1.4}$$

(cf. for instance [5], [2], and list of references in the books [6], [7]). On the other hand, we say that *extinction* occurs for $(1.1)_-$, (1.3) if there exists $T^* < +\infty$ such that the solution of $(1.1)_-$, (1.3) under consideration satisfies $u(x, t) \equiv 0$ for $t \geq T^*$. The infimum of such times T^*, T, is then named as the *extinction time*. A point x_0 is termed as extinction point if there exist sequences $\{x_n\}$, $\{t_n\}$ such that

$$\lim_{n\to\infty} x_n = x_0 , \quad \lim_{n\to\infty} t_n = T \text{ and } u(x_n, t_n) > 0$$

for any n. Again, dropping the diffusion term in $(1.1)_-$ indicates that, in order to obtain extinction, one needs the assumption

$$\int_0^{\varepsilon} \frac{d\eta}{Q(\eta)} < \infty \text{ for any } \varepsilon > 0 . \tag{1.4'}$$

Conditions under which blow-up (resp. extinction) occur in a finite number of points have been extensively discussed in the literature; see [8]— [11], [6], [7] (resp. [12]). The description of the asymptotics of solutions near blow-up points (resp. extinction points) at the blow-up time (resp. the extinction time) is an interesting problem in the general theory of non-linear evolution equations, and as such, it has been the object of considerable effort. As of now, a complete picture is only available for the one-dimensional, semilinear case where $\varphi(s) = s$, $Q(s) = s^p$ with $p > 1$ or $Q(s) = e^s$ in $(1.1)_+$ (resp. $Q(s) = s^p$ with $0 < p < 1$ in $(1.1)_-$). Let us specialize to the blow-up case $(1.1)_+$ with $Q(s) = s^p$, $p > 1$, for definiteness. It is then known that, to the first approximation, blow-up behaviour is self-similar, in the sense that the following result holds. If $x = 0$ is any blow-up point and $u(x, t)$ blows up at $t = T$, then

$$\lim_{t\to T} (T - t)^{1/(p-1)} u(x(T - t)^{1/2}, t) = (p - 1)^{1/(p-1)} , \tag{1.5}$$

uniformly in sets $|x| \leq C$ with $C > 0$

(cf. [13] — [17]). Notice that $u_T(x, t) = [(p-1)(T-t)]^{1/(p-1)}$ is an explicit self-similar solution of $(1.1)_+$ under our current assumptions. Different behaviours appear when high-order asymptotics are considered, and a complete classification of them is now available. It was first obtained formally by the method of matched asymptotic expansions in [18] (cf. also [19] and [20], [13] for pioneering works in this direction). Later, these results were made rigorous by M. A. Herrero and J. J. L. Velázquez in the series of papers that will be published soon. Roughly speaking, the asymptotics at blow-up (including the nature of the singularity which appears there at blow-up time) depends of the number of maxima which collapse at the blow-up point. However, only the case of a single maximum reaching up to the blow-up point at the blow-up time is generic, in the sense of being preserved under small perturbations of initial values; a similar situation exists for some quasilinear heat equations, see references in [6, Chapter IV]). This corresponds to an asymptotic behaviour given by

$$\text{(1.6)} \qquad \lim_{t\to T} (T-t)^{1/(p-1)} u\left(\xi\left((T-t)\,|\log(T-t)|\right)^{1/2}, t\right) = (p-1)^{-1/(p-1)}\left[1+\left(\frac{p-1}{4p}\right)\xi^2\right]^{-1/(p-1)}$$

uniformly on sets $|\xi| \le C$ with $C > 0$,

$$\text{(1.7)} \qquad \lim_{|x|\to 0}\left(\frac{|x|^2}{|\log|x||}\right)^{1/(p-1)} u(x, T) = \left(\frac{8p}{(p-1)^2}\right)^{1/(p-1)}$$

The reader is referred to [6], [7], [13], [14], [21], [22] for related results, as well to [12] for corresponding results for the extinction problem. For the quasi-linear equation with $\varphi(s) = s^{\sigma+1}, \sigma > 0$, $Q(s) = s^p$, we have single point blow-up for $p > \sigma + 1$ while for $p = \sigma + 1$ regional blow-up occurs, and if $p \in (1, \sigma + 1)$ there exists global blow-up, see [6] and [23] for extensive references. Notice that in these cases asymptotic behaviours are quite different from (1.6) and (1.7), and nontrivial explicit self-similar solutions are asymptotically stable. See [6, Chapter IV] for the case $p = \sigma + 1$ and [23] for $p > \sigma + 1$.

Let us state now the aim of this note. While solutions to $(1.1)_\pm$ may indeed behave in many different ways for any choice of φ and Q there, we believe that in any case only one behaviour is generic. Moreover, we expect this behaviour to be easily described by means of an easy-to-compute algorithm, to be described in the next Section. We should stress, however, that only one-sided bounds (as for instance (3.8) and (3.9) below) can be made rigorous at this stage when $\varphi(s) \neq s$, and even in such case, tight assumptions on the initial values are currently required.

§ 2. Friedman-McLeod's method

From now on, we shall reduce ourselves to the study of radial solutions of $(1.1)_\pm$. Moreover, we shall also assume that

$$\varphi(u_0) \in C^1, \quad u_0'(r) \le 0 \quad \text{for } r = |x| \ge 0,$$

so that the corresponding solution is radially symmetric and nonincreasing in $r > 0$. Following [9], we consider the auxiliary function

(2.1) $$J(r, t) = r^{N-1}\varphi'(u)\, u_r(r,t) + r^N F(u(r, t)),$$

where

(2.2) $F(u)$ is a C^2-function (depending on φ and Q) which should satisfy $F(u) > 0$ and $F'(u) > 0$ for $u > 0$.

One readily checks that $J(r, t)$ satisfies the following parabolic equation (cf. [25] for $N = 1$ and [26])

(2.3) $$J_t = \mathrm{A}\, J + q(u, r),$$

where

$$\mathrm{A}\, J = \bar{a}J_{rr} + \bar{b}J_r + \bar{c}J, \quad \bar{a} = \bar{a}(u) \equiv \varphi'(u),$$

$$\bar{b} = \bar{b}(u, r, J) \equiv \frac{\varphi''}{\varphi' r^{N-1}}(J - r^N F) - \frac{(N-1)\varphi'}{r},$$

$$\bar{c} = \bar{c}(u, r, J) \equiv -\frac{N\varphi'' F}{\varphi'} - 2F' - \frac{r^{2-N}F''J}{\varphi'} + \frac{2r^2 FF''}{\varphi'} \pm \frac{(\varphi' Q)'}{\varphi'},$$

and

$$q = q(u, r) \equiv r^N F^2\left\{\frac{\varphi''}{\varphi'}\left(N \mp \frac{Q}{F}\right) + 2(\log F)' \mp \left(\frac{Q}{F}\right)'\right\} - r^{N+2}\frac{F^2F''}{\varphi'}.$$

It then follows from the Maximum Principle (cf. [9], [25]) that, under some conditions on φ, Q and F, $J(r, t)$ has a constant sign for any $t > 0$ for which $u(r, t)$ is well-defined. In particular, consider equation $(1.1)_+$ and suppose that $u_0(r)$ is such that

(2.4a) $$J(r, 0) \le 0 \text{ for } r \ge 0;$$

(2.4b) $$I_+(u) \equiv \frac{\varphi''}{\varphi'}\left[N - \frac{Q(u)}{F(u)}\right] + 2[\log F(u)]' - \left[\frac{Q(u)}{F(u)}\right]' \le 0 \text{ for } u > 0;$$

(2.4c) $$F''(u) \ge 0 \text{ for } u > 0.$$

Then

(2.5) $J(r, t) \le 0$ for any $r \ge 0$ and any $t \ge 0$ for which $u(r, t)$ is defined.

On the other hand, consider equation $(1.1)_-$, and suppose that $u_0(r)$ is such that

(2.6a) $$J(r, 0) \ge 0 \text{ for } r \ge 0;$$

(2.6b) $$I_-(u) \equiv \frac{\varphi''}{\varphi'}\left[N + \frac{Q(u)}{F(u)}\right] + 2[\log F(u)]' + \left[\frac{Q(u)}{F(u)}\right]' \ge 0 \text{ for } u > 0;$$

(2.6c) $$F''(u) \le 0 \text{ for } u > 0.$$

Then

(2.7) $$J(r, t) \ge 0 \text{ for any } r \ge 0 \text{ and any } t \ge 0.$$

We shall analyze now some consequences of inequalities (2.5), (2.7). Suppose first that $u(r, t)$ solves $(1.1)_+$ and (2.5) holds. Integrating this last inequality yields

(2.8) $$2 \int_{u(r,t)}^{u(0,t)} \frac{\varphi'(s)}{F(s)} ds \ge r^2 \text{ for } r > 0, \quad t \in (0, T).$$

Therefore, if

(2.9) $$H(u) \equiv 2 \int_u^\infty \frac{\varphi'(s)}{F(s)} ds < \infty \text{ for } u > 0,$$

it follows from (2.8) that

(2.10) $$u(r, t) \le H^{-1}(r^2) \text{ for } r > 0, \ t \in (0, T).$$

In this case $r = 0$ is a single blow-up point. Furthermore, if the behaviour of $u(0, t)$ for $t \approx T$ is known, (2.8) provides an upper bound for $u(r, t)$ when $r \approx 0$ and $t \approx T$. On the other hand, recalling the definition of $J(r, t)$ in (2.1), it follows that if we divide in (2.5) by r^N and let $r \to 0$, we obtain

$$(\varphi'(u)u_r)_r + F(u) \le 0 \text{ at } r = 0$$

and, since by regularity

$$N(\varphi'(u)u_r)_r = u_t - Q(u) \text{ at } r = 0,$$

we see that

(2.11) $$u_t \le Q(u) - NF(u) \text{ at } r = 0, \text{ for } t \in (0, T).$$

By integrating (2.11), we derive a lower bound for $u(0, t)$ when $t \approx T$.

The previous results have immediate counterparts for equation $(1.1)_-$. In this case, if (2.7) holds, (2.8) is to be replaced by

(2.12) $$2 \int_{u(r,t)}^{u(0,t)} \frac{\varphi'(s)}{F(s)} ds \le r^2 \text{ for } r > 0, \quad t \in (0, T).$$

We may use (2.12) to derive a lower bound for the support of $u(r, t)$ near the extinction time. Indeed, since $u_{rr}(0, t) \le 0$ and $u_r(0, t) \equiv 0$, we have that $u_t(0, t) \le -Q(u(0, t))$ in $(0, T)$, whence

(2.13) $$u(0, t) \ge \Phi^{-1}(T - t) \text{ in } (0, T),$$

where

$$\Phi(u) = \int_0^u \frac{d\eta}{Q(\eta)}$$

(cf. (1.4′)). Therefore, if

$$\int_0^{\varepsilon} \frac{\varphi'(s)}{F(s)}\, ds < +\infty \quad \text{for any} \quad \varepsilon > 0, \tag{2.14}$$

we deduce from (2.13) that as $t \to T$

$$\operatorname{supp} \; u(r, t) \equiv \{r \geq 0 \mid u(r, t) > 0\} \supseteq \{r < S(t)\},$$

where

$$S^2(t) = 2 \int_0^{\Phi^{-1}(T-t)} \frac{\varphi'(s)}{F(s)}\, ds .$$

Finally, the analogue of (2.11) reads now

$$u_t \geq -Q(u) - NF(u) \text{ at } r = 0 \text{ for } t \in (0, T), \tag{2.15}$$

and it yields the upper bound of $u(0, t)$. It is interesting to compare this with the lower one given by the simple inequality $u'(0, t) \leq -Q(u(0, t))$.

§ 3. Selecting $F(u)$ in $(1.1)_+$

It is apparent that the precise form of the above estimates depends crucially on the choice of $F(u)$ in (2.1), which has to be made in such a way that (2.4) (resp. (2.6) in the case $(1.1)_-$) holds. This has been done for various values of φ and Q (cf. for instance [7], [13], [14], [25] — [28] for the blow-up case, and [12] for extinction problems). We shall now comment on a formal procedure to perform such selection for rather general choices of φ and Q. Consider first the blow-up case $(1.1)_+$. Bearing in mind (2.4b), we examine the first order ODE

$$\frac{\varphi''}{\varphi'}\left[N - \frac{Q(u)}{F(u)}\right] + 2\,[\log F(u)]' - \left[\frac{Q(u)}{F(u)}\right]' = 0 \quad \text{for} \quad u > 0 . \tag{3.1}$$

Setting $z(u) = Q(u)/F(u)$, (3.1) gives

$$\left(1 + \frac{2}{z}\right) z' = 2\frac{Q'}{Q} - \frac{\varphi''}{\varphi}(z - N) \quad \text{for} \quad u > 0 . \tag{3.2}$$

In addition to (1.2), we shall assume that

$$\lim_{u \to \infty} Q(u) = +\infty, \quad Q'(u) > 0, \quad \varphi''(u) \geq 0 \quad \text{for large} \quad u > 0, \tag{3.3}$$

and the following (finite or infinite) limit exists

$$L = \lim_{u \to +\infty} \frac{[\log \varphi'(u)]'}{[\log Q(u)]'} .$$

Suppose now that $L = 0$ in (3.3). Then

$$\frac{\varphi''}{\varphi'}(u) \ll \frac{Q'}{Q}(u) \quad \text{as} \quad u \to \infty, \quad \text{and} \quad \lim_{u \to \infty} z(u) = \infty .$$

Thus for large values of $u > 0$, (3.2) is asymptotically equivalent to

$$z' = 2\frac{Q'}{Q} - \frac{\varphi''}{\varphi'} z ,$$

whence

$$z(u) \approx \frac{2}{\varphi'(u)} \int_1^u \varphi'(s)\, [\log Q(s)]' ds \quad \text{for} \quad u >> 1.$$

Recalling the definition of $z(u)$, we are thus led to the choice

(3.4) $$F(u) \approx F_0(u) = Q(u)\, \varphi'(u) \Big\{ 2 \int_1^u \varphi'(s)\, [\log Q(s)]' ds \Big\}^{-1}$$

as $u \to \infty$, provided that $L = 0$ in (3.3). Notice that we are not saying that $F_0(u)$ in (3.4) will always satisfy (2.4b), although it can be readily checked that (2.4b) holds for $u >> 1$ with the above choice of F_0 if

$$\lim_{u\to\infty} \frac{[\log \varphi'(u)]' z_0(u)}{[\log Q(u)]'} < \frac{4}{N+2} \quad \text{with} \quad z_0(u) = \frac{Q(u)}{F_0(u)} .$$

Instead, we are assuming that $F(u)$ can be selected so that (2.4) holds and its asymptotic behaviour for large values of u is given by (3.4) in the case $L = 0$. If $L > 0$, it turns out that, as $u \to \infty$, (3.2) is asymptotically equivalent to

$$\left(1 + \frac{2}{z}\right) z' = \frac{Q'}{Q}\left[2 - L(z - N)\right] ,$$

and it is easy to see that any solution of this equation satisfies

$$z(u) \to \frac{2 + LN}{L} \quad \text{as} \quad u \to \infty .$$

We then expect that $F(u)$ can be selected so that

(3.5) $$F(u) = \left(N + \frac{2}{L}\right)^{-1} Q(u) \quad \text{when } L > 0 .$$

Example 1. Set $\varphi'(s) \equiv 1$ in $(1.1)_+$. In this case, $L = 0$ in (3.3), and therefore we expect

$$F(u) \approx F_0(u) = \frac{Q(u)}{2 \log Q(u)} \quad \text{as} \quad u \to \infty .$$

In particular, we have

(3.6a) $$F_0(u) = \frac{u^p}{2p \log u} \quad \text{if} \quad Q(u) = u^p , \quad p > 1$$

(cf. [13], [14], [28]).

(3.6b) $$F_0(u) = \frac{e^u}{2u} \quad \text{if} \quad Q(u) = e^u$$

(cf. [7], [27] — [29]).

(3.6c) $$F_0(u) = \frac{1}{2}u(\log u)^{p-1} \quad \text{if} \quad Q(u) = (1+u)[\log(1+u)]^p, \; p > 1$$

(cf. [28]).

Notice that these functions satisfy (2.2) and (2.4c) for $u >> 1$. Furthermore, in view of (2.9), single-point blow-up will occur if

(3.7) $$\int^{\infty} \frac{\log Q(s)}{Q(s)} ds < +\infty,$$

which certainly holds in cases (3.6a, b). Consider the power-like case in (3.6a). It then follows from (2.8) and (2.10) (with F replaced by F_0 and $t = T$) that the final profile should satisfy

(3.8) $$u(r, T) \leq \left(\frac{(p-1)^2}{8p}\right)^{-1/(p-1)} r^{-2/(p-1)} |\log r|^{1/(p-1)} + \\ + O(r^{-2/(p-1)} |\log r|^{(2-p)/(p-1)} \log|\log r|) \quad \text{for} \quad r > 0 \quad \text{small enough}$$

(compare with (1.7) above). Moreover, whenever (1.5) holds (which is the case for different classes of solutions $u(r, t)$ when $N \geq 1$), (2.8) yields

(3.9) $$u(r, t) \leq (T-t)^{-1/(p-1)}\left\{(p-1) + \frac{(p-1)^2}{4p}\eta^2 + o(1)\right\}^{-1/(p-1)} \text{as } t \to T,$$

$\eta = r[(T-t)|\log(T-t)|]^{-1/2}$, uniformly on compact subsets in η, a result to be compared with (1.6). Recalling the result already known for the one-dimensional case (of Section 1 herein), we expect single point blow-up satisfying (3.8), (3.9) (with equality replacing the inequality sign there) to be the generic blow-up behaviour for solutions of $(1.1)_+$ for which (1.5) holds. Notice that for the given choice of $F(u)$ inequality (2.11) takes the form $u' \leq Q(u)\{1 - N/[2\log Q(u)]\}$ as $t \to T$, and integrating near $t = T$ with $Q(u) = u^p$ yields the explicit lower bound

$$u(0, t) \geq [(p-1)(T-t)]^{-1/(p-1)}\{1 + N/[2p|\log(T-t)|]\}.$$

When $Q(u) = e^u$, (3.8) is to be replaced by

(3.10) $$u(r, T) \leq 2|\log r| + \log|\log r| + \log 8 + o(1) \text{ as } r \to 0,$$

whereas instead of (3.9) there holds

(3.11) $$u(r, t) \leq -\log(T-t) - \log\left(1 + \frac{\eta^2}{4}\right) - \frac{\log|\log(T-t)|}{|\log(T-t)|} + \\ + O\left(\frac{1}{|\log(T-t)|}\right) \text{ as } t \to T$$

uniformly on subsets $\eta = r[(T-t)|\log(T-t)|]^{-1/2} \leq C < +\infty$, provided that

$$\lim_{t \to T}\left[u(x(T-t)^{1/2}, t) + \log(T-t)\right] = 0,$$

uniformly on compact sets $|x| \leq C < \infty$ (this is the analogue of (1.5) in the current case). Again, we expect single-point blow-up with the behaviour specified by the right-hand sides of (3.10), (3.11), to be then the generic situation in the exponential case. Inequality (2.11) implies $u(0, t) \geq |\log(T-t)| + N/[2|\log(T-t)|]$ as $t \to T$. Notice that the asymptotic behaviour given in the right-hand side of (3.11) is proved by J. Bebernes, S. Bricher and V. A. Galaktionov to be asymptotically stable with respect to small perturbations of the coefficients of the semilinear equation $u_t = \Delta u + e^u$.

When $Q(u) = (1+u)[\log(1+u)]^p$ with $p > 1$, (3.7) holds for $p > 2$. Actually single point blow-up is expected only in this case, whereas blow-up on a region of finite measure (regional blow-up) is expected for $p = 2$ (see also [30]). and blow-up in R^N (global blow-up) is expected for $1 < p < 2$ (cf. [6], [28], [31], [32]).

Example 2. We now turn our attention to quasi-linear equations, where $\varphi'(s) \not\equiv 1$. For instance, consider

$$u_t = \operatorname{div}(u^\sigma \nabla u) + e^u, \tag{3.12}$$

where $\sigma > 0$ is a fixed constant. Then $L = 0$ in (3.3), and

$$F_0(u) \approx \frac{\sigma}{2} e^u \quad \text{for } u \gg 1 .$$

This leads to the following estimate corresponding to single point blow-up:

$$u(r, t) \leq \left| \log \left(\frac{\sigma r^2}{4} |\log r^2|^{-\sigma} \right) \right| [1 + o(1)]$$

for $r > 0$ small enough, and $t \approx T$. If instead of (3.12), we have

$$u_t = \operatorname{div}(u^\sigma \nabla u) + u^\beta \text{ with } \sigma > 0, \quad \beta > 1 ,$$

it follows that $L = \sigma/\beta > 0$, whence

$$F_0(u) = \frac{\sigma}{N\sigma + 2\beta} u^\beta \quad \text{for} \quad u \gg 1.$$

Then, if $\beta > \sigma + 1$, (2.10) yields (cf. [25], [26])

$$u(r, t) \leq \left\{ \frac{\sigma [\beta - (\sigma+1)]}{2(N\sigma + 2\beta)} r^2 \right\}^{-1/[\beta - (\sigma+1)]} \quad \text{for} \quad r > 0 \text{ small, } t \approx T .$$

This is now the single-point blow-up case. If $\beta = \sigma + 1$ (resp. $1 < \beta < \sigma + 1$), we expect regional blow-up (resp. global blow-up) to occur; cf. [6, Chapter IV] and [33]. Consider finally the equation

$$u_t = \operatorname{div}(e^u \nabla u) + u^\beta \text{ with } \beta > 1.$$

In this case $L = +\infty$ in (3.3), and therefore $F_0(u) = u^\beta/N$ for $u \gg 1$. Since (2.9) does not hold, we do not expect single point blow-up to occur. Actually, (2.10) gives now

$$u(r, t) \leq u(0, t) - \frac{r^2}{2N} u^\beta(0, t)\, e^{-u(0, t)} [1 + o(1)]$$

as $t \to T$ uniformly on compact subsets, which strongly suggest the existence of global blow-up.

§ 4. The absorption case

In view of our previous analysis, a crucial point towards understanding the asymptotic behaviour of solutions of $(1.1)_-$ near the extinction time consists in a suitable choice of function $F(u)$ satisfying (2.2) and (2.6). As in § 3, we shall assume that such a choice is possible, and proceed to derive formally the behaviour of such $F(u)$ for $u > 0$ small enough. To this end, we shall assume that (3.3) holds with limits $u \to \infty$ replaced by $u \to 0$. Since $z = Q/F$ solves now

$$z'\left(1-\frac{2}{z}\right)+\frac{\varphi''}{\varphi'}\left(N+z\right)=-\frac{2Q'}{Q} \quad \text{as} \quad u \to 0,$$

we readily see that, if $L=0$ in (3.3) with $u \to 0$, we may expect

$$F(u) \approx F_0(u) = Q(u)\,\varphi'(u)\left\{2\int_u^1 \varphi'(s)\,[\log Q(s)]'ds\right\}^{-1} \tag{4.1}$$

for $0 < u << 1$. When $L > 0$, we take

$$F(u) \approx F_c(u) = \frac{Q(u)\,\varphi'(u)}{c} \quad \text{for } 0 < u << 1, \tag{4.2}$$

where $c > 0$ is fixed, but otherwise arbitrary. A routine computation shows then that all the required assumptions on $F_c(u)$ hold for $0 < u << 1$.

Example 3. Set $\varphi(s) \equiv s$ in $(1.1)_-$. Then $L=0$ in (3.3) with $u \to 0$, and therefore we expect

$$F(u) \approx F_0(u) = \frac{Q(u)}{2\,|\log Q(u)|} \quad \text{for} \quad 0 < u << 1.$$

In particular, we have

$$F_0(u) = \frac{u^p}{2p\,|\log u|} \quad \text{if} \quad Q(u) = u^p, \quad 0 < p < 1, \tag{4.3a}$$

$$F_0(u) = \frac{1}{2}\,u\,|\log u|^{\alpha-1} \quad \text{if} \quad Q(u) = u\,|\log u|^{\alpha}, \quad \alpha > 1. \tag{4.3b}$$

Notice that these functions satisfy (2.2) and (2.6c) for $0 < u << 1$. In the case considered in (4.3a), (2.12) with (2.13) yield the following estimate

$$u(r,t) \geq [(1-p)(T-t)]^{1/(1-p)}\left(1-\frac{\eta^2}{\eta_0^2}\right)_+^{1/(1-p)}[1+o(1)] \text{ as } t \to T \tag{4.4}$$

where $\eta = r[(T-t)|\log(T-t)|]^{-1/2}$, and convergence is uniform on sets $0 < \eta \leq c < \eta_0$ with $\eta_0 = 2[p(1-p)^{-1}]^{1/2}$ The fact that this is the actual asymptotic behaviour whenever $u_0(x)$ has a single maximum has been recently proved by M. A. Herrero and J. J. L. Velázquez. Notice that inequality (2.15) provides the explicit upper bound

$$u(0,t) \leq [(1-p)(T-t)]^{1/(1-p)}\{1 + N/[2p\,|\log(T-t)|]\} \text{ as } t \to T.$$

Consider now $(1.1)_-$ with the absorption term given in (4.3b). Then (2.14) holds if and only if $\alpha > 2$, whereas (1.4) is satisfied whenever $\alpha > 1$. When $\alpha > 2$, we obtain

$$u(r,t) \geq \exp\left(-\left\{[(\alpha-1)(T-t)]^{1/(\alpha-1)}\left(1-\frac{\xi^2}{\xi_0^2}\right)_+^{1/(\alpha-2)}\right\}^{-1}[1+o(1)]\right)$$

as $t \to T$,

where $\xi = r(T-t)^{-m}$, $m = \dfrac{(\alpha-2)}{[2(\alpha-1)]}$, and the above estimate is uniform on sets $0 \leq \xi \leq c < \xi_0$ with $\xi_0 = 2(\alpha-2)^{-1/2}(\alpha-1)^m$. When $1 < \alpha < 2$, we obtain as a lower bound a function which is positive everywhere, namely

$$u(r,t) \geq \exp\left(-\left\{[(\alpha-1)(T-t)]^{1/(\alpha-1)}\left(1+\frac{\xi^2}{\xi_*^2}\right)_+^{1/(\alpha-2)}\right\}^{-1}[1+o(1)]\right)$$

as $t \to T$

uniformly on sets $0 \leq \xi \leq c < +\infty$, where $\xi_* = 2(2-\alpha)^{-1/2}(\alpha-1)^m$. This indicates that single point extinction is confined to the parameter range $\alpha > 2$.

Finally, we notice that for a general semilinear equation, (2.14) implies that the condition (cf. (1.4′))

$$\int_0^\varepsilon \frac{\log Q(s)}{Q(s)}\, ds = +\infty$$

yields global extinction, i. e., $u(x,t) > 0$ everywhere near $t = T$.

Example 4. To conclude, we consider the equation

$$u_t = \operatorname{div}(u^\sigma \nabla u) - u^p \text{ with } \sigma > 0, \quad 0 < p < 1. \tag{4.5}$$

We then have $L > 0$ in (3.3), whence the choice $F_c(u) = u^{p+\sigma}/c$ for $0 < u << 1$. Assumptions (2.2) and (2.6c) hold then for $0 < u << 1$ provided that $p + \sigma \leq 1$. Notice that we obtain in this case

$$u(r,t) \geq [(1-p)(T-t)]^{1/(1-p)}\left(1-\frac{\zeta^2}{\zeta_0^2}\right)_+^{1/(1-p)} \tag{4.6}$$

as $t \to T$, where $\zeta = r(T-t)^{-1/2}$, $\zeta_0^2 = 2c$ and (4.6) holds uniformly on sets $0 \leq \zeta \leq b < \zeta_0$. A comparison of (4.4) and (4.6) suggests the existence of nontrivial boundary layers when $\sigma \to 0$ in (4.5).

§ 5. Concluding remarks

1. We expect that the method introduced by Friedman and McLeod in [9] will describe generic critical behaviours of solutions of $(1.1)_\pm$ near blow-up or extinction times. More precisely, we conjecture that, if the behaviour of $u(0,t)$ as $t \to T$ near a blow-up or extinction point is known, all the information about the corresponding

asymptotics on small compact subsets near the origin is encoded in the first order ODE

$$\varphi'(u)u_r + r\,F(u) = 0 \text{ for } r > 0 \text{ as } t \to T,$$

where an optimal choice of $F(u)$ is to be done, as indicated in § 3 and 4 above.

2. The same approach has been used in [34] (see also [35] with one-dimensional analysis) for the equation with gradient-like diffusion $u_t = \mathrm{div}(|\nabla u|^\sigma \nabla u) + u^\beta$, and single point was proved to exist if $\sigma > 0, \beta > \sigma + 1$. Different parabolic equations with nonlocal terms have been considered in [36] and by C. J. Budd, J. W. Dold and V. A. Galaktionov.

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