



A Weak-Type Expression of the Orlicz Modular

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Abstract. An equivalent expression of Orlicz modulars in terms of measure of level sets of difference quotients is established. The result in a sense complements the famous Maz'ya–Shaposhnikova formula for the fractional Gagliardo–Slobodeckij seminorm and its recent extension to the setting of Orlicz functions.

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1. Introduction

The fractional order Sobolev spaces $W^{s,p}(\mathbb{R}^N)$, $p \in [1, \infty)$, $s \in (0, 1)$, endowed with the Gagliardo–Slobodeckij seminorm, which is defined for smooth compactly supported functions u as

$$|u|_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right)^p \frac{1}{|x - y|^N} dx dy,$$

have played an important role in the theory of partial differential equations and its applications for a long time (see the introductory section of [5]). Much as it is tempting to think that

$$\lim_{s \rightarrow 1^-} |u|_{s,p}^p \approx \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \quad (1)$$

or

$$\lim_{s \rightarrow 0^+} |u|_{s,p}^p \approx \int_{\mathbb{R}^N} |u(x)|^p dx,$$

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the Gagliardo–Slobodeckij seminorm notoriously fails to capture these limiting cases—to that end, it is sufficient to consider any nonconstant $u \in C_0^\infty(\mathbb{R}^N)$ and observe that $|u|_{s,p}^p$ converges to ∞ as $s \rightarrow 1^-$ or $s \rightarrow 0^+$. Nevertheless, it was discovered around 20 years ago that these “defects” can be, in a sense, “fixed” by introducing certain compensatory factors. Namely, for every $u \in C_0^\infty(\mathbb{R}^N)$, a special case of what is now often called the Bourgain–Brezis–Mironescu formula [3] tells us that

$$\lim_{s \rightarrow 1^+} (1-s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right)^p \frac{1}{|x - y|^N} dx dy = C(N, p) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx. \tag{2}$$

Moreover, Maz’ya and Shaposhnikova proved in [8] that

$$\lim_{s \rightarrow 0^+} s \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right)^p \frac{1}{|x - y|^N} dx dy = C(N, p) \int_{\mathbb{R}^N} |u(x)|^p dx. \tag{3}$$

Recently, a completely different approach, not involving integration of fractional difference quotients at all, to repairing (1) was taken by Brezis, Van Schaftingen and Yung. They proved in [4] that, instead of introducing a compensatory factor, the limit as $s \rightarrow 1^-$ can be recovered if the strong L^p norm of fractional difference quotients is replaced by the weak $L^{p,\infty}$ quasi-norm. More precisely, they obtained the following result. Let $p \in [1, \infty)$ and $u \in C_0^\infty(\mathbb{R}^N)$ and define

$$E_{\lambda,1} = \left| \left\{ (x, y) \in \mathbb{R}^{2N} : x \neq y, \frac{|u(x) - u(y)|^p}{|x - y|^{N+p}} \geq \lambda^p \right\} \right|_{2N},$$

where $|\cdot|_{2N}$ stands for the Lebesgue measure on \mathbb{R}^{2N} . In [4] it was shown that

$$c(N, p) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \leq \sup_{\lambda > 0} \lambda^p E_{\lambda,1} \leq C(N) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda^p E_{\lambda,1} = C(N, p) \int_{\mathbb{R}^N} |\nabla u(x)|^p dx.$$

Following this innovatory approach, Gu and Yung established in [7] other, possibly even more unanticipated, formulae. They complement the Maz’ya–Shaposhnikova formula (3) in the same way the result of Brezis, Van Schaftingen and Yung complements the Bourgain–Brezis–Mironescu formula (2). Namely the result of [7] asserts that, for every $u \in L^p(\mathbb{R}^N)$, $p \in [1, \infty)$,

$$c(N) \int_{\mathbb{R}^N} |u(x)|^p dx \leq \sup_{\lambda > 0} \lambda^p E_{\lambda,0} \leq C(N) \int_{\mathbb{R}^N} |u(x)|^p dx \tag{4}$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda^p E_{\lambda,0} = C(N) \int_{\mathbb{R}^N} |u(x)|^p dx, \tag{5}$$

where

$$E_{\lambda,0} = \left| \left\{ (x, y) \in \mathbb{R}^{2N} : x \neq y, \frac{|u(x) - u(y)|^p}{|x - y|^N} \geq \lambda^p \right\} \right|_{2N}.$$

The classical results (2) and (3) were recently considerably strengthened in [1, 2, 6] by replacing the p -th power in the integrals with Orlicz functions, thus allowing for non-polynomial growth.

The aim of this short paper is to similarly extend the new developments of [7]; i.e., (4) and (5), by replacing the p -th power with a general Orlicz function globally satisfying the Δ_2 condition. Therefore, we express the Orlicz modular in terms of measures of certain level sets, without using any integral. Our proof technique is based on the argument presented in [7], appropriately extended to the Orlicz framework.

In what follows, we introduce some basic notations and definitions, needed for understanding the setting we will be considering in our main result. A proper detailed treatment of Orlicz functions and classes may be found, e.g., in [9].

A *Young function* $\Phi: [0, \infty) \rightarrow [0, \infty)$ is any continuous convex function vanishing at 0. Note that Young functions are nondecreasing. We say that a Young function Φ (globally) satisfies the Δ_2 condition if there is $k > 0$ such that $\Phi(2t) \leq k\Phi(t)$, for every $t > 0$. Then, necessarily $k \geq 2$, which follows from the convexity of Φ . We denote by $\Delta_2(\Phi)$ the infimum over all such k .

Given a Young function Φ , we say that a measurable function $u: \mathbb{R}^N \rightarrow \mathbb{R}$ belongs to the *Orlicz class* \mathcal{L}^Φ , and write $u \in \mathcal{L}^\Phi$, if

$$\int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx < \infty.$$

If Φ satisfies the Δ_2 condition, $u \in \mathcal{L}^\Phi$ implies

$$\int_{\mathbb{R}^N} \Phi(\gamma|u(x)|) \, dx < \infty, \quad \text{for every } \gamma > 0.$$

As usual, ω_N denotes the volume of the unit ball in \mathbb{R}^N .

2. Main Result

Theorem 2.1. *Let Φ be a Young function satisfying the Δ_2 condition. Let $u \in \mathcal{L}^\Phi$ and for every $t > 0$ define*

$$E_t = \left\{ (x, y) \in \mathbb{R}^{2N} : x \neq y, \frac{\Phi(|u(x) - u(y)|)}{|x - y|^N} \geq \Phi(t) \right\}.$$

Then,

$$2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx = \lim_{t \rightarrow 0^+} \Phi(t) |E_t|_{2N}. \tag{6}$$

Furthermore,

$$2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx \leq \sup_{t>0} \Phi(t) |E_t|_{2N} \leq 2\omega_N \Delta_2(\Phi) \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx. \tag{7}$$

Proof. For every $t > 0$, define the set

$$H_t = \{(x, y) \in E_t : |y| > |x|\}$$

and observe that, thanks to symmetry, it satisfies $|H_t|_{2N} = \frac{1}{2} |E_t|_{2N}$.

At first, we are going to suppose that u has compact support; i.e., there exists $R > 0$ such that

$$\text{supp } u \subset B_R.$$

Notice that, if $(x, y) \in H_t$, then necessarily $x \in B_R$, otherwise we would have $x, y \in \mathbb{R}^N \setminus B_R$ and thus $u(x) = u(y) = 0$, which would imply $(x, y) \notin H_t$.

For a fixed $x \in B_R$, define the sets

$$H_{t,x} = \{y \in \mathbb{R}^N : (x, y) \in H_t\} = \left\{ y \in \mathbb{R}^N : |y| > |x|, \frac{\Phi(|u(x) - u(y)|)}{|x - y|^N} \geq \Phi(t) \right\}$$

and

$$H_{t,x,R} = H_{t,x} \setminus B_R = \left\{ y \in \mathbb{R}^N : |y| > R, \frac{\Phi(|u(x)|)}{|x - y|^N} \geq \Phi(t) \right\}.$$

Obviously, we have

$$H_{t,x,R} \subset H_{t,x} \subset H_{t,x,R} \cup B_R. \tag{8}$$

The first inclusion together with the definition of $H_{t,x,R}$ implies

$$|H_{t,x}|_N \geq |H_{t,x,R}|_N \geq \omega_N \frac{\Phi(|u(x)|)}{\Phi(t)} - \omega_N R^N, \tag{9}$$

while the second inclusion in (8) implies

$$|H_{t,x}|_N \leq \omega_N \frac{\Phi(|u(x)|)}{\Phi(t)} + \omega_N R^N. \tag{10}$$

Since $x \in B_R$ was arbitrarily chosen, we may integrate (9) and (10) over B_R with respect to x and multiply by $\Phi(t)$ to get

$$\begin{aligned} \omega_N \int_{B_R} \Phi(|u(x)|) \, dx - \Phi(t) \omega_N^2 R^{2N} &\leq \Phi(t) \int_{B_R} |H_{t,x}|_N \, dx \\ &\leq \omega_N \int_{B_R} \Phi(|u(x)|) \, dx + \Phi(t) \omega_N^2 R^{2N}. \end{aligned}$$

Recalling that u is supported in B_R and $|H_t|_{2N} = \frac{1}{2} |E_t|_{2N}$, we may further rewrite this as

$$\begin{aligned} 2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx - 2\Phi(t) \omega_N^2 R^{2N} &\leq \Phi(t) |E_t|_{2N} \\ &\leq 2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx + 2\Phi(t) \omega_N^2 R^{2N}. \end{aligned} \tag{11}$$

Letting $t \rightarrow 0^+$, we obtain (6).

Now we are going to extend the result beyond compactly supported functions. Suppose that $u: \mathbb{R}^N \rightarrow \mathbb{R}$ is measurable. For any fixed $t > 0$, the set E_t satisfies

$$E_t \subset \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi(2|u(x)|)}{|x - y|^N} \geq \Phi(t) \right\} \cup \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi(2|u(y)|)}{|x - y|^N} \geq \Phi(t) \right\}. \tag{12}$$

Indeed, if $(x, y) \in \mathbb{R}^{2N}$ is not contained in either of the two sets on the right-hand side, then, by monotonicity and convexity of Φ ,

$$\begin{aligned} \Phi(|u(x) - u(y)|) &\leq \Phi(|u(x)| + |u(y)|) \\ &\leq \frac{1}{2}\Phi(2|u(x)|) + \frac{1}{2}\Phi(2|u(y)|) \\ &< \Phi(t)|x - y|^N, \end{aligned}$$

hence $(x, y) \notin E_t$. This shows (12).

Using the symmetry of the two sets on the right-hand side of (12), we obtain

$$\begin{aligned} |E_t|_{2N} &\leq 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{\{(x,y) \in \mathbb{R}^{2N} : |x-y|^N \leq \frac{\Phi(2|u(x)|)}{\Phi(t)}\}}(x, y) \, dy \, dx \\ &= 2 \frac{\omega_N}{\Phi(t)} \int_{\mathbb{R}^N} \Phi(2|u(x)|) \, dx, \end{aligned}$$

hence

$$\sup_{t>0} \Phi(t) |E_t|_{2N} \leq 2\omega_N \int_{\mathbb{R}^N} \Phi(2|u(x)|) \, dx. \tag{13}$$

Notice that neither the assumption nor the Δ_2 condition of Φ has been used yet, so this estimate in fact holds for any measurable u and any Young function Φ . If Φ satisfies the Δ_2 condition, (13) readily implies the second inequality in (7).

Assume that $u \in \mathcal{L}^\Phi$. Choose $R > 0$ and define

$$u_R = u\chi_{B_R}, \quad v_R = u - u_R. \tag{14}$$

Furthermore, choose $t > 0$, $\lambda \in (0, 1)$ and define

$$A_1 = \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi\left(\frac{1}{\lambda}|u_R(x) - u_R(y)|\right)}{|x - y|^N} \geq \Phi(t) \right\}$$

and

$$A_2 = \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi\left(\frac{1}{1-\lambda}|v_R(x) - v_R(y)|\right)}{|x - y|^N} \geq \Phi(t) \right\}.$$

Then, $E_t \subset A_1 \cup A_2$. Similarly as before, this can be seen using monotonicity and convexity of Φ to get

$$\begin{aligned} \Phi(|u(x) - u(y)|) &\leq \Phi(|u_R(x) - u_R(y)| + |v_R(x) - v_R(y)|) \\ &\leq \lambda\Phi\left(\frac{1}{\lambda}|u_R(x) - u_R(y)|\right) + (1 - \lambda)\Phi\left(\frac{1}{1 - \lambda}|v_R(x) - v_R(y)|\right), \end{aligned}$$

from which the inclusion follows.

Observe that the set A_1 is obtained by replacing u with $\frac{u_R}{\lambda}$ in the definition of E_t . Since $\frac{u_R}{\lambda}$ is compactly supported and belongs to \mathcal{L}^Φ (since Φ satisfies the Δ_2 condition), we may use the previously obtained estimate (11) with A_1 and $\frac{u_R}{\lambda}$ in place of E_t and u , respectively, to get

$$\Phi(t) |A_1|_{2N} \leq 2\omega_N \int_{\mathbb{R}^N} \Phi \left(\frac{|u_R(x)|}{\lambda} \right) dx + 2\Phi(t)\omega_N^2 R^{2N}.$$

Analogously, applying (13) to the function $\frac{v_R}{1-\lambda}$ in place of u (A_2 plays the role of E_t for this function), we get

$$\Phi(t) |A_2|_{2N} \leq 2\omega_N \int_{\mathbb{R}^N} \Phi \left(\frac{2|v_R(x)|}{1-\lambda} \right) dx.$$

As $E_t \subset A_1 \cup A_2$, this gives

$$\Phi(t) |E_t|_{2N} \leq 2\omega_N \left[\int_{\mathbb{R}^N} \Phi \left(\frac{|u_R(x)|}{\lambda} \right) dx + \Phi(t)\omega_N R^{2N} + \int_{\mathbb{R}^N} \Phi \left(\frac{2|v_R(x)|}{1-\lambda} \right) dx \right].$$

Since $|u_R| \leq |u|$, $|v_R| \leq |u|$, $u \in \mathcal{L}^\Phi$ and Φ satisfies the Δ_2 condition, both integrals above are finite regardless the choice of λ , and the second integral (with a fixed λ) vanishes as $R \rightarrow \infty$ by the dominated convergence theorem. Hence, consecutively letting $t \rightarrow 0^+$, $R \rightarrow \infty$ and $\lambda \rightarrow 1^-$, we finally obtain

$$\limsup_{t \rightarrow 0^+} \Phi(t) |E_t|_{2N} \leq 2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) dx. \tag{15}$$

It remains to show the opposite inequality for the lower limit. Fix $R > 0$, $\lambda \in (0, 1)$ and define u_R, v_R as in (14). Then $|u| - |v_R| = |u_R|$ and, by convexity of Φ , for any $(x, y) \in \mathbb{R}^{2N}$ we have

$$\frac{1}{\lambda} \Phi(\lambda|u_R(x) - u_R(y)|) - \frac{1-\lambda}{\lambda} \Phi \left(\frac{\lambda}{1-\lambda} |v_R(x) - v_R(y)| \right) \leq \Phi(|u_R(x) - u_R(y)|). \tag{16}$$

For any $t > 0$ define

$$A_3 = \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi(\lambda|u_R(x) - u_R(y)|)}{|x - y|^N} \geq \Phi(t) \right\}$$

and

$$A_4 = \left\{ (x, y) \in \mathbb{R}^{2N} : \frac{\Phi \left(\frac{\lambda}{1-\lambda} |v_R(x) - v_R(y)| \right)}{|x - y|^N} \geq \Phi(t) \right\}.$$

Whenever $(x, y) \in A_3 \setminus A_4$, we have

$$\begin{aligned} \Phi(t) &= \frac{1}{\lambda} \Phi(t) - \frac{1-\lambda}{\lambda} \Phi(t) \\ &< \frac{1}{|x - y|^N} \left(\frac{1}{\lambda} \Phi(\lambda|u_R(x) - u_R(y)|) - \frac{1-\lambda}{\lambda} \Phi \left(\frac{\lambda}{1-\lambda} |v_R(x) - v_R(y)| \right) \right) \\ &\leq \frac{\Phi(|u_R(x) - u_R(y)|)}{|x - y|^N}, \end{aligned}$$

where the last inequality follows from (16). This shows that $(x, y) \in E_t$. Thus, we have $E_t \supset A_3 \setminus A_4$.

We proceed analogously as before, realizing that A_3 plays the role of E_t for the compactly supported function λu_R , we use (11) to obtain

$$\Phi(t) |A_3|_{2N} \geq 2\omega_N \left[\int_{\mathbb{R}^N} \Phi(\lambda |u_R(x)|) \, dx - \Phi(t)\omega_N R^{2N} \right].$$

Similarly, an appropriate interpretation of (13) yields

$$\Phi(t) |A_4|_{2N} \leq 2\omega_N \int_{\mathbb{R}^N} \Phi \left(\frac{2\lambda}{1-\lambda} |v_R(x)| \right) \, dx.$$

Hence,

$$\begin{aligned} \Phi(t) |E_t|_{2N} &\geq \Phi(t) (|A_3|_{2N} - |A_4|_{2N}) \\ &\geq 2\omega_N \left[\int_{\mathbb{R}^N} \Phi \left(\frac{|u_R(x)|}{\lambda} \right) \, dx - \Phi(t)\omega_N R^{2N} - \int_{\mathbb{R}^N} \Phi \left(\frac{2|v_R(x)|}{1-\lambda} \right) \, dx \right]. \end{aligned}$$

Letting $t \rightarrow 0^+$, $R \rightarrow \infty$ and $\lambda \rightarrow 1^-$, in this order, now yields

$$\liminf_{t \rightarrow 0^+} \Phi(t) |E_t|_{2N} \geq 2\omega_N \int_{\mathbb{R}^N} \Phi(|u(x)|) \, dx. \tag{17}$$

Once again, the Δ_2 condition of Φ as well as the assumption $u \in \mathcal{L}^\Phi$ are both required in this step. This clearly implies the first inequality in (7). Finally, combining (17) with (15), we arrive at (6), and so the proof is complete. \square

Remark 2.2. Applying the theorem to $\Phi(t) = t^p$, $p \in [1, \infty)$, we recover [7, Theorem 1] with the same multiplicative constants.

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Data Availability The manuscript has no associated data.

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