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The full group of automorphisms of non-orientable unbordered Klein surfaces of topological genus 3, 4 and 5

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Abstract An important problem in the study of Riemann and Klein surfaces is determining their full automorphism groups. Up to now only very partial results are known, concerning surfaces of low genus or families of surfaces with special properties. This paper deals with non-orientable unbordered Klein surfaces. In this case the solution of the problem is known for surfaces of genus 1 and 2, and for hyperelliptic surfaces. Here we explicitly obtain the full automorphism group of all surfaces of genus 3, 4 and 5.

Keywords Non-orientable surface · Klein surface · Automorphism group · Symmetric crosscap number

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The authors wish to dedicate this paper to the memory of Professor José Javier Etayo (1926–2012). He was our teacher, as well as of many thousands of students throughout forty years in the Universidad Complutense.

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1 Introduction and preliminaries

A Klein surface X is a compact surface endowed with a dianalytic structure [1]. Klein surfaces may be seen as a generalization of Riemann surfaces including bordered and non-orientable surfaces. An orientable unbordered Klein surface is a Riemann surface. Given a Klein surface X of topological genus g with k boundary components the number $p = \eta g + k - 1$ is called the algebraic genus of X, where $\eta = 2$ if X is an orientable surface and $\eta = 1$ otherwise.

In the study of Klein surfaces and their automorphism groups the non-Euclidean crystallographic groups (NEC groups, in short) play an essential role. An NEC group Γ is a discrete subgroup of $\mathcal{G}$ (the full group of isometries of the hyperbolic plane $\mathcal{H}$) with compact quotient $\mathcal{H}/\Gamma$. For each Klein surface X with $p \geq 2$ there exists an NEC group Γ, such that $X = \mathcal{H}/\Gamma$ [24]. An alternative approach by means of graph theory can be seen, for instance, in [26].

A finite group G of order N is an automorphism group of a Klein surface $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. Every finite group G acts as an automorphism group of non-orientable Klein surfaces without boundary. The minimum genus of these surfaces is called the symmetric crosscap number of G and it is denoted by $\widetilde{\sigma}(G)$. Such a surface of topological genus $g \geq 3$ has at most $84(g-2)$ automorphisms. Hence for each g there is a finite number of groups acting on surfaces of genus g.

The aim of this work is determining the full automorphism group of the non-orientable unbordered Klein surfaces of topological genus 3, 4 and 5. The paper is organized as follows. Section 2 is devoted to obtain all groups having symmetric crosscap number 4 or 5, what is itself an important task. In Sect. 3 we obtain all groups acting on surfaces of each of the genera 3, 4 and 5. Finally in Sect. 4 we determine which of them are the full automorphism groups of these surfaces.

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [18]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}),\ i = 1, \ldots, k\}), \tag{1.1}$$

where $g, k, r, m_i, n_{i,j}$ are integers satisfying $g, k, r \geq 0$, $m_i \geq 2$, $n_{i,j} \geq 2$. The number g is the topological genus of X. The sign determines the orientability of X. The numbers m_i are the *proper periods*. The brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of X. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \ldots, n_{i,s_i})$ also called *link-periods*. We will denote by $[-]$ and $(-)$ the cases when $r = 0$ and $s_i = 0$, respectively.

The signature determines a presentation [27] of Γ by generators x_i $(i = 1, \ldots, r)$; e_i $(i = 1, \ldots, k)$; $c_{i,j} (i = 1, \ldots, k;\ j = 0, \ldots, s_i)$; a_i, b_i $(i = 1, \ldots, g)$ if σ has sign '+'; and d_i $(i = 1, \ldots, g)$ if σ has sign '−'. These generators satisfy the following relations: $x_i^{m_i} = 1$; $c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1$; $e_i^{-1}c_{i,0}e_ic_{i,s_i} = 1$; and finally $\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_ib_ia_i^{-1}b_i^{-1}) = 1$ (if σ has sign '+'), or $\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1$ (if σ has sign '−').

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide reflections.

Every NEC group Γ with signature (1.1) has associated a fundamental region whose area $\mu(\Gamma)$, called the *area of the group*, is:

$$\mu(\Gamma) = 2\pi\left(\eta g + k - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{i,j}}\right)\right), \tag{1.2}$$

with $\eta = 2$ or 1 depending on the sign '+' or '−' in the signature. An NEC group with signature (1.1) actually exists if and only if the right hand side of (1.2) is greater than 0. We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the *reduced area* of Γ.

If Γ is a subgroup of an NEC group Λ of finite index N, then Γ is an NEC group as well and the Riemann–Hurwitz formula holds, $|\Gamma| = N|\Lambda|$.

Let X be a non-orientable Klein surface of topological genus $g \geq 3$ without boundary. Then by [24] there exists an NEC group Γ with signature:

$$\sigma(\Gamma) = (g, -, [-], \{-\}), \tag{1.3}$$

such that $X = \mathcal{H}/\Gamma$.

2 The groups of symmetric crosscap number 4 and 5

Every finite group G may act as an automorphism group of some non-orientable unbordered Klein surfaces [3, Theorem 2.5]. The minimum genus of these surfaces is called the *symmetric crosscap number* of G, and it is denoted by $\tilde{\sigma}(G)$. Let $X = \mathcal{H}/\Gamma$ be a non-orientable unbordered Klein surface on which G acts as an automorphism group. Then there exists another NEC group Λ such that $G = \Lambda/\Gamma$. From the Riemann–Hurwitz formula we have $g - 2 = o(G)\,|\Lambda|$, where $o(G)$ denotes the order of G. Then

$$\tilde{\sigma}(G) \leq g = 2 + o(G)\,|\Lambda|,$$

and so obtaining the symmetric crosscap number is equivalent to finding a group Λ and an epimorphism $\theta : \Lambda \to G$ such that $\Gamma = \ker\theta$ is a surface NEC group (that is, without elements of finite orders) and $G = \theta(\Lambda^+)$, where Λ^+ is the subgroup consisting of the orientation-preserving elements of Λ, see [25], and minimal $|\Lambda|$.

The groups having symmetric crosscap numbers 1 and 2 have been classified by Tucker [26]. The groups of genus 1 are C_n, D_n, A_4, S_4 and A_5. We have two families of groups of genus 2, $C_2 \times C_n$, $n > 2$ even, and $C_2 \times D_n$, n even. It is known that there exists no group of symmetric crosscap number 3 [19].

Conder at a conference in Castro-Urdiales in 2010 announced that using computing software, he had obtained the groups of symmetric crosscap number up to 65, in terms of their 'SmallGroupLibrary' description. He kindly sent us this unpublished list. However, this output gives information neither on the algebraic structure of the groups nor on the epimorphism θ which are necessary in order to obtain the full automorphism groups of surfaces. Hence in this section we explicitly determine all groups having symmetric crosscap numbers 4 and 5.

The symmetric crosscap number of groups belonging to several infinite families, as well as of the groups of order lesser than 32, has already been obtained. These results will be quoted precisely while we will use them.

For the groups of symmetric crosscap number 1 and 2 another parameter has been defined in [16]. We denote by $\tilde{\tau}(G)$ the least genus $g \geq 3$ such that the group G acts on non-orientable unbordered Klein surfaces of genus g. In particular, $\tilde{\tau}(C_n) = 3$ for $n = 2, 3, 4, 6$; there exists no group C_n with $\tilde{\tau}(C_n) = 4$, and $\tilde{\tau}(C_n) = 5$ for $n = 5, 8, 10$, see [3] or [16]. These results allow us to restrict the possible orders of elements of the groups acting on surfaces of genus $g \leq 5$.

In the next Table we list all signatures of NEC groups Λ whose proper periods and link-periods subject to the restrictions imposed in the previous paragraph and whose reduced area satisfies $|\Lambda| < 1/12$. We also indicate the signature of the canonical Fuchsian subgroup Λ^+ of Λ. All these data will be used later on.

2.1 Groups of symmetric crosscap number 4

Theorem 1 *The groups of symmetric crosscap number* 4 *are* $C_2 \times A_4$ *and* $C_2 \times S_4$.

Proof Recall that if G is an automorphism group of a non-orientable unbordered Klein surface of genus 4 it can be expressed as Λ/Γ, where Γ has the signature $(4, -, [-], \{-\})$. Then the order of G is

$$o(G) = 2/|\Lambda|.$$

We distinguish two cases according to if $o(G) \leq 24$ or $o(G) > 24$. The unique group of order up to 24 with symmetric crosscap number 4 is $C_2 \times A_4$ [16].

Now suppose that $o(G) > 24$, then $|\Lambda| < 1/12$. Since G is an image of Λ^+ and as we have seen above the elements of G can only have orders 2, 3, 4 and 6, we only need to check groups Λ in Table 1 such that Λ^+ has signature $(0, +, [2, 4, 6], \{-\})$ or $(0, +, [3, 3, 4], \{-\})$. In both cases $|\Lambda| = 1/24$ and so $o(G) = 48$.

May and Zimmerman proved in [21] that there are just two groups of order 48 for which the corresponding group Λ^+ has the required signatures. The first group $C_2 \times S_4$ has symmetric crosscap number 4 (see [13]). The next group is called J_{48} and it has a presentation given by generators A and B, and relations $A^3 = B^3 = (AB)^4 = (AB^2)^3 = 1$. It is called $(3, 3 \mid 3, 4)$ in [10]. The groups $(3, 3 \mid 3, k)$ were firstly studied by Edington in [11]. He proved that such a group contains an Abelian subgroup of order k^2 generated by two elements of order k. In particular $(3, 3 \mid 3, 4)$ contains a subgroup with structure $C_4 \times C_4$. Hence $\tilde{\sigma}((3, 3 \mid 3, 4)) \geq \tilde{\sigma}(C_4 \times C_4) = 10$, see [17] for the last equality.

Table 1 The relevant NEC groups with small reduced area

Λ	$\lvert\Lambda\rvert$	Λ^+
$(0, +, [-], \{(2, 4, 10)\})$	$3/40$	$(0, +, [2, 4, 10], \{-\})$
$(0, +, [-], \{(2, 5, 6)\})$	$1/15$	$(0, +, [2, 5, 6], \{-\})$
$(0, +, [-], \{(3, 3, 5)\})$	$1/15$	$(0, +, [3, 3, 5], \{-\})$
$(0, +, [3], \{(5)\})$	$1/15$	$(0, +, [3, 3, 5], \{-\})$
$(0, +, [-], \{(2, 4, 8)\})$	$1/16$	$(0, +, [2, 4, 8], \{-\})$
$(0, +, [-], \{(2, 5, 5)\})$	$1/20$	$(0, +, [2, 5, 5], \{-\})$
$(0, +, [5], \{(2)\})$	$1/20$	$(0, +, [2, 5, 5], \{-\})$
$(0, +, [-], \{(2, 4, 6)\})$	$1/24$	$(0, +, [2, 4, 6], \{-\})$
$(0, +, [-], \{(3, 3, 4)\})$	$1/24$	$(0, +, [3, 3, 4], \{-\})$
$(0, +, [3], \{(4)\})$	$1/24$	$(0, +, [3, 3, 4], \{-\})$
$(0, +, [-], \{(2, 3, 10)\})$	$1/30$	$(0, +, [2, 3, 10], \{-\})$
$(0, +, [-], \{(2, 4, 5)\})$	$1/40$	$(0, +, [2, 4, 5], \{-\})$
$(0, +, [-], \{(2, 3, 8)\})$	$1/48$	$(0, +, [2, 3, 8], \{-\})$

Remark 1 Table III in [9] gives the list of non-orientable regular maps whose genera range from 3 to 30. This list includes two maps of genus 4, both with 48 automorphisms. This fact leads up to question whether there are more groups with symmetric crosscap number 4 (see [22, Problem 3]) and was understood as the existence of two groups of order 48 with symmetric crosscap number 4, [13, Section 3]. We are going to prove that in fact the automorphism group of each map is isomorphic to $C_2 \times S_4$.

The respective groups are described in [9] by a presentation with generators R, S, T and relations:

(i) $T^2 = (RT)^2 = (TS)^2 = (RS)^2 = R^4 = TS^{-1}RS^{-1}R^{-2} = S^6 = 1$,
(ii) $T^2 = (RT)^2 = (TS)^2 = (RS)^2 = R^4 = (RS^{-2})^2 = S^6 = S^2RS^{-1}R^{-2}T = 1$.

In order to deal with the first presentation, let us choose the elements $R = (1, 3, 4, 2)$, $S = X(1, 2, 3)$ and $T = (2, 3)$ in the group $C_2 \times S_4$, where X denotes the generator of C_2. Since $S^2 = (1, 3, 2)$, $S^3 = X$, and $\langle(1, 3, 4, 2), (1, 3, 2)\rangle = S_4$, these elements generate $C_2 \times S_4$. Besides, T has order 2, R has order 4, S has order 6, $RT = (1, 2)(3, 4)$ has order 2, $TS = X(1, 2)$ has order 2, $RS = X(3, 4)$ has order 2, and finally

$$TS^{-1}RS^{-1}R^{-2} = (2, 3)X(1, 3, 2)(1, 3, 4, 2)X(1, 3, 2)(1, 4)(2, 3) = 1.$$

Hence $C_2 \times S_4$ has the generators R, S, T, which satisfy the relations (i).

It is easy to check that if we exchange $T = (2, 3)$ for $T = X(2, 3)$ then the elements R, S, T generate $C_2 \times S_4$ and satisfy the relations (ii). Thus both groups of order 48 in [9] are isomorphic to $C_2 \times S_4$. It means that Theorem 1 agrees with the result in [9] and gives the answer to the problem 3 in [22].

Table 2 The groups of order 36 with known symmetric crosscap number

G	$\tilde{\sigma}(G)$	References
$C_3 \times C_{12}$	23	Gromadzki [17]
$C_6 \times C_6$	26	Gromadzki [17]
$C_6 \times D_3$	14	Etayo and Martínez [13]
$C_3 \times A_4$	14	Etayo and Martínez [15]
$C_3 \times DC_3$	17	Etayo and Martínez [15]
DC_9	20	May [19]
$D_3 \times D_3$	5	Etayo and Martínez [14]

2.2 Groups of symmetric crosscap number 5

This case is a little more involved than the previous one. In the proof of the next result we shall distinguish two cases according to if the order of the group is or is not greater than 36, that is if the reduced area of the corresponding group Λ is or is not lesser than $1/12$.

Theorem 2 *There are eight groups with symmetric crosscap number* 5, *which are* $C_3 \times C_3$, $((3, 3, 3; 2))$, $C_3 \times D_3$, $\langle 5, 4, 2\rangle$, $D_3 \times D_3$, $(4, 4 \mid 2, 3)$, $(2, 4, 6; 2)$ *and* S_5.

Proof Let G be an automorphism group of a surface of genus 5. Then $G = \Lambda/\Gamma$, where Γ has the signature $(5, -, [-], \{-\})$, and so $o(G) = 3/|\Lambda|$.

First we consider the case when $o(G) \leq 36$. The symmetric crosscap number of the groups of order lesser than 32 was obtained in [16]. There are the following groups among them with symmetric crosscap number 5: $C_3 \times C_3$, $((3, 3, 3; 2))$, $C_3 \times D_3$ and $\langle 5, 4, 2\rangle$.

Now suppose $o(G) = 32$, then $|\Lambda| = 3/32$. This implies that the signature of Λ contains proper periods or link-periods equal to 16 or 32. By arguments given in Sect. 2, such elements cannot act on surfaces of genus 5.

All groups of orders 33, 34 and 35 are cyclic or dihedral and so their symmetric crosscap number is 1.

Next we deal with groups of order 36. There are fourteen such groups. The cyclic and the dihedral groups have symmetric crosscap number 1, and $\tilde{\sigma}(C_2 \times C_{18}) = 2$. So it remains eleven groups. Seven of them have been already studied and we present them in Table 2.

The next group which we have to consider is $G = C_2 \times ((3, 3, 3; 2))$. Since the second factor contains a subgroup $C_3 \times C_3$, the group G contains a subgroup $C_3 \times C_6$ whose symmetric crosscap number is 11, so that G must be rejected.

The three remaining groups are described in [23] by generators and relations as follows:

$$Gp(1) = \langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = a^{-1}\rangle,$$
$$Gp(2) = \langle a, b, c \mid a^2 = b^2 = c^9 = [a, b] = 1, c^{-1}ac = b, c^{-1}bc = ab\rangle,$$
$$Gp(3) = \langle a, b, c \mid a^3 = b^3 = c^4 = [a, b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle.$$

Table 3 Candidate groups G with $O(G) > 36$

G	$o(G)$	Λ^+	Λ
$(4, 10 \mid 2, 2)$	40	$(0, +, [2, 4, 10], \{-\})$	$(0, +, [-], \{(2, 4, 10)\})$
$(2, 4, 6; 2)$	72	$(0, +, [2, 4, 6], \{-\})$	$(0, +, [-], \{(2, 4, 6)\})$
S_5	120	$(0, +, [2, 4, 5], \{-\})$	$(0, +, [-], \{(2, 4, 5)\})$

The group $Gp(2)$ must be rejected because if contains elements of order 9. The group $Gp(3)$ has only one element of order 2, see [20]. This implies that the signature of the corresponding NEC group Λ cannot have link-periods and there exists no such a group with $|\Lambda| = 1/12$.

We are going to prove that $\tilde{\sigma}(Gp(1))$ is 5. This group is denoted in [10] by $(4, 4 \mid 2, 3)$ and it was already studied in [11], where a more suitable presentation is given as follows:

$$\langle t_1, t_2 \mid t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^3 = 1\rangle.$$

Consider the NEC group Λ with signature $(0, +, [4], \{(3)\})$ and define $\theta : \Lambda \to (4, 4 \mid 2, 3)$ by

$$\theta(x_1) = t_1, \theta(e_1) = t_1^3, \theta(c_{1,0}) = t_1 t_2, \theta(c_{1,1}) = t_1^2 t_2 t_1^3.$$

Then $\theta(c_{1,0}c_{1,1}) = t_2^2 t_1^2$ has order 3. The elements $\theta(x_1)$ and $\theta(c_{1,0}c_{1,1})$ belong to $\theta(\Lambda^+)$. So the order of $\theta(\Lambda^+)$ is a multiple of 12. All groups of order 12 except A_4 have elements of order 6 which do not exist in $(4, 4 \mid 2, 3)$; and the group A_4 has not elements of order 4. Hence $\theta(\Lambda^+)$ has not order 12 and so $\theta(\Lambda^+)$ must be the full group $(4, 4 \mid 2, 3)$. Since $|\Lambda| = 1/12$ we conclude that $\tilde{\sigma}((4, 4 \mid 2, 3)) = 5$, and we have finished the analysis of the groups of $o(G) \leq 36$.

Now let $o(G) > 36$. Then $G = \Lambda/\Gamma$ with $|\Lambda| < 1/12$. So we need to check the signatures appearing in Table 1. The candidates satisfying the condition on Λ^+ were obtained in [23]. We present them in Table 3 together with relevant information about the groups Λ and Λ^+.

We study the three groups separately. We begin with the group $G = (4, 10 \mid 2, 2)$. It has a presentation (see [10]) with generators A and B, and relations $A^4 = B^{10} = (AB)^2 = (A^{-1}B)^2 = 1$. In order to prove that G is not an image of the group Λ with signature $(0, +, [-], \{(2, 4, 10)\})$, observe that the element A^2 is central in G. Then it is easy to check that the elements of G of order 2 are A^2, B^5, A^2B^5, and AB^i and A^3B^i with i odd. Calculating the products of any pair of these elements we obtain that the product has order 10 only if the factors are respectively AB^j and A^3B^k. However, there exists no element of order 2 such that simultaneously its product with AB^j have order 2, and the product with A^3B^k have order 4, or conversely. So the corresponding link-periods cannot be 2 and 4, and so the group G must be rejected since there does not exist any epimorphism θ from Λ onto G.

Next let $G = (2, 4, 6; 2)$. In [10], Coxeter gives a description of this group as the subgroup of S_6 generated by the permutations $A = (1, 6)(2, 5)(3, 4)$ and $B =$

$(1,2)(3,4,5,6)$. Then the elements $(AB^2)^3 = (1,2)(3,4)(5,6)$, $B(AB^2)^2A = (3,5)$ and $(AB^2)^3A = (1,5)(2,6)$ belong to G. We define the desired epimorphism θ from Λ with signature $(0, +, [-], \{(2,4,6)\})$ onto G by means of

$$\theta(c_{1,0}) = \theta(c_{1,3}) = (1,5)(2,6),\ \theta(c_{1,1}) = (1,2)(3,4)(5,6),\ \theta(c_{1,2}) = (3,5).$$

Then $\theta(c_{1,0}c_{1,1}) = A$ and $\theta(c_{1,1}c_{1,2}) = B$. So $\theta(\Lambda^+) = G$ and since $|\Lambda| = 1/24$ we have proved that $\tilde{\sigma}((2,4,6;2)) = 5$.

Finally the group S_5 has symmetric crosscap number 5 what was proved in [22].

So we have considered all groups and checked which of them have symmetric crosscap number 5.

3 Groups acting on surfaces of genus 3, 4 and 5

Let G be a finite group. Recall that $\tilde{\sigma}(G)$ is the least integer g such that G is an automorphism group of a non-orientable unbordered Klein surface of genus g. Then $\tilde{\sigma}(G) \leq 2$ if and only if G is C_n, D_n, $C_2 \times C_n$, $C_2 \times D_n$, A_4, S_4 or A_5. For these groups, we denote by $\tilde{\tau}(G)$ the least integer $g \geq 3$ such that G is an automorphism group of a non-orientable unbordered Klein surface of genus g.

Now let $g \geq 3$. All groups G satisfying $\tilde{\sigma}(G) = g$ or $\tilde{\tau}(G) = g$ act on a surface of genus g. The other possible automorphism groups of such a surface are those satisfying $3 \leq \tilde{\sigma}(G) < g$ or $3 \leq \tilde{\tau}(G) < g$.

In this section we determine all groups acting on surfaces of genera $g = 3$, 4 or 5. There is no group such that $\tilde{\sigma}(G) = 3$, whilst those having $\tilde{\sigma}(G) = 4$ or 5 have been obtained in Sect. 2. In [16] the parameter $\tilde{\tau}(G)$ was obtained for all groups. We divide our study into three parts according to the values of g. In each case we place a Table with the signature of Λ such that $G = \Lambda/\Gamma$ and a diagram with the relations of inclusion of the groups acting on surfaces of a given genus. We shall use them later on.

3.1 Surfaces of genus 3

Proposition 1 *The groups acting on non-orientable unbordered Klein surfaces of genus* 3 *are* C_2, C_3, C_4, C_6, $C_2 \times C_2$, D_3, D_4 *and* D_6.

Proof There is no group G with symmetric crosscap number 3 and the groups satisfying $\tilde{\tau}(G) = 3$ were obtained in [16].

3.2 Surfaces of genus 4

Proposition 2 *The groups acting on non-orientable unbordered Klein surfaces of genus* 4 *are:*

(i) $C_2 \times A_4$ *and* $C_2 \times S_4$;

(ii) $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, $C_2 \times D_4$, A_4 *and* S_4;
and

(iii) C_2, C_3, C_4, C_6, $C_2 \times C_2$, D_3, D_4 *and* D_6.

Proof (i) Both groups are those having symmetric crosscap number 4 (see Theorem 1).

(ii) The groups G satisfying $\tilde{\tau}(G) = 4$ are $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, $C_2 \times D_4$, A_4 and S_4, as shown in [16].

(iii) We need to check that the groups satisfying $\tilde{\tau}(G) = 3$ act on surfaces of genus 4. Both D_4 and D_6 are subgroups of $C_2 \times S_4$ which is the automorphism group of some surface of genus 4. Hence these two groups and their subgroups do act on surfaces of genus 4.

Diagram 1 presents the relations of inclusion among the groups listed in Proposition 2.

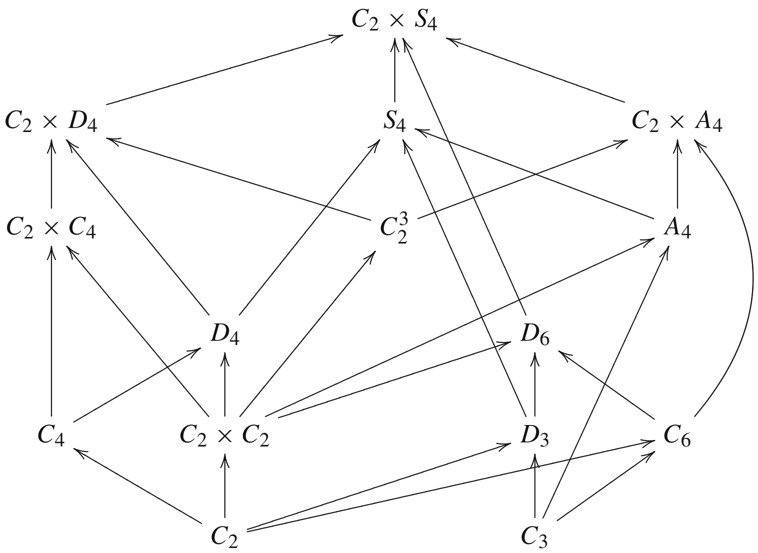

Diagram 1

In Table 4 we give for each group G the signature(s) of Λ such that $G = \Lambda/\Gamma$.

3.3 Surfaces of genus 5

Proposition 3 *The groups acting on non-orientable unbordered Klein surfaces of genus 5 are*

(i) $C_3 \times C_3$, $((3,3,3;2))$, $C_3 \times D_3$, $\langle 5,4,2 \rangle$, $D_3 \times D_3$, $(4,4 \mid 2,3)$, $(2,4,6;2)$ *and* S_5;

(ii) C_5, C_8, C_{10}, D_5, D_8, D_{10} *and* A_5;
and

Table 4 Groups acting on surfaces of genus 4

G	$o(G)$	Λ
C_2	2	$(0, +, [2, 2, 2, 2], \{(-)\})$
		$(0, +, [2, 2], \{(-), (-)\})$
		$(0, +, [-], \{(-), (-), (-)\})$
		$(1, -, [2, 2, 2, 2], \{-\})$
		$(1, -, [2, 2], \{(-)\})$
		$(1, -, [-], \{(-), (-)\})$
		$(2, -, [2, 2], \{-\})$
		$(2, -, [-], \{(-)\})$
		$(3, -, [-], \{-\})$
		$(1, +, [-], \{(-)\})$
C_3	3	$(2, -, [3], \{-\})$
C_4	4	$(0, +, [4, 4], \{(-)\})$
		$(1, -, [4, 4], \{-\})$
$C_2 \times C_2$	4	$(0, +, [2, 2, 2], \{(-)\})$
		$(0, +, [2, 2], \{(2, 2)\})$
		$(0, +, [2], \{(2, 2, 2, 2)\})$
		$(0, +, [-], \{(2, 2, 2, 2, 2, 2)\})$
		$(1, -, [2, 2, 2], \{-\})$
		$(1, -, [2], \{(-)\})$
		$(0, +, [2], \{(-)(-)\})$
C_6	6	$(0, +, [2, 6], \{(-)\})$
		$(1, -, [2, 6], \{-\})$
D_3	6	$(0, +, [2, 2], \{(3)\})$
		$(0, +, [-], \{(-), (3)\})$
		$(1, -, [-], \{(3)\})$
$C_2 \times C_4$	8	$(0, +, [2, 4], \{(-)\})$
		$(0, +, [4], \{(2, 2)\})$
$C_2 \times C_2 \times C_2$	8	$(0, +, [2], \{(2, 2, 2)\}$
		$(0, +, [-], \{(2, 2, 2, 2, 2)\})$
D_4	8	$(0, +, [2, 4], \{(-)\})$
		$(0, +, [4], \{(2, 2)\})$
		$(0, +, [2, 2], \{(2)\})$
D_6	12	$(0, +, [-], \{(2, 2, 2, 6)\})$
		$(0, +, [2], \{(2, 6)\})$
A_4	12	$(1, -, [2, 3], \{-\})$
$C_2 \times D_4$	16	$(0, +, [-], \{(2, 2, 2, 4)\})$
S_4	24	$(0, +, [4], \{(3)\})$
		$(0, +, [2], \{(2, 3)\})$
		$(0, +, [-], \{(2, 2, 2, 3)\})$
$C_2 \times A_4$	24	$(0, +, [6], \{(2)\})$
$C_2 \times S_4$	48	$(0, +, [-], \{(2, 4, 6)\})$

(iii) C_2, C_3, C_4, $C_2 \times C_2$, C_6, D_3, D_4, D_6, A_4 *and* S_4.

Proof The lists (i) and (ii) contain all groups G with $\tilde{\sigma}(G) = 5$ and $\tilde{\tau}(G) = 5$ which were obtained in Theorem 2 and [16] respectively.

Now we consider the groups acting on surfaces of genus 4 and the relations of inclusion presented on Diagram 1.

We start with $G = C_2 \times C_4$. If G acts on surfaces of genus 5 then $G = \Lambda/\Gamma$ with $|\Lambda| = 3/8$. Then either the signature of Λ has a proper period 8 or a link-period 4. But G has not elements of order 8, and it has not elements of order 2 with product of order 4. Hence $C_2 \times C_4$ does not act on surfaces of genus 5, and so neither $C_2 \times D_4$ nor $C_2 \times S_4$ act on surfaces of genus 5.

The groups S_4 and D_6 are subgroups of S_5. Thus both of them are automorphism groups of surfaces of genus 5.

The remaining maximal subgroup of $C_2 \times S_4$ is $G = C_2 \times A_4$. If this one acts on surfaces of genus 5 it can be expressed as $G = \Lambda/\Gamma$ with $|\Lambda| = 1/8$. Since G has not elements of order 4 or 8, this group must be rejected.

The maximal subgroups of $C_2 \times A_4$ are A_4, C_6 and $C_2 \times C_2 \times C_2$. The first two are subgroups of S_5. The group $C_2 \times C_2 \times C_2$ is discarded by means of a similar argument that was used for $C_2 \times C_4$ since it only has elements of order 2.

As a consequence the groups acting on surfaces of genus 4 that do not act on genus 5 are $C_2 \times C_4$, $C_2 \times D_4$, $C_2 \times S_4$, $C_2 \times A_4$ and $C_2 \times C_2 \times C_2$.

The next Diagram presents all relations of inclusion among groups given in Proposition 3.

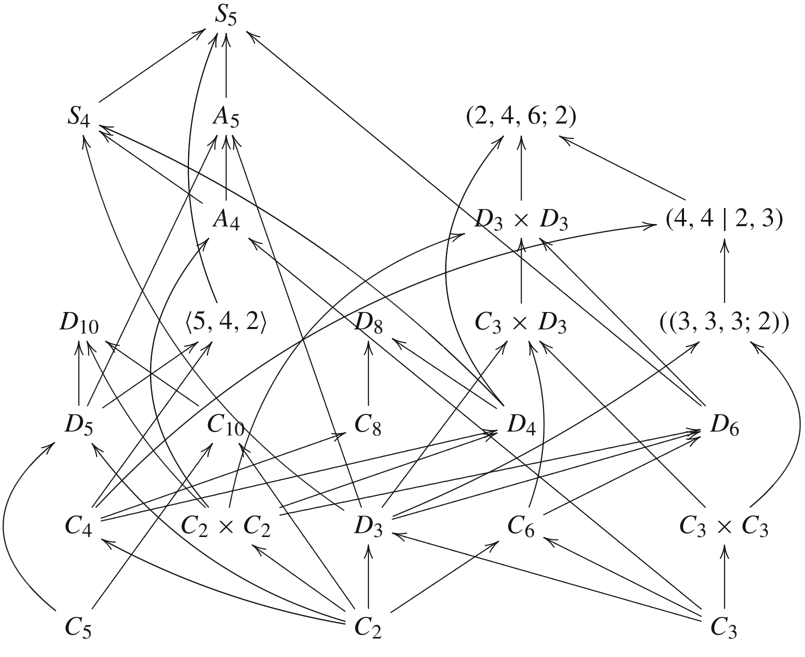

Diagram 2

In Table 5 we give for each group G the signature(s) of Λ such that $G = \Lambda/\Gamma$.

Table 5 Groups acting on surfaces of genus 5

G	$o(G)$	Λ
C_2	2	$(0, +, [2, 2, 2, 2, 2], \{(-)\})$
		$(0, +, [2, 2, 2], \{(-), (-)\})$
		$(0, +, [2], \{(-), (-), (-)\})$
		$(1, -, [2, 2, 2], \{(-)\})$
		$(1, -, [2], \{(-), (-)\})$
		$(2, -, [2], \{(-)\})$
		$(1, +, [2], \{(-)\})$
C_3	3	$(1, -, [3, 3, 3], \{-\})$
		$(3, -, [-], \{-\})$
C_4	4	$(0, +, [2, 2, 4], \{(-)\})$
		$(1, -, [2, 2, 4], \{-\})$
		$(2, -, [4], \{-\})$
		$(0, +, [4], \{(-), (-)\})$
		$(1, -, [4], \{(-)\})$
$C_2 \times C_2$	4	$(0, +, [2, 2], \{(2, 2, 2)\})$
		$(0, +, [2], \{(2, 2, 2, 2, 2)\})$
		$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2)\})$
		$(0, +, [-], \{(2, 2, 2), (-)\})$
		$(1, -, [-], \{(2, 2, 2)\})$
C_5	5	$(1, -, [5, 5], \{-\})$
C_6	6	$(0, +, [2], \{(-), (-)\})$
		$(1, -, [2], \{(-)\})$
D_3	6	$(0, +, [2, 3], \{(3)\})$
		$(0, +, [2], \{(3, 3, 3)\})$
		$(0, +, [2, 2, 2], \{(-)\})$
		$(0, +, [2], \{(-), (-)\})$
		$(1, -, [2], \{(-)\})$
C_8	8	$(0, +, [2, 8], \{(-)\})$
D_4	8	$(0, +, [-], \{(2, 2, 2, 2, 4)\})$
		$(0, +, [2], \{(2, 2, 4)\})$
$C_3 \times C_3$	9	$(1, -, [3, 3], \{-\})$
C_{10}	10	$(0, +, [2, 5], \{(-)\})$
D_5	10	$(0, +, [2, 5], \{(-)\})$
		$(0, +, [2], \{(5, 5)\})$
D_6	12	$(0, +, [-], \{(2, 2, 2, 2, 2)\})$
		$(0, +, [2], \{(2, 2, 2)\})$
A_4	12	$(1, -, [-], \{(2)\})$
D_8	16	$(0, +, [-], \{(2, 2, 2, 8)\})$
$C_3 \times D_3$	18	$(0, +, [6], \{(3)\}$
		$(0, +, [2, 3], \{(-)\})$

Table 5 continued

G	$o(G)$	Λ
$((3,3,3;2))$	18	$(0,+,[2],\{(3,3)\})$
D_{10}	20	$(0,+,[-],\{(2,2,2,5)\})$
$\langle 5,4,2\rangle$	20	$(0,+,[4],\{(5)\})$
S_4	24	$(0,+,[2],\{(2,4)\})$
$D_3\times D_3$	36	$(0,+,[-],\{(2,2,2,3)\})$
		$(0,+,[-],\{(2,6,6)\})$
$(4,4\mid 2,3)$	36	$(0,+,[4],\{(3)\})$
A_5	60	$(0,+,[-],\{(2,5,5)\})$
$(2,4,6;2)$	72	$(0,+,[-],\{(2,4,6)\})$
S_5	120	$(0,+,[-],\{(2,4,5)\})$

Remark 2 The converse problem is determining the spectrum of a given group G, that is finding the values of g such that G is an automorphism group of a surface of genus g.

It is easy to check that C_2, C_3, C_4, C_6, $C_2\times C_2$, D_3, D_4, D_6 act on surfaces of genus g for every $g\geq 3$. Consider for instance $C_2\times C_2$ generated by X and Y. Let $g\geq 3$ and take an NEC group Λ with the signature $(0,+,[-],\{(2,\overset{g+2}{\ldots},2)\})$. Define a homomorphism θ from Λ onto $C_2\times C_2$ by $\theta(c_{1,0})=\theta(c_{1,g+2})=X$, $\theta(c_{1,2i})=Y$, $\theta(c_{1,2i+1})=XY$ for $0<2i, 2i+1<g+2$. Then $\theta(c_{1,0}c_{1,1})=Y$, $\theta(c_{1,1}c_{1,2})=X$, and so $\theta(\Lambda^+)=C_2\times C_2$. The reduced area of Λ is $(g-2)/4$, and so $C_2\times C_2$ acts on a surface of genus g.

This result corrects [7, Remark 3.3], where the signatures with link-periods were omitted, and hence only a part of the spectrum of $C_2\times C_2$ was obtained.

4 Full groups of automorphisms

Let $X=\mathcal{H}/\Gamma$ be a non-orientable unbordered Klein surface. If an automorphism group $G=\Lambda/\Gamma$ acting on X is not the full automorphism group, then Λ is properly contained with finite index in another NEC group Λ', which normalizes Γ. An NEC group is said to be maximal if there does not exist another NEC group containing it properly.

A signature σ is said to be maximal if for every NEC group Λ' with signature σ' containing an NEC group Λ with signature σ and such that $d(\Lambda)=d(\Lambda')$, it is to say, the Teichmüller spaces of Λ and Λ' have the same dimension, then $\sigma=\sigma'$.

It was proved in [8] that given a maximal signature σ, there exists a maximal NEC group Λ with signature σ. Besides, if σ is not maximal then the pair (σ,σ') of signatures of Λ and Λ' is called a normal pair if Λ is a normal subgroup in Λ', and a non-normal pair if Λ is a non-normal subgroup in Λ'.

The complete lists of normal and non-normal pairs were obtained respectively in [2,12]. They are crucial in this section. By above remarks, if a finite group G can be written as Λ/Γ and the signature of Λ does not appear in either of the lists, then G

is the full group of automorphisms of the surface $X = \mathcal{H}/\Gamma$. On the other hand if the signature of Λ is not maximal then the action of G can possibly be extended to a group G' containing G. These cases will be handled separately.

If the signature σ of Λ is not maximal we take the pair (σ, σ') and consider a group Λ' with signature σ'. The action of G extends to a group G' if for every epimorphism $\theta : \Lambda \to G$ with kernel Γ, there exists an epimorphism $\theta' : \Lambda' \to G'$ whose restriction to Λ is θ. Otherwise, the action of G does not extend and G is the full automorphism group of some surface.

So far only partial results have been obtained. Firstly, in [4] the full groups of automorphism of hyperelliptic surfaces of genus $g \geq 3$ were obtained. Secondly, if G is a cyclic group, it was determined in [6] when the action of G extends to another group G'. We will quote both results while we will use them.

4.1 Surfaces of genus 3

The surfaces of topological genus 3 have algebraic genus 2 and so they are all hyperelliptic. Their automorphism groups were obtained in [4, Theorem 3.5] what leads to

Theorem 3 *Let X be a non-orientable unbordered Klein surface of topological genus 3. Then its full automorphism group is one of the following five groups:* C_2, $C_2 \times C_2$, D_3, D_4 *and* D_6. *Furthermore, for each group, there exists a surface of genus* 3 *on which the group acts as the full automorphism group.*

For completeness let us observe that the three groups in Proposition 1 which are missing in Theorem 3, namely C_3, C_4 and C_6, extend their actions to D_3, D_4 and D_6, respectively.

4.2 Surfaces of genus 4

Let X be a non-orientable unbordered Klein surface of topological genus 4. Then X can be expressed as $\mathcal{H}/\Gamma$, where Γ is an NEC group with signature $(4, -, [-], \{-\})$. If G is an automorphism group of X then $G = \Lambda/\Gamma$, where Λ is another NEC group. For each group G, we examine the signatures of Λ given in Table 4. If one of them is maximal then clearly G is the full automorphism group of X. Otherwise, we must check if there exists an action of G with a non-maximal signature which cannot be extended to the action of some group G' such that G is a subgroup of G'. Only when it does not hold, G is not the full automorphism group of X.

Theorem 4 *Let X be a non-orientable unbordered Klein surface of topological genus 4. Then its full automorphism group is one of the following groups:* C_2, $C_2 \times C_2$, D_3, $C_2 \times C_2 \times C_2$, D_4, D_6, $C_2 \times D_4$, S_4 *and* $C_2 \times S_4$. *Furthermore, for any such a group G, there exists a surface of genus* 4 *on which G acts as the full automorphism group.*

Proof By using the arguments explained above, we check which groups listed in Preposition 2 are the full automorphism group of a surface of topological genus 4. We divide our study according to the type of results used in each case.

(a) It is already known that each one of the groups: C_2, $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, D_6 and $C_2 \times D_4$, is the full automorphism group of some hyperelliptic surfaces, see [4, Theorem 3.5]. Observe that this theorem holds also for even genus g.

(b) Now we shall prove that the cyclic groups C_3, C_4 and C_6 extend to D_3, D_4 and D_6 respectively.

If $G = C_3$ then Λ has the signature $(2, -, [3], \{-\})$. By [6, Table 1, case 2] the action of C_3 extends to D_3.

Now let $G = C_4$. Then Λ has the signature $(0, +, [4, 4], \{(-)\}$ or $(1, -, [4, 4], \{-\})$. By [6, Table 1, cases 12 and 8] the action of C_4 extends to D_4.

Finally let $G = C_6$. Then the group Λ has the signature $(0, +, [2, 6], \{(-)\}$ or $(1, -, [2, 6], \{-\})$.

Now by [6, Table 1, cases 11 and 7] the action of C_6 extends to D_6. None of the groups C_3, C_4, C_6 is the full automorphism group of a surface of genus 4.

From now on the remaining groups must be handled separately.

(c) Let $G = D_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = 1\rangle$. Consider Λ with signature $(0, +, [2, 2], \{(3)\})$. We define $\theta : \Lambda \to D_3$ by $\theta(x_1) = A$, $\theta(x_2) = B$, $\theta(e_1) = BA$, $\theta(c_{1,0}) = A$, $\theta(c_{1,1}) = B$. Because the signature σ of Λ does not appear in the pairs (σ, σ') of [2] and [12], it is maximal and so is the group Λ, hence there exist surfaces whose full automorphism group is D_3.

(d) Now let $G = C_2 \times C_4$. For this group G the group Λ can have two signatures, which are $(0, +, [2, 4], \{(-)\})$ and $(0, +, [4], \{(2, 2)\})$. Both signatures appear in pairs (σ, σ') with $\sigma' = (0, +, [-], \{(2, 2, 2, 4)\})$ in [2]. We are going to prove that in both cases the action of $C_2 \times C_4$ extends to $C_2 \times D_4$. We denote by X and Y the generators of $C_2 \times C_4$, and use the presentation of $C_2 \times D_4$ with generators X, Y and A where $A^2 = 1$, $AX = XA$ and $(AY)^2 = 1$.

First we consider the signature $(0, +, [2, 4], \{(-)\})$. The group Λ' has signature $(0, +, [-], \{(2, 2, 2, 4)\})$ and so it is generated by the reflections $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$, $c'_{1,3}$ and $c'_{1,4}$. The generators of the subgroup Λ can be expressed in terms of the generators of Λ' as follows

$$x_1 = c'_{1,2}c'_{1,3},\ x_2 = c'_{1,3}c'_{1,4},\ e_1 = c'_{1,4}c'_{1,2},\ c_{1,0} = c'_{1,1}.$$

The epimorphism $\theta : \Lambda \to C_2 \times C_4$ is determined by $\theta(x_1) = X$, $\theta(x_2) = Y$, $\theta(e_1) = Y^{-1}X$ and $\theta(c_{1,0})$ is one of X, or Y^2 or XY^2. Now we construct an epimorphism θ' from Λ' onto $C_2 \times D_4$ whose restriction to Λ is θ. We define

$$\theta'(c'_{1,0}) = \theta'(c'_{1,4}) = AY, \quad \theta'(c'_{1,1}) = X \text{ or } Y^2 \text{ or } XY^2,$$
$$\theta'(c'_{1,2}) = XA \text{ or } Y^2A \text{ or } XY^2A, \quad \theta'(c'_{1,3}) = A.$$

where the suitable alternative depends on the choice of $\theta(c_{1,0})$. Thus the action of $C_2 \times C_4$ extends to $C_2 \times D_4$.

Now we see that the same happens with the second possible signature of Λ, which is $(0, +, [4], \{(2, 2)\})$. The generators of the subgroup Λ can be expressed in terms of the generators of Λ' as follows

$$x_1 = c'_{1,3}c'_{1,4}, \quad e_1 = c'_{1,4}c'_{1,3}, \quad c_{1,0} = c'_{1,1}, \quad c_{1,1} = c'_{1,2}, \quad c_{1,2} = c'_{1,3}c'_{1,1}c'_{1,3}.$$

Now the epimorphism θ is defined by $\theta(x_1) = Y$, $\theta(e_1) = Y^{-1}$, $\theta(c_{1,0}) = X$, $\theta(c_{1,1}) = Y^2$ and $\theta(c_{1,2}) = X$. We now define θ' from Λ' onto $C_2 \times D_4$ as follows

$$\theta'(c'_{1,0}) = \theta'(c'_{1,4}) = AY, \quad \theta'(c'_{1,1}) = X, \quad \theta'(c'_{1,2}) = Y^2, \quad \theta'(c'_{1,3}) = A.$$

The restriction of θ' to Λ is θ. Hence in any case the action of $C_2 \times C_4$ extends to $C_2 \times D_4$.

(e) Let $G = D_4$ and consider the signature $\sigma = (0, +, [2, 2], \{(2)\})$. It does not appear in the lists in [2] and [12], hence it is maximal. The action on D_4 cannot be extended.

(f) Let now $G = A_4$ with presentation $\langle X, Y \mid X^2 = Y^3 = (XY)^3 = 1\rangle$. Without loss of generality we may suppose $X = (1, 2)(3, 4)$ and $Y = (1, 2, 3)$. We are going to see that the action of A_4 extends to S_4. In this case Λ has signature $\sigma = (1, -, [2, 3], \{-\})$. The pair (σ, σ') appears in the list in [2] for $\sigma' = (0, +, [2], \{(2, 3)\}$. Let Λ' be an NEC group with this signature. This group is generated by an elliptic element x'_1 of order 2 and three reflections $c'_{1,0}$, $c'_{1,1}$ and $c'_{1,2}$. The generators of Λ can be expressed in terms of the generators of Λ' as follows

$$d_1 = x'_1c'_{1,0}, \quad x_1 = c'_{1,0}c'_{1,1}, \quad x_2 = c'_{1,1}c'_{1,2}.$$

The epimorphism θ is determined by $\theta(d_1) = Y$, $\theta(x_1) = X$ and $\theta(x_2) = XY$. We define the epimorphism $\theta' : \Lambda' \to S_4$ whose restriction to Λ is θ by

$$\theta'(x'_1) = Z, \quad \theta'(e'_1) = Z, \quad \theta'(c'_{1,0}) = ZY, \quad \theta'(c'_{1,1}) = ZYX, \quad \theta'(c'_{1,2}) = ZY^2.$$

where Z is the transposition $(2, 3)$. So the action of A_4 extends to S_4.

(g) Let $G = S_4$. The last two signatures in Table 4 are maximal and so the action of S_4 corresponding to them cannot be extended.

(h) Let now $G = C_2 \times A_4$. Let A be the generator of C_2 and take the presentation of A_4 which was given in case (f). We are going to prove that the action of $C_2 \times A_4$ extends to $C_2 \times S_4$. The group Λ has the signature $\sigma = (0, +, [6], \{(2)\})$. The pair (σ, σ') where $\sigma' = (0, +, [-], \{(2, 6, 4)\})$ appears in the list in [2]. The group Λ' is generated by the reflections $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$ and $c'_{1,3}$. The generators of Λ can be expressed in terms of the generators of Λ' as follows

$$x_1 = c'_{1,1}c'_{1,2}, \quad e_1 = c'_{1,2}c'_{1,1}, \quad c_{1,0} = c'_{1,2}c'_{1,3}c'_{1,2}, \quad c_{1,1} = c'_{1,3}.$$

The epimorphism θ is determined by $\theta(x_1) = AY$, $\theta(e_1) = Y^2A$, $\theta(c_{1,0}) = X$, $\theta(c_{1,1}) = YXY^2$. We define the epimorphism $\theta' : \Lambda' \to C_2 \times S_4$ whose restriction to Λ is θ by

$$\theta'(c'_{1,0}) = YXY^2, \quad \theta'(c'_{1,1}) = T, \quad \theta'(c'_{1,2}) = ATY, \quad \theta'(c'_{1,3}) = YXY^2,$$

where T is the transposition $(1,3)$. So the action of $C_2 \times A_4$ extends to $C_2 \times S_4$.

(i) Finally let $G = C_2 \times S_4$. This group is the maximal one in the family of groups acting of surfaces of genus 4.

4.3 Surfaces of genus 5

This last subsection is devoted to surfaces of genus 5. Let X be a non-orientable unbordered Klein surface of topological genus 5. Then X can be expressed as $\mathcal{H}/\Gamma$, where Γ is an NEC group with the signature $(5, -, [-], \{-\})$. As before, when G is an automorphism group of X, then $G = \Lambda/\Gamma$ where Λ is another NEC group.

Theorem 5 *Let X be a non-orientable unbordered Klein surface of topological genus 5. Then its full automorphism group is one of the following groups:* C_2, C_3, C_4, $C_2 \times C_2$, D_3, D_4, D_6, D_8, $C_3 \times D_3$, $((3,3,3;2))$, D_{10}, $\langle 5,4,2\rangle$, S_4, $D_3 \times D_3$, $(2,4,6;2)$ *and* S_5. *Furthermore, for each such a group G, there exists a surface of genus 5 on which G acts as the full automorphism group.*

Proof The proof will be similar to the proof of the previous theorem. However, this time we must consider twenty five groups listed in Proposition 3 and the signatures from Table 5.

(a) Each of the groups C_2, $C_2 \times C_2$, D_4, D_8 and D_{10} is the full automorphic group of some hyperelliptic surface [4, Theorem 3.5].

(b) Next we consider the other cyclic groups listed in Proposition 3. First consider $G = C_3$ and an NEC group Λ of signature $(1, -, [3,3,3], \{-\})$. This signature does not appear in the lists of [2] and [12], and there exists an epimorphism θ from Λ onto C_3 defined by $\theta(d_1) = X, \theta(x_1) = X, \theta(x_2) = X, \theta(x_3) = X^2$.

Consider $G = C_4$ and an NEC group Λ with the signature $(0, +, [2,2,4], \{(-)\})$. This signature does not appear in the lists of [2] and [12], and there exists an epimorphism θ from Λ onto C_4 defined by $\theta(x_1) = X^2, \theta(x_2) = X^2, \theta(x_3) = X, \theta(e_1) = X^3$ and $\theta(c_{1,0}) = X^2$.

Hence for both, C_3 and C_4, there exist surfaces having them as the full automorphism groups.

The action of the remaining cyclic groups extends to the corresponding dihedral groups, check the signature of Λ in Table 5 and then apply [6]. For C_5 see case 7 in Table 1 of that paper; for C_6, see cases 10 and 5; and for the groups C_8 and C_{10}, see case 11.

From now on the remaining groups must be handled separately.

(c) Let $G = D_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = 1\rangle$. Consider Λ with signature $(0, +, [2], \{(3,3,3)\})$. We define $\theta : \Lambda \to D_3$ by $\theta(x_1) = A, \theta(e_1) = A, \theta(c_{1,0}) = B$, $\theta(c_{1,1}) = A$, $\theta(c_{1,2}) = B$ and $\theta(c_{1,3}) = ABA$. Because the signature σ of Λ does not appear in the pairs (σ, σ') of [2] and [12], the signature σ is maximal and so is the group Λ, hence there exist surfaces whose full group of automorphisms is D_3.

(d) Let $G = C_3 \times C_3$. We denote by X and Y the generators in G. We are going to prove that the action of $C_3 \times C_3$ extends to the action of the group $((3,3,3;2))$. The group Λ has the signature $(1, -, [3,3], \{-\})$. The list in [2] includes the pair (σ, σ'), where $\sigma' = (0, +, [2], \{(3,3)\})$. The group Λ' with signature σ' is generated by x_1',

e'_1, $c'_{1,0}$, $c'_{1,1}$ and $c'_{1,2}$. We express now the generators of Λ as follows

$$d_1 = x'_1 c'_{1,0}, \quad x_1 = c'_{1,0} c'_{1,1}, \quad x_2 = c'_{1,1} c'_{1,2}.$$

The epimorphism θ from Λ onto $C_3 \times C_3$ is determined by $\theta(d_1) = XY, \theta(x_1) = X$, $\theta(x_2) = Y$. We describe the group ((3, 3, 3; 2)) as a semidirect product of $C_3 \times C_3$ and C_2, where the generator Z of C_2 satisfies the relations $ZYZ = Y^{-1}$, $ZXZ = X^{-1}$. We define the epimorphism $\theta' : \Lambda' \to ((3, 3, 3; 2))$, whose restriction to Λ is θ, by

$$\theta'(x'_1) = XYZ, \quad \theta'(e'_1) = XYZ, \quad \theta'(c'_{1,0}) = Z, \quad \theta'(c'_{1,1}) = X^2 Z, \quad \theta'(c'_{1,2}) = ZXY.$$

Hence the action of $C_3 \times C_3$ extends to the group ((3, 3, 3; 2)).

(e) Let now $G = D_5 = \langle A, B \mid A^2 = B^2 = (AB)^5 = 1\rangle$. There are two possibilities for the signature of Λ. They are $(0, +, [2, 5], \{(-)\})$ and $(0, +, [2], \{(5, 5)\})$. Both signatures appear in pairs (σ, σ') in the list of [2], for $\sigma' = (0, +, [-], \{(2, 2, 2, 5)\})$. In both cases we shall extend the action to D_{10}.

First consider the signature $\sigma = (0, +, [2, 5], \{(-)\})$. The group Λ' with signature σ' is generated by the reflections $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$, $c'_{1,3}$ and $c'_{1,4}$. The generators of Λ are expressed by

$$x_1 = c'_{1,1} c'_{1,0}, \quad x_2 = c'_{1,4} c'_{1,3}, \quad e_1 = c'_{1,3} c'_{1,1}, \quad c_{1,0} = c'_{1,2}.$$

The epimorphism θ from Λ onto D_5 is given by $\theta(x_1) = A, \theta(x_2) = AB, \theta(e_1) = B$ and $\theta(c_{1,0}) = B$. We now define the epimorphism $\theta' : \Lambda' \to D_{10} = C_2 \times D_5$ whose restriction to Λ is θ. Call X the generator of C_2.

$$\theta'(c'_{1,0}) = AX, \quad \theta'(c'_{1,1}) = X, \quad \theta'(c'_{1,2}) = B, \quad \theta'(c'_{1,3}) = BX, \quad \theta'(c'_{1,4}) = AX.$$

Now we deal with the second signature $\sigma = (0, +, [2], \{(5, 5)\})$. In this case the generators of Λ are expressed by

$$x_1 = c'_{1,1} c'_{1,2}, \quad e_1 = c'_{1,2} c'_{1,1}, \quad c_{1,0} = c'_{1,3}, \quad c_{1,1} = c'_{1,4}, \quad c_{1,2} = c'_{1,1} c'_{1,3} c'_{1,1}.$$

The epimorphism θ from Λ onto D_5 is given by $\theta(x_1) = A, \theta(e_1) = A, \theta(c_{1,0}) = A, \theta(c_{1,1}) = B$ and $\theta(c_{1,2}) = A$. Now let X be the generator of C_2 in the group $D_{10} = C_2 \times D_5$. Then we can define the epimorphism $\theta' : \Lambda' \to D_{10}$ whose restriction to Λ is θ by

$$\theta'(c'_{1,0}) = B, \quad \theta'(c'_{1,1}) = X, \quad \theta'(c'_{1,2}) = AX, \quad \theta'(c'_{1,3}) = A, \quad \theta'(c'_{1,4}) = B.$$

Hence in both cases the action of D_5 extends to D_{10}.

(f) Next we deal with $G = D_6$. The signature $\sigma = (0, +, [-], \{(2, 2, 2, 2, 2)\})$ does not appear in the lists in [2] and [12]. Let Λ be a NEC group with signature σ and let θ be a homomorphism from Λ onto D_6 given by $\theta(c_{1,0}) = A, \theta(c_{1,1}) = (AB)^3$, $\theta(c_{1,2}) = B, \theta(c_{1,3}) = ABABA, \theta(c_{1,4}) = (AB)^3, \theta(c_{1,5}) = A$. Then D_6 acts on a surface of genus 5 and this action does not extend.

(g) Let $G = A_4$. We are going to prove that the action of A_4 extends to S_4. The group Λ has the signature $\sigma = (1, -, [-], \{(2)\})$. The list in [2] includes the pair (σ, σ') where $\sigma' = (0, +, [2], \{(2, 4)\})$. The group Λ' with signature σ' is generated by x'_1, e'_1, $c'_{1,0}$, $c'_{1,1}$ and $c'_{1,2}$. We express the generators of Λ by

$$d_1 = x'_1 c'_{1,1}, \quad e_1 = c'_{1,1} x'_1 c'_{1,1} x'_1, \quad c_{1,0} = c'_{1,1} c'_{1,2} c'_{1,1}, \quad c_{1,1} = c'_{1,2}.$$

The epimorphism θ from Λ onto A_4 is defined without loss of generality by $\theta(d_1) = (1, 2, 3)$, $\theta(e_1) = (1, 2, 3)$, $\theta(c_{1,0}) = (1, 2)(3, 4)$ and $\theta(c_{1,1}) = (1, 4)(2, 3)$. We define the epimorphism $\theta' : \Lambda' \to S_4$ whose restriction to Λ is θ by

$$\theta'(x'_1) = \theta'(e'_1) = (1, 2), \ \theta'(c'_{1,0}) = (1, 3)(2, 4), \ \theta'(c'_{1,1}) = (1, 3), \ \theta'(c'_{1,2}) = (1, 4)(2, 3)$$

Hence the action of A_4 extends to S_4.

(h) The following group to consider is $G = C_3 \times D_3$. We denote the generator of C_3 by X, and the generators of order 2 in D_3 by A and B. Take the group Λ with signature $\sigma = (0, +, [2, 3], \{(-)\})$. The list in [2] includes the pair (σ, σ') where $\sigma' = (0, +, [-], \{(2, 2, 2, 3)\})$. The group Λ' with the signature σ' is generated by $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$, $c'_{1,3}$ and $c'_{1,4}$. The generators of Λ must be expressed by

$$x_1 = c'_{1,21} c'_{1,3}, \quad x_2 = c'_{1,3} c'_{1,4}, \quad e_1 = c'_{1,4} c'_{1,2}, \quad c_{1,0} = c'_{1,1},$$

The epimorphism θ from Λ onto G is defined by $\theta(x_1) = A$, $\theta(x_2) = XAB$, $\theta(e_1) = X^2 B$ and $\theta(c_{1,0}) = B$. Then an epimorphism $\theta' : \Lambda' \to D_3 \times D_3$ whose restriction to Λ is θ must satisfy

$$\theta'(c'_{1,0}) = \theta'(c_{1,4}) = Y, \quad \theta'(c'_{1,1}) = B, \quad \theta'(c'_{1,2}) = Y X^2 B, \quad \theta'(c'_{1,3}) = Y X^2 B A,$$

where Y is an element of order 2 such that $YXY = X^{-1}$ and Y commutes with A and B. However, the element YX^2BA has order 6 and it cannot be the image of a reflection. Hence this action of $C_3 \times D_3$ does not extend to $D_3 \times D_3$.

(i) Let now $G = ((3, 3, 3; 2))$. The signature of Λ is $\sigma = (0, +, [2], \{(3, 3)\})$. The unique pair (σ, σ') in [2] with index 2 corresponds to $\sigma' = (0, +, [-], \{(2, 2, 2, 3)\})$. This signature does not appear in Table 5 for the group $(4, 4 \mid 2, 3)$ which is the unique group in Diagram 2 containing G with index 2. Hence the action of the group $((3, 3, 3; 2))$ does not extend.

(j) Consider now $G = \langle 5, 4, 2 \rangle$. In this case Λ has signature $\sigma = (0, +, [4], \{(5)\})$, and the group G is generated by the elements $X = (1, 2, 3, 4, 5)$ and $Y = (1, 3, 4, 2)$. We check in Diagram 2 that the unique group containing G is $G' = S_5$. According to Table 5 the signature of Λ' for G' is $\sigma' = (0, +, [-], \{(2, 4, 5)\})$. If the action of G always extends to G', then every group with signature σ is a subgroup of a group of signature σ'.

Call Λ^+ and Λ'^+ their respective canonical Fuchsian subgroups. Then Λ^+ must be a subgroup of Λ'^+. In [5, Theorem 5.2.vi] it is proved that this implies that the conjugates of $X^{-1}Y^{-1}(XY)^{-2}$ generate a normal subgroup K of index 20 in G. The element $X^{-1}Y^{-1}(XY)^{-2}$ is the permutation $(1, 2, 5, 4)$ and so its conjugates cannot generate $K = \{1\}$.

Hence the action of G does not extend to S_5.

(k) Now $G = S_4$. The unique group in Diagram 2 containing S_4 is S_5. There exists no pair (σ, σ') in [2] and [12] with index 5. Hence the action of S_4 does not extend.

(l) Consider $G = D_3 \times D_3$. We call A and B, and C and D the generators of order 2 of the two factors of G. Let Λ have signature $(0, +, [-], \{(2, 2, 2, 3)\})$ and consider the epimorphism θ from Λ onto G defined by $\theta(c_{1,0}) = AC$, $\theta(c_{1,1}) = A$, $\theta(c_{1,2}) = D$, $\theta(c_{1,3}) = BD$ and $\theta(c_{1,4}) = AC$. Since the signature of Λ is maximal the action of $D_3 \times D_3$ does not extend.

(m) Now let $G = (4, 4 \mid 2, 3)$. We are going to prove that the action of G extends to the group $(2, 4, 6; 2)$. The group Λ has signature $\sigma = (0, +, [4], \{(3)\})$. The list in [2] includes the pair (σ, σ') where $\sigma' = (0, +, [-], \{(2, 4, 6)\})$. The group Λ' with signature σ' is generated by $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$ and $c'_{1,3}$. We express the generators of Λ by

$$x_1 = c'_{1,2}c'_{1,1}, \quad e_1 = c'_{1,1}c'_{1,2}, \quad c_{1,0} = c'_{1,3}, \quad c_{1,1} = c'_{1,2}c'_{1,1}c'_{1,3}c'_{1,1}c'_{1,2}.$$

The epimorphism from Λ onto G was determined in the proof of Theorem 2. The group G is isomorphic to the subgroup of S_6 generated by the elements $t_1 = (1, 2, 3, 4)(5, 6)$ and $t_2 = (1, 2)(3, 4, 5, 6)$, [10, page 98]. Recall that the group $(2, 4, 6; 2)$ is generated by $A = (1, 6)(2, 5)(3, 4)$ and $B = (1, 2)(3, 4, 5, 6) = t_2$. Take $R = AB(AB^2)^2 = (2, 4)$. We define the epimorphism $\theta' : \Lambda' \to (2, 4, 6; 2)$ whose restriction to Λ is θ by

$$\begin{aligned}&\theta'(c'_{1,0}) = \theta'(c'_{1,3}) = t_1t_2 = (2, 4)(3, 5),\\ &\theta'(c'_{1,1}) = R = (2, 4), \theta'(c'_{1,2}) = t_1R = (1, 4)(2, 3)(5, 6).\end{aligned}$$

Observe that $\theta'((c'_{1,1}c'_{1,2}c'_{1,0}c'_{1,1})^3) = A$ and $\theta'((c'_{1,0}c'_{1,2})^3c'_{1,1}c'_{1,2}c'_{1,0}c'_{1,2}) = B$. Hence the action of $(4, 4 \mid 2, 3)$ extends to $(2, 4, 6; 2)$.

(n) Now consider $G = A_5$. We are going to prove that the action of G extends to the group S_5. The group Λ has signature $\sigma = (0, +, [-], \{(2, 5, 5)\})$. The list in [2] includes the pair (σ, σ') where $\sigma' = (0, +, [-], \{(2, 4, 5)\})$. The group Λ' with signature σ' is generated by $c'_{1,0}$, $c'_{1,1}$, $c'_{1,2}$ and $c'_{1,3}$. We express the generators of Λ by

$$c_{1,0} = c'_{1,1}c'_{1,2}c'_{1,1}, \quad c_{1,1} = c'_{1,2}, \quad c_{1,2} = c'_{1,3}, \quad c_{1,3} = c'_{1,1}c'_{1,2}c'_{1,1}.$$

The epimorphism θ from Λ onto G is defined by $\theta(c_{1,0}) = (1, 5)(2, 4)$, $\theta(c_{1,1}) = (1, 4)(2, 5)$, $\theta(c_{1,2}) = (1, 3)(4, 5)$ and $\theta(c_{1,3}) = (1, 5)(2, 4)$. We define the epimorphism $\theta' : \Lambda' \to S_5$ whose restriction to Λ is θ by

$$\theta'(c'_{1,0}) = (1, 3)(4, 5), \ \theta'(c'_{1,1}) = (4, 5), \ \theta'(c'_{1,2}) = (1, 4)(2, 5), \ \theta'(c'_{1,3}) = (1, 3)(4, 5).$$

Hence the action of A_5 extends to S_5.

Table 6 The full groups of automorphisms of surfaces of genus 3,4, and 5

Group G	Order $o(G)$	Group G	Order $o(G)$
Genus 3		Genus 5	
C_2	2	C_2	2
$C_2 \times C_2$	4	C_3	3
D_3	6	C_4	4
D_4	8	$C_2 \times C_2$	4
D_6	12	D_3	6
		D_4	8
Genus 4		D_6	12
C_2	2	D_8	16
$C_2 \times C_2$	4	$C_3 \times D_3$	18
D_3	6	$((3, 3, 3; 2))$	18
$C_2 \times C_2 \times C_2$	8	D_{10}	20
D_4	8	$\langle 5, 4, 2 \rangle$	20
D_6	12	S_4	24
$C_2 \times D_4$	16	$D_3 \times D_3$	36
S_4	24	$(2, 4, 6; 2)$	72
$C_2 \times S_4$	48	S_5	120

(ñ) Finally the groups $(2, 4, 6; 2)$ and S_5 are maximal among the groups acting on surfaces of genus 5. So they are the full automorphism group of the surfaces on which they act.

For the convenience of the reader we collect the results in Theorems 3, 4 and 5 in Table 6.

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