UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Generalized Takagi Functions

Funciones de Takagi Generalizadas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Jesús Llorente Jorge

Directores

Juan Ferrera Cuesta
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PROGRAMA DE DOCTORADO EN INVESTIGACIÓN MATEMÁTICA

GENERALIZED TAKAGI FUNCTIONS

Funciones de Takagi Generalizadas

Memoria para optar al grado de
Doctor en Matemáticas
presentada por

Jesús Llorente Jorge

bajo la dirección de

Juan Ferrera Cuesta *y* **Javier Gómez Gil**

Madrid, 2024

Financial Support. This work was funded partially by the UCM CT63/19-CT6419 grant and by the Spanish MCIU project PGC2018-097286-B-I00.

A mis padres,
Inés e Ignacio.

AGRADECIMIENTOS

Hace ya más de doce años que empecé a estudiar el doble grado en Ingeniería Informática y Matemáticas en la Complutense. Nunca pensé que tuviese una etapa formativa tan larga, ni tampoco tan condicionada. Aquel martes 29 de julio de 2014 mi madre se suicidó mientras me echaba la siesta, y a partir de ahí, la vida siguió como siguen las cosas que no tienen mucho sentido. Ese día no solamente se fue mi madre, sino también la persona que había sido hasta mis veinte años. Atrás quedaron los ideales, las ilusiones y las ganas de comerse el mundo.

A mis dieciséis años le decía a mi madre que algún día sería Doctor en Matemáticas con el único objetivo de ver su cara de susto cuando pronunciaba las palabras “matemáticas” y “doctor”. Pues en ese momento, no sabía nada de matemáticas ni lo que significaba “ser doctor”. También es cierto que esta tesis es el último tema que recuerdo haber hablado con mi madre, y de algún modo, terminar esta tesis supone dejar de vivir con la esperanza de que algún día mi madre podrá ver que soy Doctor en Matemáticas y ya no tendrá que asustarse más. No puedo continuar levantándome cada mañana y seguir preguntándome si todo ha sido un sueño.

Esta tesis supone soltar la mano de mi madre y empezar a vivir con su recuerdo de una manera distinta. Es momento de dejar de sobrevivir. Quizás, la única ilusión de aquel chico de veinte años que todavía hoy sigue vigente es el sueño de convertirse en profesor de matemáticas en la universidad. Ahora me doy cuenta de que conseguirlo o no, es irrelevante. Lo realmente importante es el camino, y sólo se hace camino al andar.

A mis casi veinte y diez, veinte y quince dicen que aparento, estoy convencido de que yo soy el que soy gracias a todas las personas que han dejado su huella a lo largo de mi vida. Dos de las personas que más han influido en este camino han sido mis directores de tesis, Juan y Javier. Creo que no hay muchos matemáticos que tengan una comprensión tan global y profunda de las matemáticas como la que tienen ellos. Es realmente impresionante la facilidad que tienen para saber “por donde van los tiros” en un problema de matemáticas. Siempre estaré agradecido por su paciencia y por todo el tiempo que me han dedicado. A lo largo de estos años he pasado por varias depresiones, y ellos han sido una enorme motivación para salir en busca de la luz, incluso en los momentos de más oscuridad. Son un espejo en el que mirarme tanto a nivel matemático como a nivel personal.

Siempre estaré muy agradecido Richard Aron por transmitirme su “espíritu joven” y su pasión por las matemáticas durantes tres meses que estuve visitándole en Kent. Fue una de las mejores experiencias de mi vida.

También me gustaría mostrar mi agradecimiento a Zoltán Buczolich. Creo que es uno de los mejores matemáticos del mundo a día de hoy y he tenido la gran suerte de poder aprender de él durante mi estancia en Budapest. Estoy muy agradecido por su fantástica hospitalidad y la dedicación que ha tenido conmigo todo este tiempo. Tanto él como su mujer, Krisztina, son unas personas excepcionales que me han hecho sentir “como en casa” durante mi estancia.

A lo largo de estos años, he tenido la oportunidad de conocer a grandes matemáticos y personas. Estoy muy agradecido por haber compartido tantos buenos momentos con Bruce Hanson. Una persona fantástica por la que siento una gran admiración, tanto matemática como personal.

También me gustaría dar las gracias a los diferentes profesores que he tenido a lo largo de estos años, y que me han trasmitido su cariño y su buen quehacer matemático: Jesús Jaramillo, Juan Seoane, Gustavo Muñoz, Miguel García Bravo, Angelines Prieto, José Luis González Llavona, Fernando Cobos y Pilar Cembranos.

Por último, y no menos importante, me gustaría agradecer con el corazón a mi psicóloga Montse Peña, y mi profesor de inglés Brandon Salo. Esta tesis no hubiera sido posible sin ellos.

Jesús Llorente Jorge
17 de enero de 2024

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RESUMEN

La función de Takagi es, probablemente, el ejemplo más sencillo de una función continua no derivable en ningún punto. En notación moderna, la función de Takagi $T : [0,1] \to \mathbb{R}$ se define como

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi\left(2^n x\right) \tag{T}$$

siendo $\phi(x)$ la distancia del punto x al entero más cercano. Es importante destacar que la definición original dada por T. Takagi en [102] es completamente diferente a la presentada anteriormente. Probablemente, este hecho junto con el aislamiento de Japón a principios del siglo XX, justifican el paso inadvertido de la función de Takagi en el mundo occidental. Como consecuencia, la función de Takagi fue redescubierta a lo largo del siglo XX por numerosos autores como B. van der Waerden [107], R. Tambs-Lyche [104], G. de Rham [94] y T. H. Hildebrandt [68] entre otros.

La función de Takagi posee sorprendentes propiedades desde diferentes perspectivas. Por esta razón, ha sido estudiada por un extenso número de matemáticos dando lugar a una gran cantidad de generalizaciones con el objetivo de extender las propiedades intrínsecas de la función de Takagi a una familia más amplia de funciones. En la literatura, estas generalizaciones reciben el nombre de *funciones de Takagi generalizadas*.

El objetivo de esta memoria es presentar los resultados obtenidos como consecuencia del estudio de ciertas propiedades para diferentes funciones de Takagi generalizadas. A continuación, exponemos la estructura de los capítulos que conforman dicha memoria.

En el Capítulo 1 se presentan las diferentes familias de funciones que se consideran en este trabajo. Todas ellas tienen como denominador común a la función de Takagi, que escribiremos como

$$T(x) = \sum_{n=1}^{\infty} g_n(x)$$

siendo $g_n(x) = \operatorname{dist}(x, D_n)$ la distancia de x al conjunto $D_n = \left\{k2^{n-1} : k = 0, 1 \ldots, 2^{n-1}\right\}$. Este conjunto recibe el nombre de *números diádicos de orden n*. En este primer capítulo detallaremos la relación existente entre la función de Takagi y la expansión binaria de los números pertenecientes al intervalo $[0,1]$. Esto nos permitirá establecer los cimientos sobre

los que descansan las funciones de Takagi generalizadas que introduciremos a continuación: la *Clase de Takagi*, la *función de Takagi-Van der Waerden* y la *Clase Generalizada*. Además, presentaremos las características intrínsecas de cada una de ellas, y mencionaremos los resultados obtenidos por otros autores en relación a las mismas, con el único objetivo de poner en valor y contextualizar los resultados que presentamos en esta memoria.

El Capítulo 2 tiene por objetivo presentar los resultados obtenidos en relación a las propiedades de diferenciabilidad de orden dos de las funciones de la Clase de Takagi. Esta generalización de la función de Takagi fue introducida por M. Hata y M. Yamaguti en el año 1984 (ver [64]). Está formada por todas las funciones $T_w : [0,1] \to \mathbb{R}$ definidas como

$$T_w(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

siendo $w = (w_n)_n$ una sucesión de pesos tal que $(2^{-n} w_n)_n \in \ell_1$, y $g_n(x) = \operatorname{dist}(x, D_n)$ la distancia del punto x al conjunto $D_n = \left\{k2^{n-1} : k = 0, 1 \ldots, 2^{n-1}\right\}$. En su artículo, M. Hata y M. Yamaguti probaron que la Clase de Takagi es un subespacio cerrado de $C[0,1]$ isomorfo a ℓ_1.

Probablemente, la Clase de Takagi es la generalización más famosa y estudiada de la función de Takagi. Además, contiene numerosas funciones que habían sido estudiadas anteriormente en la literatura, como por ejemplo, la función de G. Faber o las funciones de A. S. Besicovitch y H. D. Ursell (ver [44] y [24]).

Entre los diversos resultados existentes sobre las funciones de la Clase de Takagi, el teorema obtenido por N. Kôno (ver [79]) es quizás el resultado más conocido:

Teorema 1. *Para cada función de la Clase de Takagi se verifica:*

(1) T_w *es absolutamente continua con derivada*

$$T'_w(x) = \sum_{n=1}^{\infty} w_n g'_n(x)$$

en casi todo punto, si y solo si $w \in \ell_2$.

(2) T_w *es derivable en un conjunto no numerable de medida nula y el rango de la derivada es todo* $\mathbb{R}$*, si y solo si* $w \in c_0 \setminus \ell_2$.

(3) T_w *es no derivable en ningún punto, si y solo si* $w \notin c_0$.

Este resultado determina completamente la diferenciabilidad de las funciones de la Clase de Takagi en términos de la pertenencia de sucesión de pesos $w = (w_n)_n$ a cierto espacio de sucesiones.

En el Capítulo 2 estudiamos algunas propiedades de diferenciabilidad de orden dos para las funciones de la Clase de Takagi. En particular, caracterizaremos cuando una función de dicha clase es convexa o cóncava en términos de una condición sobre la sucesión de pesos. Además, investigaremos cuando T_w tiene un desarrollo de Taylor de orden dos en un punto, o satisface la propiedad de Stepanoff de order dos en un punto. Veremos que el estudio de estas propiedades debe realizarse necesariamente bajo la hipótesis de que $w = (w_n)_n$ es una sucesión perteneciente a ℓ_1. En este caso, la función T_w es derivable en todo punto $x \notin D$ con derivada dada por

$$T'_w(x) = \sum_{n=1}^{\infty} w_n g'_n(x)$$

Además, es immediato ver que la función T_w es Lipschitz si y solo si $w \in \ell_1$. Con respecto a la convexidad, probamos el siguiente resultado:

Teorema 2. *Sea $w \in \ell_1$. La función T_w es convexa a trozos si y solo si existe $n_0 \geq 1$ tal que*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0}$$

es una sucesión no positiva y no decreciente.

A continuación, caracterizamos cuando la función T_w tiene un desarrollo de Taylor de orden dos en un punto $x \notin D$ en términos de la sucesión de pesos y de una propiedad relacionada con la expansión binaria de dicho punto. Otra forma de estudiar la diferenciabilidad de orden dos en un punto $x \notin D$ consiste en analizar la existencia de la derivada de la función $T'^{+}_w : [0,1] \to \mathbb{R}$. Cuando $w \in \ell_1$, esta función está totalmente definida y es continua en todo punto $x \notin D$. Por tanto, podemos estudiar la existencia del límite

$$\left(T'^{+}_w\right)'(x) = \lim_{y \to x} \frac{T'^{+}_w(y) - T'^{+}_w(x)}{y - x}$$

en un punto $x \notin D$. Probamos el siguiente resultado:

Teorema 3. *Sea $w \in \ell_1$. La función T_w tiene un desarrollo de Taylor de orden dos en un punto $x \notin D$, si y solo si, la función T'^{+}_w es derivable en x.*

Asimismo, investigamos cuando la función T_w tiene un desarrollo de Taylor de orden dos en casi todo punto de $[0,1]$. Veremos que la sucesión $\beta = (\beta_n)_n$ definida como

$$\beta_n = 2^n \left(-w_n + \sum_{j=n+1}^{\infty} w_j\right)$$

juega un papel fundamental en el estudio de dicha propiedad. Demostraremos este sorprendente teorema:

Teorema 4. *Sea $w \in \ell_1$. Las siguientes afirmaciones son equivalentes:*

(1) T_w tiene un desarrollo de Taylor de orden dos en casi todo punto.

(2) $T_w'^{+}$ tiene variación acotada.

(3) $T_w'^{+}$ es derivable en casi todo punto.

(4) $\beta \in \ell_1$.

(5) T_w es la diferencia de dos funciones convexas pertenecientes a la Clase de Takagi.

Además, probaremos que si la sucesión $(2^n w_n)_n$ converge, entonces el conjunto de puntos donde T_w posee un desarrollo de Taylor de orden dos tiene medida nula y dimensión de Hausdorff uno, si y solo si, la sucesión $\beta \notin \ell_1$.

La condición de Stepanoff de orden dos en un punto es más debil que la existencia de desarrollo de Taylor de orden dos en un punto. Procediendo de manera análoga, caracterizamos también cuando T_w satisface dicha condición para un punto $x \notin D$ en términos de la sucesión de pesos y de una propiedad relacionada con la expansión binaria de dicho punto.

Con respecto a la diferenciabilidad de orden dos en un punto diádico, obtenemos el siguiente resultado:

Teorema 5. *Sea $w \in \ell_1$. Para cada función T_w se verifica:*

(1) T_w tiene un desarrollo de Taylor de orden dos en un punto $x \in D$, si y solo si T_w es derivable en x y la sucesión $(2^n w_n)_n$ converge.

(2) T_w satisface la condición de Stepanoff orden dos en un punto $x \in D$, si y solo si T_w es derivable en x y $\limsup_n 2^n |w_n| < \infty$.

Los resultados obtenidos nos permiten construir funciones de la Clase de Takagi con ciertas propiedades preescritas. Por ejemplo, sea $\alpha \in \mathbb{R}$ tal que $\alpha > 1$ y consideramos la sucesión de pesos $w = (w_n)$ dada por $w_n = \alpha^{-n}$ para todo n. Si $\alpha \geq 2$ entonces T_w es cóncava a trozos y por tanto, satisface la condición de Stepanoff en casi todo punto. Además, si $\alpha > 2$ entonces

$$T_w'(x) = \sum_{n=1}^{\infty} \frac{1-2\varepsilon_n}{\alpha^n} = \frac{1}{\alpha-1} - 2\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\alpha^n}, \quad \varepsilon_n \in \{0,1\}$$

para todo $x \notin D$. En consecuencia, el rango de T_w' es un conjunto de Cantor con dimensión de Hausdorff $\log 2/\log\alpha$. Sin embargo, si $1 < \alpha < 2$ entonces T_w satisface la condición de Stepanoff de orden dos en ningún punto.

Un resultado de A. P. Calderón and A. Zygmund afirma que si una función $f:\mathbb{R}\to\mathbb{R}$ satisface la condición de Stepanoff de orden dos en casi todo punto, entonces tiene desarrollo de Taylor de orden dos en casi todo punto (ver Teorema 5 de [33]). Los conjuntos de puntos en los que se satisfacen dichas propiedades no necesariamente coinciden. Para sucesión $w=(w_n)$ definida por $w_n=\frac{(-1)^n}{n^2 2^n}$ para todo n, se tiene que T_w tiene desarrollo de Taylor de orden dos en casi todo punto, y en consecuencia satisface la condición de Stepanoff de orden dos en casi todo punto. Sin embargo, construiremos un punto $x\in[0,1]$ tal que T_w satisface la condición de Stepanoff de orden dos en x pero no tiene desarrollo de Taylor de orden dos en dicho punto.

Los resultados que aparecen en dicho capítulo han sido publicados en [53].

Tanto el Capítulo 3 como el Capítulo 4 tratan sobre la función de Takagi-Van der Waerden. Como hemos comentado anteriormente, la función de Takagi fue redescubierta por varios autores a lo largo de la primera mitad del siglo XX. Algunos de estos autores, construyeron una función similar a la de Takagi pero sustituyendo la base dos que aparece en la definición (T) por otro número entero mayor que dos.

En 1930, B. L. van der Waerden redescubrió la función de Takagi utilizando base 10, mientras que E. Landau la redescubrió empleando base 4 en 1934 (ver [107] y [83]).

Hasta donde sabemos, este tipo de función definida mediante una base entera $r\geq 2$ arbitraria aparece por primera vez en el trabajo de F. A. Behrend, quién prueba su diferenciabilidad en ningún punto (ver [21]).

Dado un número entero $r\geq 2$, la función de Takagi-Van der Waerden $f_r:[0,1]\to\mathbb{R}$ se define como

$$f_r(x)=\sum_{n=1}^{\infty} g_n(x)$$

donde $g_n(x)=\operatorname{dist}(x,D_n)$ es la distancia del punto x al conjunto $D_n=\left\{kr^{n-1}:k=0,1\ldots,r^{n-1}\right\}$. Además, denotamos por D el conjunto de los número r-ádicos del intervalo unidad, es decir, $D=\bigcup_{n=1}^{\infty} D_n$. Del mismo modo que para la función de Takagi, en la Sección 1.3 del Capítulo 1 establecemos la relación existente entre la función f_r y la expansión en base r de los números del intervalo unidad.

Para cada n denotamos por $\widetilde{D}_n$ el conjunto de todos los puntos medios entre dos puntos consecutivos de D_n, es decir

$$\widetilde{D}_n=\left\{\frac{x+y}{2}:x,y\in D_n \text{ con } (x,y)\cap D_n=\emptyset\right\},$$

y consideramos $\widetilde{D}=\bigcup_{n=1}^{\infty}\widetilde{D}_n$. Cuando r es par, tenemos que $\widetilde{D}_n\subset D_{n+1}$ para todo n, y por tanto, $\widetilde{D}\subset D$. Sin embargo, cuando r es impar, tenemos que $\widetilde{D}_n\subset\widetilde{D}_{n+1}$ para todo n, y $\widetilde{D}\cap D=\emptyset$. De algún modo, estos hechos están detrás de la diferente naturaleza que tiene la función de Takagi-Van der Waerden según r sea un número par o impar. Esto quedará reflejado en los diferentes resultados que obtendremos a lo largo de los capítulos mencionados anteriormente.

En el Capítulo 3 estudiamos la existencia de derivadas infinitas para la función de Takagi-Van der Waerden. En particular, probamos que dicha función no tiene derivada lateral finita en ningún punto. Para demostrar este resultado, generalizamos el argumento utilizado por F. S. Cater para obtener el resultado correspondiente a la función de Takagi (ver [34]).

A continuación, caracterizamos el conjunto de puntos en los que la función de Takagi-Van der Waerden tiene una derivada lateral infinita en términos de una condición sobre la expansión en base r de dichos puntos. Este problema fue resuelto para la función de Takagi por M. Krüppel (ver [81]), y de manera independiente, por P. C. Allaart y K. Kawamura (ver [11]).

Comenzamos probando el siguiente resultado:

Teorema 6. *Sea $r \geq 2$ un entero. Se verifica:*

(1) Si $x \in D$, entonces $f_r'^{+}(x) = +\infty$ y $f_r'^{-}(x) = -\infty$.

(2) Si r es impar y $x \in \widetilde{D}$, entonces $f_r'^{+}(x) = -\infty$ y $f_r'^{-}(x) = +\infty$.

De alguna manera, este resultado nos dice que la función de Takagi-Van der Waerden tiene cúspides de la forma $\bigvee$ en cada punto r-ádico. Además, cuando r es impar, tiene también cúspides de la forma $\bigwedge$ en cada punto de $\widetilde{D}$. Este es un primer atisbo del diferente comportamiento según r sea par o impar.

Con respecto al caso cuando r es par, tenemos este resultado:

Teorema 7. *Sea $r \geq 2$ un número entero par y sea $x \notin D$. Entonces, se verifica:*

(1) $f_r'^{+}(x) = +\infty$ si y solo si $\sum_{k=1}^{\infty} g_k'(x) = +\infty$.

(2) $f_r'^{-}(x) = -\infty$ si y solo si $\sum_{k=1}^{\infty} g_k'(x) = -\infty$.

Detrás de este resultado está la propiedad de que $\widetilde{D}_n \subset D_{n+1}$ para todo n, y por tanto, se tiene $\widetilde{D} \subset D$ cuando r es un número par. No es posible obtener un resultado análogo para el caso impar. Veremos que para el punto

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n},$$

siendo $\varepsilon_n = 0$ si $n = 10^k$ para algún k, y $\varepsilon_n = 1$ en otro caso, se tiene que $\sum_{k=1}^{\infty} g_k'(x) = +\infty$ y

$$d^+ f_3(x) := \liminf_{h \to 0^+} \frac{f_3(x+h) - f_3(x)}{h} = -\infty$$

.

Con el objetivo de abordar este problema desde un enfoque general, introducimos la siguiente notación. Para cada punto $x \notin D \cup \widetilde{D}$ con expresión en base r dada por $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$, escribimos el conjunto $\{i : \varepsilon_i \neq (r-1)/2\}$ como una sucesión creciente $(i_n)_n$. Observemos que si r es un número par, entonces $i_n = n$ para todo n. Probaremos el siguiente resultado:

Teorema 8. *Sea $r \geq 2$ un número entero y sea $x \notin D \cup \widetilde{D}$. Entonces, $f_r'^{+}(x) = +\infty$ si y solo si*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n)\right) = +\infty. \tag{D}$$

Como era de esperar, cuando r es un número par, la condición (R) es equivalente a la convergencia a $+\infty$ de la serie de las derivadas.

De manera análoga, para un punto $x \notin D \cup \widetilde{D}$ con expresión en base r dada por $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$, escribimos el conjunto $\{m : \varepsilon_m \neq 0\}$ como una sucesión creciente $(m_n)_n$.

Teorema 9. *Sea $r \geq 2$ un número entero y sea $x \notin D \cup \widetilde{D}$. Entonces, $f_r'^{-}(x) = +\infty$ si y solo si*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1} - m_n) + \log_r(m_{n+1} - m_n)\right) = +\infty. \tag{I}$$

El término logarítmico que aparece en (R) y (L) es especialmente llamativo. En la demostración de dichos resultados podemos apreciar cómo surge este término de manera natural. Además, ilustraremos la naturaleza de ambas condiciones mediante varios ejemplos.

Finalmente, los resultados para el caso en que la derivada lateral es $-\infty$ en un punto $x \notin D \cup \widetilde{D}$, se obtienen aplicando los teoremas previos al punto $1 - x$ y haciendo uso de la ecuación $f_r(x) = f_r(1-x)$.

El teorema de Denjoy-Young-Saks (ver [30]) nos dice que el conjunto de puntos donde la función de Takagi-Van der Waerden posee una derivada infinite tiene medida de Lebesgue nula. La última parte del Capítulo 3 está dedicada a probar que dicho conjunto tiene dimensión de Hausdorff uno. Este resultado fue probado para la función de Takagi por P. C. Allaart y K. Kawamura (ver [11]). Las técnicas empleadas por dichos autores no pueden ser extrapoladas al caso de una base r arbitraria, por lo que nos veremos obligados a abordar dicho problema utilizando un enfoque distinto.

Los resultados que aparecen en dicho capítulo han sido publicados en [54].

En el Capítulo 4 estudiamos la subdiferenciabilidad y la superdiferenciabilidad de la función de Takagi-Van der Waerden. Ambas propiedades pertenecen al denominado análisis "nonsmooth". Recordemos que la subdiferential de una función semicontinua inferiormente $f : \mathbb{R} \to \mathbb{R}$ en un punto $x \in \mathbb{R}$ se define como el conjunto $\partial^- f(x)$ formado por todos $\xi \in \mathbb{R}$ tales que

$$\liminf_{h \to 0} \frac{f(x+h) - f(x) - \xi h}{|h|} \geq 0.$$

Entre las diferentes propiedades de la subdiferencial de una función $f : \mathbb{R} \to \mathbb{R}$, cabe destacar dos de ellas: el conjunto $\{x \in \mathbb{R} : \partial^- f(x) \neq \emptyset\}$ es denso en $\mathbb{R}$ (ver Teorema 4.21 de [47]), y el cardinal del conjunto $\partial^- f(x)$ es menor o igual que uno en casi todo punto $x \in \mathbb{R}$ (ver Teorema 4.4.3 de [59]).

En 2011, P. Góra y R. J. Stern probaron que la función de Takagi es una caso extremo en relación a las dos propiedades mencionadas anteriormente. Más precisamente, obtuvieron que si x es un número diádico del intervalo unidad entonces $\partial^- T(x) = \mathbb{R}$, y en otro caso, la subdiferencial es vacía (ver [61]). En otras palabras, el conjunto de puntos donde la subdiferencial es no vacía es "lo más pequeño posible", es decir, un conjunto numerable; mientras que la subdiferencial en dichos puntos es "lo más grande posible", es decir, todo $\mathbb{R}$.

Posteriormente, Juan Ferrera y Javier Gómez Gil definieron una función de tipo Takagi-Van der Waerden a partir de un conjunto denso y numerable ee un espacio de Hilbert arbitrario y probaron que esta función así definida tiene el mismo comportamiento subdiferencial que la función de Takagi.

Aunque el resultado de Juan Ferrera y Javier Gómez Gil incluye el caso de la función de Takagi-van der Waerden, debido a la importancia de dicho resultado, comenzamos el Capítulo 4 dando una prueba alternativa del mismo para el caso concreto de la función f_r. Esta prueba es nueva, y además, hace uso de las herramientas desarrolladas a lo largo del Capítulo 3. Esto nos permite obtener un argumento más simple y conciso que el empleado por P. Góra y R. J. Stern para la función de Takagi.

De algún modo, el concepto de superdiferencial es complementario al concepto de subdiferencial. Recordemos que la superdiferencial de una función semicontinua superiormente $f : \mathbb{R} \to \mathbb{R}$ en un punto $x \in \mathbb{R}$ puede definirse como

$$\partial^+ f(x) = -\partial^-(-f)(x).$$

Además, es conocido que una función $f : \mathbb{R} \to \mathbb{R}$ es derivable en un punto x, si y solo si tanto $\partial^-(f)(x)$ como $\partial^+ f(x)$ son no vacíos.

En 2019, Juan Ferrera y Javier Gómez Gil caracterizaron el conjunto de puntos en los que la superdiferencial de la función de Takagi es no vacía (ver [50]). En el Capítulo 4 describimos la superdiferencial de la función de Takagi-Van der Waerden en un punto a través de la expansión en base r de dicho punto.

Con respecto al caso en que r es impar, obtenemos el siguiente resultado:

Teorema 10. *Sea $r \geq 3$ un número entero impar. Si $x \in \widetilde{D}$ entonces $\partial^+ f_r(x) = \mathbb{R}$, mientras que $\partial^+ f_r(x) = \emptyset$ en otro caso.*

En 1984, Y. Baba (ver [16]) probó que si r es impar entonces la función f_r alcanza su máximo valor únicamente en el punto $x = 1/2$ y

$$f_r(1/2) = \frac{r}{2(r-1)}.$$

Como consecuencia del resultado anterior, cuando r es impar tenemos que la función de Takagi-Van der Waerden tiene un máximo local en x si y solo si $x \in \widetilde{D}$.

El comportamiento superdiferencial de la función f_r cambia drásticamente cuando r es un número par. Probaremos el siguiente teorema.

Teorema 11. *Sea $r \geq 2$ un número entero par y sea $x \in [0,1]$ con expansión en base r dada por $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$. Entonces, $\partial^+ f_r(x) \neq \emptyset$ si y solo si existe un número $m_0 = m_0(x) \geq 1$ tal que*

(1) $\varepsilon_{m_0+2i} = \frac{r}{2} - 1$ y $\varepsilon_{m_0+2i+1} \geq \frac{r}{2}$, o bien

(2) $\varepsilon_{m_0+2i} = \frac{r}{2}$ y $\varepsilon_{m_0+2i+1} \leq \frac{r}{2} - 1$.

para todo $i \geq 0$. En tal caso, si existe un número $n_0 \geq m_0$ tal que $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ y $\varepsilon_{n_0+2i+1} \geq \frac{r}{2}$ para todo $i \geq 0$, o bien $\varepsilon_{n_0+2i} = \frac{r}{2}$ y $\varepsilon_{n_0+2i+1} \leq \frac{r}{2} - 1$ para todo $i \geq 0$, entonces

$$\partial^+ f_r(x) = \begin{cases} G'_{n_0-1}(x) + [0,1] & si \quad \varepsilon_{n_0} = \frac{r}{2} - 1, \\ G'_{n_0-1}(x) + [-1,0] & si \quad \varepsilon_{n_0} = \frac{r}{2}. \end{cases}$$

En otro caso, se tiene que

$$\partial^+ f_r(x) = \left\{G'_{m_0-1}(x)\right\}.$$

Este resultado nos permitirá caracterizar el conjunto de máximos locales de la función de Takagi-Van der Waerden cuando r es par. En este caso, Y. Baba también describió el conjunto de puntos en los que la función f_r alcanza su máximo valor en términos de la expansión en base r^2 de dichos puntos. Además, demostró que dicho conjunto es un conjunto de Cantor con dimensión de Hausdorff 1/2.

Haciendo uso este último resultado, veremos que el conjunto de puntos donde la superdiferencial de la función f_r es no vacía tiene también dimensión de Hausdorff 1/2. Asimismo, probaremos que la medida de Hausdorff 1/2 dimensional de dicho conjunto es infinito.

Finalizamos el Capítulo 4 probamos el siguiente resultado:

Teorema 12. *Sea $r \geq 2$ un número entero par. La medida de Hausdorff $1/2$-dimensional del conjunto de puntos donde f_r alcanza su máximo valor es $1/\sqrt{r+1}$.*

Es importante destacar que los resultados obtenidos en este capítulo completan el estudio de los puntos extremos de la función de Takagi-Van der Waerden iniciado por Y. baba y J. P. Kahane. Todos los resultados que aparecen en dicho capítulo han sido publicados en [55].

El último capítulo de esta memoria trata sobre la denominada Clase Generalizada. Fue introducida por Juan Ferrera y Javier Gómez Gil en el año 2018 (ver [49]), y contiene todas las familias de funciones que se han presentado anteriormente.

Para cada sucesión estrictamente creciente $\mathbf{r} = (r_n)_n$ de números enteros no negativos tales que $r_1 = 1$ y r_{n+1} divide a r_n, la Clase Generalizada está formada por todas las funciones $f_{\mathbf{r},w} : [0,1] \to \mathbb{R}$ definidas como

$$f_{\mathbf{r},w}(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

donde $w = (w_n)_n$ es una sucesión de pesos tal que $\left(r_n^{-1} w_n\right)_n \in \ell_1$, y $g_n(x)$ es la distancia del punto x al conjunto $D_n = \left\{k r_n^{-1} : k = 0, \ldots, r_n\right\}$. Es importante observar que si $r_n = 2^{n-1}$ para todo n, entonces se obtiene la clase de Takagi. Del agún modo, la Clase Generalizada incorpora las propiedades intrínsecas de todas las familias de funciones estudiadas en esta memoria.

Comenzamos el Capítulo 5 estudiando algunas propiedades globales de las funciones pertenecientes a la Clase Generalizada. En particular, obtenemos el siguiente resultado.

Teorema 13. *Sea $f_{\mathbf{r},w}$ una función de la Clase Generalizada. Se verifica:*

(1) *$f_{\mathbf{r},w}$ es Lipschitz si y solo si $w \in \ell_1$. En este caso, la norma Lipschitz de $f_{\mathbf{r},w}$ viene dada por $\|w\|_1$.*

(2) *$f_{\mathbf{r},w}$ tiene la propiedad de Hölder de orden $0 < \alpha < 1$, si y solo si*

$$\limsup_n \frac{|w_n|}{r_n^{1-\alpha}} < \infty.$$

Como consecuencia, tenemos que si $w \in \ell_\infty$ entonces $f_{\mathbf{r},w}$ tiene la propiedad de Hölder para todo $0 < \alpha < 1$. Hasta donde sabemos, los resultados relativos a la propiedad de Hölder son nuevos para la Clase de Takagi.

El resto del Capítulo 5 está dedicado a estudiar el concepto de diferenciabilidad aproximada para las funciones de la Clase Generalizada. Este concepto es una generalización del concepto de diferenciabilidad clásico. Fue introducido por A. Y. Khinchin en el año 1914 (ver [78]), y ha adquirido una gran relevancia en del análisis moderno a lo largo de los últimos años. Probaremos el siguiente resultado para las funciones de la Clase Generalizada:

Teorema 14. *Si la función* $f_{\mathbf{r},\mathbf{w}}$ *es approximadamente diferenciable en* $x \in [0,1]$, *entonces la serie*

$$\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$$

converge.

Haciendo uso de los resultados obtenidos por Juan Ferrera y Javier Gómez Gil en [49], veremos que la diferenciabilidad clásica y aproximada en un punto son propiedades equivalentes para las funciones de la Clase Generalizada.

En 2020, Juan Ferrera and Javier Gómez Gil probaron que la función de Takagi es aproximadamente diferenciable en ningún punto (ver [51]). Además de su interés per se, la relevancia de este resultado radica en la dificultad de encontrar en la literatura ejemplos sencillos de funciones satisfaciendo dicha propiedad. Terminamos el Capítulo 5 con el siguiente resultado:

Teorema 15. *La función* $f_{\mathbf{r},w}$ *es aproximadamente diferenciable en ningún punto, si y solo si* $w \notin c_0$.

Todos los resultados que aparecen en este último capítulo han sido publicados en [56].

ABSTRACT

The Takagi function is probably the simplest example of a continuous nowhere differentiable function. In modern notation, it is defined as

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}\phi\left(2^n x\right), \quad x \in [0,1], \tag{T}$$

where $\phi(x)$ is the distance from the point x to the nearest integer. It is important to note that the original definition given by T. Takagi in [102] is entirely different from the one presented above. Probably this, along with Japan's isolation in the early 20th century, led to the overlooked status of the Takagi function for several decades in the Western World. As a result, the Takagi function was rediscovered throughout the 20th century by numerous authors such as B. L. Van der Waerden [107], R. Tambs-Lyche [104], G. de Rham [94], and T. H. Hildebrandt [68], among others.

The Takagi function exhibits remarkable properties from different perspectives. For this reason, it has been extensively studied by a large number of mathematicians, leading to numerous generalizations aiming to extend the intrinsic properties of the Takagi function to a broader family of functions. In the literature, these generalizations are referred to as "generalized Takagi functions."

The goal of this thesis is to present the results obtained as a consequence of studying certain properties for different generalized Takagi functions. Below, we outline the structure of the chapters comprising this thesis.

The families of functions considered in this work are presented in Chapter 1. All of them share the Takagi function as a common denominator, which we write as

$$T(x) = \sum_{n=1}^{\infty} g_n(x), \quad x \in [0,1],$$

where $g_n(x) = \operatorname{dist}(x, D_n)$ is the distance from x to the set $D_n = \left\{k2^{n-1} : k = 0, 1, \ldots, 2^{n-1}\right\}$. This set is known as the "dyadic numbers of order" n. In the first chapter, we detail the relationship between the Takagi function and the dyadic numbers of the unit interval. This allows us to establish the foundation for the generalized Takagi functions introduced

later: the *Takagi Class*, the *Takagi-Van der Waerden function* and the *Generalized Class*. Additionally, we present the intrinsic characteristics of each family of functions, while mentioning the results obtained by other authors in relation to them. This is done solely to highlight and contextualize the results presented in this thesis.

Chapter 2 is devoted to presenting the results obtained regarding the second order differentiability properties of the functions in the Takagi Class. This generalization of the Takagi function was introduced by M. Hata and M. Yamaguti in 1984 (see [64]). It consists of all functions $T_w : [0,1] \to \mathbb{R}$ defined as

$$T_w(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights such that $(2^{-n} w_n)_n \in \ell_1$, and $g_n(x) = \operatorname{dist}(x, D_n)$ is the distance from the point x to the set $D_n = \{k2^{n-1} : k = 0, 1 \ldots, 2^{n-1}\}$. In their article, M. Hata and M. Yamaguti proved that the Takagi Class is a closed subspace of $C[0,1]$ isomorphic to ℓ_1.

It is likely that the Takagi Class is the most famous and studied generalization of the Takagi function. Moreover, it contains numerous examples that had been studied earlier in the literature, such as G. Faber's function or the functions of A. S. Besicovitch and H. D. Ursell (see [44] and [24]).

Among the various results on the functions in the Takagi Class, the theorem obtained by N. Kôno (see [79]) is perhaps the most well-known:

Theorem 1. *The following statements hold:*

(1) T_w *is absolutely continuous with derivative*

$$T'_w(x) = \sum_{n=1}^{\infty} w_n g'_n(x)$$

almost everywhere if and only if $w \in \ell_2$.

(2) T_w *is differentiable at an uncountable set of measure zero, and the range of the derivative is all of* $\mathbb{R}$ *if and only if* $w \in c_0 \setminus \ell_2$.

(3) T_w *is nowhere differentiable if and only if* $w \notin c_0$.

This result completely determines the differentiability of the functions in the Takagi Class in terms of whether or not the sequence of weights $w = (w_n)_n$ belongs to a certain space of sequences.

In Chapter 2, we study some second order differentiability properties of the functions in the Takagi Class. In particular, we characterize when a function from this Class is convex or concave in terms of a condition on the sequence of weights $w = (w_n)_n$. Additionally, we investigate when T_w has Taylor expansion of order two at a point or satisfies the Stepanoff condition of order two at a point. We show that the study of these properties must be carried out under the assumption that $w = (w_n)_n$ is a sequence belonging to ℓ_1. In this case, the function T_w is differentiable at every point $x \notin D$ with a derivative given by

$$T'_w(x) = \sum_{n=1}^{\infty} w_n g'_n(x)$$

Furthermore, it is observed that the function T_w is Lipschitz if and only if $w \in \ell_1$. Regarding convexity, we will prove the following result:

Theorem 2. *Let $w \in \ell_1$. The function T_w is piecewise convex if and only if there exists $n_0 \geq 1$ such that*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0}$$

is a non-positive and non-decreasing sequence.

Next, we characterize when the function T_w has Taylor expansion of order two at a point $x \notin D$ in terms of the sequence of weights and a property related to the binary expansion of such a point. Another way to study second order differentiability at a point $x \notin D$ is to analyze the existence of the derivative of the function $T_w'^{+} : [0,1] \to \mathbb{R}$. When $w \in \ell_1$, this function is well-defined and continuous at every point $x \notin D$. Therefore, we can study the existence of the limit

$$\left(T_w'^{+}\right)'(x) = \lim_{y \to x} \frac{T_w'^{+}(y) - T_w'^{+}(x)}{y - x}$$

at a point $x \notin D$. We will prove the following result:

Theorem 3. *The function T_w has Taylor expansion of order two at a point $x \notin D$ if and only if the function $T_w'^{+}$ is differentiable at x.*

Similarly, we investigate when the function T_w has Taylor expansion of order two almost everywhere in $[0,1]$. We show that the sequence $\beta = (\beta_n)_n$ defined as

$$\beta_n = 2^n \left(-w_n + \sum_{j=n+1}^{\infty} w_j\right)$$

plays a fundamental role in studying this property. We will demonstrate this surprising theorem:

Theorem 4. *Let $w \in \ell_1$. Then, the following statements are equivalent:*

(1) T_w has Taylor expansion of order two almost everywhere.

(2) $T_w'^{+}$ has bounded variation.

(3) $T_w'^{+}$ is differentiable almost everywhere.

(4) $\beta \in \ell_1$.

(5) T_w is the difference of two piecewise convex functions of the Takagi Class.

Furthermore, we will prove that if the sequence $(2^n w_n)_n$ converges, then the set of points where T_w has Taylor expansion of order two has measure zero and Hausdorff dimension one, if and only if $\beta \notin \ell_1$.

The Stepanoff condition of order two at a point is weaker than having Taylor expansion of order two at a point. Proceeding analogously, we also characterize when T_w satisfies this condition for a point $x \notin D$ in terms of the sequence of weights and a property related to the binary expansion of such a point.

Concerning second order differentiability at a dyadic point, we will obtain the following result:

Theorem 5. *Let $w \in \ell_1$. Then, the following statements hold:*

(1) T_w has Taylor expansion of order two at a point $x \in D$ if and only if T_w is differentiable at x and the sequence $(2^n w_n)_n$ converges.

(2) T_w satisfies the Stepanoff condition of order two at a point $x \in D$ if and only if T_w is differentiable at x and $\limsup_n 2^n |w_n| < \infty$.

The results obtained allow us to construct functions in the Takagi Class with certain predefined properties. For example, let $\alpha \in \mathbb{R}$ such that $\alpha > 1$, and consider the sequence of weights $w = (w_n)$ given by $w_n = \alpha^{-n}$ for every n. If $\alpha \geq 2$, then T_w is piecewise concave and consequently, satisfies the Stepanoff condition of order two almost everywhere. Moreover, if $\alpha > 2$, then

$$T_w'(x) = \sum_{n=1}^{\infty} \frac{1-2\varepsilon_n}{\alpha^n} = \frac{1}{\alpha - 1} - 2\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\alpha^n}, \quad \varepsilon_n \in \{0,1\}$$

for every $x \notin D$. Therefore, the range of T_w' is a Cantor set with Hausdorff dimension $\log 2/\log \alpha$. However, if $1 < \alpha < 2$ then T_w satisfies nowhere the Stepanoff condition of order two.

A result by A. P. Calderón and A. Zygmund states that if a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the Stepanoff condition of order two almost everywhere, then it has Taylor expansion of order two almost everywhere (see Theorem 5 in [33]). The sets of points where these properties are satisfied do not necessarily coincide. For the sequence $w=(w_n)$ defined by $w_n=\frac{(-1)^n}{n^2 2^n}$ for every n, T_w has Taylor expansion of order two almost everywhere and consequently, satisfies the Stepanoff condition of order two almost everywhere. However, we will construct a point $x\in[0,1]$ such that T_w satisfies the Stepanoff condition of order two at x but does not have Taylor expansion of order two at such a point.

The results presented in this chapter have been published in [53].

Both Chapter 3 and Chapter 4 deal with the Takagi-Van der Waerden function. As mentioned earlier, the Takagi function was rediscovered by several authors throughout the first half of the 20th century. Some of these authors constructed a function similar to the Takagi function but replaced the base two in the definition (T) with another integer greater than two.

In 1930, B. L. van der Waerden rediscovered a version of the Takagi function using base 10, while E. Landau rediscovered it using base 4 in 1934 (see [107] and [83]).

To the best of our knowledge, this type of function defined with an arbitrary integer base $r\geq 2$ first appeared in the work of F. A. Behrend, who proved its nowhere differentiability (see [21]).

Given an integer $r\geq 2$, the Takagi-Van der Waerden function $f_r:[0,1]\to\mathbb{R}$ is defined as

$$f_r(x)=\sum_{n=1}^{\infty} g_n(x),$$

where $g_n(x)=\operatorname{dist}(x,D_n)$ is the distance from the point x to the set $D_n=\left\{kr^{n-1}:k=0,1\ldots,r^{n-1}\right\}$. Additionally, we denote by D the set of all r-adic numbers in the unit interval, that is, $D=\bigcup_{n=1}^{\infty} D_n$. Similar to what we did with the Takagi function, in Section 1.3 of Chapter 1, we establish the relationship between the function f_r and the base r expansion of the numbers in the unit interval.

For each n, we denote by $\widetilde{D}_n$ the set of all middle points between two consecutive points of D_n given by

$$\widetilde{D}_n=\left\{\frac{x+y}{2}:x,y\in D_n \text{ with } (x,y)\cap D_n=\emptyset\right\},$$

and we consider $\widetilde{D}=\bigcup_{n=1}^{\infty}\widetilde{D}_n$. When r is even, we have $\widetilde{D}_n\subset D_{n+1}$ for every n, and hence $\widetilde{D}\subset D$. However, when r is odd, we have $\widetilde{D}_n\subset\widetilde{D}_{n+1}$ for every n, and $\widetilde{D}\cap D=\emptyset$. In a way, these facts underlie the different nature of the Takagi-Van der Waerden function depending on whether r is even or odd. This will be reflected in the different results obtained throughout the aforementioned chapters.

In Chapter 3, we study the existence of infinite derivatives for the Takagi-Van der Waerden function. In particular, we prove that this function has no finite one-sided derivatives at any point. To demonstrate this result, we generalize the argument used by F. S. Cater to obtain the corresponding result for the Takagi function (see [34]).

Next, we characterize the set of points at which the Takagi-Van der Waerden function has an infinite one-sided derivative in terms of a condition on the base-r expansion of these points. This problem was solved for the Takagi function by M. Krüppel (see [81]), and independently by P. C. Allaart and K. Kawamura (see [11]).

We start by proving the following result:

Theorem 6. *Let $r \geq 2$ be an integer. The following statements hold:*

(1) If $x \in D$, then $f_r'^{+}(x) = +\infty$ and $f_r'^{-}(x) = -\infty$.

(2) If r is odd and $x \in \widetilde{D}$, then $f_r'^{+}(x) = -\infty$ and $f_r'^{-}(x) = +\infty$.

In a sense, this result tells us that the Takagi-Van der Waerden function has cusps pointing downwards at every r-adic point. Additionally, when r is odd, it also has cusps pointing upwards at every point of $\widetilde{D}$. This is a first glimpse of the different behavior depending on whether r is even or odd.

Regarding the case when r is even, we will prove the following result:

Theorem 7. *Let $r \geq 2$ be an even integer and $x \notin D$. Then, the following statements hold:*

(1) $f_r'^{+}(x) = +\infty$ if and only if $\sum_{k=1}^{\infty} g_k'(x) = +\infty$.

(2) $f_r'^{-}(x) = -\infty$ if and only if $\sum_{k=1}^{\infty} g_k'(x) = -\infty$.

Behind this result is the property that $\widetilde{D}_n \subset D_{n+1}$ for every n, and hence $\widetilde{D} \subset D$ provided that r is an even number. A similar result is not possible for the odd case. We will see that for the point

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n},$$

where $\varepsilon_n = 0$ if $n = 10^k$ for some k, and $\varepsilon_n = 1$ otherwise, we have $\sum_{k=1}^{\infty} g_k'(x) = +\infty$ and

$$d_+ f_3(x) := \liminf_{h \to 0^+} \frac{f_3(x+h) - f_3(x)}{h} = -\infty.$$

To address this problem from a general approach, we introduce the following notation. For each point $x \notin D \cup \widetilde{D}$ with a base-r expansion given by $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$, we write the set $\{i : \varepsilon_i \neq (r-1)/2\}$ as an increasing sequence $(i_n)_n$. Note that if r is an even number, then $i_n = n$ for every n. We will prove the following result:

Theorem 8. *Let $r \geq 2$ be an integer, and let $x \notin D \cup \widetilde{D}$. Then, $f_r'^{+}(x) = +\infty$ if and only if*

$$\lim_{n\to\infty} \left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n) \right) = +\infty. \tag{R}$$

As expected, when r is an even number, condition (R) is equivalent to the divergence to $+\infty$ of the series of derivatives.

Similarly, for a point $x \notin D \cup \widetilde{D}$ with a base-r expansion given by $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$, we write the set $\{m : \varepsilon_m \neq 0\}$ as an increasing sequence $(m_n)_n$.

Theorem 9. *Let $r \geq 2$ be an integer, and let $x \notin D \cup \widetilde{D}$. Then, $f_r'^{-}(x) = +\infty$ if and only if*

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1} - m_n) + \log_r(m_{n+1} - m_n) \right) = +\infty. \tag{L}$$

The logarithmic term in (R) and (L) is particularly interesting. In the proof of these results, we will see how this term naturally arises. Additionally, we will illustrate the nature of both conditions with several examples.

Finally, the results for the case when the one-sided derivative is $-\infty$ at a point $x \notin D \cup \widetilde{D}$ are obtained by applying the previous theorems to the point $1 - x$ and using the symmetry equation $f_r(x) = f_r(1-x)$.

The Denjoy-Young-Saks theorem (see [30]) gives us that the set of points where the Takagi-Van der Waerden function has an infinite derivative has Lebesgue measure zero. The last part of Chapter 3 is dedicated to proving that this set has Hausdorff dimension one. This result was proven for the Takagi function by P. C. Allaart and K. Kawamura (see [11]). The techniques used by these authors cannot be extrapolated to the case of an arbitrary base r, so we will have to address this problem using a different approach.

The results presented in this chapter have been published in [54].

In Chapter 4, we study the subdifferentiability and superdifferentiability of the Takagi-Van der Waerden function. Both properties belong to the so-called “nonsmooth" analysis. Recall that the subdifferential of a lower semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ is defined as the set $\partial^- f(x)$ consisting of all $\xi \in \mathbb{R}$ such that

$$\liminf_{h\to 0} \frac{f(x+h) - f(x) - \xi h}{|h|} \geq 0.$$

Among the different properties of the subdifferential of a function $f : \mathbb{R} \to \mathbb{R}$, two noteworthy ones are: the set $\{x \in \mathbb{R} : \partial^- f(x) \neq \emptyset\}$ is dense in $\mathbb{R}$ (see Theorem 4.21 in [47]), and the cardinality of the set $\partial^- f(x)$ is less than or equal to one almost everywhere in $\mathbb{R}$ (see Theorem 4.4.3 in [59]).

In 2011, P. Góra and R. J. Stern proved that the Takagi function is an extreme case with respect to the two properties mentioned earlier. More precisely, they obtained that if x is a dyadic number in the unit interval then $\partial^- T(x) = \mathbb{R}$, and the subdifferential is empty otherwise (see [61]). Roughly speaking, the set of points where the subdifferential is non-empty is "as small as possible", which is a countable dense set. Whereas, the subdifferential at those points is as "large as possible", which is all of $\mathbb{R}$.

Subsequently, Juan Ferrera and Javier Gómez Gil defined a Takagi-Van der Waerden type function from a dense and countable set in an arbitrary Hilbert space and proved that this function, as defined, has the same subdifferential behavior as the Takagi function (see [48]).

Although the result of Juan Ferrera and Javier Gómez Gil includes the case of the Takagi-Van der Waerden function, due to the importance of this result, we start Chapter 4 by providing an alternative proof for the specific case of the function f_r. This proof is new and makes use of the tools developed throughout Chapter 3. This allows us to obtain a simpler and a more concise argument than the one used by P. Góra and R. J. Stern for the Takagi function.

In a way, the concept of superdifferentiability is complementary to the concept of sub-differentiability. Recall that the superdifferential of an upper semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ can be defined as

$$\partial^+ f(x) = -\partial^-(-f)(x).$$

Additionally, it is known that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x if and only if both $\partial^-(f)(x)$ and $\partial^+ f(x)$ are non-empty.

In 2019, Juan Ferrera and Javier Gómez Gil characterized the set of points where the superdifferential of the Takagi function is non-empty (see [50]). In Chapter 4, we describe the superdifferential of the Takagi-Van der Waerden function at a point by means of its base-r expansion.

Regarding the case where r is odd, we obtain the following result:

Theorem 10. *Let $r \geq 3$ be an odd integer. If $x \in \widetilde{D}$ then $\partial^+ f_r(x) = \mathbb{R}$, and $\partial^+ f_r(x) = \emptyset$ otherwise.*

In 1984, Y. Baba (see [16]) proved that if $r$is odd then the function f_r attains its maximum value only at the point $x = 1/2$ and

$$f_r(1/2) = \frac{r}{2(r-1)}.$$

As a consequence of our result, when r is odd, the Takagi-Van der Waerden function has a local maximum at x if and only if $x \in \widetilde{D}$.

The superdifferentiability behavior of the function f_r changes dramatically when r is an even number. We will prove the following theorem:

Theorem 11. *Let $r \geq 2$ be an even integer, and let $x \in [0,1]$ with base-r expansion given by $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$. Then, $\partial^+ f_r(x) \neq \emptyset$ if and only if there exists a number $m_0 = m_0(x) \geq 1$ such that*

(1) $\varepsilon_{m_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{m_0+2i+1} \geq \frac{r}{2}$, or

(2) $\varepsilon_{m_0+2i} = \frac{r}{2}$ and $\varepsilon_{m_0+2i+1} \leq \frac{r}{2} - 1$

for all $i \geq 0$. In such a case, if there exists $n_0 \geq m_0$ such that either $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{n_0+2i+1} \geq \frac{r}{2}$ for all $i \geq 0$, or $\varepsilon_{n_0+2i} = \frac{r}{2}$ and $\varepsilon_{n_0+2i+1} \leq \frac{r}{2} - 1$ for all $i \geq 0$, then

$$\partial^+ f_r(x) = \begin{cases} G'_{n_0-1}(x) + [0,1] & \text{if} \quad \varepsilon_{n_0} = \frac{r}{2} - 1, \\ G'_{n_0-1}(x) + [-1,0] & \text{if} \quad \varepsilon_{n_0} = \frac{r}{2}. \end{cases}$$

Otherwise we have

$$\partial^+ f_r(x) = \left\{ G'_{m_0-1}(x) \right\}.$$

This result will allow us to characterize the set of local maxima of the Takagi-Van der Waerden function when r is even. In this case, Y. Baba also described the set of points where the function f_r attains its maximum value in terms of the base-r^2 expansion of those points. Additionally, he showed that this set is a Cantor set with Hausdorff dimension $1/2$.

Using this last result, we will see that the set of points where the superdifferential of the function f_r is non-empty also has a Hausdorff dimension of $1/2$. Furthermore, we will prove that the $1/2$-dimensional Hausdorff measure of this set is infinite.

In the final part of Chapter 4, we will prove the following result:

Theorem 12. *Let $r \geq 2$ be an even integer. The $1/2$-dimensional Hausdorff measure of the set where f_r attains its maximum value is $1/\sqrt{r+1}$.*

It is important to highlight that the results obtained in this chapter complete the study of the extreme points of the Takagi-Van der Waerden function initiated by Y. Baba and J. P. Kahane. All the results presented in this chapter have been published in [55].

The last chapter of this dissertation deals with the so-called Generalized Class. It was introduced by Juan Ferrera and Javier Gómez Gil in 2018 (see [49]), and it includes all the families of functions presented earlier.

Given a strictly increasing sequence $\mathbf{r} = (r_n)_n$ of non-negative integers such that $r_1 = 1$ and r_{n+1} divides r_n, the Generalized Class is formed by all functions $f_{\mathbf{r},w} : [0,1] \to \mathbb{R}$ defined as

$$f_{\mathbf{r},w}(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights such that $\left(r_n^{-1} w_n\right)_n \in \ell_1$, and $g_n(x)$ is the distance from the point x to the set $D_n = \left\{k r_n^{-1} : k = 0, \dots, r_n\right\}$. It is important to note that if $r_n = 2^{n-1}$ for every n, then we obtain the Takagi class. In a way, the Generalized Class incorporates the intrinsic properties of all the families of the functions studied in this dissertation.

We begin Chapter 5 by studying some global properties of functions belonging to the Generalized Class. In particular, we will obtain the following result.

Theorem 13. *Let $f_{\mathbf{r},w}$ be a function of the Generalized Class. The following statements hold:*

(1) $f_{\mathbf{r},w}$ is Lipschitz if and only if $w \in \ell_1$. In this case, the Lipschitz norm of $f_{\mathbf{r},w}$ is given by $\|w\|_1$.

(2) $f_{\mathbf{r},w}$ is Hölder continuous of order $0 < \alpha < 1$ if and only if

$$\limsup_n \frac{|w_n|}{r_n^{1-\alpha}} < \infty.$$

As a consequence, if $w \in \ell_\infty$, then $f_{\mathbf{r},w}$ is Hölder continuous of order α for every $0 < \alpha < 1$. To the best of our knowledge, the results regarding the Hölder property are even new for the Takagi Class.

The rest of Chapter 5 is dedicated to studying the concept of approximate differentiability for functions in the Generalized Class. This concept is a generalization of classical differentiability. It was introduced by A. Y. Khinchin in 1914 (see [78]), and it has gained significant relevance in modern analysis in recent years. We will prove the following result for functions in the Generalized Class:

Theorem 14. *If the function $f_{\mathbf{r},\mathbf{w}}$ is approximately differentiable at $x \in [0,1]$, then the series*

$$\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$$

converges.

By using the results obtained by Juan Ferrera and Javier Gómez Gil in [49], we will see that classical and approximate differentiability at a point are equivalent properties for the functions in the Generalized Class.

In 2020, Juan Ferrera and Javier Gómez Gil proved that the Takagi function is nowhere approximately differentiable (see [51]). Besides its intrinsic interest, the relevance of this result lies in the difficulty of finding simple examples in the literature of functions satisfying this property. We will conclude Chapter 5 with the following result:

Theorem 15. *The function* $f_{\mathbf{r},w}$ *is nowhere approximately differentiable if and only if* $w \notin c_0$*.*

All the results presented in this last chapter have been published in [56].

CHAPTER 1

GENERALIZED TAKAGI FUNCTIONS

“A theory of the history of
mathematics I prefer is an
interesting and enjoyable theory
of history”

Teiji Takagi (1875 – 1960)

In 1872 K. Weierstrass presented his example of a continuous nowhere differentiable function to the Prussian Academy of Sciences, commonly known today as the Weierstrass function (see [109] and [110]). It was defined as

$$W(x) = \sum_{k=0}^{\infty} a^k \cos\left(b^k \pi x\right), \quad x \in [0,1] \tag{1.1}$$

where a is a real number with $0 < a < 1$ and b is an odd integer satisfying $ab > 1 + 3\pi/2$.

Although the Bohemian mathematician B. Bolzano had constructed a function of this type much earlier (see [73]), the publication of the Weierstrass function caused a tremendous stir among the mathematicians of this period. This was due to the common belief that a continuous function must have derivatives at a significant set of points. This belief was certainly fueled by A. M. Ampère, who attempted to give a “proof” of this “fact” in 1806 (see [13]).

After the discovery of the Weierstrass example, many other mathematicians also constructed examples of continuous nowhere differentiable functions. For instance, we can mention H. A. Schwarz [96], U. Dini [40], D. Hilbert [67], T. Takagi [102], B. van der Waerden [107] and S. Banach [17] among others. We refer the reader to [74] and [37] for more information about continuous nowhere differentiable functions and related topics.

The Takagi function is probably the simplest example of a continuous nowhere differentiable function. It was introduced by Teiji Takagi in 1903 (see [102]). He was born in an agricultural area of the Grifu prefecture, located in a mountainous region in central Japan. At that time, Japan was isolated mathematically. He obtained his doctorate in 1903 with a thesis supervised by D. Hilbert, where he proved a conjecture of L. Kronecker (see [103]). In 1932 he was vice-president of the International Congress of Mathematicians celebrated in Zurich, and he was also a member of the selection committee for the first Fields Medal. Nowadays, he is considered the father of Class Field Theory, and he is credited with bringing modern mathematics to Japan at the beginning of the twentieth century (see [71] and [90]). May these words serve as a tribute to Professor Teiji Takagi, whose function is the seed of this thesis.

Figure 1.1: Teiji Takagi

In this first chapter we will be walking through the different families of functions found throughout this thesis. In addition, we will go into greater detail regarding the way they are constructed and at the same time, we will establish the notation that will be used throughout the entire work.

1.1 The Takagi function

In the literature, the Takagi function $T : [0,1] \to \mathbb{R}$ is defined as

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi\left(2^n x\right) \tag{1.2}$$

where $\phi(x)$ denotes the distance from the point x to the nearest integer. Teiji Takagi expressed his function in a different way, since he defined it in terms of the binary expansion of a point belonging to the unit interval. This fact together with the social and economic isolation of Japan at the beginning of the twentieth century are probably the main reasons why the Takagi function went unnoticed in the Western World. This led other authors to rediscover the Takagi function throughout the first half of the twentieth century. For instance, we can mention G. de Rham (see [94]), J. R. Trollope (see [106]) and R. Tambs-Lyche (see [104]), among others.

Although the definition (1.2) is the most common expression for the Takagi function, it has been defined in different ways throughout the existing literature (see Section 4 of [10]). The authors more or less chose the most convenient definition according to the property of the Takagi function that interested them.

In this thesis we will use the following definition of the Takagi function, which was introduced for the first time by Juan Ferrera and Javier Gómez Gil in their work [48]. For every n we consider the set of all dyadic numbers of order n given by

$$D_n = \left\{ \frac{k}{2^{n-1}} : k = 0, 1 \ldots, 2^{n-1} \right\}$$

and we observe that $D_n \subset D_{n+1}$ for every n. The Takagi function can be defined as

$$T(x) = \sum_{n=1}^{\infty} g_n(x) \tag{1.3}$$

where $g_n(x) = d(x, D_n)$ denotes the distance from the point x to the set D_n. We also denote by D the set of all dyadic points belonging to the unit interval, which is given by $D = \bigcup_{n=1}^{\infty} D_n$.

1.1.1 The binary expansion and the function g_n

Throughout this thesis we will frequently use the relation between the binary expansion of a point belonging to the unit interval, and the value of the function g_n at such a point. Now, we establish such a relationship.

Every real number $x \in [0,1]$ can be written in terms of its binary expansion given by

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^n} = 0.\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n \ldots \qquad \text{with} \qquad \varepsilon_n \in \{0,1\}.$$

The binary expansion of a point is unique except for the dyadic numbers different from zero and one. The points belonging to this set, namely $D \setminus \{0,1\}$, have two possible representations: one ending in all zeros and the other ending in all ones. In this case, we choose the representation ending in all zeros and consequently, the binary expansion of a point belonging to $D \setminus \{0,1\}$ consists of a finite number of digits different from zero. More precisely, for every $n \geq 2$ we have $x \in D_n \setminus \{0,1\}$ if and only if $\varepsilon_k(x) = 0$ for all $k \geq n$.

For every n the interval $[0,1]$ is divided into 2^{n-1} subintervals of equal length whose endpoints are exactly the points belonging to D_n. Each of those subintervals will be referred to as a dyadic interval of order n and it can be written as (x_n, y_n) with $x_n, y_n \in D_n$ and

$(x_n, y_n) \cap D_n = \emptyset$. Furthermore, each of them unequivocally determines the first $n-1$ digits of the binary expansion of the points contained within them. More precisely, we have that $x \in (x_n, y_n)$ if and only if $\varepsilon_k(x) = \varepsilon_k(x_n)$ for all $k \leq n-1$.

For every n the function g_n is a piecewise affine map whose nodes are located at the points belonging to the set D_{n+1}. This means that g_n is affine on each dyadic interval of order $n+1$. Moreover, we have that $g_n(x) = 0$ if and only if $x \in D_n$, whereas $g_n(x) = 2^{-n}$ if and only if $x \in D_{n+1} \setminus D_n$. In addition, we must observe that the function g_n is periodic with period $2^{-(n-1)}$. This type of function is informally known as a “tent” or “sawtooth” map.

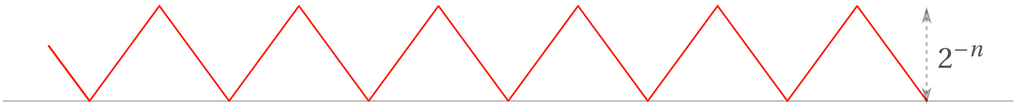

Figure 1.2: The graph of function g_n

Once we have determined the value of the function g_n at a point belonging to D_{n+1}, we consider the case when $x \in [0,1] \setminus D_{n+1}$. It is immediately observed that it lies between two consecutive dyadic points of order n, namely we denote by $a_n = a_n(x)$ the largest element of D_n smaller than x and by $b_n = b_n(x)$ the least element of D_n larger than x. We note that $b_n = a_n + 2^{-(n-1)}$. Thus we may write

$$g_n(x) = \operatorname{dist}(x, D_n) = \min\{x - a_n, b_n - x\}.$$

Nonetheless, we can go further than this (see Figure 1.3 below). We denote by c_n the midpoint of the interval (a_n, b_n). It is worth noting that $c_n \in D_{n+1}$, and hence $g_n(c_n) = 2^{-n}$. We have that $g_n(x) = x - a_n$ if and only if $a_n < x < c_n$, meanwhile we have $g_n(x) = b_n - x$ if and only if $c_n < x < b_n$. Consequently, we must know the position of the point x with regard to c_n. However, this position is determined solely by the value of the n-th digit in the binary expansion of x. More precisely, we have $\varepsilon_n = 0$ if and only if $a_n < x < c_n$, whereas $\varepsilon_n = 1$ if and only if $c_n < x < b_n$. Since $c_n \in D_{n+1}$, we obtain that $[a_{n+1}, b_{n+1}] = [a_n, c_n]$ if and only if $\varepsilon_n = 0$, meanwhile $[a_{n+1}, b_{n+1}] = [c_n, b_n]$ if and only if $\varepsilon_n = 1$.

In light of the previous discussion, we may determine the relation between the binary expansion of a point and the value of the function g_n' at such a point.

Firstly, if $x \in D_{n+1} \setminus D_n$ then $g_k'(x) \in \{-1, 1\}$ for all $1 \leq k \leq n-1$, $g_n'^{+}(x) = -1$ and $g_n'^{-}(x) = 1$, whereas $g_k'^{+}(x) = 1$ and $g_k'^{-}(x) = -1$ for all $k \geq n+1$.

In contrast to this, for a point $x \notin D$ we have that the derivative of the function g_n exists and $g_n'(x) \in \{-1, 1\}$ for all indices n. Moreover, we may write

$$g_n'(x) = 1 - 2\varepsilon_n(x)$$

for every n. It is noteworthy that the previous formula is actually the definition of the n-th Rademacher function. In 1987, N. Kôno was the first author who realized this fact (see [79]).

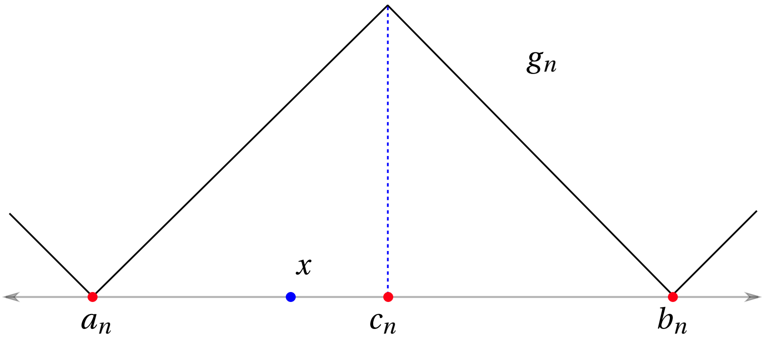

Figure 1.3: The graph of function g_n locally at a point

For every n the function g_n is symmetric about the line $x = 1/2$. Hence, we have that $g_n(x) = g_n(1-x)$ for all $x \in [0,1]$, and $g_n'(x) = -g_n'(1-x)$ for all $x \in [0,1] \setminus D_{n+1}$.

For a given dyadic interval $[x_n, y_n]$ of order n, that is $x_n, y_n \in D_n$ with $(x_n, y_n) \cap D_n = \emptyset$, we have that g_k is affine on $[x_n, y_n]$ for every $k < n$. Since such an interval determines unequivocally the first $n-1$ digits of the binary expansion of the points contained within it, we have that $g_k'^{+}(x_n) = g_k'(x)$ for all $k \leq n-1$ and for every $x \in (x_n, y_n)$.

1.1.2 The graph of the Takagi function

The graph of the Takagi function has been studied by many authors. We present below some of its main features. We refer the reader to the excellent surveys [10] and [82] for more information about the graph of the Takagi function and related topics.

The Takagi function can be written as a limit of piecewise affine functions. We have

$$T(x) = \lim_{n\to\infty} G_n(x)$$

where $G_n = g_1 + \cdots + g_n$ is a polygonal function whose nodes are located at the points of D_{n+1}. These functions converge uniformly to the Takagi function and they approximate it monotonically from below, namely

$$G_1(x) \leq G_2(x) \leq \cdots \leq G_n(x) \leq \cdots \leq T(x)$$

for all $x \in [0,1]$. Furthermore, if $x \in D_{n+1}$ for some n, then $T(x) = G_n(x)$ since $g_k(x) = 0$ for all $k \geq n+1$. In particular, we have $T(0) = T(1) = 0$.

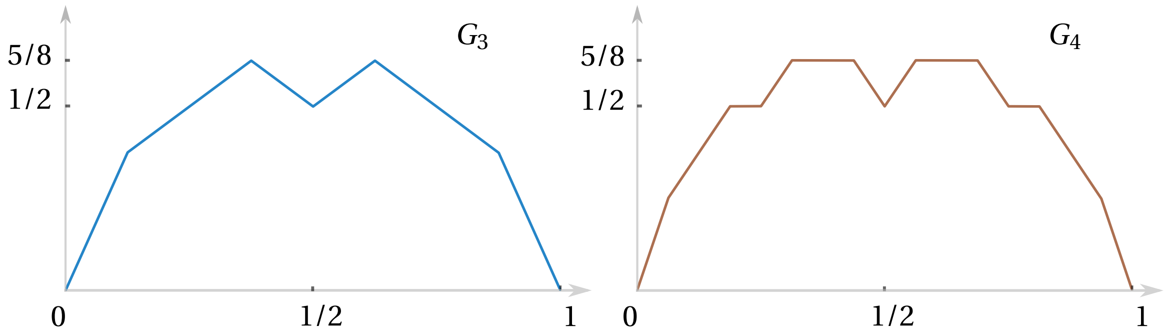

Figure 1.4: The graph of G_3 and G_4

As outlined above, for every n we have $g_n(x) = g_n(1-x)$ for all $x \in [0,1]$, so we obtain that the graph of the Takagi function is symmetric about the line $x = 1/2$. It satisfies the symmetry equation

$$T(x) = T(1-x) \tag{1.4}$$

for all $x \in [0,1]$. Furthermore, the Takagi function is the unique continuous function that satisfies (1.4) and

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{T(x)}{2}$$

for all $x \in [0,1]$ (see Theorem 4.1 of [82]).

The Takagi function also satisfies the following "self-affine" property: for every n and $x \in D_n$ we have

$$T\left(x + \frac{y}{2^{n-1}}\right) = T(x) + \frac{1}{2^{n-1}}\left[T(y) + y\sum_{k=1}^{n-1} g_k'^{+}(x)\right]$$

for all $y \in [0,1]$ (see Theorem 4.2 of [82]). Roughly speaking, it means that the graph of the Takagi function above the dyadic interval of order n given by $\left[x, x+2^{-(n-1)}\right]$ is a miniature version of a "tilted" Takagi function, reduced by a factor $2^{-(n-1)}$ and shifted by $T(x)$.

In 1959, J. P. Kahane investigated the extreme points of the Takagi function (see [75]). Among his results, he showed that the maximum value of the Takagi function is $2/3$ and described the set of points where this maximum is attained in terms of the binary expansion of such points. In addition, he also obtained that the graph of the Takagi function has local minima at the dyadic numbers.

It is well-known that the graph of a Lipschitz function $f : [0,1] \to \mathbb{R}$ has Hausdorff dimension one. Although the Takagi function is very far from being a Lipschitz function, R. D. Mauldin and S. C. Williams proved that the graph of the Takagi function has Hausdorff dimension one (see [89]). Later, J. M. Anderson and L. D. Pitt showed that it has σ-finite linear measure (see [14]).

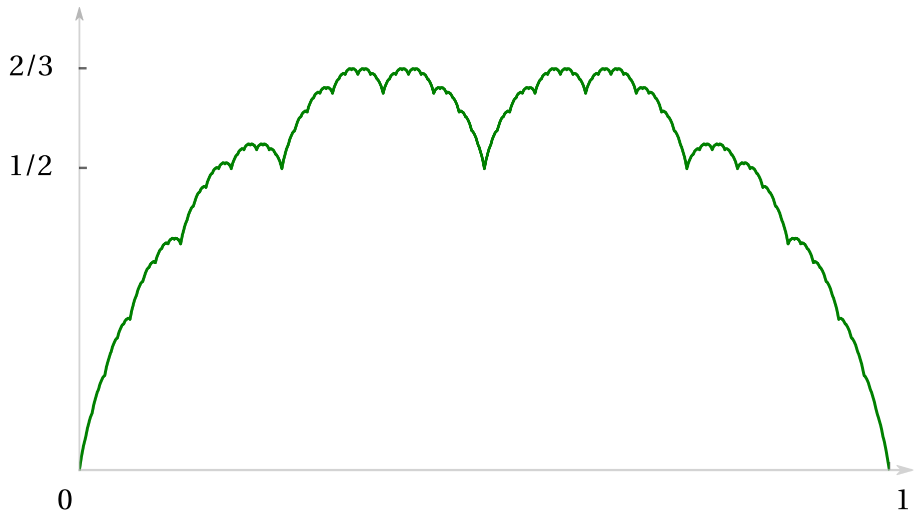

Figure 1.5: The graph of the Takagi function

In 2008, Z. Buczolich decomposed the graph of the Takagi function into two sets: an irregular 1-set satisfying that its projection onto the y-axis is of Lebesgue measure zero and its projection onto the x-axis has Lebesgue measure one; whereas the remaining part of the graph can be covered by countably many graphs of monotone functions (see [32]). We refer the reader to [46] for an introduction to the theory of 1-sets.

As a consequence of such a decomposition, Z. Buczolich obtained that for almost every ordinate $y \in [0,2/3]$ the level set $L_y = \{x \in [0,1] : T(x) = y\}$ is finite. The cardinalities of the finite level sets have been deeply investigated by P. C. Allaart, who proved that any even positive integer is the cardinality of some level set (see [6]). Furthermore, he also showed that the set of ordinates whose level set is uncountably large, is residual in the range of the Takagi function (see [4] and [5]). Finally, we would also point out that E. de Amo, I. Bhouri, M. Díaz Carrilo and J. Fernández-Sánchez proved that the Hausdorff dimension of each level set is less than or equal to 1/2 (see [12]).

1.2 The Takagi class

One of the most famous generalizations of the Takagi function is the so-called Takagi class. This family of functions was introduced by M. Hata and M. Yamaguti in 1984 and since then, it has caught the eye of a large number of mathematicians (see [64]).

The motivation of M. Hata and M. Yamaguti to introduce the Takagi Class comes from the point of view of dynamical systems. In their previous work [65], they obtained that for $b = 2$ and any $0 < a < 1$ the Weierstrass function (1.1) is the solution of a functional equation involving the iterations of a certain chaotic one-dimensional dynamical system. In [64], they were interested in finding the solutions of such a functional equation when considering the one-dimensional dynamical system given by the tent map function $\psi(x) = 2g_1(x)$.

As above, for every n we denote by D_n the set of all dyadic numbers of order n given by $D_n = \{k2^{-(n-1)} : k = 0, 1 \dots, 2^{-(n-1)}\}$, and $g_n(x)$ denotes the distance from the point x to the set D_n. We also denote by D the set of all dyadic numbers of the unit interval, which is given by $D = \bigcup_{n=1}^{\infty} D_n$.

The Takagi class is the set of all functions $T_w : [0,1] \to \mathbb{R}$ defined by

$$T_w(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights satisfying $(2^{-n} w_n)_n \in \ell_1$. The study of the function T_w will be done in the interval $[0,1]$; although, for the sake of convenience, we will occasionally take advantage of the fact that the function T_w can be extended periodically to the whole real line.

As a first result, M. Hata and M. Yamaguti proved that the Takagi Class is a closed subspace of the Banach space $C[0,1]$ of all continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the supremum-norm, and it is isomorphic to ℓ_1. By looking at the Schauder expansion of the function T_w in the Faber-Schauder system of $C[0,1]$ (see [76]), they also obtained that each function T_w is the unique continuous solution of the discrete boundary value problem

$$f\left(\frac{k}{2^{n-1}}\right) + f\left(\frac{k+1}{2^{n-1}}\right) - 2f\left(\frac{2k+1}{2^n}\right) = \frac{-w_n}{2^{n-1}}$$

for $k = 0, \dots, 2^{n-1} - 1$ and $n \in \mathbb{N}$, with $f(1) = f(0) = 0$.

Apart from the Takagi function, the Takagi class contains many examples of functions that have been studied throughout the existing literature. For instance, G. Faber and J. P. Kahane investigated the modulus of continuity and the non-differentiability of functions in the Takagi Class given by highly lacunary series (see [44] and [75]). In addition, A. S. Besicovitch and H. D. Ursell also studied the Hausdorff dimension of the graphs of such functions in the Takagi class (see [24]).

When we consider the sequence $w = (w_n)_n$ given by $w_n = 2^{(n-1)(1-\alpha)}$ with $\alpha > 0$, we obtain the so-called Takagi-Landsberg function. According to B. Mandelbrot (see page 246 of [19]), this generalization of the Takagi function was introduced by G. Landsberg (see [84]). In 1992, F. Ledrappier proved that the Hausdorff dimension of the graph of this function is $2 - \alpha$ for almost every $0 < \alpha < 1$. When $1 \leq \alpha \leq 2$, this function becomes the extreme case with regard to an inequality concerning approximately midconvex functions (see [28], [66] and [101]).

Another particular subfamily of the Takagi Class is formed by the so-called signed Takagi functions. It consists of all the functions T_w where the sequence $w = (w_n)_n$ satisfies $w_n \in \{-1,1\}$ for every n. In 2013, P. C. Allaart investigated the level sets of the functions belonging to this subfamily and generalized the existing results for the Takagi function. Additionally, he determined the extreme values of such a family of functions (see [7]).

Following this line of thought, P. C. Allaart introduced a random version of the signed Takagi function. It is obtained by considering each $w_n \in \{-1,1\}$ as an independent, identically distributed random variable with $P(w_n = 1) = p$ and $P(w_n = -1) = 1 - p$ where $0 \leq p \leq 1$. He determined the probability distributions of the maximum value of such a function in terms of the sequence $w = (w_n)_n$, as well as the size of the set of points where this maximum is attained (see [8]).

With respect to the foregoing, it should be mentioned that the set of points where a function of the Takagi Class attains its maximum value has recently been characterized by X. Han and A. Shied (see [62]).

In 1987 N. Kôno proved Theorem 2.1 below, which completely determines the differentiability behavior of the functions of the Takagi Class depending on whether the sequence of weights $w = (w_n)_n$ belongs, or does not belong to ℓ_2 and c_0 (see [79]). In order to obtain such a result, N. Kôno expressed the function T_w in terms of the Rademacher system and took advantage of several probabilistic techniques. He also studied the uniform and local modulus of continuity of a function in the Takagi Class. These results were later improved by P. C. Allaart in [9]. Finally, N. Kôno also showed that the Takagi Class only contains one family of functions which is smooth.

1.2.1 Smoothness in the Takagi Class

The concept of smoothness was first considered by Riemann in his classical paper on trigonometric series and it was deeply investigated by A. Zygmund in 1945 (see [113]). A continuous function $f : [0,1] \to \mathbb{R}$ is smooth if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x+h) + f(x-h) - 2f(x)| \leq \varepsilon h$$

for all $0 < h < \delta$ and $x \in [0,1]$. According to A. Zygmund, the origin of this terminology comes from the fact that the graph of a smooth function cannot have angular points. In his work, N. Kôno proved that the only smooth function of the Takagi Class is obtained by taking $w_n = M2^{-n}$ for every n with $M \in \mathbb{R}$. In this case, T_w is a polynomial given by $T_w(x) = Mx(1-x)$ for all $x \in [0,1]$.

A. Zygmund also studied a weaker condition named quasi-smoothness. A continuous function $f:[0,1]\to\mathbb{R}$ is quasi-smooth if there exist $M>0$ and $\delta>0$ such that

$$|f(x+h)+f(x-h)-2f(x)|\le Mh$$

for all $0<h<\delta$ and $x\in[0,1]$. In contrast with the above, the graph of a quasi-smooth function can have angular points but can have no cusps. In 1977, Z. Ciesielski (see [38]) proved that a continuous function $f:[0,1]\to\mathbb{R}$ is quasi-smooth if and only if it satisfies

$$\sup_{n\in\mathbb{N}}\ \sup_{x\in D_{n+1}\setminus\{0,1\}}\frac{\left|f(x+2^{-n})+f(x-2^{-n})-2f(x)\right|}{2^{-n}}<+\infty. \tag{1.5}$$

Concerning the functions in the Takagi Class, S. Abbott, J. M. Anderson and L. D. Pitt showed that the so-called alternating Takagi function, which is obtained by taking $w_n=(-1)^n$ for every n, is a quasi-smooth function (see [1]). Here, we characterize when a function in the Takagi Class is a quasi-smooth function. As far as we know, these results concerning the quasi-smoothness property are new.

First, we observe that for every point $x\in D_{n+1}\setminus\{0,1\}$ there exists an integer $k\le n$ such that $x\in D_{k+1}\setminus D_k$. Therefore, we may rewrite condition (1.5) as

$$\sup_{n\in\mathbb{N}}\ \sup_{1\le k\le n}\ \sup_{x\in D_{k+1}\setminus D_k}\frac{\left|f(x+2^{-n})+f(x-2^{-n})-2f(x)\right|}{2^{-n}}<+\infty.$$

Lemma 1.1. *Let $n\in\mathbb{N}$. For every $1\le k\le n$ and $x\in D_{k+1}\setminus D_k$ we have*

$$\frac{|T_w(x+2^{-n})+T_w(x-2^{-n})-2T_w(x)|}{2^{-(n-1)}}=\left|-w_k+w_{k+1}+\cdots+w_n\right|.$$

Proof. Since x, $x+2^{-n}$ and $x-2^{-n}$ belong to D_{n+1} we have

$$T_w\left(x+2^{-n}\right)+T_w\left(x-2^{-n}\right)-2T_w(x)=\sum_{j=1}^{n}w_j\left(g_j(x+2^{-n})+g_j(x-2^{-n})-2g_j(x)\right)$$

The function g_j is affine on $(x-2^{-n},x+2^{-n})$ for every $1\le j\le k-1$ and consequently

$$g_j\left(x+2^{-n}\right)+g_j\left(x-2^{-n}\right)-2g_j(x)=0$$

for every $0\le j\le k-1$. As is customary, we denote $a_k=x-2^{-k}$ and $b_k=x+2^{-k}$, so we have that x is the midpoint between a_k and b_k. Hence,

$$g_k\left(x+2^{-n}\right)+g_k\left(x-2^{-n}\right)-2g_k(x)=b_k-\left(x+2^{-n}\right)+x-2^{-n}-a_k-2^{-(k-1)}=\frac{-1}{2^{n-1}}.$$

Furthermore,

$$g_j\left(x+2^{-n}\right)+g_j\left(x-2^{-n}\right)-2g_j(x)=g_j\left(x+2^{-n}\right)+g_j\left(x-2^{-n}\right)=\frac{1}{2^{n-1}}.$$

for every $k+1\le j\le n$. This gives us the result. □

The space ***bs*** consists of all sequences $x = (x_n)_n$ such that

$$\sup_{n\in\mathbb{N}} \left| \sum_{j=1}^{n} x_j \right| < +\infty.$$

It is a Banach space isometrically isomorphic to ℓ_∞. The following result follows immediately from Lemma 1.1 and Z. Ciesielski 's result.

Theorem 1.2. *A function T_w is quasi-smooth if and only if $w \in$ **bs**.*

This completes the study of the smoothness property of the functions in the Takagi Class, initiated by N. Kôno.

1.3 The Takagi-Van der Waerden function

As previously mentioned in the opening of Section 1.1, a version of the Takagi function has been rediscovered by several authors throughout the twentieth century. Some of those authors used a base different from the base two that appears in the definition of the Takagi function.

In 1930, B. L. van der Waerden rediscovered the Takagi function by using base ten instead of base two (see [107]), whereas E. Landau provided a variant of the Takagi function by using base four instead of base two in 1934 (see [83]).

The Takagi-Van der Waerden function is defined similarly to the Takagi function. Let $r \geq 2$ be an integer. For every $n \geq 1$ we consider the set of all r-adic numbers of order n given by

$$D_n = \left\{ \frac{k}{r^{n-1}} : k = 0, 1 \ldots, r^{n-1} \right\}$$

and we observe that $D_n \subset D_{n+1}$ for every n. The Takagi-Van der Waerden function $f_r : [0,1] \to \mathbb{R}$ is defined as

$$f_r(x) = \sum_{n=1}^{\infty} g_n(x)$$

where $g_n(x) = \operatorname{dist}(x, D_n)$ denotes the distance from x to the set D_n. Note that f_2 is the Takagi function. In the literature, the function f_{10} is known as the Van der Waerden function and f_4 is known as the Landau function.

Although the study of the function f_r will be done in the unit interval; for the sake of convenience, sometimes we will take advantage of the fact that f_r can be extended periodically to the whole real line. Therefore, the function f_r becomes an even function.

As far as we know, this family of functions first appeared in the work of F. A. Behrend (see [21]), who proved that f_r is a continuous nowhere differentiable function. In addition, the article of Y. Baba is the first occasion in which the function f_r is referred to as the Takagi-Van der Waerden function (see [16]).

1.3.1 The base-r expansion and the function g_n

The relation between the base-r expansion of a point of the unit interval, and the value of the function g_n at such a point will play an important role in Chapter 3 and Chapter 4. We proceed similarly to what we did in the previous section. We denote by D the set of all r-adic numbers in the unit interval, which is given by $D = \bigcup_{n=1}^{\infty} D_n$.

Every real number $x \in [0,1]$ can be written in terms of its base-r expansion given by

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{r^n} = 0.\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots. \qquad \varepsilon_n \in \{0,1,\ldots,r-1\}.$$

The base-r expansion of a point is unique except for those points belonging to the set $D \setminus \{0,1\}$, which have two possible representations: one ending in all zeros and the other ending in all $(r-1)$-digits. In this case, we choose the representation ending in all zeros and consequently, for every $n \geq 2$ we have $x \in D_n \setminus \{0,1\}$ if and only if $\varepsilon_k(x) = 0$ for all $k \geq n$.

For every n the interval $[0,1]$ is divided into r^{n-1} subintervals of equal length whose endpoints are exactly the points belonging to D_n. Each of those subintervals receives the name of r-adic interval of order n and it takes the form of (x_n, y_n) where $x_n, y_n \in D_n$ and $(x_n, y_n) \cap D_n = \emptyset$. In addition, every r-adic interval of order n unequivocally determines the first $n-1$ digits of the base-r expansion of the points contained within it. More precisely, we have that $x \in (x_n, y_n)$ if and only if $\varepsilon_k(x) = \varepsilon_k(x_n)$ for all $k \leq n-1$.

Up to this point, there are no significant changes with regard to the binary expansion. However, the behavior of the function g_n will begin to change. In order to provide a uniform approach, we introduce the following notation.

For every $n \geq 1$ we consider the set of all middle points between two consecutive points of D_n given by

$$\widetilde{D}_n = \left\{ \frac{x+y}{2} : x, y \in D_n \text{ with } (x,y) \cap D_n = \emptyset \right\},$$

which will be referred to as the set of middle points of order n. Moreover, we denote by $\widetilde{D}$ the set of all middle points of the unit interval, that is $\widetilde{D} = \bigcup_{n=1}^{\infty} \widetilde{D}_n$. When considering the Takagi function case, we have that $\widetilde{D}_n = D_{n+1} \setminus D_n$ for every n.

For every n the function g_n is a piecewise affine map whose nodes are located at the points of $D_n \cup \widetilde{D}_n$, and it is periodic with period $r^{-(n-1)}$. Moreover, we have that $g_n(x) = 0$ if and only if $x \in D_n$, meanwhile $g_n(x) = r^{-(n-1)}/2$ if and only if $x \in \widetilde{D}_n$. Note that if r is odd then we have $x \in \widetilde{D}_n$ if and only if $\varepsilon_k(x) = (r-1)/2$ for all $k \geq n$, and consequently

$$g_k(x) = \frac{1}{2r^{k-1}}$$

for all $k \geq n$.

For a point $x \notin D_n \cup \widetilde{D}_n$ we have that it lies between two consecutive r-adic points of order n, namely we denote by $a_n = a_n(x)$ the largest element of D_n smaller than x and by $b_n = b_n(x)$ the least element of D_n larger than x. We observe that $b_n = a_n + r^{-(n-1)}$. We also denote by $c_n = c_n(x)$ the midpoint between a_n and b_n, that is $c_n = (a_n + b_n)/2$.

We are interested in determining the location of the point $x \notin D_n \cup \widetilde{D}_n$ with respect to the center c_n by means of its base-r expansion. This location is determined solely by the value of the n-digit in the base-r expansion of x. The interval $[a_n, b_n]$ is divided into r closed subintervals $J_0^n, \dots, J_{r-1}^n$ of r^{-n} in length and the set $D_{n+1} \cap [a_n, b_n]$ is formed by the endpoints of such subintervals. In addition, the interval $[a_{n+1}, b_{n+1}]$ will be one of such subintervals determined by the n-th digit in the base-r representation of the point x. More accurately, we have $[a_{n+1}, b_{n+1}] = J_k^n$ if and only if $\varepsilon_n(x) = k$, which leads us to differentiate between the case when r is even and the case when r is odd for the purpose of knowing the exact position of the point x with respect to the center c_n.

When r is even, the interval $[a_n, b_n]$ is divided into an even number of subintervals, and hence the center c_n is the right-endpoint of the subinterval corresponding to the digit $(r-2)/2$ and it is also the left-endpoint of the subinterval corresponding to the digit $r/2$. Of course, we have $c_n \in D_{n+1}$. Therefore, we conclude that $g_n(x) = x - a_n$ if and only if $0 \leq \varepsilon_n \leq (r-2)/2$ and $g_n(x) = b_n - x$ if and only if $r/2 \leq \varepsilon_n \leq r-1$.

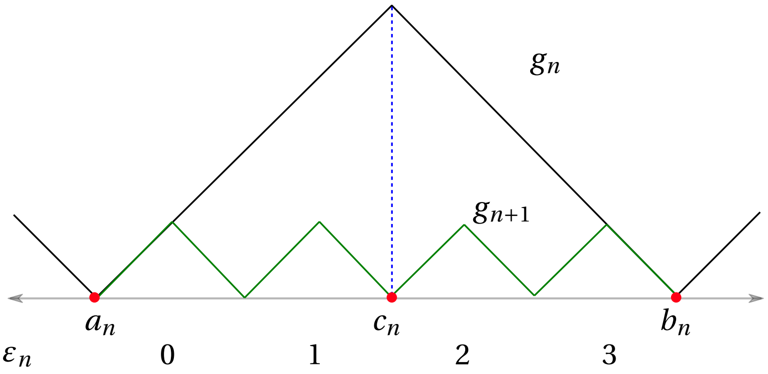

Figure 1.6: Example for $r = 4$

When r is odd, the center c_n is not an endpoint of any of the subintervals $J_0^n, \dots, J_{r-1}^n$ but it is the center of the subinterval corresponding to the digit $(r-1)/2$, because the interval

$[a_n, b_n]$ is divided into an odd number of subintervals. Therefore, $c_n \notin D_{n+1}$ but $c_n \in \widetilde{D}_{n+1}$. We have $g_n(x) = x - a_n$ provided that $0 \leq \varepsilon_n < (r-1)/2$ meanwhile $g_n(x) = b_n - x$ provided that $(r-1)/2 < \varepsilon_n \leq r-1$. However, when $\varepsilon_n(x) = (r-1)/2$ nothing can be said about the location of x with respect to c_n. In such a case, this information will be given by the first digit that is subsequent to ε_n in the base-r expansion of x and is different from $(r-1)/2$.

This leads us to define

$$k_n = k_n(x) = \min\{k \geq n : \varepsilon_k \neq (r-1)/2\} \tag{1.6}$$

and it is finite since $x \notin \widetilde{D}_n$. Thus, when $\varepsilon_n = (r-1)/2$ we have that $g_n(x) = x - a_n$ if and only if $\varepsilon_{k_n} < (r-1)/2$ meanwhile $g_n(x) = b_n - x$ if and only if $\varepsilon_{k_n} > (r-1)/2$. Finally, it should be acknowledged that the definition of k_n also makes sense when r is even. However, we have $k_n = n$ for every n because the value $(r-1)/2$ is not a digit in such a case.

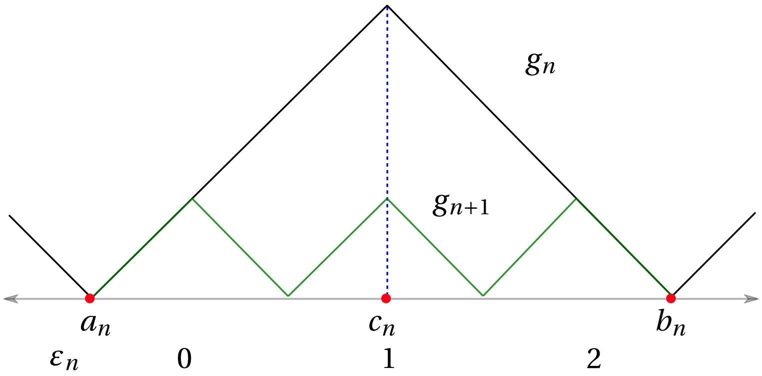

Figure 1.7: Example for $r = 3$

In light of the previous discussion, some remarks are required. We have that $\widetilde{D}_n \subset D_{n+1}$ for every n provided that r is an even integer. However, when r is odd, we have $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n and $D \cap \widetilde{D} = \emptyset$.

Now, we may determine the relation between the base-r expansion of a point, and the value of the derivative of g_n at such a point.

If $x \in \widetilde{D}_n \setminus \widetilde{D}_{n-1}$ for some n, then $g_n'^{+}(x) = -1$ and $g_n'^{-}(x) = 1$ but $g_k'(x) \in \{-1, 1\}$ for all $1 \leq k < n$. Moreover, for a point $x \in D_n \setminus D_{n-1}$ we have $g_k'^{+}(x) = 1$ and $g_k'^{-}(x) = -1$ for all $k \geq n$, whereas $g_k'(x) \in \{-1, 1\}$ for all $1 \leq k < n-1$. In this latter case, when r is even it may happen that $x \in \widetilde{D}_{n-1}$ so then $g_{n-1}'^{+}(x) = -1$ and $g_{n-1}'^{-}(x) = 1$, whereas $g_{n-1}' \in \{-1, 1\}$ otherwise.

However, the derivative of the function g_n at a point $x \notin D \cup \widetilde{D}$ exists for all indices n, and $g_n'(x) \in \{-1, 1\}$ for every n. Furhermore, we have $g_n'(x) = 1$ if and only if $\varepsilon_{k_n} < (r-1)/2$, meanwhile $g_n'(x) = -1$ if and only if $\varepsilon_{k_n} > (r-1)/2$, where $k_n = \min\{k \geq n : \varepsilon_k \neq (r-1)/2\}$.

Finally, we observe that for every n the function g_n is symmetric about the line $x = 1/2$. Hence, for every n we have that $g_n(x) = g_n(1-x)$ for all $x \in [0,1]$, and $g_n'(x) = -g_n'(1-x)$ for every $x \notin D \cup \widetilde{D}$. As mentioned earlier, sometimes we will take advantage of the fact that the Takagi-Van der Waerden function can be extended periodically to the whole line. In such a case, we also have $g_n'(x) = -g_n'(-x)$ for every $x \notin D \cup \widetilde{D}$.

1.3.2 The graph of the Takagi-Van der Waerden function

The Takagi-Van der Waerden function can also be written as a limit of piecewise affine functions. We have

$$f_r(x) = \lim_{n\to\infty} G_n(x)$$

where $G_n = g_1 + \cdots + g_n$ is a polygonal function whose nodes are located precisely at the points belonging to the set $D_n \cup \widetilde{D}_n$. As before, we have

$$G_1(x) \le G_2(x) \le \cdots \le G_n(x) \le \cdots \le f_r(x)$$

for all $x \in [0,1]$. Furthermore, if $x \in D_{n+1}$ for some n, then $f_r(x) = G_n(x)$ since $g_k(x) = 0$ for all $k \ge n+1$. In particular, we have $f_r(0) = f_r(1) = 0$.

The graph of the Takagi-Van der Waerden function is symmetric about line $x = 1/2$. In other words, it satisfies the symmetry equation $f_r(x) = f_r(1-x)$ for all $x \in [0,1]$.

In 1984, Y. Baba computed the maximum value of the Takagi-Van der Waerden function. When r is odd, this maximum value is only attained at $x = 1/2$. However, when r is even, the set of points where it is attained is a Cantor set of Hausdorff dimension 1/2 and he described this set in terms of the base-r^2 expansion of its points (see [16]). He also characterized the Takagi-Van der Waerden function as the unique continuous solution of certain functional equations. In this sense, he generalized the results obtained by J. P. Kahane, M. Hata and M. Yamaguti for the Takagi function (see [75] and [65]).

More recently, P. C. Allaart has examined the level sets of the Takagi-Van der Waerden function (see [3]). He proved that for almost every ordinate y in the range of f_r the level set $L_y = \{x \in [0,1] : f_r(x) = y\}$ is finite, whereas for almost every $x \in [0,1]$ the level set corresponding to $f_r(x)$ is uncountable. He also proved that the Hausdorff dimension of any level set of f_r is at most 1/2 whenever r is even. However, this problem remains unsolved when r is odd. In this sense, he generalized the results obtained by Z. Buczolich and de Amo et al. for the Takagi function (see [32] and [12]).

Throughout this thesis, we will be able to appreciate the different nature the Takagi-Van der Waerden function has depending on whether r is odd or even.

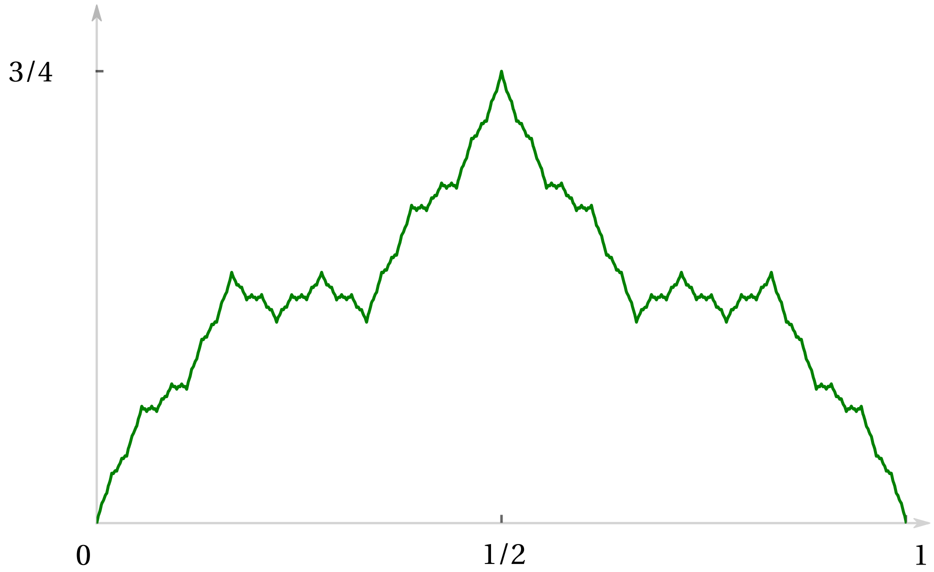

Figure 1.8: The graph of the function f_3

1.4 The Generalized Class

The Generalized Class was introduced for the first time by J. Ferrera and J. Gómez-Gil in the year 2018 (see [49]). It contains all the families of functions that we have previously studied in this thesis. In a sense, this generalization incorporates the intrinsic characteristics of the functions we have presented in this thesis so far.

When we were dealing with the Takagi-Van der Waerden function, we outlined that every r-adic interval $[x_n, y_n]$ of order n was divided into r closed subintervals of r^{-n} in length and the set $D_{n+1} \cap [x_n, y_n]$ consists of the endpoints of such subintervals. Observe that this behavior remains the same for every index n.

For the Generalized function that we are about to introduce, every interval of the form $[x_n, y_n]$ with $x_n, y_n \in D_n$ and $(x_n, y_n) \cap D_n = \emptyset$ is divided into ρ_n closed subintervals of equal length and the set $D_{n+1} \cap [x_n, y_n]$ consists of the endpoints of such subintervals. Here, the number of intervals in which an interval of the form $[x_n, y_n]$ is divided, depends on the index n.

In order to formalize this idea, we proceed as follows. Let $\mathbf{r} = (r_n)_n$ be a strictly increasing sequence of non-negative integers such that $r_1 = 1$ and r_n divides r_{n+1} for every n. We define the set

$$D_n = \left\{ \frac{k}{r_n} : k = 0, \ldots, r_n \right\}$$

for every $n \geq 1$. We observe that $D_n \subset D_{n+1}$ since r_n divides r_{n+1} for every n. With respect to the foregoing, it is important to highlight that we actually have $r_{n+1} = r_n \rho_n$ for every n. The Generalized function $f_{\mathbf{r}} : [0,1] \to \mathbb{R}$ is defined as

$$f_{\mathbf{r}}(x) = \sum_{n=1}^{\infty} g_n(x)$$

where $g_n(x) = \operatorname{dist}(x, D_n)$ denotes the distance from the point x to the set D_n. We also denote $D = \bigcup_{n=1}^{\infty} D_n$. As might be expected, we obtain the Takagi-Van der Waerden function f_r by taking $r_n = r^{n-1}$ for every n with $r \geq 2$ an integer.

As is customary, for every n we consider the set of all middle points between two consecutive points of D_n given by

$$\widetilde{D}_n = \left\{ \frac{x+y}{2} : x, y \in D_n \text{ with } (x,y) \cap D_n = \emptyset \right\},$$

which will be referred to as the set of middle points order n. Moreover, we denote by $\widetilde{D}$ the set of all middle points of the unit interval, that is $\widetilde{D} = \bigcup_{n=1}^{\infty} \widetilde{D}_n$.

Again, when we were dealing with the Takagi-Van der Waerden function we had that $\widetilde{D}_n \subset D_{n+1}$ for every n provided that r is an even integer, whereas $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n whenever r is an odd integer. For the Generalized function, for each index n we have $\widetilde{D}_n \subset D_{n+1}$ provided that ρ_n is an even integer, meanwhile $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ whenever ρ_n is an odd integer.

For every $n \geq 1$ the function g_n is a piecewise affine map whose nodes are located at the points belonging to the set $D_n \cup \widetilde{D}_n$, and it is periodic with period $1/r_n$. Moreover, we have that $g_n(x) = 0$ if and only if $x \in D_n$, meanwhile $g_n(x) = r_n^{-1}/2$ if and only if $x \in \widetilde{D}_n$. In addition, for each interval of the form $[x_n, y_n]$ with $x_n, y_n \in D_n$ and $(x_n, y_n) \cap D_n = \emptyset$, we have that g_k is affine on $[x_n, c_n]$ and on $[c_n, y_n]$ for all $1 \leq k \leq n$, where c_n is the midpoint of the interval $[x_n, y_n]$.

For a point $x \notin D_n \cup \widetilde{D}_n$, we denote by $a_n = a_n(x)$ the biggest element of D_n smaller than x and by $b_n = b_n(x)$ the least element of D_n bigger than x. Observe that $b_n = a_n + r_n^{-1}$. We also denote by $c_n = c_n(x)$ the midpoint between a_n and b_n, that is $c_n = (a_n + b_n)/2$. Recall that $g_n(x) = x - a_n$ if and only if $a_n \leq x \leq c_n$, whereas $g_n(x) = b_n - x$ if and only if $c_n \leq x \leq b_n$.

In the same line of thought as before, we may associate each Generalized function to a certain representation system for numbers belonging to the unit interval. However, this is not required for our purposes. We only need to take into consideration that for a point $x \notin D_n \cup \widetilde{D}_n$ we have $g_n'(x) = 1$ if and only if $a_n \leq x \leq c_n$, whereas $g_n'(x) = -1$ otherwise.

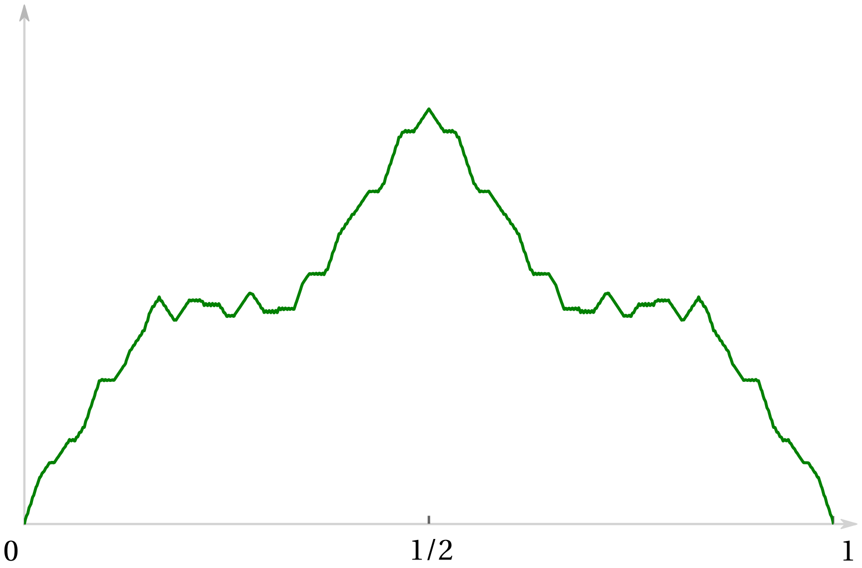

Figure 1.9: A Generalized function where $r_{n+1} = 4r_n$ whenever n is even, and $r_{n+1} = 3r_n$ otherwise

Following the same reasoning as M. Hata and M. Yamaguti, we introduce the so-called Generalized Class. For a given strictly increasing sequence of non-negative integers $\mathbf{r} = (r_n)_n$ satisfying that $r_1 = 1$ and r_n divides r_{n+1} for every n, the Generalized Class is formed by all the functions $f_{\mathbf{r},w} : [0,1] \to \mathbb{R}$ defined as

$$f_{\mathbf{r},w}(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights such that $(r_n^{-1} w_n)_n \in \ell_1$. This is an obvious generalization of the Takagi Class. When $\mathbf{r} = (r^{n-1})_n$ with $r \geq 2$ an integer, we obtain the so-called Takagi-Van der Waerden Class.

The Generalized Class will be investigated regarding Chapter 5. Nonetheless, there is still plenty of work to do regarding this generalization. Further research will be undertaken in the future.

CHAPTER 2

SECOND ORDER DIFFERENTIABILITY IN THE TAKAGI CLASS

The publication of M. Hata and M. Yamaguti's article drew the attention of many authors, who investigated several properties of the functions in the Takagi Class. In 1987, N. Kôno carried out a deep study of the differentiability properties of the functions in such a Class (see [79]). He obtained the following striking result:

Theorem 2.1. *For each function T_w belonging to the Takagi Class we have:*

(1) T_w is absolutely continuous with the derivative

$$T'_w(x) = \sum_{n=1}^{\infty} w_n g'_n(x)$$

almost everywhere, if and only if $w \in \ell_2$.

(2) T_w is derivable at an uncountable null set and the range of the derivative is all of $\mathbb{R}$, if and only if $w \in c_0 \setminus \ell_2$.

(3) T_w is nowhere derivable, if and only if $w \notin c_0$.

This reveals that three qualitatively different cases may arise depending on the sequence of weights we are considering. It is interesting to mention that this type of theorem can also be found in a very different setting. For instance, P. D. Lax showed that Pólya's space-filling curve, which maps the unit interval onto a solid right triangle, has the same three qualitative cases that appear in Kôno's theorem depending on the size of the smaller acute angle of the triangle (see [85]).

Behind Kôno's theorem lies the fact that a function T_w is derivable at $x \notin D$ if and only if the series of the derivatives $\sum_{n=1}^{\infty} w_n g_n'(x)$ converges. In such a case, we have

$$T_w'(x) = \sum_{n=1}^{\infty} w_n g_n'(x). \tag{2.1}$$

In 2020, Juan Ferrera and Javier Gómez Gil extended Kôno's theorem to a more general family of functions named the Generalized Takagi Class (see [49]). Under the assumption that $w \in c_0$, they obtained that the lateral derivatives of a function T_w are given by

$$T_w'^{+} = \sum_{j=1}^{\infty} w_j g_j'^{+} \quad \text{and} \quad T_w'^{-} = \sum_{j=1}^{\infty} w_j g_j'^{-}.$$

This brings us back to the behavior of the function g_n, which is a piecewise affine map whose nodes are located precisely at the points of D_{n+1}. In light of the discussion in Section 1.1 of Chapter 1, we recall the following result:

Lemma 2.2. *Let $x \in [0,1]$ with binary expansion given by $x = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n}$. For every n, we have the following:*

(1) When $x \in D_{n+1}$, we have:

- *(a) $g_n(x) = 0$ if and only if $x \in D_n$.*
- *(b) $g_n(x) = 2^{-n}$ if and only if $x \in D_{n+1} \setminus D_n$.*
- *(c) $g_n'^{+}(x) = 1$ and $g_n'^{-}(x) = -1$ if and only if $x \in D_n$.*
- *(d) $g_n'^{+}(x) = -1$ and $g_n'^{+}(x) = 1$ if and only if $x \in D_{n+1} \setminus D_n$*

(2) When $x \notin D_{n+1}$, we have:

- *(a) $g_n(x) = x - a_n$ and $g_n'(x) = 1$, if and only if $\varepsilon_n = 0$.*
- *(b) $g_n(x) = b_n - x$ and $g_n'(x) = -1$ if and only if $\varepsilon_n = 1$.*

In light of Kôno's theorem, it is natural to wonder what can be said when considering the case $w \in \ell_1$. That is the issue we will try to address in this chapter. In the sequel, and unless expressly stated otherwise, we assume that $w \in \ell_1$.

It is immediately observed that T_w is Lipschitz if and only if $w \in \ell_1$, and moreover the Lipschitz norm is given by $\|w\|_1$. Moreover, the assumption $w \in \ell_1$ yields that the series of the derivatives at every non-dyadic point converges, and consequently, the function T_w is derivable at every $x \notin D$ with the derivative given by (2.1). Concerning the differentiability at a dyadic point, we have that if T_w is differentiable at a point $x \in D_{n+1} \setminus D_n$ then

$$T_w'^{+}(x) = \sum_{j=1}^{n-1} w_j g_j'(x) - \Big(w_n - \sum_{j=n+1}^{\infty} w_j \Big), \text{ and}$$
$$T_w'^{-}(x) = \sum_{j=1}^{n-1} w_j g_j'(x) + \Big(w_n - \sum_{j=n+1}^{\infty} w_j \Big),$$

which implies

$$w_n = \sum_{j=n+1}^{\infty} w_j. \tag{2.2}$$

In addition, concerning the differentiability of the function T_w at every dyadic point, we have the following result:

Lemma 2.3. *The following statements are equivalent:*

1. *There is $M \in \mathbb{R}$ such that $w_n = M2^{-n}$ for every n.*
2. *T_w is a polynomial given by $T_w(x) = Mx(1-x)$ for some $M \in \mathbb{R}$.*
3. *T_w is derivable at every $x \in D$.*

Proof. That (1) implies (2) follows from the following elementary differential equation:

$$T_w'(x) = M \sum_{n=1}^{\infty} \frac{1-2\varepsilon_n}{2^n} = M(1-2x)$$

for every $x \notin D$. That (2) implies (3) is immediate. Finally, in order to see that (3) implies (1), we must observe that if T_w is differentiable at every $x \in D$, then by (2.2) we get $w_n = \sum_{j=n+1}^{\infty} w_j$ for every n . It is not hard to see that this gives

$$M = \sum_{j=1}^{\infty} w_j = \frac{M}{2} + \frac{M}{2^2} + \cdots + \frac{M}{2^{n-1}} + 2w_n$$

for every n, and the result follows immediately. □

It is interesting to recall that N. Kôno also proved that the only smooth function of the Takagi Class is obtained by taking $w_n = M2^{-n}$ for every n with $M \in \mathbb{R}$ (see Subsection 1.2.1). The proof of the following lemma is straightforward. It is enough to observe that if $x, y \in (k2^{-n}, (k+1)2^{-n})$ for some $n \geq 1$ and $k \in \{0, \dots, 2^n - 1\}$ then $\varepsilon_j(x) = \varepsilon_j(y)$, and hence $g_j'(x) = g_j'(y)$ for every $1 \leq j \leq n$.

Lemma 2.4. *Let $w \in \ell_1$. Then, $T_w'^+$ and $T_w'^-$ are continuous at every $x \notin D$. Moreover $T_w'^+$ (respectively, $T_w'^-$) is right continuous (respectively, left continuous) at every $x \in D$.*

Once we have determined the first order differentiability properties of the functions in the Takagi Class provided that $w \in \ell_1$, we will investigate several second order differentiability properties throughout this chapter. In this sense, our results will reveal a fourth qualitatively different case. Furthermore, we will prove that the assumption $w \in \ell_1$ constitutes the necessary framework where the study of such properties must be done.

2.1 Second order differentiability properties to study

Firstly, we investigate the convexity of a function in the Takagi Class. An important theorem due to A. D. Alexandrov states that a convex function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable almost everywhere (see [2] or Section 3.11 of [92]). It is well-known that if a function is twice differentiable at a point then it has Taylor expansion of order two at such a point. The converse assertion is not true.

A function $f : \mathbb{R} \to \mathbb{R}$ has a Taylor expansion of order two at x, whenever there exist two numbers d_x and A_x such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - d_x h - A_x h^2}{h^2} = 0.$$

It is immediately observed that if a function has a Taylor expansion of order two at x then it is derivable at x and we have $d_x = f'(x)$. Hence we may rewrite the condition as

$$A_x = \lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h}{h^2}.$$

If a function has a Taylor expansion of order two at a point x then it satisfies the Stepanoff condition of order two at such a point. This property is named after the Russian mathematician M. W. Stepanoff, who proved that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable almost everywhere in the set of points x satisfying

$$\limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} < +\infty.$$

This last condition is more commonly known today as the Stepanoff condition of order one (see [99] and [100]).

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Stepanoff condition of order two at x provided that there exists a constant d_x such that

$$\limsup_{h \to 0} \frac{|f(x+h) - f(x) - d_x h|}{h^2} < +\infty.$$

As above, it is immediately observed that if a function satisfies the Stepanoff condition of order two at x then it is derivable at x and we have $d_x = f'(x)$. Of course, a function that satisfies the Stepanoff condition of order two at a point, does not necessarily have a Taylor expansion of order two at such a point (see Example 2.41 below).

Although the Stepanoff condition of order two is a weaker property than a Taylor expansion of order two, a celebrated result of A. P. Calderón and A. Zygmund states that if a function satisfies the Stepanoff condition of order two almost everywhere then it has a Taylor expansion of order two almost everywhere, but not necessarily at the same points (see Example 2.40 below).

Another way to view order two differentiability at a point $x \notin D$ is to consider the derivative of $T_w'^+ : [0,1] \to \mathbb{R}$. This function is defined everywhere and it is continuous at every $x \notin D$ by Lemma 2.4. Hence, for every $x \notin D$ we may study the existence of

$$\left(T_w'^+\right)'(x) = \lim_{y \to x} \frac{T_w'^+(y) - T_w'^+(x)}{y - x}.$$

Observe also that the existence of $(T_w'^+)'(x)$ is equivalent to the existence of the limit

$$\lim_{y \notin D, y \to x} \frac{T_w'(y) - T_w'(x)}{y - x}$$

and if it exists then it is equal to$(T_w'^+)'(x)$. The nontrivial implication follows since $(T_w'^+)'(x)$ is right continuous everywhere by Lemma 2.4.

For an arbitrary function the existence of the derivative of the right-hand derivative at a point, and the existence of a Taylor expansion of order two at a point are independent conditions, since there exist derivable functions which have Taylor expansion of order two at points where they are not twice derivable; and although if a function has second derivative at x, then it has a Taylor expansion of order two at x, it is not clear that a similar result holds if we only require that the right derivative is derivable at x. The reason is that the Mean Value Theorem is not true for the right derivative.

2.2 Convexity

We investigate the convexity of a function in the Takagi Class. Since $w \in \ell_1$ we have that T_w is Lipschitz on $[0,1]$ and hence, it is absolutely continuous on $[0,1]$. Let $0 \leq a < b \leq 1$. The Fundamental Theorem of Calculus for Lebesgue Integrals (see [57]) gives

$$T_w(x) = \int_a^x T_w'(t)\,dt = \int_a^x T_w'^+(t)\,dt$$

for every $x \in (a,b)$. Therefore, if $T_w'^+$ is non-decreasing on (a,b) then T_w is convex on (a,b) (see Section 1.6. of [92]). Conversely, if T_w is convex on (a,b) then

$$T_w'^-(x) \leq T_w'^+(x) \leq T_w'^-(y) \leq T_w'^+(y) \tag{2.3}$$

whenever $x, y \in (a,b)$ and $x<y$. In particular, both $T_w'^+$ and $T_w'^-$ are non-decreasing on (a,b) (see Theorem 1.3.3. of [92]). In summary, the function T_w is convex on (a,b) if and only if $T_w'^+$ is non-decreasing on (a,b). This allows us to obtain the first result of this section.

Proposition 2.5. *Let $w \in \ell_1$. Then, the function T_w is piecewise convex if and only if there exists $n_0 \geq 1$ such that*

$$w_n \leq \sum_{j=n+1}^{\infty} w_j \leq 0 \tag{2.4}$$

for every $n \geq n_0$.

Proof. First, we assume that T_w is piecewise convex on $[0,1]$. Let n be big enough such that T_w is convex while restricted to $[x_n, y_n]$ for some $x_n, y_n \in D_n$ such that $(x_n, y_n) \cap D_n = \emptyset$. We denote by c_n the midpoint of such interval. We have

$$T_w'^+(c_n) = \sum_{j=1}^{n-1} w_j g_j'(c_n) - w_n + \sum_{j=n+1}^{\infty} w_j \quad \text{and}$$
$$T_w'^-(c_n) = \sum_{j=1}^{n-1} w_j g_j'(c_n) + w_n - \sum_{j=n+1}^{\infty} w_j.$$

Since $T_w'^-(c_n) \leq T_w'^+(c_n)$ by (2.3), we obtain that $w_n \leq \sum_{j=n+1}^{\infty} w_j$. Furthermore, we have that $T_w'^+(x_n) \leq T_w'^-(y_n)$ by (2.3) again. Since

$$T_w'^+(x_n) = \sum_{j=1}^{n-1} w_j g_j'^+(x_n) + \sum_{j=n}^{\infty} w_j \quad \text{and}$$
$$T_w'^-(y_n) = \sum_{j=1}^{n-1} w_j g_j'^-(y_n) - \sum_{j=n}^{\infty} w_j,$$

we get $\sum_{j=n}^{\infty} w_j \leq 0$ because $g_j'^+(x_n) = g_j'^-(y_n)$ for every $1 \leq j \leq n-1$.

Conversely, we assume that Condition (2.4) holds for some $n_0 \geq 1$. It is enough to prove that T_w is convex while restricted to each connected component of $[0,1] \setminus D_{n_0}$. Let I be one of those connected components and we prove that $T_w'^+$ is non-decreasing on I. Indeed, if $x, y \in I$ and $x<y$ then there exists $m \geq n_0$ such that $\varepsilon_j(x) = \varepsilon_j(y)$ for $j<m$, whereas $\varepsilon_m(x) = 0$ and $\varepsilon_m(y) = 1$. Hence

$$\begin{aligned} T_w'^+(y) - T_w'^+(x) &= 2\sum_{n \geq m} w_n \big(\varepsilon_n(x) - \varepsilon_n(y)\big) \\ &= -2w_m + 2\sum_{n>m} w_n \big(\varepsilon_n(x) - \varepsilon_n(y)\big) \geq 2\left(-w_m + \sum_{n>m} w_n\right) \geq 0. \end{aligned}$$

□

Observe that a sequence $w = (w_n)_n$ satisfies Condition (2.4) for some $n_0 \geq 1$, if and only if

$$\sum_{j=n}^{\infty} w_j \leq 2 \sum_{j=n+1}^{\infty} w_j \leq 0$$

for every $n \geq n_0$, and equivalently

$$2^n \sum_{j=n}^{\infty} w_j \leq 2^{n+1} \sum_{j=n+1}^{\infty} w_j \leq 0$$

for every $n \geq n_0$. Therefore, we obtain the following result:

Theorem 2.6. *Let $w \in \ell_1$. Then, the function T_w is piecewise convex if and only if there exists $n_0 \geq 1$ such that*

the sequence

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0} \tag{CX}$$

is non-positive and non-decreasing.

Proceeding in the same way, we obtain the following result concerning the concavity of a function in the Takagi Class.

Theorem 2.7. *Let $w \in \ell_1$. Then, the function T_w is piecewise concave if and only if there exists $n_0 \geq 1$ such that the sequence*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_{n \geq n_0} \tag{CC}$$

is non-negative and non-increasing.

It should be pointed out that if a sequence $w = (w_n)_n$ satisfies the Condition (CX) or (CC) for some $n_0 \geq 1$, then the sequence $(2^n w_n)_n$ converges since it can be written as a difference of two convergent sequences.

Corollary 2.8. *Let $w \in \ell_1$. If the function T_w is piecewise convex or piecewise concave, then the sequence $(2^n w_n)_n$ converges.*

2.3 Taylor expansion of order two at a point

The aim of this section is to prove that the existence of a Taylor expansion of order two at a non-dyadic point, and the differentiability of the lateral derivative at such a point are properties that agree for the functions in the Takagi Class. Furthermore, we may characterize them in terms of the sequence $w = (w_n)_n$ and the binary expansion of such a non-dyadic point. Before proceeding, we observe the following reduction of the problem:

Lemma 2.9. *Let $x \notin D$ and $w \in \ell_1$. Then, we have*

$$\lim_{h\to 0} \frac{T_w(x+h) - T_w(x) - T'_w(x)h}{h^2} = A_x$$

if and only if

$$\lim_{h\to 0} \frac{T_v(x+h) - T_v(x) - T'_v(x)h}{h^2} = 0$$

where $v = (v_n)_n$ is defined by $v_n = w_n + A_x 2^{-n}$.

Proof. It is enough to observe that Lemma 2.3 gives that $T_v(z) = T_w(z) + A_x(z - z^2)$ for all $z \in [0,1]$, and the polynomial $P(z) = A_x(z - z^2)$ satisfies $P''(z) = -2A_x$. □

We deduce some consequences of the existence of a Taylor expansion of order two at a point.

Proposition 2.10. *Let $w \in \ell_1$. If T_w has a Taylor expansion of order two at a point $x \in [0,1]$, then $A_x = -\lim_n 2^n w_n$.*

Proof. By Lemma 2.9 we have

$$\lim_{h\to 0} \frac{T_v(x+h) - T_v(x) - T'_v(x)h}{h^2} = 0 \tag{2.5}$$

where $v = (v_n)_n$ is defined by $v_n = w_n + A_x 2^{-n}$. Since $w_n = -A_x 2^{-n} + v_n$, it is enough to prove that $\lim_n v_n 2^n = 0$.

Assume first that $x \notin D$. For every n, we consider $x \in (a_n, b_n)$ with $a_n, b_n \in D_n$ and $(a_n, b_n) \cap D_n = \emptyset$. From (2.5) and

$$\left|\frac{T_v(b_n) - T_v(a_n) - T'_v(x)(b_n - a_n)}{(b_n - a_n)^2}\right| \le \left|\frac{T_v(b_n) - T_v(x) - T'_v(x)(b_n - x)}{(b_n - x)^2}\right| + \left|\frac{T_v(a_n) - T_v(x) - T'_v(x)(a_n - x)}{(a_n - x)^2}\right|$$

we deduce

$$\lim_n \frac{T_v(b_n) - T_v(a_n) - T'_v(x)(b_n - a_n)}{(b_n - a_n)^2} = 0.$$

Since

$$\frac{T_v(b_n)-T_v(a_n)-T_v'(x)(b_n-a_n)}{(b_n-a_n)^2} = 2^{2(n-1)}\sum_{j=n}^{\infty} v_j\big(-g_j'(x)(b_n-a_n)\big)$$
$$= -2^{n-1}\sum_{j=n}^{\infty} v_j g_j'(x)$$

we have

$$\lim_n 2^n \sum_{j=n}^{\infty} v_j g_j'(x) = 0,$$

and consequently

$$\lim_n 2^n v_n g_n'(x) = \lim_n 2^n \sum_{j=n}^{\infty} v_j g_j'(x) - \frac{1}{2}\lim_n 2^{n+1}\sum_{j=n+1}^{\infty} v_j g_j'(x) = 0,$$

which implies that $\lim_n 2^n v_n = 0$.

Finally, when $x \in D$ it is enough to observe that

$$\frac{T_v(x+2^{-n})-T_v(x)-T_v'(x)2^{-n}}{2^{-2n}} = -2^n \sum_{j=n+1}^{\infty} v_j$$

provided that n is big enough, and the result follows as above. □

In view of Proposition 2.10 above, keep in mind that if the function T_w is piecewise convex, then the sequence $(2^n w_n)_n$ converges by Corollary 2.8. The existence of the limit $L = \lim_n 2^n w_n$ is equivalent to each of the following statements:

$$\begin{aligned} &(1)\quad w_n = L2^{-n} + v_n \text{ with } \lim_n 2^n v_n = 0.\\ &(2)\quad \lim_n 2^n \textstyle\sum_{k=n+1}^{\infty} w_k = L. \end{aligned} \tag{2.6}$$

Regarding the dyadic case, we have the following result:

Proposition 2.11. *Let $x \in D$ and $w \in \ell_1$. Assume that* $\lim_n 2^n w_n$ *exists. Then, T_w has a Taylor expansion of order two at x if and only if T_w is derivable at x.*

Proof. Assume that $x \in D_{n+1} \setminus D_n$ for some n. Recall that if T_w has a Taylor expansion of order two at x, then T_w is derivable at x. Conversely, suppose that T_w is derivable at x. We have

$$w_n = \sum_{j=n+1}^{\infty} w_j \quad \text{and} \quad T_w'(x) = \sum_{j=1}^{n-1} w_j g_j'(x). \tag{2.7}$$

By Lemma 2.3 and the first statement of (2.6), we may assume that $\lim_m 2^m w_m = 0$. Given $\varepsilon > 0$ there exists $m_0 \geq 1$ such that $|w_m| < \varepsilon 2^{-m}$ for every $m \geq m_0$. For $h \in \mathbb{R}$ small enough there exists $m \geq \max\{m_0, n\}$ satisfying that $2^{-(m+1)} < |h| \leq 2^{-m}$. We have that $g_k(x+h) - g_k(x) - g_k'(x)h = 0$ since g_k is affine between x and $x+h$ for every $k < n$. Moreover, $g_n(x+h) - g_n(x) = -|h|$ since $x \in D_{n+1} \setminus D_n$ and we also have $g_j(x+h) = |h|$ for every $n+1 \leq j \leq m$. Thus, by (2.7) we get

$$\begin{aligned}\frac{T_w(x+h) - T_w(x) - T_w'(x)h}{h^2} &= \frac{1}{h^2}\left[w_n(g_n(x+h) - g_n(x)) + \sum_{j=n+1}^{\infty} w_j g_j(x+h)\right] \\ &= \frac{1}{h^2}\left[-|h| w_n + \sum_{j=n+1}^{\infty} w_j g_j(x+h)\right] \\ &= \frac{1}{h^2}\sum_{j=n+1}^{\infty} w_j (g_j(x+h) - |h|) \leq \frac{1}{|h|}\sum_{j=m+1}^{\infty} |w_j| \leq 2\varepsilon.\end{aligned}$$

□

Regarding the non-dyadic case, our main concern is to characterize when the function T_w has a Taylor expansion of order 2 at a non-dyadic point in terms of a property that the binary expansion of such a point must satisfy. To do so, we introduce the following notation.

As is customary, for a point $x \notin D$ we consider its binary expansion given by $x = \sum_n \varepsilon_n 2^{-n}$ with $\varepsilon_n \in \{0,1\}$. We define the set

$$\mathcal{N}_x = \{n \in \mathbb{N} : \varepsilon_n \neq \varepsilon_{n+1}\}.$$

and for every n we define $\ell_n = \ell_n(x)$ as the nonnegative integer satisfying

$$\varepsilon_n = \cdots = \varepsilon_{n+\ell_n} \neq \varepsilon_{n+\ell_n+1},$$

namely ℓ_n is the length of the run of digits equal to ε_n starting at ε_n. Here $\ell_n = 0$ if $\varepsilon_n \neq \varepsilon_{n+1}$, so we have that $\mathcal{N}_x$ can also be defined as the set of indices n such that $\ell_n = 0$.

Theorem 2.14 below is the main result of this section. It states that the existence of a Taylor expansion of order two at a point $x \notin D$ is equivalent to the existence of the derivative of $T_w'^{+}$ at such a point. The following lemma will enable us to simplify its proof.

Lemma 2.12. *Let $x \notin D$ and $\mathcal{N}_x = \{n \in \mathbb{N} : \varepsilon_n \neq \varepsilon_{n+1}\}$. For every $n \in \mathcal{N}_x$ we consider $h_n = g_n'(x)2^{-(n+\ell_{n+1})}$ and the following statements hold:*

(1) $g_j(x+h_n) - g_j(x) - g_j'(x)h_n = 0$ *for every* $j < n$.

(2) $g_n(x+h_n) + g_j(x+h_n) = 2^{-n} = g_n(x) + g_j(x)$ *for every* $n+1 \leq j \leq n+\ell_{n+1}$.

(3) $g_j(x+h_n) - g_j(x) - g_j'(x)h_n = -(g_n(x+h_n) - g_n(x) - g_n'(x)h_n)$ *for every* $n+1 \leq j \leq n+\ell_{n+1}$.

(4) $-2|h_n| < g_n(x+h_n) - g_n(x) - g_n'(x)h_n < -|h_n|$.

(5) $g_j(x+h_n) = g_j(x)$ *for every* $j > n + \ell_{n+1}$.

Proof. For every n, we consider $x \in (a_n, b_n)$ with $a_n, b_n \in D_n$ and $(a_n, b_n) \cap D_n = \emptyset$. We also denote by c_n the midpoint of the interval (a_n, b_n).

(1) It follows since $x, x+h_n \in (a_n, b_n)$, and consequently g_j is affine between x and $x+h_n$ for every $j < n$.

(2) If $\varepsilon_n(x) = 0$, which implies $g_n'(x) = 1$, then $\varepsilon_{n+1}(x) = 1$ because $n \in \mathcal{N}_x$. Moreover, we have $\varepsilon_{n+1}(x) = \cdots = \varepsilon_{n+\ell_{n+1}}(x) = 1$ and hence

$$g_n(x) + g_j(x) = x - a_n + (c_n - x) = c_n - a_n = 2^{-n}$$

for every $n+1 \le j \le n+\ell_{n+1}$. Now, since $\varepsilon_n(x+h_n) = 1$ and $\varepsilon_{n+1}(x+h_n) = \cdots = \varepsilon_{n+\ell_{n+1}}(x+h_n) = 0$ we obtain

$$g_n(x+h_n) + g_j(x+h_n) = b_n - (x+h_n) + x + h_n - c_n = b_n - c_n = 2^{-n}.$$

The case when $\varepsilon_n = 1$ is similar.

(3) It is an immediate consequence of case (2) since $g_n'(x) = -g_j'(x)$ for every $n+1 \le j \le n+\ell_{n+1}$.

(4) If $\varepsilon_n(x) = 0$ then $\varepsilon_n(x+h_n) = 1$ and hence

$$\begin{aligned} g_n(x+h_n) - g_n(x) - g_n'(x)h_n &= b_n - (x+h_n) - (x - a_n) - h_n \\ &= 2(c_n - x) - 2h_n \le -h_n \end{aligned}$$

since $c_n - x \le 2^{-(n+1+\ell_{n+1})}$. If $\varepsilon_n(x) = 1$ then $\varepsilon_n(x+h_n) = 0$ and hence

$$\begin{aligned} g_n(x+h_n) - g_n(x) - g_n'(x)h_n &= x + h_n - a_n - (b_n - x) + h_n \\ &= 2(x - c_n) - 2|h_n| \le -|h_n| \end{aligned}$$

since $x - c_n \le 2^{-(n+1+\ell_{n+1})}$.

(5) It follows since the function g_j is periodic with period $2^{-(j-1)}$ for every $j > n + \ell_{n+1}$.

□

Lemma 2.13. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. If $A = \{x \in (0,1) : f'(x)$ exists$\}$, $x \in A$ and there exists*

$$\lim_{\substack{y\to x\\ y\in A}} \frac{f'(y)-f'(x)}{y-x} = L,$$

then

$$\lim_{y\to x} \frac{f(y)-f(x)-f'(x)(y-x)}{(y-x)^2} = \frac{L}{2}.$$

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(y)-f'(x)-L(y-x)| < \varepsilon|y-x|$$

if $y \in A$ and $|y-x| < \delta$. The absolute continuity of f implies

$$\frac{\left|f(y)-f(x)-f'(x)(y-x)-\frac{1}{2}L(y-x)^2\right|}{(y-x)^2} = \frac{\left|\int_x^y \left(f'(t)-f'(x)-L(t-x)\right) dt\right|}{(y-x)^2} \le \varepsilon.$$

□

With respect to the sequence $w = (w_n)_n$, as we saw above in Proposition 2.10, if T_w has a Taylor expansion of order two at some point then the sequence $(2^n w_n)_n$ converges. This fact allows us to define the sequence $\beta = (\beta_n)_n$ as

$$\beta_n = 2^n \left(-w_n + \sum_{j=n+1}^{\infty} w_j\right)$$

for every n. The second statement of (2.6) yields that this sequence converges to 0.

Theorem 2.14. *Let $w \in \ell_1$, $x \notin D$ and $\mathcal{N}_x = \{n \in \mathbb{N} : \varepsilon_n \neq \varepsilon_{n+1}\}$. The following statements are equivalent:*

(1) $(T_w'^{+})'(x)$ exists.

(2) T_w has a Taylor expansion of order two at x.

(3) The sequence $(2^n w_n)_n$ converges and

$$\lim_{n\in\mathcal{N}_x} 2^{\ell_{n+1}} \beta_n = 0. \tag{2.8}$$

Proof. That (1) implies (2) follows from Lemma 2.13 since T_w is absolutely continuous.

We prove that (2) implies (3). Suppose that T_w has a Taylor expansion of order two at x. From Proposition 2.10 we know that $A_x = -\lim_n 2^n w_n$. Without loss of generality, we may assume that $\lim_n 2^n w_n = 0$. This is due to the fact that if $(2^n w_n)_n$ does not converge to zero, then by Lemma 2.9 we have

$$\lim_{h\to 0} \frac{T_v(x+h) - T_v(x) - T_v'(x)h}{h^2} = 0$$

where $v = (v_n)_n$ is defined by $v_n = w_n + A_x 2^{-n}$. Thus, we obtain $\lim_n 2^n v_n = 0$. Now, we observe that if we prove (2.8) for the sequence v then we get the same result for w since

$$-w_n + \sum_{j=n+1}^{\infty} w_j = -v_n + \sum_{j=n+1}^{\infty} v_j$$

for every n.

Let us consider $n \in \mathcal{N}_x$. If $\ell_{n+1} = 0$ we have that $2^{\ell_{n+1}}\beta_n = \beta_n$ and we know that $\lim_n \beta_n = 0$. If $\ell_{n+1} > 0$ then we take $h_n = g_n'(x) 2^{-(n+\ell_{n+1})}$ and we have

$$\begin{aligned}
&\frac{T_w(x+h_n) - T_w(x) - T_w'(x)h_n}{h_n^2} \\
&= 2^{2(n+\ell_{n+1})} \sum_{j=n}^{n+\ell_{n+1}} w_j\big(g_j(x+h_n) - g_j(x) - h_n g_j'(x)\big) - 2^{n+\ell_{n+1}} \sum_{j=n+1+\ell_{n+1}}^{\infty} w_j g_j'(x) \\
&= 2^{2(n+\ell_{n+1})}\big(g_n(x+h_n) - g_n(x) - h_n g_n'(x)\big)\left(w_n - \sum_{j=n+1}^{n+\ell_{n+1}} w_j\right) - 2^{n+\ell_{n+1}} \sum_{j=n+1+\ell_{n+1}}^{\infty} w_j g_j'(x)
\end{aligned}$$

where we have used the statements (1), (3) and (5) of Lemma 2.12. Now, we define

$$s_n = -2^{n+\ell_{n+1}}\left(g_n(x+h_n) - g_n(x) - h_n g_n'(x)\right),$$

we have that $s_n \in (1,2)$ by statement (4) of Lemma 2.12. Hence, taking limits with $\ell_{n+1} > 0$ we obtain

$$\begin{aligned}
0 &= \lim_{n\in\mathcal{N}_x} 2^{\ell_{n+1}} s_n 2^n \left(-w_n + \sum_{j=n+1}^{n+\ell_{n+1}} w_j\right) \\
&= \lim_{n\in\mathcal{N}_x} s_n 2^{\ell_{n+1}}\beta_n - \lim_{n\in\mathcal{N}_x} s_n 2^{n+\ell_{n+1}} \sum_{j=n+1+\ell_{n+1}}^{\infty} w_j = \lim_{n\in\mathcal{N}_x} s_n 2^{\ell_{n+1}}\beta_n.
\end{aligned}$$

Therefore we conclude $\lim_{n\in\mathcal{N}_x} 2^{\ell_{n+1}}\beta_n = 0$.

We prove that (3) implies (1). By Lemma 2.9 we may assume that $\lim_n 2^n w_n = 0$, which implies $\lim_n 2^n \sum_{j=n}^{\infty}|w_j| = 0$. We will prove that $(T_w'^+)'(x) = 0$. Let $2^{-(n+1)} < h \le 2^{-n}$ and suppose that n is big enough such that $\varepsilon_j = 0$ for some $j < n$. If $\varepsilon_n = 0$ then

$$\left|\frac{T_w'^+(x+h) - T_w'^+(x)}{h}\right| = \left|\sum_{j=n}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| \le 2^{n+2}\sum_{j=n}^{\infty}|w_j|$$

which tends to 0 as $n \to \infty$. If $\varepsilon_n = 1$, then we denote by $m(n)$ the maximum of the set $\{j < n : \varepsilon_j = 0\}$. We have that $m(n) \in \mathcal{N}_x$ and $m(n) + 1 + \ell_{m(n)+1} \geq n$. Therefore,

$$\begin{aligned}
&\left|\frac{T_w'^{+}(x+h) - T_w'^{+}(x)}{h}\right| = \left|\sum_{j=m(n)}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| \\
&\leq \left|\sum_{j=m(n)}^{n} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| + \left|\sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| \\
&= \frac{2}{h}\left|-w_{m(n)} + \sum_{j=m(n)+1}^{n} w_j\right| + \left|\sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| \\
&\leq 2^{n+2}\left|-w_{m(n)} + \sum_{j=m(n)+1}^{\infty} w_j\right| + 2^{n+2}\sum_{j=n+1}^{\infty} |w_j| + \left|\sum_{j=n+1}^{\infty} w_j \frac{g_j'(x+h) - g_j'(x)}{h}\right| \\
&\leq 2^{n+2}\left|-w_{m(n)} + \sum_{j=m(n)+1}^{\infty} w_j\right| + 2^{n+3}\sum_{j=n+1}^{\infty} |w_j| \\
&\leq 2^{m(n)+\ell_{m(n)+1}+3}\left|-w_{m(n)} + \sum_{j=m(n)+1}^{\infty} w_j\right| + 2^{n+3}\sum_{j=n+1}^{\infty} |w_j| \\
&= 2^{\ell_{m(n)+1}+3}|\beta_{m(n)}| + 2^{n+3}\sum_{j=n+1}^{\infty} |w_j|
\end{aligned}$$

which tends to 0 as $n \to \infty$. The proof for $h < 0$ is similar. □

It should be noted that if we are assuming the hypotheses of Theorem 2.14 for a point $x \notin D$, then $(T_w'^{+})'(x) = 2A_x$ by Lemma 2.13 where

$$A_x = \lim_{h \to 0} \frac{T_w(x+h) - T_w(x) - T_w'(x)h}{h^2} = -\lim_{n} 2^n w_n$$

by Proposition 2.10.

As a consequence of Theorem 2.14, we can always find non-dyadic points at which T_w does not have a Taylor expansion of order two, unless we have $\beta_n = 0$ eventually. If the latter is the case, then there is a constant M such that $w_n = M2^{-n}$ eventually. Nevertheless, the convergence of the sequence $(2^n w_n)_n$ guarantees that there are many points where T_w has a Taylor expansion of order two.

For every $N \in \mathbb{N}$ we define the set

$$\mathscr{A}_N = \{x \notin D : \ell_n(x) \leq N \text{ for every } n\}.$$

and we also define

$$\mathscr{A} = \bigcup_{N=1}^{\infty} \mathscr{A}_N.$$

Thus, $\mathscr{A}$ is the set of all $x \notin D$ such that the sequence $(\ell_n)_n$ is bounded. It is not difficult to see that $\mathscr{A}$ is a Lebesgue null set with Hausdorff dimension one (see [26] for instance). We observe that if the sequence $(2^n w_n)_n$ converges then Condition (2.8) holds for every point of the set $\mathscr{A}$. Therefore, we have the following result:

Corollary 2.15. *If $w = (w_n)_n$ satisfies that $(2^n w_n)_n$ converges, then T_w has a Taylor expansion of order two at a Lebesgue null set with Hausdorff dimension one.*

2.4 How big is the set where T_w has a Taylor expansion of order two?

It is well-known that if a function has bounded variation, then it is derivable almost everywhere (see [57]). In particular, if $T_w'^+$ has bounded variation then $T_w'^+$ is derivable almost everywhere, which is equivalent to the existence of a Taylor expansion of order two almost everywhere by Theorem 2.14. Of course this situation happens when T_w is piecewise concave or piecewise convex, and more generally when T_w is the difference of two piecewise convex functions. For the functions in the Takagi class, we will see that the converse assertion is true in the following strong sense: $T_w'^+$ has bounded variation if and only if T_w is the difference of two piecewise convex functions belonging to the Takagi class (see Corollary 2.23 below).

Recall that if we have a partition of the interval $[0,1]$ given by $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, then the variation of a function $f : [0,1] \to \mathbb{R}$ over the partition P is defined by

$$V(f,P) = \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|.$$

The total variation of f over $[0,1]$ is defined by

$$V(f) = \sup_{P \subset [0,1]} V(f,P)$$

where the supremum is taken over all partitions P of the interval $[0,1]$. The function f has bounded variation if $V(f) < \infty$. We refer the reader to [57] for more information about functions of bounded variation.

Our next goal is to characterize when $T_w'^+$ has bounded variation in terms of a property that the sequence $w = (w_n)_n$ must satisfy.

Since the set D of all dyadic numbers is dense in $[0,1]$ and $T_w'^+$ is continuous at every $x \notin D$, the total variation of $T_w'^+$ on $[0,1]$ is given by

$$V(T_w'^+) = \sup_{P \subset D} V(T_w'^+, P)$$

where the supremum is taken over all partitions P of the interval $[0,1]$ whose elements are dyadic numbers. For a given partition $P \subset D$ we have that $P \subset D_n$ for some n, so we must compute the value $V(T_w'^+, D_n)$, namely the variation of $T_w'^+$ with respect to the partition given by the set D_n. To do so, we introduce the following notation.

For every n we denote by $\mathscr{F}_n$ the family of connected components of $[0,1] \setminus D_n$. We should note that every interval in $\mathscr{F}_n$ has an endpoint belonging to $D_n \setminus D_{n-1}$, whereas the other endpoint belongs to some set D_k with $k < n$. Thus, we denote by $\mathscr{F}_{n,k} \subset \mathscr{F}_n$ the family of intervals with one endpoint belonging to $D_k \setminus D_{k-1}$ where $D_0 = \emptyset$. We have that $\mathscr{F}_n = \bigcup_{k=1}^{n-1} \mathscr{F}_{n,k}$. It is worth pointing out that $\mathscr{F}_{n,1}$ consists of two intervals, and $\mathscr{F}_{n,k}$ consists of 2^{k-1} intervals provided that $1 < k < n$.

Although the study of the function T_w takes place in the interval $[0,1]$, we may consider that the function T_w is extended periodically to the whole real line. Therefore, we have

$$T_w'^+(1) = \sum_{k=1}^{\infty} w_k.$$

Lemma 2.16. *For the partition $\mathscr{P}_n = D_n$ we have*

$$V(T_w'^+, D_n) = \sum_{(a,b)\in\mathscr{F}_n} \left|T_w'^+(b) - T_w'^+(a)\right| = 2^{n-1}|w_{n-1}|$$
$$+ 2|w_1 + \cdots + w_{n-1}| + \sum_{k=2}^{n-1} 2^{k-1}| - w_{k-1} + w_k + \cdots + w_{n-1}|.$$

Proof. Let $(a,b) \in \mathscr{F}_n$ and assume that $(a,b) \in \mathscr{F}_{n,k}$ with $k \geq 2$.

Firstly, we consider the case $a \in D_n \setminus D_{n-1}$ and $b \in D_k \setminus D_{k-1}$. Note that there are exactly 2^{k-2} such pairs. We have

$$\left|T_w'^+(b) - T_w'^+(a)\right| = \left|\sum_{j=k-1}^{n-1} w_j\big(g_j'^+(b) - g_j'^+(a)\big)\right|$$

since $g_j'^+(b) = g_j'^+(a) = 1$ for every $j \geq n$, and $a, b \in J$ for some interval $J \in \mathscr{F}_{k-1}$, which implies that $g_j'^+(a) = g_j'^+(b)$ provided that $j < k-1$. For the rest of the indices we have that $g_{k-1}'^+(b) = -1$ and $g_{k-1}'^+(a) = 1$, meanwhile $g_j'^+(b) = 1$ and $g_j'^+(a) = -1$ for every $k \leq j < n$. Hence, we get

$$\left|T_w'^+(b) - T_w'^+(a)\right| = 2| - w_{k-1} + w_k + \cdots + w_{n-1}|.$$

Secondly, we consider the case $b \in D_n \setminus D_{n-1}$ and $a \in D_k \setminus D_{k-1}$. As before, note that there are exactly 2^{k-2} such pairs. We have $g_j'^+(b) = g_j'^+(a) = 1$ for every $j \geq n$ and $g_j'^+(a) = g_j'^+(b)$ for every $j < k-1$ as above. However, now we have $g_{k-1}'^+(b) = g_{k-1}'^+(a) = -1$ and $g_j'^+(b) = g_j'^+(a) = 1$ for every $k \leq j < n-1$, which gives

$$\left|T_w'^+(b) - T_w'^+(a)\right| = |w_{n-1}(g_{n-1}'^+(b) - g_{n-1}'^+(a))| = | - 2w_{n-1}|.$$

Finally, when $k = 1$ we have

$$\left|T_w'^+(2^{-n+1}) - T_w'^+(0)\right| = |-2w_{n-1}|$$

and

$$|T_w'^+(1) - T_w'^+(1-2^{-n+1})| = 2|w_1 + \cdots + w_{n-1}|.$$

The result follows by adding all items. $\square$

Proposition 2.17. *The function $T_w'^+$ has bounded variation if and only if*

$$\limsup_n \left(2^{n-1}|w_{n-1}| + \sum_{k=2}^{n-1} 2^{k-1}|-w_{k-1} + w_k + \cdots + w_{n-1}|\right) < +\infty. \tag{2.9}$$

Proof. The necessity of this condition follows easily from Lemma 2.16. Conversely, we first observe that condition (2.9) implies $w \in \ell_1$. As outlined before, we have

$$V(T_w'^+) = \sup_{P \subset D} V(T_w'^+, P)$$

and for a given a partition $P \subset D$, we have that $P \subset D_n$ for some n. Therefore, the result follows from Lemma 2.16 since $2|w_1 + \cdots + w_{n-1}| \le 2\,\|w\|_1$. $\square$

As we saw in Theorem 2.14, if $(T_w'^+)'(x)$ exists for some point $x \notin D$ then the sequence $(2^n w_n)_n$ converges. In such a case, the second statement of (2.6) yields

$$\lim_n 2^n w_n = \lim_n 2^n \sum_{k=n+1}^{\infty} w_k.$$

Furthermore, since

$$\left|\sum_{k=1}^{n} |\beta_k| - \sum_{k=1}^{n} 2^k \left|-w_k + \sum_{j=k+1}^{n} w_j\right|\right| \le \sum_{k=1}^{n} 2^k \left|\sum_{j=n+1}^{\infty} w_j\right| \le 2^{n+1}\left|\sum_{j=n+1}^{\infty} w_j\right|$$

we obtain

$$\sum_{k=2}^{\infty} 2^{k-1}|-w_{k-1} + w_k + \cdots + w_{n-1}| < +\infty$$

if and only if the series

$$\sum_{k=1}^{\infty} |\beta_k| = \sum_{k=1}^{\infty} 2^k \left|-w_k + \sum_{j=k+1}^{\infty} w_j\right|$$

converges. Additionally, we observe that the sequence $(2^n w_n)_n$ converges provided that $\beta \in \ell_1$. Therefore, we have obtained the following result:

Theorem 2.18. *The function $T_w'^+$ has bounded variation if and only if $\beta \in \ell_1$.*

Recall that if $T_w'^+$ has bounded variation then it is derivable almost everywhere, which is equivalent to saying that T_w has a Taylor expansion of order two almost everywhere by Theorem 2.14. The following result reveals that the converse assertion is true for the functions in the Takagi Class.

Theorem 2.19. *The function T_w has a Taylor expansion of order two almost everywhere if and only if $\beta \in \ell_1$.*

Proof. If $\beta \in \ell_1$ then $T_w'^+$ has bounded variation, and consequently T_w has a Taylor expansion of order 2 a.e. Conversely, assume that T_w has a Taylor expansion of order two almost everywhere. We know that the sequence $(2^n w_n)$ converges by Theorem 2.14, and hence we obtain $\beta \in c_0$. For the sake of contradiction assume that $\beta \notin \ell_1$. For every n, we define the set

$$\mathscr{B}_n = \left\{x \in [0,1] : \ell_n(x) \geq \varrho_n,\ \varepsilon_{n-1}(x) \neq \varepsilon_n(x)\right\}$$

where $\varrho_n = -\log_2 |\beta_{n-1}|$. We also define

$$\mathscr{B} = \bigcap_{n=1}^{\infty} \left(\bigcup_{j=n}^{\infty} \mathscr{B}_j \right) = \limsup_n \mathscr{B}_n.$$

In Example 6.6 of [26], a refinement of the second Borel-Cantelli lemma is used to prove that for a sequence of positive reals $(t_n)_n$, the set $\{x \in [0,1] : \ell_n(x) \geq t_n$ infinitely often $\}$ has measure one provided that the series $\sum_{n=1}^{\infty} 2^{-t_n} = \infty$. Therefore, we conclude that $\mathscr{B}$ has measure one since the series $\sum_{n=1}^{\infty} 2^{-\varrho_n}$ diverges.

Finally, we observe that if $x \in \mathscr{B}$ then $\ell_n(x) \geq \varrho_n$ and $n-1 \in \mathscr{N}_x$ for infinitely many indices n, and hence

$$\limsup_{n \in \mathscr{N}_x} 2^{\ell_{n+1}} |\beta_n| \geq 1.$$

By Theorem 2.14 we get that T_w fails to have a Taylor expansion of order two at every point of $\mathscr{B}$, which is a set of measure one. This contradicts our assumption. □

As we may note, the condition $\beta \in \ell_1$ plays an important role in our framework. It is a condition that only involves the sequence of weights $w = (w_n)_n$. We can go further than this. We can ask what kind of property of the sequence w is behind the condition $\beta \in \ell_1$. We will see that this assumption means that the sequence w can be written as the difference of two sequences, each of them satisfying the Condition (CX) that we described in Section 2.2. Recall that a sequence $u = (u_n)$ satisfies the Condition (CX) if there exists $n_0 \geq 1$ such that the sequence

$$\left(2^n \sum_{k=n}^{\infty} u_k\right)_{n \geq n_0} \tag{CX}$$

is non-positive and non-decreasing. Furthermore, recall that Theorem 2.6 states that T_w is piecewise convex if and only if the sequence w satisfies the Condition (CX). Therefore, we conclude that $\beta \in \ell_1$ if and only if T_w is the difference of two convex functions that belong to the Takagi Class. For our purposes we may assume that $n_0 = 1$.

Lemma 2.20. *The sequence $w = (w_n)_n$ is the difference of two sequences satisfying the Condition* (CX) *if and only if the sequence*

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_n$$

is the difference of two non-positive non-decreasing sequences.

Proof. If $w = u - v$ where $u = (u_n)_n$ and $v = (v_n)_n$ are two sequences satisfying the Condition (CX) then

$$2^n \sum_{k=n}^{\infty} w_k = 2^n \sum_{k=n}^{\infty} u_k - 2^n \sum_{k=n}^{\infty} v_k$$

for every n, and the result follows immediately.

Conversely, if $u = (u_n)_n$ and $v = (v_n)_n$ are non-positive and non-decreasing sequences such that

$$2^n \sum_{k=n}^{\infty} w_k = u_n - v_n$$

for every n, then

$$w_n = \left(2^{-n} u_n - 2^{-(n+1)} u_{n+1}\right) - \left(2^{-n} v_n - 2^{-(n+1)} v_{n+1}\right),$$

for every n. It is immediate to see that both sequences $\left(2^{-n} u_n - 2^{-(n+1)} u_{n+1}\right)_n$ and $\left(2^{-n} v_n - 2^{-(n+1)} v_{n+1}\right)_n$ satisfy the Condition (CX). □

Lemma 2.21. *A sequence (c_n) is the difference of two non-decreasing and non-positive sequences $(a_n)_n$ and $(b_n)_n$, if and only if*

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| < +\infty.$$

Proof. If $(a_n)_n$ and $(b_n)_n$ are non-decreasing and non-positive sequences such that $c_n = a_n - b_n$ for every n, then

$$|c_n - c_{n+1}| \leq |a_n - a_{n+1}| + |b_n - b_{n+1}| = a_{n+1} - a_n + b_{n+1} - b_n,$$

and hence

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| \leq -a_1 + \lim_n a_n - b_1 + \lim_n b_n < +\infty.$$

Conversely, assume that

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| < +\infty$$

and consequently, the sequence $(c_n)_n$ is convergent. We define

$$a_m = -|\lim_n c_n| - \sum_{n=m}^{\infty} |c_n - c_{n+1}| \qquad \text{and} \qquad b_m = a_m - c_m$$

for every m. It is clear that $c_m = a_m - b_m$, and that $(a_m)_m$ is non-positive and non-decreasing. Moreover, we have

$$\begin{aligned} b_m &= -|\lim_n c_n| - \sum_{n=m}^{\infty} |c_n - c_{n+1}| - c_m \\ &= -\sum_{n=m}^{\infty} (c_n - c_{n+1}) - \lim_n c_n - |\lim_n c_n| - \sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq 0 \end{aligned}$$

and

$$b_m - b_{m+1} = c_{m+1} - c_m - |c_m - c_{m+1}| \leq 0$$

for every m, which gives us the result. □

Proposition 2.22. *The sequence $w = (w_n)_n$ is the difference of two sequences satisfying the Condition* (CX) *if and only if $\beta \in \ell_1$.*

Proof. We have that $w = u - v$ where u and v are two sequences satisfying the Condition (CX) if and only if the sequence

$$\left(2^n \sum_{k=n}^{\infty} w_k\right)_n$$

is the difference of two non-decreasing and non-positive sequences by Lemma 2.20, which is equivalent to saying that

$$\sum_{n=1}^{\infty} 2^n \left| w_n - \sum_{k=n+1}^{\infty} w_k \right| = \sum_{n=1}^{\infty} \left| 2^n \sum_{k=n}^{\infty} w_k - 2^{n+1} \sum_{k=n+1}^{\infty} w_k \right| < +\infty$$

by Lemma 2.21. □

Theorem 2.23. *The function T_w is the difference of two piecewise convex functions of the Takagi class if and only if $\beta \in \ell_1$.*

Collecting the results that we have presented above we obtain the following striking result:

Theorem 2.24. *Let $w \in \ell_1$. Then, the following statements are equivalent:*

(1) T_w has a Taylor expansion of order two almost everywhere.

(2) $T_w'^{+}$ has bounded variation.

(3) $T_w'^{+}$ is derivable almost everywhere.

(4) $\beta \in \ell_1$.

(5) T_w is the difference of two piecewise convex functions of the Takagi class.

Finally, we characterize when $\beta \in \ell_1$ in terms of a more manageable condition. To do so, we define the sequence $\alpha = (\alpha_n)_n$ as

$$\alpha_n = 2^{n+1} w_{n+1} - 2^n w_n$$

for every n.

Lemma 2.25. *We have that $\beta \in \ell_1$ if and only if $\alpha \in \ell_1$.*

Proof. Since

$$2\beta_n - \beta_{n+1} = 2^{n+1}\Big(- w_n + \sum_{j=n+1}^{\infty} w_j + w_{n+1} - \sum_{j=n+2}^{\infty} w_j\Big) = 2\alpha_n \tag{2.10}$$

for every n, we obtain that $\alpha \in \ell_1$ provided that $\beta \in \ell_1$.

Now, assume that $\alpha \in \ell_1$. Then, for $1 \le p < q$ we have

$$2^q w_q - 2^p w_p = \sum_{n=p}^{q-1} \left(-2^n w_n + 2^{n+1} w_{n+1}\right) = \sum_{n=p}^{q-1} \alpha_n = \sum_{n=1}^{q-1} \alpha_n - \sum_{n=1}^{p-1} \alpha_n$$

which implies that $(2^n w_n)_n$ is a Cauchy sequence and therefore it converges. The second statement of 2.6 yields that $\beta \in c_0$. Finally, using the equality (2.10) we get

$$2\sum_{n=1}^{N} |\beta_n| \le \sum_{n=1}^{N} |\beta_{n+1}| + 2\sum_{n=1}^{N} |\alpha_n| = \sum_{n=2}^{N+1} |\beta_n| + 2\sum_{n=1}^{N} |\alpha_n|,$$

and consequently

$$\sum_{n=1}^{N} |\beta_n| \le |\beta_{N+1}| - |\beta_1| + 2\sum_{n=1}^{N} |\alpha_n|.$$

which implies that $\beta \in \ell_1$. □

2.4.1 A dichotomy

In the sequel, we denote by $\mathscr{L}$ the Lebesgue measure on $\mathbb{R}$. As stated in Corollary 2.15, if the sequence of weights $w = (w_n)_n$ satisfies that $(2^n w_n)_n$ converges, then T_w has Taylor expansion of order two at a Lebesgue null set with Hausdorff dimension one. Furthermore, we know that T_w has Taylor expansion of order two almost everywhere if and only if the sequence $\beta = (\beta_n)_n$ belongs to ℓ_1. We may ask ourselves if we can say more when $\beta \notin \ell_1$ but $(2^n w_n)_n$ converges. The answer is “no” and the reason for this is the following result:

Lemma 2.26. *The set $\mathscr{T}$ of those points x such that T_w has a Taylor expansion of order two at x is either a null set or has measure one.*

Proof. Recall that we have denoted by $\mathscr{F}_n$ the family of connected components of $[0,1] \setminus D_n$. Let I_1 and I_2 be two different intervals of $\mathscr{F}_n$ and we consider the translation τ that satisfies $\tau(I_1) = I_2$. We have that T_w has a Taylor expansion of order two at $x \in I_1$ if and only if T_w has a Taylor expansion of order two at $\tau(x)$. This is because this property does not depend on the first n digits of the binary expansion of x, and we know that x and $\tau(x)$ have the same digits ε_k for every $k > n$. From this fact we deduce

$$\mathscr{L}(I \cap \mathscr{T}) = 2^{-(n-1)}\mathscr{L}(\mathscr{T}) = \mathscr{L}(I)\mathscr{L}(\mathscr{T})$$

for every I in $\mathscr{F}_n$.

We may express every interval $J \subset [0,1]$ as a disjoint union of a countable number of intervals belonging to $\mathscr{F}$ plus a countable set. Hence $\mathscr{L}(J \cap \mathscr{T}) = \mathscr{L}(J)\mathscr{L}(\mathscr{T})$ for every interval J. This implies that the density of $\mathscr{T}$ at a point $x \in [0,1]$ is given by

$$\lim_{r \to 0^+} \frac{\mathscr{L}((x-r, x+r) \cap \mathscr{T})}{\mathscr{L}((x-r, x+r))} = \mathscr{L}(\mathscr{T})$$

and the Lebesgue density theorem (see [43]) yields that $\mathscr{T}$ has measure either zero or one. $\square$

We obtain the following result:

Proposition 2.27. *If the sequence $(2^n w_n)_n$ converges, then the set of points where T_w has a Taylor expansion of order two is a null set with Hausdorff dimension one if and only if $\beta \notin \ell_1$.*

In view of the results, we conclude that two mutually exclusive alternatives arise when the sequence $(2^n w_n)_n$ converges:

(1) $T_w'^{+}$ has bounded variation and equivalently, T_w has a Taylor expansion of order two almost everywhere, or

(2) $T_w'^{+}$ does not have bounded variation and equivalently T_w has a Taylor expansion of order two only on a null set of Hausdorff dimension one.

2.5 The Stepanoff condition of order two

The Stepanoff condition is the weakest property that we are examining. Our first result reveals some requirements for the sequence of weights $w = (w_n)_n$ in order for T_w to satisfy a Stepanoff condition of order two at some point. In particular, if T_w satisfies a Stepanoff condition of order two at a point, then we obtain that $w \in \ell_1$. This justifies why the study of the second order differentiability properties that we are considering in this chapter must be done under the assumption $w \in \ell_1$.

Lemma 2.28. *Let $w = (w_n)_n$ be a sequence such that $(w_n 2^{-n})_n \in \ell_1$. If T_w satisfies a Stepanoff condition of order two at a point $x \in [0,1]$, then there exists $M > 0$ such that $2^n |w_n| \leq M$ for every n. In particular, we have $w \in \ell_1$.*

Proof. It is enough to prove that there is $K > 0$ such that $2^n |w_n| \leq K$ eventually. Let $M > 0$ and $\delta > 0$ satisfying

$$|T_w(x+h) - T_w(x) - T_w'(x)h| \leq Mh^2 \tag{2.11}$$

provided that $0 < |h| < \delta$.

Assume first that $x \notin D$. For every n we consider $x \in (a_n, b_n) \in \mathscr{F}_n$. We also denote by c_n the midpoint of such an interval. For every n such that $2^{-(n-1)} < \delta$ we take $h_n = 2(c_n - x)$ and we have the following:

(1) $|h_n| < 2^{-(n-1)}$.

(2) $g_k(x+h_n) - g_k(x) - g_k'(x)h_n = 0$ for every $k < n$.

(3) $g_k(x+h_n) = g_k(x)$ for every $k \geq n$.

Therefore,

$$|T_w(x+h_n) - T_w(x) - T_w'(x)h_n| = \left| -h_n \sum_{k=n}^{\infty} w_k g_k'(x) \right|$$

and by using (2.11) we obtain

$$|w_n| = \left| \sum_{k=n}^{\infty} w_k g_k'(x) - \sum_{k=n+1}^{\infty} w_k g_k'(x) \right| \leq M|h_n| + M|h_{n+1}| < \frac{3M}{2^n}$$

for every n such that $2^{-(n-1)} < \delta$.

Finally, assume that $x \in D$. For every n such that $2^{-(n-1)} < \delta$ we take $h_n = 2^{-(n-1)}$ and then, the proof follows in a similar way as above by replacing g'_k with g'^{+}_k. □

Proposition 2.29 below characterizes when T_w satisfies a Stepanoff condition of order two at a dyadic point. Recall that if T_w satisfies a Stepanoff condition of order two at $x \in D_{n+1} \setminus D_n$, then it must be derivable at such a point. Since

$$T'^{+}_w(x) = \sum_{j=1}^{n-1} w_j g'_j(x) - \left(w_n - \sum_{j=n+1}^{\infty} w_j \right) \quad \text{and} \quad T'^{-}_w(x) = \sum_{j=1}^{n-1} w_j g'_j(x) + \left(w_n - \sum_{j=n+1}^{\infty} w_j \right),$$

we deduce that

$$w_n = \sum_{j=n+1}^{\infty} w_j \tag{2.12}$$

is a necessary condition for T_w to satisfy a Stepanoff condition of order two at x.

Proposition 2.29. *Let $w \in \ell_1$ and $x \in D$. Then, T_w satisfies a Stepanoff condition of order two at x if and only if T_w is derivable at x and there exists $M > 0$ such that $2^n |w_n| \le M$ for every n.*

Proof. Assume that $x \in D_{n+1} \setminus D_n$ for some n. By Lemma 2.28 and the previous comment it is enough to prove the sufficiency part. Since T_w is derivable at x we have

$$T'_w(x) = \sum_{j=1}^{n-1} w_j g'_j(x).$$

We consider $2^{-(m+1)} \le |h| < 2^{-m}$ for some $m \ge n$ and we have that $g_k(x+h) - g_k(x) - g'_k(x)h = 0$ since g_k is affine between x and $x+h$ for every $k < n$, so we get

$$\begin{aligned} |T_w(x+h) - T_w(x) - T'_w(x)h| &= w_n(g_n(x+h) - g_n(x)) + \sum_{j=n+1}^{\infty} w_j g_j(x+h) \\ &= -|h| w_n + \sum_{j=n+1}^{\infty} w_j g_j(x+h) = \sum_{j=n+1}^{\infty} w_j \big(g_j(x+h) - |h| \big) \\ &= \sum_{j=m+1}^{\infty} w_j \big(g_j(x+h) - |h| \big) \le |h| \sum_{j=m+1}^{\infty} |w_j| \le 2Mh^2 \end{aligned}$$

which gives us the result. □

Concerning the non-dyadic case, our purpose is to characterize when the function T_w satisfies a Stepanoff condition of order two at a non-dyadic point in terms of a property that the binary expansion of such a point must satisfy. We will be working with the notation that we introduced when considering the Taylor expansion of order two.

Lemma 2.30. *Let $w \in \ell_1$. If T_w satisfies a Stepanoff condition of order two at $x \notin D$, then there exists $M > 0$ such that $2^{\ell_{n+1}}|\beta_n| \leq M$ for every $n \in \mathcal{N}_x$.*

Proof. By using Lemma 2.28 we may take $K > 0$ and $\delta > 0$ such that $|w_n| \leq K2^{-n}$ for every n, and $|T_w(x+h) - T_w(x) - T_w'(x)h| \leq Kh^2$ provided that $0 < |h| < \delta$.

For every $n \in \mathcal{N}_x$ such that $2^{-n} < \delta$ we take $h_n = g_n'(x)2^{-(n+\ell_{n+1})}$ and we have

$$\begin{aligned} K2^{-2(n+\ell_{n+1})} &\geq |T_w(x+h_n) - T_w(x) - T_w'(x)h_n| \\ &= \left| \sum_{j=n}^{n+\ell_{n+1}} w_j(g_j(x+h_n) - g_j(x) - g_j'(x)h_n) - h_n \sum_{j=n+1+\ell_{n+1}}^{\infty} w_j g_j'(x) \right| \end{aligned}$$

by Lemma 2.12. Hence we obtain

$$\begin{aligned} \left| \sum_{j=n}^{n+\ell_{n+1}} w_j\big(g_j(x+h_n) - g_j(x) - g_j'(x)h_n\big) \right| &\leq K2^{-2(n+\ell_{n+1})} + \left| h_n \sum_{j=n+1+\ell_{n+1}}^{\infty} w_j g_j'(x) \right| \\ &\leq 2K2^{-2(n+\ell_{n+1})}. \end{aligned}$$

Now, we denote $r_n = -2^{n+\ell_{n+1}}(g_n(x+h_n) - g_n(x) - g_n'(x)h_n)$ and by Lemma 2.12 we have that $r_n \in (1,2)$. From Lemma 2.12 again we obtain

$$\begin{aligned} 2^{2(n+\ell_{n+1})} \left| \sum_{j=n}^{n+\ell_{n+1}} w_j(g_j(x+h_n) - g_j(x) - g_j'(x)h_n) \right| &= r_n 2^{n+\ell_{n+1}} \left| -w_n + \sum_{j=n+1}^{n+\ell_{n+1}} w_j \right| \\ \geq r_n 2^{\ell_{n+1}}|\beta_n| - r_n 2^{n+\ell_{n+1}} \sum_{j=n+1+\ell_{n+1}}^{\infty} |w_j| &\geq 2^{\ell_{n+1}}|\beta_n| - 2^{n+1+\ell_{n+1}} \sum_{j=n+1+\ell_{n+1}}^{\infty} |w_j|. \end{aligned}$$

Finally, we conclude

$$2^{\ell_{n+1}}|\beta_n| \leq 2K + 2^{n+\ell_{n+1}+1} \sum_{j=n+\ell_{n+1}+1}^{\infty} |w_j| \leq 4K$$

which gives us the result. □

Lemma 2.31. *Let $x \notin D$. If there exists $M > 0$ such that*

(1) $2^n|w_n| \leq M$ for every n, and

(2) $2^{\ell_{n+1}}|\beta_n| \leq M$ for every $n \in \mathcal{N}_x$,

then T_w satisfies a Stepanoff condition of order two at x.

Proof. For every n we consider $x \in (a_n, b_n) \in \mathcal{F}_n$. We also denote by c_n the midpoint of such interval.

We consider first the case when $h > 0$. We have $2^{-(n+1)} < h \le 2^{-n}$ for some n, and we may assume that n is big enough such that $\varepsilon_{j_1} = 0$ and $\varepsilon_{j_2} = 1$ for some $j_1, j_2 < n$.

If $\varepsilon_n = 0$ then $x, x+h \in (a_n, b_n)$ and consequently g_j is affine on $[x, x+h]$ for every $j < n$, so we get

$$\begin{aligned}\left|\frac{T_w(x+h) - T_w(x) - hT'_w(x)}{h^2}\right| &= \frac{1}{h^2}\left|\sum_{j=n}^{\infty} w_j\big(g_j(x+h) - g_j(x) - hg'_j(x)\big)\right| \\ &\le \frac{2}{h}\sum_{j=n}^{\infty}|w_j| \le 2^{n+2}\sum_{j=n}^{\infty}\frac{M}{2^j} = 8M.\end{aligned}$$

If $\varepsilon_n = 1$ then we denote $m(n) = \max\{j < n : \varepsilon_j = 0\}$ and we have that $m(n) \in \mathcal{N}_x$ and $m(n) + 1 + \ell_{m(n)+1} \ge n$. We obtain

$$\left|T_w(x+h) - T_w(x) - hT'_w(x)\right| = \left|\sum_{j=m(n)}^{\infty} w_j\big(g_j(x+h) - g_j(x) - hg'_j(x)\big)\right|$$

since $x \in (a_{m(n)}, c_{m(n)})$ which implies that $x + h \in (a_{m(n)}, b_{m(n)})$. In order to estimate the right side of the formula, proceeding similarly to what we did in Lemma 2.12, we observe

$$g_j(x+h) - g_j(x) - hg'_j(x) = -\big(g_{m(n)}(x+h) - g_{m(n)}(x) - hg'_{m(n)}(x)\big)$$

for every $m(n) < j \le n$. Hence we have

$$\begin{aligned}&\left|\sum_{j=m(n)}^{\infty} w_j\big(g_j(x+h) - g_j(x) - hg'_j(x)\big)\right| \\ &\le \left|g_{m(n)}(x+h) - g_{m(n)}(x) - hg'_{m(n)}(x)\right|\left|w_{m(n)} - \sum_{j=m(n)+1}^{n} w_j\right| \\ &+ \left|\sum_{j=n+1}^{\infty} w_j\big(g_j(x+h) - g_j(x) - hg'_j(x)\big)\right| \\ &\le 2h\left|w_{m(n)} - \sum_{j=m(n)+1}^{n} w_j\right| + 2h\sum_{j=n+1}^{\infty}|w_j| \le \frac{2h}{2^{m(n)}}|\beta_{m(n)}| + 4h\sum_{j=n+1}^{\infty}|w_j| \\ &\le 8h^2 2^{\ell_{m(n)+1}}|\beta_{m(n)}| + 8Mh^2 \le 16Mh^2\end{aligned}$$

which implies

$$\limsup_{h\to 0^+}\frac{|T_w(x+h) - T_w(x) - hT'_w(x)|}{h^2} < +\infty.$$

The case when $h < 0$ is similar. □

We have proved the following result:

Theorem 2.32. *The function T_w satisfies a Stepanoff condition of order two at $x \notin D$ if and only if there exists $M > 0$ such that*

(1) $2^n|w_n| \leq M$ *for every n, and*

(2) $2^{\ell_{n+1}}\beta_n \leq M$ *for every $n \in \mathscr{N}_x$.*

Now, we take a look at the set of points where T_w satisfies a Stepanoff condition of order two. At the end of Section 2.3 we defined the set

$$\mathscr{A} = \bigcup_{N=1}^{\infty} \mathscr{A}_N.$$

where $\mathscr{A}_N = \{x \notin D : \ell_n(x) \leq N \text{ for every } n\}$. Recall that $\mathscr{A}$ is a Lebesgue null set with Hausdorff dimension 1. Observe that if the sequence of weights $w = (w_n)$ satisfies that there is $M > 0$ such that $2^n|w_n| \leq M$ for every n, then the second statement of Theorem 2.32 holds for every point of the set $\mathscr{A}$. Hence, we have the following result:

Corollary 2.33. *If there exists $M > 0$ such that $2^n|w_n| \leq M$ for every n, then T_w satisfies a Stepanoff condition of order two at a Lebesgue null set with Hausdorff dimension one.*

The result of A. P. Calderón and A. Zygmund mentioned above together with Theorem 2.24 gives the following result:

Theorem 2.34. *The function T_w satisfies a Stepanoff condition of order two almost everywhere if and only if $\beta \in \ell^1$.*

Proceeding in the same way as we did in Subsection 2.4.1, we deduce the next result analogous to Proposition 2.27.

Proposition 2.35. *If there exists $M > 0$ such that $2^n|w_n| \leq M$ for every n, then the set of points where T_w satisfies a Stepanoff condition of order two is a null set with Hausdorff dimension one if and only if $\beta \notin \ell^1$.*

2.6 Final examples

The results we have obtained throughout this chapter allow us to construct functions in the Takagi Class with prescribed properties. We end this chapter by presenting some of them. It is worth noting that examples of functions with such properties are hard to find in the literature and, in general, they are not as simple as the examples from the Takagi Class.

It is well-known that every convex or concave function is differentiable at all but countably many points (see Theorem 1.3.3 of [92] for instance). In the following example, the function T_w is piecewise concave when $\alpha > 2$ and it is differentiable at every point except for the dyadics. Recall that a function T_w is differentiable at $x \in D_{n+1} \setminus D_n$ if and only if

$$w_n = \sum_{j=n+1}^{\infty} w_j.$$

Example 2.36. Let $\alpha \in \mathbb{R}$ be such that $\alpha > 1$. We consider the sequence $w = (w_n)$ defined by $w_n = \alpha^{-n}$ for every n. If $\alpha \geq 2$ then T_w is piecewise concave by Theorem 2.7, and hence it satisfies a Stepanoff condition of order two almost everywhere. Moreover, for $\alpha > 2$ we have

$$T'_w(x) = \sum_{n=1}^{\infty} \frac{1-2\varepsilon_n}{\alpha^n} = \frac{1}{\alpha-1} - 2\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\alpha^n}, \quad \varepsilon_n \in \{0,1\}$$

for every $x \notin D$. The range of T'_w is a Cantor-like set of Hausdorff dimension $\log 2/\log \alpha$ (see Chapter 4 of [41]).

However, if $1 < \alpha < 2$ then by Lemma 2.28 we get that T_w does not satisfy the Stepanoff condition of order two anywhere.

Example 2.37. For the sequence $w = (w_n)$ defined by $w_n = (n+1)(n2^n)^{-1}$ for every n, we have that T_w is piecewise concave by Theorem 2.7.

Example 2.38. For the sequence $w = (w_n)$ defined by $w_n = \frac{(-1)^n}{n^2 2^n}$, the function T_w has a Taylor expansion of order two almost everywhere by Theorem 2.24 since $\beta \in \ell_1$. Observe that T_w is neither convex nor concave by Theorem 2.6 and Theorem 2.7. However, by Theorem 2.24 again we know that it can be written as the difference of two piecewise concave functions, as $T_u - T_v$ where $u = (u_n)_n$ is given by $u_n = \frac{1}{n^2 2^n}$ if n is even and $u_n = 0$ for n odd, whereas $v = (v_n)_n$ is given by $v_n = \frac{1}{n^2 2^n}$ if n is odd and $v_n = 0$ for n even.

Example 2.39. For the sequence $w = (w_n)$ defined by $w_n = \frac{(-1)^n}{n 2^n}$, the function T_w has a Taylor expansion of order two only on a null set of Hausdorff dimension one by Proposition 2.27.

The result of P. Calderón and A. Zygmund yields that having a Taylor expansion of order two almost everywhere is equivalent to satisfying the Stepanoff condition of order two almost everywhere. However, the set of points where a function satisfies a Stepanoff condition of order two may not agree with the set of points where it has Taylor expansion of order two. The following example illustrates this fact.

Example 2.40. For the sequence $w = (w_n)$ defined by $w_n = \frac{(-1)^n}{n^2 2^n}$ for every n, the function T_w has a Taylor expansion of order two almost everywhere, and it satisfies the Stepanoff condition of order two almost everywhere. However, there exist points x such that T_w satisfies the Stepanoff condition of order two at x but does not have a Taylor expansion of order two at x.

Indeed, let us consider the point $x = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n}$ with $\varepsilon_n \in \{0,1\}$ defined in the following way: $\varepsilon_n = 1$ if $n = n_m$ for some m, and $\varepsilon_n = 0$ otherwise, where we define recursively the increasing sequence $(n_m)_m$ by $n_1 = 1$, and $n_{m+1} = n_m + [2\log_2 n_m] + 2$.

The sequence w satisfies $\lim_n 2^n w_n = 0$ and $\beta \in \ell^1$. The point x satisfies that if $n \geq 2$ then $n \in \mathscr{N}_x = \{n : \varepsilon_n \neq \varepsilon_{n+1}\}$ if and only if either $n = n_m$ or $n = n_m - 1$ for some m. Moreover, we have $\ell_{n_m} = 0$ and $\ell_{n_m+1} = [2\log_2 n_m]$. Hence, we obtain $\lim_m 2^{\ell_{n_m}} \beta_{n_m-1} = 0$, but

$$2^{\ell_{n_m+1}} |\beta_{n_m}| = 2^{[2\log_2 n_m]} |\beta_{n_m}| \geq \frac{1}{2} n_m^2 |\beta_{n_m}| \geq \frac{1}{2} n_m^2 2^{n_m} |w_{n_m}| = \frac{1}{2}$$

which is bounded but does not converge to 0.

Example 2.41. Let us consider the sequence w defined by $w_n = \frac{(-1)^n}{2^n}$. Then, T_w has nowhere a Taylor expansion of order two but it satisfies the Stepanoff condition of order two at every point of the set $\mathscr{A}$ defined above.

CHAPTER 3

INFINITE DERIVATIVES OF THE TAKAGI-VAN DER WAERDEN FUNCTIONS

When it comes to studying a continuous nowhere differentiable function, it is natural to ponder the existence, or non existence, of one-sided derivatives and infinite derivatives. In a sense, this can be thought of as a measure of the degree of non-differentiability of such a function.

G. C. Young (1868–1944) studied this issue in detail for the classical Weierstrass function in the year 1916 (see [112] or Theorem 3.5.5 of [74]). She proved that it has cusps pointing upwards, which are points with a right-hand derivative equal to $-\infty$ and a left-hand derivative equal to $+\infty$, at a countable dense set. Whereas, it has cusps pointing downwards, which are points with a right-hand derivative equal to $+\infty$ and a left-hand derivative equal to $-\infty$, at another countable dense set in $[0,1]$. She also obtained that the classical Weierstrass function does not have infinite derivatives, a fact already known to K. Weierstrass.

Perhaps, the most extreme case is that given by A. S. Besicovitch, which admits neither finite nor infinite unilateral derivatives at any point (see [22] or [23]). Nowadays, a function satisfying this property is called a Besicovitch function and we refer to the survey [36] for more information about this topic.

In 1933, T. H. Hildebrandt rediscovered the Takagi function and he provided another proof of its nowhere differentiability (see [68]). An Editor's note affixed to his paper suggested characterizing those points (if any) where the Takagi function has an infinite derivative.

Three years later, E. G. Begle (1914–1978) and W. L. Ayres (1905–1976) proved that the

Takagi function has right-hand derivative equal to $+\infty$ and left-hand derivative equal to $-\infty$ at every dyadic point of the interval $[0,1]$. Regarding the case when $x \in [0,1]$ is not a dyadic point, they showed that $T'^{+}(x) = +\infty$ if the series of the derivatives

$$\sum_{n=1}^{\infty} g_n'(x) \tag{3.1}$$

converges to $+\infty$. They assumed that the proof for the case $T'^{-}(x) = +\infty$ was similar and they wrongly claimed that the derivative of the Takagi function at a non-dyadic point is $+\infty$ whenever the series of the derivatives at such point converges to $+\infty$ (see [20]). There is no evidence in the literature that this mistake was ever noticed until a few years ago.

In 2010, M. Krüppel, unaware of Begle and Ayres' article, published a counterexample to their claim (see Section 7.2 of [80]). He provided a point of the unit interval by means of its binary expansion such that the series of the derivatives at such a point converges to $+\infty$ but the lower-left Dini derivative of the Takagi function at such a point is equal to $-\infty$. This counterexample will be thoroughly examined later in Example 3.12.

P. C. Allaart and K. Kawamura (see [11]) and M. Krüppel (see [81]) independently characterized the set of points where the Takagi function possesses an infinite derivative. On the one hand, they obtained the "if and only if" statement concerning the case when the right-hand derivative of the Takagi function at a non-dyadic point is equal to $+\infty$. On the other hand, taking advantage of the fact that every non-dyadic point $x \in [0,1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{m_n}}$$

where (m_n) is a strictly increasing sequence of positive integers determined uniquely by x, they proved that $T'^{-}(x) = +\infty$ if and only if

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1} - m_n) + \log_2(m_{n+1} - m_n) \right) = +\infty. \tag{3.2}$$

In connection with the base two expansion of the point x, let us observe that m_n corresponds to the position of the n-th 1-digit in such a expansion. Finally, by using the symmetry of the Takagi function, that is $T(x) = T(1-x)$ for every $x \in [0,1]$, they obtained similar results regarding the case when the right-hand and the left-hand derivative of the Takagi function at a non-dyadic point is equal to $-\infty$.

Moreover, P. C. Allaart and K. Kawamura showed that the set of points where the Takagi function possesses an infinite derivative has Hausdorff dimension one.

The purpose of this chapter is to present a thorough study of the one-sided derivatives of the Takagi-Van der Waerden function. Thus, we answer the question posed by P. C. Allaart and K. Kawamura in the survey [10], which suggested determining those points where the Takagi-Van der Waerden function has an infinite derivative.

3.1 Behavior of the series of the derivatives

As was mentioned earlier, the right-hand derivative of the Takagi function at a non-dyadic point is equal to $+\infty$ if and only if the series of the derivatives (3.1) at such a point converges to $+\infty$.

We continue by examining Condition (3.2), which characterizes when the left-hand derivative of the Takagi function at a non-dyadic point is equal to $+\infty$. Recall that for a non-dyadic point $x \in [0,1]$ with binary expansion $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\dots$ we have $g_n'(x) = 1 - 2\varepsilon_n(x)$ for every n. It should be noted that Condition (3.2) implies

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} g_k'(x) = +\infty.$$

In addition, for every n we have $\varepsilon_{m_n}(x) = 1$ and $\varepsilon_{m_{n+1}}(x) = 1$, whereas if $m_n < j < m_{n+1}$ then $\varepsilon_j(x) = 0$. We have $g'_{m_n}(x) = g'_{m_{n+1}}(x) = -1$ and $g_j'(x) = 1$ for every $m_n < j < m_{n+1}$, which yields

$$\sum_{k=1}^{m_n} g_j'(x) < \sum_{k=1}^{j} g_k'(x) \leq \sum_{k=1}^{m_{n+1}} g_k'(x) + 1$$

whenever $m_n < j < m_{n+1}$. Therefore, Condition (3.2) implies that the series of the derivatives (3.1) at such a point converges to $+\infty$.

On account of the role the convergence to $+\infty$ of the series of the derivatives plays in the Takagi function case, in order to obtain similar results for the Takagi-Van der Waerden function, it seems quite natural to investigate the appearance of the base-r expansion of a point at which the series of the derivatives tends to $+\infty$ or $-\infty$. To do so, we should bear in mind the relation between the base-r expansion of a point of the unit interval, and the value of the function g_n at such a point. We discussed this relationship in Section 1.3 of Chapter 1 and it can be summarized in the following result:

Lemma 3.1. *Let $r \geq 2$ be an integer and let $x \in [0,1]$ with base-r expansion given by $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$. For every n, we have the following:*

(1) *When $x \in D_n \cup \widetilde{D}_n$, we have:*

   (a) $g_n(x) = 0$ *if and only if* $x \in D_n$.

   (b) $g_n(x) = r^{-(n-1)}/2$ *if and only if* $x \in \widetilde{D}_n$.

  (c) *$g_n'^{+}(x) = 1$ and $g_n'^{-}(x) = -1$ if and only if $x \in D_n$.*

  (d) *$g_n'^{+}(x) = -1$ and $g_n'^{-}(x) = 1$ if and only if $x \in \widetilde{D}_n$.*

(2) *When $x \notin D_n \cup \widetilde{D}_n$, we have:*

  (a) *$g_n(x) = x - a_n$ and $g_n'(x) = 1$, if and only if $\varepsilon_{k_n} < (r-1)/2$,*

  (b) *$g_n(x) = b_n - x$ and $g_n'(x) = -1$, if and only if $\varepsilon_{k_n} > (r-1)/2$,*

  where $k_n = \min\{k \geq n : \varepsilon_k \neq (r-1)/2\}$.

For a point belonging to $D \cup \widetilde{D}$ the derivative of the function g_n at such a point will not exist for large enough n. Hence, the study of the behavior of the series of the derivatives only makes sense for a point $x \notin D \cup \widetilde{D}$. By virtue of Lemma 3.1 we may grasp the nature of the base-r expansion of a point not belonging to $D \cup \widetilde{D}$ whose series of derivatives converges to $+\infty$ or $-\infty$. To do so, we introduce the following notation. For a point $x \notin D \cup \widetilde{D}$ we define

$$O_m(x) = \#\left\{n \leq m : \varepsilon_{k_n} < (r-1)/2\right\}, \quad \text{and} \quad I_m(x) = \#\left\{n \leq m : \varepsilon_{k_n} > (r-1)/2\right\}$$

where # stands for the cardinality of a set. Roughly speaking, we may say that $O_m(x)$ counts the number of indices j's for which the digit ε_j contributes "+1", meaning that $g_j'(x) = +1$, and $I_m(x)$ the number of indices j's for which the digit ε_j contributes "−1", that is $g_j'(x) = -1$, among the first m digits of the base r expansion of x. We may write

$$\sum_{n=1}^{m} g_n'(x) = O_m(x) - I_m(x) \tag{3.3}$$

and the next result is immediate.

Lemma 3.2. *Let $r \geq 2$ be an integer and let $x \notin D \cup \widetilde{D}$ be written as $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$ where $\varepsilon_n \in \{0, \ldots, r-1\}$. Then, we have the following:*

(1) *$\sum_{n=1}^{\infty} g_n'(x) = +\infty$ if and only if $\lim_{n\to\infty} O_n(x) - I_n(x) = +\infty$, and*

(2) *$\sum_{n=1}^{\infty} g_n'(x) = -\infty$ if and only if $\lim_{n\to\infty} O_n(x) - I_n(x) = -\infty$.*

Using the notation $O_m(x)$ and $I_m(x)$ is motivated by the case $r = 2$, where $O_m(x)$ is the number of 0's and $I_m(x)$ is the number of 1's among the first m binary digits of x. In this case, we have

$$I_m(x) = \sum_{n=1}^{m} \varepsilon_n, \quad O_m(x) = m - I_m(x)$$

and the finite sum (3.3) reflects the excess of zeros over ones in the first m digits of the base two expansion of the point.

Recall that $\mathscr{L}$ stands for the Lebesgue measure on $\mathbb{R}$.

Theorem 3.3. *Let $r \geq 2$ be an integer. Then, the sets $A^+ = \left\{x \in [0,1] : \sum_{n=1}^{\infty} g_n'(x) = +\infty\right\}$ and $A^- = \left\{x \in [0,1] : \sum_{n=1}^{\infty} g_n'(x) = -\infty\right\}$ have Lebesgue measure zero.*

Proof. We consider the function $S : [0,1] \to [0,1]$ defined by $S(x) = 1 - x$. We have that $g_n(x) = g_n(S(x))$ and $g_n'(x) = -g_n'(S(x))$ for all n and all $x \notin D \cup \widetilde{D}$. Furthermore, we also observe that $x \notin D \cup \widetilde{D}$ if and only if $1 - x \notin D \cup \widetilde{D}$. The sets A^+ and A^- are obviously disjoint, and they have the same measure since $A^+ = S\left(A^-\right)$.

For every $n \geq 1$ and $k = 0, \ldots, r^n - 1$ we denote $J_k^n = (kr^{-n}, (k+1)r^{-n})$. We have

$$J_k^n \cap A^+ = \frac{k}{r^n} + \left(J_0^n \cap A^+\right)$$

and hence $\mathscr{L}\left(J_k^n \cap A^+\right) = \mathscr{L}\left(J_0^n \cap A^+\right)$ for every $k = 0, \ldots, r^n - 1$, which implies

$$\mathscr{L}\left(J_k^n \cap A^+\right) = \frac{1}{r^n}\mathscr{L}\left(A^+\right) = \mathscr{L}\left(J_k^n\right)\mathscr{L}\left(A^+\right).$$

for every $k = 0, \ldots, r^n - 1$.

We may write every interval $I \subset [0,1]$ as a disjoint union of a countable number of intervals J_k^n plus a null countable set. Therefore, $\mathscr{L}\left(I \cap A^+\right) = \mathscr{L}(I)\mathscr{L}\left(A^+\right)$ for every interval $I \subset [0,1]$. This implies that the density of A^+ at a point $x \in [0,1]$ is given by

$$\lim_{r \to 0^+} \frac{\mathscr{L}\left((x-r, x+r) \cap A^+\right)}{\mathscr{L}\left((x-r, x+r)\right)} = \mathscr{L}\left(A^+\right)$$

and the Lebesgue density theorem (see [43]) gives that A^+ has measure either 0 or 1. However, it cannot be 1 because A^+ and A^- are disjoint sets and they have the same measure. We conclude $\mathscr{L}\left(A^-\right) = \mathscr{L}\left(A^+\right) = 0$. □

3.2 One-sided derivatives and the series of the derivatives

In this section we investigate the one-sided derivative of the Takagi-Van der Waerden function at a point. To do so, we address the study of the right-hand derivative of the Takagi-Van der Waerden function at a point $x \notin D \cup \widetilde{D}$ and then, the analogous result for the left-hand derivative is obtained by applying the previous result for the point $1 - x$ and using the symmetry equation $f_r(x) = f_r(1-x)$.

We also shed light on the relationship between the behavior of the Dini derivatives of the Takagi-Van der Waerden function at a point, and the convergence to $+\infty$ or $-\infty$ of the series of the derivatives at such a point. This allows us to understand how the partial sum of the series of the derivatives at a point not belonging to $D \cup \widetilde{D}$ appears in a natural way while carrying out this investigation. Furthermore, we will see the different nature the Takagi-Van der Waerden function has depending on whether r is odd or even.

Recall that the four Dini derivatives of a function $f:\mathbb{R}\to\mathbb{R}$ at a point $x\in\mathbb{R}$ are defined as follows.

(1) Upper right Dini derivative:

$$D^{+}f(x)=\limsup_{h\to 0^{+}}\frac{f(x+h)-f(x)}{h}. \tag{3.4}$$

(2) Lower right Dini derivative:

$$d^{+}f(x)=\liminf_{h\to 0^{+}}\frac{f(x+h)-f(x)}{h}. \tag{3.5}$$

(3) Upper left Dini derivative:

$$D^{-}f(x)=\limsup_{h\to 0^{-}}\frac{f(x+h)-f(x)}{h}. \tag{3.6}$$

(4) Lower left Dini derivative:

$$d^{-}f(x)=\liminf_{h\to 0^{-}}\frac{f(x+h)-f(x)}{h}. \tag{3.7}$$

The four Dini derivatives exist at every point for every function defined on the real line and they are named after the Italian mathematician U. Dini. It is immediate to see that a function is derivable at a point if and only if all four Dini derivates are equal at that point, and it has a one-sided derivative at a point if the two Dini derivates from that side are equal. We refer to the book [105] for more information about the Dini derivatives and their applications.

As already stated in the opening of this chapter, Begle and Ayres (see [20]) proved that the Takagi function has cusps pointing downwards at every dyadic point of the unit interval. Although not explicitly stated, this result was extended to the Takagi-Van der Waerden function case by Juan Ferrera and Javier Gómez Gil (see [48]). In a broader context, for an integer $r\geq 2$ they obtained that

$$D^{-}f_r(x)=-\infty \quad \text{and} \quad d^{+}f_r(x)=+\infty$$

whenever $x\in D$, which gives us the following result.

Proposition 3.4. *Let $r\geq 2$ be an integer. If $x\in D$, then $f_r'^{+}(x)=+\infty$ and $f_r'^{-}(x)=-\infty$.*

When r is even, we have $\widetilde{D}_n \subset D_{n+1}$ and consequently, $\widetilde{D} \subset D$. However, if r is odd then $\widetilde{D} \cap D = \emptyset$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n. In the latter case, the result concerning the one-sided derivatives of the Takagi-Van der Waerden function at a point of $\widetilde{D}$ is an immediate consequence of Theorem 4.2 in Chapter 4, which yields

$$D^+ f_r(x) = -\infty \quad \text{and} \quad d^- f_r(x) = +\infty$$

whenever $x \in \widetilde{D}$.

Proposition 3.5. *Let $r \geq 3$ be an odd integer. If $x \in \widetilde{D}$, then $f_r'^+(x) = -\infty$ and $f_r'^-(x) = +\infty$.*

The previous results reveal that the Takagi-Van der Waerden function has cusps pointing downwards at every r-adic point, and when r is odd, it has cusps pointing upwards at every middle-point. It is interesting to compare this property with the nature of the classical Weierstrass function.

It should be recalled that every point $x \notin D \cup \widetilde{D}$ lies between two consecutive r-adic numbers of order n, and that we have denoted them by $a_n = \max\{y \in D_n : y < x\}$ and $b_n = \min\{y \in D_n : y > x\}$. Recall that $\varepsilon_k(a_n) = \varepsilon_k(x)$ for every $k \leq n-1$ and $\varepsilon_k(a_n) = 0$ for every $k \geq n$. We also denote by c_n the midpoint of the interval $[a_n, b_n]$.

The following lemma reveals how the partial sum of the series of the derivatives at a point $x \notin D \cup \widetilde{D}$ comes into play when considering the difference quotients of the Takagi-Van der Waerden function over the intervals $[a_n, x]$ and $[x, b_n]$.

Lemma 3.6. *Let $r \geq 2$ be an integer and $x \notin D \cup \widetilde{D}$. For every n, we have*

$$\frac{f_r(b_n) - f_r(x)}{b_n - x} \leq \sum_{k=1}^{n} g_k'(x) \leq \frac{f_r(a_n) - f_r(x)}{a_n - x},$$

where $a_n = \max\{y \in D_n : y < x\}$ and $b_n = \min\{y \in D_n : y > x\}$.

Proof. It is enough to observe that

$$\begin{aligned}\frac{f_r(b_n) - f_r(x)}{b_n - x} &= \sum_{k=1}^{n} \frac{g_k(b_n) - g_k(x)}{b_n - x} - \sum_{k=n+1}^{\infty} \frac{g_k(x)}{b_n - x} \\ &\leq \sum_{k=1}^{n} \frac{g_k(b_n) - g_k(x)}{b_n - x} \leq \sum_{k=1}^{n} g_k'(x)\end{aligned}$$

where the last inequality follows because for every $1 \leq k \leq n$ we have that the function g_k is 1-Lipschitz and is affine on $[x, b_n]$ provided that $g_k'(x) = -1$. The other inequality is similar. □

For every n we recall that g_n is a piecewise affine map with nodes located at the points of $D_n \cup \widetilde{D}_n$ and consequently, for a point $x \notin D \cup \widetilde{D}$ the function g_k is affine on $[x, x+d_n]$ for every $1 \leq k \leq n$ where

$$d_n = d_n(x) = \min\{y - x : y \in D_n \cup \widetilde{D}_n \text{ and } y > x\}.$$

Since the distance between two consecutive points of $D_n \cup \widetilde{D}_n$ is $r^{-(n-1)}/2$ it follows that $0 < d_n < r^{-(n-1)}/2$ and if $d_{n+1} < d_n$ then we necessarily have $d_n > r^{-n}/2$.

In order to prove Lemma 3.7 below, for every $n \geq 1$ we are interested in describing when the situation $d_n = d_{n+1}$ holds in terms of the base r expansion of $x \notin D \cup \widetilde{D}$. A brief reflection on it suffices to reveal that $d_n = d_{n+1}$ if and only if one of the following situations occurs:

- **Situation 1:** $\varepsilon_n = r-1$ and $\varepsilon_{k_{n+1}} > (r-1)/2$. Here, we have $x + d_n = x + d_{n+1} \in D_n \subset D_{n+1}$.

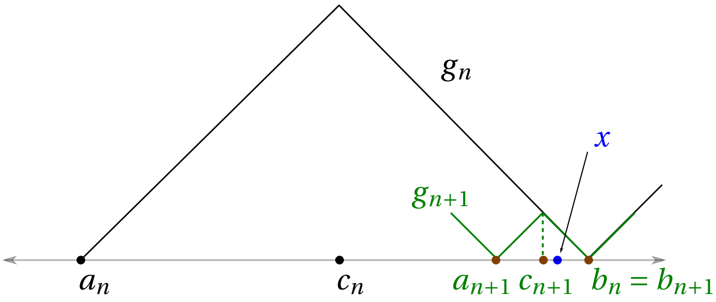

Figure 3.1: Situation 1

- **Situation 2:** $\varepsilon_n = (r-2)/2$ and $\varepsilon_{n+1} \geq r/2$. This case can only happen when $r \geq 2$ is an even integer, and hence we have $x + d_n = x + d_{n+1} \in \widetilde{D}_n \subset D_{n+1}$.

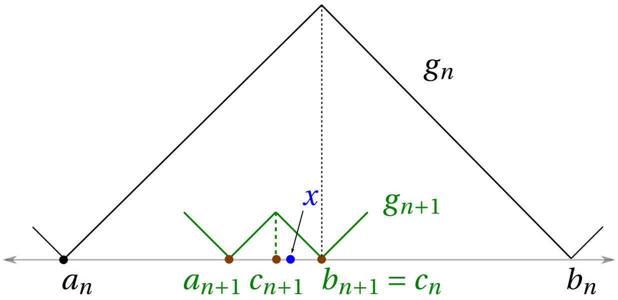

Figure 3.2: Situation 2

- **Situation 3:** $\varepsilon_n = (r-1)/2$ and $\varepsilon_{k_{n+1}} < (r-1)/2$. This case can only happen when $r \geq 2$ is an odd integer, and hence we have $x + d_n = x + d_{n+1} \in \widetilde{D}_n \subset \widetilde{D}_{n+1}$.

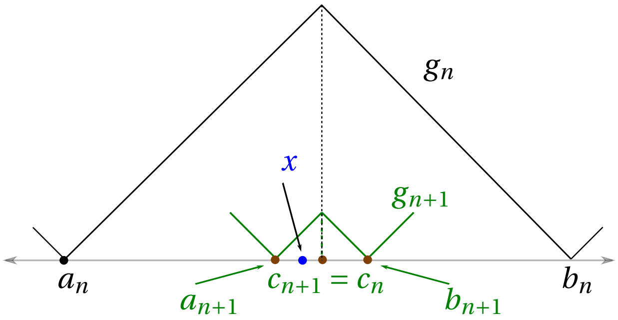

Figure 3.3: Situation 3

Furthermore, for a point $x \notin D \cup \widetilde{D}$ we have that $(x+d_n)_n$ is a non-increasing sequence of points belonging to $D \cup \widetilde{D}$ that converges to x as $n \to \infty$. We may extract a strictly decreasing subsequence $(x+d_{n_k})_k$ converging to x such that if $n_k < n \leq n_{k+1}$ then

$$d_n = d_{n_{k+1}} \quad \text{and} \quad x + d_{n_{k+1}} \in D_n \cup \widetilde{D}_n.$$

where we consider $n_0 = 0$. Furthermore, since $d_{n_k+1} < d_{n_k}$ we get

$$\frac{1}{2r^{n_k}} < d_{n_k} < \frac{1}{2r^{n_k-1}}.$$

Lemma 3.7. *Let $r \geq 2$ be an integer and $x \notin D \cup \widetilde{D}$. There exist decreasing sequences (h_n) and (h'_n) converging to 0 such that*

$$\frac{f_r(x+h_n) - f_r(x)}{h_n} \geq \sum_{j=1}^{n} g'_j(x) - \frac{r}{r-1} - 1, \text{ and}$$
$$\frac{f_r(x-h'_n) - f_r(x)}{-h'_n} \leq \sum_{j=1}^{n} g'_j(x) + \frac{r}{r-1} + 1.$$

Proof. Firstly, we observe

$$\begin{aligned}\frac{f_r(x+d_{n_k}) - f_r(x)}{d_{n_k}} &= \sum_{j=1}^{n_k} \frac{g_j(x+d_{n_k}) - g_j(x)}{d_{n_k}} + \sum_{j=n_k+1}^{\infty} \frac{g_j(x+d_{n_k}) - g_j(x)}{d_{n_k}} \\ &\geq \sum_{j=1}^{n_k} g'_j(x) - \frac{1}{2d_{n_k}} \sum_{j=n_k+1}^{\infty} \frac{1}{r^{j-1}} \geq \sum_{j=1}^{n_k} g'_j(x) - \frac{r}{r-1} \end{aligned} \tag{3.8}$$

where we have used that $d_{n_k} > r^{-n_k}/2$ and that g_j is affine on $[x, x+d_{n_k}]$ for every $1 \leq j \leq n_k$.

Now, we construct the sequence $(h_n)_n$ as outlined below. Let $n \geq 1$ be fixed. If $n = n_k$ for some k then $h_n = d_{n_k}$ and the result follows immediately by (3.8), meanwhile for $n_k < n < n_{k+1}$ we distinguish two cases:

If $x+d_{n_{k+1}} = x+d_{n_k+1} \in D_{n_{k+1}}$ then we have either $x+d_{n_k+1} \in \widetilde{D}_{n_k+1}$, which implies $g'_{n_k+1}(x)=1$ (see Figure 3.2), or $x+d_{n_k+1} \in D_{n_k+1}$ which implies $g'_{n_k+1}(x)=-1$ (see Figure 3.1). In both cases, we take $h_n = d_{n_k}$, which yields $g'_j(x)=-1$ for $n_k+2 \le j \le n_{k+1}$ and therefore, by (3.8) we get

$$\frac{f_r(x+h_n)-f_r(x)}{h_n} \ge \sum_{j=1}^{n_k} g'_j(x) - \frac{r}{r-1} \ge \sum_{j=1}^{n} g'_j(x) - 1 - \frac{r}{r-1}.$$

If $x+d_{n_{k+1}} \in \widetilde{D}_{n_{k+1}}$ then we take $h_n = d_{n_{k+1}}$, which implies $g'_j(x)=1$ for $n_k < j \le n_{k+1}$ and hence, again by (3.8) we obtain

$$\frac{f_r(x+h_n)-f_r(x)}{h_n} \ge \sum_{j=1}^{n_{k+1}} g'_j(x) - \frac{r}{r-1} \ge \sum_{j=1}^{n} g'_j(x) - \frac{r}{r-1}.$$

This proves the first inequality. The second inequality for a point $x \notin D \cup \widetilde{D}$ is obtained by applying the previous one at $1-x$ and using the property $f_r(x) = f_r(1-x)$. □

The previous result together with Lemma 3.6 allows us to deduce some links between the Dini derivatives of the Takagi-Van der Waerden function and the behavior of the series of the derivatives at a point.

Proposition 3.8. *Let $r \ge 2$ be an integer and $x \notin D \cup \widetilde{D}$. Then, the following statements hold:*

1. *If either $f_r'^{+}(x) = +\infty$ or $f_r'^{-}(x) = +\infty$, then $\sum_{k=1}^{\infty} g'_k(x) = +\infty$.*
2. *If either $f_r'^{+}(x) = -\infty$ or $f_r'^{-}(x) = -\infty$, then $\sum_{k=1}^{\infty} g'_k(x) = -\infty$.*
3. *If $\sum_{k=1}^{\infty} g'_k(x) = +\infty$ then $D^+ f_r(x) = D^- f_r(x) = +\infty$.*
4. *If $\sum_{k=1}^{\infty} g'_k(x) = -\infty$ then $d_+ f_r(x) = d_- f_r(x) = -\infty$.*

Proposition 3.8 yields that the Takagi-Van der Waerden function has cusps only at the points belonging to $D \cup \widetilde{D}$. A similar result was obtained by G. C. Young for the classical Weierstrass function (see [112]).

Although the following result can be obtained as a consequence of the Denjoy-Young-Saks theorem (see [30]), it also follows from Theorem 3.3, and statements (1) and (2) of Proposition 3.8.

Corollary 3.9. *Let $r \ge 2$ be an integer. Then, the sets $\{x : f_r'^{+}(x) = +\infty\}$, $\{x : f_r'^{-}(x) = +\infty\}$, $\{x : f_r'^{+}(x) = -\infty\}$ and $\{x : f_r'^{-}(x) = -\infty\}$ are null sets for the Lebesgue measure.*

The following result shows that Proposition 3.8 can be improved when $r \geq 2$ is an even integer. In this case, we have $\widetilde{D}_n \subset D_{n+1}$ for every n and this plays a major role in the proof of such a result.

Theorem 3.10. *Let $x \notin D \cup \widetilde{D}$. If $r \geq 2$ is an even integer, then the following statements hold:*

(1) $f_r'^{+}(x) = +\infty$ *if and only if* $\sum_{k=1}^{\infty} g_k'(x) = +\infty$.

(2) $f_r'^{-}(x) = -\infty$ *if and only if* $\sum_{k=1}^{\infty} g_k'(x) = -\infty$.

Proof. It suffices to prove the first statement since $f_r'^{-}(x) = -\infty$ if and only if $f_r'^{+}(1-x) = +\infty$. Let $d_{n_{k+1}} < h \leq d_{n_k}$ for some k. By using that g_j is affine on $[x, x+h]$ for every $1 \leq j \leq n_k$ we get

$$
\begin{aligned}
\frac{f_r(x+h)-f_r(x)}{h} &\geq \sum_{j=1}^{n_k} g_j'(x) + \sum_{j=n_k+1}^{n_{k+1}} \frac{g_j(x+h)-g_j(x)}{h} - \frac{1}{2h}\sum_{j=n_{k+1}+1}^{\infty} \frac{1}{r^{j-1}} \\
&= \sum_{j=1}^{n_k} g_j'(x) + \sum_{j=n_k+1}^{n_{k+1}} \frac{g_j(x+h)-g_j(x)}{h} - \frac{1}{2hr^{n_{k+1}}}\frac{r}{r-1} \\
&\geq \sum_{j=1}^{n_k} g_j'(x) + \sum_{j=n_k+1}^{n_{k+1}} \frac{g_j(x+h)-g_j(x)}{h} - \frac{r}{r-1} \qquad (3.9)
\end{aligned}
$$

where the last inequality follows from $d_{n_{k+1}} > r^{-n_{k+1}}/2$ since $d_{n_{k+1}+1} < d_{n_{k+1}}$.

If $n_{k+1} > n_k + 1$ then we have either $x + d_{n_{k+1}} = x + d_{n_k+1} \in D_{n_k+1}$ or $x + d_{n_{k+1}} \in \widetilde{D}_{n_k+1}$. In the former case, we have $g_j'(x) = -1$ for every $n_k + 1 \leq j \leq n_{k+1}$ (see Figure 3.1) . In the latter case, since $r \geq 2$ is an even integer we have $\widetilde{D}_{n_k+1} \subset D_{n_k+2}$ and hence, we get $g_{n_k+1}'(x) = 1$ and $g_j'(x) = -1$ for every $n_k + 2 \leq j \leq n_{k+1}$ (see Figure 3.2). In both cases, we obtain

$$
\frac{f_r(x+h)-f_r(x)}{h} \geq \sum_{j=1}^{n_{k+1}} g_j'(x) - 2 - \frac{r}{r-1}. \qquad (3.10)
$$

Otherwise, that is $n_{k+1} = n_k + 1$, we also obtain (3.10) and it follows immediately from (3.9). This proves the result. □

The following example illustrates that Theorem 3.10 does not hold for the function f_3. This example demonstrates the importance of r parity while dealing with these properties. Roughly speaking, the point given in the next example contains long chains of “ones” separated by a zero digit. This means that such a point is “very close” to the center of a 3-adic interval, or better said, it is “very close” to the center of many 3-adic intervals because we have $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n. Therefore, the behavior of the difference quotients change fiercely when considering a positive increment that is sufficiently small. Let us observe that similar examples exist when $r \geq 2$ is odd and the next one somehow reveals how the digit $(r-1)/2$ makes a difference when considering the odd case.

Example 3.11. Let $r = 3$. We consider the point

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n},$$

where $\varepsilon_n = 0$ if $n = 10^k$ for some k, and $\varepsilon_n = 1$ otherwise. From Lemma 3.1 it follows that $g_n'(x) = 1$ for every n, and hence $\sum_{k=1}^{\infty} g_k'(x) = +\infty$. However, we have $d_+ f_3(x) = -\infty$.

Proof. For every k we take

$$h_k = \frac{1}{3^{10^k - 1}} + \frac{1}{3^{10^k}} = \frac{4}{3^{10^k}}$$

and we will compute the quotient

$$\frac{f_3(x+h_k) - f_3(x)}{h_k}$$

by looking at the base r expansion of x and $x + h_k$. Firstly, because

$$a_{10^{k-1}} = \sum_{n=1}^{10^{k-1}-1} \frac{\varepsilon_n}{3^n} < x < x + h_k < \sum_{n=1}^{10^{k-1}-1} \frac{\varepsilon_n}{3^n} + \sum_{n=10^{k-1}}^{\infty} \frac{1}{3^n} = c_{10^{k-1}}$$

we get that g_j is affine on $[x, x+h_k]$ for every $1 \leq j \leq 10^{k-1}$. Furthermore, for every $n \geq 10^k + 1$ the function g_n is periodic of period $3^{-(n-1)}$, and consequently $g_n(x) = g_n(x+h_k)$ for every $n \geq 10^k + 1$. From these facts we obtain

$$\begin{aligned} \frac{f_3(x+h_k) - f_3(x)}{h_k} &= \sum_{n=1}^{10^k} \frac{g_n(x+h_k) - g_n(x)}{h_k} = \sum_{n=1}^{10^{k-1}} g_n'(x) + \sum_{n=10^{k-1}+1}^{10^k} \frac{g_n(x+h_k) - g_n(x)}{h_k} \\ &= 10^{k-1} + \sum_{n=10^{k-1}+1}^{10^k} \frac{g_n(x+h_k) - g_n(x)}{h_k} \end{aligned} \tag{3.11}$$

Furthermore, if $10^{k-1} + 1 \leq n \leq 10^k - 1$ then $\varepsilon_{k_n}(x) = 0$ and $\varepsilon_{k_n}(x+h_k) = 2$, which yields

$$\frac{g_n(x+h_k) - g_n(x)}{h_k} = \frac{b_n - (x+h_k) - (x - a_n)}{h_k} = \frac{2(c_{10^k} - x)}{h_k} - 1$$

by Lemma 3.1, where we recall that c_{10^k} is the midpoint between a_n and b_n. Continuing from (3.11) and as

$$c_{10^k} - x \leq \frac{1}{2} \cdot \frac{1}{3^{10^k - 1}}$$

we conclude

$$\frac{f_3(x+h_k) - f_3(x)}{h_k} \leq 10^{k-1} - \frac{1}{4} \cdot (9 \cdot 10^{k-1} - 1) + 1 = \frac{-5 \cdot (10^{k-1} - 1)}{4}$$

which tends to $-\infty$ as $k \to \infty$. □

As previously mentioned in the opening of this chapter, Begle and Ayres (see [20]) wrongly claimed that the Takagi function has an infinite derivative at a non-dyadic point whenever the series of the derivatives at such point converges to infinity. The next example, provided by M. Krüppel in Section 7.2. of [80], shows that the derivative of the Takagi function at a point need not exist even if the series of the derivatives at such a point converges to infinity.

Krüppel's argument takes advantage of a formula for the value of the Takagi function at a dyadic point, and the difference quotients of the Takagi function over dyadic intervals are considered. Below we lay out the argument in a slightly different way with the goal of gaining a better understanding. Roughly speaking, the binary expansion of the point contains long chains of zeros separated by the digit one, which means that such a point is "very close" to a dyadic point and hence, the behavior of the difference quotients change dramatically when considering a negative increment that is small enough.

Example 3.12. Let $r = 2$. We consider the point

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n},$$

where $\varepsilon_n = 1$ if $n = 4^k$ for some k, and $\varepsilon_n = 0$ otherwise. We certainly have $\sum_{k=1}^{\infty} g_k'(x) = +\infty$. However, $f_2'(x) \neq +\infty$ since $d_- f_2(x) = -\infty$.

Proof. For every $k \geq 2$ we take $h_k = 2^{-(4^k-3)}$ and we proceed in a similar way as in Example 3.11. We have

$$a_{4^{k-1}} = \sum_{n=1}^{4^{k-1}-1} \frac{\varepsilon_n}{2^n} < x - h_k < x < \sum_{n=1}^{4^{k-1}-1} \frac{\varepsilon_n}{2^n} + \frac{1}{2^{4^{k-1}-1}} = b_{4^{k-1}}$$

and hence, the function g_j is affine on $[x - h_k, x]$ for every $1 \leq j \leq 4^{k-1} - 1$. Furthermore, for every $n \geq 4^k - 2$ the function g_n is periodic of period $2^{-(n-1)}$ and consequently, $g_n(x) = g_n(x - h_k)$ for every $n \geq 4^k - 2$. This yields

$$\begin{aligned}\frac{f_2(x-h_k) - f_2(h_k)}{-h_k} &= \sum_{n=1}^{4^k-3} \frac{g_n(x-h_k) - g_n(x)}{-h_k} = \sum_{n=1}^{4^{k-1}-1} g_n'(x) + \sum_{n=4^{k-1}}^{4^k-3} \frac{g_n(x-h_k) - g_n(x)}{-h_k} \\ &= 4^{k-1} - 1 - 2(k-2) + \sum_{n=4^{k-1}}^{4^k-3} \frac{g_n(x-h_k) - g_n(x)}{-h_k}. \qquad (3.12)\end{aligned}$$

Since $\varepsilon_{4^{k-1}}(x) = 1$ and $\varepsilon_{4^{k-1}}(x - h_k) = 0$ we get by Lemma 3.1

$$g_{4^{k-1}}(x - h_k) - g_{4^{k-1}}(x) = x - h_k - a_{4^{k-1}} - (b_{4^{k-1}} - x) = 2(x - c_{4^{k-1}}) - h_k,$$

meanwhile if $4^{k-1}+1 \le n \le 4^k-3$ then $\varepsilon_n(x)=0$ and $\varepsilon_n(x-h_k)=1$ which gives

$$g_n(x-h_k)-g_n(x)=c_{4^{k-1}}-(x-h_k)-(x-c_{4^{k-1}})=-\left[2(x-c_{4^{k-1}})-h_k\right]$$

by Lemma 3.1 again. Continuing from (3.12) and as $c_{4^{k-1}}-x \le 2^{-(4^k-1)}$ we conclude

$$\begin{aligned}\frac{f_2(x-h_k)-f_2(x)}{-h_k} &= 4^{k-1}-2k+3+\left[\frac{2(x-c_{4^{k-1}})}{-h_k}+1\right]\left(4-3\cdot 4^{k-1}\right)\\ &= 4^{k-1}-2k+3+\left[\frac{2(c_{4^{k-1}}-x)}{h_k}+1\right]\left(4-3\cdot 4^{k-1}\right)\\ &\le 4^{k-1}-2k+3+\frac{3}{2}\left(4-3\cdot 4^{k-1}\right)=-\frac{7}{2}4^{k-1}-2k+9\end{aligned}$$

which tends to $-\infty$ as $k \to \infty$. □

Finally, it should be underlined that a brief description of Krüppel's counterexample can be also found in Section 2 of [11].

3.2.1 A finite one-sided derivative at no point

In 1982 P. Billingsley gave a simple and elegant proof of the nowhere differentiability of the Takagi function (see [25]). Two years later, F. S. Cater modified the argument given by Billingsley in order to prove that the Takagi function has a finite one-sided derivative at no point (see [34]). A generalization of Cater's argument together with the tools we have developed in this section, allow us to obtain a similar result for the Takagi-Van der Waerden function.

Proposition 3.13. *Let $r \ge 2$ be a integer. Then, the Takagi-Van der Waerden function has finite one-sided derivative at no point.*

Proof. Let $x \in [0,1]$. If $x \in D$ then $f_r'^{+}(x)=+\infty$ and $f_r'^{-}(x)=-\infty$ by Proposition 3.4, meanwhile if r is odd and $x \in \widetilde{D}$ then $f_r'^{+}(x)=-\infty$ and $f_r'^{-}(x)=+\infty$ by Proposition 3.5. From now on, we assume $x \notin D \cup \widetilde{D}$. For the sake of contradiction, suppose $f_r'^{+}(x)=L \in \mathbb{R}$. For every n, we recall that $b_n=\min\{z \in D_n : z > x\}$ and we consider

$$y_n = b_n + \frac{1}{2r^{n-1}},$$

which belongs to $\widetilde{D}_n$. We write

$$f_r(b_n)-f_r(x)=(L+t_n)(b_n-x), \quad \text{and} \quad f_r(y_n)-f_r(x)=(L+s_n)(y_n-x).$$

Hence

$$\begin{aligned} f_r(y_n)-f_r(b_n) &= \left[f_r(y_n)-f_r(x)\right]-\left[f_r(b_n)-f_r(x)\right] = (L+s_n)(y_n-x)-(L+t_n)(b_n-x) \\ &= L(y_n-b_n)+s_n(y_n-x)-t_n(b_n-x) = \frac{L}{2r^{n-1}}+s_n(y_n-x)-t_n(b_n-x) \end{aligned}$$

and consequently,

$$2r^{n-1}\left[f_r(y_n)-f_r(b_n)\right] = L+2r^{n-1}s_n(y_n-x)-2r^{n-1}t_n(b_n-x).$$

Since $0<b_n-x<r^{-(n-1)}$ and $0<y_n-x<3/2r^{n-1}$ we get

$$\left|2r^{n-1}\left[f_r(y_n)-f_r(b_n)\right]-L\right| \le 3|s_n|+2|t_n|$$

which yields

$$\lim_n 2r^{n-1}\left[f_r(y_n)-f_r(b_n)\right] = L \tag{3.13}$$

because $\lim_n s_n = \lim_n t_n = 0$. However, we have

$$2r^{n-1}\left[f_r(y_n)-f_r(b_n)\right] = \sum_{j=1}^{\infty}\frac{g_j(y_n)-g_j(b_n)}{y_n-b_n} = \sum_{j=1}^{n} g_j'(x)-\sum_{j=n+1}\frac{g_j(y_n)}{y_n-b_n}$$

where we have used that the function g_j is affine on $[b_n, y_n]$ for every $1\le j\le n$. If r is even then $g_j(y_n)=0$ for every $j\ge n+1$ since $\widetilde{D}_m\subset D_{m+1}$ for every m. If r is odd then

$$\frac{1}{y_n-b_n}\sum_{j=n+1}^{\infty} g_j(y_n) = 2r^{n-1}\sum_{j=n+1}^{\infty}\frac{1}{2r^{j-1}} = \frac{1}{r-1}$$

since $\widetilde{D}_m\subset\widetilde{D}_{m+1}$ for every m. In both cases, the fact that

$$\sum_{j=1}^{n} g_j'(x)$$

cannot converge to a finite limit contradicts (3.13). The argument for the left-hand derivative is similar. □

3.3 Characterization of the points with an infinite derivative

This section is devoted to presenting a condition that the base r expansion of a point not belonging to $D\cup\widetilde{D}$ must satisfy for the existence of an infinite one-sided derivative at such a point. Theorem 3.15 deals with the right-hand derivative, meanwhile Theorem 3.16 concerns the left-hand derivative. It should be pointed out that the proofs of both theorems are similar in structure and Examples 3.11 and 3.12 gave us a clear hint as to how these proofs should be executed. The following result is used for proving both theorems.

Lemma 3.14. *Let $r \geq 2$ be an integer and $\ell \geq 2$. Then, there exists $x_0 \in (0, \ell)$ satisfying that $r^{-x_0}(\ell - x_0) + x_0 < \log_r \ell + 3$.*

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) = r^{-x}(\ell - x) + x - \frac{\log \ell}{\log r}$. We prove that the function φ attains its minimum at a point $x_0 \in (0, \ell)$ and $\varphi(x_0) < 3$. Indeed, it is clear that $\varphi'(x) < 0$ when $x \leq 0$ and $\varphi'(x) > 0$ when $x \geq \ell$. Therefore, φ attains its minimum at a point $x_0 \in (0, \ell)$ and $\varphi'(x_0) = -r^{-x_0}(\ell - x_0)\log r + 1 - r^{-x_0} = 0$. Hence, we have $r^{x_0} - 1 = (\ell - x_0)\log r$ and we obtain

$$\begin{aligned}\varphi(x_0)\log r &= r^{-x_0}(l - x_0)\log r + x_0 \log r - \log l = r^{-x_0}(l - x_0)\log r + \log \frac{r^{x_0}}{\ell}\\ &= 1 - r^{-x_0} + \log \frac{r^{x_0}}{\ell} = 1 - r^{-x_0} + \log \frac{1 + (\ell - x_0)\log r}{\ell}\\ &< 1 + \log(1 + \log r) < 1 + \log r < 3\log r\end{aligned}$$

which gives us the result. ☐

When $r \geq 2$ is an even integer, Theorem 3.10 states that the right-hand derivative of the Takagi-Van der Waerden function at a point not belonging to $D \cup \widetilde{D}$ is $+\infty$ if and only the series of the derivatives converges to $+\infty$ at such a point. Example 3.11 shows that this theorem does not hold when $r \geq 2$ is an odd integer, however it leads us to focus our attention on the digit $(r-1)/2$ in order to obtain the corresponding result for this case.

For a point $x \notin D \cup \widetilde{D}$ with base-r expansion $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$ we arrange the infinite set $\{i : \varepsilon_i \neq (r-1)/2\}$ as an increasing sequence $(i_n)_n$. We note that $i_n = n$ for every n whenever $r \geq 2$ is even, and hence Theorem 3.15 below is equivalent to Theorem 3.10 in this case. At first glance, the logarithmic term that appears in Condition (3.14) below may seem a little surprising. For this reason, we will finish this section with some examples in order to gain a better understanding of its meaning.

We advise the reader to recall the discussion in Section 3.2 from before, where we described when the situation $d_n = d_{n+1}$ holds in terms of the base r expansion of $x \notin D \cup \widetilde{D}$. It is repeatedly used throughout the proof of the following result.

Theorem 3.15. *Let $r \geq 2$ be an integer and $x \notin D \cup \widetilde{D}$. Then, $f_r'^{+}(x) = +\infty$ if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n)\right) = +\infty. \tag{3.14}$$

Proof. We only have to prove the case when r is odd, because if r is even then $i_n = n$ for every n, so Condition (3.14) becomes equivalent to the convergence to $+\infty$ of the series of the derivatives, and hence the result follows from Theorem 3.10.

We assume first that Condition (3.14) holds. Let $r^{-(p+1)} \leq 2h < r^{-p}$ for some non-negative integer p and let n be such that $i_n < p \leq i_{n+1}$. In the sequel we denote

$$\Delta_k(h) = \frac{g_k(x+h) - g_k(x)}{h}$$

and let us observe that $\Delta_k(h) \in [-1,1]$ for every k. Now, if $s \geq p$ then

$$\left|\frac{f_r(x+h) - f_r(x)}{h} - \sum_{k=1}^{s} \Delta_k(h)\right| = \left|\sum_{k=s+1}^{\infty} \Delta_k(h)\right| \leq \frac{1}{2h} \sum_{k=s+1}^{\infty} \frac{1}{r^{k-1}} \leq \frac{r^2}{r-1}. \tag{3.15}$$

Let $1 \leq k \leq i_n$. If $\varepsilon_{i_n} \neq r-1$ then, by the discussion in Section 3.2 it follows that $d_{i_n+1} < d_{i_n}$ and therefore $d_k \geq d_{i_n} > r^{-i_n}/2 \geq r^{-(p-1)}/2 > h$. Hence, g_k is affine on $[x, x+h]$ for every $1 \leq k \leq i_n$ and consequently $\Delta_k(h) = g_k'(x)$ for every $1 \leq k \leq i_n$. If $\varepsilon_{i_n} = r-1$ then we take $k_0 = \max\{k < i_n : \varepsilon_k < r-1\}$. If the set $\{k < i_n : \varepsilon_k < r-1\}$ is empty, then $k_0 = 0$. By the discussion in Section 3.2, we have $d_{k_0} > d_{k_0+1}$ and $d_{k_0} > r^{-k_0}/2 \geq r^{-(i_n+1)}/2 > h$. Therefore, g_k is affine on $[x, x+h]$ for every $1 \leq k \leq k_0$ and hence $\Delta_k(h) = g_k'(x)$ for every $1 \leq k \leq k_0$. Since $g_k'(x) = -1$ provided that $k_0 < k \leq i_n$, we get $\Delta_k(h) \geq g_k'(x)$ for every $k_0 < k \leq i_n$.

Thus, in both cases we obtain

$$\sum_{k=1}^{i_n} \Delta_k(h) \geq \sum_{k=1}^{i_n} g_k'(x). \tag{3.16}$$

In particular, if $p = i_{n+1} = i_n + 1$ then by (3.15) and (3.16) we get

$$\frac{f_r(x+h) - f_r(x)}{h} = \sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{i_{n+1}} g_k'(x) - 2 - \frac{r^2}{r-1}$$

and we are done. Therefore, in the sequel we suppose $i_{n+1} - i_n > 1$.

If $\varepsilon_{i_{n+1}} > (r-1)/2$ then $g_k'(x) = -1$ for all $i_n < k \leq i_{n+1}$ and hence, by (3.15) and (3.16), we obtain

$$\sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{i_{n+1}} g_k'(x) - \frac{r^2}{r-1}.$$

If $\varepsilon_{i_{n+1}} < (r-1)/2$ then $x + d_k \in \widetilde{D}_k$ for all $i_n < k \leq i_{n+1}$ and, by the discussion in Section 3.2, we obtain that $d_{i_n+1} = \cdots = d_{i_{n+1}} < r^{-(i_{n+1}-1)}/2$. If $h \leq d_{i_{n+1}}$ then

$$\sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{i_{n+1}} g_k'(x) - \frac{r^2}{r-1}.$$

If $h > d_{i_{n+1}}$ we have that $x + d_k < x + h \leq x + r^{-p}/2$ and then

$$g_k(x+h) - g_k(x) = x + d_k + \frac{1}{2r^{k-1}} - (x+h) - \left(x - \left(x + d_k - \frac{1}{2r^{k-1}}\right)\right) = 2d_k - h \tag{3.17}$$

for every k such that $i_n < k \le p$. By the discussion in Section 3.2, we have $d_{i_{n+1}} > d_{i_{n+1}+1}$ and then $d_{i_{n+1}} > r^{-i_{n+1}}/2$. Therefore,

$$\begin{aligned}\sum_{k=i_n+1}^{p} \Delta_k(h) \ge \sum_{k=i_n+1}^{p} \frac{d_k}{h} - 1 \ge (p-i_n)(-1+r^{p-i_{n+1}}) &= -(p-i_n)+(p-i_n)r^{p-i_{n+1}} \\ &= -(i_{n+1}-i_n)-(p-i_{n+1})+(i_{n+1}-i_n)r^{p-i_{n+1}}-(i_{n+1}-p)r^{p-i_{n+1}} \\ &\ge -(i_{n+1}-i_n)+\log_r(i_{n+1}-i_n)-(i_{n+1}-p)r^{p-i_{n+1}} \\ &\ge -(i_{n+1}-i_n)+\log_r(i_{n+1}-i_n)-1. \end{aligned} \tag{3.18}$$

Finally, we conclude

$$\sum_{k=1}^{\infty} \Delta_k(h) \ge \sum_{k=1}^{i_n} g'_k(x) - (i_{n+1}-i_n) + \log_r(i_{n+1}-i_n) - 1 - \frac{r^2}{r-1}$$

which gives us the result.

Conversely, let us suppose that $f_r'^{+}(x) = +\infty$. Without loss of generality, we may assume $i_{n+1} - i_n \ge 5$, since otherwise

$$\sum_{k=1}^{i_n} g'_k(x) - (i_{n+1}-i_n) + \log_r(i_{n+1}-i_n) \ge \sum_{k=1}^{i_n} g'_k(x) - 4$$

and the result follows from Proposition 3.8.

If $\varepsilon_{i_{n+1}} > (r-1)/2$ then, by the discussion in Section 3.2, we have $d_{i_{n+1}-1} > d_{i_{n+1}}$ and hence $d_{i_{n+1}-1} > r^{-(i_{n+1}-1)}/2$. Therefore, if we take $2h_{n+1} = r^{-(i_{n+1}-1)}$ then $\Delta_k(h_{n+1}) = g'_k(x)$ for every $1 \le k \le i_{n+1}-1$ and, by (3.15) we obtain

$$\sum_{k=1}^{\infty} \Delta_k(h_{n+1}) \le \sum_{k=1}^{i_{n+1}} g'_k(x) + \frac{r^2}{r-1}. \tag{3.19}$$

If $\varepsilon_{i_{n+1}} < (r-1)/2$ then

$$d_{i_{n+1}} < d_{i_n} \quad \text{and} \quad d_{i_{n+1}} < \frac{1}{2r^{i_{n+1}-1}}$$

by discussion 3.2. Assume $i_n < p \le i_{n+1}-3$. As we have seen in (3.17), if we take $h_{n+1} > 0$ satisfying $r^{-(p+1)} \le 2h_{n+1} = r^{-(p+\alpha)} < r^{-p}$ for certain $\alpha > 0$ that will be specified below, then

$$\sum_{k=1}^{p} \Delta_k(h_{n+1}) = \sum_{k=1}^{i_n} g'_k(x) + (p-i_n)\left(\frac{2d_{i_{n+1}}}{h_{n+1}} - 1\right) \le \sum_{k=1}^{i_n} g'_k(x) + (p-i_n)\left(r^{p+\alpha-i_{n+1}+2} - 1\right). \tag{3.20}$$

Let x_0 be as in Lemma 3.14 for $\ell = i_{n+1} - i_n - 3$ and let $p = i_{n+1} - 3 - [x_0]$ where $[x_0]$ stands for the integer part of x_0. It is clear that $i_n < p \le i_{n+1} - 3$. We take $\alpha = [x_0] + 1 - x_0$ and recall that $2h_{n+1} = r^{-(p+\alpha)}$. Therefore, $x_0 = -(p+\alpha - i_{n+1} + 2)$ and $\ell - x_0 = p - i_n + [x_0] - x_0$. From Lemma 3.14 we get

$$(p - i_n + [x_0] - x_0) r^{p+\alpha-i_{n+1}+2} + x_0 \le \log_r(i_{n+1} - i_n - 3) + 3$$

and consequently,

$$\begin{aligned}(p - i_n)\left(r^{p+\alpha-i_{n+1}+2} - 1\right) &\le (p - i_n + [x_0] - x_0)\left(r^{p+\alpha-i_{n+1}+2} - 1\right) \\ &= (p - i_n + [x_0] - x_0) r^{p+\alpha-i_{n+1}+2} + x_0 - (p - i_n + [x_0]) \\ &\le \log_r(i_{n+1} - i_n - 3) + 3 - (i_{n+1} - i_n - 3) \\ &\le -(i_{n+1} - i_n) + \log_r(i_{n+1} - i_n) + 6.\end{aligned}$$

This together with (3.15) and (3.20) yields

$$\frac{f_r(x + h_{n+1}) - f_r(x)}{h_{n+1}} \le \sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n) + 6 + \frac{r^2}{r-1}. \tag{3.21}$$

Finally, by (3.19) and (3.21) we conclude

$$\frac{f_r(x + h_{n+1}) - f_r(x)}{h_{n+1}} = \sum_{k=1}^{\infty} \Delta_k(h_{n+1}) \le \sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n) + 6 + \frac{r^2}{r-1}$$

for every n, and this gives the result. □

It is worth noting that $i_n = 10^n$ for every n when considering the point given in Example 3.11, and hence

$$\lim_{n\to\infty}\left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_r(i_{n+1} - i_n)\right) = \lim_{n\to\infty}\left(2i_n - i_{n+1} + \log_r(i_{n+1} - i_n)\right) = -\infty$$

which gives $f_3'^{+}(x) \neq +\infty$ by Theorem 3.15.

Regarding the left-hand derivative of the Takagi-Van der Waerden function, we should recall that Allaart and Kawamura (see [11]) and Krüppel (see [81]) proved that $T'^{-}(x) = +\infty$ if and only if

$$\lim_{n\to\infty}\left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1} - m_n) + \log_2(m_{n+1} - m_n)\right) = +\infty. \tag{3.22}$$

where m_n is the position of the n-th 1-digit in the binary expansion of the non-dyadic point x. This together with Example 3.12 lead us to introduce the following notation.

For a point $x \notin D \cup \widetilde{D}$ with base-r expansion $x = \varepsilon_1\varepsilon_2\varepsilon_3\ldots$ we arrange the infinite set $\{m : \varepsilon_m \neq 0\}$ as an increasing sequence $(m_n)_n$. This allows us to present the Theorem 3.16 below, and as in Theorem 3.15, we must emphasize the appearance of the logarithmic term in Condition (3.23).

Theorem 3.16. *Let $r \geq 2$ be an integer and $x \notin D \cup \widetilde{D}$. Then, $f_r'^{-}(x) = +\infty$ if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1}-m_n) + \log_r(m_{n+1}-m_n)\right) = +\infty. \tag{3.23}$$

Proof. We assume first that Condition (3.23) holds. Let $r^{-(p+1)} \leq 2h < r^{-p}$ for some non-negative integer p and let n such that $m_n \leq p < m_{n+1}$. As we did in Theorem 3.15, we denote

$$\Delta_k(h) = \frac{g_k(x-h)-g_k(x)}{-h}$$

and if $s \geq p$ then

$$\left|\sum_{k=s+1}^{\infty} \Delta_k(h)\right| \leq \frac{r^2}{r-1}. \tag{3.24}$$

Let $1 \leq k \leq m_n$ and recall that a_k denotes the biggest element of D_k smaller than x, that is $a_k = \sum_{j=1}^{k-1} \varepsilon_j r^{-j}$. If $g_k'(x) = 1$ then $g_k(x) = x - a_k$, and $g_k(x-h) = x-h-a_k$ since $h < r^{-p} \leq r^{-m_n} \leq x - a_{m_n} \leq x - a_k$, where the third inequality holds because $\varepsilon_{m_n} \neq 0$. Therefore,

$$\Delta_k(h) = g_k'(x) = 1 \tag{3.25}$$

and hence we obtain

$$\sum_{k=1}^{m_n} \Delta_k(h) \geq \sum_{k=1}^{m_n} g_k'(x). \tag{3.26}$$

In particular, if $m_{n+1} = m_n + 1$ then, by (3.24) and (3.26) we get

$$\sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{m_n} g_k'(x) - \frac{r^2}{r-1}$$

and we are done. Therefore, in the sequel we suppose $m_{n+1} - m_n > 1$.

Let $m_n < k \leq p$. We have $g_k'(x) = 1$ and $a_k = a_p$. On the one hand, if $x - a_p > h$ then $g_k(x-h) = x-h-a_p$ and $\Delta_k(h) = g_k'(x) = 1$, which gives

$$\sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{m_n} g_k'(x) - \frac{r^2}{r-1}$$

by (3.24) and (3.26). On the other hand, if $x - a_p \leq h$ then $g_k(x-h) = a_p - (x-h)$ and consequently

$$\begin{aligned}\Delta_k(h) &= \frac{a_p-(x-h)-(x-a_p)}{-h} = -1 + \frac{2(x-a_p)}{h} \\ &= -1 + \frac{2}{h}\sum_{j=p}^{\infty}\frac{\varepsilon_j}{r^j} = -1 + \frac{2}{h}\sum_{j=m_{n+1}}^{\infty}\frac{\varepsilon_j}{r^j}\end{aligned} \tag{3.27}$$

since $\varepsilon_{m_{n+1}} \neq 0$, which yields

$$\Delta_k(h) \geq -1 + \frac{2}{hr^{m_{n+1}}} \geq -1 + 4r^{p-m_{n+1}} \geq -1 + r^{p-m_{n+1}}.$$

Proceeding in a similar way as we did in (3.18), we obtain

$$\sum_{k=m_n+1}^{p} \Delta_k(h) \geq (p-m_n)\left(-1 + r^{p-m_{n+1}}\right) \geq -(m_{n+1}-m_n) + \log_r(m_{n+1}-m_n) - 1$$

and finally, we conclude

$$\sum_{k=1}^{\infty} \Delta_k(h) \geq \sum_{k=1}^{m_n} g_k'(x) - (m_{n+1}-m_n) + \log_r(m_{n+1}-m_n) - 1 - \frac{r^2}{r-1}$$

which gives us the result.

Conversely, let us suppose that $f_r'^{-}(x) = +\infty$. Without loss of generality, we may assume that $m_{n+1} - m_n \geq 6$, since otherwise

$$\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1}-m_n) + \log_r(m_{n+1}-m_n) \geq \sum_{k=1}^{m_n} g_k'(x) - 5$$

and the result follows from statement (2) of Proposition 3.8.

Let $m_n < p \leq m_{n+1} - 4$ and we take $h_{n+1} > 0$ satisfying $r^{-(p+1)} \leq 2h_{n+1} = r^{-(p+\alpha)} < r^{-p}$ for a certain $\alpha > 0$ that will be specified below. Let us note that if $\Delta_k(h_{n+1}) = g_k'(x)$ for some k, then $\Delta_j(h_{n+1}) = g_j'(x)$ for all $1 \leq j \leq k$. As we have seen in (3.25), if $g_{m_n}'(x) = 1$ then $\Delta_{m_n}(h_{n+1}) = g_{m_n}'(x) = 1$ and observe that if $\varepsilon_{m_n} = (r-1)/2$ then $g_{m_n}'(x) = 1$ since $\varepsilon_{m_n+1} = 0$. Therefore, only the case when $\varepsilon_{m_n} > (r-1)/2$ remains to be considered:

If $\varepsilon_{m_n} > r/2$ then $g_{m_n}'(x) = -1$ and $g_{m_n}(x - h_{n+1}) = a_{m_n} + r^{-(m_n-1)} - (x - h_{n+1})$ since

$$h_{n+1} < \frac{1}{2r^p} \leq \frac{1}{2r^{m_n}} \leq \frac{\varepsilon_{m_n}}{r^{m_n}} - \frac{1}{2r^{m_n-1}} \leq x - a_{m_n} - \frac{1}{2r^{m_n-1}},$$

which gives $\Delta_{m_n}(h_{n+1}) = g_{m_n}'(x) = -1$.

If $\varepsilon_{m_n} = r/2$, which occurs only when r is even, we distinguish the following cases: If $\varepsilon_{m_n-1} = r/2$ then $\Delta_{m_n-1}(h_{n+1}) = g_{m_n-1}'(x) = -1$ since

$$h_{n+1} < \frac{1}{2r^p} \leq \frac{1}{2r^{m_n}} \leq \frac{\varepsilon_{m_n}}{r^{m_n}} = \frac{\varepsilon_{m_n-1}}{r^{m_n-1}} + \frac{\varepsilon_{m_n}}{r^{m_n}} - \frac{1}{2r^{m_n-2}} \leq x - a_{m_n-1} - \frac{1}{2r^{m_n-2}},$$

meanwhile if $\varepsilon_{m_n-1} \neq r/2$ then the function g_{m_n-1} is affine on $[x-h_{n+1}, x]$ because $\varepsilon_{m_n} = r/2$, and hence $\Delta_{m_n-1}(h_{n+1}) = g_{m_n-1}'(x)$. From the previous argument we deduce

$$\sum_{k=1}^{m_n-1} \Delta_k(h_{n+1}) = \sum_{k=1}^{m_n-1} g_k'(x). \tag{3.28}$$

Now, let us consider $m_n < k \leq p$. We have $x - a_k = x - a_p \leq h_{n+1}$ and, as we have seen in (3.27), we also get

$$\Delta_k(h_{n+1}) = -1 + \frac{2(x-a_p)}{h_{n+1}}.$$

This together with (3.28) yield

$$\begin{aligned}\sum_{k=1}^{p} \Delta_k(h_{n+1}) &= \sum_{k=1}^{m_n-1} g_k'(x) + \Delta_{m_n}(h_{n+1}) + (p-m_n)\left(-1 + \frac{2(x-a_p)}{h_{n+1}}\right) \\ &\leq \sum_{k=1}^{m_n} g_k'(x) + 2 + (p-m_n)\left(r^{p+\alpha-m_{n+1}+3} - 1\right),\end{aligned}$$

where the last inequality holds since $a_p = a_{m_{n+1}}$.

Once we come to this point, we proceed as we did in Theorem 3.15 in order to choose the appropriate value of α and consequently, the value of h_{n+1}. Then, the result follows as in Theorem 3.15. □

It is worth noting that for the point defined in Example 3.11, that is

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n},$$

where $\varepsilon_n = 0$ if $n = 10^k$ for some k, and $\varepsilon_n = 1$ otherwise. We have $f_3'^{-}(x) = +\infty$.

Finally, we will take advantage of the previous theorems together with the symmetry equation to deduce the following result concerning the case when f_r has derivative $-\infty$ at a point. To do so, for every $x \notin D \cup \widetilde{D}$ with base r expansion $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$ we arrange the infinite set $\{p : \varepsilon_p \neq r-1\}$ as an increasing sequence $\{p_n\}_n$.

Corollary 3.17. *Let $r \geq 2$ be an integer and $x \notin D \cup \widetilde{D}$. The following statements hold:*

1. $f_r'^{+}(x) = -\infty$ *if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{p_n} g_k'(x) + (p_{n+1}-p_n) - \log_r(p_{n+1}-p_n)\right) = -\infty.$$

2. $f_r'^{-}(x) = -\infty$ *if and only if*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{i_n} g_k'(x) + (i_{n+1}-i_n) - \log_r(i_{n+1}-i_n)\right) = -\infty.$$

Proof. It is enough to observe that

$$1 - x = \sum_{k=1}^{\infty} \frac{(r-1)-\varepsilon_k}{r^k}$$

and the result follows immediately from the previous theorems. □

3.3.1 Some examples

Theorems 3.15 and 3.16 together with Corollary 3.17 completely characterize the set of points where the Takagi-Van der Waerden function has an infinite one-sided derivative. In particular, they generalize Theorem 3.1. of [11] and Proposition 4.5. of [80] for the Takagi function. Since the conditions that appear in such results may look a bit mysterious, we set out below some examples in order to provide further insight into these conditions.

Example 3.18. Let $r = 3$ and we consider the point

$$x = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{3^{2^n}} = \sum_{n=1}^{\infty} \frac{1}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^{2^n}}$$

Then, we have $f_3'(x) = +\infty$.

Proof. Firstly, using the notation of Theorem 3.15, we have $i_n = 2^n$ since $\varepsilon_{2^n}(x) = 0$ and $\varepsilon_m = 1$ if $m \neq 2^n$ for every n. From Lemma 3.1 it follows $g_m'(x) = 1$ for every m, and hence

$$\begin{aligned} &\lim_{n\to\infty} \left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_3(i_{n+1} - i_n) \right) \\ &= \lim_{n\to\infty} \left(2^n - (2^{n+1} - 2^n) + \log_3(2^{n+1} - 2^n) \right) = +\infty \end{aligned} \tag{3.29}$$

which implies $f_3'^{+}(x) = +\infty$ by Theorem 3.15.

Secondly, using the notation of Theorem 3.16, we have $m_{n+1} - m_n \leq 2$ and consequently

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m_n} g_k'(x) - (m_{n+1} - m_n) + \log_3(m_{n+1} - m_n) \right) = +\infty,$$

which implies $f_3'^{-}(x) = +\infty$ by Theorem 3.16. □

The previous example shows the role the term $\log_r(i_{n+1} - i_n)$ plays in this matter, since without that term the limit (3.29) would be zero.

Example 3.19. Let $r = 3$ and we consider the point

$$x = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{3^{3^k}}.$$

Then, we have $f_3'^{+}(x) \neq \pm\infty$ and $f_3'^{-}(x) = +\infty$.

Proof. From Lemma 3.1 it follows that $g_k'(x) = 1$ for every k. This implies $f_3'^{+}(x) \neq -\infty$ by statement (2) of Proposition 3.8. Furthermore, using the notation of Theorem 3.15, we have $i_n = 3^n$ for every n and hence

$$\lim_{n\to\infty} \left(\sum_{k=1}^{i_n} g_k'(x) - (i_{n+1} - i_n) + \log_3(i_{n+1} - i_n) \right) = -\infty.$$

Therefore, $f_3'^{+}(x) \neq \pm\infty$. In order to see that $f_3'^{-}(x) = +\infty$, using the notation of Theorem 3.16 we have $m_{n+1} - m_n \leq 2$ for every n, and the result follows by Theorem 3.16. □

3.4 Hausdorff dimension of the set of points with an infinite derivative

Once it is known that the set of points where the Takagi-Van der Waerden function has an infinite derivative has Lebesgue measure zero (see Corollary 3.9), it is natural to ask about the Hausdorff dimension of such a set.

Following this line of thought, Allaart and Kawamura (see [11]) proved that the set of points where the Takagi function possesses an infinite derivative has Hausdorff dimension one. This motivates us to address this issue for the Takagi-Van der Waerden function. In order to prove their result, Allaart and Kawamura realized that $\{x \in [0,1] : T'(x) = \infty\}$ contains the sets $\{x \in [0,1] : d_1(x) = \alpha\}$ for $0 < \alpha < 1/2$, where $d_1(x)$ denotes the density of the point $x = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n}$, $\varepsilon_n \in \{0,1\}$ defined by

$$d_1(x) = \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \varepsilon_n.$$

A result proved by H. G. Eggleston (see [42]) states that the Hausdorff dimension of the set $\{x \in [0,1] : d_1(x) = \alpha\}$ is

$$\frac{-\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha)}{\log 2} \tag{3.30}$$

and hence, the Hausdorff dimension of the set $\{x \in [0,1] : T'(x) = \infty\}$ is greater than or equal to the supremum of (3.30) when $0 < \alpha < 1/2$, which gives us the result.

The previous technique can be used to obtain the corresponding result for the Takagi-Van der Waerden function when $r \geq 2$ is even, but it is not trivial to extend this technique in a simple way when considering the odd case. This justifies the approach we adopt below.

In Section 3.1, for a point $x \notin D \cup \widetilde{D}$ with base-r expansion given by

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{r^n}, \quad \varepsilon_n \in \{0, \dots, r-1\} \tag{3.31}$$

we have defined the numbers

$$O_m(x) = \#\left\{n \leq m : \varepsilon_{k_n} < (r-1)/2\right\}, \quad \text{and} \quad I_m(x) = \#\left\{n \leq m : \varepsilon_{k_n} > (r-1)/2\right\}$$

where $k_n = \min\{k \geq n : \varepsilon_k \neq (r-1)/2\}$. In addition, we recall that $a_m = a_m(x)$ is the biggest element of D_m smaller than or equal to x which is given by

$$a_m = \sum_{k=1}^{m-1} \frac{\varepsilon_j}{r^j}.$$

Let $m \geq 3$ be an odd integer. We define the set

$$B_m = \left\{ z = \sum_{j=1}^{m} \frac{\varepsilon_j}{r^j} > 0 : O_m(z) - I_m(z) \geq 1, \varepsilon_m \neq (r-1)/2 \right\}$$

and we observe $B_m \subset D_{m+1} \setminus \{0,1\}$. Moreover, for every $k = 0,1,2,\ldots$ we also define

$$A_{m,k} = \left\{ x \in [0,1] : r^{mk} a_{m(k+1)+1} - \left[r^{mk} a_{m(k+1)+1} \right] \in B_m \right\}$$

where $[x]$ stands for the greatest integer smaller than or equal to x.

It should be acknowledged that if $x \in A_{m,k}$ with base r expansion given by (3.31) then

$$r^{mk} a_{m(k+1)} - \left[r^{mk} a_{m(k+1)} \right] = \sum_{n=mk+1}^{mk+m} \frac{\varepsilon_n}{r^{n-mk}} \in B_m$$

which yields $\varepsilon_{mk+m} \neq (r-1)/2$ and we have

$$O_m \left(\sum_{n=mk+1}^{mk+m} \varepsilon_n r^{-n+mk} \right) - I_m \left(\sum_{n=mk+1}^{mk+m} \varepsilon_n r^{-n+mk} \right) = \sum_{n=mk+1}^{mk+m} g_n'(x) \qquad (3.32)$$

by equation (3.3). To summarize, a point belongs to $A_{m,k}$ if and only if the $(k+1)$-th block of m digits long of its base r expansion satisfies the condition in the definition of B_m. Finally, we define

$$A_m = \bigcap_{k=0}^{\infty} A_{m,k}$$

and, continuing in the same line of thought, we may conclude that the binary expansion of any point belonging to A_m consists of a sequence of m-digits blocks where each of them satisfies the condition in the definition of B_m.

The approach we have adopted in the following lemmas is based on looking at the set A_m as a self-similar set and computing its similarity dimension, which coincides with its Hausdorff dimension whenever the open set condition is fulfilled.

We recall that a mapping $\varphi : [0,1] \to [0,1]$ is a similarity of ratio $0 < r < 1$ if $|\varphi(x) - \varphi(y)| = r|x-y|$ for all $x, y \in [0,1]$. It is well-known that a finite family of similarities $\varphi_1, \varphi_2, \ldots, \varphi_p$ with ratios $r_1, r_2, \ldots, r_p$ respectively, determines a unique non-empty compact subset F of $[0,1]$ satisfying

$$F = \bigcup_{n=1}^{p} \varphi_n(F).$$

This set receives the name of attractor for such a collection of similarities. Since it is a union of a number of smaller copies of itself, it is called a self-similar set and its similarity dimension is defined as the number s that satisfies

$$\sum_{n=1}^{p} r_n^s = 1. \qquad (3.33)$$

It is also well-known that, under fairly general conditions, the Hausdorff dimension of the set F agrees with its similarity dimension. More precisely, if the family of similarities $\varphi_1, \varphi_2, \ldots, \varphi_p$ fulfils the "open set condition", that is, there exists a non-empty bounded open set V such that

$$\bigcup_{n=1}^{p} \varphi_n(V) \subset V \tag{3.34}$$

with this union disjoint, then the self-similar set F has Hausdorff dimension given by its similarity dimension. We refer to the books [41] and [45] for details on self-similar sets and their similarity dimension.

Lemma 3.20. *Let $r \geq 3$ be an odd integer. If $m \geq 3$ is an odd integer, then the Hausdorff dimension of the set A_m is greater than or equal to*

$$1 - \frac{\log r + 1}{m \log r}.$$

Proof. For every m we partition the interval $[0,1]$ into r^m equal length intervals and we denote each of them by $J_d = [d, d + r^{-m}]$ where $d \in D_{m+1} \setminus \{1\}$. We also define the function $\varphi_d : [0,1] \to J_d$ by $\varphi_d(x) = d + x r^{-m}$, which is a similarity of ratio r^{-m}. We claim that the set A_m is the attractor for the family of similarities $\left(\varphi_d\right)_{d \in B_m}$, that is, A_m is the unique non-empty compact set that satisfies

$$A_m = \bigcup_{d \in B_m} \varphi_d(A_m) \tag{3.35}$$

Indeed, we observe first that if $y \in \varphi_d(A_m)$ for some $d \in B_m$, then there is $x \in A_m$ such that $\varphi_d(x) = y$. Moreover, if $x \in A_{m,k}$ for some k then $\varphi_d(x) \in A_{m,k+1}$, and hence $y \in A_m$. Secondly, if $x \in A_m$ then $y = r^m (x - a_{m+1})$ belongs to A_m as well, and consequently $\varphi_{a_{m+1}}(y) = x$. This proves (3.35).

The set A_m is a self-similar set and by (3.33), its similarity dimension is the unique non-negative number s satisfying $\#B_m = r^{ms}$. Furthermore, the family of similarities $\left(\varphi_d\right)_{d \in B_m}$ satisfies the "open set condition" (3.34) for $V = (0,1)$ since

$$\bigcup_{d \in B_m} \varphi_d\left((0,1)\right) \subset (0,1)$$

and consequently, the Hausdorff dimension of the set A_m is given by its similarity dimension.

Now, we compute the cardinality of the set B_m. As we did in the proof of Theorem 3.3, we consider the symmetry function $S : [0,1] \to [0,1]$ defined by $S(x) = 1 - x$ and we have that $S(J_d) \cap J_d = \emptyset$ for every $d \in B_m$ because the m-th digit of d is different from $(r-1)/2$. In addition, for $d_1, d_2 \in D_{m+1} \setminus \{1\}$ it is easy to see that $S(J_{d_1}) = J_{d_2}$ if and only if $d_2 = S(d_1) - r^{-m}$. In this case, we make the following observations:

- The m-th digit in the base-r expansion of d_1 is $(r-1)/2$ if and only if the m-th digit in the base-r expansion of d_2 is $(r-1)/2$.
- Assume that the m-th digit in the base-r expansion of d_1 is not $(r-1)/2$. Then, $d_1 \in B_m$ if and only if $d_2 \notin B_m \cup \{0\}$.

From these facts we deduce

$$\#B_m = \frac{r^{m-1}(r-1)}{2} - 1$$

and hence, the Hausdorff dimension of A_m is

$$\frac{\log(\#B_m)}{m\log r} = \frac{\log\left(\frac{r^{m-1}(r-1)}{2} - 1\right)}{m\log r} \geq \frac{\log\left(r^{m-1} - 1\right)}{m\log r} \geq \frac{(m-1)\log r - 1}{m\log r} = 1 - \frac{\log r + 1}{m\log r}.$$

□

Lemma 3.21. *Let $r \geq 2$ be an even integer. If $m \geq 3$ is an odd integer, then the Hausdorff dimension of the set A_m is greater than or equal to*

$$1 - \frac{\log 2 + 1}{m\log r}.$$

Proof. It is enough to observe $\#B_m = r^m/2 - 1$ provided that r is even, and the same argument that we used in Lemma 3.20 gives the result. □

Theorem 3.22. *Let $r \geq 2$ be an integer. The set $\{x : f_r'(x) = +\infty\}$ has Lebesgue measure zero and Hausdorff dimension one.*

Proof. As previously mentioned in the opening of the section, this set has Lebesgue measure zero by Corollary 3.9. Concerning the Hausdorff dimension, this is a consequence of the fact that

$$\left(A_m \setminus (D \cup \widetilde{D})\right) \subset \left\{x : f_r'(x) = +\infty\right\}$$

for every m. Indeed, if $x \notin D \cup \widetilde{D}$ belongs to A_m then $x \in A_{m,k}$ for every k and by (3.32), we get

$$\sum_{n=1}^{\infty} g_n'(x) = \sum_{k=0}^{\infty} \sum_{n=mk+1}^{m(k+1)} g_n'(x) \geq \sum_{k=0}^{\infty} 1 = +\infty.$$

Furthermore, according to Theorem 3.15 and Theorem 3.16, for every n we have $m_{n+1} - m_n < 2m$ because $0 \notin B_m$, meanwhile $i_{n+1} - i_n \leq m$ when r is odd and $i_{n+1} - i_n = 1$ when r is even. Finally, the result follows from Lemmas 3.20 and 3.21. □

The next result follows from Theorem 3.22 by using the symmetry equation $f_r(x) = f_r(1-x)$.

Theorem 3.23. *Let* $r \geq 2$ *be an integer. The set* $\{x : f_r'(x) = -\infty\}$ *has Lebesgue measure zero and Hausdorff dimension one.*

CHAPTER 4

NONSMOOTH ANALYSIS OF THE TAKAGI-VAN DER WAERDEN FUNCTIONS

In the absence of differentiability, it is very natural to wonder about properties involving nonsmooth concepts. Among the wide variety of nonsmooth concepts that can be found in the literature (see [93] or [59]), the concept of the (Fréchet) subdifferential has proved to be very efficient in the non-differentiable framework; it has been deeply investigated during the last few years. In turn, a rich and practical calculus has come to the forefront when working with the subdifferential (see [47] or [39]).

By the same token, S. X. Wang studied the subdifferential of the classical Weierstrass function. He obtained that the subdifferential is non-empty only on a countable dense set, and that the subdifferential at each point of that set is all of $\mathbb{R}$ (see Example 3.7.11 of [108]). We must point out that S. X. Wang's result was based on the work of K. M. Garg, who had previously described the Dini derivatives of the Weierstrass function (see Section 4 of [60]).

Among the properties of the subdifferential of a function $f : \mathbb{R} \to \mathbb{R}$, we must mention that the set of points with non-empty subdifferential is dense in the domain of f (see Theorem 4.21 of [47]), and the cardinality of the subdifferential of f is less than or equal to one at almost every point in the domain of f (see Theorem 4.4.3 of [59]). In light of these properties, the Weierstrass function is an extreme example with regard to these two properties of the subdifferential; the set of points where the subdifferential is non-empty is as "small as possible", which is a countable dense set. Whereas, the subdifferential at those points is as "large as possible", which is all of $\mathbb{R}$.

In 2011, P. Góra and R. J. Stern proved that the Takagi function shares the same property as the Weierstrass function when it comes to subdifferentiability behavior. They obtained that the subdifferential of the Takagi function is $\mathbb{R}$ at every dyadic point; whereas, it is empty otherwise (see [61]). In this sense, the Takagi function is an extreme case for the subdifferential and it is a much simpler example for this property than the Weierstrass function.

Seven years later, Juan Ferrera and Javier Gómez Gil showed that such an extreme subdifferentiability behavior can also be obtained for certain functions defined on a separable real Hilbert space (see [48]). For every dense and countable subset D of a separable real Hilbert space $\mathscr{H}$, they constructed a Takagi-Van der Waerden type function with non-empty subdifferential only on the set D, and its subdifferential is $\mathscr{H}$ at every point of D. As might be expected, their result includes the Takagi-Van der Waerden function as a particular case.

In consideration of the subdifferentiability behavior of the Takagi-Van der Waerden function, it seems natural to wonder about its superdifferentiability behavior. Roughly speaking, the superdifferential is the "complementary concept" to the subdifferential since a function is differentiable at a point if and only if the superdifferential and the subdifferential at such a point are both non-empty. Following this line of thought, Juan Ferrera and Javier Gómez Gil described the superdifferential of the Takagi function at a point in terms of its binary expansion (see [50]).

The purpose of this chapter is to present a characterization of the superdifferential of the Takagi-Van der Waerden function at a point in terms of the base-r expansion of such a point. Of course, this extends the previous result obtained by Juan Ferrera and Javier Gómez Gil for the Takagi function. As a consequence of our results, we can state that the Takagi-Van der Waerden function has empty subdifferential and superdifferential almost everywhere.

We take advantage of the fact that if a function has a local maximum at a point then zero is an element of the superdifferential. We also characterize the set of points where the Takagi-Van der Waerden function attains a local maximum. This extends the results obtained by J. P. Kahane concerning the local maxima of the Takagi function (see [75]), and completes the study of the extrema of the Takagi-Van der Waerden function initiated by Y. Baba (see [16]). This latter author proved that the set of global maxima of the Takagi-Van der Waerden function has Hausdorff dimension $1/2$ whenever r is even. We also compute the $1/2$- dimensional Hausdorff measure of such a set, which was not even known for the Takagi function.

4.1 An extreme case for the subdifferential

The aim of this section is to give a new demonstration of the subdifferentiability behavior of the Takagi-Van der Waerden function. It is based on the arguments that Juan Ferrera and Javier Gómez Gil used and the tools we have developed in Chapter 3. It should be underlined that the argument we present below is simpler and more concise than the argument used by P. Góra and R. S. Stern for the Takagi function.

A function $f:\mathbb{R}\to\mathbb{R}$ is lower semicontinuous at $x\in\mathbb{R}$ provided that

$$f(x)\le \liminf_{y\to x} f(y),$$

while it is upper semicontinuous at x whenever

$$f(x)\ge \limsup_{y\to x} f(y).$$

Of course, the function f is continuous at x if and only if it is both upper and lower semicontinuous at x.

For a lower semicontinuous function $f:\mathbb{R}\to\mathbb{R}$ and a point $x\in\mathbb{R}$, the subdifferential of f at x, denoted by $\partial^- f(x)$, is defined as the set of $\xi\in\mathbb{R}$ that satisfy

$$\liminf_{h\to 0}\frac{f(x+h)-f(x)-\xi h}{|h|}\ge 0.$$

The function f is said to be subdifferentiable at x whenever $\partial^- f(x)\neq\emptyset$.

The subdifferential of a function at a point can be characterized in terms of the upper left Dini derivative (3.6) and the lower right Dini derivative (3.5). More precisely, a lower semicontinuous function $f:\mathbb{R}\to\mathbb{R}$ is subdifferentiable at $x\in\operatorname{dom} f$ if and only if

$$D^- f(x)\le d^+ f(x) \quad\text{and}\quad [D^- f(x), d^+ f(x)]\cap\mathbb{R}\neq\emptyset. \tag{4.1}$$

In such a case, we have $\partial^- f(x)=[D^- f(x), d^+ f(x)]\cap\mathbb{R}$.

Theorem 4.1. *Let $r\ge 2$ be an integer. If $x\in D$, then $\partial^- f_r(x)=\mathbb{R}$. Otherwise, $\partial^- f_r(x)=\emptyset$.*

Proof. If $x\in D$, then there exists an integer $n_0\ge 1$ such that $x\in D_{n_0}\setminus D_{n_0-1}$ where $D_0=\emptyset$. For every $\xi\in\mathbb{R}$ we choose an integer $n\ge 1$ such that $n>2n_0+|\xi|$. If $|h|<r^{-(n-1)}/2$ then $g_k(x+h)=|h|$ for all $n_0\le k\le n$, and hence

$$\begin{aligned} f_r(x+h)-f_r(x)-\xi h &= \sum_{k=1}^{n_0-1} g_k(x+h)-g_k(x)+\sum_{k=n_0}^{\infty} g_k(x+h)-\xi h\\ &\ge \sum_{k=1}^{n_0-1} g_k(x+h)-g_k(x)+\sum_{k=n_0}^{n} g_k(x+h)-\xi h\\ &\ge -(n_0-1)|h|+(n-n_0+1)|h|-|\xi||h|\ge |h| \end{aligned}$$

which implies $\xi \in \partial^- f_r(x)$. Thus, $\partial^- f_r(x) = \mathbb{R}$. As a matter of fact, we have proved that the function $f_r(y) - \xi y$ attains a local minimum at x.

When $r \geq 2$ is an even integer, we have $\widetilde{D} \subset D$. However, when r is odd, $D \cap \widetilde{D} = \emptyset$. In this latter case, if $x \in \widetilde{D}$ then there exists an integer $n_1 \geq 1$ such that $x \in \widetilde{D}_{n_1} \setminus \widetilde{D}_{n_1-1}$ where $\widetilde{D}_0 = \emptyset$. For every $n \geq 2n_1$ we take $a_n = x - r^{-(n-1)}/2$, which belongs to D_n, and we have $g_k(a_n) - g_k(x) = -(x - a_n)$ for all $n_1 \leq k \leq n$. Thus,

$$\begin{aligned}
\frac{f_r(a_n) - f_r(x)}{a_n - x} &= \sum_{k=1}^{n} \frac{g_k(a_n) - g_k(x)}{a_n - x} - \frac{1}{a_n - x} \sum_{k=n+1}^{\infty} g_k(x) \\
&= \sum_{k=1}^{n} \frac{g_k(a_n) - g_k(x)}{a_n - x} + \frac{1}{x - a_n} \sum_{k=n+1}^{\infty} \frac{1}{2r^{k-1}} \\
&= \sum_{k=1}^{n_1-1} \frac{g_k(a_n) - g_k(x)}{a_n - x} + (n - n_1 + 1) + \frac{1}{r-1} \geq n - 2n_1 + \frac{1}{r-1}
\end{aligned}$$

which goes to $+\infty$ as $n \to \infty$. This yields $D^- f_r(x) = +\infty$, and hence $\partial^- f_r(x) = \emptyset$ by (4.1).

In the sequel, we suppose $x \notin D \cup \widetilde{D}$. We have Lemma 3.6 in Chapter 3, which states that for every n we have

$$\frac{f_r(b_n) - f_r(x)}{b_n - x} \leq \sum_{k=1}^{n} g_k'(x) \leq \frac{f_r(a_n) - f_r(x)}{a_n - x},$$

where $a_n = \max\{y \in D_n : y < x\}$ and $b_n = \min\{y \in D_n : y > x\}$. This implies

$$\begin{aligned}
d^+ f_r(x) &\leq \liminf_n \frac{f_r(b_n) - f_r(x)}{b_n - x} \leq \liminf_n \sum_{k=1}^{n} g_k'(x), \text{ and} \\
D^- f_r(x) &\geq \limsup_n \frac{f_r(a_n) - f_r(x)}{a_n - x} \geq \limsup_n \sum_{k=1}^{n} g_k'(x).
\end{aligned}$$

Therefore, if $\limsup_n \sum_{k=1}^n g_k'(x) = +\infty$ then $D^- f_r(x) = +\infty$, meanwhile if $\liminf_n \sum_{k=1}^n g_k'(x) = -\infty$ then $d^+ f_r(x) = -\infty$. In both cases, we get $\partial^- f_r(x) = \emptyset$ by (4.1). If both limits are finite, then

$$\limsup_n \sum_{k=1}^{n} g_k'(x) \geq \liminf_n \sum_{k=1}^{n} g_k'(x) + 1$$

since $\sum_{k=1}^n g_k'(x)$ cannot converge to a finite number. Hence,

$$D^- f_r(x) \geq \limsup_n \sum_{k=1}^{n} g_k'(x) \geq \liminf_n \sum_{k=1}^{n} g_k'(x) + 1 \geq d^+ f_r(x) + 1 > d^+ f_r(x)$$

which yields $\partial^- f_r(x) = \emptyset$ by (4.1). □

4.2 Superdifferential analysis

We turn now to the superdifferential analysis of the Takagi-Van der Waerden function. We refer to the books [47] and [91] for information about the superdifferential and related topics.

For an upper semicontinuous function $f:\mathbb{R}\to\mathbb{R}$ and a point $x\in\mathbb{R}$, the superdifferential of f at x, denoted by $\partial^+ f(x)$, is defined as the set of $\xi\in\mathbb{R}$ that satisfy

$$\limsup_{h\to 0}\frac{f(x+h)-f(x)-\xi h}{|h|}\leq 0.$$

The function f is said to be superdifferentiable at x whenever $\partial^+ f(x)\neq\emptyset$.

It is worth noting that a function f is superdifferentiable at x if and only if $-f$ is subdifferentiable at x, and in this case we have

$$\partial^+ f(x) = -\partial^-(-f)(x). \tag{4.2}$$

Many subdifferentiability results can be transformed into superdifferentiability ones via the above formula. For instance, $\partial^+ f(x)$ is a closed convex subset of $\mathbb{R}$ and if f attains a local maximum at x, then $0\in\partial^+ f(x)$. In addition, if f is differentiable at x then $\partial^+ f(x)=\{f'(x)\}$.

As a matter of fact, a continuous function f is differentiable at x if and only if both $\partial^- f(x)$ and $\partial^+ f(x)$ are non-empty. In such a case, we have $\partial^- f(x)=\partial^+ f(x)=\{f'(x)\}$ (see Proposition 4.8 of [47]).

A characterization of the superdifferential in terms of the Dini derivatives can also be obtained from the corresponding one for the subdifferential (4.1) and by using formula (4.2). To be more specific, an upper semicontinuous function $f:\mathbb{R}\to\mathbb{R}$ is superdifferentiable at $x\in\mathbb{R}$ if and only if

$$D^+ f(x)\leq d_- f(x) \quad \text{and} \quad [D^+ f(x), d_- f(x)]\cap\mathbb{R}\neq\emptyset. \tag{4.3}$$

In such a case, we have $\partial^+ f(x)=[D^+ f(x), d_- f(x)]\cap\mathbb{R}$.

Now, recall that the Takagi-Van der Waerden function can be extended periodically to the whole real line, and hence it becomes an even function. For every $x\notin D\cup\widetilde{D}$ we have $g_n'(x)=-g_n'(-x)$ for every n, and

$$\begin{aligned} D^+ f_r(x) &= \limsup_{h\to 0^+}\frac{f_r(x+h)-f_r(x)}{h} = \limsup_{h\to 0^-}\frac{f_r(-x+h)-f_r(-x)}{-h} \\ &= -\liminf_{h\to 0^-}\frac{f_r(-x+h)-f_r(-x)}{h} = -d_- f_r(-x). \end{aligned} \tag{4.4}$$

4.2.1 The case when r is odd

When $r \geq 3$ is odd, we have to bear in mind that $\widetilde{D} \cap D = \emptyset$ and $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n.

Theorem 4.2. *Let $r \geq 3$ be an odd integer. If $x \in \widetilde{D}$ then the function $f_r(z) - z\xi$ attains a local maximum at x for every $\xi \in \mathbb{R}$, and in particular we have $\partial^+ f_r(x) = \mathbb{R}$, otherwise $\partial^+ f_r(x) = \emptyset$.*

Proof. Firstly, if $x \in D$ then $\partial^+ f_r(x) = \emptyset$ necessarily, because $\partial^- f_r(x) = \mathbb{R}$ by Theorem 4.1 and the function f_r is nowhere differentiable.

Secondly, if $x \in \widetilde{D}$ then there exists an index $n_1 \geq 1$ such that $x \in \widetilde{D}_{n_1} \setminus \widetilde{D}_{n_1-1}$. For every $\xi \in \mathbb{R}$ we choose an integer $n \geq 1$ satisfying that $n > 2n_1 + |\xi|$. Now, if $|h| < r^{-(n-1)}/2$ then $g_k(x+h) - g_k(x) = -|h|$ for every $n_1 \leq k \leq n$, and $g_k(x+h) - g_k(x) \leq 0$ since $g_k(x) = r^{-(k-1)}/2$ for every $k > n$. Therefore, we obtain

$$\begin{aligned} f_r(x+h) - f_r(x) - \xi h &\leq \sum_{k=1}^{n_1-1} (g_k(x+h) - g_k(x)) - (n-n_1)|h| + |\xi||h| \\ &\leq |h|(2n_1 + |\xi| - n - 1) \leq 0 \end{aligned}$$

which yields $\xi \in \partial^+ f_r(x)$ and hence $\partial^+ f_r(x) = \mathbb{R}$. It also proves that $f_r(z) - z\xi$ attains a local maximum at x.

Thirdly, if $x \notin D \cup \widetilde{D}$ then we may consider the function

$$\varphi(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2} \frac{1}{r^{n-1}} - g_n(z) \right) = \frac{r}{2(r-1)} - f_r(z),$$

which is a Generalized Takagi-Van der Waerden function associated to the set $\widetilde{D}$ in the sense defined in [48], and by Theorem 1.2. of [48] we have $\partial\varphi(x) = \emptyset$. Thus the result follows immediately since $\partial\varphi(x) = -\partial^+ f_r(x)$. However, for the sake of self-containment, we give below another demonstration. For every n, we recall that $k_n(x) = \min\{k \geq n : \varepsilon_k \neq (r-1)/2\}$ and we consider

$$x_n^* := \begin{cases} 2c_{k_n+1} - x & \text{if } g'_{k_n+1}(x) = -1 \\ 2a_{k_n+1} - x & \text{if } g'_{k_n+1}(x) = 1. \end{cases}$$

We observe that $x_n^* < x$. In the former case, x_n^* is the symmetric point to x about the point c_{k_n+1} and we obtain $g_j(2c_{k_n+1} - x) = g_j(-x) = g_j(x)$ for every $j \geq k_n + 1$ since $2c_{k_n+1} = a_{k_n+1} + b_{k_n+1}$. In the latter case, x_n^* is the symmetric point to x about the point a_{k_n+1} and we have $g_j(2a_{k_n+1} - x) = g_j(-x) = g_j(x)$ for every $j \geq k_n + 1$. In both cases, we get

$$f_r(x_n^*) - f_r(x) = \sum_{j=1}^{k_n} (g_j(x_n^*) - g_j(x)).$$

If $g'_{k_n+1}(x) = -1$, then $x_n^* = 2c_{k_n+1} - x \in (a_{k_n+1}, b_{k_n+1})$ and $(a_{k_n+1}, b_{k_n+1}) \cap \widetilde{D}_j = \emptyset$ for every $j \le k_n$, so we get

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G'_{k_n}(x). \tag{4.5}$$

Otherwise, there is an integer $m \le k_n$ such that $a_m < a_{m+1} = \cdots = a_{k_n+1}$ for some $m \le k_n$, which implies $\varepsilon_{m+1} = \cdots = \varepsilon_{k_n} = 0$. Since $x_n^* = 2a_{m+1} - x \in (a_{m+1} - r^{-k_n}/2, a_{m+1})$ and $g_j(x_n^*) = g_j(x)$ for all $j \ge m+1$ we get

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G'_m(x) \le G'_{k_n}(x). \tag{4.6}$$

From (4.5) and (4.6) we obtain

$$d_- f_r(x) \le \liminf_n \frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} \le \liminf_n G'_{k_n}(x).$$

Since $G'_{k_n}(-x) = -G'_{k_n}(x)$ we get

$$D^+ f_r(x) = -d_- f_r(-x) \ge -\liminf_n G'_{k_n}(-x) = \limsup_n G'_{k_n}(x).$$

Therefore,

$$d_- f_r(x) \le \liminf_n G'_{k_n}(x) \le \limsup_n G'_{k_n}(x) \le D^+ f_r(x).$$

Finally, if $\partial^+ f_r(x) \ne \emptyset$, then by (4.3), we deduce that $\lim_n G'_{k_n}(x)$ exists. However, this limit can not exist since

$$G'_{k_{n+1}}(x) - G'_{k_n}(x) = g'_{k_{n+1}}(x)(k_{n+1} - k_n)$$

and $\lim_n k_n = +\infty$. This proves the result. □

As a consequence of the previous theorem we have that when r is odd the function f_r has a local maximum at x if and only if $x \in \widetilde{D}$, and as this set is countable, the Hausdorff dimension of the local maxima set is zero.

4.2.2 The case when r is even

When $r \ge 2$ is even, we recall that $\widetilde{D}_n \subset D_{n+1}$ for every n, and hence $\widetilde{D} \subset D$. For a point $x \in D$ we have $\partial^- f_r(x) = \mathbb{R}$ by Theorem 4.1, and consequently $\partial^+ f_r(x) = \emptyset$ because the function f_r is nowhere differentiable. Therefore, it only remains to study the case when $x \notin D$. Our approach is based on determining how the Dini derivatives involved in the definition of the superdifferential (4.3) are related to $\liminf_n G'_n(x)$ and $\limsup_n G'_n(x)$.

Lemma 4.3. *Let $r \ge 2$ be an even integer and $x \notin D$. Then,*

$$d_- f_r(x) \le \liminf_n G_n'(x) + 1, \quad \text{and} \quad D^+ f_r(x) \ge \limsup_n G_n'(x) - 1.$$

Proof. For every n we consider $x_n^* = 2a_{n+1} - x$, which is less than x, and we have

$$d_- f_r(x) \le \liminf_n \frac{f_r(x_n^*) - f_r(x)}{x_n^* - x}.$$

We observe that $g_j(x_n^*) = g_j(-x) = g_j(x)$ for every $j \ge n+1$, and consequently

$$f_r(x_n^*) - f_r(x) = \sum_{j=1}^{n} (g_j(x_n^*) - g_j(x)).$$

If $a_{n+1} = c_n$ then $g_n'(x) = -1$ and $2a_{n+1} = 2c_n = a_n + b_n \in D_n$, which yields $g_n(x_n^*) = g_n(x)$. Thus,

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_{n-1}'(x) = G_n'(x) + 1.$$

If $a_n < a_{n+1} \ne c_n$ then

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_n'(x).$$

If $a_m < a_{m+1} = \cdots = a_n = a_{n+1}$ for some $m < n$ then $g_j'(x) = 1$ for every $m+1 \le j \le n$. Hence,

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_{m-1}'(x) = G_m'(x) + 1 < G_n'(x) + 1$$

provided that $a_{m+1} = c_m$, and

$$\frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} = G_m'(x) < G_n'(x)$$

otherwise.

From these facts we deduce

$$d_- f_r(x) \le \liminf_n \frac{f_r(x_n^*) - f_r(x)}{x_n^* - x} \le \liminf_n G_n'(x) + 1.$$

Since $g_j'(x) = -g_j'(-x)$ for all $j \ge 1$ we obtain

$$D^+ f_r(x) = -d_- f_r(-x) \ge -\liminf_n G_n'(-x) - 1 = \limsup_n G_n'(x) - 1.$$

□

Lemma 4.4. *Let $r \ge 2$ be an even integer and $x \notin D$. Then,*

$$g_n(x) + g_{n+1}(x) \le \frac{r}{2r^n}$$

for every n. Furthermore, equality holds if and only if one of the following situations arises:

(1) $\varepsilon_n = \frac{r}{2} - 1$ *and* $\varepsilon_{n+1} \geq \frac{r}{2}$*, or*

(2) $\varepsilon_n = \frac{r}{2}$ *and* $\varepsilon_{n+1} \leq \frac{r}{2} - 1$.

Proof. Firstly, we prove $g_1(x) + g_2(x) \leq 1/2$ for all $x \in [0,1]$. Indeed, since $g_1 + g_2$ is differentiable with null derivative on $\left(\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}\right)$ we have that it is constant on $\left[\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}\right]$. Moreover, due to the periodicity of g_2 it is clear that $\left[\frac{1}{2} - \frac{1}{2r}, \frac{1}{2} + \frac{1}{2r}\right]$ is the set where $g_1 + g_2$ attains its maximum. Moreover, $g_1(1/2) + g_2(1/2) = 1/2$.

In addition, we have that $x \in \left[\frac{1}{2} - \frac{1}{2r}, \frac{1}{2}\right]$ if and only if $\varepsilon_1(x) = \frac{r}{2} - 1$ and $\varepsilon_2(x) \geq \frac{r}{2}$, and $x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2r}\right]$ if and only if $\varepsilon_1(x) = \frac{r}{2}$ and $\varepsilon_2(x) \leq \frac{r}{2} - 1$.

Recall that the functions g_1 and g_2 may be extended periodically to the whole real line, so we have $g_{m+1}(x) = r^{-m} g_1(r^m x)$ for every m and we get

$$\begin{aligned} g_n(x) + g_{n+1}(x) &= \frac{1}{r^{n-1}} g_1(r^{n-1}x) + \frac{1}{r^n} g_1(r^n x) \\ &= \frac{1}{r^{n-1}} g_1(r^{n-1}x) + \frac{1}{r^{n-1}} g_2(r^{n-1}x) \leq \frac{1}{2r^{n-1}}. \end{aligned}$$

Finally, observe that for a point $x \in [0,1]$ we have $r^{n-1}x = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{n-1}.\varepsilon_n\varepsilon_{n+1}\ldots$. The periodicity of g_1 and g_2 gives that $g_1(r^{n-1}x) = g_1(0.\varepsilon_n\varepsilon_{n+1}\ldots)$ and $g_2(r^{n-1}x) = g_2(0.\varepsilon_n\varepsilon_{n+1}\ldots)$, so we get the result. ☐

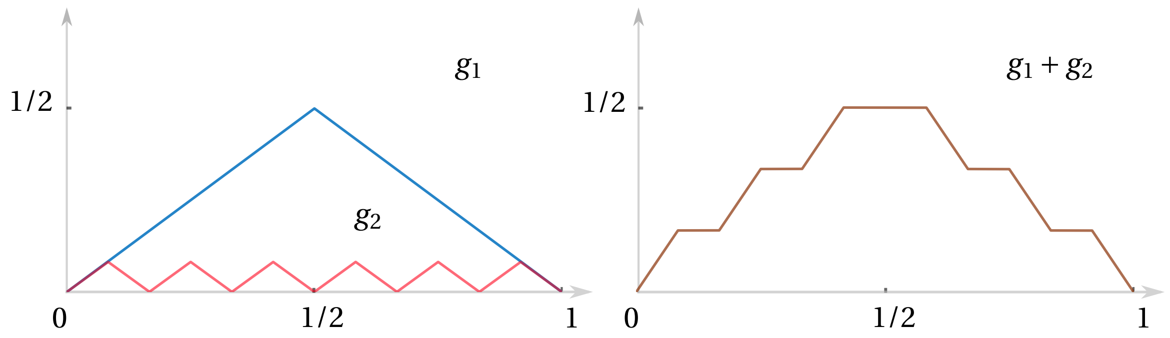

Figure 4.1: Functions $g_1(x)$ and $g_2(x)$ for $r = 6$.

Lemma 4.3 provides a necessary condition that a point $x \notin D$ with non-empty superdifferential must satisfy. Indeed, if $\limsup_n G_n'(x) = +\infty$ then $D^+ f_r(x) = +\infty$ and if $\liminf_n G_n'(x) = -\infty$ then $d_- f_r(x) = -\infty$ by Lemma 4.3. In both cases we obtain $\partial^+ f_r(x) = \emptyset$. When both limits are finite, we necessarily have

$$\limsup_n G_n'(x) \geq \liminf_n G_n'(x) + 1.$$

In addition to this, if $\limsup_n G'_n(x) - \liminf_n G'_n(x) > 2$ then

$$D^+ f_r(x) - d_- f_r(x) \geq \limsup_n G'_n(x) - \liminf_n G'_n(x) - 2 > 0,$$

and by (4.3), we obtain $\partial^+ f_r(x) = \emptyset$ again. Therefore, if $\partial^+ f_r(x) \neq \emptyset$ then

$$\limsup_n G'_n(x) - \liminf_n G'_n(x) \in \{1,2\}.$$

Our next result unveils the appearance of the base-r expansion of a point with non-empty superdifferential.

Theorem 4.5. *Let $r \geq 2$ be an even integer and $x = \sum_{n=1}^{\infty} \varepsilon_n r^{-n}$ with $\varepsilon_n \in \{0,1,\dots,r-1\}$. If $\partial^+ f_r(x) \neq \emptyset$ then there exists $m_0 \geq 1$ such that for all $i \geq 0$ one of the following situations arises:*

$$\begin{aligned} &(1) \quad \varepsilon_{m_0+2i} = \tfrac{r}{2} - 1 \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \geq \tfrac{r}{2}, \textit{ or} \\ &(2) \quad \varepsilon_{m_0+2i} = \tfrac{r}{2} \quad \textit{and} \quad \varepsilon_{m_0+2i+1} \leq \tfrac{r}{2} - 1. \end{aligned} \qquad (S_{m_0})$$

Proof. We denote $S = \limsup_n G'_n(x)$ and $I = \liminf_n G'_n(x)$. As previously mentioned, we necessarily have that S and I are finite and $S - I \in \{1,2\}$. Thus, there exists an integer $m \geq 1$ such that $I \leq G'_n(x) \leq S$ for all $n \geq m-1$. Without loss of generality we may assume that $G'_{m-1}(x) = I+1$ and we have $|G'_n(x) - G'_{m-1}(x)| \leq 1$ for all $n \geq m$. In particular, we have $G'_{m+2i+1}(x) = I+1$ and necessarily for all $i \geq 0$ either

(a) $\varepsilon_{m+2i} \leq \frac{r}{2} - 1$ and $\varepsilon_{m+2i+1} \geq \frac{r}{2}$, or

(b) $\varepsilon_{m+2i} \geq \frac{r}{2}$ and $\varepsilon_{m+2i+1} \leq \frac{r}{2} - 1$.

If there are infinitely many indices $(i_j)_{j\geq 1}$ such that $\varepsilon_{m+2i_j} < \frac{r}{2} - 1$, then $\varepsilon_{m+2i_j+1} \geq \frac{r}{2}$. We consider $y_j^* = 2b_{m+2i_j+1} - x > x$ and we have

$$\frac{f_r(y_j^*) - f_r(x)}{y_j^* - x} = G'_{m+2i_j}(x) = G'_{m+2i_j-1}(x) + 1 = I + 2,$$

so we obtain $D^+ f_r(x) \geq I+2$. By Lemma 4.3 we obtain $D^+ f_r(x) \geq I+2 > I+1 \geq d_- f_r(x)$ which implies $\partial^+ f_r(x) = \emptyset$.

Otherwise, suppose there are infinitely many indices $(i_j)_{j\geq 1}$ such that $\varepsilon_{m+2i_j} > \frac{r}{2}$. Then we have that $-x$ satisfies that $\varepsilon_{m+2i_j}(-x) < \frac{r}{2} - 1$ for all $j \geq 1$, so we are under the previous situation. Thus, we obtain $D^+ f_r(-x) > d_- f_r(-x)$ and hence $d_- f_r(x) = -D^+ f_r(-x) < -d_- f_r(-x) = D^+ f_r(x)$, which implies $\partial^+ f_r(x) = \emptyset$.

If we are not under either of the previous situations then there exists $m_0 \geq m$ such that Property (S_{m_0}) is satisfied. □

Our purpose is to prove that a point has non-empty superdifferential if and only if such a point satisfies Property (S_{m_0}). To do so, we calculate the Dini derivatives $d_- f_r(x)$ and $D^+ f_r(x)$ for each point $x \notin D$ fulfilling Property (S_{m_0}).

Proposition 4.6. *Let $r \geq 2$ be an even integer and $x \notin D$ be satisfying Property (S_{m_0}). Then,*

$$d_- f_r(x) = \liminf_n G'_n(x) + 1 \quad \text{and} \quad D^+ f_r(x) = \limsup_n G'_n(x) - 1.$$

Proof. We denote $S = \limsup_n G'_n(x)$ and $I = \liminf_n G'_n(x)$. Without loss of generality we may assume that $G'_{m_0-1}(x) = I+1$ and consequently, we have $G'_{m_0+2i-1}(x) = I+1$ for all $i \geq 0$.

If $x^* \in (a_{m_0}, x)$, then we consider the least index $n > m_0$ such that $\varepsilon_{n-1}(x) \geq \frac{r}{2}$ and $x^* \notin (a_n, b_n)$. Thus, $g'_{n-1}(x) = -1$ and $G'_{n-2}(x) = G'_{n-1}(x) + 1 \geq I + 1$ where the inequality holds since $G'_j(x) \geq I$ for all $j \geq m_0$. Hence we have

$$\begin{aligned}\frac{f_r(x^*) - f_r(x)}{x^* - x} &= G'_{n-2}(x) + \frac{1}{x^* - x} \sum_{j=n-1}^{\infty} (g_j(x^*) - g_j(x)) \\ &\geq I + 1 + \frac{1}{x - x^*} \sum_{j=n-1}^{\infty} (g_j(x) - g_j(x^*)).\end{aligned}$$

From Lemma 4.4 we get

$$g_{m_0+2i}(x) + g_{m_0+2i+1}(x) - (g_{m_0+2i}(x^*) + g_{m_0+2i+1}(x^*)) \geq 0$$

for all $i \geq 0$, and consequently we obtain

$$\sum_{j=m_0+2p}^{\infty} (g_j(x) - g_j(x^*)) = \sum_{i=p}^{\infty} \left[g_{m_0+2i}(x) + g_{m_0+2i+1}(x) - (g_{m_0+2i}(x^*) + g_{m_0+2i+1}(x^*)) \right] \geq 0$$

where $p \geq 0$ is any integer. With the previous fact in mind, we distinguish two cases: if $n = m_0 + 2p + 1$ for some $p \geq 0$, then

$$\frac{f_r(x^*) - f_r(x)}{x^* - x} \geq I + 1.$$

If $n = m_0 + 2p$ for some $p \geq 0$, then $G'_{n-1}(x) = G'_{m_0-1}(x) = I+1$. As $g'_{n-1}(x) = -1$ we necessarily have $g'_{n-2}(x) = 1$, which implies $G'_{n-2}(x) = I + 2$. Hence we get

$$\begin{aligned}\frac{f_r(x^*) - f_r(x)}{x^* - x} &\geq G'_{n-2}(x) + \frac{g_{n-1}(x^*) - g_{n-1}(x)}{x^* - x} + \sum_{j=m_0+2p}^{\infty} \frac{1}{x^* - x}(g_j(x^*) - g_j(x)) \\ &\geq G'_{n-2}(x) + \frac{g_{n-1}(x^*) - g_{n-1}(x)}{x^* - x} \geq I + 1\end{aligned}$$

which implies $d_- f_r(x) = I + 1$ by Lemma 4.3. The other equality follows from the fact $D^+ f_r(x) = -d_- f_r(-x)$. □

Theorem 4.7. *Let $r \geq 2$ be an even integer and $x \notin D$ be satisfying Property (S_{m_0}). The following statements hold:*

(1) If there exists $n_0 \geq m_0$ such that either $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{n_0+2i+1} \geq \frac{r}{2}$ for all $i \geq 0$, or $\varepsilon_{n_0+2i} = \frac{r}{2}$ and $\varepsilon_{n_0+2i+1} \leq \frac{r}{2} - 1$ for all $i \geq 0$, then

$$\partial^+ f_r(x) = \begin{cases} G'_{n_0-1}(x) + [0,1] & \textit{if} \quad \varepsilon_{n_0} = \frac{r}{2} - 1, \\ G'_{n_0-1}(x) + [-1,0] & \textit{if} \quad \varepsilon_{n_0} = \frac{r}{2}. \end{cases}$$

(2) Otherwise we have

$$\partial^+ f_r(x) = \{G'_{m_0-1}(x)\}.$$

Proof. If there exists $n_0 \geq m_0$ such that either $\varepsilon_{n_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{n_0+2i+1} \geq \frac{r}{2}$ for all $i \geq 0$, or $\varepsilon_{n_0+2i} = \frac{r}{2}$ and $\varepsilon_{n_0+2i+1} \leq \frac{r}{2} - 1$ for all $i \geq 0$, then $g'_n(x) + g'_{n+1}(x) = 0$ for all $n \geq n_0$. Therefore,

$$G'_{n_0+2i}(x) = G'_{n_0}(x) + \sum_{j=1}^{i} (g'_{n_0+2j-1}(x) + g'_{n_0+2j}(x)) = G'_{n_0}(x) = G'_{n_0-1}(x) + g'_{n_0}(x)$$

and

$$G'_{n_0+2i+1}(x) = G'_{n_0-1}(x) + \sum_{j=0}^{i} (g'_{n_0+2j}(x) + g'_{n_0+2j+1}(x)) = G'_{n_0-1}(x)$$

for all $i \geq 0$. If $\varepsilon_{n_0} = \frac{r}{2} - 1$ then $g'_{n_0}(x) = 1$, which implies $\liminf_n G'_n(x) = G'_{n_0-1}(x)$ and $\limsup_n G'_n(x) = G'_{n_0-1}(x) + 1$. Furthermore, Proposition 4.6 gives that $D^+ f_r(x) = G'_{n_0-1}(x)$ and $d_- f_r(x) = G'_{n_0-1}(x) + 1$, so we get

$$\partial^+ f_r(x) = G'_{n_0-1}(x) + [0,1].$$

If $\varepsilon_{n_0} = \frac{r}{2}$ then $g'_{n_0}(x) = -1$, which implies $\liminf_n G'_n(x) = G'_{n_0-1}(x) - 1$ and $\limsup_n G'_n(x) = G'_{n_0-1}(x)$. As above, Proposition 4.6 gives that $D^+ f_r(x) = G'_{n_0-1}(x) - 1$ and $d_- f_r(x) = G'_{n_0-1}(x)$, so we get

$$\partial^+ f_r(x) = G'_{n_0-1}(x) + [-1,0].$$

Now, assume that condition (1) is not fulfilled. We have $g'_{m_0+2i}(x) + g'_{m_0+2i+1}(x) = 0$ for all $i \geq 0$, so we obtain

$$G'_{m_0+2i+1}(x) = G'_{m_0-1}(x) + \sum_{j=0}^{i} (g'_{m_0+2j}(x) + g'_{m_0+2j+1}(x)) = G'_{m_0-1}(x)$$

and

$$G'_{m_0+2i}(x) = G'_{m_0+2i-1}(x) + g'_{m_0+2i}(x) = G'_{m_0-1}(x) + g'_{m_0+2i}(x)$$

for all $i \geq 0$. Furthermore, due to the fact that x does not fulfill condition (1) we obtain $\liminf_i g'_{m_0+2i}(x) = -1$ and $\limsup_i g'_{m_0+2i}(x) = 1$, which yields $\liminf_n G'_n(x) = G'_{m_0-1}(x) - 1$ and $\limsup_n G'_n(x) = G'_{m_0-1}(x) + 1$. By Proposition 4.6 we get $D^+ f_r(x) = G'_{m_0-1}(x)$ and $d_- f_r(x) = G'_{m_0-1}(x)$, which implies $\partial^+ f_r(x) = \{G'_{m_0-1}(x)\}$, and this completes the proof of the result. ☐

In summary, Theorem 4.5 together with Theorem 4.7 yields that a point $x \in [0,1]$ at which f_r has non-empty superdifferential if and only if x satisfies Property (S_{m_0}): there exists $m_0 = m_0(x) \geq 1$ such that we have either

(1) $\varepsilon_{m_0+2i} = \frac{r}{2} - 1$ and $\varepsilon_{m_0+2i+1} \geq \frac{r}{2}$, or

(2) $\varepsilon_{m_0+2i} = \frac{r}{2}$ and $\varepsilon_{m_0+2i+1} \leq \frac{r}{2} - 1$

for all $i \geq 0$. It should be pointed out that if a point x satisfies Property (S_{m_0}) then the value of $G'_{m_0-1}(x)$ is independent of the choice of m_0.

In 1959, J. P. Kahane studied the extreme points of the Takagi function (see [75]). Among other results, Kahane proved that the Takagi function has a local maximum at a point $x = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n}$, with $\varepsilon_n \in \{0,1\}$, if and only if there is an integer $m \geq 1$ such that

$$G'_{m-1}(x) = 0 \quad \text{and} \quad \varepsilon_{m+2i} + \varepsilon_{m+2i+1} = 1$$

for all $i \geq 0$. In this regard, the results obtained in this section allow us to extend Kahane's result to the Takagi-Van der Waerden function.

Corollary 4.8. *Let $r \geq 2$ be an even integer and $x \in [0,1]$. Then, f_r has a local maximum at x if and only if there is an integer $m_1 \geq 1$ such that $G'_{m_1-1}(x) = 0$ and for all $i \geq 0$ we have either*

$$\varepsilon_{m_1+2i} = \frac{r}{2} - 1 \;\; \textit{and} \;\; \varepsilon_{m_1+2i+1} \geq \frac{r}{2}, \;\; \textit{or} \quad \varepsilon_{m_1+2i} = \frac{r}{2} \;\; \textit{and} \;\; \varepsilon_{m_1+2i+1} \leq \frac{r}{2} - 1.$$

Proof. If there is such an integer $m_1 \geq 1$, then for every $y \in (a_{m_1}, b_{m_1})$ we have

$$f_r(x) - f_r(y) = (x-y)G'_{m_1-1}(x) + \sum_{j=m_1}^{\infty} g_j(x) - g_j(y) \geq 0$$

by Lemma 4.4, so we get that f_r has a local maximum at x.

Conversely, assume that f_r has a local maximum at x. This implies $0 \in \partial^+ f_r(x)$ and by Theorem 4.5, there is an integer $m_0 \geq 1$ such that x satisfies Property (S_{m_0}). According to Theorem 4.7 we have two possibilities: If the second statement of Theorem 4.7 holds, then $\partial^+ f_r(x) = \{G'_{m_0-1}(x)\}$ and hence, $G'_{m_0-1}(x) = 0$. Otherwise, let $n_0 \geq m_0$ be as in the first statement of Theorem 4.7. We consider the case when $\varepsilon_{n_0} = \frac{r}{2} - 1$ and the remaining one is similar.

Since $0 \in \partial^+ f_r(x)$ we have that $G'_{n_0-1}(x) \in \{-1, 0\}$. Suppose that $G'_{n_0-1}(x) = -1$. Let $\varepsilon > 0$ such that $f_r(x) \geq f_r(y)$ for every $y \in (x-\varepsilon, x+\varepsilon)$.

If there exists an integer $k \geq 0$ such that $\varepsilon_{n_0+2k+1+2i} = \frac{r}{2}$ for all $i \geq 0$, then we take $m_1 = n_0 + 2k + 1$ and $G'_{n_0+2k}(x) = G'_{n_0}(x) = 0$. Otherwise, we may take $i_1 \geq 0$ sufficiently large such that $\varepsilon_{n_0+2i_1+1} > \frac{r}{2}$ and

$$y = x + \sum_{j=0}^{\infty} \left(\frac{r}{2} - \varepsilon_{n_0+2i_1+2j+1}\right) r^{-(n_0+2i_1+2j+1)}$$

belongs to $(x-\varepsilon, x)$. Since $G'_{n_0+2i_1}$ is zero on $[y, x]$ we have $G_{n_0+2i_1}(x) = G_{n_0+2i_1}(y)$, and by Lemma 4.4 we get

$$\begin{aligned} f_r(y) &= G_{n_0+2i_1}(x) + \sum_{k=1}^{\infty} g_{n_0+2i_1+2k-1}(y) + g_{n_0+2i_1+2k}(y) \\ &> G_{n_0+2i_1}(x) + \sum_{k=1}^{\infty} g_{n_0+2i_1+2k-1}(x) + g_{n_0+2i_1+2k}(x) = f_r(x), \end{aligned}$$

which is a contradiction. This gives that $G'_{n_0-1}(x) = 0$ necessarily. □

4.3 Hausdorff measure of the points with non-empty superdifferential when r is even

J. P. Kahane also showed that the maximum value of the Takagi function is 2/3 and the set of points where the Takagi function attains its maximum value consists of all $x \in [0,1]$ whose binary expansion $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$ satisfies

$$\varepsilon_{2i} + \varepsilon_{2i+1} = 1$$

for all $i \geq 0$. This is equivalent to saying that x has a base-4 expansion consisting only of digits 1 and 2. Thus, the global maxima set of the Takagi function is a Cantor set of Hausdorff dimension 1/2 (see Theorem 3.1 of [10]).

In 1984, Y. Baba extended the Kahane result mentioned above to the Takagi-Van der Waerden function (see [16]). When $r \geq 3$ is an odd integer, Y. Baba proved that the Takagi-Van der Waerden function attains its global maximum only at the point $x = 1/2$ and

$$f_r(1/2) = \frac{r}{2(r-1)}.$$

When $r \geq 2$ is an even integer, Y. Baba obtained that the maximum value of the Takagi-Van der Waerden function is

$$\frac{r^2}{2(r^2-1)}$$

and characterized those points where the Takagi-Van der Waerden function attains its maximum value in terms of their base-r^2 expansion. More precisely, every number $x \in [0,1]$ has a representation in the form

$$x = \sum_{n=1}^{\infty} \frac{\omega_n(x)}{r^{2n}} = 0.\omega_1\omega_2 \ldots \omega_n \ldots \tag{4.7}$$

where $\omega_n \in \{0,1,\ldots,r^2-1\}$ for every n. Then, the Takagi-Van der Waerden function has a global maximum at a point $x \in [0,1]$ if and only if its base-r^2 expansion (4.7) satisfies

$$\frac{r^2-r}{2} \leq \omega_n \leq \frac{r^2+r-2}{2} \tag{4.8}$$

for every n. Thus, the global maxima set of the Takagi-Van der Waerden function is a Cantor-like set where at each step the remaining r^2-adic intervals are divided into r^2 subintervals of equal length and the r^2-r outside subintervals are removed from each of them (see Figure 4.2). Y. Baba obtained that the Hausdorff dimension of the global maxima set is $1/2$.

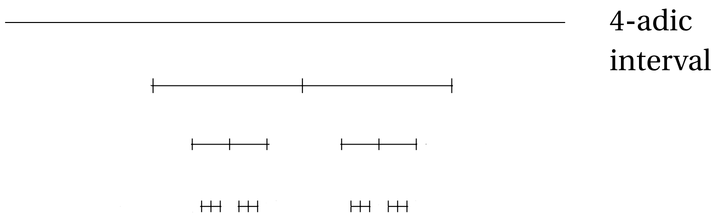

Figure 4.2: Global maxima set for $r = 2$

As might be anticipated, Y. Baba's result can also be obtained from Lemma 4.4. Indeed, the Takagi-Van der Waerden function attains its maximum value at a point $x \in [0,1]$ with base-r expansion $x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$, if and only if either

(1) $\varepsilon_{2n-1} = \frac{r}{2} - 1$ and $\varepsilon_{2n} \geq \frac{r}{2}$, or

(2) $\varepsilon_{2n-1} = \frac{r}{2}$ and $\varepsilon_{2n} \leq \frac{r}{2} - 1$

for every n. In such a case, by Lemma 4.4 we get

$$f_r(x) = \sum_{n=1}^{\infty} g_{2n-1}(x) + g_{2n}(x) = \sum_{n=1}^{\infty} \frac{r}{2r^{2n-1}} = \frac{r^2}{2(r^2-1)}.$$

We denote by E_r the global maxima set of the Takagi-Van der Waerden function and by $\mathscr{A}_r$ the set of points where the Takagi-Van der Waerden function has non-empty superdifferential. By Theorem 4.5 and Theorem 4.7 we may write

$$\mathscr{A}_r = \bigcup_{m=1}^{\infty} \left(D_m \setminus \{1\} + \frac{1}{r^{m-1}} E_r \right) \tag{4.9}$$

and consequently, the set $\mathscr{A}_r$ has Hausdorff dimension $1/2$.

Theorem 4.9. *Let $r \geq 2$ be an even integer. Then, the $1/2$-dimensional Hausdorff measure of the set $\mathscr{A}_r$ is infinity.*

Proof. For a set $F \subset \mathbb{R}$ we denote by $\mathscr{H}^{1/2}(F)$ the $1/2$-dimensional Hausdorff measure of the set F. In view of (4.9), for every m we calculate

$$\mathscr{H}^{1/2}\left(D_m \setminus \{1\} + \frac{1}{r^{m-1}} E_r \right) = \#(D_m \setminus \{1\})\, \mathscr{H}^{1/2}\left(\frac{1}{r^{m-1}} E_r \right) = \frac{r^{m-1}}{\sqrt{r^{m-1}}} \mathscr{H}^{1/2}(E_r)$$

where the first equality holds since $D_m \setminus \{1\} + r^{-(m-1)} E_r$ is a disjoint union of $\#(D_m \setminus \{1\})$ smaller similar copies of E_r. From the standard theory of self-similar sets (see Section 9.2 of [45]) it follows that $\mathscr{H}^{1/2}(E_r)$ is finite and strictly positive, and hence

$$\mathscr{H}^{1/2}(\mathscr{A}_r) \geq \mathscr{H}^{1/2}\left(D_m \setminus \{1\} + \frac{1}{r^{m-1}} E_r \right) \geq \sqrt{r^{m-1}} \mathscr{H}^{1/2}(E_r)$$

for every m, which gives us the result. ☐

Taking advantage of Corollary 4.8 and proceeding in a similar way as above, the following result is obtained:

Proposition 4.10. *Let $r \geq 2$ be an even integer. Then, the set of points where f_r attains a local maximum has Hausdorff dimension $1/2$ and its $1/2$-Hausdorff measure is infinity.*

4.3.1 Hausdorff measure of the global maxima set

As previously mentioned, Y. Baba obtained that the Hausdorff dimension of the global maxima set is $1/2$ when r is even. In addition, we must observe that the exact value of the $1/2$-dimensional Hausdorff measure of such a set is not needed in the proof of Theorem 4.9. However, it has not been calculated and up until now, it even was not known for the Takagi function.

This last part of the chapter is devoted to proving that the $1/2$-dimensional Hausdorff measure of the global maxima set E_r is $1/\sqrt{r+1}$ provided that r is even (see Theorem 4.13

below). To do so, we need to have a better understanding of how this set is constructed. We recall that a point $x \in [0,1]$ belongs to E_r if and only if its base-r^2 expansion $x = 0.\omega_1\omega_2\omega_3\ldots$ satisfies

$$\frac{r^2-r}{2} \leq \omega_n \leq \frac{r^2+r-2}{2} \tag{4.10}$$

for every n.

For every $n \geq 1$ we consider the set

$$A_n = \left\{ \sum_{k=1}^{n} \frac{\omega_k}{r^{2k}} : \frac{r^2-r}{2} \leq \omega_k \leq \frac{r^2+r-2}{2} \right\}$$

and we arrange this set as $A_n = \{x_0, \ldots, x_{r^n-1}\}$ satisfying that $x_0 < x_1 < \cdots < x_{r^n-1}$. Furthermore, we may write

$$x_0 = \frac{r^2-r}{2} \sum_{k=1}^{n} \frac{1}{r^{2k}} \tag{4.11}$$

and for $0 \leq i = \sum_{k=0}^{n-1} \varepsilon_k r^k < r^n$ where $0 \leq \varepsilon_k < r$ we have

$$x_i = x_0 + \sum_{k=0}^{n-1} \frac{\varepsilon_k}{r^{2(n-k)}}. \tag{4.12}$$

For every $0 \leq i \leq j < r^n$ we define the closed interval

$$I_{i,j}^n = \left[x_i + \frac{r^2-r}{2} \sum_{k=n+1}^{\infty} \frac{1}{r^{2k}}, x_j + \frac{r^2+r-2}{2} \sum_{k=n+1}^{\infty} \frac{1}{r^{2k}} \right].$$

It is worth noting that the left endpoint of $I_{i,j}^n$ is the smallest element of E_r that is bigger than x_i and the right endpoint of $I_{i,j}^n$ is the biggest element of E_r that is smaller than $x_j + r^{-2n}$. Of course, we have

$$E_r = \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{r^n-1} I_{i,i}^n, \tag{4.13}$$

which implies that E_r is a compact set.

The intervals $I_{i,j}^n$ will play an important role in the proof of Theorem 4.13 and we are interested in obtaining a lower estimate for their length. The following two results are devoted to this.

Lemma 4.11. *Let $r \geq 2$ be an even integer and let $m \geq 0$ be an integer. Then, the function $F \colon [1-r, r-1]^m \times [1, r-1] \to \mathbb{R}$ defined by*

$$F(y_0, \ldots, y_m) = (r+1) \sum_{k=0}^{m} y_k r^{2k} - \left(\sum_{k=0}^{m} y_k r^k + 1 \right)^2 + 1 \tag{4.14}$$

is nonnegative.

Proof. When $m = 0$ we have $F(y_0) = y_0(r-1-y_0) \geq 0$, so we suppose $m \geq 1$. Since F is a concave function, it necessarily attains its minimum at some vertex of the simplex $[1-r, r-1]^m \times [1, r-1]$. Let $V = \{1-r, r-1\}^m \times \{1, r-1\}$ be the set of vertices of such simplex.

If $\boldsymbol{y} = (y_0, \ldots, y_m) \in V$ and $y_m = 1$ then

$$\sum_{k=0}^{m} y_k r^k = r^m + \sum_{k=0}^{m-1} y_k r^k \leq r^m + (r-1) \sum_{k=0}^{m-1} r^k = 2r^m - 1$$

and

$$(r+1) \sum_{k=0}^{m} y_k r^{2k} \geq (r+1) r^{2m} - (r-1)(r+1) \sum_{k=0}^{m-1} r^{2k} = r^{2m+1} + 1$$

so we get $F(\boldsymbol{y}) \geq r^{2m+1} - 4r^{2m} + 2$, which is greater than or equal to zero provided that $r \geq 4$. The case when $r = 2$ is included in the next case.

If $\boldsymbol{y} = (y_0, \ldots, y_m) \in V$ and $y_m = r-1$ then we distinguish two cases: if there exists $p \in \{1, \ldots, m\}$ such that $y_m = \cdots = y_p = r-1$ and $y_{p-1} = 1-r$, then

$$\sum_{k=0}^{m} y_k r^k \leq \sum_{k=p}^{m} (r-1) r^k - (r-1) r^{p-1} + (r-1) \sum_{k=0}^{p-2} r^k = r^{m+1} - 1 - 2(r-1) r^{p-1}$$

and

$$(r+1) \sum_{k=0}^{m} y_k r^{2k} \geq (r+1) \left[\sum_{k=p}^{m} (r-1) r^{2k} - (r-1) \sum_{k=0}^{p-1} r^{2k} \right] = r^{2(m+1)} - 2r^{2p} + 1$$

so we get

$$\begin{aligned} F(\boldsymbol{y}) &\geq r^{2(m+1)} - 2r^{2p} + 1 - \left[r^{m+1} - 2(r-1) r^{p-1} \right]^2 + 1 \\ &= \left[2r^{m+1} - 2(r-1) r^{p-1} \right] 2(r-1) r^{p-1} - 2r^{2p} + 2 \\ &= 4(r-1) r^{m+p} - 4(r-1)^2 r^{2p-2} - 2r^{2p} + 2 \\ &= 4(r-1) r^{m+p} - 2r^{2p}(3 + 2r^{-2}) + 8r^{2p-1} + 2 \geq 0. \end{aligned}$$

Otherwise, we have

$$F(r-1, \ldots, r-1) = r^{2m+2} - 1 + 1 - (r^{m+1})^2 = 0.$$

□

Proposition 4.12. *Let $r \geq 2$ be an even integer and $n \in \mathbb{N}$. For every $0 \leq i \leq j < r^n$, the length of the interval $I_{i,j}^n$ is greater than or equal to*

$$\frac{(j-i+1)^2}{r^{2n}(r+1)}.$$

Proof. Firstly, we have

$$|I_{i,i}^n| = (r-1)\sum_{k=n+1}^{\infty}\frac{1}{r^{2k}} = \frac{1}{r^{2n}(r+1)} \tag{4.15}$$

and consequently, it only remains to prove the result for $i < j$. Let

$$i = \sum_{k=0}^{n-1}\varepsilon_k r^k, \quad j = \sum_{k=0}^{n-1}\varepsilon'_k r^k, \quad \text{with} \quad 0 \le \varepsilon_k, \varepsilon'_k < r.$$

Taking advantage of (4.11) and (4.12), we write

$$|I_{i,j}^n| = x_j - x_i + (r-1)\sum_{k=n+1}^{\infty}\frac{1}{r^{2k}} = \sum_{k=0}^{n-1}\frac{\varepsilon'_k - \varepsilon_k}{r^{2(n-k)}} + \frac{1}{r^{2n}(r+1)}.$$

Now, we consider $m = \max\{k : \varepsilon'_k - \varepsilon_k \neq 0\}$ and let F be the function defined by (4.14). We have

$$r^{2n}(r+1)|I_{i,j}^n| - (j-i+1)^2 = F(\varepsilon'_0 - \varepsilon_0, \ldots, \varepsilon'_m - \varepsilon_m) \ge 0$$

by Lemma 4.11. □

In Theorem 4.13 below, we calculate the 1/2-dimensional Hausdorff measure of the global maxima set E_r by means of its own definition. For the reader's convenience, we recall the definition of the s-dimensional Hausdorff measure of a set and we refer to the books [41] and [45] for more information about it and related topics.

Let $A \subset \mathbb{R}$ be a set and $s \ge 0$. The diameter of A is denoted by $|A| = \sup\{|x-y| : x, y \in A\}$. For each $\delta > 0$, we recall that a δ-cover of a set $A \subset \mathbb{R}$ is a countable collection of sets $\{U_i\}$ with diameters $0 < |U_i| \le \delta$ that cover A and we define

$$\mathscr{H}_\delta^s(A) = \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } A\right\}.$$

Note that we can take each U_i to be an open interval. The s-dimensional Hausdorff measure of A is given by

$$\mathscr{H}^s(A) = \lim_{\delta \to 0^+}\mathscr{H}_\delta^s(A).$$

This limit always exists and the limiting value can be 0 or ∞.

Theorem 4.13. *Let $r \ge 2$ be an even integer. Then, the $1/2$-dimensional Hausdorff measure of the global maxima set of f_r is $1/\sqrt{r+1}$.*

Proof. In view of (4.13) and (4.15), for every $n \ge 1$ we take the intervals $I_{i,i}^n$ for $i = 0, \ldots, r^n - 1$ as a δ_n-cover of E_r where

$$\delta_n = (r-1)\sum_{k=n+1}^{\infty}\frac{1}{r^{2k}} = \frac{1}{r^{2n}(r+1)},$$

and we get

$$\mathcal{H}^{1/2}_{\delta_n}(E_r) \le r^n \sqrt{\delta_n} = \frac{1}{\sqrt{r+1}}.$$

As δ_n tends to 0 as $n \to \infty$, we obtain that $\mathcal{H}^{1/2}(E_r) \le 1/\sqrt{r+1}$.

Now, it only remains to prove the other inequality. Let $\{U_k\}_{k=1}^{\infty}$ be a covering of E_r consisting of open intervals. The compactness of E_r tells us that it is covered by a finite number of them, so suppose $E_r \subset \bigcup_{k=1}^m U_k$ with $E_r \cap U_k \neq \emptyset$ for $k = 1, \dots, m$. Without loss of generality we may suppose that $U_i \cap U_j \cap U_k = \emptyset$ for any three different indices $1 \le i, j, k \le m$ and we have $U_i \not\subset U_j$ and $U_j \not\subset U_i$ for $i \neq j$. We also may assume that for $k = 1, \dots, m$ the endpoints of U_k not belong to E_r.

We claim that if we take an n sufficiently large, then for every $k = 1, \dots, m$ there are a pair of indices $0 \le i_k \le j_k < r^n$ such that

(1) $I^n_{i_k, j_k} \subset U_k$.

(2) The family of intervals $\{I^n_{i_k, j_k}\}_{k=1}^m$ is pairwise disjoint and covers E_r.

Indeed, let $I = (a, b)$ and $J = (c, d)$ be two intervals of $U_1, \dots, U_m$ such that $\overline{I} \cap \overline{J} \neq \emptyset$. We may assume that $\overline{I} \cap \overline{J} = [c, b]$. If $c \neq b$ then we may pick $x \in [c, b] \setminus E_r$ since E_r does not contain any interval, and otherwise we take $x = c = b$. Furthermore, as E_r is a closed set there is $\alpha > 0$ such that $(x - \alpha, x + \alpha) \subset \overline{I} \cup \overline{J} \setminus E_r$. Then, we have

$$\big((a, x-\alpha) \cup (x+\alpha, d)\big) \cap E_r = \big(I \cup J\big) \cap E_r.$$

Repeating this process for the $m-1$ possible intersections we obtain a new covering of E_r, named $I_1, \dots I_m$, formed by open intervals such that $\overline{I}_j \cap \overline{I}_i = \emptyset$ for $i \neq j$ and $I_k \subset U_k$ for every $k = 1, \dots, m$. Now, let $\delta > 0$ be the minimum of the distance between the intervals $I_1, \dots I_m$. If we take n sufficiently large that $\delta > 1/r^{2n}(r+1)$ then every interval of the form $I^n_{i,i}$ is contained in only one interval of $I_1, \dots I_m$. Hence, for every $k = 1, \dots, m$ if we denote $i_k = \min\{i : I^n_{i,i} \subset I_k\}$ and $j_k = \max\{j : I^n_{j,j} \subset I_k\}$ then we have $i_k \le j_k$ and $I^n_{i_k, j_k} \subset I_k$. Since $I_1, \dots I_m$ are pairwise disjoint intervals and they cover E_r, we obtain that the family of intervals $\{I^n_{i_k, j_k}\}_{k=1}^m$ is pairwise disjoint and it covers E_r. This proves the claim.

Therefore, if we conveniently reorder the family $\{I^n_{i_k, j_k}\}_{k=1}^m$, then it can be written as

$$I^n_{0, p_1}, I^n_{p_1+1, p_2}, \dots, I^n_{p_{m-1}+1, r^n-1}$$

and consequently, by Lemma 4.12 we get

$$\begin{aligned}\sum_{k=1}^{m}\sqrt{|U_k|} &\geq \sum_{k=1}^{m}\sqrt{|I^n_{i_k,j_k}|}\\ &\geq \frac{1}{r^n\sqrt{r+1}}(p_1+1+p_2-p_1+\cdots+r^n-p_{m-1}-1)=\frac{1}{\sqrt{r+1}}\end{aligned}$$

which gives us the result. □

CHAPTER 5

GLOBAL PROPERTIES AND APPROXIMATE DIFFERENTIABILITY IN THE GENERALIZED CLASS

For a given strictly increasing sequence of non-negative integers $\mathbf{r} = (r_n)_n$ satisfying that $r_1 = 1$ and r_n divides r_{n+1} for every n, the Generalized Class is formed by all the functions $f_{\mathbf{r},w} : [0,1] \to \mathbb{R}$ defined as

$$f_{\mathbf{r},w}(x) = \sum_{n=1}^{\infty} w_n g_n(x)$$

where $w = (w_n)_n$ is a sequence of weights such that $\left(r_n^{-1} w_n\right)_n \in \ell_1$, and $g_n(x)$ denotes the distance from the point x to the set $D_n = \left\{k r_n^{-1} : k = 0, \dots, r_n\right\}$. A detailed introduction to this family of functions is developed in Section 1.4 of Chapter 1.

Keep in mind the Takagi class is obtained by taking $r_{n+1} = 2r_n$ for every n and such a class has been deeply investigated in the literature (see Section 1.2 of Chapter 1). In a sense, this earlier study of the properties of the functions belonging to the Generalized Class, was guided in the beginning by the well-known results for the Takagi Class.

Following this line of thought, we begin this chapter by studying some global properties of the functions belonging to the Generalized Class. We characterize when a function of the Generalized Class is Lipschitz, or it satisfies the Hölder property of any order $0 < \alpha < 1$. As far as we know, this last result is even new for the functions belonging to the Takagi class.

Once the survey of the global properties is completed, we devote our full attention to the study of approximate differentiability in the Generalized Class. It should be mentioned that such a study is the harvest of an improvement in the techniques developed in the work [52], where we obtained that a function belonging to the Generalized Class is nowhere differentiable if and only if the sequence of weights does not belong to c_0.

The notion of approximate differentiability is a generalization of the concept of differentiability obtained by replacing the ordinary limit by an approximate limit (see Section 5.2 below). It was introduced by A. Y. Khinchin in the paper [78], which was presented for the first time at a meeting of a student mathematical club on 6 November 1914. This notion has been deeply investigated over the years, and it plays an important role in various questions of real analysis such as, for example, in the theory of Lusin-type properties of functions (see [43] and [58]).

A function $f:\mathbb{R}\to\mathbb{R}$ has the Lusin property of class C^1 if for every $\varepsilon>0$ there is a function $g:\mathbb{R}\to\mathbb{R}$ of class C^1 such that the set $\left\{x\in\mathbb{R}: f(x)\neq g(x)\right\}$ has Lebesgue measure less than ε. F. C. Liu and W. S. Tai obtained that a function $f:\mathbb{R}\to\mathbb{R}$ has the Lusin property of class C^1 if and only if it is approximately differentiable almost everywhere.

In 1951, H. Whitney stated, without giving any demonstration, that the Takagi-Van der Waerden function does not have the Lusin property of class C^1 for a sufficiently large $r\geq 2$ (see [111]). In 2003, J. B. Brown and G. Kozlowski proved that for every integer $r\geq 2$ the Takagi-Van der Waerden function agrees with no function of class C^1 on any set of positive measure (see [29]).

From these facts we deduce that the Takagi-Van der Waerden function is not approximately differentiable almost everywhere. In 2020, Juan Ferrera and Javier Gómez Gil proved a stronger result for the Takagi function, namely it is nowhere approximately differentiable (see [51]). A typical (in the Baire category sense) continuous function $f:\mathbb{R}\to\mathbb{R}$ is nowhere approximately differentiable (see [72]) . However, the examples of such functions in the existing literature are more complicated than the Takagi function (see [72] or [88]).

Coming back to the Generalized Class, we prove that differentiability and approximate differentiability are equivalent properties when dealing with a function belonging to the Generalized Class. As a consequence, we obtain that a function in such a Class is nowhere approximately differentiable if and only if the sequence of weights does not belong to c_0. This may be considered an improvement of Kôno's theorem via the approximate differentiability.

5.1 Global properties

In Chapter 2, we outlined that a function of the Takagi class T_w is Lipschitz if and only if $w\in\ell_1$, and the Lipschitz norm is given by $\|w\|_1$. We extend this result to the Generalized Takagi-Van der Waerden class.

Proposition 5.1. *The function $f_{\mathbf{r},w}$ is Lipschitz if and only if $w \in \ell_1$. If this is the case, the Lipschitz norm of $f_{\mathbf{r},w}$ is $\|w\|_1$.*

Proof. If $w \in \ell_1$ then

$$|f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)| \leq \sum_{k=1}^{\infty} |w_k||g_k(y) - g_k(x)| \leq \|w\|_1 |y-x|.$$

Conversely, assume that $f_{\mathbf{r},w}$ is Lipschitz for some Lipschitz constant $K \geq 0$. For every n, we define the set C_n as the closure of

$$\left\{x \in [0,1] \setminus (D_n \cup \widetilde{D}_n) : g_k'(x) w_k = |w_k| \text{ for every } k \leq n\right\}.$$

Observe that C_n is a finite union of closed intervals of length $r_n^{-1}/2$, whose endpoints belong to $D_n \cup \widetilde{D}_n$.

Since $\frac{1}{r_{n+1}} \leq \frac{1}{2r_n}$ we may take two consecutive points $x, y \in C_n \cap D_{n+1}$ with $x < y$ and we have

$$f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x) = \sum_{k=1}^{n} w_k(g_k(y) - g_k(x)) = \sum_{k=1}^{n} w_k g_k'(z)(y-x)$$

with $z \in (x,y)$. Hence we get

$$\sum_{k=1}^{n} |w_k|(y-x) = |f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)| \leq K(y-x),$$

and therefore $\sum_{k=1}^{n} |w_k| \leq K$. Letting n to infinity we obtain the result. □

Along with the Lipschitz property we may consider the Hölder property. For $0 < \alpha < 1$, recall that a function $f : [0,1] \to \mathbb{R}$ is Hölder continuous of order α if there is $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|^{\alpha}$ for all $x, y \in [0,1]$. It must be noted that if a function $f : [0,1] \to \mathbb{R}$ is Lipschitz then it is Hölder continuous of order α for any $0 < \alpha < 1$. The converse assertion is not true; the Takagi-Van der Waerden function is not Lipschitz since it is nowhere differentiable; nonetheless, it is Hölder continuous of order α for any $0 < \alpha < 1$ (see [97] and [98]).

Proposition 5.2. *The function $f_{\mathbf{r},w}$ is Hölder continuous of order $0 < \alpha < 1$ if and only if there exists $K > 0$ such that $|w_n| \leq K r_n^{1-\alpha}$ for every n.*

Proof. We assume first that $f_{\mathbf{r},w}$ is Hölder continuous of order $0 < \alpha < 1$ for some $M \geq 0$. As we did in Proposition 5.1, for every n we define the set C_n as the closure of

$$\left\{x \in [0,1] \setminus (D_n \cup \widetilde{D}_n) : g_k'(x) w_k = |w_k| \text{ for every } k \leq n\right\}$$

and recall that C_n is a finite union of closed intervals of length $r_n^{-1}/2$, whose endpoints belong to $D_n \cup \widetilde{D}_n$. Let I be one of the closed intervals of C_n. We take $x = \min I \cap D_{n+1}$ and $y = \max I \cap D_{n+1}$. We have

$$y - x = \frac{1}{2r_n} - \frac{\gamma_n}{2r_{n+1}}$$

where $\gamma_n = 0$ provided that r_{n+1}/r_n is even and $\gamma_n = 1$ otherwise. Since $r_{n+1} \geq 2r_n$ we obtain that $y - x \geq \frac{1}{4r_n}$ and hence

$$\sum_{k=1}^{n} |w_k| = \frac{|f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)|}{y-x} \leq \frac{M}{(y-x)^{1-\alpha}} \leq 4^{1-\alpha} M r_n^{1-\alpha}.$$

Conversely, assume that there is $K > 0$ such that $|w_n| \leq K r_n^{1-\alpha}$ for every n. Let $x \in [0,1]$ and $h \in \mathbb{R}$ such that $x + h \in [0,1]$. Let $n \in \mathbb{N}$ be such that $1/r_{n+1} \leq |h| < 1/r_n$. Hence we get

$$\begin{aligned}\sum_{k=1}^{n} |w_k||g_k(x+h) - g_k(x)| &\leq |h| \sum_{k=1}^{n} |w_k| \leq \frac{|h|^{\alpha}}{r_n^{1-\alpha}} \sum_{k=1}^{n} |w_k| \\ &\leq K|h|^{\alpha} \sum_{k=1}^{n} \left(\frac{r_k}{r_n}\right)^{1-\alpha} \leq Q_1 |h|^{\alpha}\end{aligned}$$

for some constant $Q_1 > 0$. In addition, we have

$$\begin{aligned}\sum_{k=n+1}^{\infty} |w_k||g_k(x+h) - g_k(x)| &\leq \frac{1}{2} \sum_{k=n+1}^{\infty} \frac{|w_k|}{r_k} \leq \frac{K}{2} \sum_{k=n+1}^{\infty} \frac{1}{r_k^{\alpha}} \\ &= \frac{K}{2} \frac{1}{r_{n+1}^{\alpha}} \sum_{k=n+1}^{\infty} \left(\frac{r_{n+1}}{r_k}\right)^{\alpha} \leq \frac{Q_2}{r_{n+1}^{\alpha}} \leq Q_2 |h|^{\alpha}\end{aligned}$$

for some constant $Q_2 > 0$. The result follows by adding both items. □

Corollary 5.3. *If $w \in \ell_\infty$, then $f_{\mathbf{r},w}$ is Hölder continuous of order α for every $0 < \alpha < 1$.*

5.2 The approximate derivative

The notion of approximate derivative of a function at a point relies on the concept of approximate limit, which is a generalization of the ordinary concept of limit.

For a function $f : \mathbb{R} \to \mathbb{R}$, we say that $l \in \mathbb{R}$ is the approximate limit of f at a point x, written

$$\operatorname{ap}\lim_{y \to x} f(y) = l,$$

if for every $\varepsilon > 0$ we have

$$\lim_{r \to 0^+} \frac{\mathscr{L}\big(\{y \in (x-r, x+r) : |f(y) - l| \geq \varepsilon\}\big)}{2r} = 0.$$

It is worth mentioning that we could have defined the approximate limit in terms of the concept of a density point for a given Lebesgue measurable subset of $\mathbb{R}$ (see [77] and Proposition 1.5 of [58]). Moreover, it must be noted that an approximate limit, if it exists, is unique (see Theorem 1.36 of [43]).

The concept of approximate limit allows us to obtain a generalization of continuity, named approximate continuity. A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at a point x provided that

$$\operatorname{ap}\lim_{y\to x} f(y) = f(x).$$

It is well-known that Lebesgue measurable functions can be described in terms of approximate continuity. More precisely, a function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if and only if it is approximately continuous almost everywhere (see Theorem 5.2 of [30]).

The approximate derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x, denoted by $\operatorname{ap}f'(x)$, is defined as

$$\operatorname{ap}\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$$

provided either that this limit exists or is infinite. When $\operatorname{ap}f'(x)$ is finite, we say f is approximately differentiable at x. If this is the case, then f is also approximately continuous at x. Whereas, if f is differentiable at x, then f is approximately differentiable at x and $\operatorname{ap}f'(x) = f'(x)$.

The following auxiliary lemma will be very useful for our purposes. For the sake of completeness, we present its proof below.

Lemma 5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $x \in \mathbb{R}$ and suppose that f is approximately differentiable at x with approximate derivative $apf'(x)$. Then, for any real number $\lambda > apf'(x)$ we have*

$$\lim_{r\to 0^+} \frac{\mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \frac{f(y)-f(x)}{y-x} \geq \lambda\right\}\right)}{2r} = 0.$$

Similarly, for any real number $\mu < apf'(x)$ we have

$$\lim_{r\to 0^+} \frac{\mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \frac{f(y)-f(x)}{y-x} \leq \mu\right\}\right)}{2r} = 0.$$

Proof. We prove the first statement and the proof of the remaining one is completely analogous, so it is left to the reader. Let $\varepsilon_0 > 0$ be such that $\operatorname{ap}f'(x) + \varepsilon_0 < \lambda$. Since f approximately differentiable at x we know that

$$\lim_{r\to 0^+} \frac{\mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \left|\frac{f(y)-f(x)}{y-x} - \operatorname{ap}f'(x)\right| \geq \varepsilon_0\right\}\right)}{2r} = 0.$$

The result follows immediately because

$$\begin{aligned}
&\mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \left|\frac{f(y)-f(x)}{y-x} - \mathrm{ap}f'(x)\right| \geq \varepsilon_0\right\}\right)\\
&\geq \mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \frac{f(y)-f(x)}{y-x} \geq \mathrm{ap}f'(x) + \varepsilon_0\right\}\right)\\
&\geq \mathscr{L}\left(\left\{y \in (x-r, x+r) \setminus \{x\} : \frac{f(y)-f(x)}{y-x} \geq \lambda\right\}\right).
\end{aligned}$$

□

The following result is an immediate consequence of Lemma 5.4 above.

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $x \in \mathbb{R}$ and suppose that f is approximately differentiable at x with approximate derivative $apf'(x)$. Let $(I_n)_n$ be a decreasing sequence of intervals with positive length such that $\bigcap_n I_n = \{x\}$. Then, for any real number $\lambda > apf'(x)$ we have*

$$\lim_n \frac{\mathscr{L}\left(\left\{y \in I_n : \frac{f(y)-f(x)}{y-x} \geq \lambda\right\}\right)}{\mathscr{L}(I_n)} = 0.$$

Similarly, for any real number $\mu < apf'(x)$ we have

$$\lim_n \frac{\mathscr{L}\left(\left\{y \in I_n : \frac{f(y)-f(x)}{y-x} \leq \mu\right\}\right)}{\mathscr{L}(I_n)} = 0.$$

5.3 Approximate differentiability in the Generalized class

Firstly, we investigate the approximate differentiability of a function in the Generalized Class at a point $x \notin D \cup \widetilde{D}$. As is customary, for every n we denote by $a_n = a_n(x)$ the largest element of D_n smaller than x and by $b_n = b_n(x)$ the least element of D_n larger than x. The midpoint of the interval (a_n, b_n) is denoted by $c_n = c_n(x)$. For every n we have $x \in (a_n, c_n)$ provided that $g_n'(x) = 1$, meanwhile $x \in (c_n, b_n)$ otherwise. Accordingly, we denote that interval by $J_n(x) = J_n$ whenever $x \in J_n$.

The following result will play an important role in the proof of Proposition 5.7 below.

Lemma 5.6. *Let $0 < \delta \leq 2^{-4}$ and $x \notin D \cup \widetilde{D}$. For every n such that either $r_{n+1}/r_n \geq 3$, or $r_{n+1} = 2r_n$ and $|x - c_{n+1}| \geq 2^{-4}\mathscr{L}(J_n)$, there exist two intervals $I_1, I_2 \subset J_n$ and an affine map $L : I_1 \to I_2$ satisfying the following properties:*

(1) x is an endpoint of I_1.

(2) $\mathscr{L}(I_1) = \mathscr{L}(I_2) = \delta\mathscr{L}(J_n)$.

(3) *$dist(I_1, I_2) \geq 2^{-3}\mathscr{L}(J_n)$.*

(4) *$g_k(y) = g_k(L(y))$ for every $y \in I_1$ and $k \geq n+1$.*

Proof. Firstly, we assume $r_{n+1}/r_n \geq 3$. Therefore, there exist at least two more points $z_1, z_2 \in J_n$ such that $g_{n+1}(x) = g_{n+1}(z_1) = g_{n+1}(z_2)$. Moreover, one of the points fulfilling that property, which we denote by z, satisfies

$$\frac{1}{3}\mathscr{L}(J_n) < |x - z| \leq \frac{2}{3}\mathscr{L}(J_n).$$

The affine map L must satisfy $L(x) = z$, and it will be a translation provided that $g'_{n+1}(x) = g'_{n+1}(z)$ and a point reflection otherwise. The intervals I_1 and I_2 are defined such that $L(I_1) = I_2$, x and z are endpoints of I_1 and I_2 respectively, $\mathscr{L}(I_1) = \mathscr{L}(I_2) = \delta\mathscr{L}(J_n)$, and $I_1, I_2 \subset J_n$. This last condition is always possible since

$$2\delta\mathscr{L}(J_n) + \frac{2}{3}\mathscr{L}(J_n) < \mathscr{L}(J_n).$$

Properties (1) and (2) are immediate, meanwhile property (3) holds since $|x - z| \geq \frac{1}{3}\mathscr{L}(J_n)$ and $\mathscr{L}(I_1) = \mathscr{L}(I_2) \leq \frac{1}{2^4}\mathscr{L}(J_n)$, which implies

$$\operatorname{dist}(I_1, I_2) \geq \frac{1}{3}\mathscr{L}(J_n) - \frac{1}{2^3}\mathscr{L}(J_n) \geq \frac{1}{2^3}\mathscr{L}(J_n).$$

Property (4) is also immediate because if L is a translation then $L(x) = x + \frac{m}{r_{n+1}}$ for some m, whereas if L is a point reflection then it is a point reflection over a point belonging to $D_{n+1} \cup \widetilde{D}_{n+1}$.

Secondly, we assume $r_{n+1} = 2r_n$ and $|x - c_{n+1}| \geq 2^{-4}\mathscr{L}(J_n)$. Then, the affine map L is a point reflection over c_{n+1}, and using the same reasoning as previously gives the result.

□

Proposition 5.7. *Let $x \notin D \cup \widetilde{D}$. If $f_{\mathbf{r},w}$ is approximately differentiable at x, then the series $\sum_{k=1}^{\infty} w_k g'_k(x)$ converges and*

$$apf'_{\mathbf{r},w}(x) = \sum_{k=1}^{\infty} w_k g'_k(x).$$

Proof. For every $y \neq x$ we denote

$$\Delta(y) = \frac{f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)}{y - x}$$

and for every n we write

$$\delta_n(y) = \sum_{k=n+1}^{\infty} w_k \frac{g_k(y) - g_k(x)}{y - x}.$$

It is important to observe that if $y \in J_n$ then

$$\Delta(y) = \sum_{k=1}^{n} w_k g_k'(x) + \delta_n(y).$$

We also denote $d_{ap} = \mathrm{ap} f_{\mathbf{r},w}'(x)$ and we take $M = |d_{ap}| + 1$. For every n we define

$$E_n = \{z \in J_n : |\Delta(z)| \geq M\}$$

and by Lemma 5.5 we have

$$\lim_n \frac{\mathscr{L}(E_n)}{\mathscr{L}(J_n)} = 0. \tag{5.1}$$

For the sake of contradiction, we suppose that $\sum_{k=1}^{\infty} w_k g_k'(x)$ does not converge or $d_{ap} \neq \sum_{k=1}^{\infty} w_k g_k'(x)$. Hence, we have either d_{ap} is bigger than $\liminf_n \sum_{k=1}^{n} w_k g_k'(x)$, or smaller than $\limsup_n \sum_{k=1}^{n} w_k g_k'(x)$. We assume that we are in the first situation and the second one is completely analogous. Therefore, there are numbers α and β such that

$$\liminf_n \sum_{k=1}^{n} w_k g_k'(x) < \beta < \alpha < d_{ap}$$

and consequently, there exist infinitely many indices n such that

$$\sum_{k=1}^{n} w_k g_k'(x) < \beta. \tag{5.2}$$

If there are infinitely many indices n satisfying (5.2) together with $r_{n+1} = 2r_n$ and $|x - c_{n+1}| < \frac{1}{2^4}\mathscr{L}(J_n)$, then we consider $S : J_n \to J_n$ the point reflection over c_{n+1} and the set

$$A_n^- = \{y \in J_n : \delta_n(y) \leq 0\}.$$

For every $y \in J_n$ such that $|y - c_{n+1}| \geq |x - c_{n+1}|$, we have either y or $S(y)$ belongs to A_n^-, and hence

$$\mathscr{L}(A_n^-) \geq \frac{1}{2}\left(1 - 2^{-3}\right)\mathscr{L}(J_n) > \frac{1}{4}\mathscr{L}(J_n).$$

Moreover, if $y \in A_n^-$ then $\Delta(y) \leq \beta < \alpha$, so we get

$$\frac{\mathscr{L}\left(\{y \in J_n : \Delta(y) \leq \alpha\}\right)}{\mathscr{L}(J_n)} \geq \frac{\mathscr{L}(A_n^-)}{\mathscr{L}(J_n)} \geq \frac{1}{4}$$

which contradicts Lemma 5.5.

Otherwise, there are infinitely many indices n satisfying (5.2) such that either $r_{n+1} \geq 3r_n$, or $r_{n+1} = 2r_n$ and $|x - c_{n+1}| \geq \frac{1}{2^4}\mathscr{L}(J_n)$. Now, we invoke Lemma 5.6 with

$$0 < \delta < \min\left\{\frac{\alpha - \beta}{2^3(|\beta| + M)}, \frac{1}{2^4}\right\} \tag{5.3}$$

and we obtain two intervals $I_1, I_2 \subset J_n$ and an affine map $L : I_1 \to I_2$ satisfying the four properties indicated in Lemma 5.6. Recall that

$$E_n = \left\{ z \in J_n : |\Delta(z)| \geq M \right\}$$

and let $y \in I_2 \setminus L(E_n)$. We denote

$$\lambda_n(y) = \frac{L^{-1}(y) - x}{y - x}$$

and we have

$$\begin{aligned}
\delta_n(y) &= \frac{1}{y-x} \sum_{k=n+1}^{\infty} w_k (g_k((y)) - g_k(x)) \\
&= \frac{1}{y-x} \sum_{k=n+1}^{\infty} w_k (g_k\big(L^{-1}(y)\big) - g_k(x)) \\
&= \lambda_n(y) \sum_{k=n+1}^{\infty} w_k \frac{g_k\big(L^{-1}(y)\big) - g_k(x)}{L^{-1}(y) - x} \\
&= \lambda_n(y) \Big(\Delta(L^{-1}(y)) - \sum_{k=1}^{n} w_k g_k'(x) \Big).
\end{aligned}$$

Since $\mathscr{L}(I_1) = \mathscr{L}(I_2) = \delta \mathscr{L}(J_n)$ and $\operatorname{dist}(I_1, I_2) \geq 2^{-3} \mathscr{L}(J_n)$ we obtain $|\lambda_n(y)| < 2^3 \delta$, so we get

$$\begin{aligned}
\Delta(y) &= \big(1 - \lambda_n(y)\big) \sum_{k=1}^{n} w_k g_k'(x) + \lambda_n(y) \Delta\big(L^{-1}(y)\big) \\
&\leq \big(1 - \lambda_n(y)\big) \sum_{k=1}^{n} w_k g_k'(x) + 2^3 \delta M \\
&\leq \big(1 - \lambda_n(y)\big) \beta + 2^3 \delta M
\end{aligned}$$

where we have used that $L^{-1}(y) \notin E_n$. Therefore, we obtain

$$\Delta(y) \leq \beta + 2^3 \delta |\beta| + 2^3 \delta M = \beta + 2^3 \delta (|\beta| + M) \leq \alpha$$

by (5.3). Thus, we conclude $\Delta(y) \leq \alpha$ for every $y \in I_2 \setminus L(E_n)$, so we have

$$\frac{\mathscr{L}\big(\{ y \in J_n : \Delta(y) \leq \alpha \}\big)}{\mathscr{L}(J_n)} \geq \frac{\delta \mathscr{L}(J_n) - \mathscr{L}(E_n)}{\mathscr{L}(J_n)} = \delta - \frac{\mathscr{L}(E_n)}{\mathscr{L}(J_n)}$$

and by (5.1) we obtain

$$\limsup_n \frac{\mathscr{L}\big(\{ y \in J_n : \Delta(y) \leq \alpha \}\big)}{\mathscr{L}(J_n)} \geq \delta > 0,$$

which contradicts Lemma 5.5. $\square$

Juan Ferrera and Javier Gómez Gil carried out a deep study of the differentiability properties of a wide family of functions, which contains the Generalized Takagi-Van der Waerden class. As a consequence of Theorem 2.10 of [49], we have that $f_{\mathbf{r},w}$ is derivable at a point $x \notin D \cup \widetilde{D}$ if and only if the series $\sum_{k=1}^{\infty} w_k g_k'(x)$ converges. In such a case, we have

$$f_{\mathbf{r},w}'(x) = \sum_{k=1}^{\infty} w_k g_k'(x).$$

Proposition 5.7 above allows us to improve their result via the approximate differentiability.

Theorem 5.8. *For every $x \notin D \cup \widetilde{D}$ the following statements are equivalent:*

(1) $f_{\mathbf{r},w}$ is approximately differentiable at x.

(2) $f_{\mathbf{r},w}$ is differentiable at x.

(3) The series $\sum_{k=1}^{\infty} w_k g_k'(x)$ converges.

If this is the case, then

$$f_{\mathbf{r},w}'(x) = apf_{\mathbf{r},w}'(x) = \sum_{k=1}^{\infty} w_k g_k'(x).$$

The remainder of this section will deal with the case when $x \in D \cup \widetilde{D}$. We will slightly modify the argument that we used in Proposition 5.7 in order to obtain the corresponding result for this case. Moreover, taking advantage of the results obtained by Juan Ferrera and Javier Gómez Gil in their paper [49], we prove an analogous result to Theorem 5.8 above.

Proposition 5.9. *Let $x \in D$. If $f_{\mathbf{r},w}$ is approximately differentiable at x, then both series $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ and $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ converge. Moreover,*

$$apf_{\mathbf{r},w}'(x) = \sum_{k=1}^{\infty} w_k g_k'^{+}(x) = \sum_{k=1}^{\infty} w_k g_k'^{-}(x).$$

Proof. There exists an index $n_0 \geq 1$ such that $x \in D_{n_0} \setminus D_{n_0-1}$ and we observe

$$\sum_{k=n_0}^{\infty} w_k g_k'^{+}(x) = -\sum_{k=n_0}^{\infty} w_k g_k'^{-}(x)$$

so both series converge or diverge simultaneously. We consider the case concerning $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ and the case concerning $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ is similar. As before, we denote $d_{ap} = \mathrm{ap}f_{\mathbf{r},w}'(x)$ and we take $M = |d_{ap}| + 1$. Moreover, for every $y \neq x$ we write

$$\Delta(y) = \frac{f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)}{y - x}.$$

For the sake of contradiction, we suppose that $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ does not converge or $d_{ap} \neq \sum_{k=1}^{\infty} w_k g_k'^{+}(x)$. We have either d_{ap} is bigger than $\liminf_n \sum_{k=1}^{n} w_k g_k'^{+}(x)$ or smaller than $\limsup_n \sum_{k=1}^{n} w_k g_k'^{+}(x)$. We assume that we are in the first situation and the second one is completely analogous. Therefore, there are numbers α and β such that

$$\liminf_n \sum_{k=1}^{n} w_k g_k'^{+}(x) < \beta < \alpha < d_{ap},$$

and consequently, there exist infinitely many indices n such that

$$\sum_{k=1}^{n} w_k g_k'^{+}(x) < \beta.$$

For all those indices n, we may assume that $x \in D_n$ and we take $z_{n+1} \in D_{n+1}$ with $z_{n+1} > x$ such that $(x, z_{n+1}) \cap D_{n+1} = \emptyset$. Furthermore, we consider the interval $I_{n+1} = \big(z_{n+1} - \delta r_{n+1}^{-1}, z_{n+1}\big)$ where

$$0 < \delta < \min\left\{\frac{3(\alpha-\beta)}{4(|\beta|+M)}, \frac{1}{4}\right\} \tag{5.4}$$

and we define $E_{n+1} = \big\{z \in [x, z_{n+1}] : |\Delta(z)| \geq M\big\}$. From Lemma 5.5 we get

$$\lim_n r_{n+1} \mathscr{L}(E_{n+1}) = 0. \tag{5.5}$$

Let $y \in I_{n+1} \setminus S(E_{n+1})$ where S is the point reflection over the midpoint of (x, z_{n+1}) and we denote

$$\lambda_n(y) = \frac{S^{-1}(y) - x}{y - x}.$$

We know $S^{-1}(y) - x \leq \delta r_{n+1}^{-1}$. Since $\delta < 1/4$ we have $y - x \geq \frac{3}{4} r_{n+1}^{-1}$ and consequently, $0 < \lambda_n(y) \leq \frac{4}{3}\delta$. Hence,

$$\begin{aligned}
\Delta(y) &= \sum_{k=1}^{n} w_k g_k'^{+}(x) + \sum_{k=n+1}^{\infty} w_k \frac{g_k(y) - g_k(x)}{y - x} \\
&= \sum_{k=1}^{n} w_k g_k'^{+}(x) + \lambda_n(y) \sum_{k=n+1}^{\infty} w_k \frac{g_k\big(S^{-1}(y)\big) - g_k(x)}{S^{-1}(y) - x} \\
&= \big(1 - \lambda_n(y)\big) \sum_{k=1}^{n} w_k g_k'^{+}(x) + \lambda_n(y) \Delta\big(S^{-1}(y)\big) \\
&\leq \big(1 - \lambda_n(y)\big)\beta + \frac{4}{3}\delta M
\end{aligned}$$

where we have used that $S^{-1}(y) \notin E_{n+1}$. Therefore, we obtain

$$\Delta(y) \leq \beta + |\beta|\lambda_n(y) + \frac{4}{3}\delta M \leq \beta + \frac{4}{3}\delta(|\beta| + M) \leq \alpha$$

by (5.4). Thus, we conclude $\Delta(y) \le \alpha$ for every $y \in I_{n+1} \setminus S(E_{n+1})$, so we have

$$\frac{\mathscr{L}\big(\{y \in [x, z_{n+1}] : \Delta(y) \le \alpha\}\big)}{r_{n+1}^{-1}} \ge \frac{\delta r_{n+1}^{-1} - \mathscr{L}\big(E_{n+1}\big)}{r_{n+1}^{-1}} = \delta - r_{n+1}\mathscr{L}\big(E_{n+1}\big)$$

and by (5.5) we obtain

$$\limsup_n r_{n+1}\mathscr{L}\big(\{y \in [x, z_{n+1}] : \Delta(y) \le \alpha\}\big) \ge \delta > 0$$

which contradicts Lemma 5.5. □

Proposition 5.10. *Let $x \in \widetilde{D} \setminus D$. If $f_{\mathbf{r},w}$ is approximately differentiable at x, then both series $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ and $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ converge. Moreover,*

$$apf'_{\mathbf{r},w}(x) = \sum_{k=1}^{\infty} w_k g_k'^{+}(x) = \sum_{k=1}^{\infty} w_k g_k'^{-}(x).$$

Proof. For every n, recall that if r_{n+1}/r_n is even then $\widetilde{D}_n \subset D_{n+1}$, whereas if r_{n+1}/r_n is odd then $\widetilde{D}_n \subset \widetilde{D}_{n+1}$. Let n_0 be the first index such that $x \in \widetilde{D}_{n_0}$. Since $x \notin D$ we have that r_{n+1}/r_n is an odd number for every $n \ge n_0$, and consequently $x \in \widetilde{D}_n$ for every $n \ge n_0$. Therefore,

$$\sum_{k=n_0}^{\infty} w_k g_k'^{+}(x) = -\sum_{k=n_0}^{\infty} w_k g_k'^{-}(x)$$

so both series converge or diverge simultaneously. We consider the case concerning $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ and the case concerning $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ is similar. As is customary, we denote $d_{ap} = \mathrm{ap}f'_{\mathbf{r},w}(x)$ and we take $M = |d_{ap}| + 1$. Moreover, for every $y \neq x$ we write

$$\Delta(y) = \frac{f_{\mathbf{r},w}(y) - f_{\mathbf{r},w}(x)}{y - x}.$$

For the sake of contradiction, we suppose that $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ does not converge or $d_{ap} \neq \sum_{k=1}^{\infty} w_k g_k'^{-}(x)$. Proceeding as we did in Proposition 5.9, we assume that there are numbers α and β such that

$$\liminf_n \sum_{k=1}^{n} w_k g_k'^{-}(x) < \beta < \alpha < d_{ap}$$

and consequently, there exist infinitely many indices n such that

$$\sum_{k=1}^{n} w_k g_k'^{-}(x) < \beta.$$

For all those indices n, we may assume that $x \in \widetilde{D}_n$ and we take $z_{n+1} = x - r_{n+1}^{-1}$. Moreover, we consider the interval $I_{n+1} = \big(z_{n+1} - \delta r_{n+1}^{-1}, z_{n+1}\big)$ with

$$0 < \delta < \min\left\{\frac{3(\alpha - \beta)}{4\delta(|\beta| + M)}, \frac{1}{4}\right\} \tag{5.6}$$

and we define $E_{n+1} = \{z \in [z_{n+1}, x] : |\Delta(z)| \geq M\}$. Let $y \in I_{n+1} \setminus \tau(E_{n+1})$ where τ is the translation $\tau : [z_{n+1}, x] \to [z_{n+1} - r_{n+1}^{-1}, z_{n+1}]$. Now, we observe that $y - x \geq \frac{3}{4} r_{n+1}^{-1}$ and g_k is affine on $I_{n+1}$for every $k \leq n$, so we have

$$\begin{aligned}\Delta(y) &= \sum_{k=1}^{n} w_k g_k'^{-}(x) + \sum_{k=n+1}^{\infty} w_k \frac{g_k(y) - g_k(x)}{y - x} \\ &= \sum_{k=1}^{n} w_k g_k'^{-}(x) + \frac{\tau^{-1}(y) - x}{y - x} \sum_{k=n+1}^{\infty} w_k \frac{g_k\big(\tau^{-1}(y)\big) - g_k(x)}{\tau^{-1}(y) - x} \\ &\leq \beta + \frac{4}{3}\delta(|\beta| + M) \leq \alpha\end{aligned}$$

by (5.6), and the proof follows as in Proposition 5.9. □

For a point $x \in D \cup \widetilde{D}$ we take $n_0 \geq 1$ as the first index such that $x \in D_{n_0} \cup \widetilde{D}_{n_0}$. Since we have either $\widetilde{D}_n \subset D_{n+1}$ or $\widetilde{D}_n \subset \widetilde{D}_{n+1}$ for every n, it follows that $g_k'^{+}(x) = -g_k'^{-}(x)$ for all $k \geq n_0$. Therefore, if both series $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ and $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ converge and agree with $\mathrm{ap} f'_{\mathbf{r},w}(x)$, we have

$$\mathrm{ap} f'_{\mathbf{r},w}(x) - \sum_{k=1}^{n_0-1} w_k g_k'(x) = \sum_{k=n_0}^{\infty} w_k g_k'^{+}(x) = -\sum_{k=n_0}^{\infty} w_k g_k'^{+}(x)$$

which implies

$$\sum_{k=n_0}^{\infty} w_k g_k'^{+}(x) = \sum_{k=n_0}^{\infty} w_k g_k'^{-}(x) = 0. \tag{5.7}$$

When $x \in D \setminus \widetilde{D}$ or $x \in D \cap \widetilde{D}$, Proposition 2.15 and Proposition 2.16 of [49] yield that $f_{\mathbf{r},w}$ is derivable at x if and only if Condition (5.7) holds. The same reasoning done by the authors in Proposition 2.15 of [49] gives the same result for the remaining case $x \in \widetilde{D} \setminus D$. Thus, we obtain the following result.

Theorem 5.11. *For every $x \in D \cup \widetilde{D}$ the following statements are equivalent:*

(1) $f_{\mathbf{r},w}$ is approximately differentiable at x.

(2) $f_{\mathbf{r},w}$ is derivable at x.

(3) Both series $\sum_{k=1}^{\infty} w_k g_k'^{+}(x)$ and $\sum_{k=1}^{\infty} w_k g_k'^{-}(x)$ converge and agree.

If this is the case, then

$$f'_{\mathbf{r},w}(x) = ap f'_{\mathbf{r},w}(x) = \sum_{k=1}^{n_0-1} w_k g_k'(x)$$

where n_0 is the first index such that $x \in D_{n_0} \cup \widetilde{D}_{n_0}$.

Theorem 5.8 together with Theorem 5.11 yields that if $f_{\mathbf{r},w}$ is approximately differentiable at some point of $[0,1]$, then $w \in c_0$. As a consequence of Theorem 2.13 of [49], if $w \in c_0 \setminus \ell_1$ then $f_{\mathbf{r},w}$ is derivable at an uncountable set and the range of the derivative is $\mathbb{R}$. Whereas, if $w \in \ell_1$ then $f_{\mathbf{r},w}$ is Lipchitz by Proposition 5.1, and consequently it is derivable almost everywhere by Rademacher's Theorem. From these facts, we obtain an improvement of Kôno's theorem via the approximate differentiability.

Corollary 5.12. *The function* $f_{\mathbf{r},w}$ *is nowhere approximately differentiable if and only if* $w \notin c_0$.

CONCLUSIONES

La función de Takagi y sus generalizaciones han sido estudiadas por numerosos autores a lo largo de los años. En esta tesis se resuelven algunas cuestiones que se han planteado a lo largo de los años sobre determinadas propiedades de las funciones de Takagi generalizadas.

El Teorema de Kôno es probablemente el resultado más sorprendente y conocido para las funciones de la Clase de Takagi. Este teorema ilustra tres diferentes comportamiendos de diferenciabilidad de las funciones de la Clase de Takagi. En el Capítulo 2 abordamos el estudio de algunas propiedades de diferenciabilidad de orden dos para las funciones de Takagi. Quizás, el resultado más sorprendente que obtenemos, es el Teorema 2.24 en el que probamos que un conjunto de propiedades, en principio distintas para funciones arbitrarias, son equivalentes para las funciones de la Clase de Takagi y que pueden caracterizarse en términos de un condición sencilla sobre la sucesión de pesos. Para obtener este resultado, se desarrollan técnicas nuevas en las que el desarrollo binario de los puntos del intervalo $[0,1]$ juega un papel fundamental. Los resultados obtenidos en este capítulo pueden ver como un cuarto comportamiendo cualitativo que no estaba contemplado en el clásico teorema de Kôno.

En [11], P. C. Allaart and K. Kawamura caracterizaron el conjunto de puntos en los que la función de Takagi tiene una derivada infinita. En el survey [10] realizado por ambos autores, propusieron como problema abierto caracterizar dicho conjunto de puntos para la función de Takagi-Van der Waerden. La solución a este problema es presentada en el Capítulo 3 y realizamos dicha caracterización en términos de la expansión en base r de los puntos. Esta fue la primera vez en la que pudimos atisbar la importancia de la paridad de r en el estudio de esta propiedad. Esto supuso el desarrollo de nuevas técnicas que nos permitieran abordar dicho problema desde un enfoque global. Una generalización immediata de la función de Takagi-Van der Waerden es la denominada *función Generalizada* que presentamos en el Capítulo 1. A la luz de los resultados obtenidos en Capítulo 3, parece natural preguntarnos si es posible caracterizar dichos puntos para la función Generalizada en términos de la expansión de tales puntos en cierto sistema de representación numérico que podríamos asociar a cada función Generalizada. El estudio de dicho sistema de numeración sería un problema interesante en sí mismo.

En 2011, P. Góra y R. J. Stern probaron que la función de Takagi tiene un comportamiento muy singular desde el punto de vista de la subdiferenciabilidad. Esto llevó a Juan Ferrera y Javier Gómez a estudiar el comportamiendo de dicha función desde la perspectiva de la superdiferencial. En vista de los resultados que obtuvieron, nos planteamos extender este estudio para la función de Takagi-Van der Waerden. Además, debido a la relación existente entre la superdiferencial y los máximos locales, podríamos caracterizar el conjunto de máximos locales de la función de Takagi-Van der Waerden. Esto permitiría completar el estudio de los puntos extremos de dicha función que había iniciado Y. Baba en 1984. Cuando r es par, Y. Baba probó que el conjunto de puntos donde la función de Takagi-Van der Waerden alcanza su máximo global es un conjunto de Cantor de dimensión de Hausdorff $1/2$. Esto nos llevó a preguntarnos sobre la medida de Hausdorff $1/2$-dimensional de dicho conjunto. En el Capítulo 4 damos respuestas a todas estas cuestiones. Para ello, utilizamos técnicas provenientes del análisis nonsmooth y de la geometría fractal.

En 2020, Juan Ferrera and Javier Gómez Gil probaron que la función de Takagi es aproximadamente diferenciable en ningún punto. En un primer momento, nos planteamos si sería posible mejorar el teorema de Kôno probando que las funciones de la Clase de Takagi son aproximadamente diferenciables en ningún punto si y solo si la sucesión de pesos no está en c_0. Sin embargo, los argumentos que emplearon Juan Ferrera and Javier Gómez Gil en su artículo estaban basados en que las funciones G_n, que aproximan a la función de Takagi por debajo, tienen pendientes enteras. Por tanto, el estudio de dicho problema requería el desarrollo de nuevas técnicas. Los nuevos argumentos que obtuvimos tenían un fuerte componente geométrico que estaba basado el la regularidad de los puntos diádicos. De algún modo, las funciones de la Clase Generalizada también tienen ese componente y pudimos obtener una versión del teorema de Kôno en términos de la diferenciabilidad aproximada. Esto nos lleva a plantearnos si es posible obtener un teorema de Kôno en su totalidad para la clase Generalizada. En [49], ambos autores intentaron obtener una versión de Kôno en un contexto más general, pero sus resultados solamente son aplicables cuando r_{n+1}/r_n es un número par eventualmente. De nuevo, vemos la importancia de la paridad en relación a este tipo de propiedades.

Como podemos observar, en esta tesis se han resuelto diversos problemas sobre las funciones de Takagi generalizadas que abren la puerta a otros problemas relacionados más generales, para los que las técnicas que se han desarrollado a lo largo de dicha tesis constituyen un primer paso hacia su investigación.

CONCLUSIONS

The Takagi function and its generalizations have been deeply investigated by numerous authors over the years. In this thesis, some questions raised regarding certain properties of the generalized Takagi functions have been addressed.

Kôno's theorem is arguably the most surprising and well-known result for functions in the Takagi Class. This theorem illustrates three different behaviors of differentiability for functions in such a Class. In Chapter 2, we delve into the study of some second order differentiability properties for the functions in the Takagi Class. Perhaps the most surprising result obtained is Theorem 2.24, where we prove that several properties are equivalent for the functions in the Takagi Class. However, for arbitrary functions these properties are not even related. Moreover, they can be characterized in terms of a simple condition on the sequence of weights. To achieve this result, new techniques are developed in which the binary expansion of the points in the interval $[0,1]$ plays a fundamental role. The results obtained in this chapter can be seen as a fourth qualitative behavior that was not considered in the classic Kôno's theorem.

In [11], P. C. Allaart and K. Kawamura characterized the set of points where the Takagi function has an infinite derivative. In the survey [10] conducted by both authors, they posed an open problem to characterize this set of points for the Takagi-Van der Waerden function. The solution to this problem is presented in Chapter 3, and we perform this characterization in terms of the base-r expansion of such points. This was the first time we glimpsed the importance of the r parity while dealing with this property. This led to the development of new techniques to approach this problem from a global perspective. An immediate generalization of the Takagi-Van der Waerden function is the so-called *Generalized Function*, presented in Chapter 1. In light of the results obtained in Chapter 3, it seems natural to ask if it is possible to characterize these points for the Generalized Function in terms of the expansion of such points in a certain numerical representation system that we could associate with each Generalized Function. The study of such a numbering system would be an interesting problem in itself.

In 2011, P. Góra and R. J. Stern proved that the Takagi function has a very singular subdifferentiability behavior. This led Juan Ferrera and Javier Gómez to study the behavior of this function from the superdifferential perspective. In view of the results they obtained, we set out to extend this study to the Takagi-Van der Waerden function. Additionally, due to the relationship between the superdifferential and local maxima of a function, we characterized the set of local maxima of the Takagi-Van der Waerden function. This completes the study of the extreme points of this function that Y. Baba initiated in 1984. When r is even, Y. Baba also proved that the set of points where the Takagi-Van der Waerden function attains its global maximum is a Cantor set with Hausdorff dimension $1/2$. This led us to inquire about the Hausdorff measure $1/2$-dimensional of this set. In Chapter 4, we provide answers to all these questions. To do so, we use techniques from nonsmooth analysis and fractal geometry.

In 2020, Juan Ferrera and Javier Gómez Gil proved that the Takagi function is nowhere approximately differentiable. Initially, we wondered if it would be possible to improve Kôno's theorem by proving that the functions of the Takagi Class are nowhere approximately differentiable if and only if the sequence of weights does not belong to c_0. However, the arguments used by Juan Ferrera and Javier Gómez Gil in their article were based on the fact that the polygonal functions G_n, which approximate the Takagi function from below, have integer slopes. Therefore, the study of this problem required the development of new techniques. The arguments we obtained have a strong geometric component based on the regularity of the dyadic points. In a way, the Generalized Class functions also have that component, and we were able to obtain a version of Kôno's theorem via the approximate differentiability. This leads us to consider whether it is possible to obtain a complete version of Kôno's theorem for the Generalized Class. In [49], both authors attempted to obtain a Kôno's type theorem in a more general context, but their results are only applicable when r_{n+1}/r_n is eventually an even number. Again, we see the importance of parity in relation to this type of property.

As we can see, in this thesis, various problems about generalized Takagi functions have been solved, opening the door to other related, more general problems. The techniques developed throughout this thesis constitute a first step toward their investigation.

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