

The Volatility-Return Relationship: Insights from Linear and Non-Linear Quantile Regressions

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Abstract

This paper examines the asymmetric relationship between price and implied volatility and the associated extreme quantile dependence using a linear and non-linear quantile regression approach. Our goal is to demonstrate that the relationship between the volatility and market return, as quantified by Ordinary Least Square (OLS) regression, is not uniform across the distribution of the volatility-price return pairs using quantile regressions. We examine the bivariate relationships of six volatility-return pairs, namely: CBOE VIX and S&P 500, FTSE 100 Volatility and FTSE 100, NASDAQ 100 Volatility (VXN) and NASDAQ, DAX Volatility (VDAX) and DAX 30, CAC Volatility (VCAC) and CAC 40, and STOXX Volatility (VSTOXX) and STOXX. The assumption of a normal distribution in the return series is not appropriate when the distribution is skewed, and hence OLS may not capture a complete picture of the relationship. Quantile regression, on the other hand, can be set up with various loss functions, both parametric and non-parametric (linear case) and can be evaluated with skewed marginal-based copulas (for the non-linear case), which is helpful in evaluating the non-normal and non-linear nature of the relationship between price and volatility. In the empirical analysis we compare the results from linear quantile regression (LQR) and copula based non-linear quantile

regression known as copula quantile regression (CQR). The discussion of the properties of the volatility series and empirical findings in this paper have significance for portfolio optimization, hedging strategies, trading strategies and risk management, in general.

Keywords: Return Volatility relationship, quantile regression, copula, copula quantile regression, volatility index, tail dependence.

JEL Codes: C14, C58, G11,

1 Introduction

The quantification of the relationship between the changes in stock index returns and changes in the associated volatility index serves is important given that it serves as the basis for hedging. This relationship is mostly quantified as being asymmetric (Badshah, 2012; Dennis, Mayhew and Stivers, 2006; Fleming, Ostdiek and Whaley, 1995; Giot, 2005; Hibbert, Daigler and Dupoyet, 2008; Low, 2004; Whaley, 2000; Wu, 2001). An asymmetric relationship means that the negative change in the stock market has a higher impact on the volatility index than a positive change, or vice-versa. The asymmetric volatility-return relationship has been pointed out in two hypotheses, that is, the leverage hypothesis (Black, 1976; Christie, 1982) and the volatility feedback hypothesis (Campbell and Hentschel, 1992).

In a call/put option contract time to maturity and strike price form its basic characteristics, the other inputs, namely risk free rate and dividend payout can be decided easily (Black and Scholes, 1973). When pricing an option, the expected volatility over the life of the option becomes a critical input, and it is also the only input which is not directly observed by market participants. In an actively traded market, volatility can be calculated by inverting the chosen option pricing formula for the observed market price of the option. This volatility calculated by inverting the option pricing formula is known as implied volatility. With an increasing focus on risk modelling in modern finance, modelling and predicting asset volatility, along with its dependence with the underlying asset class has become an important research topic.

Any changes in volatility will likely lead to movements in stock market prices. For example, an expected rise in volatility will lead to a decline in stock market prices. The volatility indices are used for option pricing and hedging calculations, and their change is reflected in the corresponding stock markets. Financial risk is mostly composed of rare or extreme events that result in high risk, and they reside in the tails of the return distribution. In option pricing, rare or extreme events result in volatility skew patterns (Liu, Pan and Wang, 2005).

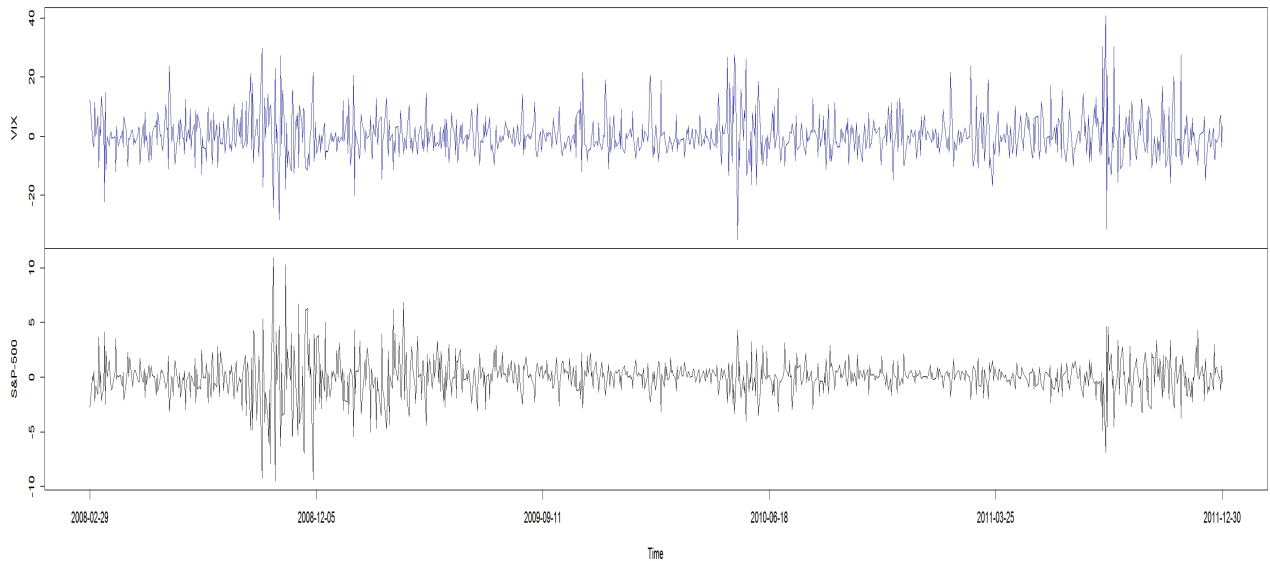


Figure 1: Time Series Plot VIX and S&P 500

Ordinary least squares regression (OLS) is the most widely used method for quantifying a relationship between two classes of assets or return distributions in the finance literature. Figure 1 shows the logarithmic return series of VIX and S&P 500 stock indices for the years 2008-2011. The time series plot shows that the VIX index changes according to the changes in S&P 500. We employ two cases of quantile regression (linear and non-linear) to evaluate the asymmetric volatility-return relationship between changes in the volatility index (VIX, VFTSE, VXN, VDAX, VSTOXX and VCAC) and corresponding stock index return series (S&P 500, FTSE 100, NASDAQ, DAX 30, STOXX and CAC 40). We focus on the daily asymmetric return-volatility relationship in this paper.

Giot (2005), Hibbert et al. (2008) and Low (2004) use OLS in their study of asymmetric return-volatility relationships across implied volatility (IV) change distributions. The construction of OLS means that it is fitted on the basis of deviations from the means of the distributions concerned, and does not reflect, given its concerned with averages, the extreme quantile relationships. Badshah (2012) extends past studies using linear quantile regression (LQR) to estimate the negative asymmetric return-volatility relationship between stock index return (S&P 500, NASDAQ, DAX 30, STOXX) and changes in volatility index return (VIX, VXN, VDAX, VSTOXX) for lower and upper quantiles which give negative and positive returns. Badshah (2012) found that negative returns have higher impacts than positive returns using a linear quantile regression framework. Kumar (2012) used LQR to examine the statistical properties of the volatility index of India and its relationship with the Indian stock market.

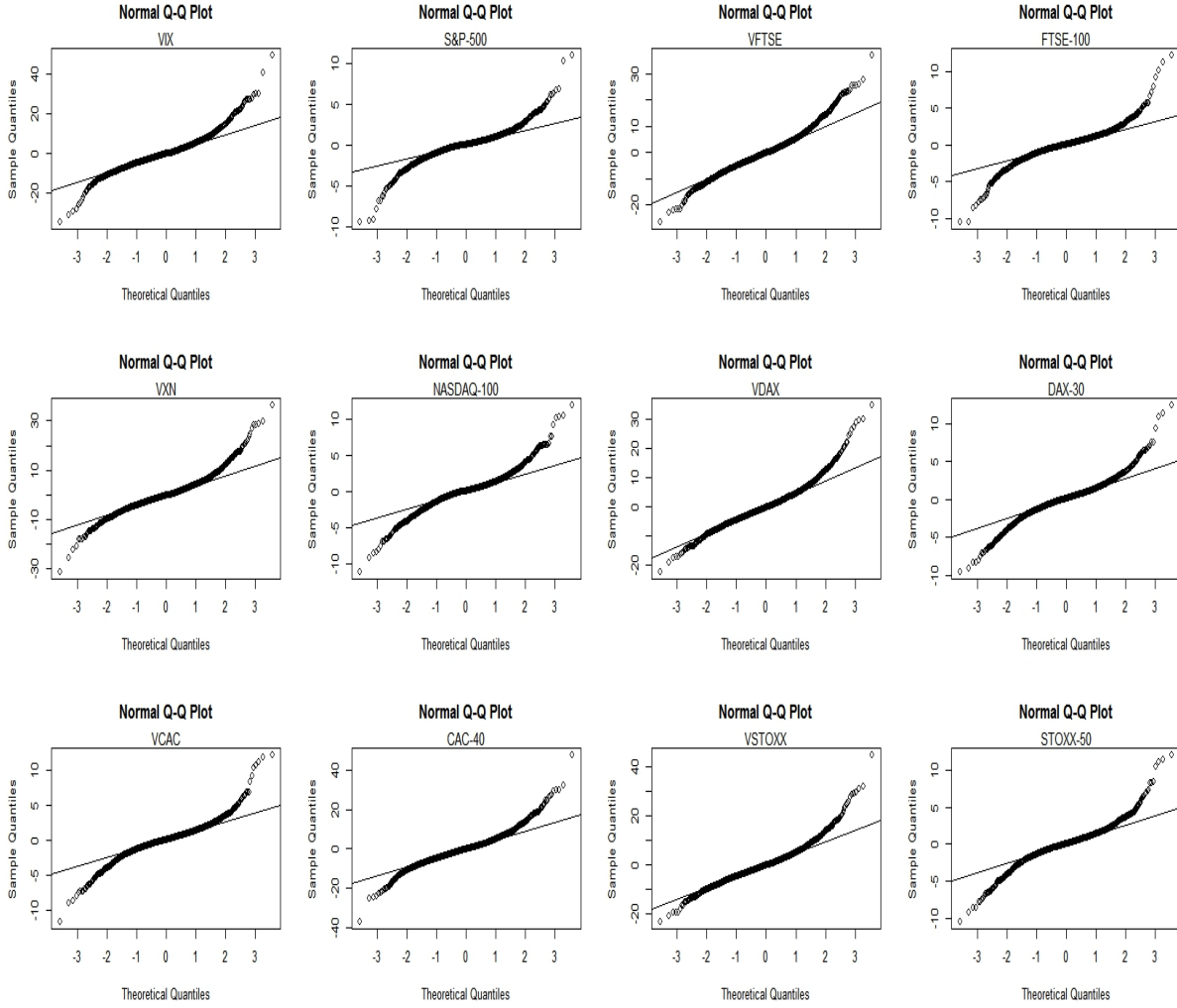


Figure 2: Q-Q Plots

Figure 2 gives the quantile-quantile plots for the data, and none of the data series shows a good fit to normal distributions. When the data distribution is not normal, quantile regression (QR) can provide more efficient estimates for return-volatility relationships (Badshah, 2012). QR can be used not only linearly but also for non-linear relationships using Copula-based models. Badshah, (2012) used QR to investigate return-volatility relationships focussed on the linear case. We extend this analysis, by considering the non-linear aspects of the relationship using copula-based non-linear quantile regression models, CQR.

The rest of the paper is as follows. In Section 2 we discuss linear quantile regression LQR, followed by non-linear quantile regression using copula CQR in Section 3. In Section 4 we describe the data sets together with the research design and methodology. We discuss the results in Section 5 and conclude in Section 6.

2 Quantile Regression

Regression analysis is undoubtedly the most widely used statistical technique in market risk modelling; and has been applied in various contexts, such as factor models to model returns or autocorrelated models to model volatility in time series. All these models are based on regression analysis in combination with different approaches and emphases.

A simple linear regression model can be written as:

$$Y_{it} = \alpha_i + \beta_i X_{it} + \varepsilon_{it}, \quad (1)$$

Equation (1) represents the dependent variable, Y_{it} , as a linear function of one or more independent variables, X_{it} , subject to a random ‘disturbance’ or ‘error’ term, ε_{it} which is assumed to be *i.i.d* and independent of X_i .

A bivariate normal distribution is assumed to apply to both the dependent and independent variables in the case of simple linear regression. The regression estimates the mean value of the dependent variable for given levels of the independent variables. For this type of regression, where the focus is on the understanding of the central tendencies in a data set, OLS is a very effective method. Nevertheless, OLS may lose its effectiveness when we try to go beyond the mean value or towards the extremes of a data set (Allen, Singh and Powell, 2010; Allen, Gerrans, Singh and Powell, 2009; Barnes and Hughes, 2002). Specifically, in the case of an unknown or arbitrary joint distribution (X_i, Y_i) , OLS does not provide all the necessary information required to quantify the conditional distribution of the dependent variable. As presented in our descriptive statistics (Section 4.1), the data set used in this analysis is not normal, and hence quantile regression may be a better choice, particularly if we want to explore the relationships in the tails of the two distributions.

Quantile Regression can be viewed as an extension of classical OLS (Koenker and Bassett, 1978). In Quantile Regression, the estimation of the conditional mean by OLS is extended to the similar estimation of an ensemble of models of various conditional quantile functions for a data distribution. Quantile Regression can better quantify the conditional distribution of $(Y|X)$. The central special case is the median regression estimator that minimises a sum of absolute errors. The estimates of remaining conditional quantile functions are obtained by minimizing an asymmetrically weighted sum of absolute errors, where the weights are the function of the quantile of interest. This makes Quantile Regression a robust technique, even in the presence of outliers. Taken together the ensemble of estimated conditional quantile functions of $(Y|X)$ offers a more complete view of the effect of covariates on the location, scale and shape of the distribution of the response variable.

For parameter estimation in Quantile Regression, quantiles, as proposed by Koenker and Bassett (1978), can be defined via an optimisation problem. To solve an OLS re-

gression problem of fitting a line through a sample mean, the process is defined as the solution to the problem of minimising the sum of squared residuals, in the same way the median quantile (0.5%) in Quantile Regression is defined through the problem of minimising the sum of absolute residuals. The symmetrical piecewise linear absolute value function assures the same number of observations above and below the median of the distribution.

The other quantile values can be obtained by minimizing a sum of asymmetrically weighted absolute residuals, thereby giving different weights to positive and negative residuals. Solving

$$\min_{\xi \in R} \sum \rho_{\tau}(y_i - \xi) \quad (2)$$

where $\rho_{\tau}(\cdot)$ is the tilted absolute value function, as shown in Figure 2.4, gives the τ th sample quantile with its solution. Taking the directional derivatives of the objective function with respect to ξ (from left to right) shows that this problem yields the sample quantile as its solution.

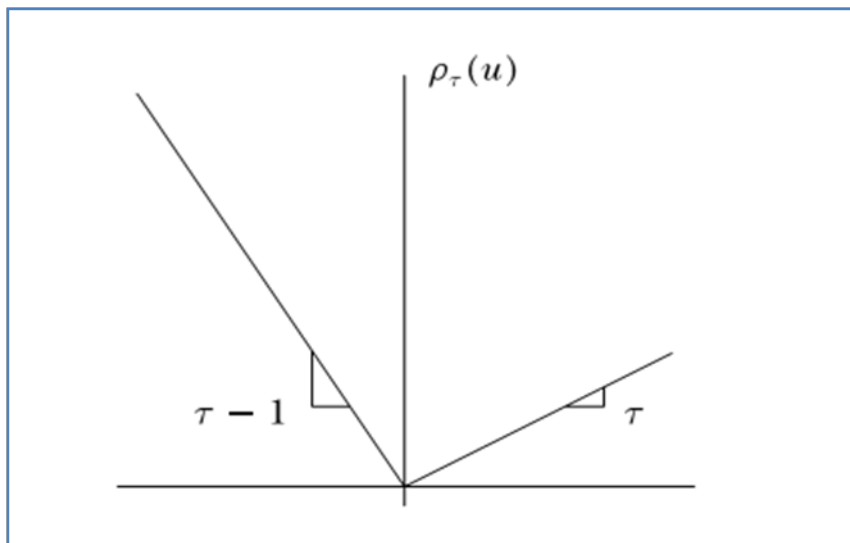


Figure 3: Quantile Regression ρ Function

After defining the unconditional quantiles as an optimisation problem, it is easy to define conditional quantiles similarly. Taking the least squares regression model, as a base, for a random sample, y_1, y_2, \dots, y_n , we solve

$$\min_{\mu \in R} \sum_{i=1}^n (y_i - \mu)^2 \quad (3)$$

which gives the sample mean, an estimate of the unconditional population mean, EY . Replacing the scalar, μ , by a parametric function $\mu(x, \beta)$, and then solving

$$\min_{\mu \in R^p} \sum_{i=1}^n (y_i - \mu(x_i, \beta))^2 \quad (4)$$

gives an estimate of the conditional expectation function $E(Y/x)$.

Proceeding the same way for Quantile Regression, to obtain an estimate of the conditional median function, the scalar ξ in the first equation is replaced by the parametric function $\xi(x_t, \beta)$, and τ is set to $1/2$. Further insight into this robust regression technique can be obtained from Koenker and Bassett's (2005) Quantile Regression monograph, or as discussed by Alexander (2008).

Quantile regression has been applied frequently in research over the past decade in various areas of econometric analysis, financial modelling and socio-economic research. These studies include Buchinsky and Leslie (1997), who analyse changing US wage structures. Buchinsky and Hunt (1999) analyse earning mobility and factors affecting the transmission of earnings across generations. Eide and Showanter (1998) study the effect of school quality on education. Financial research work using quantile regression includes Engle and Manganelli (2004) and Morillo (2000), quantifying Value-at-Risk (VaR) using quantile regression and studying option pricing using Monte Carlo simulations. Barnes and Hughes (2002) applied Quantile Regression to study the Capital Asset Pricing Model (CAPM) in their work on cross sections of stock market returns. Chan and Lakonishok (1992) applied Quantile Regression to robust measurement of size and book to market effects. Gowlland, Xiao and Zeng (2009) investigate book to market effects beyond the central tendency. Allen, Singh and Powell (2011) apply Quantile Regression to test applications of the Fama-French factor model in the DJIA 30 stocks, and explore the relative merits of estimates of factor based risk factors across quantiles, as contrasted with OLS estimates.

Other than Badshah (2012) and Kumar (2012), there is no prior work investigating the return-volatility relationship between volatility indices and corresponding market indices using quantile regression. We apply the LQR model to evaluate the return-volatility relationship, and we also test the non-linear case of CQR to further examine the nature of this important relationship.

3 Non-Linear Quantile Regression (CQR)

Bouyé and Salmon (2009) extended Koenker and Basset's (1978) idea of regression quantiles, and introduced a general approach to non linear quantile regression modelling using copula functions. Copula functions are used to define the dependence structure between the dependent and exogenous variables of interest. We first give a brief introduction to copulas, followed by an introduction to the concept of CQR.

3.1 Copula

Modelling the dependency structure within assets is a key issue in risk measurement. The most common measure for dependency, correlation, loses its effect when a dependency measure is required for distributions which are not normally distributed. Examples of deviations from normality are the presence of kurtosis or fat tails and skewness in univariate distributions. Deviations from normality also occur in multivariate distributions given by asymmetric dependence, which suggests that assets show different levels of correlation during different market conditions (Erb et al., 1994; Longin and Solnik, 2001; Ang and Chen, 2002 and Patton, 2004). Modelling dependence with correlation is not inefficient when the distribution follows the strict assumptions of normality and constant dependence across the quantiles. As it is well known in financial risk modelling, that return distributions do not necessarily follow normal distributions across quantiles, we need more sophisticated tools for modelling dependence than correlations. Copulas provide one such measure.

The statistical tool which is used to model the underlying dependence structure of a multivariate distribution is the copula function. The capability of a copula to model and estimate multivariate distributions comes from *Sklar's Theorem*, according to which each joint distribution can be decomposed into its marginal distributions and a copula C responsible for the dependence structure. Here we define a Copula using Sklar's Theorem, along with some important types of copula, adapted from Franke, Härdle and Hafner (2008).

A function $C : [0, 1]^d \rightarrow [0, 1]$ is a d dimensional copula if it satisfies the following conditions for every $u = (u_1, \dots, u_d)^\top \in [0, 1]^d$ and $j \in \{1, \dots, d\}$:

1. if $u_j = 0$ then $C(u_1, \dots, u_d) = 0$;
2. $C(1, \dots, 1, u_j, 1, \dots, 1) = u_j$;
3. for every $v = (v_1, \dots, v_d)^\top \in [0, 1]^d, v_j \leq u_j$

$$V_C(u, v) \geq 0$$

where $V_C(u, v)$ is given by

$$\sum_{i_1=1}^2 \dots \sum_{i_d=1}^2 (-1)^{i_1+\dots+i_d} C(g_{1i_1}, \dots, g_{di_d}).$$

Properties 1 and 3 state that copulae are grounded functions and that all d -dimensional boxes with vertices in $[0, 1]^d$ have non-negative C-volume. The second Property shows that the copulae have uniform marginal distributions.

Sklar's Theorem

Consider a d -dimensional distribution function, F , with marginals F_1, \dots, F_d . Then for every $x_1, \dots, x_d \in \overline{\mathbb{R}}$, a copula, C can exist with

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\}. \quad (5)$$

C is unique if F_1, \dots, F_d are continuous. If F_1, \dots, F_d , are distributions then the function F is a joint distribution function with marginals F_1, \dots, F_d .

For a joint distribution F with continuous marginals F_1, \dots, F_d , for all $u = (u_1, \dots, u_d)^\top \in [0, 1]^d$, the unique copula C is given as:

$$C(u_1, \dots, u_d) = F\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\}. \quad (6)$$

Copulae can be divided into two broad types, Elliptical Copulae-Gaussian Copula and Student's t-Copula and Archimedean Copulae-Gumbel copula, Clayton copula and Frank Copula .

Normal or Gaussian Copula

The copula derived from the n -dimensional multivariate and univariate standard normal distribution functions, Φ and Φ , is called a normal or Gaussian copula. The normal copula can be defined as

$$C(u_1, \dots, u_n; \Sigma) = \Phi(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)), \quad (7)$$

where the correlation matrix and (Σ) is the parameter for the normal copula, and $u_i = F_i(x_i)$ is the marginal distribution function.

The normal copula density is given by:

$$c(u_1, \dots, u_n; \Sigma) = |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}\xi'(\Sigma^{-1} - \mathbf{I})\xi\right) \quad (8)$$

where Σ is the correlation matrix, and $|\Sigma|$ is its determinant. $\xi = (\xi_1, \dots, \xi_n)'$, where ξ_i is the u_i quantile of the standard normal variable, X_i .

Figure 4 gives the density plot for a bivariate Gaussian copula with a correlation of 0.5. As shown in the figure, the normal copula is a symmetric copula.

Student's t-Copula

Similar to the Gaussian copula, t-copulas model the dependence structure of multivariate t-distributions. The parameters for the student t-copula are the correlation matrix and degrees of freedom. Student t-copula shows symmetrical dependence, but is higher than

Gaussian Copula Density

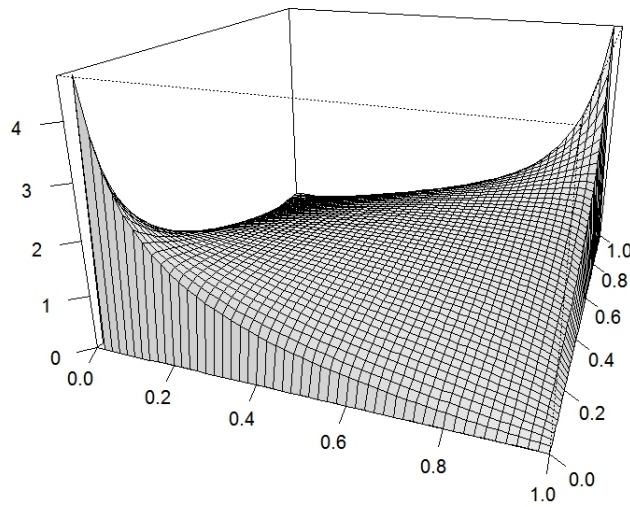


Figure 4: Density of the Gaussian copula

those in the Gaussian copula, as shown in Figure-5. Alexander (2008) considers the density functions and quantile functions of the student-t copula.

Student's t-Copula Density

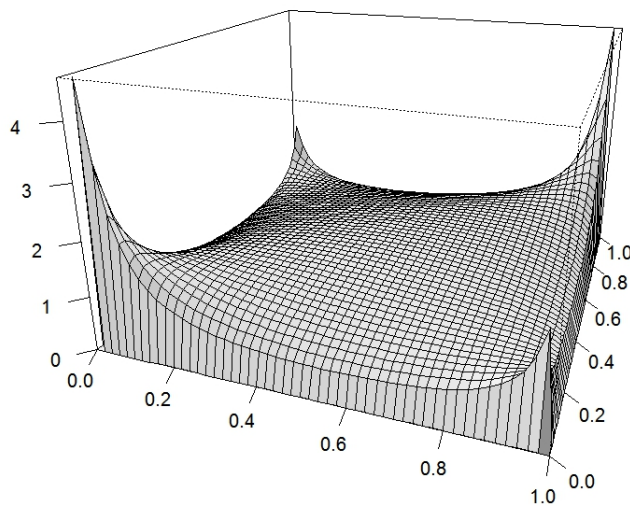


Figure 5: Density of t-copula

Archimedean Copulae

Archimedean copulae is a family of copula which is built on a generator function, with some restrictions. There can be various copulae in this family of copulae due to the various generator functions available (see Nelson (1999)). For a generator function, ϕ , the Archimedean copula can be defined as:

$$C(u_1, \dots, u_n) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_n)). \quad (9)$$

The density function is given by

$$c(u_1, \dots, u_n) = \phi^{-1}(\phi(u_1) + \dots + \phi(u_n)) \prod_{i=1}^n \phi'(u_i). \quad (10)$$

Clayton Copula

The Clayton copula, as introduced by Clayton (1978), has a generator function:

$$\phi(u) = \alpha^{-1}(u^{-\alpha} - 1), \quad \alpha \neq 0. \quad (11)$$

The inverse generator function is

$$\phi^{-1}(x) = (\alpha x + 1)^{-1/\alpha}.$$

With variation in parameter, α , the Clayton copulas capture a range of dependence. The Clayton copula is particularly helpful in capturing positive lower tail dependence. Figure 6 gives a density plot for the bivariate Clayton copula with $\alpha = 0.5$. The asymmetric lower tail dependence is evident from the figure.

Like the Normal and Student-t copula, Archimedean copula can also be used for CQR¹. Here we use only Normal and Student-t copula for our analysis as they capture both positive and negative dependence. The Clayton copula captures only positive lower tail dependence and hence is omitted.

We will not discuss further the types of copula in detail, but refer the reader to Joe (1997), Nelsen (1999), Alexander (2008) and Cheung (2009), who give a useful overview of copula for financial practitioners. The quantile functions of the copulas used in the CQR are reported in the following discussion of copula quantile regression. The quantile function of the Clayton is also given for completeness.

3.2 Copula Quantile Regression (CQR)

Bouyé and Salmon (2009) discussed copula quantile regression in detail by highlighting the properties of quantile curves. They also gave simple closed forms of the quantile

¹The example of the Clayton copula with its quantile function is given in the next subsection.

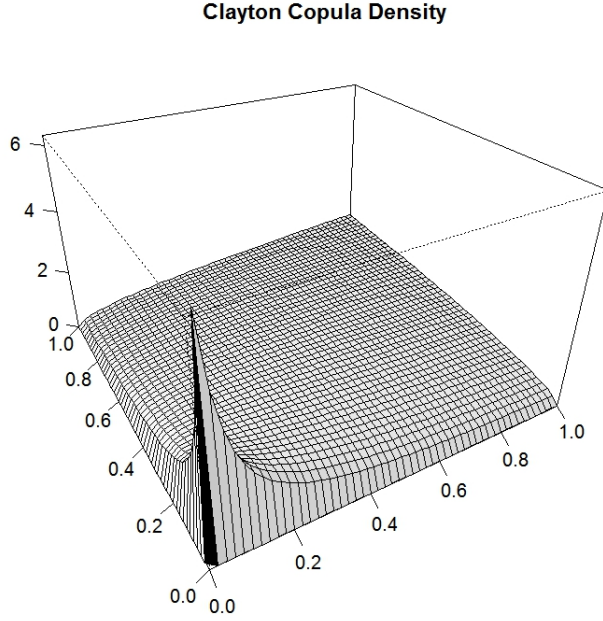


Figure 6: Density of Clayton copula

curve for major copula (normal, Student t, Joe-Clayton, and Frank) which are used in the linear quantile regression model (Equation 5) to calculate non-linear regression quantiles. Here we will give the closed form solution of the four copula quantile curves, for the sake of brevity, (refer to the original paper by Bouyé and Salmon (2009) for a detailed discussion). Alexander (2008) also gives a brief introduction of non-linear copula based quantile regressions and provides some empirical examples.

The non-linear quantile regression model is formed by replacing the linear quantile regression model (5) with the quantile curve of a copula. Every copula has a quantile curve, which may be decomposed in an explicit manner.

If we have two marginals, $F_X(x)$ and $F_Y(y)$, of x and y , with their estimated distribution parameters, we can then define a bivariate copula with certain parameters, θ .

Normal CQR

The bivariate normal copula has one parameter, the correlation ϱ , and its quantile curve can be written as:

$$y = F_Y^{-1} \left[\Phi \left(\varrho \Phi^{-1}(F_X(x)) + \sqrt{1 - \varrho^2} \Phi^{-1}(q) \right) \right]. \quad (12)$$

Student-t CQR

The Student-t copula has two parameters, the degrees of freedom, ν , and the correlation, ρ . The quantile curve of the Student-t copula is given by:

$$y = F_Y^{-1} \left[t_\nu \left(\rho t_\nu^{-1}(F_X(x)) + \sqrt{(1 - \rho^2)(\nu + 1)^{-1} (\nu + t_\nu^{-1}(F_X(x))^2)} t_{\nu+1}^{-1}(q) \right) \right]. \quad (13)$$

Clayton CQR

Clayton copula is a member of the Archimedean Copula, with a generator function having parameter, α . The quantile curve of the Clayton copula is given by:

$$y = F_Y^{-1} \left[\left((1 + F_X(x))^{-\alpha} (q^{-\alpha/(1+\alpha)} - 1) \right)^{-1/\alpha} \right]. \quad (14)$$

In order to evaluate non-linear quantile regressions using copula, for a given sample $\{(x_t, y_t)\}_{t=1}^T$, the q (or τ) quantile regression curve can be defined as $y_t = \xi(x_t, q; \hat{\theta}_q)$. The parameters $\hat{\theta}_q$ are found by solving the following optimization problem:

$$\min_{\mu \in \mathbb{R}^p} \sum \rho_q(y_t - \xi(x_t, q; \theta)). \quad (15)$$

This optimization problem can be solved by using the Quantreg package of the statistical software R, after defining the copula using copula related packages.

In this paper, we use LQR and CQR with normal or Gaussian and Student-t copula to evaluate the return-volatility relationship. We now discuss the data and methodology implemented in the following section.

4 Data and Methodology

4.1 Description of Data

In the empirical analysis, we use daily price data for market and volatility indices of six volatility-return pairs, namely, VIX and S&P 500, VFTSE and FTSE 100, VXN and NASDAQ, VDAX and DAX 30, VCAC and CAC 40, and VSTOXX and STOXX. We obtained daily prices from Datastream for a period of approximately 10 years, from 2/02/2001 to 31/12/2011. Daily percentage logarithmic returns are used for the analysis. Table 1 gives the descriptive statistics for our data set. All the data series show excess kurtosis indicating fat tails. The Jarque-Bera test statistics in Table 1 strongly reject the presence of normal distributions in the series. Given the descriptive statistics, we can conclude that all the return time series (for the market and volatility series) exhibit fat tails and are not normally distributed. The ADF test statistics also reject the presence of unit roots in the time series.

	VIX	S&P 500	VFTSE	FTSE 100	VXN	NASDAQ	VDAX	DAX 30	VCAC	CAC 40	VSTOXX	STOXX 50
Observations	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845	2845
Minimum	-35.0588	-9.4695	-26.7893	-10.5381	-31.3049	-11.1149	-22.2296	-9.6010	-11.7370	-37.1866	-23.4362	-10.4552
Quartile 1	-3.5950	-0.5659	-3.6566	-0.6881	-2.9385	-0.8043	-3.1912	-0.8592	-0.8263	-3.3499	-3.4479	-0.8517
Median	-0.3063	0.0273	-0.1913	0.0448	-0.1797	0.0462	-0.2621	0.0701	0.0313	-0.1943	-0.3179	0.0183
Arithmetic Mean	0.0022	-0.0025	0.0092	-0.0021	-0.0304	-0.0029	0.0287	0.0074	-0.0099	0.0249	0.0263	-0.0130
Quartile 3	2.9640	0.5964	3.1760	0.7634	2.4648	0.8377	2.7955	0.9174	0.8808	2.8021	2.8650	0.8792
Maximum	49.6008	10.9572	37.1670	12.2189	36.2851	11.8493	34.5301	12.3697	12.1434	47.5537	44.6046	11.9653
SE Mean	0.1177	0.0255	0.1142	0.0288	0.0973	0.0343	0.0986	0.0338	0.0331	0.1116	0.1066	0.0335
LCL Mean (0.95)	-0.2284	-0.0525	-0.2147	-0.0585	-0.2213	-0.0701	-0.1646	-0.0589	-0.0748	-0.1939	-0.1828	-0.0787
UCL Mean (0.95)	0.2329	0.0475	0.2331	0.0543	0.1605	0.0644	0.2220	0.0738	0.0550	0.2438	0.2353	0.0527
Variance	39.3801	1.8481	37.0906	2.3561	26.9609	3.3484	27.6513	3.2562	3.1183	35.4488	32.3380	3.1923
Stdev	6.2754	1.3595	6.0902	1.5350	5.1924	1.8299	5.2585	1.8045	1.7659	5.9539	5.6866	1.7867
Skewness	0.7148	-0.1798	0.5494	-0.1176	0.6461	0.0327	0.7517	-0.0518	0.0205	0.5678	0.8619	-0.0214
Kurtosis	4.8375	8.0012	2.2572	7.9493	4.4896	4.2954	3.2289	4.4147	5.8562	4.4866	3.8149	5.0474
ADF	-58.9219	-58.7627	-56.6697	-54.9693	-55.6025	-56.9344	-52.6907	-53.3281	-54.0247	-57.2406	-55.0843	-53.8264
JarqueBera	3022.9229	7618.9607	749.2105	7511.9282	2593.1979	2192.8800	1507.5219	2317.0224	4074.3108	2544.8816	2082.1950	3026.9972

Table 1: Descriptive Statistics

4.2 Methodology

In the empirical analysis we evaluate the volatility-return relationship, which can be represented by the following:

$$V_t = \alpha + \beta R_t + \varepsilon_t \quad (16)$$

where V_t is the daily logarithmic return of the volatility index and R_t gives the daily logarithmic return of the market index. α , β and ε gives the intercept, the slope coefficient, which represents the degree of association, and the error term respectively.

We will use three regression techniques in this paper, the basic linear regression model (estimated by OLS), linear quantile regression, and non-linear copula quantile regression, to quantify the return-volatility relationship for the six return-volatility pairs. The relationship quantified by OLS is around the mean of the distribution, and hence does not quantify the tail regions. In this paper, we examine if the relationship quantified by the quantile regressions are different from OLS and if they are different across the various quantiles in the distribution.

The major results from the paper are discussed in the following section.

5 Discussion of the Results

5.1 Linear Regression-OLS

We first evaluate the volatility-return relationship using OLS. As mentioned before, OLS gives the relationship around the mean of the distribution and hence omits the extreme cases. These would be the circumstances when the market is either in crisis or when it is performing exceptionally well. The relationship quantified by OLS gives the relationship between the average of the volatility and return series.

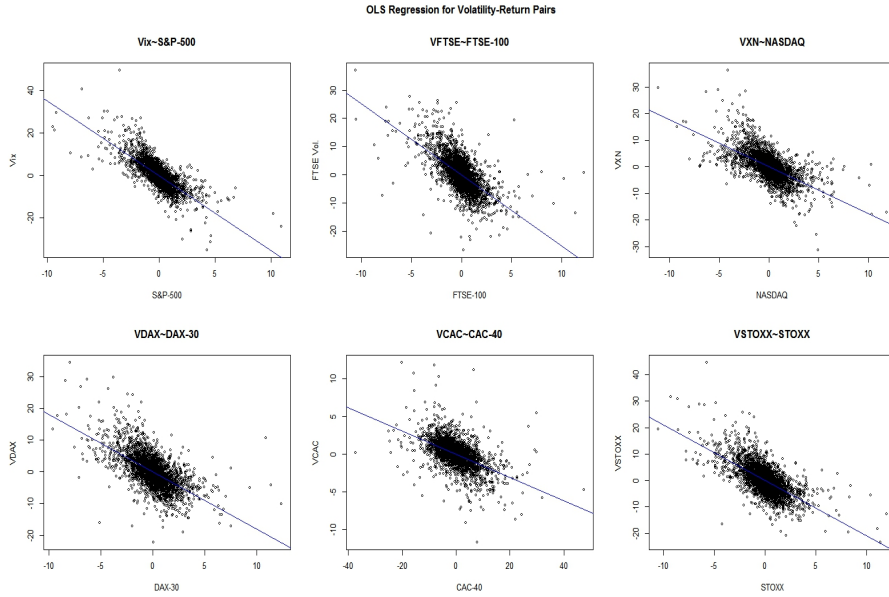


Figure 7: OLS Regression for Volatility-Return Pairs

Figure 7 gives the plot of OLS regression fit for the actual volatility-return data. The common observation in all the figures is that the regression line runs through the mean of the observations. As the regression line represents the mean behaviour, the estimated values are around the mean of the distribution and, in the case of non-normality, is not well suited to quantify relationships in the tails or other quantiles diverging from mean. Table 2 gives the point estimates of the intercept and regression coefficient for all the volatility-return pairs. The values of the regression coefficient indicate an inverse volatility return relationship. These results confirm the earlier work. The key issue is whether the nature of this relationship changes across the quantiles of the distribution.

	α	P-value	β	P-value
VIX-S&P	-0.0065	0.9325	-3.5147	0.0000
VFTSE-FTSE	0.0039	0.9646	-2.5387	0.0000
VXN-NASDAQ	-0.0355	0.6419	-1.7651	0.0000
VDAX-DAX	0.0421	0.5862	-1.8059	0.0000
VCAC-CAC	-0.0060	0.8304	-0.1549	0.0000
VSTOXX-STOXX	-0.0011	0.9893	-2.1028	0.0000

Table 2: OLS Regression Results

All the estimated β values are significant at the 1% level

5.2 Linear Quantile Regression (LQR)

In financial risk measurement, quantification of the tails plays an important role in risk modelling. OLS estimates quantify the relationship around the mean of the distribution, but QR, on the other hand, can be used to quantify the relationship across various quantiles. We use LQR to model the volatility-return relationship across the quantiles,

and focus particularly on the lower quantiles, which represent large negative returns and the risk in the market. We evaluate volatility-return relationships across seven quantiles of interest $q = \{0.01, 0.05, 0.25, 0.5, 0.75, 0.95, 0.99\}$ which include the median as well as two extremes, the lower 1% and higher 99% quantiles.

Figure 8 gives the plots for the LQR coefficient (β) for all the volatility-return pairs. It is evident from the figure that these coefficients are different across the quantiles, and hence the relationship also changes.

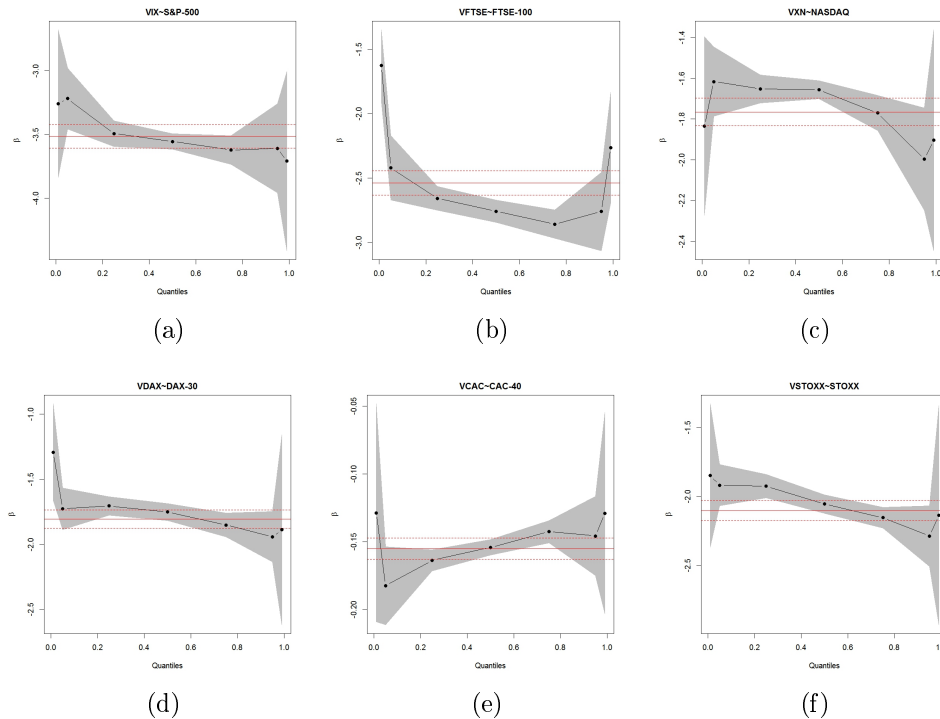


Figure 8: Volatility-Return Coefficient (β) Estimates Across Quantiles

Table 3 gives the estimates for the LQR model, with intercept, α , and slope coefficient, β , which measures the dependence of volatility on market return. The estimated dependence coefficient (β) values are significant across the quantiles, and are also not the same. The results clearly indicate that the volatility-return relationship changes across the quantiles and that they are also statistically significant.

	Quantile Regression Estimates													
	$\alpha_{0.01}$	$\beta_{0.01}$	$\alpha_{0.05}$	$\beta_{0.05}$	$\alpha_{0.25}$	$\beta_{0.25}$	$\alpha_{0.5}$	$\beta_{0.5}$	$\alpha_{0.75}$	$\beta_{0.75}$	$\alpha_{0.95}$	$\beta_{0.95}$	$\alpha_{0.99}$	$\beta_{0.99}$
VIX-S&P	-9.92	-3.26	-5.72	-3.22	-2.31	-3.49	-0.09	-3.55	1.95	-3.62	6.59	-3.61	12.06	-3.71
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VFTSE-FTSE	-11.87	-1.63	-6.84	-2.42	-2.63	-2.66	-0.15	-2.76	2.45	-2.86	7.51	-2.76	13.31	-2.26
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VXN-NASDAQ	-9.75	-1.83	-6.07	-1.62	-2.28	-1.65	-0.16	-1.66	2.00	-1.77	6.90	-2.00	12.22	-1.90
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VDAX-DAX	-10.12	-1.29	-5.96	-1.73	-2.43	-1.71	-0.06	-1.75	2.26	-1.85	6.56	-1.94	12.60	-1.89
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.43	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VCAC-CAC	-4.48	-0.13	-2.23	-0.18	-0.72	-0.16	0.02	-0.15	0.71	-0.14	2.18	-0.15	4.10	-0.13
p-value	0.00	0.01	0.00	0.00	0.00	0.00	0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00
VSTOXX-STOXX	-9.62	-1.85	-6.44	-1.92	-2.58	-1.93	-0.15	-2.05	2.34	-2.15	6.86	-2.29	12.41	-2.14
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 3: LQR Results
A p-value of ≤ 0.05 shows significance at the 5% level

5.3 Copula Quantile Regression (CQR)

LQR quantifies a linear volatility-return relationship, but CQR can be used to quantify this relationship in a non-linear framework. In CQR, the non linear volatility-return relationship is quantified by the copula quantile functions of the respective copula. We use the Normal and Student-t copulae in the following analysis.

The marginals for the bivariate CQR are assumed to be the Student-t distribution. The data are first transformed to marginals by fitting it to the standard Student-t distribution. The estimates are calculated using the Quantreg package in R.

Table 4 gives the ρ estimates for the seven quantiles for the Normal and Student-t copulae. In most of the pairs, the negative dependence is greater for low and high quantiles. The lower tail negative dependence is also higher than the upper tail negative dependence.

Figure 9 plots the estimates for the Student-t CQR for all the volatility-return pairs across the quantiles. The figure shows that the estimates have an approximate inverted U shape, except for VIX-S&P 500. The inverted U shape (higher dependence across tails) is most prominent for the VCAC-CAC 40 pair.

Normal CQR							
	$\varrho_{0.01}$	$\varrho_{0.05}$	$\varrho_{0.25}$	$\varrho_{0.5}$	$\varrho_{0.75}$	$\varrho_{0.95}$	$\varrho_{0.99}$
VIX-S&P	-0.8645	-0.8913	-0.8861	-0.8916	-0.9099	-0.8696	-0.8602
VFTSE-FTSE	-0.7735	-0.8114	-0.7925	-0.8137	-0.8287	-0.7891	-0.7497
VXN-NASDAQ	-0.8066	-0.8164	-0.7647	-0.7045	-0.8047	-0.7831	-0.7718
VDAX-DAX	-0.7584	-0.7937	-0.7322	-0.7106	-0.7841	-0.7785	-0.7479
VCAC-CAC	-0.6915	-0.7755	-0.7067	-0.5631	-0.6759	-0.7691	-0.6970
VSTOXX-STOXX	-0.8015	-0.8055	-0.7517	-0.7496	-0.8122	-0.7999	-0.7469

Student-t CQR							
	$\varrho_{0.01}$	$\varrho_{0.05}$	$\varrho_{0.25}$	$\varrho_{0.5}$	$\varrho_{0.75}$	$\varrho_{0.95}$	$\varrho_{0.99}$
VIX-S&P	-0.8726	-0.8920	-0.8814	-0.8857	-0.9030	-0.8691	-0.8617
VFTSE-FTSE	-0.7815	-0.8058	-0.7780	-0.7818	-0.8048	-0.7827	-0.7915
VXN-NASDAQ	-0.8207	-0.8074	-0.7500	-0.6930	-0.7916	-0.7806	-0.7794
VDAX-DAX	-0.7696	-0.8016	-0.7159	-0.6911	-0.7578	-0.7782	-0.7663
VCAC-CAC	-0.7868	-0.7608	-0.6436	-0.4888	-0.5959	-0.7667	-0.7661
VSTOXX-STOXX	-0.8190	-0.8065	-0.7404	-0.7322	-0.7990	-0.8027	-0.7458

Table 4: Normal and Student-t CQR Estimates
All the estimates in the table are statistically significant.

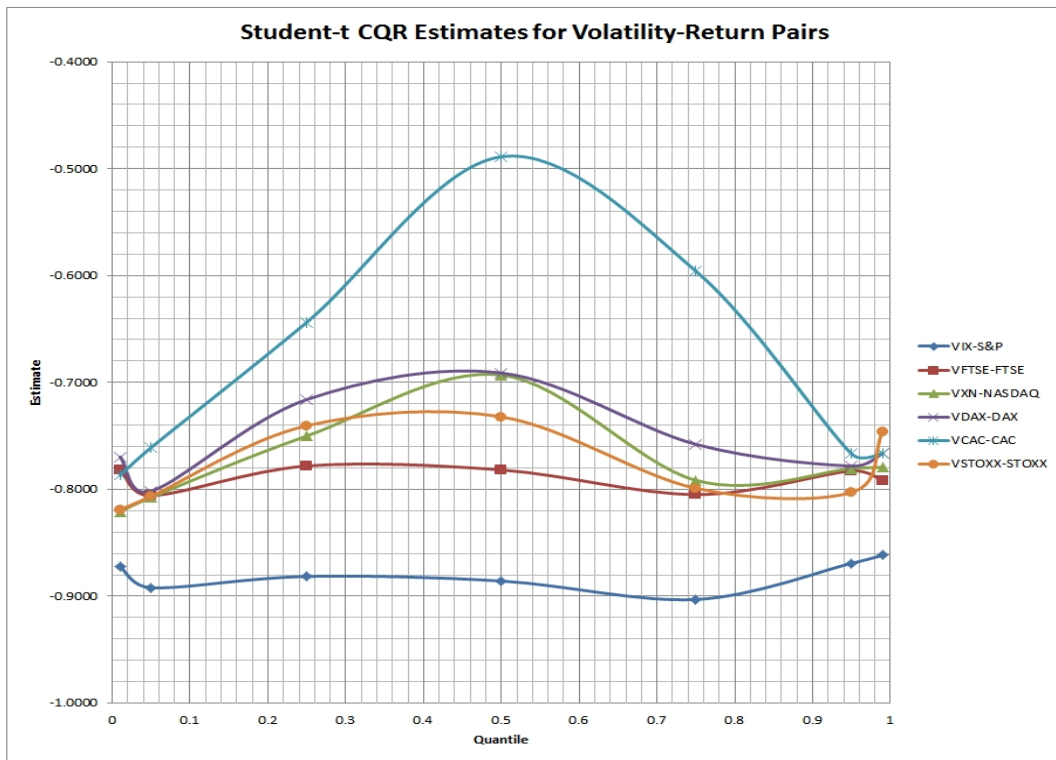
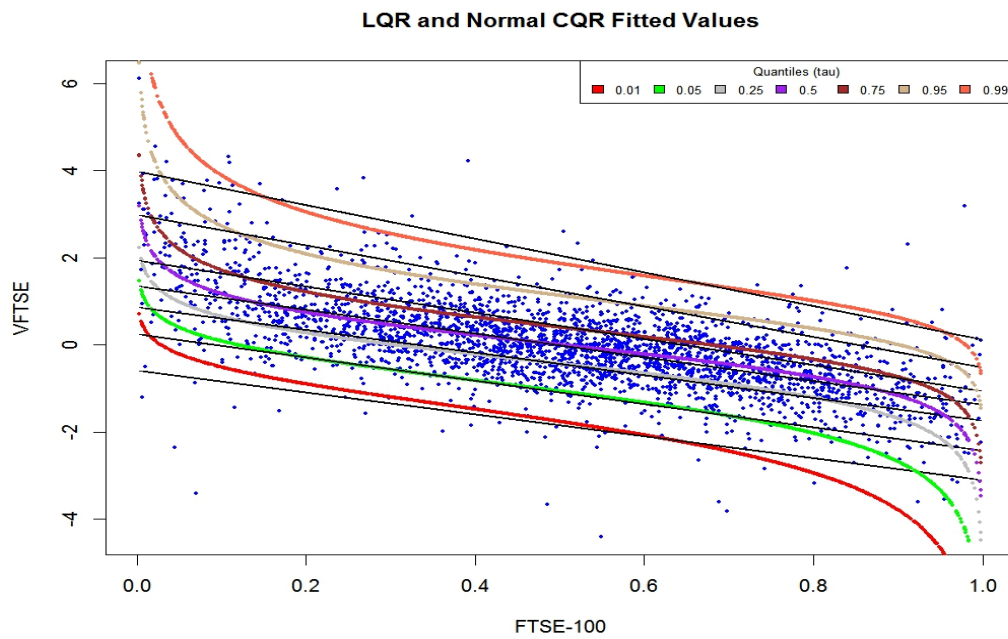


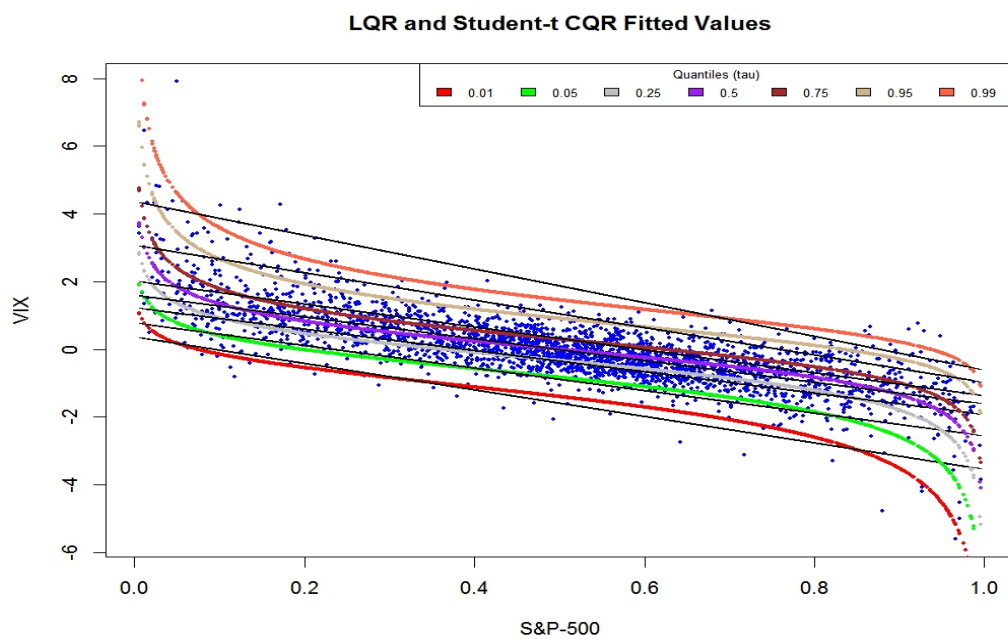
Figure 9: Student-t CQR Estimates

Another point of the analysis is to see how well the estimates from LQR and CQR fit the data. Figure 10 plots the LQR and CQR fitted values across the quantiles over the marginal data. Figure 10(a) plots the VFTSE-FTSE pair fitted values estimated from Normal CQR and LQR, and Figure 10(b) plots the VIX-S&P fitted values estimated

from the Student-t CQR and LQR. The figures show that we can model the non-linear relationship using copula in a quantile regression framework.



(a) VFTSE-FTSE Normal CQR



(b) VIX-S&P Student-t CQR

Figure 10: Fitted Values from CQR and LQR

6 Conclusion

The empirical analysis in this paper demonstrated the application of both linear and non-linear quantile regression models. We used LQR and CQR to model the inverse volatility-return relationship for six volatility-return pairs. The paper focussed on the use of copula to model non-linear quantile regression relationships which facilitate the quantification of bivariate non-linear correlation within the quantiles of the distribution. Linear regression quantifies the relationship between dependent and exogenous variables around the mean of the distribution, and hence does not quantify the relationship for the quantiles across the distribution. Quantile regression is a very useful tool for quantifying the relationship across various quantiles of a distribution.

The tails of the return distribution are of immense interest in financial risk modelling, as they represent the risk associated with the asset or the financial instrument. The volatility-return relationship and its quantification have great importance for hedging, as the change in volatility leads to changes in market prices. In this analysis we used OLS to quantify the linear volatility-return relationship around the mean, which as quantified by LQR, is not consistent for quantiles across the distribution. CQR is yet another useful tool for quantifying non-linear bivariate relationships across quantiles. The analysis conducted in this paper demonstrated that CQR better fits the actual data than LQR as it is capable of capturing non-linearities in the nature of the volatility-return relationship. The results from this analysis also support the existence of an asymmetric volatility-return relationship for the majority of the index pairs.

The empirical analysis of this paper has significance for hedging, portfolio management and risk modelling, in general. The empirical analysis can be extended further by including more copula models, such as the Frank copula and Joe-Clayton copula, amongst others, in the CQR model.

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