AN ARMA REPRESENTATION OF UNOBSERVED COMPONENT MODELS UNDER GENERALIZED RANDOM WALK SPECIFICATIONS: NEW ALGORITHMS AND EXAMPLES*

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Abstract

Among the alternative Unobserved Components formulations within the stochastic state space setting, the Dynamic Harmonic Regression (DHR) has proved particularly useful for adaptive seasonal adjustment signal extraction, forecasting and back-casting of time series. Here, we show first how to obtain ARMA representations for the Dynamic Harmonic Regression (DHR) components under several random walk specifications. Later, we uses these theoretical results to derive an alternative algorithm based on the frequency domain for the identification and estimation of DHR models. The main advantages of this algorithm are linearity, fast computing, avoidance of some numerical issues, and automatic identification of the DHR model. To compare it with other alternatives, empirical applications are provided.

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1 Introduction

During the last two decades the literature on signal extraction has been roughly based on the so-called model-based approach. Three directions have emerged: (1) one, termed the *ARIMA-model based* or "reduced" form model (see Box et al., 1978; Hillmer & Tiao, 1982; Burman, 1980; Gomez & Maravall, 1996a); (2) a second one, termed *optimal regularization* (see Akaike, 1980; Jakeman & Young, 1984; Young, 1991); and (3) a third one that begins by directly specifying the model for the components within a stochastic State Space (SS) setting. This last SS formulation was originated in the 1960s in the control engineering area and has been absorbed within the statistical literature during the last years (see Harvey, 1989; West & Harrison, 1989; Young et al., 1988; Young, 1994). In spite of some differences in the specifications, the models in these approaches are closely related. The relationship and, in some cases, the exact equivalence of these methods is discussed in Young & Pedregal (1999) within the context of optimal filter theory.

The Dynamic Harmonic Regression (DHR) model developed by Young et al. (1999) belongs to the Unobserved Components (UC) type and is formulated within the SS. Young et al. (1999) claim that this method yields asymptotically equivalent results to the aforementioned approaches if the models on which they are based are made compatible. The DHR model is based on an spectral approach under the hypothesis that the observed time series can be decomposed into several DHR components whose variances are concentrated around certain frequencies. This is an appropriate hypothesis if the observed time series has well defined spectral peaks which implies that its variance is distributed around narrow frequency bands. Basically, the method attempts to: (1) identify the spectral peaks, (2) assign a DHR component to each spectral peak, (3) estimate the hyper-parameters that control the spectral fit of each component to its corresponding spectral peak, and (4) estimate the DHR components using the Kalman Filter and the Fixed Interval Smoothing (FIS) algorithms.

In the univariate case, the DHR model can be written as an special case of the univariate UC model which has the general form:

$$y_t = T_t + C_t + S_t + e_t; \quad t = 0, 1, 2, \ldots,$$

where y_t is the observed time series, T_t is the trend or low-frequency component, C_t is the cyclical component, S_t is the seasonal component, and e_t is an irregular component normally distributed Gaussian sequence with zero mean value and variance σ_e^2, ($\{e_t\} \sim$ w.n. $N(0, \sigma_e^2)$).

In the DHR model, T_t, C_t, and S_t consist of a number of DHR components,

$s_t^{p_j}$, with the general form

$$s_t^{p_j} = a_{j_t} \cos(\omega_j t) + b_{j_t} \sin(\omega_j t), \tag{1}$$

where p_j and ω_j are, the period and the frequency associated with each jth DHR component respectively. T_t is the zero frequency term ($T_t \equiv s_t^{\infty} = a_{0t}$), while the cyclical and seasonal components are $C_t = \sum_{j=1}^{R_c} s_t^{p_j}$, and $S_t = \sum_{j=R_c+1}^{R} s_t^{p_j}$, respectively; where $\omega_j = 1/p_j,\ j = 1, \ldots, R_c$ are the cyclical frequencies, and $\omega_j,\ j = (R_c+1), \ldots, R$ are the seasonal frequencies. Hence, the complete DHR model is then

$$y_t^{dhr} = \sum_{j=0}^{R} s_t^{p_j} + e_t = \sum_{j=0}^{R} \left\{ a_{j_t} \cos(\omega_j t) + b_{j_t} \sin(\omega_j t) \right\} + e_t. \tag{2}$$

The oscillations of each DHR component are modulated by a_{j_t} and b_{j_t} which are stochastic Time Varying Parameters (TVP) within the family of the Generalized Random Walk (GRW) models (Young, 1994); therefore, non-stationarity is allowed in the various components. This DHR model can be considered a straightforward extension of the classical harmonic regression model, in which the gain and phase of the harmonic components can vary as a result of estimated temporal changes in the parameters a_{j_t} and b_{j_t}.[1]

The stochastic evolution of a_{j_t} and b_{j_t} is defined by a two dimensional stochastic state vector $\mathbf{x}_{j_t} = [l_{j_t}\ d_{j_t}]'$, where l_{j_t} and d_{j_t} are respectively the changing level and slope of the associated parameter. The evolution of $\mathbf{x}_{j_t}$ is described by a GRW process of the form

$$\mathbf{x}_{j_t} = \mathbf{F}_j \mathbf{x}_{j_{t-1}} + \mathbf{G}_j \eta_{j_t}, \quad j = 0, 1, \ldots, R, \tag{3}$$

where $\eta_{j_t} = [\nu_{j_t}, \xi_{j_t}]'$; $\{\nu_{j_t}\} \sim$ w.n. $N(0, \sigma^2_{\nu_j})$; $\{\xi_{j_t}\} \sim$ w.n. $N(0, \sigma^2_{\xi_j})$; $R = R_c + R_s$; and

$$\mathbf{F}_j = \begin{bmatrix} \alpha_j & \beta_j \\ 0 & \gamma_j \end{bmatrix}, \quad \mathbf{G}_j = \begin{bmatrix} \delta_j & 0 \\ 0 & 1 \end{bmatrix}.$$

By restricting certain values in $\mathbf{F}_j$ and $\mathbf{G}_j$, the GRW model comprises a large number of characterizations found in the signal extraction literature (Young, 1984). For instance, the Integrated Random Walk (IRW): $\alpha = \beta = \gamma = 1,\ \delta = 0$; the scalar Random Walk (RW): $\alpha = \beta = \delta = 0,\ \gamma = 1$; the Smoothed Random Walk (SRW): $0 < \alpha < 1,\ \beta = \gamma = 1,\ \delta = 0$; as well as Harvey's Local Linear Trend: $\alpha = \beta = \gamma = 1,\ \delta = 1$; and the

[1] The main difference between the DHR model and related techniques, such as Harvey's structural model, lies in the formulation of the UC model for the periodic components and the method of optimizing the hyper-parameters .

"Damped Trend": $\alpha = \beta = \delta = 1,\ 0 < \gamma < 1$ (see Harvey, 1989; Koopmans et al., 1995). Although not directly related to the main body of this paper, it is instructive to consider the nature of the prediction equations for the various GRW processes. While the RW prediction is constant at the level of the prediction origin, the SRW allows a range of intermediate possibilities between the RW and the IRW models as function of α. If we restrict our analysis to the cases $(0 \leq \alpha, \beta \leq 1,\ \gamma = 1,\ \delta = 0)$, we deal with RW, SRW, IRW specifications and also with stationary models. Then, the reduced form of (3) can be written as

$$(1 - \alpha_j L)(1 - \beta_j L) l_{j_t} = \xi_{j_{t-1}}; \quad 0 \leq \alpha_j, \beta_j \leq 1 \tag{4}$$

The method for optimizing the hyper-parameters of the model (i.e., the variances $\boldsymbol{\sigma}^2_{dhr} = [\sigma^2_0, \sigma^2_1, \ldots, \sigma^2_R]'$ of the processes $\xi_j,\ j = 0 \ldots, R$, and the variance σ^2_e of the irregular component) was formulated by Young et al. (1999) in the frequency domain, and is based upon expressions for the *pseudo-spectrum* of the full DHR model:

$$f_{dhr}\left(\omega, \boldsymbol{\sigma}^2\right) = \sum_{j=0}^{R} \sigma^2_j S_j(\omega) + \sigma^2_e; \quad \boldsymbol{\sigma}^2 = \left[\boldsymbol{\sigma}^2_{dhr}, \sigma^2_e\right]' \tag{5}$$

where $\sigma^2_j S_j(\omega)$ are the *pseudo-spectra* of the DHR components s^{p_j}, and σ^2_e is the variance of the irregular component (Young et al., 1999, p. 377).

A simple manipulation of (5) allow us to write

$$f_{dhr}\left(\omega, [\mathbf{NVR}, \sigma^2_e]\right) = \sigma^2_e \left[\sum_{j=0}^{R} NVR_j \cdot S_j(\omega) + 1\right],$$

where $\mathbf{NVR}$ is the vector with elements $NVR_j = \sigma^2_j / \sigma^2_e,\ j = 1, 2, \ldots, R$.

Young et al. (1999) propose one final simplification using the estimate of the residual white noise from an AutoRegressive (AR) model. Young et al. (1999) describe the complete DHR algorithm in the following four steps:

1. Estimate an AR(n) spectrum $f_y(\omega)$ of the observed time series and use its associated residual variance $\widehat{\sigma}^2$ as the estimation of σ^2_e. The AR order is identified by the Akaike's Information Criterion.

2. Find the Linear Least Squares estimate of the **NVR** parameter vector which minimizes the linear least squares function

$$J\left(f_y, f_{dhr}\right) = \sum_{k=1}^{m} \left[f_y(\omega_k) - f_{dhr}(\omega_k, [\mathbf{NVR}, \widehat{\sigma}^2])\right]^2; \tag{6}$$

   where $\omega_k \in [0\ \pi]$ are the m points where the *pseudo-spectra* f_y and f_{dhr} are evaluated.

3. Find the Non-linear Least Squares estimate of the **NVR** parameter vector which minimizes the non-linear least squares function

$$J_L\left(f_y, f_{dhr}\right) = \sum_{k=1}^{m} \left[\log f_y(\omega_k) - \log f_{dhr}(\omega_k, [\mathbf{NVR}, \widehat{\sigma}^2])\right]^2 \qquad (7)$$

using the result from step 2 to define the initial conditions.

4. Use the NVR estimates from step 3 to obtain the recursive forward pass (Kalman filter) and backward pass (FIS algorithm) smoothed estimates of the DHR components.

This optimization algorithm has been used extensively over the past years, in the micro-CAPTAIN DOS program, and more recently in a Matlab® toolbox under the CAPTAIN heading. As a time series/forecasting algorithm it has been used in different areas of research such as business cycle analysis (García-Ferrer & Queralt, 1998), environmental issues (Young & Pedregal, 1999), industrial turning point predictions (García-Ferrer & Bujosa-Brun, 2000), forecasting economic sectorial demand (García-Ferrer et al., 1997), etc. Additionally, the DHR model is a powerful signal extraction alternative that can compete well with the best known techniques such as the X-12 ARIMA (Findley et al., 1996), the ARIMA-model based models like SEATS/TRAMO (Gomez & Maravall, 1996b; Maravall, 1993) and the structural model STAMP program (Koopmans et al., 1995).

2 ARMA models for the DHR components

In this section it is shown that each DHR component has an AutoRegressive Moving Average (ARMA) representation and, therefore, an associated *pseudo-covariance generating function*.

The trend follows an AR(2) model:

$$(1 - \alpha_0 L)(1 - \beta_0 L)T_t = \xi_{0t-1}, \quad \{\xi_{0t}\} \sim \text{w.n. } N(0, \sigma^2_{\xi_0}); \qquad (8)$$

hence, its *pseudo-covariance generating function* is

$$\Lambda_T(z) = \frac{\sigma_0^2}{[1 \;\; -\alpha_0 z][1 \;\; -\beta_0 z][1 \;\; -\alpha_0 z^{-1}][1 \;\; -\beta_0 z^{-1}]}\blacktriangleright, \qquad (9)$$

where $\frac{1}{b}\blacktriangleright$ is the inverse[2] of the sequence b in the field of fractions of formal sequences, $\mathbb{C}((z))$. The Nyquist component also follows an AR(2) model

$$(1 + \alpha_R L)(1 + \beta_R L)s^2 = \xi_{Rt-1}, \quad \{\xi_{Rt}\} \sim \text{w.n. } N(0, \sigma^2_{\xi_R}) \qquad (10)$$

[2]See Section A in the Appendix.

and therefore its *pseudo-covariance generating function* is

$$\Lambda_T(z) = \frac{\sigma_R^2}{[1 \quad +\alpha_R z][1 \quad +\beta_R z][1 \quad +\alpha_R z^{-1}][1 \quad +\beta_R z^{-1}]} \tag{11}$$

For the case of the remaining cyclical and seasonal components two propositions are shown[3]. The first one states that for each cyclical and seasonal component s^{p_j} there is a sequence $\Lambda_{s^{p_j}} \in \mathbb{C}(z)$ such as its extended Fourier transform[4], $\mathcal{FE}$, is the *pseudo-spectrum* of s^{p_j}. The second one shows the existence of an ARMA model whose *pseudo-covariance generating function* is $\Lambda_{s^{p_j}}$. Consequently, the *pseudo-spectrum*[5] of the ARMA model is the *pseudo-spectrum* of s^{p_j}. The *pseudo-spectrum* for these components are given by[6]:

$$f_{s^{p_j}}(\omega) = \mathcal{FE}\left(\Lambda_{s^{p_j}}(z)\right) = \frac{1}{2}\left[f_a(\omega - \omega_j) + f_a(\omega + \omega_j)\right], \quad \omega_j \in (0, \pi).$$

It follows that each *pseudo-spectrum* $f_{s^{p_j}}$ can be stated as

$$f_{s^{p_j}}(\omega) = \sigma_j^2 \begin{bmatrix} & \frac{1}{2}\frac{1}{(1+\alpha^2-2\alpha\cos(\omega-\omega_j))(1+\beta^2-2\beta\cos(\omega-\omega_j))} \\ + & \frac{1}{2}\frac{1}{(1+\alpha^2-2\alpha\cos(\omega+\omega_j))(1+\beta^2-2\beta\cos(\omega+\omega_j))} \end{bmatrix}. \tag{12}$$

Finally, the *pseudo-covariance generating function* for the irregular component is $\Lambda_e(z) = \sigma_e^2$.

The consequence of the previous results is that we can write the DHR model $y_t^{dhr} = \sum_{j=0}^{R} s_t^{p_j} + e_t$, as a sum of $(R+1)$ ARMA models plus a white noise process $\{e_t\}$. The specific ARMA model for each DHR component depends on the type of GRW processes followed by its a_j and b_j parameters. In all cases, however, the modulus of the AR roots are always α_j^{-1} and β_j^{-1} [see Equation (3)]. Table 1 shows the corresponding ARMA models for the DHR components under different GRW specifications: AR, RW, SRW, and IRW. Finally, Table 2 shows the alternative ARMA specifications for the different components: trend, cyclical and seasonal, and the Nyquist component.

3 The new BGF estimation algorithm

In the original NVR optimization algorithm two questions arise. First, the logarithmic transformation is used because it produces a more clearly located

[3]The propositions and their proofs can be seen in Section B in the Appendix.

[4]The extended Fourier transform $\mathcal{FE}$ is the application that corresponds to each fraction of finite sequences $p * (q)^{-1} \blacktriangleright$ the fraction $\mathcal{F}(p)/\mathcal{F}(q)$ where $\mathcal{F}(x)$ is the Fourier transform of x. For more details see Bujosa et al. (2001)

[5]Here we define the *pseudo-spectrum* of an ARMA processes as the extended Fourier transform of its *pseudo-covariance generating function* (see Bujosa et al., 2001).

[6](see Bujosa, 2000)

Component	AR and RW $\alpha_j = 0;\ 0 < \beta_j \leq 1$	SRW and IRW $0 \leq \alpha_j \leq 1;\ \beta_j = 1$
Trend T	AR(1)	AR(2)
Nyquist s^2	AR(1)	AR(2)
Cyclical or seasonal s^{p_j}	s^4 $(\omega_j = \pi/2)$: AR(2) Remaining components : ARMA(2,1)	ARMA(4,2)

Table 1: Summary of ARMA models of the components.

and defined optimum, so improving the estimation of the hyper-parameters; hence, the original algorithm uses a non-linear objective function. Second, when minimizing the objective functions in (6) and (7), we need to avoid the regions around the poles[7]. Our proposal is to estimate the NVR hyper-parameters in the frequency domain by minimizing a linear objective function. To do so, a linear algebraic transformation of (6) capable of eliminating the poles in $f_{dhr}(\omega, \boldsymbol{\sigma}^2)$ and $f_y(\omega)$ it is needed.

3.1 A linear algebraic transformation

In the optimization processes we seek the vector $\boldsymbol{\sigma}^2$ that minimizes[8]

$$\min_{[\boldsymbol{\sigma}^2]\in\mathbb{R}^{R+1}} \left\| f_y(\omega) - f_{dhr}\left(\omega, \boldsymbol{\sigma}^2\right) \right\|. \tag{13}$$

It has been shown that the DHR components follow non-stationary ARMA processes; therefore, $f_{dhr}(\omega, \boldsymbol{\sigma}^2)$ has poles. In order to find a solution of Equation (13) we need to eliminate the AR roots on the unit circle (AR unit roots). Using the ARMA representation of the DHR components $s_t^{p_j}$ we have

$$s_t^{p_j} = \frac{\theta_j(L)}{\varphi_j(L)} \blacktriangleright \xi_{j_{t-1}}, \quad \{\xi_{j_t t}\} \sim \text{w.n. } N(0, \sigma^2_{\xi_{j_t}}).$$

Substituting $s_t^{p_j}$ in Equation (2) we obtain an alternative expression of the DHR model

$$y_t^{dhr} = \sum_{j=1}^{R} \frac{\theta_j(L)}{\varphi_j(L)} \blacktriangleright \xi_{j_{t-1}} + e_t.$$

[7]Since the DHR models are non-stationary, their spectral peaks are *poles*. Roughly speaking, a pole is a point in the real line, say ω_0, such that $f(\omega)$ approaches infinity as ω approaches ω_0.

[8]Young et al. (1999) simplify the problem using the residual variance $\widehat{\sigma}^2$ from the fitted AR model, as estimation of σ_e^2, and then dividing by $\widehat{\sigma}^2$, so they seek the vector $\mathbf{NVR} = [1,\ NVR_0, \ldots,\ NVR_R]$, where $NVR_j = \sigma_j^2/\widehat{\sigma}^2$.

GRW model	Trend $\omega_0 = 0$	Cyclical and seasonal components $0 < \omega_j < \pi$	Nyquist component $\omega_j = \pi$
General model $(0 \leq \alpha,\ \beta \leq 1)$	$(1 - (\alpha_0 + \beta_0)L + \alpha_0\beta_0 L^2)T_t = \xi_{0t-1}$	$(\phi_j^{\alpha}(L) * \phi_j^{\beta}(L)) s^{p_j}{}_t = \left(\sqrt{\frac{\alpha_j \beta_j \cos(2\omega_j)}{\gamma_j^\star \eta_j^\star}}\right)(1 - \theta_j^1 L - \theta_j^2 L^2)\xi_{jt-1}$	$(1 + (\alpha_R + \beta_R)L + \alpha_R\beta_R L^2)s_t^2 = \xi_{Rt-1}$
Random Walk (RW) $(\alpha=0,\ \beta=1)$	$(1 - L)T_t = \xi_{0t-1}$	$\phi_j^{\beta}(L) s_t^{p_j} = \sqrt{1 + \sin(\omega_j)}\left(1 - \frac{\cos(\omega_j)}{1+\sin(\omega_j)} L\right)\xi_{jt-1}$	$(1 + L)s_t^2 = \xi_{Rt-1}$
Smoothed Random Walk (SRW) $(0<\alpha<1,\ \beta=1)$	$(1 - (1 + \alpha_0)L + \alpha_0 L^2)T_t = \xi_{0t-1}$	$(\phi_j^{\alpha}(L) * \phi_j^{\beta}(L)) s^{p_j}{}_t = \left(\sqrt{\frac{\alpha_j \cos(2\omega_j)}{\gamma_j^\star \eta_j^\star}}\right)(1 - \theta_j^1 L - \theta_j^2 L^2)\xi_{jt-1}$	$(1 + (1 + \alpha_R)L + \alpha_R L^2)s_t^2 = \xi_{Rt-1}$
Integrated Random Walk (IRW) $(\alpha=\beta=1)$	$(1 - 2L + L^2)T_t = \xi_{0t-1}$	$(\phi_j^{\alpha}(L) * \phi_j^{\beta}(L)) s^{p_j}{}_t = \left(\sqrt{\frac{\cos(2\omega_j)}{\gamma_j^\star \eta_j^\star}}\right)(1 - \theta_j^1 L - \theta_j^2 L^2)\xi_{jt-1}$	$(1 + 2L + L^2)s_t^2 = \xi_{Rt-1}$

$\phi_j^{\alpha}(L) = [1 - \alpha e^{i\omega_j} L] * [1 - \alpha e^{-i\omega_j} L] = [1 - 2\alpha\cos(\omega_j)L + \alpha^2 L^2]$;
$\phi_j^{\beta}(L) = [1 - \beta e^{i\omega_j} L] * [1 - \beta e^{-i\omega_j} L] = [1 - 2\beta\cos(\omega_j)L + \beta^2 L^2]$;
$\gamma_j^\star$, y $\eta_j^\star$ are given in Equation (32), and $\theta_j^1 = \gamma_j^\star + \eta_j^\star$; $\theta_j^2 = -\gamma_j^\star \eta_j^\star$.

Table 2: ARMA specification for the DHR components.

Therefore, the *pseudo-spectrum* of the DHR model is given by

$$f_{dhr}\left(\omega, \boldsymbol{\sigma}^2\right) = \sum_{j=1}^{R} \sigma_j^2 \frac{\theta_j(e^{-i\omega})\theta_j(e^{i\omega})}{\varphi_j(e^{-i\omega})\varphi_j(e^{i\omega})} + \sigma_e^2; \tag{14}$$

and $S_j(\omega) = \frac{\theta_j(e^{-i\omega})\theta_j(e^{i\omega})}{\varphi_j(e^{-i\omega})\varphi_j(e^{i\omega})}$.

Young et al. (1999) suggest the use of an AR spectrum as the estimation for $f_y(\omega)$. Then, if $B_y(L)$ denotes the AR polynomial fitted to the observed time series, $f_y(\omega)$ can be substituted by

$$\frac{\widehat{\sigma}^2}{B_y(e^{-i\omega})B_y(e^{i\omega})},$$

where $\widehat{\sigma}^2$ is the residual variance of the AR model. Hence, minimizing (13) is equivalent to

$$\min_{[\boldsymbol{\sigma}^2]\in\mathbb{R}^{R+1}} \left\| \frac{\widehat{\sigma}^2}{B_y(e^{-i\omega})B_y(e^{i\omega})} - \left[\sum_{j=0}^{R} \sigma_j^2 \frac{\theta_j(e^{-i\omega})\theta_j(e^{i\omega})}{\varphi_j(e^{-i\omega})\varphi_j(e^{i\omega})} + \sigma_e^2 \right] \right\|. \tag{15}$$

In order to align the spectral peaks of the DHR components with those of the estimated AR spectrum $f_y(\omega)$, the components can be chosen so that the full DHR model has all the unit roots of $B_y(L)$. Then, we can split each polynomial $\varphi_j(z)$ in $\varphi_j(z) = \phi_j(z) * \Phi_j(z)$, where $\Phi_j(z)$ has the unit roots and $\phi_j(z)$ has the remaining roots. Multiplying (15) by

$$\Psi(\omega) = \prod\nolimits_{h=0}^{R} \Phi_h(e^{-i\omega})\Phi_h(e^{i\omega}),$$

we have

$$\min_{[\boldsymbol{\sigma}^2]\in\mathbb{R}^{R+1}} \left\| \frac{\widehat{\sigma}^2\Psi(\omega)}{B_y(e^{-i\omega})B_y(e^{i\omega})} - \sum_{j=0}^{R} \sigma_j^2 \frac{\theta_j(e^{-i\omega})\theta_j(e^{i\omega}) \prod_{j\neq h} \Phi_h(e^{-i\omega})\Phi_h(e^{i\omega})}{\phi_j(e^{-i\omega})\phi_j(e^{i\omega})} - \sigma_e^2\Psi(\omega) \right\| \tag{16}$$

(cf. Bell, 1984, equations 1.4, 1.5 y 1.6).

Hence, the new proposed algorithm minimizes

$$\min_{\boldsymbol{\sigma}^2\in\mathbb{R}^{R+2}} \left\| \Psi(\omega) \cdot \left[f_y(\omega) - f_{dhr}\left(\omega, \boldsymbol{\sigma}^2\right) \right] \right\|. \tag{17}$$

This objective function is linear and can be evaluated in the whole range $[-\pi, \pi]$ because $(\Psi(\omega) \cdot f_y(\omega))$ and $(\Psi(\omega) \cdot f_{dhr}(\omega, \boldsymbol{\sigma}^2))$ do not have poles. Moreover, Equation (17) can be minimized by Ordinary Least Squares (OLS) to obtain the estimation of $\boldsymbol{\sigma}^2 = [\boldsymbol{\sigma}^2_{dhr}, \sigma^2_e]'$, so simplifying the estimation algorithm.

3.2 Improving the spectral fitting

If the order p of $B_y(L)$ is large enough, $B_y(L)$ has *additional roots* that are not included in the DHR model. These additional roots produce *additional spectral peaks* in the AR spectrum, $f_y(\omega)$, but these peaks are not associated with any spectral peak of the *pseudo-spectrum* of the DHR model, $f_{dhr}(\omega, \boldsymbol{\sigma}^2)$.

Because the *pseudo-spectra* are semidefinite positive functions, they are *non-orthogonal functions*. Therefore the additional spectral peaks affect the spectral fitting of the DHR components. The magnitude of this influence depends on the modulus of each additional root and on the location of the additional spectral peak. For example, when Young et al. (1999) add a medium-term into the DHR model and use an AR(54) spectrum they find that: *"The main problem with this high-order AR(54) spectrum is that ... it injects obviously spurious peaks and distortions ... making estimation of the NVR parameters more difficult ... "*. In order to overcome the problem, Young et al. (1999) concatenate a low-order spectrum with a high-order spectrum, *"using the higher-order AR spectrum to define the lower-frequency cyclical band of the spectrum, and the lower-order spectrum to specify the higher-frequency seasonal behavior"*.

Here we propose a different approach. In order to avoid the effect of the additional peaks in the spectral fitting of the DHR model, we fit these spurious peaks with additional components. By fitting the spurious peaks we isolate the spectral fitting of the DHR model from the *distortions* due to the spurious peaks. Therefore, a two stage procedure is proposed.

3.2.1 First stage

In the first stage, the vector of variances $\boldsymbol{\sigma}^2_{dhr}$ is estimated using additional components. For each *additional peak* an additional component is included (the models for this additional components are explained in the next section).

Let $f_{ac}(\omega, \boldsymbol{\sigma}^2_{ac})$ be the *pseudo-spectrum* of the sum of the additional components:

$$f_{ac}(\omega, \boldsymbol{\sigma}^2_{ac}) = \sum_{h=R+1}^{k} \sigma_h^2 S_h(\omega), \tag{18}$$

$$\boldsymbol{\sigma}^2_{ac} = \left[\sigma^2_{R+1}, \sigma^2_{R+2}, \ldots, \sigma^2_k\right], \tag{19}$$

where $\sigma_h^2 S_h(\omega)$ is the *pseudo-spectrum* of the hth *additional component*; $\boldsymbol{\sigma}^2_{ac}$ is the vector of the variances of the innovations of the *additional components*; and $k+1$ is the number of spectral peaks of $f_y(\omega)$.

In the first stage,

$$\min_{[\boldsymbol{\sigma}^2_{dhr}, \boldsymbol{\sigma}^2_{ac}] \in \mathbb{R}^{k+1}} \left\| \Psi(\omega) \cdot \left[f_y(\omega) - \sum_{j=0}^{R} \sigma_j^2 S_j(\omega) - f_{ac}(\omega, \boldsymbol{\sigma}^2_{ac}) \right] \right\| \tag{20}$$

is minimized by OLS, and the estimated variances of the innovations of the DHR components $\widehat{\boldsymbol{\sigma}}^2_{dhr}$ are obtained.

3.2.2 Second stage

In the second stage, the variance of the irregular component σ_e^2 is estimated by minimizing

$$\min_{\sigma_e^2 \in \mathbb{R}} \left\| \Psi(\omega) \cdot \left[f_y(\omega) - \sum_{j=0}^{R} \widehat{\sigma}_j^2 S_j(\omega) - \sigma_e^2 \right] \right\| \tag{21}$$

by OLS, using the estimated values $\widehat{\boldsymbol{\sigma}}^2_{dhr}$ from the first stage. Finally, we compute $\widehat{\boldsymbol{\sigma}}^2 = [\widehat{\boldsymbol{\sigma}}^2_{dhr}, \widehat{\sigma}_e^2]'$, and $\mathbf{NVR}' = \widehat{\boldsymbol{\sigma}}^2_{dhr}/\widehat{\sigma}_e^2$. Note that the two stage algorithm described above is linear and does no require skipping any region around the poles.

4 Identification algorithm

With the new algorithm described above the variances and the NVR hyperparameters are estimated by unrestricted OLS. Then, if the identification of the DHR model is incorrect, the new estimation algorithm might provide negative values for the estimated variances! For this reason we need a good DHR model specification. A good specification should provide a DHR model with an spectrum of similar shape as the shape of the spectrum of the observed time series[9]. In this section we propose a simultaneous identification and estimation algorithm.

4.1 Selecting the DHR components from $B_y(L)$

Our identification procedure consists of two steps: firstly, we identify the AR roots of $B_y(L)$ associated with the frequencies of the components to

[9]Although it is possible to obtain the structural model whose spectrum equals the AR-spectrum expanding $1/B_y(L)$ with partial fractions, in most cases these partial fractions do no belong to the family of ARMA models of Table 2, so this would imply to move away from the DHR framework.

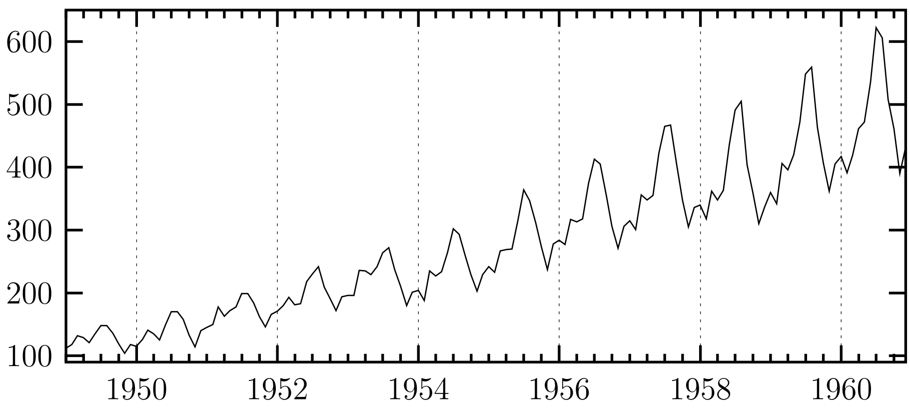

Figure 1: Airline Passenger (AP) series.

be estimated with the DHR model (usually the trend, and the seasonal); and secondly, for each frequency, we choose the DHR model whose α and β parameters are equal to the modulus of the AR roots of $B_y(L)$ associated to that frequency. We will illustrate this procedure using the famous Airline Passenger (AP) series from Box & Jenkins (1970).

4.1.1 First step

This monthly series shows a clear trend and seasonal patterns. For this reason, the "*a priori*" DHR model should have DHR components associated to the frequencies $\omega_j = 0,\ 2\pi/12,\ 2\pi/6,\ 2\pi/4,\ 2\pi/3,\ 2\pi/2.4,\ 2\pi/2$, so the model should explain the oscillation of the time series around $P_j = \infty,\ 12,\ 6,\ 4,\ 3,\ 2.4,\ 2$ periodicities.

An AR(16) model is fitted to the AP series[10]. The roots of the AR polynomial $B_y(L)$ fitted to the series appear in Table 3. Some of them are close to the P_j periodicities (e.g., 2.39, 5.97, 4.02,…, ∞). These are the AR roots associated with the DHR components.

In order to decide whether or not an AR root is associated with the jth DHR component of periodicity P_j we use a simple criterion. We fix a range of frequencies $\pm\epsilon$ radians around each $\omega_j = 2\pi/P_j$. If the frequency ω associated with the AR root lies inside any range, i.e., if $|\omega_j - \omega| \leq \epsilon$, then the AR root is associated with the jth DHR component. The default (heuristic) value ϵ, used in our program for the seasonal components is $2\pi/125 = 0.05$

[10]The procedure of how to choose this AR(16) order it is explained in the next Subsection.

Roots		Period	$\min_j \lvert\omega_j - \omega\rvert$	Norm	DHR Component model
-0.77	$\pm 0.12i$	2.101	0.150	0.78	—
-0.85	$\pm 0.49i$	2.397	**0.003**	0.98	RW
-0.50	$\pm 0.87i$	3.008	**0.006**	1.01	RW
0.01	$\pm 1.00i$	4.025	**0.010**	1.00	RW
0.10	$\pm 0.24i$	5.349	0.127	0.26	—
0.50	$\pm 0.87i$	5.974	**0.004**	1.01	RW
0.88	$\pm 0.50i$	12.038	**0.002**	1.01	RW
1.01		∞	**0**	1.01	SRW ($\alpha = 0.86$)
0.86		∞	**0**	0.86	

Table 3: Roots of the AR(16) polynomial $B_y(L)$ fitted to the AP series.

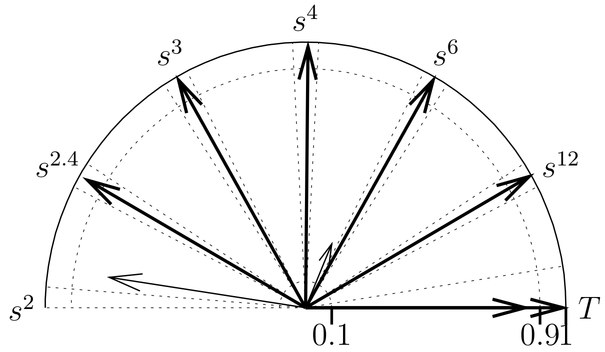

Figure 2: AR-roots.

radians, and for the trend $\epsilon = 2\pi/36 = 0.17$ radians. The range for the trend component is wider in order to incorporate the roots associated with cyclical periods[11] in the trend. This allow us to estimate trend-cyclical components.

The cases where the condition is fulfilled appear in bold in the third column of Table 3; and correspond to the roots that lie inside the regions around each ω_j in Figure 2. In this example, there are no AR roots associated to the Nyquist component (s^2), there are two AR roots associated with the trend (T), and there is one pair of conjugated AR roots associated with each one of the remaining DHR components. There are also two pairs of conjugated AR roots that are not associated with any DHR component. Therefore, the DHR model for the AP series includes the T, s^{12}, s^6, s^4, s^3 and $s^{2.4}$ components as Young et al. (1999) suggest.

[11]Longer than three years for monthly data.

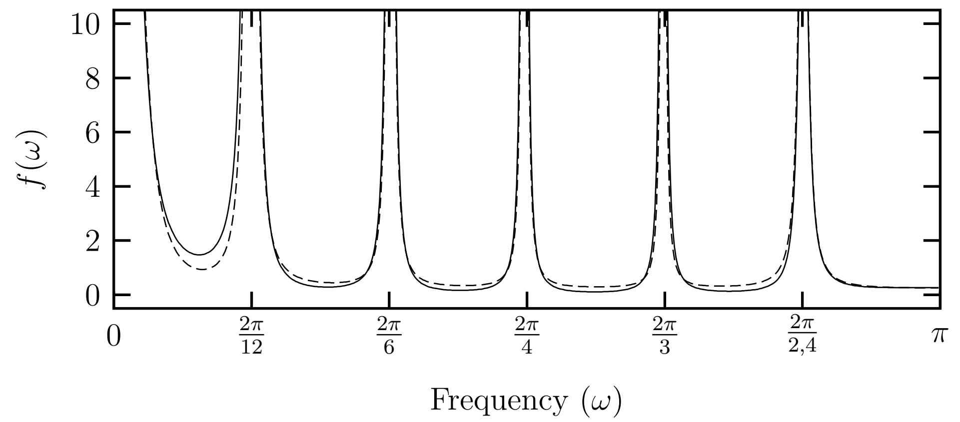

Figure 3: Spectral fitting of the DHR model (dotted) to the AR(16)-spectrum of the AP series (solid).

4.1.2 Second step

The most powerful spectral peaks of $f_y(\omega)$ are due to the AR roots whose modulus are close to one. If we use DHR models with the same AR roots, the *pseudo-spectrum* of the DHR model should have a similar shape that the AR-spectrum. Therefore, given the DHR components of the model (step one), the GRW processes for each component are chosen so that their α_j and β_j parameters are equal to the inverse of the modulus[12] of the AR roots of $B_y(L)$. For the spurious peaks we use as *additional models* the corresponding partial fractions from the expansions of $1/B_y(L)$. The spectral fitting achieved with this procedure is shown in Figure 3.

Young et al. (1999) suggest IRW models for the trend and the seasonal components of the AP series. With the new identification criterion we identify a SRW ($\alpha = 0,86$) model for the trend, and RW models for the seasonal components (see Table 3).

4.2 Selecting the order of the AR-spectrum.

The identification procedure of the DHR model depends on the estimated AR polynomial $B_y(L)$. Young et al. (1999) use the Akaike's Information Criterion to identify the order p of $B_y(L)$. Since our estimation procedure is very fast, it is possible to use a wide range of orders p and identify and estimate one DHR

[12]When these modulus are close to one, an α_j and/or β_j parameters equal to one can be imposed.

	BGF			Captain		
DHR model	Trend: SRW ($\alpha = 0.86$) Seasonals: RW			Trend: IRW Seasonals: IRW		
	AR(16)			AR(14)		
Mega-Flops	0.1033			0.3447		
$\widehat{NVR}$	T	$=$	0.0203415	T	$=$	0.0033760
	s^{12}	$=$	0.0667478	s^{12}	$=$	0.0000275
	s^{6}	$=$	0.0212145	s^{6}	$=$	0.0000041
	s^{4}	$=$	0.0086650	s^{4}	$=$	0.0000036
	s^{3}	$=$	0.0058846	s^{3}	$=$	0.0000027
	$s^{2.4}$	$=$	0.0487536	$s^{2.4}$	$=$	0.0000017
$\widehat{\boldsymbol{\sigma}^2}$	$\widehat{\sigma}^2_e$	$=$	26.03930	$\widehat{\sigma}^2_e$	$=$	106.80761
	$\widehat{\sigma}^2_T$	$=$	0.52968	$\widehat{\sigma}^2_T$	$=$	0.36058
	$\widehat{\sigma}^2_{s^{12}}$	$=$	1.73807	$\widehat{\sigma}^2_{s^{12}}$	$=$	0.00294
	$\widehat{\sigma}^2_{s^{6}}$	$=$	0.55241	$\widehat{\sigma}^2_{s^{6}}$	$=$	0.00044
	$\widehat{\sigma}^2_{s^{4}}$	$=$	0.22563	$\widehat{\sigma}^2_{s^{4}}$	$=$	0.00039
	$\widehat{\sigma}^2_{s^{3}}$	$=$	0.15323	$\widehat{\sigma}^2_{s^{3}}$	$=$	0.00029
	$\widehat{\sigma}^2_{s^{2.4}}$	$=$	1.26951	$\widehat{\sigma}^2_{s^{2.4}}$	$=$	0.00018

Table 4: Estimation results for the AP series with the Captain and the BGF algorithms.

model for each AR polynomial $B_y(L)$. Among the alternative DHR models it is possible to select one of them under certain criteria. A criterion that provides good results with minimum numerical cost is to choose the DHR model whose residual spectrum, i.e., the transformed difference between the AR-spectrum and the sum of *pseudo-spectra* of the DHR components

$$\Psi(\omega) \cdot f_y(\omega) - \Psi(\omega) \cdot \sum\nolimits_{j=0}^{R} \widehat{\sigma}_j^2 S_j(\omega),$$

has the shape closest to the shape of a transformed white noise spectrum $\Psi(\omega)$. The results for the AP series example have been obtained following the last criterion. The results obtained with the Captain and the BGF algorithms are shown in Table 4. Note the differences in the identification process. In order to compare the number of millions of floating point operations (Mega-Flops), we have counted only the estimation of the parameters, given the order of the AR polynomial $B_y(L)$. The estimated DHR components are shown in Figure 4.

As as second empirical example we have used the Spanish Industrial Production Index (IPI) series[13]. The BGF algorithm selects an AR(24) polynomial for the log of the series. The roots associated with DHR components are

[13] The Spanish IPI data, from January 1975 to March 2001 period, have been obtained from the Instituto Nacional de Estadística.

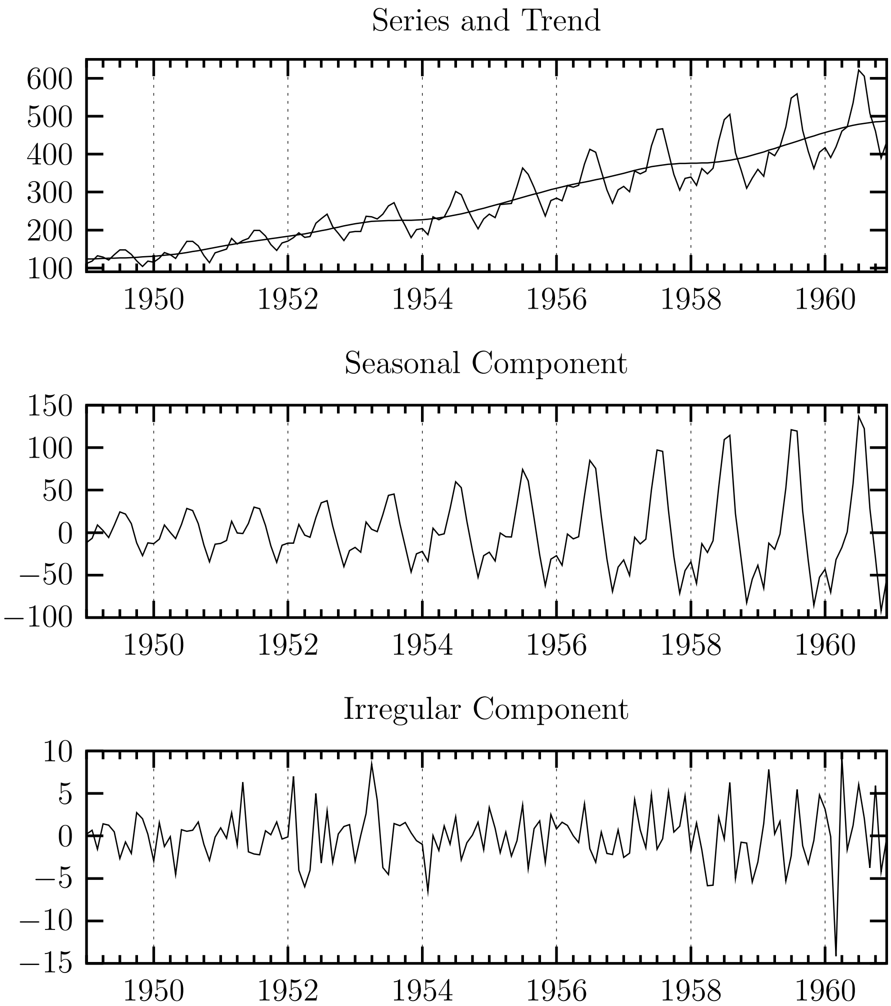

Figure 4: The estimated unobserved components for the AP series with the BGF algorithm.

Roots		Period	$\min_j \lvert\omega_j - \omega\rvert$	Norm	DHR Component model
-1.00		2.00	**0**	1.00	RW
-0.86	$\pm 0.50i$	2.40	**0.0004**	1.00	RW
-0.50	$\pm 0.86i$	3.00	**0.0008**	1.00	RW
	$\pm 1.00i$	4.00	**0.0003**	1.00	RW
0.50	$\pm 0.86i$	6.00	**0.0006**	1.00	RW
0.86	$\pm 0.50i$	12.03	**0.0013**	1.00	RW
0.95	$\pm 0.14i$	42.85	**0.1466**	0.96	IRW
1.00		∞	**0**	1.00	

Table 5: Some roots of the AR(24) polynomial $B_y(L)$ fitted to the log of the Spanish IPI series associated to the DHR components.

shown in Table 5. Note that there are one pair of complex roots associated with cycles of period 42.8 (longer than three years), and therefore, this pair is associated to the trend (or trend-cycle) component suggesting an IRW processes. Consequently, the identification process suggest a DHR model with an IRW trend and RW seasonal components[14]. *We should remark that the only input information used by the BGF algorithm in both examples is the raw time series data and the periodicity of the time series, i.e., monthly, quarterly, etc.*

In order to compare the results we have estimated the same DHR models with both the Captain and the BGF algorithms. The results are shown in Table 6. Note that the main observed difference is in the estimated variance of the irregular component $\widehat{\sigma}_e^2$. Some preliminary Montecarlo experiments have shown that the variance of the residual white noise from an AR model overestimates σ_e^2. Therefore, the estimated Noise Variance Ratio (NVR)s with the BGF algorithm tend to be bigger than the estimated values with Captain. The estimated DHR components with BGF algorithm are shown in Figure 5.

5 Conclusions

Among the available stochastic Unobserved Components alternatives, the Dynamic Harmonic Regression (DHR) model has been used extensively over the past years in different areas of research such as business cycle analysis, environmental issues, industrial turning points predictions, forecasting economic sectorial demand, etc. Additionally, the DHR model is a powerful

[14]This is exactly the same specification found in García-Ferrer & Bujosa-Brun (2000) for this variable using a similar data set.

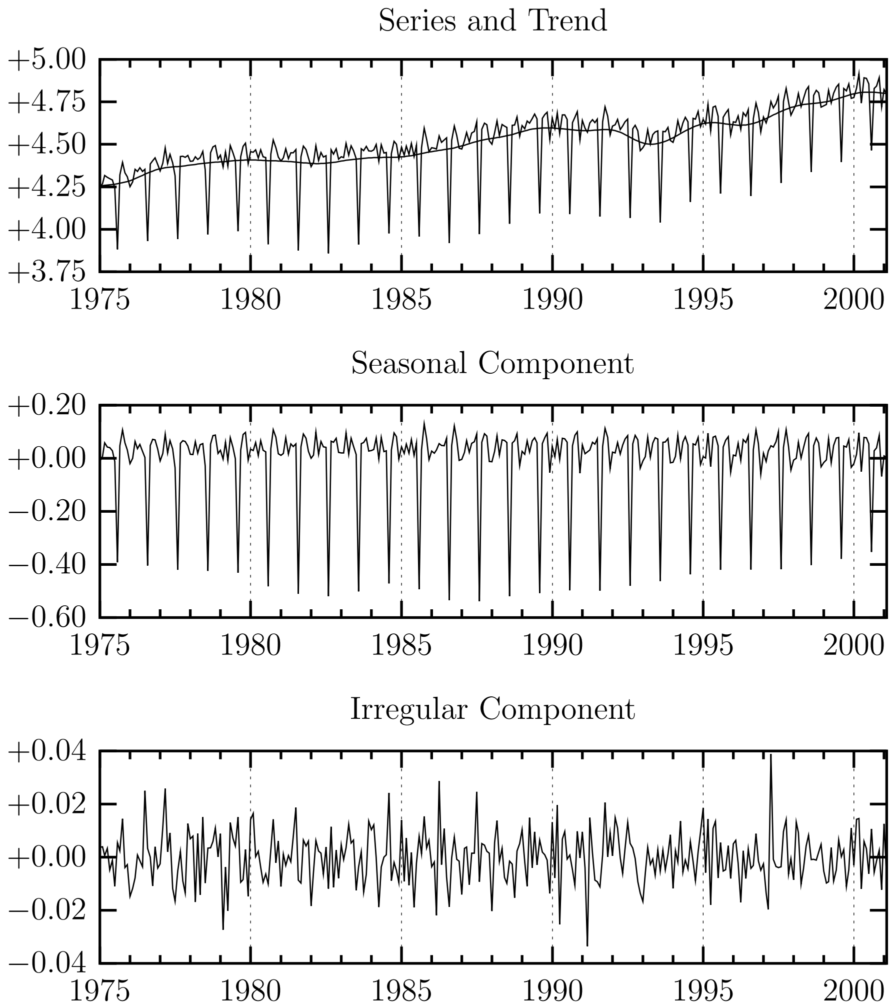

Figure 5: The estimated unobserved components for the log of the Spanish IPI series, 1975.1–2001.3.

	BGF			Captain		
DHR model	Trend: IRW Seasonals: RW			Trend: IRW Seasonals: RW		
	AR(24)			AR(32)		
Mega-Flops	1.5972			1.8119		
$\widehat{NVR}$	T	$=$	0.0087524	T	$=$	0.0023893
	s^{12}	$=$	0.0333327	s^{12}	$=$	0.0053092
	s^{6}	$=$	0.0108833	s^{6}	$=$	0.0058329
	s^{4}	$=$	0.0276207	s^{4}	$=$	0.0072667
	s^{3}	$=$	0.0744102	s^{3}	$=$	0.0239450
	$s^{2.4}$	$=$	0.0238528	$s^{2.4}$	$=$	0.0151660
	s^{2}	$=$	0.0174329	s^{2}	$=$	0.0046043
$\widehat{\boldsymbol{\sigma}^2}$	$\widehat{\sigma}^2_e$	$=$	3.4411e-04	$\widehat{\sigma}^2_e$	$=$	1.0936e-03
	$\widehat{\sigma}^2_T$	$=$	3.0118e-06	$\widehat{\sigma}^2_T$	$=$	2.6130e-06
	$\widehat{\sigma}^2_{s^{12}}$	$=$	1.1470e-05	$\widehat{\sigma}^2_{s^{12}}$	$=$	5.8062e-06
	$\widehat{\sigma}^2_{s^{6}}$	$=$	3.7451e-06	$\widehat{\sigma}^2_{s^{6}}$	$=$	6.3789e-06
	$\widehat{\sigma}^2_{s^{4}}$	$=$	9.5047e-06	$\widehat{\sigma}^2_{s^{4}}$	$=$	7.9469e-06
	$\widehat{\sigma}^2_{s^{3}}$	$=$	2.5606e-05	$\widehat{\sigma}^2_{s^{3}}$	$=$	2.6186e-05
	$\widehat{\sigma}^2_{s^{2.4}}$	$=$	8.2080e-06	$\widehat{\sigma}^2_{s^{2.4}}$	$=$	1.6586e-05
	$\widehat{\sigma}^2_{s^{2}}$	$=$	5.9989e-06	$\widehat{\sigma}^2_{s^{2}}$	$=$	5.0353e-06

Table 6: Estimation results for log of the Spanish IPI series with the Captain and the BGF algorithms.

signal extraction alternative that can compete well with the best known techniques. The oscillations of each DHR component are modulated by stochastic time varying parameters within the family of Generalized Random Walk (GRW) models suggested by Young many years ago. Interestingly, by restricting certain values in the matrices of the state space representation, the GRW model comprises a large number of characterizations found in the signal extraction literature.

In the first part of this paper we have shown that each DHR component has an AutoRegressive Moving Average (ARMA) representation. In particular, we have shown that for each cyclical and seasonal component there is a sequence such as its Extended Fourier Transform is the *pseudo-spectrum* of the component. We have also shown the existence of an ARMA model whose *pseudo-covariance generating function* is, precisely, the aforementioned sequence. The consequence of the previous results is that we can write the DHR model as a sum of certain ARMA models plus a white noise process.

In the second part of the paper we propose an alternative algorithm to estimate the model hyper-parameters that makes uses of a linear algebraic transformation in order to eliminate the poles in the original objective function. Once we remove this problem, Ordinary Least Squares can be used.

The algorithm provides simultaneous identification (GRW model for trend and seasonal components) and estimation of the hyper-parameters . It is worth nothing that the only input information required by the BGF algorithm is the raw time series data and information about its periodicity, i.e., monthly, quarterly, etc. This is a real advantage over existing alternatives, that requires additional input information from the researcher's side.

Two final comments regarding future developments. First, we have not tried yet to analyze the forecasting performance of the new algorithm. So far, given the similarities with other DHR models used in the past, we should not expect large differences in forecasting. Only when trend models differ considerably should we expect the prediction results to be different. Second, our results can be easily extended to some other well known alternatives mentioned earlier as far as they can be treated as special cases of Generalized Random Walk specifications. These, should be logical lines of future research.

Appendix

A Inverse $(b)^{-1\blacktriangleright}$

Because we deal with non-stationary models it is necessary to use an inverse of the sequences that provides a well defined *pseudo-covariance generating function*, $\Lambda_T(z)$. Should we define the *cograde* of a non-null sequence b as the biggest integer index that verify $j < cograde(b) \Rightarrow b_j = 0$, we can define the inverse sequence of a non-null sequence b with $cograde(b) = k$ as

$$(b_j)^{-1\blacktriangleright} \equiv \left(\frac{1}{b}\blacktriangleright\right)_j \equiv \begin{cases} 0 & \text{if} \quad j < -k \\ \frac{1}{b_k} & \text{if} \quad j = -k \\ \frac{-1}{b_k}\sum_{r=-k}^{j-1} a_r b_{j+k-r} & \text{if} \quad j > -k \end{cases}$$

(for more details see Bujosa et al., 2001).

B Propositions

Proposition B.1. *For each $0 < \omega_j < \pi$, there is a sequence $\Lambda_{s^{p_j}}(L) \in \mathbb{C}(z)$ whose extended Fourier transform is the* pseudo-spectrum $f_{s^{p_j}}(\omega)$ *of*

Equation (12),

$$\Lambda_{s^{p_j}}(z) = \sigma_j^2 \frac{\left\{1+2\alpha_j\beta_j+\alpha_j^2+\beta_j^2+\alpha_j^2\beta_j^2\right\}-\left\{\alpha_j+\beta_j+\alpha_j\beta_j^2+\alpha_j^2\beta_j\cos(\omega_j)\right\}\left(z+z^{-1}\right)+\left\{\alpha_j\beta_j\cos(2\omega_j)\right\}\left(z^2+z^{-2}\right)}{\varphi_j(z) * \varphi_j(z^{-1})} \blacktriangleright, \tag{22}$$

where

$$\varphi_j(z)=\left[1-\{2(\alpha_j+\beta_j)\cos\omega_j\}z+\{\alpha_j^2+\beta_j^2+4\alpha_j\beta_j\cos^2(\omega_j)\}z^2-\{2(\alpha_j\beta_j^2+\alpha_j^2\beta_j)\cos(\omega_j)\}z^3+\{\alpha_j^2\beta_j^2\}z^4\right]. \tag{23}$$

Proof. We proceed backwards. Substituting $2\cos x$ by $e^{ix} + e^{-ix}$ in (12), factorizing, and then substituting e^{-ix} by z, we obtain the sequence $\Lambda_{s^{p_j}}(L)$

$$\Lambda_{s^{p_j}}(z) = \sigma_j^2/2 \cdot \frac{\left[1-\alpha_j e^{\overline{i\omega_j}}\frac{1}{z}\right]\left[1-\beta_j e^{\overline{i\omega_j}}\frac{1}{z}\right]\left[1-\alpha_j e^{i\omega_j}z\right]\left[1-\beta_j e^{i\omega_j}z\right]+\left[1-\alpha_j e^{i\omega_j}\frac{1}{z}\right]\left[1-\beta_j e^{i\omega_j}\frac{1}{z}\right]\left[1-\alpha_j e^{\overline{i\omega_j}}z\right]\left[1-\beta_j e^{\overline{i\omega_j}}z\right]}{\left[1-\alpha_j e^{i\omega_j}\frac{1}{z}\right]\left[1-\beta_j e^{i\omega_j}\frac{1}{z}\right]\left[1-\alpha_j e^{\overline{i\omega_j}}z\right]\left[1-\beta_j e^{\overline{i\omega_j}}z\right]\left[1-\alpha_j e^{\overline{i\omega_j}}\frac{1}{z}\right]\left[1-\beta_j e^{\overline{i\omega_j}}\frac{1}{z}\right]\left[1-\alpha_j e^{i\omega_j}z\right]\left[1-\beta_j e^{i\omega_j}z\right]} \blacktriangleright. \tag{24}$$

Operating and substituting $e^{ix} + e^{-ix}$ by $2\cos x$, we finally obtain Equation (22). □

Proposition B.2. *For each $0 < \omega_j < \pi$, there is an ARMA model whose* pseudo-covariance generating function *is the sequence $\Lambda_{s^{p_j}}(L) \in \mathbb{C}(z)$ from Equation (22) of Proposition B.1.*

Proof. The proof for the **AR part** is straight forward from Equation (23) and is simply

$$\varphi_j(L) = \phi_j^{\alpha}(L) * \phi_j^{\beta}(L), \tag{25}$$

where

$$\begin{array}{ll} \phi_j^{\alpha}(L) = [1 - 2\alpha_j\cos(\omega_j)L + \alpha_j^2 L^2] & = [1 - \alpha_j e^{i\omega_j}L][1 - \alpha_j e^{-i\omega_j}L] \\ \phi_j^{\beta}(L) = [1 - 2\beta_j\cos(\omega_j)L + \beta_j^2 L^2] & = [1 - \beta_j e^{i\omega_j}L][1 - \beta_j e^{-i\omega_j}L] \end{array}.$$

The proof for the **moving average part** is much more tedious. We search the Moving Average (MA) polynomial $\theta_j(L)$ such that $\theta_j(z)\theta_j(z^{-1})$ equals the numerator in (24). Substituting z by L, $\frac{1}{z}$ by F, and operating on the numerator in (24) we can obtain the expresion

$$\begin{array}{l} (1 - \alpha_j^{-1}e^{i\omega_j}L)(1 - \beta_j^{-1}e^{i\omega_j}L)(1 - \alpha_j e^{i\omega_j}L)(1 - \beta_j e^{i\omega_j}L)(\alpha_j\beta_j e^{\overline{2i\omega_j}})F^2 + \\ (1 - \alpha_j^{-1}e^{\overline{i\omega_j}}L)(1 - \beta_j^{-1}e^{\overline{i\omega_j}}L)(1 - \alpha_j e^{\overline{i\omega_j}}L)(1 - \beta_j e^{\overline{i\omega_j}}L)(\alpha_j\beta_j e^{2i\omega_j})F^2 \end{array}. \tag{26}$$

It is not difficult to prove that if x is a root of (26) then, $1/x$ is also a root. It follows that $\theta_j(z)\theta_j(z^{-1})$ can be divided by

$$(1 - \gamma L)(1 - \gamma^{-1}L)(1 - \eta L)(1 - \eta^{-1}L).$$

Some caracteristics of γ and η are known. Because the *pseudo-spectra* of the DHR models are positive definite none of the MA roots has unit modulus; and because $\theta_j(z)$ is real, if γ is not real and $|\gamma| \neq 1$, then $\eta = \overline{\gamma}$. So, two scenarios are possible. In the first one, there are two real roots with modulus greater than one and their inverses, in the second one, there are four complex roots, and for each one of them there are its inverse, its complex pair, and the inverse of its complex pair.

We need to find the constant λ and the coeficients γ and η that verify that $\theta_j(L)\theta_j(F)$ equals (26), and

$$\theta_j(L)\theta_j(F) = \lambda F^2\ (1-\gamma L)(1-\gamma^{-1}L)(1-\eta L)(1-\eta^{-1}L). \qquad (27)$$

Therefore, the general form of the MA should be

$$\theta_j(L) = \sqrt{\lambda}\ (1-\gamma_j^\star L)(1-\eta_j^\star L), \qquad (28)$$

where $\gamma_j^\star$ and $\eta_j^\star$ are inside the unit circle; and λ is a constant.

On the one hand; ignoring λF^2 in (27), and operating, it can be obtain the polynomial

$$(L^2-(\gamma+\gamma^{-1})L+1)(L^2-(\eta+\eta^{-1})L+1),$$

or $(L^2+\delta L+1)(L^2+\rho L+1)$, where $\delta = -(\gamma+\gamma^{-1})$ and $\rho = -(\eta+\eta^{-1})$. This polynomial is equivalent to:

$$L^4+(\delta+\rho)L^3+(\delta\rho+2)L^2+(\delta+\rho)L+1,$$

where δ and ρ verify

$$\gamma^2+\delta\gamma+1=0; \quad \eta^2+\rho\eta+1=0. \qquad (29)$$

If the fourth order polynomial $aL^4+bL^3+cL^2+dL+e$ is divided by $L^4+(\delta+\rho)L^3+(\delta\rho+2)L^2+(\delta+\rho)L+1$ we obtained:

$$\begin{array}{ccccc|l} aL^4+ & bL^3+ & cL^2+ & dL+ & e & L^4+\ (\delta+\rho)L^3+\ (\delta\rho+2)L^2+\ (\delta+\rho)L+\ 1 \\ \hline aL^4+ & a(\delta+\rho)L^3+ & a(\delta\rho+2)L^2+ & a(\delta+\rho)L+ & a & a \\ & r3L^3+ & r2L^2+ & r1L+ & r0 & \end{array},$$

where $r_3 = b-a(\delta+\rho)$; $r_2 = c-a(\delta\rho+2)$; $r_1 = d-a(\delta+\rho)$; $r_0 = e-a$.

Since a necessary condition for the remainer to be zero is

$$\begin{array}{rl} 0 = & r_3 = b-a(\delta+\rho) \\ 0 = & r_2 = c-a(\delta\rho+2) \end{array},$$

δ and ρ should verify

$$\delta = \frac{-b \pm \sqrt{b^2 + 4a(2a-c)}}{-2a}; \quad \rho = \frac{b}{a} - \delta. \tag{30}$$

On the other hand, the roots of Equation (26) are the roots of

$$e^{-2i\omega_j} - (A+B)e^{-i\omega_j}L + (2 + A \cdot B)L^2 - (A+B)e^{i\omega_j}L^3 + e^{2i\omega_j}L^4 \quad + \\ e^{2i\omega_j} - (A+B)e^{i\omega_j}L + (2 + A \cdot B)L^2 - (A+B)e^{-i\omega_j}L^3 + e^{-2i\omega_j}L^4,$$

where $A = \alpha_j + \alpha_j^{-1}$ y $B = \beta_j + \beta_j^{-1}$.

If we substitute $e^{i\omega_j} + e^{-i\omega_j}$ by Ω_j we can find that

$$\underbrace{(\Omega_j^2 - 2)}_{e} - \underbrace{(A+B)\Omega_j}_{d} L + \underbrace{(4 + 2AB)}_{c} L^2 - \underbrace{(A+B)\Omega_j}_{b} L^3 + \underbrace{(\Omega_j^2 - 2)}_{a} L^4. \tag{31}$$

Therefore, $2a - c = 2(\Omega_j^2 - 2) - 4 - 2AB$, and $-b = -(A+B)\Omega_j$, Substituting in (30) we find that

$$\delta = \frac{(A+B)\Omega_j \pm \sqrt{(A+B)^2\Omega_j^2 + 8(\Omega_j^2 - 2)(\Omega_j^2 - 4 - 2(AB))}}{-2(\Omega_j^2 - 2)}.$$

Finally, using Equation (29), we have found that:

$$\gamma = \frac{-\delta \pm \sqrt{\delta^2 - 4}}{2}; \ \eta = \frac{-\rho \pm \sqrt{\rho^2 - 4}}{2}. \tag{32}$$

So, given the values of α_j, β_j y ω_j, it is posible to calculate γ and η. The constant λ is

$$\lambda = \frac{\alpha_j \beta_j \cos(2\omega_j)}{\gamma_j^\star \eta_j^\star}, \tag{33}$$

where $\gamma_j^\star$ and $\eta_j^\star$ are the roots inside the unit circle (see Equation (28)). $\square$

Combining equations (23), (28), (32), and (33) we can write the equivalent ARMA model for the $s_t^{p_j}$ component as

$$\varphi_j(L) s^{p_j}{}_t = \left(\sqrt{\frac{\alpha_j \beta_j \cos(2\omega_j)}{\gamma_j^\star \eta_j^\star}} \right) (1 - \theta_j^1 L - \theta_j^2 L^2) \xi_{j\,t-1}, \quad \{\xi_{j\,t}\} \sim \text{w.n. } N(0, \sigma_{\xi_j}^2). \tag{34}$$

Corollary. *The* pseudo-covariance generating function *of each cyclical or seasonal component $s_t^{p_j}$ is given by*

$$\Lambda_{s^{p_j}}(z) = \left(\sigma_j^2 \frac{\alpha_j \beta_j \cos(2\omega_j)}{\gamma_j^\star \eta_j^\star} \right) \frac{[1 \quad -\theta_j^1 z \quad -\theta_j^2 z^2][1 \quad -\theta_j^1 z^{-1} \quad -\theta_j^2 z^{-2}]}{\varphi_j(z) * \varphi_j(z^{-1})} \tag{35}$$

where θ_j^1, θ_j^2, $\gamma_j^\star$, y $\eta_j^\star$ are given in Equation (28), and $\varphi_j(z)$ is provided by Equation (23).

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