SMOOTH SURJECTIONS WITHOUT SURJECTIVE RESTRICTIONS

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Abstract. Let $f : E \to F$ be a surjective mapping between two real or complex Banach spaces, with f having some strong differentiability properties. We investigate when there is a smaller Banach space $G \subset E$ such that the restriction of f to G remains surjective.

1. Introduction

Let E and F be Banach spaces, with E having larger dimension or larger density character than F. Suppose that $f : E \to F$ is a smooth mapping that is onto F. Our interest here is in investigating when there is a smaller closed subspace $G \subset E$ such that $f|_G$ remains surjective. Our discussion will be divided into three parts. We first consider the general case where E and F are real spaces with $\dim(E) = \infty$ and $\dim(F) < \infty$. Then, in Section 2, we investigate the case where E and F are again real spaces, with E being non-separable and $\dim(F) \geq 2$. Finally, in Section 3, we concentrate on finite-dimensional spaces and we stress the difference between the real and complex cases.

Recall that, given $1 \leq m \leq \infty$, a Banach space E is said to be C^m-*smooth* if there exists a function in $C^m(E)$ with bounded nonempty support. Let us begin with a general observation.

Theorem 1. *Let E be any infinite-dimensional Banach space, and $n \in \mathbb{N}$. Then there exists a C^∞ smooth map $f : E \to \mathbb{R}^n$ such that the following hold:*

1. *f is surjective,*
2. *when restricted to any finite-dimensional subspace of E, f is not surjective.*

Proof. Since E is infinite-dimensional, E admits a basic sequence $\{u_j\}$. Consider the associated coefficient functionals $\{u_j^*\}$, which are defined on the closed linear span of $\{u_j\}$. We may assume that $\|u_j\| = 1$ and $\|u_j^*\| \leq M$, for some constant $M > 0$. We can extend $\{u_j^*\}$ to functionals in E^* with the same norm.

We prove first the case $n = 1$. Consider the linear operator $T : E \to c_0$ defined by

$$T(x) = \left(\frac{u_j^*(x)}{j} \right)_{j \in \mathbb{N}}.$$

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It is clear that $\{T(j\,u_j)\} = \{e_j\}$ is the unit vector basis of c_0. For each j, consider the ball $B_j = B(e_j, \frac{1}{4^j})$ in c_0, and set $W_j = T^{-1}(B_j)$. Note that every point in E has a neighborhood intersecting at most one member of the family $\{W_j\}$. Since c_0 is C^∞-smooth (see, e.g. [3]), for each j we can find a nonzero C^∞-smooth function $\varphi_j : c_0 \to \mathbb{R}$, with support contained in the ball B_j. For each j define $h_j = \varphi_j \circ T$, which is a nonzero C^∞-smooth function on E with support contained in W_j. In particular, for each j the image $h_j(W_j)$ contains a nontrivial interval $[a_j, b_j]$, and therefore by composing with a suitable C^∞-smooth function $\theta_j : \mathbb{R} \to \mathbb{R}$ we can obtain a C^∞-smooth function $g_j = \theta_j \circ h_j : E \to \mathbb{R}$ with support contained in W_j and such that $g_j(W_j) = [-j, j]$, for every j. Then define $g : E \to \mathbb{R}$ by setting

$$g(x) = \begin{cases} g_j(x) & \text{if } x \in W_j, \\ 0 & \text{otherwise.} \end{cases}$$

Since the family $\{W_j\}$ is locally finite, g is a C^∞-smooth function, and it takes E onto $\mathbb{R}$.

Now suppose that X is a closed subspace of E such that $g(X) = \mathbb{R}$. For each j there exists $v_j \in X$ with $g(v_j) = j$, and by construction there exists some $k_j \geq j$ such that $v_j \in W_{k_j}$. Thus $T(v_j) \in B_{k_j}$ for every j. By choosing a subsequence, we may assume that $\{k_j\}$ is strictly increasing, and therefore $\{e_{k_j}\}$ is equivalent to the unit vector basis of c_0. Now we have that

$$\sum_{j=1}^{\infty} \|T(v_j) - e_{k_j}\| \leq \sum_{j=1}^{\infty} \frac{1}{4^{k_j}} < 1.$$

As a consequence, the sequence $\{T(v_j)\}$ is a basic sequence in c_0 (see e.g. Theorem 6.18 in [6]). In particular, the space X is infinite-dimensional.

Next we discuss the case $n > 1$. Consider the projection $\pi : E \to E$ given by

$$\pi(x) = u_1^*(x)u_1 + \cdots + u_{n-1}^*(x)u_{n-1}.$$

Since the subspace $E_0 = \ker(\pi)$ of E is infinite-dimensional, from the previous case we obtain a C^∞-smooth surjection $g : E_0 \to \mathbb{R}$ such that, for every finite-dimensional subspace G of E_0, the restriction $g|_G$ is not surjective. Define $f : E \to \mathbb{R}^n$ by

$$f(x) = (u_1^*(x), \ldots, u_{n-1}^*(x), g(x - \pi(x))).$$

It is clear that f is a a C^∞-smooth surjection. Suppose now that X is a closed subspace of E such that $f(X) = \mathbb{R}^n$. For each $t \in \mathbb{R}$ there is some $x_t \in X$ such that $f(x_t) = (0, \ldots, 0, t)$. Then $x_t \in E_0$ and $t = g(x_t)$. Therefore $g(X \cap E_0) = \mathbb{R}$, so that X is infinite-dimensional.

□

On the other hand, the following remark shows that any continuous, surjective function $f : E \to \mathbb{R}$ remains surjective at least on some separable subspace of E.

Remark 2. Let E be any infinite-dimensional Banach space, and suppose that $f : E \to \mathbb{R}$ is a continuous onto map. For each $k \in \mathbb{Z}$ there exists some $x_k \in E$ such that $f(x_k) = k$. Then the closed linear subspace G of E spanned by $\{x_k : k \in \mathbb{Z}\}$ is separable, and $f(G)$ is a connected subset of $\mathbb{R}$ containing $\mathbb{Z}$, so that $f(G) = \mathbb{R}$.

2. Main results

The section is devoted to seeing that there is no result analogous to Remark 2 if we consider mappings $f : E \to F$, where E is C^∞-smooth, say, and F has dimension at least 2. Note that, in particular, we can choose $E = c_0(\Gamma)$ and $F = \mathbb{R}^2$ in Theorem 4 below. For background information about the existence of smooth surjections between Banach spaces, we refer the reader to [2].

We first describe a general class of Banach spaces to which our results apply. By the *cozero set* of a real function $f : E \to \mathbb{R}$ we mean as usual the set $coz(f) = \{x \in E : f(x) \neq 0\}$. Now let κ be a cardinal number and $1 \leq m \leq \infty$. We say that a Banach space E has C^m*-cellularity* $\geq \kappa$ if there exists a locally finite family of disjoint cozero-sets of functions in $C^m(E)$, with cardinality κ. It is not difficult to see that every Banach space E has C^∞-cellularity $\geq \aleph_0$. Indeed, consider a nonzero continuous linear functional $\xi : E \to \mathbb{R}$ and, for each $n \in \mathbb{N}$, consider a C^∞-smooth function $\theta_n : \mathbb{R} \to \mathbb{R}$ such that $coz(\theta_n) = (n - \frac{1}{4}, n + \frac{1}{4})$. Then $\{coz(\theta_n \circ \xi)\}_{n\in\mathbb{N}}$ is a locally finite sequence of disjoint cozero-sets of C^∞ functions on E. Later on, we will show some examples of Banach spaces with C^m-cellularity $\geq \kappa$, where $\kappa > \aleph_0$. We give in Lemma 3 below a characterization of this property. Recall that the *density character* of a topological space is the smallest cardinal of a dense subset. Note in particular that, by Lemma 3, every C^m-smooth Banach space with density character $\geq \kappa$ has also C^m-cellularity $\geq \kappa$.

Lemma 3. *Let E be a Banach space, and consider a cardinal $\kappa > \aleph_0$. Then the following conditions are equivalent:*

- (a) *E has C^m-cellularity $\geq \kappa$.*
- (b) *There exist a set Γ with cardinality κ and a C^m-smooth mapping $\phi : E \to c_0(\Gamma)$, whose range contains the unit vector basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of $c_0(\Gamma)$.*
- (c) *There exist a C^m-smooth Banach space Z and a C^m-smooth mapping $\phi : E \to Z$, whose range has density character $\geq \kappa$.*

Proof. (a) $\Rightarrow$ (b): Let $\{W_\gamma\}_{\gamma\in\Gamma}$ be a disjoint and locally finite family of subsets of E, where each $W_\gamma = coz(\phi_\gamma)$ for some $\phi_\gamma \in C^m(E)$, and Γ has cardinality κ. Without loss of generality, we may assume that for each γ there is a point $x_\gamma \in W_\gamma$ such that $\phi_\gamma(x_\gamma) = 1$. Then the mapping $\phi : E \to c_0(\Gamma)$ given by $\phi(x) = (\phi_\gamma(x))_{\gamma\in\Gamma}$ is well-defined and satisfies the required properties.

(b)$\Rightarrow$ (c): This is clear, since the space $c_0(\Gamma)$ is C^∞-smooth.

(c)$\Rightarrow$ (a): Let Z be a C^m-smooth Banach space and $\phi : E \to Z$ a C^m-smooth mapping, whose range $\phi(E)$ has density $\geq \kappa$. For each $n \in \mathbb{N}$, by Zorn's lemma we can choose a maximal subset S_n of $\phi(E)$ with the property that $\|u - v\| \geq \frac{1}{n}$ whenever u and v are different points in S_n. It is easy to see that $S = \cup_{n\in\mathbb{N}} S_n$ is dense in $\phi(E)$. Indeed, if $w \in \phi(E)$ is not in the closure of S, then for some n the ball $B(w, \frac{1}{n})$ does not intersect S_n, and this contradicts the maximality. Since $\phi(E)$ has density $\geq \kappa$, we deduce that, for some n_0, the set S_{n_0} has cardinality $\geq \kappa$. In this way we obtain a subset $\{u_\gamma\}_{\gamma\in\Gamma} \subset \phi(E)$, where Γ has cardinality κ, such that $\|u_\gamma - u_{\gamma'}\| \geq \frac{1}{n_0}$ if $\gamma \neq \gamma'$. Since Z is C^m-smooth, for each γ we can choose a nonzero function $h_\gamma \in C^m(Z)$ whose support is contained in the ball $B(u_\gamma, \frac{1}{4n_0})$ of Z. Note that, for each $w \in Z$, the ball $B(w, \frac{1}{4n_0})$ meets at most one of the balls $\{B(u_\gamma, \frac{1}{4n_0})\}_{\gamma\in\Gamma}$. Now for each γ consider the cozero-set $W_\gamma = coz(h_\gamma \circ \phi)$. Then the family $\{W_\gamma\}_{\gamma\in\Gamma}$ satisfies the required properties.

□

Next we give our main result:

Theorem 4. *Let E be a Banach space with C^m-cellularity $\geq 2^{\aleph_0}$ and let F be a separable Banach space with dimension ≥ 2. Then there exists a C^m smooth map $f : E \to F$ such that:*

1. *f is surjective,*
2. *when restricted to any separable subspace of E, f is not surjective.*

Proof. Let $\{W_\gamma\}_{\gamma\in\Gamma}$ be a disjoint and locally finite family of subsets of E, where each $W_\gamma = coz(\phi_\gamma)$ for some $\phi_\gamma \in C^m(E)$, and Γ has cardinality $2^{\aleph_0}$. Without loss of generality we may assume that each $\phi_\gamma \geq 0$ and $\phi_\gamma(x_\gamma) = 1$ for some $x_\gamma \in W_\gamma$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with $0 \leq \theta \leq 1$, $\theta(t) = 0$ for every $t \leq \frac{1}{6}$ and $\theta(t) = 1$ for every $t \geq \frac{1}{5}$. Now if we define $g_\gamma = \theta \circ \phi_\gamma$, we obtain that $g_\gamma \in C^m(E)$, the support of g_γ is contained in W_γ, and $g_\gamma \equiv 1$ on the nonempty open subset $V_\gamma = (\phi_\gamma)^{-1}(\frac{1}{5}, \infty)$ of W_γ. For each $k \in \mathbb{Z}$, we choose a C^∞ function $\theta_k : \mathbb{R} \to \mathbb{R}$ such that $\theta_k(t) = 0$ for every $t \leq \frac{1}{4}$ and $\theta_k([\frac{1}{3}, \frac{1}{2}]) = [k - \frac{1}{2}, k + \frac{1}{2}]$, and we define $h_{\gamma,k} = \theta_k \circ \phi_\gamma$. Then $h_{\gamma,k} \in C^m(E)$ has support contained in V_γ and the image $h_{\gamma,k}(V_\gamma)$ contains the interval $[k - \frac{1}{2}, k + \frac{1}{2}]$.

Since F is separable and has dimension ≥ 2, we may assume that $F = F_0 \times \mathbb{R}$, where F_0 is a nonzero separable Banach space. Thus F_0 has cardinality $2^{\aleph_0}$. Consider a partition $\Gamma = \cup_{k\in\mathbb{Z}}\Gamma_k$, where for each $k \in \mathbb{Z}$ the set Γ_k has cardinality $2^{\aleph_0}$, and for each k choose a bijection $\sigma_k : \Gamma_k \to F_0$.

For each $\gamma \in \Gamma$, if $\gamma \in \Gamma_k$ we define $f_\gamma : E \to F_0 \times \mathbb{R}$ by setting

$$f_\gamma(x) = \begin{cases} (g_\gamma(x) \cdot \sigma_k(\gamma),\, h_{\gamma,k}(x)) & \text{if } x \in W_\gamma, \\ (0,0) & \text{otherwise.} \end{cases}$$

Then each f_γ is C^m on E, with support contained in W_γ. Note that, if $\gamma \in \Gamma_k$, then $f_\gamma|_{V_\gamma}(x) = (\sigma_k(\gamma), h_{\gamma,k}(x))$, and therefore the image $f_\gamma(V_\gamma)$ contains the set $\{\sigma_k(\gamma)\} \times [k - \frac{1}{2}, k + \frac{1}{2}]$.

Since the family $\{W_\gamma\}_{\gamma\in\Gamma}$ is disjoint, we can now define a mapping $f : E \to F_0 \times \mathbb{R}$ by setting

$$f(x) = \begin{cases} f_\gamma(x) & \text{if } x \in W_\gamma \text{ for some } \gamma \in \Gamma, \\ (0,0) & \text{otherwise.} \end{cases}$$

Since $\{W_\gamma\}$ is locally finite, the mapping f is C^m on E, and the support of f is contained in $\cup_{\gamma\in\Gamma}W_\gamma$. By our previous comments, we have that for each $k \in \mathbb{Z}$, the set $F_0 \times [k - \frac{1}{2}, k + \frac{1}{2}]$ is contained in $f(\cup_{\gamma\in\Gamma_k}V_\gamma)$. Therefore f is onto $F_0 \times \mathbb{R}$.

On the other hand, suppose that X is a subset of E such that $f|_X : X \to F_0 \times \mathbb{R}$ is onto. Then for each $v \in F_0$ there exists some $x_v \in X$ such that $f(x_v) = (v, 1)$. By the construction, since the second coordinate of $f(x_v)$ is nonzero, there exists some $\gamma \in \Gamma$ such that $x_v \in V_\gamma$. Then $\gamma \in \Gamma_k$ for some k and by the definition of f_γ we have that $v = \sigma_k(\gamma)$. Therefore

$$F_0 = \bigcup_{k\in\mathbb{Z}} \sigma_k(\{\gamma \in \Gamma_k \,:\, X \cap V_\gamma \neq \emptyset\}).$$

From a cardinality argument we obtain that there exists some $k \in \mathbb{Z}$ such that

$$\text{card}\,\{\gamma \in \Gamma_k \,:\, X \cap V_\gamma \neq \emptyset\}$$

is uncountable.

Thus X contains an uncountable family of non-empty, disjoint, open sets and so it is non-separable.

□

Remark 5. The map $f : E \to F$ constructed in Theorem 4 above satisfies in addition that $\text{rank}(Df(x)) \leq 1$, for every $x \in E$. Indeed, let $x \in E$ be given. If x does not belong to $\cup_{\gamma\in\Gamma} W_\gamma$ then f is identically 0 on a neighborhood of x, and then $\text{rank}(Df(x)) = 0$. Suppose now that $x \in W_\gamma$ for some $\gamma \in \Gamma_k$. If $x \in V_\gamma$, then the first coordinate of f is constant on a neighborhood of x, and therefore $\text{rank}(Df(x)) \leq 1$. Finally, if $x \in W_\gamma \setminus V_\gamma$ then on a neighborhood of x we have that f is of the form $(g_\gamma(x) \cdot \sigma_k(\gamma), 0)$, and since the first coordinate is contained in a one-dimensional subspace of F we deduce the result.

The following corollary is a direct consequence of Theorem 4 and Lemma 3.

Corollary 6. *Let Γ be a set with cardinality $\geq 2^{\aleph_0}$, and let E be a Banach space for which there exists a continuous linear operator $T : E \to c_0(\Gamma)$, such that the unit vector basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of $c_0(\Gamma)$ is contained in $T(E)$. Then E has C^∞-cellularity $\geq 2^{\aleph_0}$. Therefore if F is a separable Banach space with dimension ≥ 2, then there exists a C^∞ smooth map $g : E \to F$ such that:*

1. *g is surjective,*
2. *when restricted to any separable subspace of E, g is not surjective.*

We next give some consequences of Corollary 6. Note that Corollary 7 below shows that, if Γ has cardinality $\kappa > \aleph_0$, then for $1 \leq p < \infty$ the space $\ell_p(\Gamma)$ has C^∞-cellularity $\geq \kappa$. In the same way Corollary 8 shows that, for any infinite set I, the space $\ell_\infty(I)$ has C^∞-cellularity $\geq 2^{\aleph_0}$.

Corollary 7. *Let Γ be a set with cardinality $\geq 2^{\aleph_0}$, let $1 \leq p < \infty$ and let F be a separable Banach space with dimension ≥ 2. Then there exists a C^∞ smooth map $g : \ell_p(\Gamma) \to F$ such that:*

1. *g is surjective,*
2. *when restricted to any separable subspace of $\ell_p(\Gamma)$, g is not surjective.*

Proof. We only have to consider the natural inclusion $j : \ell_p(\Gamma) \hookrightarrow c_0(\Gamma)$ and apply Corollary 6 above. □

The case of spaces $\ell_\infty(I)$ follows from a result due to Rosenthal [10]:

Corollary 8. *Let I be any infinite set and let F be a separable Banach space with dimension ≥ 2. There exists a C^∞ smooth map $g : \ell_\infty(I) \to F$ such that:*

1. *g is surjective,*
2. *when restricted to any separable subspace of $\ell_\infty(I)$, g is not surjective.*

Proof. By [10], there exists a quotient map $q : \ell_\infty(I) \to \ell_2(2^I)$. Now we can apply Corollary 6 to $T = j \circ q$, where $j : \ell_p(2^I) \hookrightarrow c_0(2^I)$ is the natural inclusion. □

Note that, on the other hand, it has been proved by Hajék [5] that there is no C^2-smooth surjection from $c_0(\mathbb{N})$ onto any separable Banach space with non-trivial type. Also, assuming Martin's axiom MA_{ω_1}, Guirao, et al. have shown that there does not exist a C^2-smooth surjection from $c_0(\omega_1)$ onto ℓ_2 [4, Theorem 3.7]. It is apparently an open problem, possibly involving axiomatic set theory, as to whether Corollary 7 holds if the cardinality of the set Γ is assumed to be at least ω_1.

We provide next some comments about the hypothesis of Corollary 6, in connection with the existence of biorthogonal systems on a Banach space E. Recall that a family $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma} \subset E \times E^*$ is said to be a *biorthogonal system* if $x_\alpha^*(x_\beta) = \delta_{\alpha,\beta}$ for every α, $\beta \in \Gamma$. A biorthogonal system is said to be *fundamental* if, in addition, the closed linear span of $\{x_\gamma\}_{\gamma\in\Gamma}$ coincides with E.

Proposition 9. *Let Γ be an infinite set, and consider the following conditions for a Banach space E:*

(a) *E admits a fundamental biorthogonal system with cardinality Γ.*
(b) *There exists a continuous linear operator $T : E \to c_0(\Gamma)$, such that the unit vector basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of $c_0(\Gamma)$ is contained in $T(E)$.*
(c) *E admits a biorthogonal system with cardinality Γ.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. Suppose first that $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma}$ is a fundamental biorthogonal system in E. We can assume also that $\|x_\gamma^*\| = 1$ for every γ. Then we can define the bounded operator $T : E \to \ell_\infty(\Gamma)$ by $T(x) = (x_\gamma^*(x))_{\gamma\in\Gamma}$. Now $T(x_\gamma) = e_\gamma$ for every γ, and it is clear that T takes the linear span of $\{x_\gamma\}_{\gamma\in\Gamma}$ into $c_0(\Gamma)$. Since the system is fundamental we have that, in fact, T takes values in $c_0(\Gamma)$.

Now suppose that $T : E \to c_0(\Gamma)$ is a bounded linear operator, such that the unit vector basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of $c_0(\Gamma)$ is contained in $T(E)$. For each γ, choose $x_\gamma \in E$ with $T(x_\gamma) = e_\gamma$, and define $x_\gamma^* = e_\gamma^* \circ T$, where $\{e_\gamma^*\}_{\gamma\in\Gamma}$ are the corresponding biorthogonal functionals on $c_0(\Gamma)$. Then it is easy to see that $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma}$ is a biorthogonal system in E. ☐

We recall that, under the Continuum Hypothesis (CH), Kunen constructed an example of non-separable Banach space which admits no uncountable biorthogonal system (see Theorem 7.7 of [9]). Therefore, for this space condition (b) of Proposition 9 is not fulfilled for any uncountable Γ.

On the other hand, note that the conditions of Proposition 9 hold, and Corollary 6 applies, whenever E has density $\geq 2^{\aleph_0}$ and is WCG (weakly compactly generated) or, more generally, E has a Markushevich basis $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma}$, with $card(\Gamma) \geq 2^{\aleph_0}$. Indeed, in this case there exists a continuous, linear map $T : E \to c_0(\Gamma)$ such that $\{T(x_\gamma)\}$ is the unit vector basis of $c_0(\Gamma)$ (see [7] for details).

3. FINITE-DIMENSIONAL SPACES AND THE COMPLEX CASE

For finite-dimensional vector spaces, we may consider a related problem. Suppose that $f : E \to F$ is a smooth surjection, where $\dim(E) > \dim(F)$. Then we ask whether there always exists a proper affine subspace $G \subset E$ such that $f|_G$ remains surjective.

In the real case, we obtain a negative answer to the above question:

Theorem 10. *Suppose that $m > n$. Then there exists a surjective C^∞-smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ such that, for every proper affine subspace G of $\mathbb{R}^m$, the restriction $f|_G$ is not surjective.*

Proof. We are going to first prove the result in the case $n = 1$. In order to obtain this, consider for each $k \in \mathbb{N}$ the open ball B_k in $\mathbb{R}^m$ with center $(k, k^2, \dots, k^m)$ and radius $\frac{1}{k}$. Note that every affine hyperplane H in $\mathbb{R}^m$ meets at most a finite number of balls B_k. Indeed, suppose that H is given by the equation

$$a_0 + a_1 x_1 + \cdots a_m x_m = 0,$$

and let $j \in \{1, 2, \dots, m\}$ be the largest coordinate such that $a_j \neq 0$. If H meets B_k, there exists a point $x_k = (x_1^k, x_2^k, \dots, x_m^k) \in B_k$ such that

$$0 = a_0 + a_1 x_1^k + \cdots a_j x_j^k =$$

$$= a_0 + a_1 k^1 + a_2 k^2 + \cdots a_j k^j + a_1(x_1^k - k^1) + a_2(x_2^k - k^2) \cdots a_j(x_j^k - k^j).$$

Denote $\varepsilon_k = a_1(x_1^k - k) + a_2(x_2^k - k^2) \cdots a_j(x_j^k - k^j)$. Then we have that

$$a_j = -\frac{\varepsilon_k + a_0}{k^j} - \frac{a_1}{k^{j-1}} - \cdots - \frac{a_{j-1}}{k}.$$

If H meets an infinite number of B_k's, we obtain that the right-hand side of the above expression tends to 0 as k tends to ∞, and therefore $a_j = 0$, which is a contradiction.

Now for each k, consider a C^∞-smooth map $g_k : \mathbb{R}^m \to \mathbb{R}$ with support contained in B_k and whose range is the interval $[-k, k]$. Then define the C^∞-smooth map $g : \mathbb{R}^m \to \mathbb{R}$ to be equal to g_k on each B_k, and 0 otherwise. We obtain that g is surjective and, for every proper affine subspace G of $\mathbb{R}^m$, the restriction $g|_G$ is not surjective.

Now we turn to the case $n > 1$. Here we consider f defined by

$$f(x_1, \dots, x_m) = (x_1, \dots, x_{n-1}, g(x_n, x_{n+1}, \dots, x_m)),$$

where $g : \mathbb{R}^{m-n+1} \to \mathbb{R}$ is defined as in the above case. Suppose that G is an affine subspace of $\mathbb{R}^m$ such that $f|_G$ is onto $\mathbb{R}^n$. Then for each $y \in \mathbb{R}$, the vector $(0, \dots, 0, y) \in \mathbb{R}^n$ is in the range of f, and then there exists a vector of the form $(0, \dots, 0, x_n, \dots, x_m) \in G$ such that $g(x_n, \dots x_m) = y$. This means that the restriction of g to the affine subspace

$$G_0 = G \cap \left(\{0\}^{n-1} \times \mathbb{R}^{m-n+1}\right) \subset \{0\}^{n-1} \times \mathbb{R}^{m-n+1} \equiv \mathbb{R}^{m-n+1}$$

is onto, and by the properties of g we obtain that $G_0 = \{0\}^{n-1} \times \mathbb{R}^{m-n+1}$, so that $G \supset (\{0\}^{n-1} \times \mathbb{R}^{m-n+1})$. Since in particular G contains the zero vector, we have that G is a vector subspace of $\mathbb{R}^m$. Consider now the natural projection $\pi : \mathbb{R}^m \to \mathbb{R}^{n-1}$ onto the first coordinates. Then the kernel of $\pi|_G$ has dimension at least $m - n + 1$. On the other hand, for each $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ there exists $(x_1, \dots, x_n, \dots, x_m) \in G$ such that $f(x_1, \dots, x_n, \dots, x_m) = (x_1, \dots, x_{n-1}, 0)$, and therefore the range of $\pi|_G$ has dimension $n - 1$. This implies that G has dimension m, and thus $G = \mathbb{R}^m$. □

We remark that the properties described above are *lineable* (see, e.g. [1]). So, for example, one can arrange to have a countable family of functions $(f_k)_{k \in \mathbb{N}}$, each satisfying the conclusion of Theorem 10, such that any nonzero function in the vector space spanned by $\{f_k \mid k \in \mathbb{N}\}$ also satisfies this conclusion.

The rest of this article deals with the case of complex Banach spaces. The following examples will show that the situation is different if we consider *holomorphic* surjective mappings.

Example 11. *There is a surjective entire function $f : \mathbb{C}^2 \to \mathbb{C}$ whose restriction to any complex line through the origin is no longer surjective.*

To see this, it is routine to verify that the function

$$f(z,w) = \frac{e^{\frac{e^z-1}{z}w}-1}{w}z$$

is entire. We next show that none of the three types of restriction functions,
(a) $z \in \mathbb{C} \rightsquigarrow f(z(1,\lambda))$, where $0 \neq \lambda \in \mathbb{C}$,
(b) $z \in \mathbb{C} \rightsquigarrow f(z(1,0))$, and
(c) $z \in \mathbb{C} \rightsquigarrow f(w(0,1))$,
is onto.

Indeed, the mapping (a) is $z \in \mathbb{C} \rightsquigarrow [e^{\lambda(e^z-1)}-1]\frac{1}{\lambda}$ which does not contain $1/\lambda$ its range. For (b), the mapping reduces to $z \rightsquigarrow e^z - 1$ and so -1 is not in the range of this restriction, while in (c) $w \rightsquigarrow f(w(0,1)) \equiv 0$. Note that (a), (b), and (c) show that if we let $\lambda \in \mathbb{C}$ vary, then every point in $\mathbb{C}$ is the image of some $(z,w) \in \mathbb{C}^2$.

The authors are grateful to O. Blasco and J. Martínez for observing that it is possible to generalize the above construction to get many such functions $f : \mathbb{C}^2 \to \mathbb{C}$. In addition, the argument in the above example can be slightly altered to get the following somewhat more general result.

Example 12. *For each $n \geq 2$, there exists a holomorphic surjection $f : \mathbb{C}^n \to \mathbb{C}^{n-1}$ whose restriction to every proper linear subspace through the origin is no longer surjective.*

Proof. Let $g : \mathbb{C}^2 \to \mathbb{C}$ be the function in Example 11, and let $f : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the function defined by

$$f(z_1, \ldots, z_n) := (z_1, \ldots, z_{n-2}, g(z_{n-1}, z_n)).$$

Clearly f is holomorphic and surjective, and reasoning as in the proof of Theorem 10 we see that f has the desired properties. □

In particular, there are holomorphic surjections from $\mathbb{C}^3 \to \mathbb{C}^2$ and also from $\mathbb{C}^4 \to \mathbb{C}^3$ whose restriction to every proper vector subspace is no longer surjective. However, we do not know whether the same holds for holomorphic surjections $\mathbb{C}^4 \to \mathbb{C}^2$. In addition, we don't know if there are vector subspaces of holomorphic surjections satisfying either Example 11 or 12.

On the other hand, if $f : E \to \mathbb{C}$ is a surjective holomorphic function on a complex Banach space E having dimension at least 3, then there is a $2-$dimensional subspace $G \subset E$ such that $f|_G$ is surjective. To see this, let $e_1 \in E$ be any vector such that $f(e_1) \neq f(0)$. It follows from Picard's theorem that the restriction of f to the subspace generated by e_1 is either surjective or misses at most one point of $b \in \mathbb{C}$. Letting $e_2 \in E$ be such that $f(e_2) = b$, we see that the restriction of f to the span of $\{e_1, e_2\}$ is onto.

We do not know if Example 11 can be improved to yield a surjective holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ such that the restriction of f to every affine complex line is no longer surjective. In addition, we don't know if there is an analogue of Theorem 4 for vector-valued, surjective holomorphic mappings. In particular, if E is a complex Banach space and $f : E \to \mathbb{C}^2$ is holomorphic and onto, we don't know if there is a finite dimensional, or even a separable, subspace of E on which the restriction of f is surjective. One very special case, in which a separable subspace does exist, is the following: Consider $c_0(\Gamma)$ with Γ uncountable. It is known [8] that every holomorphic function $f : c_0(\Gamma) \to \mathbb{C}^n$ depends only on a countable number of coordinates. Hence if f is surjective there exists a separable subspace G of $c_0(\Gamma)$ such that $f|_G$ is surjective.

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