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SUBVARIEDAD FUNDAMENTAL RESPECTO DE UNA CORRESPONDENCIA ALGEBRAICA

FUNDAMENTAL SUBVARIETY FOR AN ALGEBRAIC CORRESPONDENCE

(Publicado en la «Revista Matemática Hispano Americana»
4.ª Serie - Tomo X - Núms. 5 y 6)

NUEVAS GRAFICAS, S. A.
MADRID - 1950

SUBVARIEDAD FUNDAMENTAL RESPECTO DE UNA CORRESPONDENCIA ALGEBRAICA

por

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Introducción.—Hemos probado [1] que si $\mathfrak{P}$ es una subvariedad irreducible de V, regular respecto de una correspondencia algebraica T, y si $\mathfrak{P}'$ es una componente irreducible de su transformada, se verificaba que $\dim.(\mathfrak{P}')=\dim.(\mathfrak{P})+a-b$, cuando era $\mathfrak{P}^*=\mathfrak{P}^{*\prime}$ ([1] pág. 225), y que $\dim.(\mathfrak{P})+a-b<\dim.(\mathfrak{P}')\leq\dim.(\mathfrak{P}_1)+a-b$, cuando $\mathfrak{P}^*\subset\mathfrak{P}^{*\prime}$ En el § 1 del presente trabajo probamos que, si $\mathfrak{P}$ es fundamental y la componente $\mathfrak{P}'$ de su transformada no lo es en la transformación inversa, se verifica que

$$\dim.(\mathfrak{P}')>\dim.(\mathfrak{P})+a-b. \tag{a}$$

Este teorema caracteriza a las subvariedades fundamentales cuando $\mathfrak{P}^*=\mathfrak{P}^{*\prime}$, y sólo en este caso.

Si $b=0$, [1], la condición (a) es característica para las subvariedades fundamentales, si, además, se cumple que no existen subvariedades fundamentales en V′ para la correspondencia inversa.

En el § 2 construímos el ideal que representa a la subvariedad fundamental respecto de una correspondencia algebraica y obtenemos como corolario que la dimensión de esta subvariedad no es mayor a la dimensión de V disminuída en dos unidades.

HIPÓTESIS Y NOTACIONES. k es un cuerpo base arbitrario con infinitos elementos; $\{x_0, \ldots, x_n\}$ e $\{y_0, \ldots, y_m\}$ son dos sucesiones de indeterminadas sobre k;

$$A=k[x_0, \ldots, x_n; y_0, \ldots, y_m]; \tag{1}$$

$$I=A(F_1(x; y), \ldots, F_a(x; y)), \tag{2}$$

siendo I un ideal primo y bihomogéneo. Representaremos por $\mathfrak{V}$ a la variedad correspondiente a I en el espacio proyectivo doble y por $(\xi_0, \ldots, \xi_n; \eta_0, \ldots, \eta_m)$ a un punto general de $\mathfrak{V}$.

(*) Trabajo presentado al Congreso Internacional de Matemáticas, celebrado en Cambridge, U. S. A., del 30 de agosto al 6 de septiembre de 1950

Pondremos, como en [1],

$$\begin{array}{l}\Omega = k(\xi_0, \ldots, \xi_n;\ \eta_0, \ldots, \eta_m),\ \Sigma = k(\xi_0, \ldots, \xi_n),\ \Sigma' = k(\eta_0, \ldots, \eta_m), \\ P^* = k[\xi_0, \ldots, \xi_n;\ \eta_0, \ldots, \eta_m],\ P = k[\xi_0, \ldots, \xi_n],\ P' = k[\eta_0, \ldots, \eta_m].\end{array} \qquad [3]$$

Representaremos por $r+1$, $s+1$, $a+1$, $b+1$, a los grados de transcendencia de Σ y Σ' sobre k y de Ω sobre Σ y Σ', respectivamente.

Llamaremos T a la correspondencia algebraica definida por $\mathfrak{B}$ y V y V′ a las variedades original e imagen de esta correspondencia. $(\xi_0, \ldots, \xi_n)$ y $(\eta_0, \ldots, \eta_m)$ son puntos generales de V y V′, respectivamente.

Sean λ_{ij}, $i, j = 0, \ldots, m$, $(m+1)^2$ indeterminadas sobre Ω, independientes entre sí, y pongamos

$$y_i = \sum_{j=0}^{m} \lambda_{ij}\, y_j^*, \qquad y_j^* = \frac{1}{|\lambda_{ij}|} \sum_{j=0}^{m} \Lambda_{ij}\, y_j, \qquad i = 0, \ldots, m \qquad [4]$$

en donde Λ_{ij} es el adjunto de λ_{ji} en (λ_{ij}).

Pondremos:

$$\left.\begin{array}{l}
K = k(\lambda_{ij}), \\
A_1 = k[\lambda_{00}, \ldots, \lambda_{m,m};\ x_0, \ldots, x_n;\ y_0, \ldots, y_m], \\
B_1 = k[\lambda_{00}, \ldots, \lambda_{m,m};\ x_0, \ldots, x_n;\ y^*_0, \ldots, y^*_m], \\
C_1 = K[x_0, \ldots, x_n;\ y^*_0, \ldots, y^*_m], \\
B_2 = k[\lambda_{00}, \ldots, \lambda_{mm};\ \xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_m], \\
C_2 = K[\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_m], \\
B_3 = \Sigma[\lambda_{00}, \ldots, \lambda_{mm};\ y^*_0, \ldots, y^*_m], \\
C_3 = K(\xi)[y^*_0, \ldots, y^*_m], \\
B_4 = k[\lambda_{00}, \ldots, \lambda_{mm};\ \xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_{a+1}], \\
C_4 = K[\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_{a+1}], \\
B_5 = \Sigma[\lambda_{00}, \ldots, \lambda_{mm};\ y^*_0, \ldots, y^*_{a+1}], \\
C_5 = K(\xi)[y^*_0, \ldots, y^*_{a+1}], \\
\hat{A} = K(\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_a)[y^*_{a+1}], \\
D = k[\lambda_{00}, \ldots, \lambda_{mm};\ \xi_0, \ldots, \xi_n], \\
E = \Sigma[\lambda_{00}, \ldots, \lambda_{mm}], \\
F = k[\lambda_{00}, \ldots, \lambda_{mm}], \\
G = k[x_0, \ldots, x_n],
\end{array}\right| \qquad [5]$$

e

$$\left.\begin{array}{l}
I_1 = A_1 I, \\
J_1 = B_1 I = B_1 I_1, \\
L_1 = C_1 I, \\
J_2 \cong J_1/B_1 (I \cap G), \\
L_2 \cong L_1/C_1 (I \cap G), \\
J_3 = B_3 J_2, \\
L_3 = C_3 L_2, \\
J_4 = J_2 \cap B_4, \\
L_4 = L_2 \cap C_4, \\
J_5 = B_5 J_4, \\
L_5 = C_5 L_4, \\
\hat{I} = \hat{A} L_5.
\end{array}\right\} \qquad [6]$$

Representaremos por letras alemanas mayúsculas a los ideales homogéneos y bihomogéneos de P, P' y P*, y por letras alemanas minúsculas a los no homogéneos. Escribiremos $\mathfrak{P}$ y $\mathfrak{p}$, respectivamente, para representar a los ideales primos homogéneos y no homogéneos, respectivamente, y emplearemos la misma letra para representar a un ideal y a su variedad correspondiente.

§ 1. Transformada de una subvariedad fundamental.

Sea $\mathfrak{P}$ una subvariedad de V, fundamental e irreducible ([1], def. 1.10) y $\mathfrak{P}'$ una componente irreducible de su transformada. Entonces, existe un d. m. p., $\mathfrak{P}^*$, de $P^* \mathfrak{P}$ que yace sobre $\mathfrak{P}$ y $\mathfrak{P}'$. Representaremos, como en [1], por $\mathfrak{P}^{*\prime}$ al d. p. m. de $P^* \mathfrak{P}'$, que es un múltiplo, propio o impropio, de $\mathfrak{P}^*$, y pondremos $\mathfrak{P}_1 = \mathfrak{P}^{*\prime} \cap P$.

TEOREMA 1. *Si* $\mathfrak{P}$ *es fundamental para* T *y* $\mathfrak{P}'$ *no lo es para* T^{-1} *se verifica que*

$$dim.(\mathfrak{P}') > dim.(\mathfrak{P}) + a - b.$$

Demostración. Sean $\dot{\Omega}$, $\ddot{\Omega}$, $\dot{\Sigma}$ y $\dot{\Sigma}'$ los cuerpos de cocientes de $P^*/\mathfrak{P}^*$, $P^*/\mathfrak{P}^{*\prime}$, $P/\mathfrak{P}$, $P'/\mathfrak{P}'$, respectivamente.

a) *Sea* $\mathfrak{P}^{*\prime}=\mathfrak{P}^{*}$. En este caso es $\dot{\Omega}=\ddot{\Omega}$ y, como $\mathfrak{P}'$ no es fundamental respecto de T^{-1}, resulta (def. 1.10, [1]) que grad. trans. $[\dot{\Omega} : \dot{\Sigma}']=b+1$. Como $\mathfrak{P}$ es fundamental respecto de T, es grad. trans. $[\dot{\Omega} : \dot{\Sigma}]=\alpha+1>a+1$. Ahora bien, grad trans. $[\dot{\Omega} : k]=$grad. trans. $[\dot{\Omega} : \dot{\Sigma}]+$grad. trans. $[\dot{\Sigma} : k]=\alpha+1+$dim. $(\mathfrak{P})+1$, y grad. trans. $[\dot{\Omega} : k]=$grad. trans. $[\dot{\Omega} : \dot{\Sigma}']+$grad. trans. $[\dot{\Sigma}' : k]=b+1+$dim. $(\mathfrak{P}')+1$.

Por tanto, dim.$(\mathfrak{P}')=$dim.$(\mathfrak{P})+\alpha-b>$dim.$(\mathfrak{P})+a-b$.

b) Si $\mathfrak{P}^{*\prime}\subset\mathfrak{P}^{*}$, es dim$(\mathfrak{P}^{*\prime})>$dim.$(\mathfrak{P}^{*})$, o bien grad. trans. $(\ddot{\Omega} : \dot{\Sigma}')=b+1>$grad. trans. $(\dot{\Omega} : \dot{\Sigma}')=\beta+1$, de donde, (7) dim.$(\mathfrak{P}')>$dim.$(\mathfrak{P})+a-\beta>$dim.$(\mathfrak{P})+a-b$, q. e. d.

Escolio. Si $\mathfrak{P}$ fuera regular tendríamos dim.$(\mathfrak{P}')=$ $=$dim.$(\mathfrak{P})$ $a-\beta$, por consiguiente, la primera desigualdad [7] muestra que *si $\mathfrak{P}$ es fundamental y $\mathfrak{P}'$ no lo es en la transformación inversa, la dimensión de $\mathfrak{P}'$ es mayor que la que tendría si $\mathfrak{P}$ fuese regular respecto de $\mathfrak{P}'$.*

El lema siguiente contiene al T. 3.12 [1].

LEMA 1. *Si* b$=0$ *y* $\mathfrak{P}'$ *no es fundamental en la correspondencia inversa se verifica que* $\mathfrak{P}^{*\prime}=\mathfrak{P}^{*}$.

Demostración. Empleando las notaciones anteriores y teniendo en cuenta las relaciones $\mathfrak{P}^{*\prime}\subseteq\mathfrak{P}^{*}$ *y* $\mathfrak{P}^{*\prime}\cap \mathrm{P}'=\mathfrak{P}^{*}\cap P'=\mathfrak{P}'$ resulta que grad. trans. $(\ddot{\Omega} : \dot{\Sigma}')\geqslant$grad. trans. $(\dot{\Omega} : \dot{\Sigma}')$; pero, como $b=0$ y $\mathfrak{P}'$ no es fundamental, resulta (Def. 1.10 [1]) que grad. trans. $(\ddot{\Omega} : \dot{\Sigma}')=1$ y, como $\dot{\Omega}$ es un cuerpo homogéneo respecto de $\dot{\Sigma}'$, resulta que grad. trans. $(\dot{\Omega} : \dot{\Sigma}')\geqslant 1$, de donde grad. trans. $(\ddot{\Omega} : \dot{\Sigma}')=$grad. trans. $(\dot{\Omega} : \dot{\Sigma}')=1$ y dim.$(\mathfrak{P}^{*\prime})=$ $=$grad trans. $(\dot{\Omega} : \dot{\Sigma}')+$grad. trans. $(\dot{\Sigma}' : k)=$dim.$(\mathfrak{P}^{*})$, q. e. d.

TEOREMA 2. *Si en la correspondencia algebraica T es* b$=0$ *y no existen subvariedades fundamentales en V' para* T^{-1}, *una condición necesaria y suficiente para que sea fundamental una subvariedad,* $\mathfrak{P}$, *de V es que dim.*$(\mathfrak{P}')>$*dim.*$(\mathfrak{P})+$a.

La necesidad ha sido probada en el T. 1, y la suficiencia es consecuencia del lema precedente y del T. 2.12 [1].

§ 2. Subvariedad fundamental respecto de una correspondencia algebraica.

De las definiciones (5) se deducen inmediatamente las siguientes relaciones:

1) $A_1 \subset B_1$.

2) $C_1 = A_{1_F} = B_{1_F}$.

3) $B_2 \cong B_1/B_1(I \cap G)$.

4) $C_2 \cong C_1/C_1(I \cap G)$, $C_2 = B_{2_F}$.

5) $B_3 = B_{2_P}$.

6) $C_3 = C_{2_D} = B_{2_D} = B_{3_D}$.

7) $B_5 = B_{4_P}$.

8) $C_5 = C_{4_D}$.

9) $\hat{A} = C_{5_{K(\xi)[y^*_0, \ldots, y^*_a]}} = C_{5_{k[\lambda; \xi; y^*_0, \ldots, y^*_a]}}$

De estas relaciones y de (5) y (6) se deducen las siguientes proposiciones:

10) I_1 es un ideal primo e $I_1 \cap A = I$.

11) J_1 es un ideal primo y $J_1 \cap A = I$.

12) $L_1 = C_1 I_1 = C_1 J_1$ es primo y, como no existen polinomios de F contenidos en I_1 ni en J_1, se verifica que $L_1 \cap A_1 = I_1$, $L_1 \cap B_1 = J_1$.

13) J_2 y L_2 son primos y $L_2 = C_2 J_2$, $J_2 = L_2 \cap B_2$.

14) *No existen polinomios de D en los ideales J_2, L_2, J_3, L_3, J_4, L_4.* Si existiese un polinomio, $f(\xi ; \lambda)$, tal que $f \equiv 0(J_2)$, existiría un polinomio, $F(x ; \lambda ; y^*)$ en J_1 tal que $F(\xi ; \lambda ; y^*) = f(\xi ; \lambda)$, de donde $F(x ; \lambda ; y^*) = f(x ; \lambda) + F_1(x ; \lambda ; y^*)$, y $F_1(\xi ; \lambda ; y^*) = 0$; de esta última relación resulta $F(x ; \lambda ; y^*) \equiv 0(B_1(I \cap G))$ y, *a fortiori*, $F_1 \equiv 0(J_1)$ y, como $F \equiv 0(J_1)$, se deduce que $f(x ; \lambda) \equiv 0(J_1)$, por tanto, $f(x ; \lambda) \equiv 0(k[x ; \lambda](I \cap G))$ e. e., $f(\xi ; \lambda) = 0$. Análoga demostración sirve para L_2. De las relaciones $J_3 = B_3 J_2 = B_4 . J_2$, y por no pertenecer ningún polinomio de P al ideal J_2, resulta que, $J_2 = J_3 \cap B_2$, y, de aquí, que no existe ningún polinomio de D en J_3. Como $L_3 \cap C_2 = L_2$, se deduce que ningún polinomio de D pertenece a L_3. Finalmente, de $J_4 = J_2 \cap B_4$ y $L_4 = L_2 \cap C_4$ se deduce que ningún polinomio de D pertenece a J_4 ni a L_4.

15) J_4 y L_4 son primos.

16) De 14) se deduce que J_3 y L_3 son ideales primos y que $J_2 = J_3 \cap B_2$, $L_2 = L_3 \cap C_2$.

17) *Dim.* $(L_3) = a+1$. Como la dimensión de L_1 (respecto de K) es igual a la de I (respecto de k), resulta dim. $(L_1) = r + a + 2$. Por otra parte, de $C_2/L_2 \cong C_1/L_1$ se deduce que dim. $(L_2) =$ dim. $(L_1) = q + 2$.

Finalmente, como la dimensión de L_3 es igual al grado de transcendencia del cuerpo de cocientes de $\hat{C}_3/L_3$ sobre $K(\xi)$ y, en virtud de 14), el cuerpo de cocientes de C_3/L_3 es igual al de C_2/L_2, resulta que dim. $(L_3) =$ grad. trans. $(C_2/L_2 : K(\xi)) =$ dim. $(L_2) -$ grad. trans. $(K(\xi) : K)) = a + 1$.

Hipótesis.—En virtud de 17), supondremos en lo que sigue que $y^*_0, \ldots, y^*_a$ son algebraicamente independientes mód. (L_3).

18) De De 6), (6) y 13) se deduce que

$$L_3 = C_3\,L_2 = C_3\,J_2 = B_{3_D}\,(J_2) = B_{3_D}\,(B_3\,J_2) = C_3\,J_3$$

y de aquí se sigue que $y^*_0, \ldots, y^*_a$ son algebraicamente independientes mód. (J_3).

19) $L_5 = C_5 J_4 = C_5 J_5$. En efecto, de 13) se deduce que $L_2 = C_2 J_2$ y, por tanto, $L_4 = C_2 J_2 \cap C_4$ y $L_5 = C_5(C_2 J_2 \cap C_4 = C_5(C_5 J_2 \cap B_4) = C_5(B_2 J_2 \cap B_4) = C_5(J_2 \cap B_4) = C_5 J_4 = C_5 J_5$.

20) De 14) y 19) se deduce $J_5 = L_5 \cap B_5$.

21) Dim. $(J_4) =$ dim. $(J_2) =$ dim. $(L_2) +$ grad. trans. $(K : k) = q + 2 + (m+1)^2$. Y dim. $(L_4) = r + a + 2$.

22) De (6) y 14) se deduce dim. $(J_5) =$ dim. $(J_4) -$ grad. trans. $(\Sigma : k) = a + 1 + (m+1)^2$.

23) De (6), 21) y 14) se deduce que dim. $(L_5) =$ dim. $(L_4) -$ grad. trans. $(K(\xi) : K) = a + 1$.

24) Como $L_5 = C_5 L_4 = C_5(L_2 \cap C_4) = C_5(L_3 \cap C\ \cap C_4) = C_5(L_3 \cap C_4)$, las $y^*_0, \ldots, y^*_a$ son algebraicamente independientes mód. (L_5) y por tanto $\hat{I}$ es primo e $\hat{I} \cap C_5 = L_5$. De aquí y de 23) resulta dim. $(\hat{I}) =$ dim. $(L_5) - (a+1) = 0$.

25) Siendo $\hat{A}$ un anillo de polinomios con una única inde-

terminada y coeficientes de un cuerpo, es euclídeo y el ideal $\hat{I}$ es principal

$$\hat{I} = (\Psi(\xi\,;\lambda\,;y^*_0, \ldots, y^*_{a+1})),$$

siendo Ψ un polinomio irreducible de $\hat{A}$, que puede suponerse perteneciente a B_4 e irreducible en B_5. En lo que sigue supondremos que Ψ satisface estas condiciones.

26) *El polinomio Ψ posee la siguiente forma:*

$$\Psi(\xi;\lambda;y^*_0, \ldots, y^*_{a+1}) \equiv H(\xi,\lambda)\,y^{*\rho}_{a+1} + a_1(\xi;\lambda;y^*_0, \ldots, y^*_a)\,y^{*\rho-1}_{a+1} +$$
$$+ \ldots + a_\rho(\xi;\lambda;y^*_0, \ldots y^*_a)$$

en donde H no contiene ninguna (y^*) *y Ψ puede suponerse homogéneo respecto de las* (ξ) *y de las* ($y^*_0, \ldots, y^*_{a+1}$). Como I es homogéneo respecto de las (x) y de las (y), lo es también $L_1 = C_1 I$ respecto de las (x) y de las (y^*), así como $L_2 \cong L_1/C_1(I \cap G)$ lo es respecto de las (ξ) e (y^*), y $L_4 = L_2 \cap C_4$ lo será respecto de las (ξ) e ($y^*_0, \ldots, y^*_{a+1}$). Siendo las ($y^*_0, \ldots, y^*_a$) algebraicamente independientes mód. L_5, es $L_5 = \hat{I} \cap C_5$ y, como hemos tomado Ψ irreducible en C_5, el ideal $C_5(\Psi)$ es primo y, en virtud de 23), es $L_5 = C_5(\Psi)$.

De 14) y (6) se deduce que $L_4 = L_5 \cap C_4$ y, por tanto, $C_4(\Psi) \subseteq L_4$. Si Ψ no fuera bihomogéneo respecto de las (ξ) y de las ($y^*_0, \ldots, y^*_{a+1}$), como L_4 lo es, todas sus componentes bihomogéneas pertenecerían a L_5 y, como las ($y^*_0, \ldots, y^*_{a+1}$) son indeterminadas sobre K(ξ), resulta que Ψ es homogéneo respecto de las (y^*). Sea $\Psi = \Psi_{\alpha_1, \beta} + \ldots + \Psi_{\alpha_s, \beta}$ una descomposición de Ψ en sus componentes bihomogéneas; entonces será $\Psi_{\alpha_i, \beta} \equiv 0(L_5)$ y $\Psi_{\alpha_i, \beta} = \dfrac{f(\xi;\lambda)}{g(\xi:\lambda)} \Psi$, de donde $\Psi \equiv 0(C_5(\Psi_{\alpha_i, \beta}))$ y $C_5(\Psi_{\alpha_i, \beta}) = C_5(\Psi)$. *En lo que sigue supondremos*, por tanto, *que Ψ es bihomogéneo respecto de las* (ξ) *y de las* (y^*).

Si $H(\xi\,;\lambda)$ contuviese alguna de las ($y^*_0, \ldots, y^*_a$), efectuando la sustitución $y^*_i = \sum\limits_{j=0}^{a+1} \bar{\lambda}_{ij}\, y^{**}_j$, $y^{**}_{a+1} = y^*_{a+1}$, $i = 0, \ldots, a$, obtendríamos

$$[H(\xi;\lambda;\bar{\lambda}_{i,a+1}) + a_1(\xi;\lambda;\bar{\lambda}_{0,a+1}, \ldots, \bar{\lambda}_{0,a+1}) + \ldots +$$
$$+ a_\rho(\xi;\lambda;\bar{\lambda}_{0,a+1}, \ldots, \bar{\lambda}_{a,a+1})]\,y^{*\mu}_{a+1} + \ldots \qquad [8]$$

en donde μ es el grado de Ψ respecto de las (y^*). Como las $\bar{\lambda}_{i,\,a+1}$ se han supuesto parámetros y el grado de H respecto de ellas es $\mu-\rho$ y el de a_i es $\mu-\rho+i$, si el coeficiente de $y_{a+1}^{**\mu}$ en (8) fuese cero lo serían también H y las a_i, $i=1, \ldots, \rho$, y también lo sería Ψ. Bastaría, por tanto, tomar las indeterminadas $y_{00}^{**}, \ldots, y_a^{**}$ en lugar de las $y^*_0, \ldots, y^*_a$.

27) $L_4 = B_{4_F} J_4 = C_4 J_4$. De (6) y 13) se deduce que $L_1 = B_{1_F} J_1$ y $L_2 = B_{2_F} J_2$, respectivamente, y, por 14), $L_4 = B_{2_F} J_2 \cap B_{4_F} = B_{4_F} (B_{2_F} J_2 \cap B_4) = B_{4_F} [(B_{2_F} J_2 \cap B_2) \cap B_4] = B_{4_F} (J_2 \cap B_4) = B_{4_F} J_4 = C_4 J_4$.

28) $J_5 = B_5(\Psi)$. En efecto, de 20) y 26) se deduce que $B_5(\Psi) \equiv 0(J_5)$. Puesto que Ψ es irreducible en B_5 y éste es un anillo de polinomios, el ideal $B_5(\Psi)$ es primo y, por 22), resulta que $J_5 = B_5(\Psi)$.

Lema 2. *Si $\mathfrak{P}$ es un ideal primo y homogéneo de P y si $D\mathfrak{P}$ es divisor de $D(H(\xi\,;\lambda))$, el ideal $B_4\mathfrak{P}$ es divisor de $B_4(\Psi)$.*

Demostración. Sea

$$(y_0^*, \ldots, y_{a+1}^*)' = R_1 (y_0^{**}, \ldots, y_{a+1}^{**})', \quad R_1 = \\ = \begin{pmatrix} \rho_{00}, & \rho_{01}, & \ldots, & \rho_{0\,a+1} \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{a+1,0}, & \rho_{a+1,1}, & \ldots, & \rho_{a+1,a+1} \end{pmatrix} \tag*{[9]}$$

en donde las (ρ_{ij}) son indeterminadas independientes sobre A_1,

$$\Lambda = \begin{pmatrix} \lambda_{00}, & \ldots, & \lambda_{0m} \\ \ldots & \ldots & \ldots \\ \lambda_{m0}, & \ldots, & \lambda_{mm} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 & O \\ O & U \end{pmatrix}, \tag*{[10]}$$

U la matriz unidad, y

$$T = \begin{pmatrix} \tau_{00}, & \ldots, & \tau_{0m} \\ \ldots & \ldots & \ldots \\ \tau_{m0}, & \ldots, & \tau_{mm} \end{pmatrix} = \Lambda R. \tag*{[11]}$$

Entonces,

$$(y_0, \ldots, y_m)' = T (y_0^{**}, \ldots, y_m^{**})', \quad y_i^{**} = y_i^*, \quad i = a+2, \ldots, m \tag*{[12]}$$

Si empleamos la sustitución (12) en lugar de la (4) y seguimos el mismo proceso empleado para obtener Ψ, obtendremos un polinomio, Ψ^*, que diferirá de Ψ en que el primero tendrá (y^{**}) y (τ) en lugar de las (y^*) y (λ) del segundo. Ahora bien, podemos obtener Ψ^* efectuando las sustituciones $\Lambda = TR^{-1}$ y (9) en $\Psi(\xi ; \lambda ; y^*_0, \ldots, y^*_{a+1})$. Efectuando en Ψ sólo esta última, obtenemos

$$\begin{aligned}\Psi(\xi; \lambda; y^*_0, \ldots, y^*_{a+1}) &= [H(\xi; \lambda)\, \rho^{\rho}_{a+1,\, a+1} + \\ &+ a_1(\xi; \lambda; \rho_{0,\, a+1}, \ldots, \rho_{a,\, a+1})\, \rho^{\rho-1}_{a+1,\, a+1} + \ldots + \\ &+ a_\rho(\xi; \lambda; \rho_{0\, a+1}, \ldots, \rho_{a,\, a+1})]\, y^{**\rho}_{a+1} + \\ &+ b_1(\xi; \lambda; \rho; y^{**}_0, \ldots, y^{**}_a)\, y^{**\rho-1}_{a+1} + \ldots \end{aligned} \qquad [13]$$

y, efectuando ahora en (13) la sustitución $\Lambda = TR^{-1}$, resulta

$$\begin{aligned}\Psi(\xi; \tau; y^{**}_0, \ldots, y^{**}_{a+1}) &= [H(\xi; \lambda)\, \rho^{\rho}_{a+1,\, a+1} + \\ &+ a_1(\xi; \lambda; \rho_{0,\, a+1}, \ldots, \rho_{a,\, a+1})\, \rho^{\rho-1}_{a+1,\, a+1} + \ldots \\ &+ a_\rho(\xi; \lambda; \rho_{0\, a+1}, \ldots, \rho_{a,\, a+1})]\, y^{**\rho}_{a+1} + \ldots \end{aligned}$$

de donde resulta que

$$\begin{aligned}H(\xi; \tau) = H(\xi; \lambda)\, \rho^{\rho}_{a+1,\, a+1} &+ a_1(\xi; \lambda; \rho_{0,\, a+1}, \ldots \rho_{a,\, a+1})\, \rho^{\rho-1}_{a+1,\, a+1} + \\ &\ldots + a_\rho(\xi; \lambda; \rho_{0\, a+1}, \ldots, \rho_{a,\, a+1}) \end{aligned} \qquad [14]$$

Ahora bien, de las hipótesis se deduce que $D(H(\xi ; \lambda)) \equiv O(D\mathfrak{P})$, por tanto, $k[\xi ; \tau](H(\xi ; \tau)) \equiv 0(k[\xi ; \tau]\mathfrak{P})$, y como $k[\xi ; \tau] \subset k[\xi ; \lambda ; \rho]$ resulta que $k[\xi ; \lambda ; \rho](H(\xi ; \tau)) \equiv 0(k[\xi ; \lambda ; \rho]\mathfrak{P})$ y como las (λ) y las (ρ) son indeterminadas sobre P, independientes entre sí, de las relaciones anteriores y de las (6) resulta que todos los coeficientes de la derecha de (14) respecto de las (λ) y de las (ρ) están contenidos en $\mathfrak{P}$. Finalmente, si en el segundo miembro de (14) se sustituyen las $(\rho_{0,\, a+1}, \ldots, \rho_{a+1,\, a+1})$ por las $(y_0^*, \ldots, y^*_{a+1})$ se obtiene $\Psi(\xi ; \lambda ; y^*_0, \ldots, y^*_{a+1})$, luego, $B_4\Psi \equiv 0(B_4\mathfrak{P})$, q. e. d.

LEMA 3. *Si* $D\mathfrak{P}$ *es un d. p.m. de* $D(H(\xi ; \lambda))$, *siendo* $\mathfrak{P}$ *un ideal primo y homogéneo de* P, *la subvariedad* $\mathfrak{P}$ *no es fundamental.*

Demostración. Si $\mathfrak{P}$ fuese fundamental, poniendo $\mathfrak{P}_2=C_2\mathfrak{P}$, como las (λ) e (y^*) son indeterminadas respecto de Σ, el idel $\mathfrak{P}_2$ sería primo y $\mathfrak{P}=\mathfrak{P}_2\cap P$, luego, si $\xi_i\equiv\bar{\xi}_i(\mathfrak{P})$, $i=0, \ldots, n$, *sería*

$$C_2/\mathfrak{P}_2 \approx K[\bar{\xi}_0, \ldots, \bar{\xi}_n\,;\, y^*_0, \ldots, y^*_m].$$

Siendo $\mathfrak{P}$ fundamental (def. 1.10, [1]) existe un d. p. m. de $(L_2, \mathfrak{P}_2)$, que representaremos por $\bar{\mathfrak{P}}_2$, tal que dim. $(\mathfrak{P}_2)$ − dim. $(\mathfrak{P}_2) > a+1$ y, por tanto, no existe ningún polinomio de $K[\xi_0, \ldots, \bar{\xi}_n\,;\, y^*_0, \ldots, y^*_{a+1}]$ contenido en $\bar{\mathfrak{P}}_2/\mathfrak{P}_2$; pero, como $L_2\equiv 0(\mathfrak{P}_2)$ y existen polinomios de $K[\xi_0, \ldots, \xi_n\,;\, y^*_0, \ldots, y^*_{a+1}]$ contenidos en L_2, Ψ entre ellos, resulta que si en todos estos polinomios se sustituyen las (ξ) por las $(\bar{\xi})$ todos ellos se anularán, luego, sus coeficientes respecto de las (λ) e (y^*) están contenidos en $\mathfrak{P}$ y $L_4=L_2\cap C_4\equiv 0(C_4\mathfrak{P})$. Como $D(H(\xi\,;\lambda))$ es un ideal principal, es puro y de dimensión r, luego $\dim._K(D\mathfrak{P})=r$ y $\dim._K(C_4\mathfrak{P})=$ $=r+a+2$. Ahora bien, de 21) y $L_4\equiv 0(C_4\mathfrak{P})$ se deduce, ya que ambos son ideales primos, que $L_4=C_4\mathfrak{P}$; pero, de 14) se deduce que $L_4\cap P=(0)$, que contradice a $C_4\mathfrak{P}\cap P=\mathfrak{P}$, q. e. d.

Definición. Representaremos por $\mathfrak{I}$ a la intersección de todos los d. p. m. de $D(H(\xi\,;\lambda))$ que no poseen bases totalmente contenidas en P.

TEOREMA 3. *La condición necesaria y suficiente para que la subvariedad $\mathfrak{P}$ de V sea fundamental para T es que*

$$\mathfrak{I}\equiv 0(D\mathfrak{P})$$

Demostración. La condición es necesaria. Supongamos que $\mathfrak{P}$ fuese fundamental y que

$$\mathfrak{I}\not\equiv 0(D\mathfrak{P})$$

Siendo $\mathfrak{P}$ fundamental, resulta, por el T. 3.10 [1], que

$$H(\xi\,;\lambda)\equiv 0(D\mathfrak{P}).$$

Si ponemos en la igualdad (11) $T=\Lambda$, $\Lambda=T$ y

$$R=\left(\begin{array}{cccc|c} 1 & 0 & \ldots & y^*_0 & \\ 0 & 1 & \ldots & y^*_1 & 0 \\ \multicolumn{4}{c|}{\ldots\ldots\ldots\ldots\ldots} & \\ 0 & 0 & \ldots & y^*_{a+1} & \\ \hline \multicolumn{4}{c|}{0} & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \end{array}\right), \qquad [16]$$

se obtiene

$$\Lambda = \mathrm{T}\mathrm{R}, \tag{17}$$

y (14) se escribirá

$$\mathrm{H}(\xi; \lambda) = \mathrm{H}(\xi; \tau)\, y^{*\rho}_{a+1} + a_1(\xi; \tau; y^*_0, \ldots, y^*_a)\, y^{*\rho-1}_{a+1} + \ldots$$
$$+ a_\rho(\xi: \lambda; y^*_0, \ldots, y^*_a) \tag{18}$$

Pongamos $\overline{\mathrm{B}}_4 = k[\tau_{00}, \ldots, \tau_{mm}; \xi_0, \ldots, \xi_n; y^*_0, \ldots, y^*_a]$. De (17) se deduce que

$$\mathrm{B}_4 \subseteq \overline{\mathrm{B}}_4 \tag{19}$$

y como

$$\mathrm{T} = \frac{1}{y^*_{a+1}}\, \Lambda \left(\begin{array}{ccccc|ccc} y^*_{a+1} & 0 & \cdots & (-1)^{a+1} & y^*_0 & & & \\ 0 & y^*_{a+1} & \cdots & (-1)^{a+2} & y^*_0 & & 0 & \\ \cdots & \cdots & \ddots & \cdots & \cdots & & & \\ 0 & 0 & \cdots & (-1)^{2a+2} & & & & \\ \hline & & & & & y^*_{a+1} & & \\ & & 0 & & & & \ddots & \\ & & & & & & & y^*_{a+1} \end{array}\right)$$

resulta que $\overline{\mathrm{B}}_4 \subset \mathrm{B}_{4(y^*_{a+1})}$ y, como y^*_{a+1} no está contenida en ningún d. p. m. de $\mathrm{B}_4(\mathrm{H}(\xi; \lambda))$, si

$$\mathrm{B}_4(\mathrm{H}(\xi; \lambda)) = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_l$$

es una descomposición normal de $\mathrm{B}_4(\mathrm{H})$ en ideales primarios, siendo $\mathfrak{P}_i$ el ideal primo correspondiente a $\mathfrak{Q}_i$, $i = 1, \ldots, l$, será

$$\mathrm{B}_{4(y^*_{a+1})}(\mathrm{H}(\xi; \lambda)) \cap \overline{\mathrm{B}}_4 = (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{\mathrm{B}}_4) \cap \ldots \cap (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{\mathrm{B}}_4) \tag{21}$$

una descomposición de $\mathrm{B}_{4(y^*_{a+1})}(\mathrm{H}) \cap \overline{\mathrm{B}}_4$ en ideales primarios, siendo $\mathrm{B}_{4(y^*_{a+1})}\mathfrak{P}_i \cap \overline{\mathrm{B}}_4$ el ideal primo correspondiente a $\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_i \cap \overline{\mathrm{B}}_4$, $i = 1, \ldots, l$.

De (18) y (21) se deduce que

$$\mathrm{B}_{4(y^*_{a+1})}(\Psi(\xi; \tau; y^*)) \cap \overline{\mathrm{B}}_4 = (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{\mathrm{B}}_4) \cap \ldots \cap (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{\mathrm{B}}_4)$$

y como

$$\mathrm{B}_{4(y^*_{a+1})} = \overline{\mathrm{B}}_{4(y^*_{a+1})} \quad \text{e} \quad y^*_{a+1} \not\equiv 0\,(\overline{\mathrm{B}}(\Psi)), \quad \text{es} \quad \overline{\mathrm{B}}_{4(y^*_{a+1})}(\Psi) \cap \overline{\mathrm{B}}_4 = \overline{\mathrm{B}}_4(\Psi)$$

y, por tanto,

$$\overline{\mathrm{B}}_4(\Psi) = (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{\mathrm{B}}_4) \cap \ldots \cap (\mathrm{B}_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{\mathrm{B}}_4)$$

Como $D(H(\xi ; \lambda)) \equiv 0(D\mathfrak{P})$ y $\mathfrak{J} \not\equiv 0(D\mathfrak{P})$, $D\mathfrak{P}$ divide a un d. p. m. de $D(H)$ que posee una base perteneciente a P y no divide a ningún d. p. m. de $D(H)$ que no posea una base perteneciente a P. Este hecho lo expresaremos brevemente diciendo que $D\mathfrak{P}$ no posee la «propiedad p» respecto de $D(H)$. Ahora bien, como $H(\xi ; \lambda)$ se transforma mediante (17) en Ψ y teniendo en cuenta la descomposición anterior de $B_{4(y^*_{a+1})}(H)$, se obtiene que $\bar{B}_4\mathfrak{P}$ no posee la propiedad p respecto de $\bar{B}_4(\Psi)$. Pero, J_4 es un d. p. m. de $B_4(\Psi)$ con la propiedad p, por tanto, $J_4 \not\equiv 0(B_4\mathfrak{P})$. De esta relación se deduce que existe un polinomio $g(\xi ; \lambda ; y^*)$ que pertenece a J_4 y que no debe de pertenecer a $B_4\mathfrak{P}$. Efectuando, si ello es preciso, una transformación lineal, tal como hemos indicado en 26), es posible suponer que g tiene la siguiente forma:

$$g(\xi; \lambda; y^*_0, \ldots, y^*_{a+1}) = g_0(\xi; \lambda)\, y^{*\mu}_{a+1} + g_1(\xi; \lambda; y^*_0, \ldots, y^*_a)\, y^{*\mu-1}_{a+1} + \\ \ldots + g_\mu(\xi; \lambda; y^*_0, \ldots, y^*_a)$$

en donde $g_0(\xi ; \lambda) \not\equiv 0(D\mathfrak{P})$ y como $g(\xi ; \lambda ; \eta^*_0, \ldots, \eta^*_{a+1}) = 0$ (en donde η^*_i es el resultado de sustituir en y^*_i las (y) por las (η)), ya que $g \equiv 0(J_4)$, se deduce que (T. 3.10) $\mathfrak{P}$ no es fundamental, lo que está en contradicción con la hipótesis.

La condición es suficiente. Supongamos que $\mathfrak{J} \equiv 0(D\mathfrak{P})$ y que $\mathfrak{P}$ no es fundamental. Como en el teorema 2.10 [1], sería posible determinar m-a formas de I, $F_1(x ; y), \ldots, F_{m-a}(x ; y)$, tales que los ideales $\Sigma[y](F_1(\xi ; y), \ldots, F_{m-a}(\xi ; y))$ y $\Sigma[y](F_1(\bar{\xi} ; y), \ldots, F_{m-a}(\bar{\xi} ; y))$ (en donde $\xi_i = \bar{\xi}_i(\mathfrak{P})$, $i = 0, \ldots, n$) tendrían la misma dimensión $a+1$. Efectuando la sustitución $y_i = \sum_{j=0}^{m} \lambda_{ij} y^*_j$ y eliminando las indeterminadas $y^*_{a+2}, \ldots, y^*_m$ entre las ecuaciones $F_1(\xi ; \Sigma\lambda_{ij} y^*_j) = 0 \ldots, F_{m-a}(\xi ; \Sigma\lambda_{ij} y^*_j = 0$, se obtiene, *l. c.*, la resultante

$$\Psi_0(\xi; \lambda; y^*_0, \ldots, y^*_{a+1}) = A(\xi; \lambda)\, y^{*\rho}_{a+1} + \ldots + a_\rho(\xi; \lambda; y^*_0, \ldots\, y^*_a)$$

Como k posee infinitos elementos, existen infinitas especializaciones de las (λ) sobre elementos de k tales que $A(\xi ; \mu) \not\equiv 0(\mathfrak{P})$ por tanto,

$$A(\xi ; \lambda) \not\equiv 0\ (D\mathfrak{P}). \qquad [21]$$

Pero, como $\Psi_0 \equiv 0(J_5)$, de 28) se deduce que

$$\Psi_0 = \frac{f(\xi; \lambda; y^*_0, \ldots . y^*_{a+1})}{g(\xi)} \Psi(\xi; \lambda; y^*_0, \ldots, y^*_{a+1}),\ f \in B_4,\ g \in P$$

de donde $g(\xi)A(\xi ; \lambda) \equiv 0(D(H(\xi ; \lambda))$ y, por tanto $g(\xi)A(\xi ; \lambda) \equiv$ $\equiv 0(\mathfrak{F})$.

Vamos a ver que *si* $\mathfrak{P}_1$ *es un d. p. m. de* $(H(\xi ; \lambda))$ *que no posee la propiedad* p *no existe ningún polinomio de* P *en* $\mathfrak{P}_1$. En efecto, como (H) es un ideal puro, por ser principal, se deduce que dim. $(\mathfrak{P}_1) =$ grad. trans. $(D : k) - 1 = (m + 1)^2 + r$. Si $f(\xi)$ fuese un polinomio tal que $f(\xi) \in P$ y $f(\xi) \equiv 0(\mathfrak{P}_1)$, $\mathfrak{P}_1$ sería un divisor de $D(f)$, y como este ideal es también puro y su dimensión es $(m + 1)^2 + r$, $\mathfrak{P}_1$ sería un d. p. m. de $D(f)$; pero, todos los d. p. m. de $D(f)$ son extensiones a D de d. p. m. de $P(f)$ y, por tanto, poseerían la propiedad *p*, en contradicción con la hipótesis.

De esta proposición y de lo que antecede, resulta

$$A(\xi; \lambda) \equiv 0(\mathfrak{F})$$

y por la hipótesis

$$A(\xi ; \lambda) \equiv 0(D\,\mathfrak{P})$$

en contradicción con (21), q. e. d.

De este teorema y del lema 3 resulta inmediatamente :

COROLARIO. *La dimensión de la subvariedad fundamental de* V *no es mayor que dim.* (V)-2.

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Junio, 1950
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FUNDAMENTAL SUBVARIETY FOR AN ALGEBRAIC CORRESPONDENCE (*)

by

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Introduction.—In our paper [1] we have shown that if $\mathfrak{P}$ is an irreducible subvariety of V, regular for an algebraic correspondence, T, and if $\mathfrak{P}'$ is an irreducible component of its transform [1], then it was verified that $\dim.(\mathfrak{P}')=\dim.(\mathfrak{P})+a-b$, when $\mathfrak{P}^*=\mathfrak{P}^{*\prime}$ ([1], pag. 225), and that $\dim.(\mathfrak{P})+a-b\leq \leq\dim.(\mathfrak{P}')\leq\dim.(\mathfrak{P}_1)+a-b$, when $\mathfrak{P}^{*\prime}\subset\mathfrak{P}^*$. In § 1 of the present paper we show that if $\mathfrak{P}$ is fundamental and the component $\mathfrak{P}'$ of its transform is not fundamental for the inverse correspondence, it is verified that

$$\dim.(\mathfrak{P}')>\dim.(\mathfrak{P})+a-b. \qquad [I]$$

This theorem gives a characteristic propriety of the fundamental subvarieties when $\mathfrak{P}^*=\mathfrak{P}^{*\prime}$ and only in this case.

If $b=0$, [1], the condition (I) is a characteristic one for the fundamental subvarieties such that their transform is not fundamental for the inverse correspondence.

We construct in § 2 the fundamental subvariety for an algebraic correspondence, that is, the subvariety that is the topological union of all fundamental subvarieties of V. As a corollary, we show that the dimension of this subvariety is most like the dimensión of V diminished by two unities.

HYPOTHESIS AND NOTATIONS. k is an arbitrary ground field with infinite elements ; $\{x_1, \ldots, x_n\}$ and $\{y_0 \ldots, y_m\}$ are two sequences of indeterminates over k ;

$$A=k[x_0, \ldots, x_n ; y_0, \ldots, y_m] ; \qquad [1]$$

$$I=[F_1(x ; y), \ldots, F_\alpha(x ; y)), \qquad [2]$$

this I is a prime and bihomogeneous ideal ; we shall denote by $\mathfrak{V}$ the variety corresponding to I in the double projective space and by $(\xi_0, \ldots, \xi_n ; \eta_0, \ldots, \eta_m)$ a general point of $\mathfrak{V}$.

(*) This paper was presented to the International Congress of Mathematiciens in the sceance of the september, 4, P. M. 1950.

We shall put, as in [1]:

$$\Omega=k(\xi_0, \ldots, \xi_n;\ \eta_0, \ldots, \eta_m),\ \Sigma=k(\xi_0, \ldots, \xi_n),\ \Sigma'=k(\eta_0, \ldots, \eta_m), \qquad [3]$$
$$P^*=k[\xi_0, \ldots, \xi_n:\ \eta_0, \ldots, \eta_m],\ P=k[\xi_0, \ldots, \xi_n],\ P'=k[\eta_0, \ldots, \eta_m].$$

We shall call $r+1$ and $s+1$ the transcendency degrees of Σ and Σ' over k, respectively, and $a+1$ and $b+1$ those of Ω over Σ and Σ', respectively.

We shall denote by T the algebraic correspondence defined by $\mathfrak{B}$; and by V and V′ the original and image varieties, with the general points $(\xi_0, \ldots, \xi_n)$ and $(\eta_0, \ldots, \eta_m)$, respectively.

Let λ_{ij}, i, $j=0, \ldots, m$, be $(m+1)^2$ indeterminates over Ω and let

$$y_i = \sum_{j=0}^{m} \lambda_{ij}\, y_j^*, \qquad y_j^* = \frac{1}{|\lambda_{ij}|} \sum_{j=0}^{m} \Lambda_{ij}\, y_j, \qquad i = 0, \ldots, m \qquad [4]$$

where Λ_{ij} is the adjoint of λ_{ji} in (λ_{ij}).

Also we shall put:

$$\left.\begin{aligned}
&K = k\,(\lambda_{ij})\\
&A_1 = k\,[\lambda_{00}, \ldots, \lambda_{m,m};\ x_0, \ldots, x_n;\ y_0, \ldots, y_m],\\
&B_1 = k\,[\lambda_{00}, \ldots, \lambda_{m,m};\ x_0, \ldots, x_n;\ y^*_0, \ldots, y^*_m],\\
&C_1 = K[x_0, \ldots, x_n;\ y_0, \ldots, y_m] = K[x_0, \ldots, x_n;\ y^*_0, \ldots, y^*_m],\\
&B_2 = k\,[\lambda_{00}, \ldots, \lambda_{m,m};\ \xi_0, \ldots, \xi_n:\ y^*_0, \ldots, y^*_m],\\
&C_2 = K[\xi_0, \ldots, \xi_n;\ y_0, \ldots, y_m] = K[\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_m],\\
&B_3 = \Sigma\,[\lambda_{00}, \ldots, \lambda_{m,m};\ y^*_0, \ldots, y^*_m],\\
&C_3 = K(\xi)\,[y_0, \ldots, y_m] = K(\xi)\,[y^*_0, \ldots, y^*_m],\\
&B_4 = k\,[\lambda_{00}, \ldots, \lambda_{m,m};\ \xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_{a+1}],\\
&C_4 = K[\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_{a+1}],\\
&B_5 = \Sigma\,[\lambda_{00}, \ldots, \lambda_{m,m};\ y^*_0, \ldots, y^*_{a+1}],\\
&C_5 = K(\xi)\,[y^*_0, \ldots, y^*_{a+1}],\\
&\hat{A} = K(\xi_0, \ldots, \xi_n;\ y^*_0, \ldots, y^*_a)\,[y^*_{a+1}],\\
&D = k\,[\lambda_{00}, \ldots, \lambda_{m,m};\ \xi_0, \ldots, \xi_n],\\
&E = \Sigma\,[\lambda_{00}, \ldots, \lambda_{mm}],\\
&F = k\,[\lambda_{00}, \ldots, \lambda_{mm}],\\
&G = k\,[x_0, \ldots, x_n].
\end{aligned}\right\} \qquad [5]$$

And

$$\left.\begin{aligned}
I_1 &= A_1\, I, \\
J_1 &= B_1\, I = B_1\, I_1, \\
L_1 &= C_1\, I, \\
J_2 &\approx J_1/B_1\,(I \cap G), \\
L_2 &\approx L_1/C_1\,(I \cap G), \\
J_3 &= B_3\, J_2, \\
L_3 &= C_3\, L_2, \\
J_4 &= J_2 \cap B_4, \\
L_4 &= L_2 \cap C_4, \\
J_5 &= B_5\, J_4, \\
L_5 &= C_5\, L_4, \\
\hat{I} &= \hat{A}\, L_5 .
\end{aligned}\right\} \qquad [6]$$

We shall denote by capital German letters the homogeneous and bihomogeneous ideals of P, P′ and P*, and by small German letters the non homogeneous ideals; we shall write $\mathfrak{P}$ and $\mathfrak{p}$ to represent the prime homogeneous and non homogeneous ideals, respectively, and we shall use the same letters to denote a prime ideal and its corresponding subvariety.

§ 1. Transform of a fundamental subvariety.

Let $\mathfrak{P}$ be an irreducible and fundamental subvariety of V ([1], def. 1.10) and $\mathfrak{P}'$ an irreducible component of its transform, then there is a m. p. d. *), $\mathfrak{P}^*$, of $P^*\,\mathfrak{P}$ which lies over $\mathfrak{P}$ and over $\mathfrak{P}'$. We shall denote, as in [1], by $\mathfrak{P}^{*\prime}$ a m. p. d. of $P^*\,\mathfrak{P}'$ which is a proper or improper multiple of $\mathfrak{P}^*$ and we shall let $\mathfrak{P}_1 = \mathfrak{P}^{*\prime} \cap P$.

THEOREM 1. *If $\mathfrak{P}$ is fundamental for T and $\mathfrak{P}'$ is not fundamental for T^{-1}, then*

$$dim.\,(\mathfrak{P}') > dim.\,(\mathfrak{P}) + a - b.$$

Proof. Let Ω, $\bar{\Omega}$, Σ and Σ' be the quotients fields of $P^*/\mathfrak{P}^*$, $P^*/\mathfrak{P}^{*\prime}$, $P/\mathfrak{P}$, $P'/\mathfrak{P}'$, respectively.

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*) We shall denote briefly by m. p. d. the minimal prime divisors.

a) *Let* $\mathfrak{P}^{*\prime}=\mathfrak{P}^{*}$. In this case it is $\dot{\Omega}=\ddot{\Omega}$ and, as $\mathfrak{P}'$ is not fundamental for T^{-1}, it result (def. 1.10, [1]) that trans. degr. $(\dot{\Omega}:\dot{\Sigma}')=b+1$. As $\mathfrak{P}$ is fundamental for T, it is trans. degr. $(\dot{\Omega}:\dot{\Sigma})=\alpha+1>a+1$. If we denote by $\dot{k}$ the ground field of Ω, it results that trans. degr. $(\dot{\Omega}:\dot{k})=$trans. degr. $(\Omega:\Sigma)+$trans. degr. $(\dot{\Sigma}:\dot{k})=\alpha+1+\text{dim.}(\mathfrak{P})+1$, and trans. degr. $(\dot{\Omega}:\dot{k})=$ $=$trans. degr. $(\dot{\Omega}:\dot{\Sigma}')+$trans. degr. $(\dot{\Sigma}':\dot{k})=b+1+\text{dim.}(\mathfrak{P}')+1$.

Hence $\text{dim.}(\mathfrak{P}')=\text{dim.}(\mathfrak{P})+\alpha-b>\text{dim.}(\mathfrak{P})+a-b$.

b) *If* $\mathfrak{P}^{*\prime}\subset\mathfrak{P}^{*}$, we have $\text{dim.}(\mathfrak{P}^{*\prime})>\text{dim.}(\mathfrak{P}^{*})$, or also, trans. degr. $(\ddot{\Omega}:\dot{\Sigma}')=b+1>$trans. degr. $(\dot{\Omega}:\dot{\Sigma}')=\beta+1$, hence, (7) $\text{dim.}(\mathfrak{P}')=\text{dim.}(\mathfrak{P})+\alpha-\beta>\text{dim.}(\mathfrak{P})+a-\beta>\text{dim.}(\mathfrak{P})+a-b$, q. e. d.

Observation. If $\mathfrak{P}$ were regular we should have $\text{dim.}(\mathfrak{P}')=$ $=\text{dim.}(\mathfrak{P})+a-\beta$, hence the first inequality (7) shows that *if* $\mathfrak{P}$ *is fundamental and* $\mathfrak{P}'$ *is not for the inverse correspondence, then the dimension of* $\mathfrak{P}'$ *is greater than the dimension that it would have if* $\mathfrak{P}$ *were regular with relation to* $\mathfrak{P}'$.

The following lemme comprise the Th. 3.12 [1].

LEMME 1. *If* b=0 *and* $\mathfrak{P}'$ *is not fundamental for the inverse correspondence it is verified that* $\mathfrak{P}^{*\prime}=\mathfrak{P}^{*}$.

Proof. Employing the foregoing notations, we obtain, of the relations $\mathfrak{P}^{*\prime}\subseteq\mathfrak{P}^{*}$ and $\mathfrak{P}^{*\prime}\cap P'=\mathfrak{P}^{*}\cap P'=\mathfrak{P}'$, that trans. degr. $(\ddot{\Omega}:\dot{\Sigma}')\geqslant$trans. degr. $(\dot{\Omega}:\dot{\Sigma}')$, but, as $b=0$ and $\mathfrak{P}'$ is not fundamental, it is obtained (def. 1.10, [1]) that trans. degr. $(\ddot{\Omega}:\dot{\Sigma}')=1$ and as $\dot{\Omega}$ is a homogeneous field with relation to $\dot{\Sigma}'$, trans. degr. $(\dot{\Omega}:\dot{\Sigma}')\geqslant 1$, hence trans. degr. $(\ddot{\Omega}:\dot{\Sigma}')=$trans.degr. $(\dot{\Omega}:\dot{\Sigma}')=1$, and $\text{dim.}(\mathfrak{P}^{*\prime})=$trans. degr. $(\ddot{\Omega}:\dot{k})=$trans. degr. $(\ddot{\Omega}:\dot{\Sigma}')+$trans. degr. $(\dot{\Sigma}':\dot{k})=$trans. degr. $(\dot{\Omega}:\dot{\Sigma}')+$trans. degr. $(\dot{\Sigma}':\dot{k})=$trns. degr. $(\dot{\Omega}:\dot{k})=\text{dim.}(\mathfrak{P}^{*})$, *q. e. d.*

THEOREM 2. *If in the algebraic correspondence* T *is* b=0 *and there are no fundamental subvarieties in* V' *for* T^{-1}, *a necessary and sufficient condition for a subvariety,* $\mathfrak{P}$, *of* V *to be fundamental for* T *is that*

$$\mathit{dim.}(\mathfrak{P}')>\mathit{dim.}(\mathfrak{P})+\text{a}.$$

The necessarity has been shown in the Th. 1 and the sufficiency is a consequence of the foregoing Lemme and the Th. 2.12 [1].

§ 2. Fundamental subvariety for an algebraic correspondence.

Of the definitions (5) follows inmediately the relations:

1) $A_1 \subset B_1$.

2) $C_1 = A_{1_F} = B_{1_F}$.

3) $B_2 \cong B_1/B_1(I \cap G)$.

4) $C_2 \cong C_1/C_1(I \cap G)$, $C_2 = P_{2_F}$

5) $B_3 = B_{2_P}$.

6) $C_3 = C_{2_D} = B_{2_D} = B_{3_D}$.

7) $B_5 = B_{4_P}$.

8) $C_5 = C_{4_D}$.

9) $\hat{A} = C_{5_{K(\xi)[y^*_0, \dots, y^*_a]}} = C_{5_{k[\lambda;\, \xi;\, y^*_0, \dots, y^*_a]}}$

From these relations and from the (5) and (6) follows the propositions:

10) I_1 is a prime ideal and $I_1 \cap A = I$.

11) J_1 is a prime ideal and $J_1 \cap A = I$.

12) $L_1 = C_1 I_1 = C_1 J_1$ is a prime ideal and, since there are not polynomials of F contained in I_1 nor J_1, it is verified that $L_1 \cap A_1 = I_1$, $L_1 \cap B_1 = J_1$.

13) J_2 and L_2 are prime ideals and $L_2 = C_2 J_2$, $J_2 = L_2 \cap B_2$.

14) *There are not any polynomial of D in J_2, L_2, J_3, L_3, J_4, L_4.* If it had a polynomial, $f(\xi ; \lambda)$, so that $f \equiv O(J_2)$ a polynomial, $F(x ; \lambda ; y^*)$, would exist in J_1 so that $F(\xi ; \lambda ; y^*) = f(\xi ; \lambda)$, hence, $F(x ; \lambda ; y^*) = f(x ; \lambda) + F_1(x ; \lambda ; y^*)$, where $F_1(\xi ; \lambda ; y^*) = 0$ and hence $F_1(x ; \lambda ; y^*) \equiv 0(B_1(I \cap G))$ and, a fortiori, $F_1 \equiv 0(J_1)$ and as $F \equiv 0(J_1)$, it follows that $f(x ; \lambda) \equiv 0(J_1)$, hence $f(x ; \lambda) \equiv 0(k[x ; y](I \cap G))$, this is, $f(\xi ; \lambda) = 0$. An analogous proof can be employed for L_2. As $J_3 = B_3 J_2 = B_{2_P} J_2$ and no polynomial of P is contained in J_2, it follows that $J_2 = J_3 \cap B_2$, hence there is not a polynomial of D contained in J_3; as $L_3 \cap C_2 = L_2$, it follows that no polinomial of D is contained in

L_3. From $J_4 = J_2 \cap B_4$ and $L_4 = L_2 \cap C_4$ it follows that there is no polynomial of D in J_4 nor L_4.

15) J_4 and L_4 are prime ideals.

16) Of 14) it follows that J_3 and L_3 are prime ideals and $J_2 = J_3 \cap B_2$, $L_2 = L_3 \cap C_2$.

17) *Dim.* $(L_3) = a + 1$. Since the dimension of L_1 with respect to K is like that of I with respect k, and $C_2/L_2 \cong C_1/C_1(I \cap G)/L_1/C_1(I \cap G) \cong C_1/L_1$, it follows that dim. $(L_2) =$ $=$ dim. $(L_1) = q + 2$. Since dim. $(L_3) =$ trans. degr. $(C_3/L_3 : K(\xi))$, and from 14) follows that the quotients field of C_2/L_2 is like the quotients field of C_3/L_3, it follows that dim. $(L_3) =$ $=$ trans. degr. $(C_2/L_2 : K(\xi)) =$ dim. $(L_2) -$ trans. degr. $(K(\xi) :$ $: K) = q + 2 - (r + 1) = a + 1$.

Hypothesis. According to 17), we shall suppose that y^*_0, ..., y^*_a are algebraically independent mod. (L_3).

18) From 6), (6) and 13) follows: $L_3 = C_3 L_2 = C_3 (C_2 J_2) =$ $= C_3 J_2 = B_{3_D}(J_2) = B_{3_D}(B_3 J_2) = B_{3_D} J_3 = C_3 J_3$. and, hence y^*_0, ..., y^*_a are algebraically independent mod. (J_3).

19) $L_5 = C_5 J_4 = C_5 J_5$. From 13) follows $L_2 = C_2 J_2$, hence $L_4 = C_2 J_2 \cap C_4$ and $L_5 = C_5 L_4 = C_5 (C_2 J_2 \cap C_4) = C_5 (C_2 J_2 \cap B_4) =$ $= C_5 (B_2 J_2 \cap B_4) = C_5 (J_2 \cap B_4) = C_5 J_4 = C_5 (B_5 J_4) = C_5 J_5$.

20) From 14) and 19) follows $J_5 = L_5 \cap B_5$.

21) Dim. $(J_4) =$ dim. $(J_2) =$ dim. $(L_2) +$ trans degr. $(K : k) =$ $= q + 2 + (m + 1)^2$. And dim. $(L_4) = r + a + 2$.

22) From (6) and 14) follows dim. $(J_5) =$ dim. $(J_4) -$ trans. degr. $(\Sigma : k) = q + 2 + (m + 1)^2 - (r + 1) = a + 1 + (m + 1)^2$.

23) From (6), 14) and 21) follows dim. $(L_5) =$ dim. $(L_4) -$ trans. degr. $(K(\xi) : K) = a + 1$.

24) Since $L_5 = C_5 L_4 = C_5(L_2 \cap C_4) = C_5(L_3 \cap C_2 \cap C_4) = C_5(L_3 \cap C_4)$, the y^*_0, ..., y^*_a are alg. indep. mod. (L_5), hence I is a prime ideal and $\hat{I} \cap C_5 = L_5$. This and 23) give dim. $(\hat{I}) =$ dim. $(L_5) - (a + 1) = 0$.

25) Since $\hat{A}$ is a polinomial ring with only one indeterminate and coefficients of a field, $K(\xi; y^*_0, \ldots, y^*_a)$, it is an euclidean ring and hence the ideal I is principal: $\hat{I} = (\Psi(\xi; \lambda; y^*_0, \ldots, y^*_{a+1})$

where Ψ is an irreducible polynomial of A, which can be supposed as belonging to B_4 and irreducible in B_5, and in the following we shall suppose that Ψ satisfies these conditions.

26) *The polynomial Ψ has the following form:*

$$\Psi(\xi; \lambda; y^*_0, \ldots, y^*_{a+1}) = H(\xi, \lambda) y^{*\rho}_{a+1} + a_1(\xi; \lambda; y^*_0, \ldots, y^*_a) y^{*\rho-1}_{a+1} +$$
$$+ \ldots + a_\rho(\xi; \lambda; y^*_0, \ldots, y^*_a)$$

where H does not contain any one (y^*) *and Ψ can be taken as homogeneous with respect to the* (ξ) *and the* ($y^*_0, \ldots, y^*_{a+1}$).

Since I is bihomogeneous with relation to the (x) and the (y) so is $L_1 = C_1 I$ too with relation to the (x) and (y^*), $L_2 \approx L_1/C_1(I \cap G)$ is so with respect to the (ξ) and the (y^*) and $L_4 = L_2 \cap C_4$ will be so with respect to the (ξ) and ($y^*_0, \ldots, y^*_{a+1}$). Since the ($y^*_0, \ldots y^*_a$) are algebraically independent mod. (L_5) it follows that $L_5 = \hat{I} \cap C_5$ and hence Ψ is supposed to be irreducible in C_5, the ideal $C_5(\Psi)$ is a prime one and, because of 23), it follows that

$$L_5 = C_5(\Psi).$$

From 14) and (6) it follows that $L_4 = L_5 \cap C_4$, and therefore $C_4(\Psi) \subseteq L_4$; if Ψ were not bihomogeneous with respect to the (ξ) and ($y^*_0, \ldots, y^*_{a+1}$), as L_4 is bihomogeneous, every bihomogeneous component of Ψ would belong to L_5 and as the ($y^*_0, \ldots, y^*_{a+1}$) are indeterminates over K(ξ), it follows that Ψ is homogeneous with relation to the (y^*). If $\Psi = \Psi_{\alpha_1 . \beta} + \ldots + \Psi_{\alpha_s . \beta}$, where $\Psi_{\alpha_i . \beta}$ is a bihomogeneous form with respec to the (ξ) and the (y^*) of degrees α_i and β, respectively, it would be $\Psi_{\alpha_i . \beta} = 0 \, (L_5)$ and $\Psi_{\alpha_i . \beta} = \dfrac{f(\xi; \lambda)}{g(\xi; \lambda)} \Psi$, therefore $\Psi \equiv 0(C_5(\Psi_{\alpha_i . \beta}))$ and $C_5(\Psi_{\alpha_i . \beta}) = C_5(\Psi)$. Hence, *in the following we shall suppose that Ψ is bihomogeneous with respect to the* (ξ) *and* (y^*).

If $H(\xi; \lambda)$ contained some of the ($y^*_0, \ldots, y^*_a$), we should make the substitution $y^*_i = \sum_{j=0}^{a+1} \bar{\lambda}_{ij} y^{**}_j$, $y^{**}_{a+1} = y^*_{a+1}$, $i = 0, \ldots, a$, and we should get

$$[H(\xi; \lambda; \bar{\lambda}_{i, a+1}) + a_1(\xi; \lambda; \bar{\lambda}_{0, a+1}, \ldots, \bar{\lambda}_{0, a+1}) + \ldots +$$
$$+ a_\rho(\xi; \lambda; \bar{\lambda}_{0, a+1}, \ldots, \bar{\lambda}_{a, a+1})] y^{*\mu}_{a+1} + \ldots \qquad [8]$$

where μ is the degree of Ψ with respect to the $(y^*{}_0, \ldots, y^*{}_{a+1})$. As the $\bar{\lambda}_{i,a+1}$ are parameters and the degree of H with respect to them is $\mu-\rho$ and that of a_i is $\mu-\rho+i$, if the coefficient of $y^{**\mu}_{a+1}$ in (8) were zero, H and a_i, $i=1, \ldots, \rho$ would be so too and therefore Ψ.

27) $L_4 = B_{4_F} J_4 = C_4 J_4$. From (6) it follows that $L_1 = B_{1_F} J_1$ and from 13) that $L_2 = B_{2_F} J_2$. Because of 14) $L_4 = B_{2_F} J_2 \cap B_{4_F} = B_{4_F}(B_{2_F} J_2 \cap B_4) = B_{4_F}[(B_{2_F} J_2 \cap B_2) \cap B_4] = B_{4_F}(J_2 \cap B_4) = B_{4_F} J_4 = = C_4 J_4$.

28) $J_5 = B_5(\Psi)$. Effectively, from 20) and 26) it follows that $B_5(\Psi) \equiv 0(J_5)$. Since Ψ is irreducible in B_5 and this is a polynomial ring, the ideal $B_5(\Psi)$ is a prime one and, because of 22), it follows that $J_5 = B_5(\Psi)$.

LEMME 2. *If $\mathfrak{P}$ is a prime and homogeneous ideal of P and $D\mathfrak{P}$ is divisor of $D(H(\xi\,;\lambda))$, then the ideal $B_4\mathfrak{P}$ is a divisor of $B_4(\Psi)$.*

Proof. Let

$$(y^*_0, \ldots, y^*_{a+1})' = R_1(y^{**}_0, \ldots, y^{**}_{a+1})', \quad R_1 =$$

$$= \begin{pmatrix} \rho_{00}, & \rho_{01}, & \ldots, & \rho_{0\,a+1} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \rho_{a+1,0}, & \rho_{a+1,1}, & \ldots, & \rho_{a+1,a+1} \end{pmatrix} \qquad [9]$$

where the (ρ_{ij}) are indeterminates over A_1,

$$\Lambda = \begin{pmatrix} \lambda_{00}, & \ldots, & \lambda_{0m} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \lambda_{m0}, & \ldots, & \lambda_{mm} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 & 0 \\ 0 & U \end{pmatrix}, \qquad [10]$$

U the unit matrix, and

$$T = \begin{pmatrix} \tau_{00}, & \ldots, & \tau_{0m} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \tau_{m0}, & \ldots, & \tau_{mm} \end{pmatrix} = \Lambda R. \qquad [11]$$

Then

$$(y_0, \ldots, y_m)' = T(y^{**}_0, \ldots, y^{**}_m)', \quad y^{**}_i = y^*_i, \quad i = a+2, \ldots, m \qquad [12]$$

If we employ the substitution (12) instead of that of (4) and we follow the same process that we have followed to obtain Ψ, we shall obtain a polynomial, Ψ^*, which will differ from Ψ in that the first one will have (y^{**}) and (τ) instead of the (y^*) and (λ) of the latter : $\Psi^*(\xi\,;\tau\,;y^{**}{}_0\,\ldots,\ y^{**}{}_{a+1})=\Psi(\xi\,;\ \lambda\,;\ y^*{}_0,\ \ldots,\ y^*{}_{a+1})$. But we can obtain Ψ^* effecting the substitutions $\Lambda=TR^{-1}$ and (9) in $\Psi(\xi\,;\lambda\,;y^*{}_0,\ \ldots,\ y^*{}_{a+1})$; effecting now in Ψ only this latter one we get

$$\begin{aligned}\Psi(\xi;\lambda;y^*_0,\ldots,y^*_{a+1}) &= [H(\xi;\lambda)\,\rho^\rho_{a+1,a+1}+\\ &+a_1(\xi;\lambda;\rho_{0,a+1},\ldots,\rho_{a,a+1})\,\rho^{\rho-1}_{a+1,a+1}+\ldots+\\ &+a_\rho(\xi;\lambda;\rho_{0\,a+1},\ldots,\rho_{a,a+1})]\,y^{**\rho}_{a+1}+\\ &+b_1(\xi;\lambda;\rho;y^{**}_0,\ldots,y^{**}_a)\,y^{**\rho-1}_{a+1}+\ldots\end{aligned}\qquad[13]$$

and executing now in (13) the substitution $\Lambda=T\,R^{-1}$ we get :

$$\begin{aligned}\Psi(\xi;\tau;y^{**}_0,\ldots,y^{**}_{a+1}) &= [H(\xi;\lambda)\,\rho^\rho_{a+1,a+1}+\\ &+a_1(\xi;\lambda;\rho_{0,a+1},\ldots,\rho_{a,a+1})\,\rho^{\rho-1}_{a+1,a+1}+\ldots\\ &+a_\rho(\xi;\lambda;\rho_{0\,a+1},\ldots,\rho_{a,a+1})]\,y^{**\rho}_{a+1}+\ldots\end{aligned}$$

of wich follows

$$\begin{aligned}H(\xi;\tau) = H(\xi;\lambda)\,\rho^\rho_{a+1,a+1}+a_1(\xi;\lambda;\rho_{0,a+1},\ldots,\rho_{a,a+1})\,\rho^{\rho-1}_{a+1,a+1}+\\ \ldots+a_\rho(\xi;\lambda;\rho_{0\,a+1},\ldots,\rho_{a,a+1})\end{aligned}\qquad[14]$$

From the hypothesis follows that $D(H(\xi\,;\lambda))\equiv 0(D\,\mathfrak{P})$, hence $k[\xi\,;\tau](H(\xi\,;\tau))\equiv 0(k[\xi\,;\tau]\,\mathfrak{P})$ and as $k[\xi\,;\tau]\subset k[\xi\,;\lambda\,;\rho]$, it follows that $k[\xi\,;\lambda\,;\rho](H(\xi\,;\tau))\equiv 0\ (k[\xi\,;\lambda\,;\rho]\,\mathfrak{P})$ and as the (λ) and (ρ) are independent indeterminates over P, considering the (6) and foregoing relations it results that, all coefficients on the right hand side of (14) with relation to the (λ) and (ρ) are contained in $\mathfrak{P}$. If the substitution of the $(\rho_{0,a+1},\ldots,\rho_{a+1,a+1})$ by the $y^*{}_0,\ \ldots,\ y^*{}_{a+1}$, respectively, is now executed in this right hand side of (14), $\Psi(\xi;\lambda;y^*_0,\ldots,y^*_{a+1})$ is obtained, hence $B_4\Psi\equiv 0\ (B_4\mathfrak{P})$, q. e. d.

LEMME 3. *If $D\mathfrak{P}$ is a m. p. d. of $D(H(\xi\,;\lambda))$, $\mathfrak{P}$ being a prime and homogeneous ideal of P, the subvariety $\mathfrak{P}$ is not fundamental.*

Proof. If $\mathfrak{P}$ were fundamental and we put $\mathfrak{P}_2 = C_2\mathfrak{P}$, since the (λ) and (y^*) are indeterminates with relation to Σ, the ideal $\mathfrak{P}_2$ would be a prime one and $\mathfrak{P} = \mathfrak{P}_2 \cap P$, hence, if $\xi_i \equiv \bar{\xi}_i(\mathfrak{P})$, $i=0, \ldots, n$, it would be $C_2/\mathfrak{P}_2 \cong K[\bar{\xi}_0, \ldots, \bar{\xi}_n; y^*_0, \ldots, y^*_m]$. Because of $\mathfrak{P}$ being fundamental (def. 1-10 [1]) there is a m. p. d. of $(L_2, \mathfrak{P}_2)$, which we will represent by $\overline{\mathfrak{P}}_2$, so that dim. $(\overline{\mathfrak{P}}_2)-$dim. $(\mathfrak{P}_2) > a+1$ and hence there is no polynomial of $K[\bar{\xi}_0, \ldots, \bar{\xi}_n; y^*_0, \ldots, y^*_{a+1}]$ contained in $\overline{\mathfrak{P}}_2/\mathfrak{P}_2$, but, since $L_2 \equiv 0\ (\mathfrak{P}_2)$ and there are polynomials of $K[\xi_0, \ldots, \xi_n; y^*_0, \ldots, y^*_{a+1}]$ contained in L_2, Ψ among them, it follows that if in all these polynomials the (ξ) are substituted by the $(\bar{\xi})$ they will all vanish, hence all their coefficients with relation to the (λ) and (y^*) are contained in $\mathfrak{P}$, hence $L_4 = L_2 \cap C_4 \equiv 0(C\mathfrak{P})$. As $D(H(\xi; \lambda))$ is a principal ideal, it is unmixed and all their m. p. d. will have dimension r, hence dim.$(D\mathfrak{P}) = r$ and dim.$(C_4\mathfrak{P}) = r+a+2$. Of 21) and $L_4 \equiv 0\ (C_4\mathfrak{P})$ follows, since both are prime ideals, that $L_4 = C_4\mathfrak{P}$, but of 14) follows $L_4 \cap P = (0)$, which is in contradiction with $C_4\mathfrak{P} \cap P = \mathfrak{P}$, q. e. d.

Definition. We shall denote with $\mathfrak{J}$ the radical of $D(H(\xi; \lambda))$ when are neglected all their, eventuals, m. p. d. which can be represented by a base contained in P.

THEOREM 3. *An irreductible subvariety, $\mathfrak{P}$, of V is fundamental for T if and only if*

$$\mathfrak{J} \equiv 0(D\mathfrak{P})$$

Proof. The condition is necessary. We shall suppose that $\mathfrak{P}$ is fundamental and that

$$\mathfrak{J} \not\equiv 0(D\mathfrak{P})$$

Since $\mathfrak{P}$ is fundamental it follows, from Th. 3.10 [1], that $H(\xi; \lambda) \equiv 0(D\mathfrak{P})$. If we put $T = \Lambda$ and $\Lambda = T$ in the equality (11) and too

$$R = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & y^*_0 & \\ 0 & 1 & \cdots & y^*_1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & y^*_{a+1} & \\ \hline & & 0 & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right), \qquad [16]$$

it will be (17) $\Lambda = TR$, and (14) will be

$$H(\xi;\lambda) = H(\xi;\tau)\, y^{*\rho}_{a+1} + a_1(\xi;\tau;y^*_0,\ldots,y^*_a)\, y^{*\rho-1}_{a+1} + \cdots$$
$$+ a_\rho(\xi;\lambda;y^*_0,\ldots,y^*_a) \qquad [18]$$

Let $\overline{B}_4 = k[\tau_{00},\ldots,\tau_{mm};\xi_0,\ldots,\xi_n;y^*_0,\ldots,y^*_{a+1}]$; from (17) it follows that (19) $B_4 \subseteq \overline{B}_4$ and as

$$T = \frac{1}{y^*_{a+1}}\Lambda\left(\begin{array}{ccccc|ccc} y^*_{a+1} & 0 & \cdots & (-1)^{a+1} & y^*_0 & & & \\ 0 & y^*_{a+1} & \cdots & (-1)^{a+2} & y^*_0 & & 0 & \\ \cdots & \cdots & \ddots & \cdots & \cdots & & & \\ 0 & 0 & \cdots & (-1)^{2a+2} & & & & \\ \hline & & 0 & & & y^*_{a+1} & \ddots & \\ & & & & & & & y^*_{a+1} \end{array}\right)$$

it follows that (19) $\overline{B}_4 \subset B_{4(y^*_{a+1})}$ and as y^*_{a+1} is not contained in any m. p. d. of $B_4(H(\xi;\lambda))$, if $B_4(H(\xi;\lambda)) = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_l$ is an irreducible descomposition of $B_4(H)$ in primary ideals and $\mathfrak{P}_i$ is the prime ideal corresponding to $\mathfrak{Q}_i$, $i=1,\ldots,l$, it will be $B_{4(y^*_{a+1})}(H(\xi;\lambda)) = B_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \ldots \cap B_{4(y^*_{a+1})}\mathfrak{Q}_l$ a descomposition of $B_{4(y^*_{a+1})}(H)$ in primary ideals and $B_{4(y^*_{a+1})}\mathfrak{P}_i$ the prime ideal corresponding to $B_{4(y^*_{a+1})}\mathfrak{Q}_i$, $i=1,\ldots,l$. From (19) it follows now that (20) $B_{4(y^*_{a+1})}(H(\xi;\lambda)) \cap \overline{B}_4 = (B_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{B}_4 \cap \ldots \cap (B_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{B}_4)$, which is a descomposition of $B_{4(y^*_{a+1})}(H) \cap \overline{B}_4$ in primary ideals and that $B_{4(y^*_{a+1})}\mathfrak{P}_i \cap \overline{B}_4$ will be the prime ideal corresponding to $B_{4(y^*_{a+1})}\mathfrak{Q}_i \cap \overline{B}_4$, $i=1,\ldots,l$.

From (18) and (20) follows $B_{4(y^*_{a+1})}(\Psi(\xi;\tau;y^*)) \cap \overline{B}_4 = (B_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{B}_4) \cap \ldots \cap (B_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{B}_4)$, and as $B_{4(y^*_{a+1})} = \overline{B}_{4(y^*_{a+1})}$ and $y^*_{a+1} \not\equiv 0\,(\overline{B}_4(\Psi))$, then $\overline{B}_{4(y^*_{a+1})}(\Psi) \cap \overline{B}_4 = \overline{B}_4(\Psi)$, hence $\overline{B}_4(\Psi) = (B_{4(y^*_{a+1})}\mathfrak{Q}_1 \cap \overline{B}_4) \cap \ldots \cap (B_{4(y^*_{a+1})}\mathfrak{Q}_l \cap \overline{B}_4)$.

Since $D(H(\xi;\lambda)) \equiv 0\,(D\mathfrak{P})$ and $\mathfrak{J} \not\equiv 0\,(D\mathfrak{P})$, $D\mathfrak{P}$ divide some m. p. d. of D(H) which has a basis belonging to P and does not divide to some m. p. d. of D(H) which does not posses this propriety. We shall say that the ideal. $\overline{B}_4\mathfrak{P}$ has the «property p» with relation to $\overline{B}_4(\Psi)$ if and only if $\overline{B}_4\mathfrak{P}$ divide to one m. p. d. of $\overline{B}_4(\Psi)$ which is not an extension to $\overline{B}_4$ of a prime ideal of P. Hence DP does not posses the «property P» with relation to D(H).

As $H(\xi ; \lambda)$ is transformed by (17) into Ψ and taking into account the foregoing descomposition of $\overline{B}_4(\Psi)$, it follows that $\overline{B}_4 \mathfrak{P}$ cannot have the propriety p with respect to $B_4(\Psi)$. But J_4 is a m. p. d. of $\overline{B}_4(\Psi)$ with the propriety p, hence $J_4 \not\equiv 0 \, (B_4 \mathfrak{P})$. From this relation it follows that there is a polynomial, $g(\xi ; \lambda ; y^*)$ belonging to J_4, which does not belong to $B_4 \mathfrak{P}$; executing, if it is necessary, a linear transformation, as we have done in 26) it is possible to suppose that g has the following form :

$$g(\xi; \lambda; y^*_0, \dots y^*_{a+1}) = g_0(\xi: \lambda)\, y^{*\mu}_{a+1} + g_1(\xi; \lambda; y^*_0, \dots, y^*_a)\, y^{*\mu-1}_{a+1} + \\ \dots + g_\mu(\xi: \lambda; y^*_0, \dots, y^*_a)$$

in which $g_0(\xi ; \lambda) \not\equiv 0 \, (D \mathfrak{P})$ and as $g(\xi ; \lambda ; \eta^*_0, \dots, \eta^*_{a+1}) = 0$ (where η^*_i is obtained substituting in y^*_i the (y) by the (η) $i = 0, \dots, m$), because $g \equiv 0 \, (J_4)$, it follows, by the Th. 3.10 [1], that $\mathfrak{P}$ is not fundamental, which is in contradiction with our hypothesis.

The condition is sufficient. We shall suppose that $\mathfrak{J} \equiv 0$ $(D \mathfrak{P})$ and that $\mathfrak{P}$ is not fundamental. As in the theorem 2.10 [1], it would be possible find $m\text{-}a$ forms of I, $F_1(x ; y), \dots, F_{m-a}(x ; y)$ so that the ideals $\Sigma[y]\,(F_1(\xi ; y) \dots, F_{m-a}(\xi ; y))$ and $\tilde{\Sigma}[y]\,(F_1(\bar{\xi} ; y), \dots, F_{m-a}(\bar{\xi} ; y))$, (where $\xi_i \equiv \bar{\xi}_i(\mathfrak{P})$, $i = 0, \dots, n$), would have the same dimension : $a+1$. By means of the substitution $y_i = \sum\limits_{j=0}^{m} \lambda_{ij} y_j^*$, and eliminating the indeterminates $y^*_{a+2}, \dots, y^*_m$ between the equations $F_1(\xi ; \Sigma \lambda_{ij} y^*_j) = 0, \dots, F_{m-a}(\xi ; \Sigma \lambda_{ij} y^*_j) = 0$, we get (1.*c.*) the resultant

$$\Psi_0(\xi; \lambda; y^*_0, \dots, y^*_{a+1}) = A(\xi; \lambda)\, y^{*\rho}_{a+1} + \dots + \bar{a}_\rho(\xi; \lambda; y^*_0, \dots, y^*_a)$$

As there are infinite elements in k, there are infinite specializations of the (λ) over elements (μ) of k such that $A(\xi ; \mu) \not\equiv 0 \, (\mathfrak{P})$, hence,

$$A(\xi ; \lambda) \not\equiv 0 \, (D \mathfrak{P}). \qquad [21]$$

But as $\Psi_0 \equiv 0 \, (J_5)$, from 28) it follows that

$$\Psi_0 = \frac{f(\xi; \lambda; y^*_0, \dots, y^*_{a+1})}{g(\xi)} \Psi(\xi; \lambda; y^*_0, \dots, y^*_{a+1})$$

(of which $g(\xi)$ $A(\xi\,;\,\lambda)\equiv 0$ $(D(H\xi\,;\,\lambda))$ and hence $g(\xi)$ $A(\xi\,;\,\lambda)\equiv 0$ $(\mathfrak{F})$.

We are going to show that *if* $\mathfrak{P}_1$ *is a m. p. d. of* $(H(\xi\,;\,\lambda))$ *which has not the propriety* p *there is no polynomial of* P *contained in* $\mathfrak{P}_1$. Certainly, as (H) is an unmixed ideal, then it is a principal one, and it follows that $\dim(\mathfrak{P}_1)=$ trans. degr. $(D:\,:k)-1=(m+1)^2+r$. If $f(\xi)$ were a polynomial so that $f(\xi)\in P$ and $f(\xi)\equiv 0$ $(\mathfrak{P}_1)$, $\mathfrak{P}_1$ would be a divisor of $D(f)$ and as this ideal is also unmixed and its dimension is $(m+1)^2+r$, $\mathfrak{P}_1$ would be a m. p. d. of $D(f)$; but all the m. p. d. of $D(f)$ are the extensions to D of the m. p. d. of $P(f)$, hence they all have the propriety p in contradiction with the hipothesis.

From this last proposition and of the foregoing follows

$$A(\xi\,;\,\lambda)\equiv 0\;(\mathfrak{F})$$

and by the hypothesis,

$$A(\xi\,;\,\lambda)\equiv 0\;(D\mathfrak{P}),$$

in contradiction with (21), q. e. d.

From the foregoing theorem and from the lemme 3 follows inmediately:

COROLLARY. *The dimension of the fundamental subvariety of* V *is not greater then dim.* (V)-2.

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June, 1950
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