New unexpected limit operators for homogenizing optimal control parabolic problems with dynamic reaction flow on the boundary of critically scaled particles

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Abstract

We study the asymptotic behavior, as $\varepsilon \to 0$, of the optimal control and the optimal state of an initial boundary value problem in a domain that is ε-periodically perforated by balls (or, equivalently it is the complementary to a set of spherical particles). On the boundary of the perforations (or of the particles) we assume a dynamic condition with a large growth coefficient in the time derivative. The control region is a possible small subregion and the cost functional includes a balance between the prize of the controls and the error with respect to a given target profile. We consider the so-called "critical case" concerning a certain relation between the structure's period, the diameter of the balls, and the growth coefficient of the boundary condition. We show that the homogenized problem contains in the limit state equation a nonlocal "strange term", given as a solution to a suitable ordinary differential equation. We prove the weak convergence of the state and the optimal control to the state and the optimal control associated with the limit cost functional which now contains an unexpected new "strange" term.

Keywords Homogenization, Critical case, optimal control, «Strange» term, Dynamic boundary condition, homogenized cost functional.
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1 Introduction

It is well-known that important problems of Chemical Engineering lead to the optimization of some cost functionals (see, e.g. [34] and the survey [26]). Here, we will consider

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the optimal control problem associated with the distributed case in which the chemical reactor consists of a fixed bed and a dynamic reaction flow on the boundary of the particles. The problem also arises in other frameworks linked to porous media in which the word "particle" must be replaced by "perforation" (see, e.g. [21], [20], [8], [28], [19], [22], [32], [1] and the many other references quoted in the monograph [9]). The state equation of our control problem is the parabolic problem with dynamic boundary conditions

$$\left\{\begin{array}{ll} \partial_t u_\varepsilon(v) - \Delta u_\varepsilon(v) = f + \chi_{\omega_\varepsilon} v, & (x,t) \in Q_\varepsilon^T, \\ \varepsilon^{-\gamma}\partial_t u_\varepsilon(v) + \partial_\nu u_\varepsilon(v) = 0, & (x,t) \in S_\varepsilon^T, \\ u_\varepsilon(v)(x,0) = 0, & x \in \Omega_\varepsilon, \\ u_\varepsilon(v)(x,0) = 0, & x \in S_\varepsilon, \\ u_\varepsilon(v)(x,t) = 0, & (x,t) \in \Gamma^T, \end{array}\right. \tag{1}$$

where we are using the notation (considering $0 < T < \infty$)

$$\begin{array}{cccc} \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, & S_\varepsilon = \partial G_\varepsilon, & \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega, & \\ Q_\varepsilon^T = \Omega_\varepsilon \times (0,T), & \Gamma^T = \partial\Omega \times (0,T), & S_\varepsilon^T = S_\varepsilon \times (0,T), & Q^T = \Omega \times (0,T), \end{array} \tag{2}$$

which will be detailed in the next section, G_ε is the set of small particles (ε-periodically distributed and homothetic to a ball) in an open bounded regular set Ω of $\mathbb{R}^n$, $n \geq 3$, $f \in L^2(Q^T)$, and the control $v \in L^2(\omega_\varepsilon^T)$ is acting on a possibly small open part ω_ε of Ω_ε (here, $\chi_{\omega_\varepsilon}$ denotes the characteristic function of ω_ε): i.e. we introduce an open domain ω such that $\overline{\omega} \subset \Omega$, and then we consider the sets

$$\omega_\varepsilon = \omega \cap \Omega_\varepsilon, \quad \omega_\varepsilon^T = \omega_\varepsilon \times (0,T), \quad \omega^T = \omega \times (0,T).$$

The parameter $\gamma > 0$ plays crucial role since in this paper we will consider the so-called "critical case" in which each particle is a translation of a small particle $a_\varepsilon G_0$, where G_0 is the unit ball and $a_\varepsilon = C_0 \varepsilon^\gamma$, with $\gamma = \frac{n}{n-2}$ and C_0 some positive constant.

Notice, since problem (1) is a linear problem, we can assume without loss of generality that the initial data are zero. We also assume $f \in L^2(Q^T)$ and the control regularity $v \in L^2(\omega_\varepsilon^T)$. The existence of a unique weak solution $u_\varepsilon(v) \in L^2(0,T; H^1(\Omega_\varepsilon, \partial\Omega))$ with $\partial_t u_\varepsilon(v) \in L^2(0,T; L^2(\Omega_\varepsilon))$, $\partial_t u_\varepsilon(v) \in L^2(0,T; L^2(S_\varepsilon))$ can be obtained by the Galerkin approximations (see [11]). The application of the abstract theory for subdifferential of convex functions also leads to some existence results, see Remark 1.

The formulation of the optimal problem ends with the definition of the cost functional $J_\varepsilon : L^2(\omega_\varepsilon^T) \to \mathbb{R}$. We assume to be given a "target" function, a given profile observed at the final time T, $u_T \in H^1(0,T; H_0^1(\Omega)) \bigcap C(\overline{Q^T})$, and we try to optimize a weighted functional making the balance between the "prize" of each control $v \in L^2(\omega_\varepsilon^T)$ and the "error with respect to the target function" at the final time T (over Ω_ε and over S_ε) but in the same time trying to have a spatial gradient of the state $u_\varepsilon(v)(x,t)$ close to the one of the target profile u_T when $t \in (0,T)$ (notice that we know that the trace of u_T

on S_ε is well defined), i.e. we consider the cost functional

$$\begin{aligned}J_\varepsilon(v)=&\frac{1}{2}\int\limits_{Q^T_\varepsilon}|\nabla u_\varepsilon(v)-\nabla u_T|^2dxdt+\frac{1}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon(v)-u_T)^2(x,T)dx\\&+\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}(u_\varepsilon(v)-u_T)^2(x,T)ds+\frac{N}{2}\int\limits_{\omega^T_\varepsilon}v^2dxdt.\end{aligned}\tag{3}$$

Some comments on the application of this optimal control problem to the study of the approximate controllability of solutions of (1) will be given later (see Remark 2). By applying different results in the literature, see [24], [17], [33], [18], it is well known that there exists a unique optimal control $v_\varepsilon\in L^2(\omega^T_\varepsilon)$ such that

$$J_\varepsilon(v_\varepsilon)=\inf_{v\in L^2(\omega^T_\varepsilon)}J_\varepsilon(v).\tag{4}$$

The main goal of this paper is to apply a homogenization process to the above optimal control problem when $\varepsilon\to 0$. As in many other formulations, the kind of homogenized limit problem depends, strongly, on the size of the particles radii $C_0\varepsilon^\alpha$, $C_0>0$ (see, e.g. [32], [1] and the exposition made in [9]). Here, we will consider the critical case in which $\alpha=\gamma=n/(n-2)$. For some different elliptic and parabolic problems, it is well-known that this critical choice leads to the emergence of a new "strange" term (the naming is due to [6]) in the effective partial differential equation (see [21], [6], [23], [35], and the monograph [9]).

In the framework of elliptic equations with a dynamic boundary condition on the boundary of the particles, it was shown that the above-mentioned "strange" term becomes a "non-local" operator obtained by solving a suitable ordinary differential equation (we refer to [10, 13]). In this paper, we will show that a new unexpected term appears in the limit cost functional (in contrast with previous results in the literature for related formulations, e.g. see [30, 27, 29, 15]). Although, the detailed statements of our results will be presented later, we summarize now that we will prove the convergence of the optimal controls $v_\varepsilon\to v_0$ strongly in $L^2(\omega^T)$, the convergence of the corresponding states (extended to Ω) $\widetilde{u}_\varepsilon\rightharpoonup u_0$ weakly in $L^2(0,T;H^1_0(\Omega,\partial\Omega))$ and $\partial_t\widetilde{u}_\varepsilon\rightharpoonup\partial_t u_0$ weakly in $L^2(Q^T)$, with the limit state problem given by

$$\begin{cases}\partial_t u_0-\Delta u_0+\mathcal{A}_n(u_0-\mathcal{B}_nH(u_0)(x,t))=f+\chi_\omega v_0, & (x,t)\in Q^T,\\ u_0(x,0)=0, & x\in\Omega,\\ u_0(x,t)=0, & (x,t)\in\Gamma^T,\end{cases}\tag{5}$$

for suitable constants $\mathcal{A}_n,\mathcal{B}_n$ and a suitable non-local in time operator $H(u_0)$, and that v_0 is the optimal control associated to the limit cost functional $J_0(v)$ (i.e. $J_0(v_0)=\inf\{J_0(v)|v\in L^2(\omega^T)\}$) that is defined by

$$\begin{aligned}J_0(v)=&\frac{1}{2}\|\nabla(u_0(v)-u_T)\|^2_{L^2(Q^T)}+\frac{1}{2}\|u_0(v)(\cdot,T)-u_T(\cdot,T)\|^2_{L^2(\Omega)}+\frac{N}{2}\|v\|^2_{L^2(\omega^T)}+\\&+\frac{C^{n-1}\omega_n}{2}\|(u_T(\cdot,T)-\mathcal{B}_nH(u_0(v))(\cdot,T))\|^2_{L^2(\Omega)}+\frac{\mathcal{A}_n}{2}\|\partial_tH(u_0(v))\|^2_{L^2(Q^T)}.\end{aligned}\tag{6}$$

A new consequence of the critical size of the particles is the presence of the last two terms of $J_0(v)$: this is not expected if we compare what arises in this critical case with similar results obtained for non-critical cases. Some comments on the application of this optimal control limit problem to the study of the approximate controllability of solutions to the problem (5) will be given later (see Remark 6).

In order to get the proof of these convergence results, we will use the extension of Pontryagin's method to the case of distributed problems (following the main ideas introduced in [24]). The delicate question is to get a priori estimates under the presence of suitable balances appearing in this critical case and to identify the limit of several auxiliary expressions. In Section 2, we give the details of the formulation of the direct problem as well as the coupled system arising in terms of the adjoint optimal state p_ε, we will show that the optimal control is given by $v_\varepsilon = -N^{-1}p_\varepsilon\chi_{\omega_\varepsilon}$. The a priori estimates allow to pass to the limit in the couple $(u_\varepsilon, p_\varepsilon)$ (and thus in the controls v_ε) is presented in Section 3. A detailed formulation of the main theorems of this paper is collected in Section 4. For the proofs, we start by characterizing the limit couple (u_0, p_0) of the couple $(u_\varepsilon, p_\varepsilon)$, which is presented in Section 5, and finally, we prove the identification of the limit cost functional $J_0(v)$ in Section 6.

2 Problem formulation and coupled system with the adjoint state

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with a smooth boundary $\partial\Omega$ (the case of $n = 2$ can be considered but requires a different approach: e.g. see Section 4.7.2 of [9] and its references). Let $Y = (-1/2, 1/2)^n$ – a unit cube centered at the coordinate's origin, and $G_0 = \{x | |x| < 1\}$. Define $\delta B = \{x | \delta^{-1}x \in B\}$ for $\delta > 0$. We set, for $\varepsilon > 0$,

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > 2\varepsilon\}.$$

By $\mathbb{Z}^n$, we denote the set of all vectors $z = (z_1, \ldots, z_n)$ with integer coordinates z_i, $i \in \overline{1, n}$. Next, we define the set

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n \,|\, \overline{G_\varepsilon^j} \subset Y_\varepsilon^j = \varepsilon Y + \varepsilon j, G_\varepsilon^j \bigcap \widetilde{\Omega}_\varepsilon \neq \emptyset\}$. We consider $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$, C_0 is a positive constant. Also, note that $|\Upsilon_\varepsilon| = d\varepsilon^{-n}$ for some constant d. Next, the set G_ε is used to define the sets in (2).

We say that the function $u_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$ with $\partial_t u_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$ and $\partial_t u_\varepsilon \in L^2(0, T; L^2(S_\varepsilon))$ is a weak solution to the initial boundary value problem (1) if $u(x, 0) = 0$ for a.e. $x \in \Omega_\varepsilon$ and a.e. $x \in S_\varepsilon$, and it satisfies the integral identity

$$\int\limits_{Q_\varepsilon^T} \partial_t u_\varepsilon \varphi dx dt + \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla \varphi dx dt + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t u_\varepsilon \varphi ds dt = \int\limits_{Q_\varepsilon^T} (f + \chi_{\omega_\varepsilon} v)\varphi dx dt \tag{7}$$

for an arbitrary function $\varphi \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$.

For a given control function $v \in L^2(\omega_\varepsilon^T)$, we will denote by $u_\varepsilon(v)$ a unique weak solution to the initial boundary-value problem (1).The existence and uniqueness of $u_\varepsilon(v)$ was shown in [11] by means of the Galerkin method. In addition, the following a priori estimate is valid

$$\|\partial_t u_\varepsilon(v)\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t u_\varepsilon(v)\|^2_{L^2(S_\varepsilon^T)} + \max_{t\in[0,T]}\|\nabla u_\varepsilon(v)(\cdot,t)\|^2_{L^2(\Omega_\varepsilon)} \leq$$
$$\leq K(\|f\|^2_{L^2(Q^T)} + \|v\|^2_{L^2(\omega_\varepsilon^T)}),$$

here and below, we suppose that K is independent of ε.

Remark 1. *The existence and uniqueness of solutions to the problem (1) can be obtained via the abstract theory of subdifferential of convex functions. Indeed, by an easy adaptation of the proof of Theorem 1.3 of [4], we know that the vectorial operator associated to this system is a self-adjoint operator A_H on the Hilbert space $H = L^2(\Omega_\varepsilon)\times L^2(S_\varepsilon)$ and thus, by Proposition 2.15 of [5], it is the subdifferential of a convex function defined through the square root of A_H. Finally, to take into account the coefficient $\varepsilon^{-\gamma}$ in the dynamic boundary condition, we apply Propositions 3.2 and 3.3 of [31] with $\mathcal{B}$ the matrix*

$$\mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & \varepsilon^{-\gamma} I \end{pmatrix}.$$

As mentioned in the Introduction, along with the problem (1), we consider the cost functional $J_\varepsilon : L^2(\omega_\varepsilon^T) \to \mathbb{R}$ that is defined in (3) for a given $u_T \in H^1(0,T;H_0^1(\Omega)) \bigcap C(\overline{Q^T})$. By well-known results [24], [17], [33], [18], there exists a unique optimal control $v_\varepsilon \in L^2(\omega_\varepsilon^T)$ such that $J_\varepsilon(v_\varepsilon) = \inf\limits_{v\in L^2(\omega_\varepsilon^T)} J_\varepsilon(v)$.

We define the adjoin problem associated to the given state problem (1) and the cost functional (3)

$$\begin{cases} \partial_t p_\varepsilon + \Delta p_\varepsilon = \Delta(u_\varepsilon - u_T), & (x,t)\in Q_\varepsilon^T, \\ \partial_\nu p_\varepsilon - \varepsilon^{-\gamma}\partial_t p_\varepsilon = \partial_\nu(u_\varepsilon - u_T), & (x,t)\in S_\varepsilon^T, \\ p_\varepsilon(x,T) = (u_\varepsilon - u_T)(x,T), & x\in\Omega_\varepsilon, \\ p_\varepsilon(x,T) = (u_\varepsilon - u_T)(x,T), & x\in S_\varepsilon, \\ p_\varepsilon(x,t) = 0, & (x,t)\in\Gamma^T, \end{cases} \tag{8}$$

We say that the function $p_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ with $\partial_t p_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$ and $\partial_t p_\varepsilon \in L^2(0,T;L^2(S_\varepsilon))$ is a weak solution to the problem (8) if $p_\varepsilon(x,T) = (u_\varepsilon - u_T)(x,T)$ for a.e. $x\in\Omega_\varepsilon$ and a.e. $x \in S_\varepsilon$ and if it satisfies the integral identity

$$-\int\limits_{Q_\varepsilon^T}\partial_t p_\varepsilon\varphi dxdt + \int\limits_{Q_\varepsilon^T}\nabla p_\varepsilon\nabla\varphi dxdt - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t p_\varepsilon\varphi dsdt = \int\limits_{Q_\varepsilon^T}\nabla(u_\varepsilon - u_T)\nabla\varphi dxdt, \tag{9}$$

for any test function $\varphi\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$. The following results are a version of the *Pontryagin's principle* for this optimality system.

Theorem 1. *Let the pair of function $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ be an optimal solution of the problem* (4)*, then $v_\varepsilon = -N^{-1}p_\varepsilon\chi_{\omega_\varepsilon}$, where p_ε is the solution to* (8)*. The converse is also true.*

Proof. Let v be an arbitrary function from $L^2(\omega_\varepsilon^T)$ and $\lambda > 0$. By u_ε^λ, we denote the solution of (4) with the control $v_\varepsilon + \lambda v$, i.e. $u_\varepsilon^\lambda = u_\varepsilon(v_\varepsilon + \lambda v)$. We use $u_\varepsilon = u_\varepsilon(v_\varepsilon)$ to simplify the notation.

Indeed, we have

$$\begin{gathered}
J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon) = \frac{1}{2}\|\nabla(u_\varepsilon^\lambda - u_T)\|^2_{L^2(Q_\varepsilon^T)} + \frac{1}{2}\|(u_\varepsilon^\lambda - u_T)(x,T)\|^2_{L^2(\Omega_\varepsilon)} + \\
+\frac{\varepsilon^{-\gamma}}{2}\|(u_\varepsilon^\lambda - u_T)(x,T)\|^2_{L^2(S_\varepsilon)} + \frac{N}{2}\|v_\varepsilon + \lambda v\|^2_{L^2(\omega_\varepsilon^T)} - \\
-\frac{1}{2}\|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q_\varepsilon^T)} - \frac{1}{2}\|(u_\varepsilon - u_T)(x,T)\|^2_{L^2(\Omega_\varepsilon)} - \\
-\frac{\varepsilon^{-\gamma}}{2}\|(u_\varepsilon - u_T)(x,T)\|^2_{L^2(S_\varepsilon)} - \frac{N}{2}\|v_\varepsilon\|^2_{L^2(\omega_\varepsilon^T)} = \\
= \frac{1}{2}\int\limits_{Q_\varepsilon^T} \nabla(u_\varepsilon^\lambda - u_\varepsilon)\nabla(u_\varepsilon^\lambda + u_\varepsilon - 2u_T)dxdt + \frac{1}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon^\lambda - u_\varepsilon)(u_\varepsilon^\lambda + u_\varepsilon - 2u_T)(x,T)dx + \\
+\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}(u_\varepsilon^\lambda - u_\varepsilon)(u_\varepsilon^\lambda + u_\varepsilon - 2u_T)(x,T)ds + \frac{N}{2}\int\limits_{\omega_\varepsilon^T}(2\lambda v_\varepsilon v + \lambda^2 v^2)dxdt
\end{gathered}$$

Next, we define the function $\theta_\varepsilon^\lambda = u_\varepsilon^\lambda - u_\varepsilon$. It is easy to see that this function is a solution to the problem

$$\begin{cases}
\partial_t\theta_\varepsilon^\lambda - \Delta\theta_\varepsilon^\lambda = \lambda v\chi_{w_\varepsilon}, & (x,t) \in Q_\varepsilon^T,\\
\varepsilon^{-\gamma}\partial_t\theta_\varepsilon^\lambda + \partial_\nu\theta_\varepsilon^\lambda = 0, & (x,t) \in S_\varepsilon^T,\\
\theta_\varepsilon^\lambda(x,0) = 0, & x \in \Omega_\varepsilon,\\
\theta_\varepsilon^\lambda(x,0) = 0, & x \in S_\varepsilon,\\
\theta_\varepsilon^\lambda(x,t) = 0, & (x,t) \in \Gamma^T,
\end{cases}$$

The solution to this problem satisfies the estimate

$$\begin{gathered}
\|\partial_t\theta_\varepsilon^\lambda\|_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t\theta_\varepsilon^\lambda\|_{L^2(S_\varepsilon^T)} + \\
+ \max_{t\in[0,T]}\|\nabla\theta_\varepsilon^\lambda(\cdot,t)\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\max_{t\in[0,T]}\|\theta_\varepsilon^\lambda(\cdot,t)\|_{L^2(S_\varepsilon)} \leq K|\lambda|\|v\|_{L^2(\omega_\varepsilon^T)}.
\end{gathered}$$

Thus, we have the following convergences as $\lambda \to 0$

$$\begin{gathered}
u_\varepsilon^\lambda \to u_\varepsilon \text{ in } L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega)), \quad \partial_t u_\varepsilon^\lambda \to \partial_t u_\varepsilon \text{ in } L^2(0,T;L^2(\Omega_\varepsilon)),\\
\partial_t u_\varepsilon^\lambda \to \partial_t u_\varepsilon \text{ in } L^2(0,T;L^2(S_\varepsilon)), \quad u_\varepsilon^\lambda(\cdot,T) \to u_\varepsilon(\cdot,T) \text{ in } L^2(\Omega_\varepsilon),\\
u_\varepsilon^\lambda(\cdot,T) \to u_\varepsilon(\cdot,T) \text{ in } L^2(S_\varepsilon).
\end{gathered}$$

We define the function $\theta_\varepsilon = \theta_\varepsilon^\lambda/\lambda$ that is a solution to the following problem

$$\begin{cases} \partial_t\theta_\varepsilon - \Delta\theta_\varepsilon = v\chi_{\omega_\varepsilon}, & (x,t) \in Q_\varepsilon^T, \\ \varepsilon^{-\gamma}\partial_t\theta_\varepsilon + \partial_\nu\theta_\varepsilon = 0, & (x,t) \in S_\varepsilon^T, \\ \theta_\varepsilon(x,0) = 0, & x \in \Omega_\varepsilon, \\ \theta_\varepsilon(x,0) = 0, & x \in S_\varepsilon, \\ \theta_\varepsilon(x,t) = 0, & (x,t) \in \Gamma^T. \end{cases}$$

Now, as usual, we divide $J_\varepsilon(v_\varepsilon+\lambda v) - J_\varepsilon(v_\varepsilon)$ by λ and pass to the limit as $\lambda \to 0$. Thus, we get

$$J_\varepsilon'(v_\varepsilon)v = \lim_{\lambda\to 0}(J_\varepsilon(v_\varepsilon+\lambda v) - J_\varepsilon(v_\varepsilon))/\lambda =$$
$$= \int\limits_{Q_\varepsilon^T} \nabla\theta_\varepsilon\nabla(u_\varepsilon - u_T)dxdt + \int\limits_{\Omega_\varepsilon} \theta_\varepsilon(x,T)(u_\varepsilon - u_T)(x,T)dx +$$
$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \theta_\varepsilon(x,T)(u_\varepsilon - u_T)(x,T)ds + N\int\limits_{\omega_\varepsilon^T} v_\varepsilon v dxdt$$

Using the definition of the function p_ε, we transform the last expression into

$$J_\varepsilon'(v_\varepsilon)v = \int\limits_{\omega_\varepsilon^T} p_\varepsilon v dxdt + N\int\limits_{\omega_\varepsilon^T} v_\varepsilon v dxdt.$$

The function v_ε is the optimal control, hence, $J_\varepsilon'(v_\varepsilon)v = 0$ for an arbitrary function $v \in L^2(\omega_\varepsilon^T)$. This implies that $v_\varepsilon = -N^{-1}p_\varepsilon\chi_{\omega_\varepsilon}$.

□

According to the above theorem, the optimality conditions for $(u_\varepsilon, p_\varepsilon)$ lead to the study of the coupled system

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = f - N^{-1}p_\varepsilon\chi_{\omega_\varepsilon}, & (x,t) \in Q_\varepsilon^T, \\ \partial_t p_\varepsilon + \Delta p_\varepsilon = \Delta(u_\varepsilon - u_T), & (x,t) \in Q_\varepsilon^T, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\partial_t u_\varepsilon = 0, & (x,t) \in S_\varepsilon^T, \\ \partial_\nu p_\varepsilon - \varepsilon^{-\gamma}\partial_t p_\varepsilon = \partial_\nu(u_\varepsilon - u_T), & (x,t) \in S_\varepsilon^T, \\ u_\varepsilon(x,0) = 0, & x \in \Omega_\varepsilon, \\ u_\varepsilon(x,0) = 0, & x \in S_\varepsilon, \\ p_\varepsilon(x,T) = (u_\varepsilon - u_T)(x,T), & x \in \Omega_\varepsilon, \\ p_\varepsilon(x,T) = (u_\varepsilon - u_T)(x,T), & x \in S_\varepsilon, \\ u_\varepsilon(x,t) = p_\varepsilon(x,t) = 0, & (x,t) \in \Gamma^T. \end{cases} \tag{10}$$

Remark 2. *By following some arguments introduced by J.-L. Lions in [25] (see also, e.g., Section 1.6 in the book [18]), it is possible to use the study of a small adaptation of the above optimal control problem to prove the "approximate controllability property" for solutions of problem (1), provided we know a result on "unique continuation" for this problem. Given* $u_T \in H^1(0,T;H_0^1(\Omega))\bigcap C(\overline{Q^T})$ *and an arbitrarily small* $\delta > 0$, *the* *"approximate controllability property" consists in finding a control $v_\delta \in L^2(\omega_\varepsilon^T)$ such that $\|u_\varepsilon(\cdot,T) - u_T(\cdot,T)\|_{L^2(\Omega_\varepsilon)} \leq \delta$ and $\|u_\varepsilon(\cdot,T) - u_T(\cdot,T)\|_{L^2(S_\varepsilon)} \leq \delta$. By using some a priori estimates on the adjoint state $p_\varepsilon(x,t)$, it can be shown that if we introduce a new parameter $\kappa > 0$ in the cost functional*

$$J_\varepsilon^\kappa(v) = \frac{\kappa}{2}\int\limits_{Q_\varepsilon^T} |\nabla u_\varepsilon(v) - \nabla u_T|^2 dxdt + \frac{\kappa}{2}\int\limits_{\Omega_\varepsilon} (u_\varepsilon(v) - u_T)^2(x,T)dx$$
$$+\frac{\kappa\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon} (u_\varepsilon(v) - u_T)^2(x,T)ds + \frac{N}{2}\int\limits_{\omega_\varepsilon^T} v^2 dxdt,$$

then a such searched control v_δ can be found by considering the set of optimal controls $v_\kappa \in L^2(\omega_\varepsilon^T)$ associated to $J_\varepsilon^\kappa(v)$ and by taking $v_\delta = v_\kappa$ for κ large enough. As a matter of facts, the presence of the gradient difference term (the first term) in $J_\varepsilon^\kappa(v)$ leads to conclude that the associated state will also satisfy the additional property $\|u_\varepsilon - u_T\|_{L^2(Q_\varepsilon^T)} \leq \delta$.

3 A priori estimates on u_ε and p_ε

Taking p_ε as a test function in the integral identity for u_ε, we get

$$\int\limits_{Q_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon dxdt + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon dsdt + \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla p_\varepsilon dxdt = \int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt - \frac{1}{N}\int\limits_{\omega_\varepsilon^T} p_\varepsilon^2 dxdt. \tag{11}$$

Now, taking u_ε as a test function in the integral identity for p_ε, we get

$$-\int\limits_{Q_\varepsilon^T} \partial_t p_\varepsilon u_\varepsilon dxdt - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} \partial_t p_\varepsilon u_\varepsilon dsdt + \int\limits_{Q_\varepsilon^T} \nabla p_\varepsilon \nabla u_\varepsilon dxdt = \int\limits_{Q_\varepsilon^T} |\nabla u_\varepsilon|^2 dxdt - \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla u_T dxdt. \tag{12}$$

Next, we subtract (11) from (12) and obtain the expression

$$-\int\limits_{Q_\varepsilon^T} \partial_t(u_\varepsilon p_\varepsilon) dxdt - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} \partial_t(u_\varepsilon p_\varepsilon) dsdt =$$
$$= \int\limits_{Q_\varepsilon^T} |\nabla u_\varepsilon|^2 dxdt - \int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt - \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla u_T dxdt + \frac{1}{N}\int\limits_{\omega_\varepsilon^T} p_\varepsilon^2 dxdt.$$

Transforming this equality, we get

$$\|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon(\cdot,T)\|^2_{L^2(S_\varepsilon)} + \frac{1}{N}\|p_\varepsilon\|^2_{L^2(\omega_\varepsilon^T)} =$$
$$= \int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt + \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla u_T dxdt + \int\limits_{\Omega_\varepsilon} u_\varepsilon(x,T)u_T(x,T)dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} u_\varepsilon(x,T)u_T(x,T)ds.$$

From here, we immediately derive

$$\|\nabla u_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon(\cdot,T)\|^2_{L^2(S_\varepsilon)} + \|p_\varepsilon\|^2_{L^2(\omega^T_\varepsilon)} \leq$$
$$\leq K(\|f\|_{L^2(Q^T)}\|p_\varepsilon\|_{L^2(Q^T_\varepsilon)} + \|\nabla u_T\|^2_{L^2(Q^T_\varepsilon)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2).$$

Then, we take p_ε as a test function in the integral identity for p_ε and get

$$\|\nabla p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} \leq K(\|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q^T_\varepsilon)} + \|(u_\varepsilon - u_T)(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} +$$
$$+\varepsilon^{-\gamma}\|(u_\varepsilon - u_T)(\cdot,T)\|^2_{L^2(S_\varepsilon)}) \leq K(\|f\|_{L^2(Q^T)}\|p_\varepsilon\|_{L^2(Q^T_\varepsilon)} + \|\nabla u_T\|^2_{L^2(Q^T_\varepsilon)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2).$$

Now, for the function from $H^1(\Omega_\varepsilon, \partial\Omega)$, we have Friedrichs's inequality

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq K\|\nabla p_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Applying it to the previous inequality, we derive

$$\|p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} \leq K(\|f\|_{L^2(Q^T)}\|p_\varepsilon\|_{L^2(Q^T_\varepsilon)} + \|\nabla u_T\|^2_{L^2(Q^T_\varepsilon)} + \max_{x\in\overline{\Omega}}|u(x,T)|^2).$$

Hence, we obtain the estimation

$$\|p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} \leq K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T_\varepsilon)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2).$$

Then, we immediately derive the estimations

$$\begin{gathered}\|\nabla u_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon(\cdot,T)\|^2_{L^2(S_\varepsilon)} + \|p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} \leq \\ \leq K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2), \\ \|\nabla p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} \leq K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2).\end{gathered} \tag{13}$$

Next, we get estimations for the time derivatives of u_ε and p_ε. Following the work [11], we construct Galerkin's approximations of u_ε and p_ε which we denote by u^m_ε and p^m_ε respectively, where $m = 1, 2, \ldots$. Note that for u^m_ε and p^m_ε, we have the same estimates as in (13) for u_ε and p_ε.

Taking $\partial_t u^m_\varepsilon$ as a test function in the equation for u^m_ε and integrating from 0 to an arbitrary $\theta \in [0,T]$, we have

$$\|\partial_t u^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t u^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} + \max_{t\in[0,T]}\|\nabla u^m_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq$$
$$\leq K\int\limits_{Q^T_\varepsilon}(|f| + \frac{1}{N}|p^m_\varepsilon|\chi_{\omega_\varepsilon})|\partial_t u^m_\varepsilon|dxdt \leq \frac{1}{2}\|\partial_t u^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + K(\|f\|^2_{L^2(Q^T)} + \|p^m_\varepsilon\|^2_{L^2(w^T_\varepsilon)}),$$

where constant K is independent of ε and m. From here, we immediately derive

$$\|\partial_t u^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t u^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} \leq K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2).$$

Passing to the limit as $m \to \infty$, we get the estimation

$$\|\partial_t u_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t u_\varepsilon\|^2_{L^2(S^T_\varepsilon)} \leq K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \max_{x\in\overline{\Omega}} |u_T(x,T)|^2).$$

Again using Galerkin's approximation, we have for a.e. $t \in (0,T)$

$$-\|\partial_t p^m_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \varepsilon^{-\gamma}\|\partial_t p^m_\varepsilon\|^2_{L^2(S_\varepsilon)} + (\nabla p^m_\varepsilon, \partial_t \nabla p^m_\varepsilon)_{L^2(\Omega_\varepsilon)} =$$
$$= -(\partial_t u^m_\varepsilon, \partial_t p^m_\varepsilon)_{L^2(\Omega_\varepsilon)} - \varepsilon^{-\gamma}(\partial_t u^m_\varepsilon, \partial_t p^m_\varepsilon)_{L^2(S_\varepsilon)} +$$
$$+((f - N^{-1}p^m_\varepsilon \chi_{\omega_\varepsilon}), \partial_t p^m_\varepsilon)_{L^2(\Omega_\varepsilon)} - (\nabla u_T, \nabla \partial_t p^m_\varepsilon)_{L^2(\Omega_\varepsilon)}.$$

Integrating this expression with respect to t from 0 to T, we get

$$-\|\partial_t p^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \frac{1}{2}\|\nabla p^m_\varepsilon(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} - \frac{1}{2}\|\nabla p^m_\varepsilon(\cdot,0)\|^2_{L^2(\Omega_\varepsilon)} - \varepsilon^{-\gamma}\|\partial_t p^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} =$$
$$= -\int\limits_{Q^T_\varepsilon} \partial_t u^m_\varepsilon \partial_t p^m_\varepsilon dxdt - \varepsilon^{-\gamma}\int\limits_{S^T_\varepsilon} \partial_t u^m_\varepsilon \partial_t p^m_\varepsilon dsdt +$$
$$+\int\limits_{Q^T_\varepsilon} (f - N^{-1}p^m_\varepsilon \chi_{\omega_\varepsilon})\partial_t p^m_\varepsilon dxdt - \int\limits_{Q^T_\varepsilon} \nabla u_T \nabla \partial_t p^m_\varepsilon dxdt =$$
$$= -\int\limits_{Q^T_\varepsilon} \partial_t u^m_\varepsilon \partial_t p^m_\varepsilon dxdt - \varepsilon^{-\gamma}\int\limits_{S^T_\varepsilon} \partial_t u^m_\varepsilon \partial_t p^m_\varepsilon dsdt +$$
$$+\int\limits_{Q^T_\varepsilon} (f - N^{-1}p^m_\varepsilon \chi_{\omega_\varepsilon})\partial_t p^m_\varepsilon dxdt + \int\limits_{Q^T_\varepsilon} \nabla \partial_t u_T \nabla p^m_\varepsilon dxdt -$$
$$-\int\limits_{\Omega_\varepsilon} \nabla u_T(x,T)\nabla p^m_\varepsilon(x,T)dx + \int\limits_{\Omega_\varepsilon} \nabla u_T(x,0)\nabla p^m_\varepsilon(x,0)dx.$$

From here, we derive the estimate

$$\|\partial_t p^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t p^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} + \frac{1}{2}\|\nabla p^m_\varepsilon(\cdot,0)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2N}\|p^m_\varepsilon(x,0)\|^2_{L^2(\omega_\varepsilon)} \leq$$
$$\leq \frac{1}{2}\|\nabla(u^m_\varepsilon - u_T)(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \|\partial_t u^m_\varepsilon\|_{L^2(Q^T_\varepsilon)}\|\partial_t p^m_\varepsilon\|_{L^2(Q^T_\varepsilon)} +$$
$$+\varepsilon^{-\gamma}\|\partial_t u^m_\varepsilon\|_{L^2(S^T_\varepsilon)}\|\partial_t p^m_\varepsilon\|_{L^2(S^T_\varepsilon)} + \|f\|_{L^2(Q^T_\varepsilon)}\|\partial_t p^m_\varepsilon\|_{L^2(Q^T_\varepsilon)} +$$
$$+\frac{1}{2N}\|(u^m_\varepsilon - u_T)(\cdot,T)\|^2_{L^2(\omega_\varepsilon)} + \|\nabla \partial_t u_T\|_{L^2(Q^T_\varepsilon)}\|\nabla p^m_\varepsilon\|_{L^2(Q^T_\varepsilon)} +$$
$$+\|\nabla u_T(\cdot,T)\|_{L^2(\Omega_\varepsilon)}\|\nabla(u^m_\varepsilon - u_T)(\cdot,T)\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_T(\cdot,0)\|_{L^2(\Omega_\varepsilon)}\|\nabla p^m_\varepsilon(\cdot,0)\|_{L^2(\Omega_\varepsilon)}.$$

Next, we get

$$\|\partial_t p^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t p^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} + \frac{1}{4}\|\nabla p^m_\varepsilon(\cdot,0)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2N}\|p^m_\varepsilon(x,0)\|^2_{L^2(\omega_\varepsilon)} \leq$$
$$\leq \frac{1}{2}\|\nabla u^m_\varepsilon(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{4}\|\partial_t p^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \frac{\varepsilon^{-\gamma}}{4}\|\partial_t p^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} +$$
$$+K(\|\partial_t u^m_\varepsilon\|^2_{L^2(Q^T_\varepsilon)} + \varepsilon^{-\gamma}\|\partial_t u^m_\varepsilon\|^2_{L^2(S^T_\varepsilon)} + \|u^m_\varepsilon(\cdot,T)\|^2_{L^2(\omega_\varepsilon)} +$$
$$+\|f\|^2_{L^2(Q^T)} + \|\nabla \partial_t u_T\|^2_{L^2(Q^T)} + \max_{t\in[0,T]} \|\nabla u_T\|^2_{L^2(\Omega)} + \max_{x\in\overline{\Omega}} |u_T(x,T)|^2).$$

Eventually, using the estimates for u_ε^m, we obtain

$$\begin{aligned} \|\partial_t p_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t p_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)} \le K(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \\ + \|\nabla \partial_t u_T\|^2_{L^2(Q^T)} + \max_{t\in[0,T]}\|\nabla u_T\|^2_{L^2(\Omega)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2), \end{aligned} \tag{14}$$

where constant K is independent of m and ε. Passing to the limit as $m \to \infty$, we get the estimate for $\partial_t p_\varepsilon$

$$\begin{aligned} \|\partial_t p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t p_\varepsilon\|^2_{L^2(S_\varepsilon^T)} \le \\ \le K(\|f\|^2_{L^2(Q^T)} + \|\nabla \partial_t u_T\|^2_{L^2(Q^T)} + \max_{t\in[0,T]}\|\nabla u_T(\cdot,t)\|^2_{L^2(\Omega)} + \max_{x\in\overline{\Omega}}|u_T(x,T)|^2). \end{aligned} \tag{15}$$

Thus, we got uniform in ε estimates for u_ε and p_ε, and their time derivatives.

There exists the extension operator $P_\varepsilon : H^1(Q_\varepsilon^T) \to H^1(Q^T)$ (see. [7], [28]) such that

$$\|P_\varepsilon(u)\|_{H^1(Q^T)} \le \|u\|_{H^1(Q_\varepsilon^T)}.$$

Let $\widetilde{u}_\varepsilon$, $\widetilde{p}_\varepsilon$ be the extensions of the functions u_ε, p_ε then the following estimations are valid

$$\|\partial_t \widetilde{u}_\varepsilon\|^2_{L^2(Q^T)} + \|\nabla \widetilde{u}_\varepsilon\|^2_{L^2(Q^T)} \le K(\|\partial_t u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}), \tag{16}$$

$$\|\partial_t \widetilde{p}_\varepsilon\|^2_{L^2(Q^T)} + \|\nabla \widetilde{p}_\varepsilon\|^2_{L^2(Q^T)} \le K(\|\partial_t p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}). \tag{17}$$

The estimations (16), (17) imply that there exist subsequences for which we preserve the notation of the original, i.e. $\widetilde{u}_\varepsilon$ and $\widetilde{p}_\varepsilon$, such that, as $\varepsilon \to 0$, we have

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup u_0 \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ \widetilde{p}_\varepsilon &\rightharpoonup p_0 \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ \partial_t \widetilde{u}_\varepsilon &\rightharpoonup \partial_t u_0 \text{ weakly in } L^2(Q_\varepsilon^T), \\ \partial_t \widetilde{p}_\varepsilon &\rightharpoonup \partial_t p_0 \text{ weakly in } L^2(Q_\varepsilon^T). \end{aligned} \tag{18}$$

Moreover, the embedding theorem implies that $\widetilde{u}_\varepsilon \to u_0$ and $\widetilde{p}_\varepsilon \to p_0$ in $L^2(Q^T)$.

4 Main results

The following theorem characterizes the limit functions u_0 and p_0 defined in (18).

Theorem 2. *Let $n \ge 3$, $a_\varepsilon = C_0\varepsilon^\gamma$, where $C_0 > 0$, $\gamma = n/(n-2)$. If the pair $(u_\varepsilon, p_\varepsilon)$ is a solution to the problem (10), then, the pair (u_0, p_0) is a solution to the system*

$$\begin{cases} \partial_t u_0 - \Delta u_0 + \mathcal{A}_n(u_0 - \mathcal{B}_n H(u_0)) = f - N^{-1}\chi_\omega p_0, & (x,t) \in Q^T, \\ -\partial_t p_0 - \Delta p_0 + \mathcal{A}_n(p_0 - \mathcal{B}_n H^*(p_0)) = & \\ -\Delta(u_0 - u_T) + \mathcal{A}_n(u_0 - \mathcal{B}_n^2 H^*(H(u_0)) - e^{\mathcal{B}_n(t-T)}u_T(x,T)), & (x,t) \in Q^T, \\ u_0(x,0) = 0, & x \in \Omega, \\ p_0(x,T) = (u_0 - u_T)(x,T), & x \in \Omega, \\ u_0(x,t) = p_0(x,t) = 0, & (x,t) \in \Gamma^T, \end{cases} \tag{19}$$

where $Q^T = \Omega \times (0,T)$, $\mathcal{A}_n = (n-2)C_0^{n-2}w_n$, $H(\varphi)(x,t)$ *is given as a solution to*

$$\begin{cases} \partial_t H(\varphi) + \mathcal{B}_n H(\varphi) = \varphi, \quad t \in (0,T), \\ H(\varphi)(x,0) = 0, \end{cases} \tag{20}$$

where $\mathcal{B}_n = (n-2)C_0^{-1}$, *and* $H^*(\varphi)(x,t)$ *is a solution to the adjoint problem*

$$\begin{cases} -\partial_t H^*(\varphi) + \mathcal{B}_n H^*(\varphi) = \varphi, \quad t \in (0,T), \\ H^*(\varphi)(x,T) = 0. \end{cases} \tag{21}$$

Remark 3. *In the problems* (20), (21), *we can view* x *as a parameter, and for a.e.* x, *we have*

$$\int_0^T H(\varphi)\psi dt = \int_0^T H(\varphi)(-\partial_t H^*(\psi) + \mathcal{B}_n H^*(\psi))dt =$$

$$= \int_0^T \mathcal{B}_n H(\varphi)H^*(\psi)dt - H(\varphi)H^*(\varphi)\Big|_0^T + \int_0^T \partial_t H(\varphi)H^*(\psi)dt =$$

$$= \int_0^T (\partial_t H(\varphi) + \mathcal{B}_n H(\varphi))H^*(\psi)dt = \int_0^T \varphi H^*(\psi)dt.$$

Thus, we have

$$\int_0^T H(\varphi)\psi dt = \int_0^T \varphi H^*(\psi)dt. \tag{22}$$

Remark 4. *The solutions to the problems* (20) *and* (21) *can be found explicitly using the standard methods. We have*

$$H(\varphi)(x,t) = \int_0^t e^{-\mathcal{B}_n(t-\tau)}\varphi(x,\tau)d\tau, \quad H^*(\varphi)(x,t) = \int_t^T e^{\mathcal{B}_n(t-\tau)}\varphi(x,\tau)d\tau. \tag{23}$$

This explains the non-local in time nature of the "strange terms" $H(u_0)(x,t)$ *and* $H^*(p_0)(x,t)$ *arising in the homogenized system (19).*

The pair of functions (u_0, p_0) can be used now to characterize the the optimal control problem is given by the homogenized state problem (5) and a suitable limit cost functional $J_0(v)$ which we will show to be given by expression (6).

Theorem 3. *Under the same conditions as in Theorem 2, we have*

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0),$$

where v_ε *is the optimal control of the problem* (1), (3), *and* v_0 *is the optimal control of the problem* (5), (6).

Remark 5. *The optimal control* v_0 *of the problem* (5), (6) *is characterized by the coupled system* (19) *and we have the relation* $v_0 = -N^{-1}\chi_\omega p_0$. *Thus, the theorem similar to Theorem 1 is also valid.*

Proof. We begin the proof of the Theorem 2 that is split into two parts. First, we derive effective equations for u_0 in Section 5. Second, based on the derived problem for u_0, we find the limit problem for p_0 on Section 6.

5 Characterization of u_0

First, we define auxiliary functions w_ε^j as a solution to the boundary value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j, \\ w_\varepsilon^j = 1, & x \in \partial G_\varepsilon^j. \end{cases} \tag{24}$$

The solution to (24) is given explicitly by

$$w_\varepsilon^j = \frac{|x - P_\varepsilon^j|^{2-n} - (\varepsilon/4)^{2-n}}{a_\varepsilon^{2-n} - (\varepsilon/4)^{2-n}}. \tag{25}$$

Next, we define

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T_{\varepsilon/4}^j}. \end{cases}$$

It is easy to see that $W_\varepsilon \in H_0^1(\Omega, \partial\Omega)$ and $W_\varepsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. The embedding theorem implies that $W_\varepsilon \to 0$ strongly in $L^2(\Omega)$.

We take $W_\varepsilon H^*(\varphi)$, where $\varphi = \psi(x)\eta(t)$ with $\psi(x) \in C_0^\infty(\Omega)$, $\eta(t) \in C^1([0,T])$, as a test function in the integral identity (7) and get

$$\int\limits_{Q_\varepsilon^T} \partial_t u_\varepsilon W_\varepsilon H^*(\varphi)dxdt + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} \partial_t u_\varepsilon H^*(\varphi)dsdt + \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla(W_\varepsilon H^*(\varphi))dxdt =$$

$$= \int\limits_{Q_\varepsilon^T} (f - N^{-1}\chi_{\omega_\varepsilon} p_\varepsilon) W_\varepsilon H^*(\varphi)dxdt.$$

Using convergences (18) and the properties of W_ε, we have

$$\lim_{\varepsilon\to0} \int\limits_{Q_\varepsilon^T} \partial_t u_\varepsilon W_\varepsilon H^*(\varphi)dxdt = 0,$$

$$\lim_{\varepsilon\to0} \int\limits_{Q_\varepsilon^T} (f - N^{-1}\chi_{\omega_\varepsilon} p_\varepsilon) W_\varepsilon H^*(\varphi)dxdt = 0,$$

$$\lim_{\varepsilon\to0} \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla(W_\varepsilon H^*(\varphi))dxdt = \lim_{\varepsilon\to0} \int\limits_{Q_\varepsilon^T} \nabla(u_\varepsilon H^*(\varphi))\nabla W_\varepsilon dxdt.$$

Thus, we get

$$\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon H^*(\varphi)dsdt=-\int\limits_{Q_\varepsilon^T}\nabla W_\varepsilon\nabla(u_\varepsilon H^*(\varphi))dxdt+\alpha_{1,\varepsilon},$$

where $\alpha_{1,\varepsilon}\to 0$ as $\varepsilon\to 0$. Then, we make the decomposition

$$\int\limits_{Q_\varepsilon^T}\nabla W_\varepsilon\nabla(u_\varepsilon H^*(\varphi))dxdt=$$

$$=\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon u_\varepsilon H^*(\varphi)dsdt+\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}\partial_\nu w^j_\varepsilon u_\varepsilon H^*(\varphi)dsdt=$$

$$=-\varepsilon C_0^{n-2}(n-2)4^{n-1}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}u_\varepsilon H^*(\varphi)dsdt+\mathcal{B}_n\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}u_\varepsilon H^*(\varphi)dsdt+\alpha_{2,\varepsilon},$$

where $\alpha_{2,\varepsilon}\to 0$ as $\varepsilon\to 0$. For the first integral, we use Lemma 1 from [35] to get

$$\lim_{\varepsilon\to 0}\varepsilon C_0^{n-2}(n-2)4^{n-1}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}u_\varepsilon H^*(\varphi)dsdt=\mathcal{A}_n\int\limits_{Q^T}u_0H^*(\varphi)dxdt=\mathcal{A}_n\int\limits_{Q^T}H(u_0)\varphi dxdt,$$

where the last equality is due to (22). Thus, we derive

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon H^*(\varphi)dsdt=\mathcal{A}_n\int\limits_{Q^T}H(u_0)\varphi dxdt-\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\mathcal{B}_n u_\varepsilon H^*(\varphi)dsdt \tag{26}$$

Then, we have

$$\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon H^*(\varphi)dsdt=\int\limits_{S_\varepsilon}(u_\varepsilon(x,T)H^*(\varphi)(x,T)-u_\varepsilon(x,0)H^*(\varphi)(x,0))ds-$$

$$-\int\limits_{S_\varepsilon^T}u_\varepsilon\partial_t H^*(\varphi)dsdt=-\int\limits_{S_\varepsilon^T}u_\varepsilon\partial_t H^*(\varphi)dsdt, \tag{27}$$

the last equality follows from the fact that $u_\varepsilon(x,0)=0$ and $H^*(\varphi)(x,T)=0$ for $x\in S_\varepsilon$.

Thus, from (26), (27), we conclude that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}u_\varepsilon(-\partial_t H^*(\varphi)+\mathcal{B}_n H^*(\varphi))dsdt=\mathcal{A}_n\int\limits_{Q^T}H(u_0)\varphi dxdt.$$

Using the definition of $H^*(\varphi)$, we get

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}u_\varepsilon\varphi dsdt=\mathcal{A}_n\int\limits_{Q^T}H(u_0)\varphi dxdt. \tag{28}$$

Now, we take φW_ε in the integral identity (7) and, similarly to the above, using (28), we derive

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon\varphi dsdt=\mathcal{A}_n\int\limits_{Q^T}u_0\varphi dxdt-\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\mathcal{B}_n\int\limits_{S_\varepsilon^T}u_\varepsilon\varphi dsdt=$$
$$=\mathcal{A}_n\int\limits_{Q^T}u_0\varphi dxdt-\mathcal{A}_n\mathcal{B}_n\int\limits_{Q^T}H(u_0)\varphi dxdt=\mathcal{A}_n\int\limits_{Q^T}(u_0-\mathcal{B}_nH(u_0))\varphi dxdt.$$

Now, we can pass to the limit as $\varepsilon\to 0$ in the integral identity (7). Doing so, we conclude that u_0 satisfies the integral identity

$$\int\limits_{Q^T}\partial_t u_0 vdxdt+\int\limits_{Q^T}\nabla u_0\nabla vdxdt+\mathcal{A}_n\int\limits_{Q^T}(u_0-\mathcal{B}_nH(u_0))vdxdt=\int\limits_{Q^T}(f-N^{-1}\chi_\omega p_0)vdxdt.$$

6 Characterization of p_0

We take $\varphi=W_\varepsilon H(\varphi)$, where $\varphi=\psi(x)\eta(t)$ with $\psi\in C_0^\infty(\Omega)$, $\eta\in C^1([0,T])$ as a test function in the integral identity (9) and get

$$\int\limits_{Q_\varepsilon^T}\nabla W_\varepsilon\nabla(p_\varepsilon H(\varphi))dxdt-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t(p_\varepsilon-u_\varepsilon)H(\varphi)dsdt=$$
$$=\int\limits_{Q_\varepsilon^T}\partial_t(p_\varepsilon-u_\varepsilon)W_\varepsilon H(\varphi)dxdt+\int\limits_{Q_\varepsilon^T}(f-N^{-1}\chi_{\omega_\varepsilon}p_\varepsilon)W_\varepsilon H(\varphi)dxdt+\beta_\varepsilon=\kappa_\varepsilon,$$

where $\beta_\varepsilon\to 0$ as $\varepsilon\to 0$. Due to the properties of u_ε, p_ε, W_ε, we have that $\kappa_\varepsilon\to 0$ as $\varepsilon\to 0$. Hence, by decomposing the first integral, we derive

$$\varepsilon^{-\gamma}\mathcal{B}_n\int\limits_{S_\varepsilon^T}p_\varepsilon H(\varphi)dsdt+\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}\partial_\nu w_\varepsilon^j p_\varepsilon H(\varphi)dsdt-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t(p_\varepsilon-u_\varepsilon)H(\varphi)dsdt=\kappa_{1,\varepsilon},\tag{29}$$

where $\kappa_{1,\varepsilon}\to 0$ as $\varepsilon\to 0$. For the last integral, we have

$$\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t(p_\varepsilon-u_\varepsilon)H(\varphi)dsdt=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t(p_\varepsilon-u_\varepsilon+u_T)H(\varphi)dsdt-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_TH(\varphi)dsdt=$$
$$=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(p_\varepsilon-u_\varepsilon+u_T)(x,T)H(\varphi)(x,T)ds-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(p_\varepsilon-u_\varepsilon+u_T)(x,0)H(\varphi)(x,0)ds-$$
$$-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_tH(\varphi)(p_\varepsilon-u_\varepsilon+u_T)dsdt-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_TH(\varphi)dsdt.$$

From the definitions of $H(\varphi)$ and p_ε, we have $H(\varphi)(x,0) \equiv 0$ and $p_\varepsilon(x,T) - u_\varepsilon(x,T) + u_T(x,T) = 0$, hence, we obtain

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t (p_\varepsilon - u_\varepsilon) H(\varphi) dsdt = -\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t u_T H(\varphi) dsdt - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t H(\varphi)(p_\varepsilon - u_\varepsilon + u_T) dsdt.$$

Thus, substituting this expression into (29), we get

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} p_\varepsilon (\mathcal{B}_n H(\varphi) + \partial_t H(\varphi)) dsdt = - \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^T \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon p_\varepsilon H(\varphi) dsdt -$$

$$- \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\partial_t u_T H(\varphi) + \partial_t H(\varphi) u_T) dsdt + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t H(\varphi) u_\varepsilon dsdt + \beta_{1,\varepsilon},$$

where $\beta_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$. Using the definition of $H(\varphi)$, we obtain

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} p_\varepsilon \varphi dsdt = - \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^T \int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon p_\varepsilon H(\varphi) dsdt -$$

$$- \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\partial_t u_T H(\varphi) + \partial_t H(\varphi) u_T) dsdt + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\varphi - \mathcal{B}_n H(\varphi)) u_\varepsilon dsdt + \beta_{1,\varepsilon}. \tag{30}$$

The limit of the first integral on the equality's right-hand side can be found using Lemma 1 from [35]. For the third integral, we use the convergence for u_ε obtained above. Let us find the limit of the second integral. Using the definition of H, we have

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\partial_t u_T H(\varphi) + \partial_t H(\varphi) u_T) dsdt = \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\partial_t u_T H(\varphi) + (\varphi - \mathcal{B}_n H(\varphi)) u_T) dsdt =$$

$$= \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (\partial_t u_T - \mathcal{B}_n u_T) H(\varphi) dsdt + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} u_T \varphi dsdt = \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (H^*(\partial_t u_T - \mathcal{B}_n u_T) + u_T) \varphi dsdt. \tag{31}$$

Using (23), we get

$$H^*(\partial_t u_T - \mathcal{B}_n u_T)(x,t) = \int\limits_t^T e^{\mathcal{B}_n (t-\tau)} (\partial_t u_T - \mathcal{B}_n u_T) d\tau =$$

$$= e^{\mathcal{B}_n (t-\tau)} u_T \Big|_t^T + \int\limits_t^T \mathcal{B}_n e^{\mathcal{B}_n (t-\tau)} u_T d\tau - \int\limits_t^T \mathcal{B}_n e^{\mathcal{B}_n (t-\tau)} u_T d\tau = e^{\mathcal{B}_n (t-T)} u_T(x,T) - u_T(x,t).$$

From here, it follows that

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} (H^*(\partial_t u_T - \mathcal{B}_n u_T) + u_T) \varphi dsdt = \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} e^{\mathcal{B}_n (t-T)} u_T(x,T) \varphi(x,t) dsdt. \tag{32}$$

Using the regularity of the function φ and u_T, we get the convergence

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dsdt = C_0^{n-1}\omega_n\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi(x,t)dxdt. \tag{33}$$

Using Lemma 1 from [35] and (31)-(33), from (30), we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} p_\varepsilon\varphi dsdt = \mathcal{A}_n\int\limits_{Q^T} p_0 H(\varphi)dxdt +$$

$$+\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}(\varphi-\mathcal{B}_nH(\varphi))u_\varepsilon dsdt - C_0^{n-1}\omega_n\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dxdt.$$

For the second integral on the right-hand side of the expression above, we use the results of the previous section and get

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T} p_\varepsilon\varphi dsdt = \mathcal{A}_n\int\limits_{Q^T} p_0 H(\varphi)dxdt +$$

$$+\mathcal{A}_n\int\limits_{Q^T}(H(u_0)\varphi-\mathcal{B}_nH(u_0)H(\varphi))dxdt - C_0^{n-1}\omega_n\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dxdt =$$

$$=\mathcal{A}_n\int\limits_{Q^T}H^*(p_0)\varphi dxdt + \mathcal{A}_n\int\limits_{Q^T}(H(u_0)-\mathcal{B}_nH^*(H(u_0)))\varphi dxdt - C_0^{n-1}\omega_n\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dxdt.$$

Again, we take φW_ε as the test function in the integral identity (9) and derive

$$-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t p_\varepsilon\varphi dsdt = -\varepsilon^{-\gamma}\mathcal{B}_n\int\limits_{S_\varepsilon^T}p_\varepsilon\varphi dsdt -$$

$$-\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}\partial_\nu w^j_\varepsilon p_\varepsilon\varphi dsdt - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon\varphi dsdt + \alpha_\varepsilon,$$

where $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$. From here, we have (note that $C_0^{n-1}\omega_n\mathcal{B}_n=\mathcal{A}_n$)

$$\lim_{\varepsilon\to 0}-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}\partial_t p_\varepsilon\varphi dsdt = -\mathcal{A}_n\mathcal{B}_n\int\limits_{Q^T}(H^*(p_0)+H(u_0)-\mathcal{B}_nH^*(H(u_0)))\varphi dxdt +$$

$$+\mathcal{A}_n\int\limits_{Q^T}e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dxdt + \mathcal{A}_n\int\limits_{Q^T}p_0\varphi dxdt - \mathcal{A}_n\int\limits_{Q^T}(u_0-H(u_0))\varphi dxdt =$$

$$=\mathcal{A}_n\int\limits_{Q^T}((p_0-\mathcal{B}_nH^*(p_0))-(u_0-\mathcal{B}_nH^*(H(u_0))))\varphi dxdt + \mathcal{A}_n\int\limits_{Q^T}e^{\mathcal{B}_n(t-T)}u_T(x,T)\varphi dxdt.$$

Thus, passing to the limit in the integral identity (9), we conclude that p_0 satisfies the integral identity

$$\begin{gathered}
-\int\limits_{Q^T} \partial_t p_0 \varphi dxdt + \int\limits_{Q^T} \nabla p_0 \nabla \varphi dxdt + \\
+\mathcal{A}_n \int\limits_{Q^T} (p_0 - \mathcal{B}_n H^*(p_0))\varphi dxdt + \mathcal{A}_n \int\limits_{Q^T} e^{\mathcal{B}_n(t-T)} u_T(x,T)\varphi dxdt = \\
= \int\limits_{Q^T} \nabla(u_0 - u_T)\nabla\varphi dxdt + \mathcal{A}_n \int\limits_{Q^T} (u_0 - \mathcal{B}_n^2 H^*(H(u_0)))\varphi dxdt.
\end{gathered} \tag{34}$$

This concludes the proof of the Theorem 2. ☐

7 Cost functional limit

Proof. Here, we give the proof of Theorem 3.

Let us find the limit of the cost functional J_ε as $\varepsilon \to 0$. As $v_\varepsilon = N^{-1} p_\varepsilon$, we have

$$\begin{gathered}
J_\varepsilon(v_\varepsilon) = \frac{1}{2} \int\limits_{Q_\varepsilon^T} |\nabla u_\varepsilon - u_T|^2 dxdt + \frac{1}{2} \int\limits_{\Omega_\varepsilon} (u_\varepsilon - u_T)^2(x,T)dx + \\
+ \frac{\varepsilon^{-\gamma}}{2} \int\limits_{S_\varepsilon} (u_\varepsilon - u_T)^2(x,T)ds + \frac{1}{2N} \int\limits_{\omega_\varepsilon^T} p_\varepsilon^2 dxdt.
\end{gathered}$$

Using integral identity (7), we transform the functional J_ε and get

$$\begin{gathered}
J_\varepsilon(v_\varepsilon) = \frac{1}{2} \int\limits_{Q_\varepsilon^T} |\nabla (u_\varepsilon - u_T)|^2 dxdt + \frac{1}{2} \int\limits_{\Omega_\varepsilon} (u_\varepsilon - u_T)^2(x,T)dx + \\
+ \frac{\varepsilon^{-\gamma}}{2} \int\limits_{S_\varepsilon} (u_\varepsilon - u_T)^2(x,T)ds + \frac{1}{2} \int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt - \\
- \frac{1}{2} \int\limits_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla p_\varepsilon dxdt - \frac{1}{2} \int\limits_{Q_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon dxdt - \frac{\varepsilon^{-\gamma}}{2} \int\limits_{S_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon dsdt.
\end{gathered}$$

Next, we have

$$-\frac{1}{2}\int\limits_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla p_\varepsilon dxdt-\frac{1}{2}\int\limits_{Q_\varepsilon^T}\partial_t u_\varepsilon p_\varepsilon dxdt-\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon^T}\partial_t u_\varepsilon p_\varepsilon dsdt=$$

$$=-\frac{1}{2}\int\limits_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla p_\varepsilon dxdt+\frac{1}{2}\int\limits_{Q_\varepsilon^T}\partial_t p_\varepsilon u_\varepsilon dxdt+\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon^T}\partial_t p_\varepsilon u_\varepsilon dsdt-$$

$$-\frac{1}{2}\int\limits_{\Omega_\varepsilon}u_\varepsilon(x,T)(u_\varepsilon-u_T)(x,T)dx-\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}u_\varepsilon(x,T)(u_\varepsilon-u_T)(x,T)ds=$$

$$=-\frac{1}{2}\int\limits_{Q_\varepsilon^T}\nabla(u_\varepsilon-u_T)\nabla u_\varepsilon dxdt-\frac{1}{2}\int\limits_{\Omega_\varepsilon}u_\varepsilon(x,T)(u_\varepsilon-u_T)(x,T)dx-$$

$$-\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}u_\varepsilon(x,T)(u_\varepsilon-u_T)(x,T)ds.$$

Thus, we can further transform the expression for the cost functional

$$J_\varepsilon(v_\varepsilon)=\frac{1}{2}\int\limits_{Q_\varepsilon^T}\nabla u_T\nabla(u_T-u_\varepsilon)dxdt+\frac{1}{2}\int\limits_{\Omega_\varepsilon}u_T(x,T)(u_T-u_\varepsilon)(x,T)dx+$$

$$+\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}u_T(x,T)(u_T-u_\varepsilon)(x,T)ds+\frac{1}{2}\int\limits_{Q_\varepsilon^T}fp_\varepsilon dxdt=$$

$$=\frac{1}{2}\int\limits_{Q_\varepsilon^T}\nabla u_T\nabla(u_T-u_\varepsilon)dxdt+\frac{1}{2}\int\limits_{Q_\varepsilon^T}fp_\varepsilon dxdt+$$

$$+\frac{1}{2}\int\limits_{\Omega_\varepsilon}u_T^2(x,T)dx-\frac{1}{2}\int\limits_{Q_\varepsilon^T}(\partial_t u_\varepsilon u_T+\partial_t u_T u_\varepsilon)dxdt+$$

$$+\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon}u_T^2(x,T)ds-\frac{\varepsilon^{-\gamma}}{2}\int\limits_{S_\varepsilon^T}(\partial_t u_\varepsilon u_T+\partial_t u_T u_\varepsilon)dxdt.$$

Now, we should pass to the limit in the obtained expressions. First, properties of u_ε,

$\partial_t u_\varepsilon$ and p_ε imply that, as $\varepsilon \to 0$,

$$\int\limits_{Q_\varepsilon^T} \nabla u_T \nabla (u_T - u_\varepsilon) dxdt \to \int\limits_{Q^T} \nabla u_T \nabla (u_T - u_0) dxdt,$$

$$\int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt \to \int\limits_{Q^T} f p_0 dxdt,$$

$$\int\limits_{Q_\varepsilon^T} (\partial_t u_\varepsilon u_T + \partial_t u_T u_\varepsilon) dxdt \to \int\limits_{Q^T} (\partial_t u_0 u_T + \partial_t u_T u_0) dxdt,$$

$$\int\limits_{\Omega_\varepsilon} u_T^2(x,T) dx \to \int\limits_{\Omega} u_T^2(x,T) dx.$$

Next, we find the limit of the integrals over S_ε based on the convergences obtained in the previous sections

$$\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t u_T u_\varepsilon dsdt = \mathcal{A}_n \int\limits_{Q^T} H(u_0) \partial_t u_T dxdt,$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^T} \partial_t u_\varepsilon u_T dsdt = \mathcal{A}_n \int\limits_{Q^T} (u_0 - \mathcal{B}_n H(u_0)) u_T dxdt,$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} u_T^2(x,T) ds = C_0^{n-1} \omega_n \int\limits_{\Omega} u_T^2(x,T) dx.$$

Combining all of the above convergences, we derive

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = \frac{1}{2} \int\limits_{Q^T} \nabla u_T \nabla (u_T - u_0) dxdt + \frac{1}{2} \int\limits_{Q^T} f p_0 dxdt +$$
$$+ \frac{1}{2} \int\limits_{\Omega} u_T^2(x,T) dx + \frac{C_0^{n-1}\omega_n}{2} \int\limits_{\Omega} u_T^2(x,T) dx - \int\limits_{Q^T} (\partial_t u_0 u_T + \partial_t u_T u_0) dxdt -$$
$$- \frac{\mathcal{A}_n}{2} \int\limits_{Q^T} H(u_0) \partial_t u_T dxdt - \frac{\mathcal{A}_n}{2} \int\limits_{Q^T} (u_0 - \mathcal{B}_n H(u_0)) u_T dxdt. \tag{35}$$

Again, we have

$$\frac{1}{2} \int\limits_{\Omega} u_T^2(x,T) dx - \frac{1}{2} \int\limits_{Q^T} (\partial_t u_0 u_T + \partial_t u_T u_0) dxdt = \frac{1}{2} \int\limits_{\Omega} u_T(x,T)(u_T - u_0)(x,T) dx. \tag{36}$$

Also, using the definition of H, we get

$$\frac{\mathcal{A}_n}{2}\int\limits_{Q^T} H(u_0)\partial_t u_T dxdt + \frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(u_0-\mathcal{B}_n H(u_0))u_T dxdt =$$
$$= \frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(\partial_t u_T H(u_0) + \partial_t H(u_0)u_T)dxdt = \frac{\mathcal{A}_n}{2}\int\limits_{Q^T}\partial_t(u_T H(u_0))dxdt =$$
$$= \frac{\mathcal{A}_n}{2}\int\limits_{\Omega} u_T(x,T)H(u_0)(x,T)dx. \tag{37}$$

From the integral identities for u_0 and p_0, we get

$$\frac{1}{2}\int\limits_{Q^T} fp_0 dxdt = \frac{1}{2}\int\limits_{Q^T}\partial_t u_0 p_0 dxdt + \frac{1}{2}\int\limits_{Q^T}\nabla u_0 \nabla p_0 dxdt +$$
$$+\frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(u_0-\mathcal{B}_n H(u_0))p_0 dxdt + \frac{1}{2N}\int\limits_{\omega^T} p_0^2 dxdt =$$
$$= \frac{1}{2}\int\limits_{\Omega} u_0(x,T)(u_0-u_T)(x,T)dx - \frac{1}{2}\int\limits_{Q^T}\partial_t p_0 u_0 dxdt + \frac{1}{2}\int\limits_{Q^T}\nabla u_0\nabla p_0 dxdt +$$
$$+\frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(p_0-\mathcal{B}_n H^*(p_0))u_0 dxdt + \frac{1}{2N}\int\limits_{\omega^T} p_0^2 dxdt =$$
$$= \frac{1}{2}\int\limits_{\Omega} u_0(x,T)(u_0-u_T)(x,T)dx + \frac{1}{2}\int\limits_{Q^T}\nabla u_0 \nabla(u_0-u_T)dxdt + \frac{1}{2N}\int\limits_{\omega^T} p_0^2 dxdt +$$
$$+\frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(u_0-\mathcal{B}_n^2 H^*(H(u_0)))u_0 dxdt - \frac{\mathcal{A}_n}{2}\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)u_0 dxdt. \tag{38}$$

Next, we substitute (36)-(38) into (35) and derive

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = \frac{1}{2}\|\nabla(u_0-u_T)\|^2_{L^2(Q^T)} + \frac{1}{2}\|(u_0-u_T)(x,T)\|^2_{L^2(\Omega)} + \frac{1}{2N}\|p_0\|^2_{L^2(\omega^T)} +$$
$$+\frac{C_0^{n-1}\omega_n}{2}\|u_T(x,T)\|^2_{L^2(\Omega)} + \frac{\mathcal{A}_n}{2}\int\limits_{Q^T}(u_0^2 - \mathcal{B}_n^2 H^2(u_0))dxdt -$$
$$-\frac{\mathcal{A}_n}{2}\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)}u_T(x,T)u_0 dxdt - \frac{\mathcal{A}_n}{2}\int\limits_{\Omega} H(u_0)(x,T)u_T(x,T)dx. \tag{39}$$

Using the definition of H and H^*, we make the transform

$$\int\limits_{Q^T} e^{\mathcal{B}_n(t-T)} u_T(x,T) u_0 dxdt = \int\limits_{Q^T} (H^*(\partial_t u_T - \mathcal{B}_n u_T)(x,t) + u_T(x,t)) u_0 dxdt =$$
$$= \int\limits_{Q^T} (\partial_t u_T - \mathcal{B}_n u_T) H(u_0) dxdt + \int\limits_{Q^T} u_T u_0 dxdt =$$
$$= \int\limits_{Q^T} (\partial_t u_T H(u_0) + (u_0 - \mathcal{B}_n H(u_0)) u_T) dxdt = \int\limits_{Q^T} (\partial_t (u_T H(u_0))) dxdt =$$
$$= \int\limits_{\Omega} u_T(x,T) H(u_0)(x,T) dxdt.$$

Substituting this into (39), we get

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = \frac{1}{2}\|\nabla(u_0 - u_T)\|^2_{L^2(Q^T)} + \frac{1}{2}\|(u_0 - u_T)(x,T)\|^2_{L^2(\Omega)} + \frac{1}{2N}\|p_0\|^2_{L^2(\omega^T)} +$$
$$+ \frac{C_0^{n-1}\omega_n}{2}\|u_T(x,T)\|^2_{L^2(\Omega)} + \frac{\mathcal{A}_n}{2}\int\limits_{Q^T} (u_0^2 - \mathcal{B}_n^2 H^2(u_0)) dxdt - \frac{\mathcal{A}_n}{2}\int\limits_{\Omega} 2u_T(x,T) H(u_0)(x,T) dx.$$

Using the definition of $H(u_0)$, we derive

$$u_0^2 - \mathcal{B}_n^2 H^2(u_0) = (\partial_t H(u_0) + \mathcal{B}_n H(u_0))^2 - \mathcal{B}_n^2 H^2(u_0) =$$
$$= (\partial_t H(u_0))^2 + 2\mathcal{B}_n \partial_t H(u_0) H(u_0).$$

Hence, we have

$$\int\limits_{Q^T} (u_0^2 - \mathcal{B}_n^2 H^2(u_0)) dxdt = \int\limits_{Q^T} (\partial_t H(u_0))^2 dxdt + \mathcal{B}_n \int\limits_{\Omega} H^2(u_0)(x,T) dx.$$

Thus, we further transform the limit of J_ε and get

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = \frac{1}{2}\|\nabla(u_0 - u_T)\|^2_{L^2(Q^T)} + \frac{1}{2}\|(u_0 - u_T)(x,T)\|^2_{L^2(\Omega)} + \frac{1}{2N}\|p_0\|^2_{L^2(\omega^T)} +$$
$$+ \frac{C_0^{n-1}\omega_n}{2}\|(u_T - \mathcal{B}_n H(u_0))(x,T)\|^2_{L^2(\Omega)} + \frac{\mathcal{A}_n}{2}\|\partial_t H(u_0)\|^2_{L^2(Q^T)} \equiv J_0(v_0).$$

This concludes the proof. □

Remark 6. *As indicated in Remark 2, the mentioned arguments by J.-L. Lions in [25] (see also, e.g., Section 1.6 in the book [18]), allow to get some results on the "approximate controllability property" for solutions of problem (5). Given* $u_T \in H^1(0,T; H_0^1(\Omega)) \bigcap C(\overline{Q^T})$ *and an arbitrarily small* $\delta > 0$, *the "approximate controllability property" consists now* *in finding a control* $v_\delta \in L^2(\omega_\varepsilon^T)$ *such that* $\|u_\varepsilon(\cdot,T) - u_T(\cdot,T)\|_{L^2(\Omega)} \leq \delta$. *We introduce a new parameter* $\kappa > 0$ *in the cost functional*

$$J_0^\kappa(v) = \frac{\kappa}{2}\|\nabla(u_0(v) - u_T)\|^2_{L^2(Q^T)} + \frac{\kappa}{2}\|u_0(v)(\cdot,T) - u_T(\cdot,T)\|^2_{L^2(\Omega)} + \frac{N}{2}\|v\|^2_{L^2(\omega^T)} + \\ + \frac{\kappa C_0^{n-1}\omega_n}{2}\|u_T(\cdot,T) - \mathcal{B}_n H(u_0(v))(\cdot,T)\|^2_{L^2(\Omega)} + \frac{\kappa\mathcal{A}_n}{2}\|\partial_t H(u_0(v))\|^2_{L^2(Q^T)}, \tag{40}$$

and if we know a result on "unique continuation" for problem (5)*, then, by using some a priori estimates on the adjoint state* $p_0(x,t)$*, it can be shown that such searched control* v_δ *can be found by considering the set of optimal controls* $v_\kappa \in L^2(\omega_\varepsilon^T)$ *associated to* $J_0^\kappa(v)$ *and by taking* $v_\delta = v_\kappa$ *for* κ *large enough. Again, the presence of the first, fourth and fifth terms in* $J_0^\kappa(v)$ *leads to conclude that the the associated state will also satisfy some additional properties:* $\|\nabla(u_0 - u_T)\|_{L^2(Q^T)} \leq \delta$, $\|u_T(\cdot,T) - \mathcal{B}_n H(u_0)(\cdot,T)\|_{L^2(\Omega)} \leq \delta$ *and* $\|\partial_t H(u_0)\|^2_{L^2(Q^T)} \leq \delta$.

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