$Q(M)$ and the depolarization index scalar metrics

Rafael Espinosa-Luna,[1,2,3,*] Eusebio Bernabeu,[2] and Gelacio Atondo-Rubio[1]

[1]Escuela de Ciencias Físico-Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria s/n, 80010 Culiacán, Sinaloa, México

[2]Applied Optics Complutense Group, Optics Department, Universidad Complutense de Madrid, Facultad de Ciencias Físicas, Ciudad Universitaria s/n, 28040 Madrid, Spain

[3]*Permanent Address: Centro de Investigaciones en ptica, A. C.*, Loma del Bosque 115, Colonia Lomas del Campestre, 37150 León, Guanajuato, México

*Corresponding author: reluna@cio.mx

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A depolarization scalar metric for Mueller matrices, named $Q(M)$, is derived from the degree of polarization. $Q(M)$ has been recently reported, and it has been deduced from the nine bilinear constraints between the sixteen elements of the Mueller–Jones matrix. We discuss the relations between $Q(M)$ and the depolarization index. © 2008 Optical Society of America

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1. Introduction

In this work the term depolarization refers to the loss in the degree of polarization as the light propagates through an optical system, where the incident light is supposed to be completely polarized. The concept of depolarization has deserved a lot of interest over the past years. Gil and Bernabeu [1,2] have defined the depolarization index as a single number metric associated with the Mueller matrix representing the linear response of the optical system. Anderson and Barakat have introduced the randomness [3] as being a characteristic of output light depolarization corresponding structurally to a decomposition of the Mueller matrix. On the other hand, the entropy is a function of the Mueller matrix elements and needs no scanning over input polarizations, as has been defined by Cloude [4]. Chipman and coworkers [5–9] and Brosseau [10] have employed recently the degree of polarization [5,11] and have defined other derived metrics as the average and the weighted degree of polarization [6–8]. The degree of polarization has been studied for a broad kind of systems and links where the diattenuation and the polarizance vectors have been analyzed [9,10]. Gil has reported an excellent work about the polarimetric properties of the Mueller matrix, where depolarization deserves a special attention [12]. Now, in this work, a depolarization metric for Mueller matrices, denoted $Q(M)$, is derived from the degree of polarization. This metric has been derived recently from the nine bilinear constraints between the sixteen elements of the Mueller–Jones matrix [13]. We apply a number of metrics, including $Q(M)$, to several Mueller matrices to test our metric. Our results indicate that $Q(M)$ is a metric sensitive to the internal nature of the Mueller matrix. Four bounds are determined for $Q(M)$, which allow us to identify a Mueller matrix as totally depolarizing, partially depolarizing, nondepolarizing diattenuating, and nondepolarizing nondiattenuating, respectively. Finally, we discuss the relations between $Q(M)$ and the depolarization index.

2. Basic Relations

The linear response of an optical system to an incident polarization Stokes vector can be expressed in terms of intensities (irradiances), through the relation

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$$S^o = MS^i \Rightarrow \begin{pmatrix} s_0^o \\ s_1^o \\ s_2^o \\ s_3^o \end{pmatrix} = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} s_0^i \\ s_1^i \\ s_2^i \\ s_3^i \end{pmatrix}, \quad (1)$$

where M is called the Mueller matrix of the system, represented as a 4×4 matrix of real elements, and S is the Stokes vector. S represents the polarization state of light, defined in terms of the orthogonal components of the electric field vector (E_p, E_s) as [11]

$$S^\alpha = \begin{pmatrix} s_0^\alpha \\ s_1^\alpha \\ s_2^\alpha \\ s_3^\alpha \end{pmatrix} = \begin{pmatrix} \langle E_p^\alpha E_p^\alpha * \rangle + \langle E_s^\alpha E_s^\alpha * \rangle \\ \langle E_p^\alpha E_p^\alpha * \rangle - \langle E_s^\alpha E_s^\alpha * \rangle \\ \langle E_p^\alpha E_s^\alpha * \rangle + \langle E_s^\alpha E_p^\alpha * \rangle \\ \pm i(\langle E_p^\alpha E_s^\alpha * \rangle - \langle E_s^\alpha E_p^\alpha * \rangle) \end{pmatrix}, \quad (2a)$$

where $\alpha = i$ (*input*), o (*output*). Angular brackets represent temporal averages, and $*$ indicates complex conjugation. The upper (lower) sign in the right hand side of s_3^α corresponds to a description of polarization states as looking to the source (propagation direction). Another very useful representation for the Stokes vectors is given in terms of the azimuthal $(0 \le \psi \le \pi)$ and the ellipticity $(-\pi/4 \le \chi \le \pi/4)$ angles of the polarization ellipse of the wave, respectively [10]:

$$S = \langle s_0 \rangle \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{pmatrix}. \quad (2b)$$

Equation (2b) is valid only for totally polarized light. A very important property associated with an optical system is its capability to depolarize light. Almost all depolarization metrics provide a single scalar number that varies from zero (output light totally depolarized) to one (output light totally polarized), and intermediate values are associated with partial polarization.

The depolarization index $DI(M)$ is defined by [1,2]

$$0 \le DI(M) = \left\{ \sum_{j,k=0}^{3} m_{jk}^2 - m_{00}^2 \right\}^{1/2} / \sqrt{3} m_{00} \le 1. \quad (3)$$

$DI(M)$ is directly related to the Mueller matrix elements only. That means the metric is applied directly to the Mueller matrix.

The degree of polarization, $DoP(M,S)$, has been defined by [5–11]

$$0 \le DoP(M,S) = \frac{\sqrt{(s_1^o)^2 + (s_2^o)^2 + (s_3^o)^2}}{s_0^o} = \frac{\left[\sum_{j=1}^{3} (m_{j0} s_0^i + m_{j1} s_1^i + m_{j2} s_2^i + m_{j3} s_3^i)^2 \right]^{1/2}}{m_{00} s_0^i + m_{01} s_1^i + m_{02} s_2^i + m_{03} s_3^i} \le 1. \quad (4)$$

$DoP(M,S)$ is directly related to both the Mueller matrix elements of the system under study and the incident Stokes vector. A note of caution must be taken into account when referring to $DoP(M,S)$. It is measured directly from the Stokes vector emerging of the system under study, and the measured value is associated with the outgoing light; however, it is inherently related to the optical response of the system, as can be noted from Eq. (4). For the case of a specific given Mueller matrix, a usual procedure is just to scan for all the possible incident Stokes vectors whose outputs can be associated with physically realizable Stokes vectors (overpolarization condition) [10], jointly with a scanning of the gain for all the incident Stokes vectors taken from the Poincaré sphere (overgain condition) [10]. These conditions can be plotted in three dimensions as a function of the incident state of polarization parametrized by the angles $0 \le \psi \le \pi$ (azimuth) and $-\pi/4 \le \chi \le \pi/4$ (ellipticity) of the polarization ellipse of the wave, respectively [10]. The $DoP(M,S)$ (gain) output usually is not a number, but a function of the Stokes parameters, which can be expressed also in terms of ψ and χ. The surface obtained for $DoP(M,S)$ (gain) takes maximum and minimum values. If the intention is just to "compare" the $DoP(M,S)$ with other single-valued numeric depolarization metrics, one possibility is that the representative value of the appropriate Stokes value be the one that produces the maximum $DoP(M,S)$ output. Experimentally, it is not easy to produce perfect circularly polarized light; it is even harder to generate an arbitrarily controlled elliptical polarization state. Under this realistic situation, we suggest using as a representative Stokes vector one of the basic six polarization states that produces the maximum output for the $DoP(M,S)$. The six basic polarization states we refer to here are the linear polarization parallel (p), perpendicular and (s) to +45 degrees (+) and to −45 degrees (−), to the incidence plane, respectively, in addition to the circular right- (r) and left-handed (l) polarization states, respectively.

In the following, we propose a new representation for the depolarization metric of the Mueller matrix. We use the decomposition of a Mueller matrix given by

$$M \equiv M_n + M_d, \quad (5)$$

where

$$M_d = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \tag{6}$$

$$M_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix}. \tag{7}$$

The degree of polarization, Eq. (4), can be written as

$$\begin{aligned} DoP(M,S) &= \frac{\left[\sum_{j=1}^{3}(m_{j0}s_0^i + m_{j1}s_1^i + m_{j2}s_2^i + m_{j3}s_3^i)^2\right]^{1/2}}{m_{00}s_0^i + m_{01}s_1^i + m_{02}s_2^i + m_{03}s_3^i} \\ &= \frac{\{(M_nS^i)^T(M_nS^i)\}^{1/2}}{\{(M_dS^i)^T(M_dS^i)\}^{1/2}}. \end{aligned} \tag{8}$$

The degree of polarization can be reduced to the following expression:

$$\{DoP\}^2 = \frac{(S^i)^T(M_n)^T(M_n)(S^i)}{(S^i)^T(M_d)^T(M_d)(S^i)} \le 1. \tag{9}$$

Note that

$$\begin{aligned} \mathrm{Tr}[(M_n)^T M_n] &= \sum_{j=1,k=0}^{3} m_{jk}^2 = \sum_{j,k=0}^{3} m_{jk}^2 - \sum_{k=0}^{3} m_{0k}^2 \\ &= \left\{\sum_{j,k=0}^{3} m_{jk}^2 - m_{00}^2\right\} - \sum_{k=1}^{3} m_{0k}^2. \end{aligned} \tag{10}$$

Using Eq. (3) in Eq. (10), it is reduced to

$$\begin{aligned} \mathrm{Tr}[(M_n)^T M_n] &= \{3m_{00}^2[DI(M)]^2\} - \sum_{k=1}^{3} m_{0k}^2 \\ &= m_{00}^2\{3[DI(M)]^2 - [D(M)]^2\}. \end{aligned} \tag{11}$$

On the other hand,

$$\mathrm{Tr}[(M_d)^T M_d] = m_{00}^2 + \sum_{k=1}^{3} m_{0k}^2 = m_{00}^2\{1 + [D(M)]^2\}, \tag{12}$$

where the diattenuation parameter, $D(M)$, has been employed [9,10],

$$0 \le D(M) = \sqrt{m_{01}^2 + m_{02}^2 + m_{03}^2}/m_{00} \le 1. \tag{13}$$

We define the metric $Q(M)$ as the ratio of Eqs. (10) or (11) and Eq. (12)[13],

$$Q(M) \equiv \frac{\sum_{j=1,k=0}^{3} m_{jk}^2}{\sum_{k=0}^{3} m_{0k}^2} = \frac{3[DI(M)]^2 - [D(M)]^2}{1 + [D(M)]^2}. \tag{14}$$

Observe that $Q(M)$ can be written in terms of the diattenuation and the polarizance parameters also [13]:

$$Q(M) = \frac{\left\{\sum_{j,k=1}^{3} m_{jk}^2\right\}/m_{00}^2 + [P(M)]^2}{1 + [D(M)]^2}, \tag{15}$$

where the polarizance parameter $P(M)$ is defined by [9,10]

$$0 \le P(M) = \sqrt{m_{10}^2 + m_{20}^2 + m_{30}^2}/m_{00} \le 1. \tag{16}$$

By using the physical limits for $DI(M)$ and for $D(M)$, the bounds on the metric $Q(M)$ can be easily shown to be [13]

$$\begin{aligned} 0 \le Q(M) &= \frac{\sum_{j=1,k=0}^{3} m_{jk}^2}{\sum_{k=0}^{3} m_{0k}^2} = \frac{3[DI(M)]^2 - [D(M)]^2}{1 + [D(M)]^2} \\ &= \frac{\left\{\sum_{j,k=1}^{3} m_{jk}^2\right\}/m_{00}^2 + [P(M)]^2}{1 + [D(M)]^2} \le 3, \end{aligned} \tag{17}$$

where $Q(M) = 0$ for a totally depolarizing optical system; $0 < Q(M) < 1$ for a partially depolarizing optical system; $1 \le Q(M) < 3$ represents a partially depolarizing system if, in addition, $0 < DI(M) < 1$, and otherwise represents a nondepolarizing diattenuating system; and $Q(M) = 3$ for a nondepolarizing nondiattenuating optical system. Note carefully that $Q(M)$ is the unique scalar metric that, by itself, provides more information about the internal nature of an optical system. This can be observed when $Q(M) = 3$, which means the system is nondepolarizing and nondiattenuating also. On the other hand, the depolarization index $DI(M)$, by itself, is only capable of identifying a system as nondepolarizing when $DI(M) = 1$, but cannot distinguish its internal nature (if in addition it is diattenuating or not).

Note carefully that nondepolarizing diattenuating systems and non-depolarizing nondiattenuating systems are in essence deterministic systems (systems described by Mueller–Jones matrices), because $DI(M) = 1$ for both of them. Gil has shown that a necessary and sufficient condition for a Mueller matrix to be derivable from a Jones matrix is fulfilled by a nondepolarizing nondiattenuating system [12]; this means that $Q(M)$ enables us to distinguish between deterministic polarization systems.

3. Results

We can test the power of $Q(M)$ against the traditional metrics $DI(M)$ and $DoP(M,S)$ by using a partial polarizer with its transmission axis along x, with a maximum transmission of 1 and a minimum transmission of b [14],

$$M = \frac{1}{2}\begin{bmatrix} 1+b & 1-b & 0 & 0 \\ 1-b & 1+b & 0 & 0 \\ 0 & 0 & 2\sqrt{b} & 0 \\ 0 & 0 & 0 & 2\sqrt{b} \end{bmatrix}. \tag{18}$$

By applying Eq. (3) to Eq. (18), we obtain

$$DI(M) = \frac{\sqrt{1+2b+b^2}}{1+b} = 1, \quad \text{for } 0 \le b \le 1, \tag{19}$$

and the conclusion is that, *according to the metric* $DI(M)$, Eq. (18) describes a nondepolarizing system.

Note very carefully that to obtain the value provided by the metric $DoP(M,S)$, Eq. (4), we need, *in addition to* Eq. (18), to use an appropriate incident Stokes vector value. We are considering the system as a black box; otherwise, by knowing it is a nondepolarizing system, any incident Stokes vector should provide the same degree of polarization equal to 1. Let us start by considering an arbitrary incident Stokes vector and applying Eq. (1) and (4) to Eq. (18); we get

$$\begin{pmatrix} s_0^o \\ s_1^o \\ s_2^o \\ s_3^o \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1+b & 1-b & 0 & 0 \\ 1-b & 1+b & 0 & 0 \\ 0 & 0 & 2\sqrt{b} & 0 \\ 0 & 0 & 0 & 2\sqrt{b} \end{bmatrix}\begin{pmatrix} s_0^i \\ s_1^i \\ s_2^i \\ s_3^i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+b)s_0^i + (1-b)s_1^i \\ (1-b)s_0^i + (1+b)s_1^i \\ 2\sqrt{b}s_2^i \\ 2\sqrt{b}s_3^i \end{pmatrix}, \tag{20}$$

$$DoP(M,S) = \frac{\sqrt{[(1-b)s_0^i/2 + (1+b)s_1^i/2]^2 + \left[\sqrt{b}s_2^i\right]^2 + \left[\sqrt{b}s_3^i\right]^2}}{(1+b)s_0^i/2 + (1-b)s_1^i/2}. \tag{21}$$

Considering a linear polarization state parallel to the x axis, then using $s_0^i = s_1^i = 1$ and $s_2^i = s_3^i = 0$ in Eq. (21), finally we obtain the depolarization value provided by this metric:

$$DoP(M) = 1 \quad \text{for } 0 \le b \le 1, \tag{22}$$

and the conclusion is that *according to* $DoP(M,S)$, Eq. (18) is associated with a nondepolarizing system.

On the other hand, applying $Q(M)$ to Eq. (18), we obtain directly, *without needing any value for the incident Stokes vector*, the result

$$Q(M) = \frac{1+4b+b^2}{1+b^2} = \begin{cases} 3 & \text{for } b = 1 \\ 1 & \text{for } b = 0 \\ 1 < Q(M) < 3 & \text{for } 0 < b < 1 \end{cases}, \tag{23}$$

and the conclusions are that the partial polarizer represented by Eq. (18) is a nondepolarizing diattenuating optical system for all the values given by $0 \le b < 1$ $[1 \le Q(M) < 3]$ and becomes a nondepolarizing nondiattenuating optical system for $b = 1$ $[Q(M) = 3]$; that means that the Mueller matrix given by Eq. (18) is not only a nondepolarizing system, as it has been reported always, but indeed it is a nondepolarizing diattenuating or a nondepolarizing nondiattenuating system, depending on the values assigned to b, Eq. (23).

Now, consider a Mueller matrix associated with the system [15],

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.001 & 0.258 & 0.01 & 0.009 \\ 0.028 & 0.01 & 0.241 & -0.015 \\ 0.064 & 0.009 & -0.015 & 0.541 \end{bmatrix}. \tag{24}$$

Proceeding in a similar way as for the analysis of Eq. (18), the following values are obtained for the metrics considered here:

$$DI(M) = 0.375, \qquad DoP(M,S_r) = 0.605, \qquad Q(M) = 0.423, \tag{25}$$

where the subscript r denotes the maximum value for the degree of polarization when a right-hand polarization is incident on the system. Note that all the metrics describe the Mueller matrix as associated with a partial depolarizing system, as was reported by the authors of Ref. [15]. Figures 1(a) and 1(b) represent the $DoP(M,S)$ and the gain, respectively, when all the incident Stokes vectors are taken from the Poincarè sphere [10]. The figures also show the maximum and the minimum values and the angles at which they occur.

Consider the following Mueller matrix [16]:

$$M = \begin{bmatrix} 0.7599 & 0.0257 & 0.1206 & -0.0576 \\ 0.0372 & 0.5285 & 0.0001 & -0.0496 \\ 0.1208 & -0.0001 & 0.6184 & 0.1920 \\ -0.0554 & -0.0572 & -0.1794 & 0.4822 \end{bmatrix}. \tag{26}$$

The single numeric values are obtained for this case:

$$DI(M) = 0.7623, \qquad DoP(M,S_{+45}) = 0.8819, \qquad Q(M) = 1.6579, \tag{27}$$

where the maximum degree of polarization is obtained for an incident Stokes vector with a +45° linear polarization state. Observe that $1 \le Q(M) < 3$ and $DI(M) < 1$; it does follow that $Q(M)$ describes

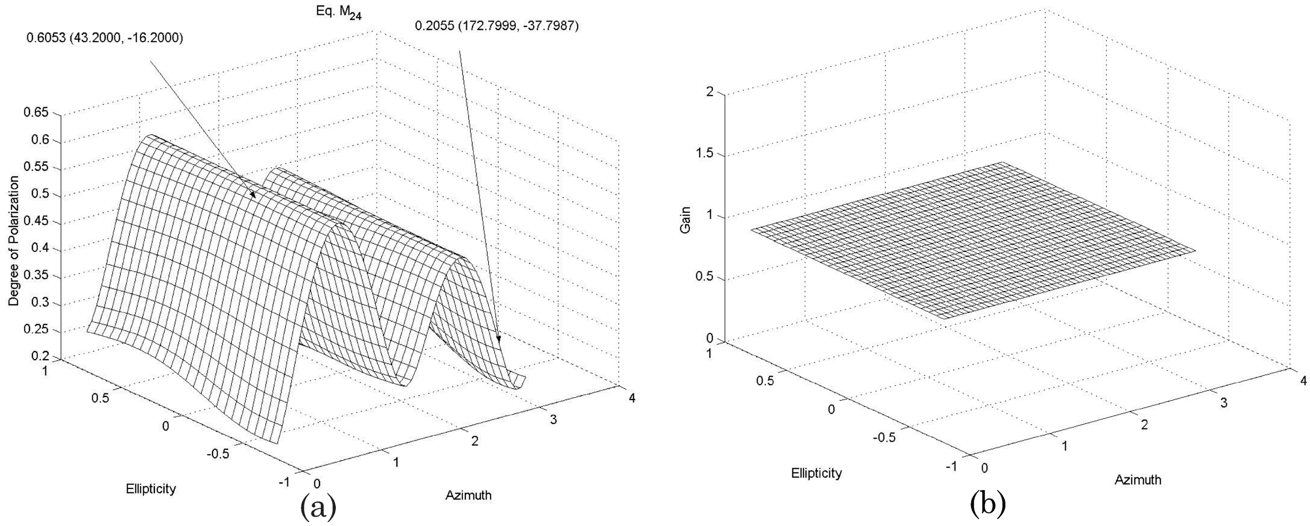

Fig. 1. (a) Output degree of polarization of the optical system described by Eq. (24) as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ. (b) Plot of the gain of the optical system described by Eq. (24) as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ.

a partially depolarizing system, a result consistent with Eqs. (3) and (4) [13,16]. Figures 2(a) and 2(b) represent the $DoP(M, S)$ and the gain, respectively, when all the incident Stokes vectors are taken from the Poincarè sphere [10]. The figures also show the maximum and the minimum values and the angles at which they occur.

In the scattering of light by particles, the degree of polarization behavior is of great interest because it is associated with the own nature of the systems under study. This is particularly important for the diagnostics of objects of a biomedical nature [17]. When DoP does not depend on the state of the incident radiation, it is called isotropic depolarization; otherwise, it is named anisotropic depolarization [17,18]. Savenkov *et al.* have reported theoretical and experimental analysis of optical systems where DoP takes on extreme values depending on the input polarization states for the same system [19].

On the other hand, to think that a visual inspection of a Mueller matrix is good enough to know that it is diattenuating or not, would be equivalent to using the same criterion to validate a polarizing from a nonpolarizing system, and so on for another optical system whose form is associated with a particular behavior (retarder, diattenuating, and depolarizing, among other possibilities). The sense of a single scalar metric is just to provide the maximum possible information about the internal nature of the system under study.

The metric $Q(M)$ can be obtained from well established relations such as the nine bilinear constraints

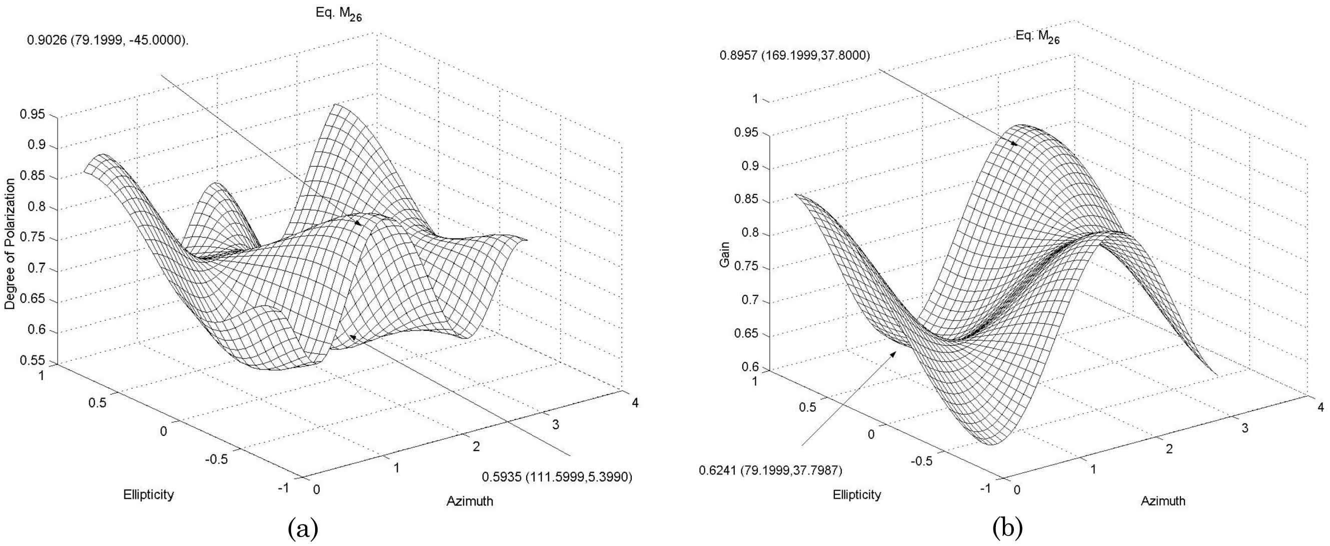

Fig. 2. (a) Output degree of polarization of the optical system described by Eq. (26) as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ. (b) Plot of the gain of the optical system described by Eq. (26) as a function of the incident state of polarization parametrized by the ellipsometric angles χ and ψ.

between the sixteen elements of the Mueller–Jones matrix [13] or from the degree of polarization (deduced here). In the following we discuss the properties of $Q(M)$ and the depolarization index $DI(M)$. Observe that the pair $Q(M)$ and $DI(M)$ provides the same internal information as the pair $DI(M)$ and $D(M)$. Remember that $DI(M)$ and $D(M)$ are used in Eq. (17) as a way to determinate the physical limits for $Q(M)$. Both $Q(M)$ and $DI(M)$ are calculated from the sixteen elements of the Mueller matrix only, Eqs. (14) and (3), respectively. Note that $Q(M)$ by itself can identify a system as a totally depolarizing $[Q(M) = 0]$, as a partially depolarizing $[0 < Q(M) < 1]$, and as a nondepolarizing nondiattenuating system $[Q(M) = 3]$, respectively. $DI(M)$ by itself can also identify a system as totally depolarizing $[DI(M) = 0]$, partially depolarizing $[0 < DI(M) < 1]$, and nondepolarizing $(DI(M) = 1)$, respectively.

On the other hand, observe that $DI(M)$ can also identify a system as nondepolarizing nondiattenuating only if $D(M)$ has been calculated and it does take the zero value. If $1 \leq Q(M) < 3$, it does describe a nondepolarizing diattenuating system if $DI(M) = 1$; otherwise, it describes a partially depolarizing system.

We believe that $Q(M)$ is an alternative metric to the existing scalar metrics $DI(M)$ and $D(M)$ and that it has a solid, well established basis that make it plausible to be employed as an alternative to the scalar metrics previously reported. To our knowledge, $Q(M)$ works properly for any Mueller matrix.

4. Conclusions

Our conclusions are that the depolarization scalar metric for Mueller matrices, $Q(M)$, and some relations with the depolarization index, the diattenuation, and the polarizance parameters, have been obtained by decomposing the Mueller matrix into two parts. We have applied some metrics, including $Q(M)$, to several systems, as a way to test our metric. We have shown that the $Q(M)$ metric is sensitive to the internal nature of the depolarizing Mueller matrices and that it can be calculated directly from the sixteen elements of the Mueller matrix, Eq. (14). Four bounds were fixed for $Q(M)$, which allows us to identify a Mueller matrix as totally depolarizing, partially depolarizing, nondepolarizing diattenuating, and nondepolarizing nondiattenuating, respectively. Finally, we have discussed the properties between $Q(M)$ and the depolarization index.

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