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PAPER

A new method to sum divergent power series: educated match

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Abstract

We present a method to sum Borel- and Gevrey-summable asymptotic series by matching the series to be summed with a linear combination of asymptotic series of known functions that themselves are scaled versions of a single, appropriate, but otherwise unrestricted, function Φ. Both the scaling and linear coefficients are calculated from Padé approximants of a series transformed from the original series by Φ. We discuss in particular the case that Φ is (essentially) a confluent hypergeometric function, which includes as special cases the standard Borel–Padé and Borel–Leroy–Padé methods. A particular advantage is the mechanism to build knowledge about the summed function into the approximants, extending their accuracy and range even when only a few coefficients are available. Several examples from field theory and Rayleigh–Schrödinger perturbation theory illustrate the method.

1. Introduction

Summation of divergent asymptotic expansions has led to a vast literature from both mathematical and physical points of view. The mathematical goal is often to assign a standard sum to a series whose coefficients satisfy certain growth conditions and whose sum satisfies certain conditions at infinity [1, 2]. The physical literature focuses on a wide range of specialized, computational methods. Especially since the work carried out in the 1970s on the coupling constant analyticity of anharmonic oscillators [3–5], two summation methods have become dominant: Padé approximants and Borel summation. Both have been found useful in fields as diverse as quantum mechanics, statistical mechanics, quantum field theory, and string theory. Padé approximants are most often directly used empirically (see, for example, the recent study on the existence of an ultraviolet zero for the six-loop beta function of the $\lambda\Phi_4^4$ theory [6]), and at times with new, alternative transformation procedures [7]. Borel summability has been rigorously proved in several instances. The analytic continuation implicit in the Borel summation process poses a practical problem that has been dealt with in essentially two ways: conformal mappings [8–10], and Borel–Padé approximants. In the latter, the analytic continuation is again performed empirically by Padé approximants of the Borel-transformed series [3, 11–13]. Most recently, Mera *et al* [14, 15] and Pedersen *et al* [16] have developed a method that uses hypergeometric functions to sum perturbation theory series using only a few terms.

Initially motivated in part by the papers of Mera *et al*, we present here a new method to build concise, explicit, analytic approximants to the Borel or Gevrey sum of an asymptotic power series. These approximants *match* the series to be summed with a linear combination of asymptotic series of known functions. The known functions are scaled versions of a single function Φ, and both the scaling and linear coefficients are readily calculated from Padé approximants of a transformed series determined by the original series and by Φ. If Φ is taken to be (essentially) a confluent hypergeometric function, the new method includes as special cases the standard Borel–Padé and Borel–Leroy–Padé summation methods. Even more important, prior additional (i.e., *educated*) knowledge about the summed function can be built into the approximants via the function Φ, sometimes dramatically extending the accuracy and range of the approximants. The 'linear combination' here is similar to the linear combination of the Janke–Kleinert resummation algorithm, which is described as 're-expanding the asymptotic expansion in a complete set of basis functions', and which is mathematically equivalent to conformal mapping techniques [17]. Our method, in contrast, is essentially linked to the theory of Padé approximants.

2. Φ-Padé approximants

Our goal is to approximate the Borel sum $\psi(z)$ of a divergent power series,

$$\psi(z) \sim \sum_{k=0}^{\infty} d_k z^k, \tag{1}$$

using *any appropriate known function* $\Phi(z)$ with its own Borel-summable series,

$$\Phi(z) \sim \sum_{k=0}^{\infty} f_k (-z)^k. \tag{2}$$

The method is at the same time hidden in, and a generalization of, the Borel–Padé summation method [13], which we briefly review.

Let us denote by $P_{n-1}(z)/Q_n(z)$ the $[n-1, n]$ Padé approximant of the Borel transform of the series (1),

$$\hat{\psi}_{\mathrm{B}}(z) = \sum_{k=0}^{\infty} \frac{d_k}{k!} z^k, \tag{3}$$

and let us assume that $Q_n(z)$ has only simple zeros. Partial fraction expansion,

$$\frac{P_{n-1}(z)}{Q_n(z)} = \sum_{j=1}^{n} \frac{r_j}{z - z_j}, \tag{4}$$

and term-by-term integration lead to the standard Borel–Padé approximant $\psi_{\mathrm{B},[n-1,n]}(z)$ to $\psi(z)$,

$$\psi_{\mathrm{B},[n-1,n]}(z) = \int_0^{\infty} \mathrm{e}^{-t} \sum_{j=1}^{n} \frac{r_j}{zt - z_j} \mathrm{d}t, \tag{5}$$

$$= \sum_{j=1}^{n} \frac{r_j}{-z_j} E_{\mathrm{Euler}}(-z/z_j), \tag{6}$$

where we define $E_{\mathrm{Euler}}(z)$ by

$$E_{\mathrm{Euler}}(z) = \int_0^{\infty} \frac{\mathrm{e}^{-t}}{1 + zt} \mathrm{d}t = z^{-1} \mathrm{e}^{1/z} E_1(1/z), \tag{7}$$

and where $E_1(1/z)$ is a standard version of the exponential integral (see chapter 5 of [18]).

The two points to note in this derivation are (i) that the $E_{\mathrm{Euler}}(z)$ in equation (7) is precisely the Borel sum of the factorially divergent Euler series [19] obtained by setting $f_k = k!$ in equation (2), and (ii) that the asymptotic expansion of $\psi_{\mathrm{B},[n-1,n]}(z)$ is identical to that of $\psi(z)$ through order z^{2n-1}, i.e., that

$$\sum_{j=1}^{n} \frac{r_j}{-z_j} E_{\mathrm{Euler}}(-z/z_j) = \sum_{k=0}^{2n-1} d_k z^k + O(z^{2n}). \tag{8}$$

In principle, the $2n$ parameters z_j and r_j could have been determined *de nouveau* from equation (8) by substituting in it the Euler series and equating coefficients.

These observations motivate the following generalization of the Borel–Padé summation method. We define the 'Φ-transform' of the series given in equation (1) by

$$\hat{\psi}_{\Phi}(z) = \sum_{k=0}^{\infty} \frac{d_k}{f_k} z^k, \tag{9}$$

and the associated new approximants $\psi_{\Phi,[n-1,n]}(z)$ to $\psi(z)$ by

$$\psi_{\Phi,[n-1,n]}(z) = \sum_{j=1}^{n} \frac{r_j}{-z_j} \Phi(-z/z_j). \tag{10}$$

As a generalization of equation (8), the approximants $\psi_{\Phi,[n-1,n]}(z)$, which depend on $2n$ parameters r_j, z_j, $(j = 1,\ldots,n)$, satisfy

$$\sum_{j=1}^{n} \frac{r_j}{-z_j} \Phi(-z/z_j) = \sum_{k=0}^{2n-1} \mathrm{d}_k z^k + O(z^{2n}), \tag{11}$$

and therefore the r_j and z_j solve the $2n$ equations

$$\sum_{j=1}^{n} \frac{r_j}{-z_j} f_k (z_j)^{-k} = d_k, \quad (k = 0, 1, \ldots, 2n - 1). \tag{12}$$

In practice, these parameters are most easily calculated from the partial fraction expansion of the $[n - 1, n]$ Padé approximant to $\hat{\psi}_\Phi(z)$, i.e.,

$$\frac{P_{n-1}(z)}{Q_n(z)} = \sum_{k=0}^{2n-1} \frac{d_k}{f_k} z^k + O(z^{2n}) = \sum_{j=1}^{n} \frac{r_j}{z - z_j}. \tag{13}$$

In other words, the z_j are the poles, for simplicity assumed to be simple, and the r_j the residues, of the $[n - 1, n]$ Padé approximant to $\hat{\psi}_\Phi(z)$. Accordingly we call $\psi_{\Phi,[n-1,n]}(z)$ the '$[n - 1, n]$ Φ-Padé approximant' to $\psi(z)$.

The Borel–Padé approximant uses no information about the sum $\psi(z)$ except for Borel summability. Generally these approximations will not be accurate over the full range of the variable z. By an 'educated' choice of $\Phi(z)$, we mean building additional knowledge about the nature of $\psi(z)$ into $\Phi(z)$, which may lead to very accurate approximations over the full range of the variable z even when only a very limited number of coefficients d_k of the original asymptotic series are available. Typical examples of prior knowledge that can be built into the Φ-Padé approximations are the large z behavior of $\psi(z)$ or perhaps the large k behavior of the coefficients d_k.

2.1. The confluent hypergeometric Φ

A prime candidate for Φ is the confluent hypergeometric function U (see chapter 13 of [18]) or, more precisely, the function

$$\Phi(z) = z^{-a} U(a, 1 + a - b, 1/z), \tag{14}$$

for which the coefficients f_k in equation (2) are

$$f_k = \frac{(a)_k (b)_k}{k!}, \tag{15}$$

where the Pochhammer symbol $(c)_k$ is defined by $(c)_k = \Gamma(c + k)/\Gamma(c)$. Note that this $\Phi(z)$ is symmetric in a and b, which is more obvious from equation (15) than from equation (14). From a theoretical point of view the confluent hypergeometric U is a natural choice for at least two reasons. (i) the Borel–Padé method is the special case $a = b = 1$, since

$$z^{-1} U(1, 1, 1/z) = z^{-1} e^{1/z} E_1(1/z), \tag{16}$$

which is the $E_{\text{Euler}}(z)$ of equation (7). (ii) Just as the Borel transform is inverted by the Laplace transform, there is a generalization (which we state without proof) that inverts the 'confluent hypergeometric transform' (see equations (9) and (15)): if

$$\hat{\psi}_\Phi(z) = \sum_{k=0}^{\infty} \frac{d_k k!}{(a)_k (b)_k} z^k, \tag{17}$$

then

$$\psi(z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \hat{\psi}_\Phi(zs) e^{-s} s^{a-1} U(1 - b, a - b + 1, s) ds. \tag{18}$$

(When $b = 1$, $U(0, a, t) = 1$, and the result is the Borel–Leroy transformation [10].) From a practical point of view, the confluent hypergeometric function (14) is also a very convenient choice, because as $z \to \infty$,

$$\Phi(z) \sim z^{-b} \frac{\Gamma(a - b)}{\Gamma(a)} + z^{-a} \frac{\Gamma(b - a)}{\Gamma(b)}, \quad (a - b \neq \text{integer}), \tag{19}$$

$$\sim z^{-a} \frac{\log(z) - 2\gamma - \psi^{(0)}(a)}{\Gamma(a)}, \quad (a = b), \tag{20}$$

where γ is Euler's constant and $\psi^{(0)}(a)$ is the polygamma function. Since the approximant $\psi_{\Phi,[n-1,n]}(z)$ depends linearly on Φ (see equation (10)), an appropriate choice of a and b permits the large z behavior (if known) of $\psi(z)$ to be built into the Φ-Padé approximants. We illustrate these general ideas with several examples and generalizations of the method.

3. Examples

3.1. Zero-dimensional ϕ^4 field theory

As the simplest example, the confluent hypergeometric $\Phi = \left(\frac{3}{2g}\right)^{3/4} U\left(\frac{3}{4}, \frac{3}{2}, \frac{3}{2g}\right)$ trivially sums the perturbative series for the partition function $Z(g)$ of zero-dimensional ϕ^4 theory [9, 10], because

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 - gx^4/4!} dx, \tag{21}$$

$$= (3/(2g))^{3/4} U(3/4,\ 3/2,\ 3/(2g)), \tag{22}$$

is equal to the Φ of equation (14) with $a = 3/4$, $b = 1/4$ and $z = 2g/3$. In fact, the $[0,\ 1]$ approximant to the asymptotic expansion of $Z(g)$,

$$Z(g) \sim \sum_{k=0}^{\infty} \frac{\Gamma\left(k + \frac{3}{4}\right)\Gamma\left(k + \frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)k!} \left(-\frac{2g}{3}\right)^k, \tag{23}$$

has $z_1 = -3/2$, $r_1 = 3/2$, and is exactly $Z(g)$.

3.2. The Euler–Heisenberg effective Lagrangian

A second physically relevant example is the Euler–Heisenberg effective Lagrangian [20, 21]. For the spinor case in a purely magnetic background,

$$\mathcal{L}(g) = \int_0^{\infty} e^{-s/g} \left(\coth s - \frac{1}{s} - \frac{s}{3}\right) \frac{ds}{s^2}, \tag{24}$$

(see equations (1.18) and (1.19) in [21]), and has the asymptotic expansion,

$$\mathcal{L}(g) \sim \sum_{k=0}^{\infty} \frac{B_{2k+4}}{(2k+4)(2k+3)(2k+2)} (2g)^{2k+2}, \tag{25}$$

$$\sim -\frac{1}{45} g^2 + \frac{4}{315} g^4 - \frac{8}{315} g^6 + \cdots, \tag{26}$$

where B_{2k+4} denote Bernouilli numbers. Standard Borel–Padé summation of equation (25) would involve Padé approximants in g^2 that lead to rational functions of s^2, i.e., *even* functions of s, that have to approximate the Borel transform, which is an *odd* function of s (essentially the non-exponential factor in the integrand of equation (24)). This parity clash can be avoided by taking

$$\Phi(z) = z^{-2} U(2,\ 2,\ 1/z), \tag{27}$$

i.e., $a = 2$, $b = 1$, and $f_k = (k+1)!$ rather than $k!$. The inverse confluent hypergeometric transform equation (18) contains the explicit factor s, so that the Φ-transform with $a = 2$ and $b = 1$ is in fact an even function of s:

$$\hat{\mathcal{L}}_{\Phi, a=2, b=1}(s) = \left(\coth s - \frac{1}{s} - \frac{s}{3}\right) \frac{1}{s^3}. \tag{28}$$

For every $n \geqslant 1$, all the poles z_j, $(j = 1,\ 2, \ldots, n)$, of the $[n-1,\ n]$ Padé approximants in s^2 to $\hat{\mathcal{L}}_{\Phi, a=2, b=1}(s)$ are negative and simple, meaning that the poles in s are paired on the imaginary axis. The resulting approximants have the form,

$$\mathcal{L}_{\Phi, a=2, b=1; [n-1, n]}(g) = \sum_{j=1}^{n} \frac{r_j}{-z_j} \frac{1}{2} (\Phi(ig/\sqrt{-z_j}) + \Phi(-ig/\sqrt{-z_j})), \tag{29}$$

with the $\Phi(z)$ given by equation (27). For example, the first Padé approximant to the Φ-transformed series is

$$-\frac{g^2}{45} \frac{21}{2} \frac{1}{g^2 + \frac{21}{2}} \sim -\frac{g^2}{45}\left(1 - 45 \frac{4}{315} \frac{1}{3!} g^2 + \cdots\right), \tag{30}$$

with $z_1 = -\frac{21}{2}$, $r_1 = -\frac{g^2}{45}\frac{21}{2}$, and the corresponding Φ-Padé approximant is

$$\mathcal{L}_{\Phi, a=2, b=1; [0,1]}(g) = -\frac{g^2}{45} \frac{1}{2} (\Phi(ig/\sqrt{21/2}) + \Phi(-ig/\sqrt{21/2})). \tag{31}$$

If expanded as a power series in g, this simple approximation reproduces the first two nonvanishing terms of equation (26), but at the same time it also captures the functional form of the large-g expansion: in fact $\mathcal{L}_{\Phi, a=2, b=1; [0,1]}(g) \sim -(7/30)\log(g)$, while the exact result is $\mathcal{L}(g) \sim -(1/3)\log(g)$ [21]. Note that the exact expansion,

$$\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\frac{1}{s^3} = \sum_{j=1}^{\infty} \frac{-2}{j^2\pi^2(j^2\pi^2 + s^2)}, \tag{32}$$

can be viewed as the '$[\infty - 1, \infty]$' Padé approximant in s^2 for the Φ-transform, from which the exact poles and residues can be read off:

$$z_j = -j^2\pi^2, \quad r_j = -\frac{2}{j^2\pi^2}. \tag{33}$$

With Φ given by equation (27), the resulting Φ-Padé infinite sum reproduces $\mathcal{L}(g)$:

$$\mathcal{L}_{\Phi,a=2,b=1;[\infty-1,\infty]}(g) = \sum_{j=1}^{\infty} \frac{-2}{j^4\pi^4} \frac{\Phi(\mathrm{i}g/(j\pi)) + \Phi(-\mathrm{i}g/(j\pi))}{2}. \tag{34}$$

We remark in passing that the coefficients $-2/(j^4\pi^4)$ give the rate of convergence of the approximants.

3.3. One-dimensional ϕ^4 field theory: the quartic anharmonic oscillator

Third, we consider one-dimensional ϕ^4 theory, i.e., the familiar x^4-perturbed anharmonic oscillator, whose Schrödinger equation is

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + gx^4\right)\Psi(x) = E(g)\Psi(x). \tag{35}$$

The first three coefficients of the ground state Rayleigh–Schrödinger perturbation series are

$$E(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots. \tag{36}$$

The coefficients $E^{(k)}$ of this Borel-summable [3] series behave like

$$E^{(k)} \sim (-1)^{k+1}\frac{2^{1/2}3^{k+\frac{1}{2}}}{\pi^{3/2}}\Gamma\left(k + \frac{1}{2}\right), \quad k \to \infty. \tag{37}$$

More important is the large-g behavior of $E(g)$, which follows from a simple scaling argument,

$$E(g) \sim g^{1/3}\varepsilon, \quad \text{as} \quad g \to \infty, \tag{38}$$

where $\varepsilon = 0.667\,986\ldots$ is the ground state energy of the purely quartic oscillator. If the $g^{1/3}$ behavior is built into Φ, then even a two-parameter [0, 1] approximant gives an excellent fit to $E(g)$ all the way from 0 to ∞. The details are elementary enough to execute by hand. Because of the sign pattern, we sum the once-subtracted series,

$$\psi(g) = \frac{E(g) - 1/2}{g}, \tag{39}$$

whose large-g behavior is $g^{-2/3}$ (then multiply by g and add 1/2 to report the results). Equation (19) shows that a suitable Φ with this behavior can be obtained by taking $a = 2/3$ and $b > a$ in equation (14). If b were then chosen to fit the exact quartic ε, its value would be $0.997\,7547\ldots$. We take $b = 1$ (Borel–Leroy–Padé but note that $a = 2/3$ is different from that implied by equation (37)). The $[0, 1]$ Padé approximant to the transformed series, which needs only the two coefficients 3/4 and $-21/8$ from the $E(g)$-series and $f_1 = 2/3$ from the Φ-series, has $z_1 = -4/21$ and $r_1 = 1/7$. The [0, 1] Φ-Padé approximant is

$$E_{\Phi,[0,1]}(g) = \frac{1}{2} + \frac{3}{4}\left(\frac{4}{21}\right)^{2/3} g^{1/3} U\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{21g}\right), \tag{40}$$

which, despite its simple origins, turns out to give remarkable agreement with $E(g)$ for all $g > 0$, as seen in figure 1. At ∞,

$$E_{\Phi,[0,1]}(g) \sim \frac{3}{4}\left(\frac{4}{21}\right)^{2/3}\Gamma\left(\frac{1}{3}\right)g^{1/3}, \tag{41}$$

$$= 0.665\,147\ldots g^{1/3}, \quad \text{as} \quad g \to \infty; \tag{42}$$

the constant $0.665\,147\ldots$ is within 0.4% of the exact quartic ε. Higher-order $[n - 1, n]$ approximants generally agree progressively better. It is clear from figure 1 in which the [0,1] Borel–Padé approximant is also plotted, how relatively simple information used to choose the function generating the match can dramatically affect the quality and range of the approximant[4].

[4] All numerical calculations have been done in extended precision using *Mathematica*, version 11.1; the commands, PadeApproximant and HypergeometricU, were particularly relevant.

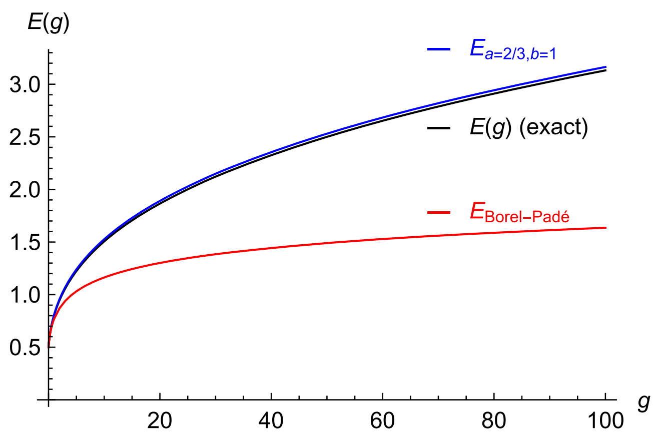

Figure 1. Exact $E(g)$ (black) and [0, 1] approximants for Borel–Padé (red) and ($a = 2/3,\ b = 1$) confluent hypergeometric Φ (blue). The confluent hypergeometric Φ approximant agrees well with the exact $E(g)$, because the $g^{1/3}$ large-g behavior is carried by the $\Phi(g)$.

3.4. Implementation of the large-order behavior of the perturbation coefficients

As an example of the versatility of the method we show how to incorporate in a simple way the asymptotic behavior of the coefficients d_k into the function Φ. We consider the β-function for the ϕ^4 theory in $d = 3$ dimensions [10], with coefficients

$$\tilde{\beta}(\tilde{g}) = 0 - \tilde{g} + \tilde{g}^2 - \frac{308}{729}\tilde{g}^3 + 0.351\,069\,5977\tilde{g}^4 \\ - 0.376\,526\,8283\tilde{g}^5 + 0.495\,54751\tilde{g}^6 - 0.749\,689\tilde{g}^7 + O(\tilde{g}^8), \tag{43}$$

and growth

$$\tilde{\beta}_k \sim (-0.147\,774\,232\ldots)^k k^{7/2} k!, \quad k \to \infty. \tag{44}$$

The [3, 4] Padé approximant for the Borel transform of $\tilde{\beta}(\tilde{g})$ has a pole on the positive axis at $\tilde{g} = 17.34418$ and consequently fails to be analytic in a strip containing the positive real axis, invalidating a possible [3, 4] Borel–Padé approximant. Stirling's formula shows that asymptotically the f_k in equation (15) go like

$$f_k \sim \frac{k! k^{a+b-2}}{\Gamma(a)\Gamma(b)}\left(1 + \frac{a^2 - a + b^2 - b + 1/6}{2k}\right), \quad \text{as} \quad k \to \infty, \tag{45}$$

so that the growth of the coefficients $\tilde{\beta}_k$ in equation (44) is matched when $a + b = 11/2$; the $1/k$-term is then minimum when $a = b = 11/4$. With this straightforward choice of a and b, and with the corresponding [3, 4] approximant to $\tilde{\beta}(\tilde{g})$, we obtain a value for the nontrivial root of the β-function of $\tilde{g}^* = 1.4192$, which is close to the value 1.4105 of [10]. But we have no estimate of the accuracy of our calculation.

4. Φ-Padé approximants for Gevrey-summable series

Next we adapt the new Φ-Padé approximant method to the cases of summable series whose coefficients d_k grow like $(mk)!$, where $m = 2, 3, \ldots$, and which are variously known as generalized Borel summable [3], m-summable or Gevrey-$1/m$ summable [2]. The $m = 2$ case is useful for summing the x^6-perturbed oscillator and the Euler–Heisenberg series (25), and $m = 3$ is useful for the x^8-perturbed oscillator, etc. We regard these series in z with $(mk)!$ growth to be series in $z^{1/m}$ with $k!$ growth, but in which the coefficients of all the fractional powers are 0. By averaging over the m-th roots of unity, from a given $(k!)$-$\Phi(z)$ (equation (2)) we can construct m appropriate 'Gevrey-$1/m$' summed series $\Phi_\mu^{(1/m)}(z)$, $\mu = 0, 1, \ldots, m - 1$. $\Phi_\mu^{(1/m)}(z)$ has the asymptotic series,

$$\Phi_\mu^{(1/m)}(z) \sim \sum_{k=0}^{\infty} f_{\mu+mk}(-z)^k, \tag{46}$$

and the explicit formula,

$$\Phi_\mu^{(1/m)}(z) = \frac{\frac{1}{m}\sum_{j=1}^{m}\omega_m^{-\mu j}\Phi(-\omega_m^{\,j} e^{\pi i/m} z^{1/m})}{(e^{\pi i/m} z^{1/m})^\mu}, \tag{47}$$

where $\omega_m = e^{2\pi i/m}$. The practical procedural consequence is that f_k gets replaced by $f_{\mu+mk}$ in equations (12) and (13). The question, which μ is appropriate, is similar to which a and b are appropriate, and the answers depend on which properties, e.g., large z, d_k for large k, etc., are most appropriate for ψ. Moreover, the same Gevrey-$1/m$

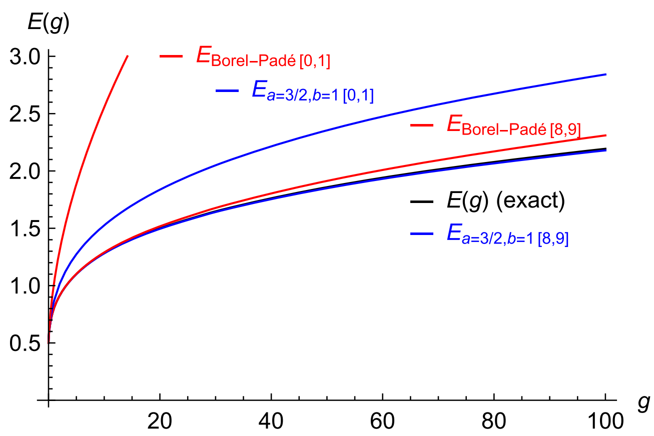

Figure 2. Exact $E(g)$ (black), Borel–Padé approximants (red), and $(a = 3/2,\ b = 1)$-Padé approximants (blue) for the x^6-perturbed anharmonic oscillator. The $g^{1/4}$ large-g behavior is carried by the $(a = 3/2,\ b = 1)$-confluent-hypergeometric-function-based $\Phi_0^{(1/2)}(g)$. The [0, 1] and [8, 9] approximants are shown. The largest relative error for the $(a = 3/2,\ b = 1)$ [8, 9] approximant occurs at $g = 100$ and is less than 0.007, which is barely distinguishable from the exact $E(g)$.

$\Phi_\mu^{(1/m)}$ can result from two different Gevrey-1 Φ's with different μ's, as illustrated in the next three equations and following remark: if, for instance,

$$\Phi(z) \sim \sum_{k=0}^{\infty} k!(-z)^k, \tag{48}$$

then

$$\Phi_0^{(1/2)}(z) \sim \sum_{k=0}^{\infty} (2k)!(-z)^k, \tag{49}$$

$$\Phi_1^{(1/2)}(z) \sim \sum_{k=0}^{\infty} (2k+1)!(-z)^k. \tag{50}$$

The Euler–Heisenberg integral discussed above, particularly equation (29), is better understood as a Gevrey-1/2 series summed by the $\mu = 0$ version of the $\Phi(z)$ given by equation (27), which is the same as the $\mu = 1$ version of $z^{-1}U(1,\ 1,\ 1/z)$ (equation (48)) given by equation (50).

4.1. The sextic anharmonic oscillator

A classic Gevrey-1/2 series is the Rayleigh–Schrödinger perturbation series for the x^6-perturbed anharmonic oscillator (i.e, the Schrödinger equation (35) with gx^4 replaced by gx^6). The first three coefficients of the ground-state energy series are

$$E(g) = \frac{1}{2} + \frac{15}{8}g - \frac{3495}{64}g^2 + \cdots. \tag{51}$$

For large k, the coefficients $E^{(k)}$ behave like

$$E^{(k)} \sim (-1)^{k+1}\left(\frac{32}{\pi^2}\right)^{k+1}\Gamma\left(2k+\frac{1}{2}\right), \quad k \to \infty, \tag{52}$$

and for large g

$$E(g) \sim g^{1/4}\varepsilon, \tag{53}$$

where ε here is the ground-state energy of the pure x^6 oscillator. To build the $g^{1/4}$ behavior into the approximants, we take (for the once-subtracted series) $\Phi(z) = z^{-3/2}U(3/2,\ 1,\ 1/z)$. Although equation (19) seems to imply that the large-z behavior would be z^{-1} rather than $z^{-3/2}$, the z^{-1} term is canceled in constructing $\Phi_0^{(1/2)}$. When the approximant for the subtracted series is multiplied by g, the remaining $(g^{1/2})^{-3/2}$ term gives $g^{1/4}$. The $[0,\ 1]$ Φ-Padé approximant, which like the x^4 case can be done by hand, yields

$$E_{\Phi,[0,1]}(g) = \frac{1}{2} + g\frac{15}{8}\Phi_0^{(1/2)}\left(\frac{30}{233g}\right). \tag{54}$$

This simple $[0,\ 1]$ approximation for the sextic oscillator, while superior to the $[0,\ 1]$ Borel–Padé approximant, is not as dramatically accurate as the analogous approximation for the quartic oscillator given in equation (40), but as n increases the accuracy of the $[n-1,\ n]$ Φ-Padé approximant increases monotonically to the point that in figure 2 it is difficult to distinguish between the exact and [8, 9]-approximant values for $0 \leqslant g \leqslant 100$. (The maximum relative error at $g = 100$ is less than 0.007.) The error in the Borel–Padé approximants is much larger.

5. Summary

In summary, the conceptualization presented here emphasizes matching the series to be summed with a linear combination of asymptotic series of known functions, cf equation (10). The known functions are scaled versions of a single function $\Phi(z)$, and the scaling and linear coefficients are calculated from the $[n - 1, n]$ Padé approximants of the transformed series generated by $\Phi(z)$. The whole idea stems from the realization that the Borel–Padé approximant has exactly that structure, but where the $\Phi(z)$ is the sum of Euler's factorially divergent power series, and from the thought that approximants would be much more accurate if $\Phi(z)$ were more appropriate for the unknown sum $\psi(z)$. Building the long-range behavior of ψ into Φ is particularly successful.

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