

Effect of ABCD transformations on beam paraxiality

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Abstract: The limits of the paraxial approximation for a laser beam under ABCD transformations is established through the relationship between a parameter concerning the beam paraxiality, the paraxial estimator, and the beam second-order moments. The applicability of such an estimator is extended to an optical system composed by optical elements as mirrors and lenses and sections of free space, what completes the analysis early performed for free-space propagation solely. As an example, the paraxiality of a system composed by free space and a spherical thin lens under the propagation of Hermite-Gauss and Laguerre-Gauss modes is established. The results show that the paraxial approximation fails for a certain feasible range of values of main parameters. In this sense, the paraxial estimator is an useful tool to monitor the limits of the paraxial optics theory under ABCD transformations.

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1. Introduction

The concept of *paraxiality* applied to optical beams refers to how accurate is the paraxial approximation (PA) for describing the propagation of a light beam through a given optical system. The ondulatory foundation of the PA is based on the assumption that the complex amplitude of the field slowly varies when compared with the fast oscillatory phase factor. In other words, if the beam is expanded in its plane wave spectrum, the PA holds if and only if the dispersion of the propagation directions of these plane waves is smaller than a determined value. From a geometrical optics viewpoint, the PA is the key assumption to be satisfied for describing the effects of an optical system, composed, for example, by mirrors and/or lenses on light rays. In this frame, rays are near or almost parallel to the optical axis of the system such that the mathematical formulation reduces to a matricial analysis, the so-called ray transfer matrix analysis (or ABCD matrix analysis) taking into account a linear relation between the output and input ray quantities (positions and slopes). The matricial operator, called ABCD matrix, indicates the system effect on the ray tracing. For astigmatic systems, such a matrix must necessarily be 4×4 so that the ABCD theory is bidimensional (two transverse directions) [1]. For non-astigmatic systems, the transfer matrix reduces to 2×2 and only one transverse dimension is necessary [2]. The foundations of the ABCD theory are in detail tackled in the most standard laser textbooks [3, 4]. The fulfillment of the PA is the basis of the development of the *paraxial wave optics* and derived branches of laser optics [3, 4]. It is within the PA framework that imaging properties of lenses and stability of resonators occur [5]. To the extent that the PA fails, the clarity of an image can suffer and unstabilities within a resonator can appear among other undesired effects. Therefore, if one is primarily concerned with minimizing the negative effects concerning the break of the PA validity, it is necessary to predict the cases where the PA could be not strictly met. However, this is not a trivial task because the paraxial-nonparaxial limit is not accurately delimited and depends on both, the optical system characteristics and propagating beam structure. In free space propagation, the PA validity problem was tackled by introducing a single parameter, the so-called paraxial estimator, quantifying the paraxiality of a laser beam [6, 7]. This parameter is based on the comparison of the energy fluxes associated with the full Helmholtz equation and its first-order approximation often called paraxial equation [8, 9]. The idea for quantifying the PA validity by mean of a single parameter was also developed by other authors [10]. In all these cases, the optical system was assumed to be the free space. No analysis on the validity limits of the PA was performed in more complex optical system including (non-astigmatic and astigmatic as well) optical elements like lens and mirrors in addition to the free space.

The lack of such an issue encouraged us to study the PA validity for laser beams under ABCD transformations by using the tool of the paraxial estimator, extending its effectiveness to systems composed by optical elements as lenses and/or mirrors, in addition to the free space. The analysis takes advantage of the relationship between the paraxial estimator and the second order moments [11], that possesses well-defined transformation properties under beam propagation through an ABCD system. The paper is structured as follows: Section 2 reviews the

approaches on the beam second-order moments and its transformation properties upon ABCD transformations. Section 3 points out the relationship between the paraxial estimator and the beam second moment matrix transformations. In Section 4, the paraxial estimator is derived for Hermite-Gauss and Laguerre-Gauss modes crossing an optical system composed by a free space section and a thin lenses. Numerical simulations to analyze the validity of the PA on those systems are also performed. Finally, Section 5 gives the concluding remarks.

2. Beam second-order moments and ABCD optical systems

Let us consider a coherent monochromatic beam that propagates (and spreads) in the $z > 0$ direction. Its complex amplitude in a transverse plane (fixed value of z) is given by the function $E(x, y)$, where we assume that the beam waist (minimum transverse size), w_m , is located at the plane $z = 0$. This plane is the input plane to a single or cascade ABCD system. The plane wave decomposition of such a beam around the z axis is done by means of the bidimensional Fourier transform (FT) [12]

$$\mathcal{E}(u, v) = \int E(x, y) e^{-2\pi i(ux + vy)} dx dy, \quad (1)$$

where $\mathcal{E}(u, v)$ is the Fourier amplitude of the beam in the conjugate space, (u, v) , with $2\pi u$ and $2\pi v$ being the transverse wavenumbers associated with the x and y coordinates, respectively. We will assume the indefinite integrals to the whole real axis if not stated otherwise. Notice that u and v are dimensional coordinates with units of $(\text{length})^{-1}$. It is a common practice to describe the beams by means of the Wigner distribution (WD) for its characterization. The WD is a real function that represents the beam in the phase-space, (x, y, u, v) , i. e., it possesses the information from both the spatial and spectral amplitudes of the beam. Consequently, the global properties of the beam can easily be extracted calculating the moments of the WD, which are indirectly measurable [13, 14].

For a beam whose complex amplitude is $E(x, y)$ in the plane $z = 0$, its WD is defined as

$$W(\mathbf{r}, \mathbf{p}) = \int \left\langle E\left(\mathbf{r} + \frac{1}{2}\mathbf{r}'\right) E^*\left(\mathbf{r} - \frac{1}{2}\mathbf{r}'\right) \right\rangle \exp(-i2\pi\mathbf{r}' \cdot \mathbf{p}) d\mathbf{r}', \quad (2)$$

where $\mathbf{r} = [x, y]^t$ and $\mathbf{p} = [u, v]^t$ are the spatial and spectral column vectors, correspondingly, t denotes the transpose operation and $\langle \cdot \rangle$ the ensemble average. Its WD moments are defined in the same way the statistical moments are defined for statistical distributions. In particular the only zero-order moment is proportional to the irradiation of the beam, while its four first-order moments define the centroid and propagation direction of the beam. Without loss of generality one can always select such an axis for the transverse plane that the first-order moments of the beam are zero. Then, we can define the normalized second-order moments in this frame of reference as

$$m_{ss'} = \frac{\int ss' W(\mathbf{r}, \mathbf{p}) d\mathbf{r} d\mathbf{p}}{\int W(\mathbf{r}, \mathbf{p}) d\mathbf{r} d\mathbf{p}}, \quad (3)$$

where s and s' are placeholders for the spatial and spectral coordinates, x, y, u , and v . These moments are widely used in beam characterization as they contain important information, as for example, the beam size and beam spread in the x and y spatial coordinates and the quality factor of the beam [15]. All the ten independent second-order moments of a beam can conveniently be arranged into a 4×4 real symmetric positive-definite matrix [16]

$$\mathbf{M} = \begin{bmatrix} \mathbb{M}_{qq} & \mathbb{M}_{qp} \\ \mathbb{M}_{qp}^t & \mathbb{M}_{pp} \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xu} & m_{xv} \\ m_{yx} & m_{yy} & m_{yu} & m_{yv} \\ m_{ux} & m_{uy} & m_{uu} & m_{uv} \\ m_{vx} & m_{vy} & m_{vu} & m_{vv} \end{bmatrix}, \quad (4)$$

where the 2×2 sub-matrices \mathbb{M} have different physical dimensions: while \mathbb{M}_{qp} is dimensionless, \mathbb{M}_{qq} has units of $(\text{length})^2$ and \mathbb{M}_{pp} of $(\text{length})^{-2}$.

The second-order moment matrix \mathbf{M} is defined only for a fixed transversal plane (in our particular case, for the input plane $z = 0$), so it may change (and in general does) as the beam propagates through an optical system to the output plane $z = z'$. In this paper we consider the propagation through any two dimensional ABCD lossless optical system, including anisotropic (astigmatic) optical systems that are described by a 4×4 matrix called characteristic ray-transformation matrix:

$$\mathbf{T} = \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix}, \quad (5)$$

with \mathbb{A} , \mathbb{B} , \mathbb{C} , and \mathbb{D} being 2×2 sub-matrices. It has to satisfy the symplectic condition [17, 18]

$$\mathbf{T}\Omega\mathbf{T}^t = \Omega, \quad \Omega = \begin{bmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{bmatrix}, \quad (6)$$

where \mathbb{O} and \mathbb{I} are the 2×2 null and identity matrices, respectively. When the beam propagates through this kind of optical systems, its second-order moment matrix transforms from \mathbf{M} in the input plane $z = 0$ to \mathbf{M}' in the output plane $z = z'$ by the relation [1]:

$$\mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^t \quad (7)$$

Notice that, in order to maintain the same physical units in \mathbf{M}' and \mathbf{M} , submatrices \mathbb{A} and \mathbb{D} have to be dimensionless, while \mathbb{B} has to have units of $(\text{length})^2$ and \mathbb{C} of $(\text{length})^{-2}$. If we have a ABCD system as a cascade of n optical elements (e.g. a sequence of spaces and lenses) whose single ray-transfer matrix are $\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_n$, it is well-known that such multi-element system is equivalent to a single optical component of ray-transfer matrix $\mathbf{M} = \mathbf{T}_n \dots \mathbf{T}_2 \mathbf{T}_1$. The above description of the second-order moments and rules of transformation through ABCD optical systems give us the machinery to quantitatively analyze the validity of the paraxial approximation. It must be emphasized that the ABCD matrices are real-valued for the cases treated here (lossless media). But complex-valued matrices are also possible that arise, for example, from optical systems with inherent apertures. The effects of finite-sized optical elements, tilt and random jitter, and distributed random inhomogeneities along the optical path were rigorously tackled in Ref. [19]. The same Authors have also treated the inclusion of stochastic fields in complex ABCD optical systems [20]. It could be of wider interest in extending the formalism developed here for including complex-valued matrices and stochastic fields. This will be done in a future work.

3. Beam paraxiality under ABCD transformations

In Ref. [6], a parameter so-called the paraxial estimator, \mathcal{P} , quantifying the validity of the paraxial approximation in free space propagation was introduced and applied in analyzing the paraxiality of Hermite-Gaussian, Laguerre-Gaussian, and Bessel-Gaussian modes. The definition of this parameter was based on the comparison between the propagation invariants of Helmholtz and paraxial equations. It was established that \mathcal{P} is a real scalar value less than one. As \mathcal{P} tends to unit, the PA fulfils more robustly for a given light beam. Working in the conjugate space, \mathcal{P} is expressed in terms of the wavelength of the light, λ , and the beam second-order moments in the conjugate space m_{uu} and m_{vv} , that are associated with the spread of the beam in the transverse plane. Explicitly [11],

$$\mathcal{P} = 1 - \frac{\lambda^2}{2} (m_{uu} + m_{vv}). \quad (8)$$

From Eq. (8), one can evaluate the paraxial estimator plane to plane using the second-order moment matrix transformation law [recall Eq. (7)]. In particular, that evaluation can be done at the input plane in which the paraxial estimator is denoted by \mathcal{P} and at the output plane in which the paraxial estimator is denoted by \mathcal{P}' . In this case, we are analyzing the validity of the PA for a given system between both planes. The paraxial estimator at the output plane $z = z'$ will be

$$\mathcal{P}' = 1 - \frac{\lambda^2}{2} (m'_{uu} + m'_{vv}) = 1 - \frac{\lambda^2}{2} \text{tr } \mathbb{M}'_{pp}, \quad (9)$$

where the prime denotes evaluation in the output plane of the optical system. Using Eq. (7) we can relate the required output moments with the input moments as follows:

$$\mathbb{M}'_{pp} = \mathbb{C}\mathbb{M}_{qq}\mathbb{C}^t + \mathbb{C}\mathbb{M}_{qp}\mathbb{D}^t + \mathbb{D}\mathbb{M}_{pq}\mathbb{C}^t + \mathbb{D}\mathbb{M}_{pp}\mathbb{D}^t. \quad (10)$$

It is important to be aware that the paraxial estimator in the exiting plane of an ABCD optical system depends on two sets of parameters: the beam characteristics, encoded in both, the wavelength and the second-order moments; and the optical system properties, encoded in the ray-transformation matrix.

The study of the validity of the PA on a given ABCD system is then reduced to calculate the trace of \mathbb{M}'_{pp} using Eq. (10) and introduce the results in Eq. (9). Also, it is obvious from Eqs. (9,10) that \mathcal{P}' does not depend on the submatrices \mathbb{A} and \mathbb{B} of the ABCD system. This fact is important because it directly yields that the single free-space propagation does not change this system paraxiality. On the contrary, the inclusion of linear optical elements as mirrors and lenses, in addition to the free space, could significantly alter the system paraxiality as it will showed below.

4. Numerical examples: paraxiality for free-space plus a thin lens under Hermite-Gauss and Laguerre-Gauss propagation

In order to show the powerfulness of the paraxial estimator for estimating the validity of the PA for ABCD systems, we will quantitatively analyze the findings of Section 3 by its application to widely-used optical systems composed by free-space and spherical thin lenses under propagation of Laguerre-Gauss (LG) and Hermite-Gauss (HG) modes. Numerical examples for single free-space (FS), single spherical thin lens (TL) and free space followed by a thin lens (FS+TL) will be given. The later is an example of a cascade system. All these systems are illustrated in Fig. 1. On the other hand, LG and HG are natural modes of the transverse distribution of the electric field within a laser cavity (for circular and rectangular symmetries) characterized by two measurable parameters of length: the wavelength λ and the transverse size parameter w_0 , this last encoded in its second moments. The HG modes in the waist transverse plane $z = 0$ are given in rectangular coordinates (x, y) by

$$E_{m,n}(x, y, 0) = E_0 H_m \left(\sqrt{2} \frac{x}{w_0} \right) H_n \left(\sqrt{2} \frac{y}{w_0} \right) \exp \left(-\frac{x^2 + y^2}{w_0^2} \right) \quad (11)$$

where E_0 is a constant with electric field dimensions and $H_{m,n}$ is the Hermite Polynomial of degree m, n . The $\text{LG}_{\ell,p}$ mode profiles can be written in cylindrical coordinates (r, ϕ, z) at the beam waist as

$$E_{\ell,p}(r, \phi, 0) = E_0 2^{\ell/2} L_p^\ell \left(2 \frac{r^2}{w_0^2} \right) \exp(i\ell\phi) \exp \left(-\frac{r^2}{w_0^2} \right) \quad (12)$$

where ϕ is the azimuthal coordinate and L_p^ℓ are the generalized Laguerre polynomials. The order-mode for both families is N , being $N = m + n$ for HG and $N = 2p + |\ell|$ for LG. The

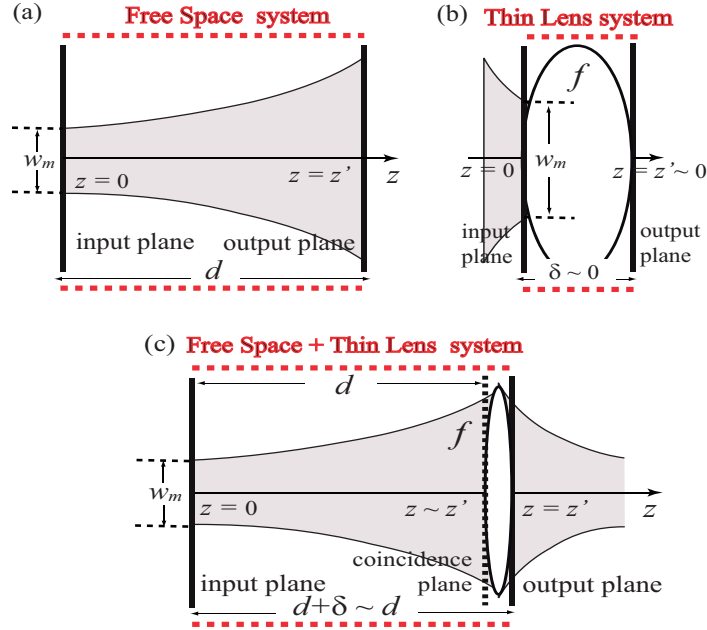


Fig. 1. Illustration of several ABCD system crossed by a light beam of waist size w_m . The systems begin at the beam waist plane ($z = 0$) and end at the output plane ($z = z'$). (a) free space propagation over a distance d . (b) spherical thin lens (of negligible thickness δ) and focal length f . (c) free space plus spherical thin lens.

mode $N = 0$, common to both families, is the fundamental Gaussian beam. This mode fulfills $w_m = w_0$, so that w_0 really represents the minimum spot size or beam waist size. But this is not the case for higher order modes. All $N \neq 0$ modes have the same transverse scale length w_0 but higher order modes use a large transverse area and the minimum spot size is approximately $w_m \approx \sqrt{N+1}w_0$ [3]. From Eqs. (11-12), one is able to calculate all the second moments of HG modes at the beam waist plane $z = 0$, \mathbf{M}_{HG} , by using Eq. (3) giving

$$\mathbf{M}_{HG} = \frac{1}{4} \begin{bmatrix} (2m+1)w_0^2 & 0 & 0 & 0 \\ 0 & (2n+1)w_0^2 & 0 & 0 \\ 0 & 0 & (2m+1)\pi^{-2}w_0^{-2} & 0 \\ 0 & 0 & 0 & (2n+1)\pi^{-2}w_0^{-2} \end{bmatrix}. \quad (13)$$

The matrix \mathbf{M}_{LG} is like Eq. (13) by replacing all $(2m+1)$ and $(2n+1)$ factors by the factor $2p+|\ell|$. With explicit values for the second moments, the paraxial estimator can be then explicitly calculated from Eq. (9) for determining the validity of the paraxial approximation in systems composed by free space and thin lenses.

4.1. Single systems: Free-space and Generalized thin lens

The free-space ABCD system is illustrated in Fig. 1 (a). The ray transformation matrix between the input ($z = 0$) and output ($z = z'$) planes is then given by

$$\mathbf{T}_F = \begin{bmatrix} \mathbb{I} & \lambda d \mathbb{I} \\ \mathbb{O} & \mathbb{I} \end{bmatrix}, \quad (14)$$

where d is the propagation distance. Applying Eq. (7) one finds that

$$\mathbf{M}' = \mathbf{T}_F \mathbf{M} \mathbf{T}_F' = \begin{bmatrix} \mathbb{M}_{qq} + d\lambda (\mathbb{M}_{qp} + \mathbb{M}_{qp}') + (d\lambda)^2 \mathbb{M}_{pp} & \mathbb{M}_{qp} + d\lambda \mathbb{M}_{pp}' \\ \mathbb{M}_{qp}' + d\lambda \mathbb{M}_{pp} & \mathbb{M}_{pp}' \end{bmatrix}. \quad (15)$$

It is clear that $\mathbb{M}_{pp}' = \mathbb{M}_{pp}$ having no changes in the second moments of the conjugate space under the free-space transformation. Consequently, the beam propagation along z leaves the paraxial estimator invariant between $z = 0$ to $z = z'$ for HG and LG modes such that

$$\mathcal{P}' = \mathcal{P} = 1 - \frac{(N+1)}{4\pi^2 \tilde{w}_0^2}, \quad (16)$$

where $\tilde{w}_0 = w_0/\lambda$ is the normalized size parameter to the wavelength. Equation (16) coincides with the result given in Ref. [6] from a direct calculus of the paraxial estimator as the ratio of Helmholtz and paraxial energy invariants. The criterion to set the paraxiality scale of \mathcal{P} follows the criterion for the validity limits of the paraxial approximation for the fundamental Gaussian mode ($N = 0$) established in most standard textbooks. For instance, after a judicious analysis, Siegman [3] concluded that the paraxial optical beams can be focused or can diverge at semi-cone angles up to 28° (0.5 rad) before significant corrections to the paraxial wave approximation become necessary. Because $\theta = \lambda/(\pi w_0)$, the use of Eq. (16) gives $\mathcal{P} = 0.94$ corresponding to $\tilde{w}_0 \sim 0.64$. From this, the scale setting for the paraxial estimator is established. The paraxial approximation may be a questionable hypotheses for \mathcal{P} -values of the order of and lower than 0.94. In the opposite limit, $\tilde{w}_0 \gg 1$, the paraxial estimator quickly tends to unit and the Gaussian beam is fully paraxial. These limits were rigorously investigated in the interesting paper of Seshadri[10]. On the other hand, higher orders modes becomes nonparaxial for higher values of \tilde{w}_0 since \mathcal{P} decreases when N increases. For example, for $N = 10$, the PA begins to be critical from $\tilde{w}_0 \sim 2.15$, or for a real value of the minimum spot size of $\tilde{w}_m \approx 7.13$. Although the free space solely does not affect the beam paraxiality, it will affect if it is part of a cascade of optical systems as we will see in further examples.

Now, we analyze an ABCD system only composed by a generalized thin lens as showed in Fig. 1 (b). Here the thin lens assumes negligible thickness so that $\delta \approx 0$. The ray transformation matrix is defined by

$$\mathbf{T}_L = \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ -\mathbb{G} & \mathbb{I} \end{bmatrix}, \quad (17)$$

where \mathbb{G} is a symmetric 2×2 matrix,

$$\mathbb{G} = \begin{bmatrix} g_x & g_{xy} \\ g_{xy} & g_y \end{bmatrix}, \quad (18)$$

and g_x , g_y , and g_{xy} are its three independent parameters with units of $(\text{length})^{-2}$. Physically, a generalized lens can be thought as a pair of cylindrical lenses with different powers that are rotated certain angle and/or reflexed along a certain line, both operations contained in the lens plane. As the second-order matrix transforms as $\mathbf{M}' = \mathbf{T}_L \mathbf{M} \mathbf{T}_L^t$, one then obtains

$$\mathbb{M}_{pp}' = \mathbb{M}_{pp} - \mathbb{G} \mathbb{M}_{qp} - (\mathbb{G} \mathbb{M}_{qp})^t + \mathbb{G} \mathbb{M}_{qq} \mathbb{G}. \quad (19)$$

Taking the trace of the \mathbb{M}_{pp}' , one finds the paraxial estimator of the beam after propagating through a generalized lens. An interesting case is the non-astigmatic spherical lens for which $g_x = g_y = 1/(\lambda f)$ and $g_{xy} = 0$, being f the focal distance of the spherical lens. In this case, the paraxial estimator for HG and LG modes takes the following form at the output plane:

$$\mathcal{P}' = \mathcal{P} - (N+1) \frac{w_0^2}{4f^2}, \quad (20)$$

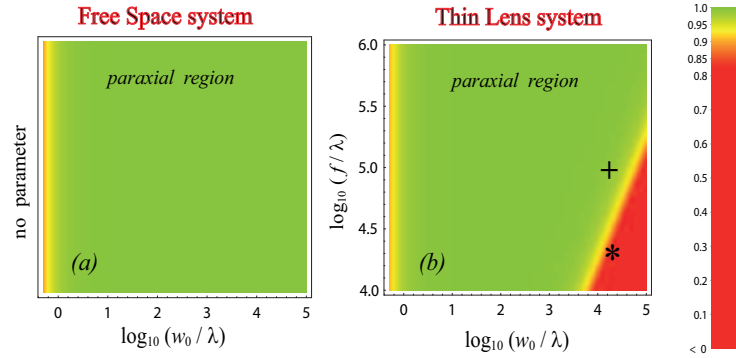


Fig. 2. A density plot of the paraxial estimator (paraxial-nonparaxial map) (a) for free space and (b) spherical thin lens systems crossed by a fundamental mode ($N = 0$) in terms of the normalized beam waist parameter \tilde{w}_0 . The vertical axis for (b) represents the focal length of the lens f (normalized to λ). There is no parameter in the vertical axis for (a). This axis was maintained with the purpose to compare both system results.

where \mathcal{P} is the paraxial estimator at the input plane given by Eq. (16). Notice that Eq. (20) has an additional term when compared with the free space propagation. Contrary to what happens for this later system, the PA validity for a spherical thin lens system becomes questionable in the limit $\tilde{w}_0 \gg 1$. In this case, the system paraxiality is dependent on the ratio (w_0/f) for a given mode N . Whenever w_0 and f are of the same order of magnitude, then the paraxial estimator may deviate from unit and the PA validity fails. Of course, this effect is more notable for higher order modes. In many practical applications such as laser scanning, laser printing and laser fusion, it is desirable to generate the smallest spot size. As the most of classic textbooks show for the case of the Gaussian beam [4], this may be achieved by use of the thickest incident beam and the shortest focal length for a fixed wavelength. In such cases, we have $w_0 \approx f$. Since the lens should capture most of the incident beam, its diameter D must be at least w_0 . The paraxial theory predicts that a large-diameter Gaussian beam with planar wavefront impinges upon a thin lens of diameter $D = w_0$ generates a focused spot of diameter $w'_0 \cong 4\lambda F_{\#}/\pi$ [4]. In this frame, Eq. (20) becomes

$$\mathcal{P}' \cong 1 - \frac{1}{4F_{\#}^2}, \quad (21)$$

where $F_{\#}^2$ is the F -number of the lens equal to the focal length divided by the diameter. In experiments, it is not surprising to have $F_{\#} \sim 1$ and, even lesser values for a lens with focal $f = 10$ mm, that is widely used to produce high focalization. In such case, a spherical thin lens system crossed by a Gaussian beam produces $\mathcal{P}' = 0.75$ and the validity of the paraxial approximation is clearly broken. The ABCD theory is no more suitable to describe this particular focusing physical phenomenon and additional nonparaxial corrections are necessary to give a more accurate result. Let us graphically illustrate the above results. Figure 2 depicts a density plot of the paraxial estimator for free space and spherical thin lens systems as indicated in the caption of the figure. \tilde{w}_0 is ranged in $[0.5, 10^5]$ and $\tilde{f} = f/\lambda$ in $[10^4, 10^6]$. A such range covers a wide range of suitable experimental values. For example, for an laser with $\lambda \approx 0.5 \mu\text{m}$, the w_0 -range lies in $[0.25 \mu\text{m}, 5 \text{ cm}]$ and the f -values lie in the interval $f \in [5 \text{ mm}, 50 \text{ cm}]$. It is clear from the graphics that the paraxiality of the spherical TL system does not depend on the focal length in the limit $\tilde{w}_0 \rightarrow 0$, being \mathcal{P}' governed by the free space formulae. However, the PA validity of the TL system could be questionable in the opposite limit $\tilde{w}_0 \gg 1$. Even though this is a well known-result, it has not been quantified previously. As an example, the symbol

“*” in Fig. 2 (b) indicates the value of the paraxial estimator of a spherical TL system with $f = 10$ mm crossed by a Gaussian beam of $w_0 = 10$ mm. A such system is out of the paraxial region ($\mathcal{P}' = 0.75$). On the contrary, a lens with $f = 50$ mm (symbol “+”) gives $\mathcal{P}' = 0.99$ so that this system is fully paraxial and, therefore, can be correctly described by the ABCD theory. This shows the powerfulness of the paraxial estimator. In fact, this single parameter, that in addition is easily computable, is a good estimator for predicting the PA validity for a ABCD system. In the following, we combine both, the free space and the spherical thin lens to form a cascade system. Its paraxiality properties could differ from the properties of their individual properties as we will see in the next subsection.

4.2. A cascade system: free space portion plus spherical thin lens

A cascade optical system is formed by two or more optical systems. Let us assume that the ABCD system is now composed by the combination of free space plus a spherical thin lens of focal f with the (beam waist) input plane placed at $z = 0$ and the output plane at the exit lens plane $z = z'$ so that the distance among both planes is $d + \delta \sim d$ as illustrated by Fig. 1 (c). First of all, we derive the paraxial estimator. The ray transformation matrix of this composite system is $\mathbf{T}_{FL} = \mathbf{T}_L \mathbf{T}_F$, where \mathbf{T}_F and \mathbf{T}_L are defined in the Eq. (14) and Eq. (17), respectively. That way, the second-order matrix transformation law under this system between input and output planes is $\mathbf{M}' = \mathbf{T}_{FL} \mathbf{M} \mathbf{T}_{FL}^t$, from where one obtains

$$\begin{aligned} \mathbf{M}'_{pp} = & \mathbf{M}_{pp} - \mathbb{G} (\mathbf{M}_{qp} + d\lambda \mathbf{M}_{pp}) - (\mathbf{M}_{qp}^t + z\lambda \mathbf{M}_{pp}) \mathbb{G} \\ & + \mathbb{G} \left[\mathbf{M}_{qq} + d\lambda (\mathbf{M}_{qp} + \mathbf{M}_{qp}^t) + (d\lambda)^2 \mathbf{M}_{pp} \right] \mathbb{G}. \end{aligned} \quad (22)$$

Calculating its trace, the paraxial estimator for a such system crossed by HG and LG modes is obtained, giving explicitly

$$\mathcal{P}' = 1 - (N+1) \left[\frac{(d/f-1)^2}{4\pi^2 \tilde{w}_0^2} + \frac{w_0^2}{4f^2} \right]. \quad (23)$$

Equation (23) depends on the separation distance between the input and output planes since \mathcal{P}' accounts the curvature phase of the beam at the lens entrance and its transformation features. If $d = 0$, the beam waist plane coincides with the lens plane and Eq. (23) reduces to Eq. (20). If one has a collimated beam exiting from the lens ($d = f$) then $\mathcal{P}' = 1 - (N+1)w_0^2/4f^2$. As the relationship between the output and input beam waist sizes is given by $w'_0 = f\lambda/\pi w_0$ (see for example Ref. [4]), then the paraxial estimator at the output plane, \mathcal{P}' , reduces to $1 - (N+1)/(4\pi \tilde{w}_0'^2)$. This justly corresponds to the free space expression for a beam with waist size w'_0 . Hence, \mathcal{P}' really estimates the paraxiality of the exiting beam supporting the relationship between the paraxial estimator and the beam second order moments. On the other hand, the paraxial estimator will give rise identical value if the input plane is symmetrically shifted from the focal plane: a exiting beam will acquire the same phase curvature if $d = f + \varepsilon$ or if $d = f - \varepsilon$, being $\varepsilon < f$. Both output beams thus corresponding to the same paraxiality.

Figure 3 depicts a density plot of \mathcal{P}' for the free space plus spherical thin lens system crossed by a Gaussian beam ($N = 0$) in terms of both the normalized beam waist parameter \tilde{w}_0 and the ratio d/f for a fixed focal length $f = 2 \times 10^4 \lambda$. The dotted line is a reference for the paraxial-nonparaxial limit at the input plane, $\mathcal{P} = 0.94$ that was included for comparative purposes. It is clear that the PA validity at the output plane is strongly dependant on the distance among the beam waist plane and the lens entrance plane. For instance, a beam with waist size of $\tilde{w}_0 = 2.0$ that is paraxial at the input plane, $\mathcal{P} = 0.99$, and symbolized by “+” in the figure, becomes nonparaxial at the output plane ($\mathcal{P}' = 0.89$) if $d/f = 5$. This same beam, now placed at $d/f = 2$ (symbolized by “*” in the figure), becomes paraxial at the output plane: $\mathcal{P}' = 0.99$.

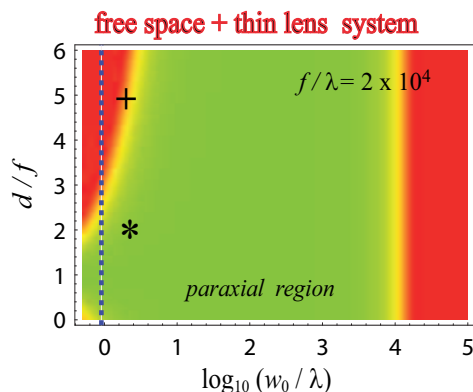


Fig. 3. A density plot of \mathcal{P}' as a function of \tilde{w}_0 and d/f for the system depicted in Fig. 1(c) crossed by the mode $N = 0$. The dotted line is a reference for the paraxial-nonparaxial limit in the input plane, with value $\mathcal{P} = 0.94$ that was included for comparative purposes.

Hence, the distance between input and output planes is critical for evaluating the PA validity. Finally, notice that around the collimation zone $d \approx f$, there would be no restrictions on the system paraxiality in the limit $\tilde{w}_0 \rightarrow 0$ since $\mathcal{P}' \simeq 1$. This would imply that an ultra-focalized beam (for instance $\tilde{w}_0 \approx 0.4$) whose paraxial estimator at the input is $\mathcal{P} = 0.84$ would give paraxial at the output plane. This apparent inconsistency is due to the failure of the paraxial approximation of the incident beam. In such case, the ABCD theory is not suitable to predict the beam characteristics at the lens exit. The fulfillment of the PA at the entrance plane is a mandatory condition in order to apply the results given by Eq. (23).

5. Conclusion

The effect of ABCD transformations on beam paraxiality was quantitatively analyzed by mean of the paraxial estimator, through its relationship with the beam second-order moments. The paraxial approximation validity of optical systems constituted by a section of free space and a spherical thin lens for Hermite-Gauss and Laguerre-Gauss propagating modes was analyzed. For a thin lens system, the PA validity could fail for large beam waist size on contrary to what happens for the free space propagation solely. Also, the beam paraxiality was found dependent on the distance between the input and output in a free space plus thin lens system. In summary, the results have shown that the beam paraxiality can suffer alterations for a feasible range of values of the beam and system parameters under ABCD transformations. Hence, the paraxial estimator is a useful tool for predicting the PA validity for single (and cascade as well) ABCD systems.

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