Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 470 (2023) 108655

FUZZY sets and systems

www.elsevier.com/locate/fss

A triangulation for pointed order polytopes

P. García-Segador [a], P. Miranda [b,*]

[a] *National Statistics Institute, Paseo de la Castellana, 183, 28071 Madrid, Spain*
[b] *Interdisciplinary Mathematical Institute, Complutense University of Madrid, Plaza de Ciencias, 3, 28040 Madrid, Spain*

Received 19 September 2022; received in revised form 22 June 2023; accepted 14 July 2023
Available online 20 July 2023

Abstract

In this paper we propose a way to triangulate a pointed order polytope. Pointed order polytopes are a generalization of order polytopes that include some important groups of polytopes appearing when bipolar scales arise in Decision Making or Game Theory, as the set of bi-capacities or the set of normalized bi-games, even for cases with restricted cooperation. Triangulating polytopes is an important and difficult problem that allows an elegant way to generate uniform random points in the polytope. For order polytopes, there exists a nice result that allows a way to triangulate this family of polytopes based on generating linear extensions. In this paper we prove a similar result for pointed order polytopes. The results in this paper allow to derive a procedure to generate random points inside a pointed order polytope that depends only on the structure of the subjacent poset, a problem that usually is simpler to tackle. In particular, this could be applied to generate bi-capacities or bi-capacities belonging to some subfamilies (e.g. k-symmetric, k-interactive, ...) in a random way.

Keywords: Poset; Polytope; Triangulation; Bi-capacities

1. Introduction

Capacities [7], also called fuzzy measures [34] or non-additive measures [11], have become an important tool in Decision Making and other related fields, such as Cooperative Game Theory. The reason for this success relies on the flexibility of capacities to model very different situations arising in practice. For example, models based on capacities extend the classical model of Expected Utility and hence they can model Ellsberg and Allais paradoxes appearing in Decision Under Uncertainty and Risk (see e.g. [16]). In the domain of Multicriteria Decision Making, capacities allow to model interactions among criteria [14], as well as situations of veto and favor [15], and hence they enrich traditional models based on weighted means. For these reasons, many papers have been devoted to these objects, both theoretical and practical (see e.g. [29,25,6,3]).

Despite this flexibility, capacities assume that there is a polar scale measuring different alternatives, i.e. focusing on the terminology of Multicriteria Decision Making, for each group of criteria, the capacity assigns a value ranging from

* Corresponding author.
E-mail addresses: pedro.garcia.segador@ine.es (P. García-Segador), pmiranda@mat.ucm.es (P. Miranda).

https://doi.org/10.1016/j.fss.2023.108655

0 (completely unsatisfactory) to 1 (completely satisfactory). However, in many practical situations, decision makers do not behave the same for positive valuations and negative valuations with respect to a criterion (see e.g. [30]). Therefore, in these situations it is necessary to consider a bi-polar scale, in which there is a *neutral* value separating positive and negative scores.

To deal with bi-polar scales, Grabisch and Labreuche have introduced the concept of bi-capacity [18,17], where in the definition it is taken into account that for an object some criteria might be completely satisfactory, some completely unsatisfactory and some have a neutral score. Since then, several papers based on bi-capacities have appeared [22,35, 23,1,31].

From a geometrical point of view, it is immediate to see that the set of capacities over a fixed referential is a convex polytope. Moreover, it has been shown [9] that this polytope is a special case of a wider class of polytopes known as order polytopes [33]. Order polytopes have the advantage that they are defined in terms of a partially ordered set (poset) and thus, many properties of the polytope can be studied in terms of the subjacent poset, a problem that is usually easier to study. Based on this fact, many results on the geometry of the set of capacities and some of its subfamilies have been obtained [2,27,13].

In order to treat the same way bi-capacities, in a recent paper [28] the concept of pointed order polytope has been introduced. This concept is a generalization of order polytope in which there is a subjacent poset and *a special element*. Hence, the same as order polytopes, the corresponding pointed order polytope can be studied in terms of the subjacent poset and the "position" of the special element. Thus defined, bi-capacities are an example of pointed order polytope where the special element is $(\emptyset, \emptyset)$. In [28], the general form of the vertices of a pointed order polytope is studied, a characterization of adjacency is given and a description of the faces is provided.

An interesting problem arising when studying a family of polytopes is to derive a procedure to generate points inside the polytope in a uniform random fashion. This is a complex problem and to deal with it, several methods have been proposed, none of them completely satisfactory for general polytopes (see e.g. methods based on Markov chains [21,2], the sweep-plane method [24], the grid method [12], and so on). See also [4] for an analysis of different procedures in different situations in the field of capacities. Hence, it is interesting to derive procedures solving the problem for different families of polytopes. In particular, for order polytopes, the triangulation method seems to be the most appropriate option. This method consists in dividing the polytope in simplices, choose one of them with probability proportional to its volume and then generate a point inside the selected simplex. This method profits the fact that sampling in simplices can be done very efficiently [32]. The *Achilles heel* of this method comes from the fact that in general it is not easy to decompose a polytope into simplices.

However, for the case of order polytopes, it has been proved that it is possible to decompose the polytope in simplices such that all of them have the same volume [26]. This serves us as motivation to think that a similar result can be derived for pointed order polytopes. This is the task that we achieve in this paper.

The rest of the paper goes as follows. First, in Section 2 we review the basic concepts about order polytopes and pointed order polytopes. Then, in Section 3 we develop a procedure to decompose in simplices any pointed order polytope satisfying that all simplices have the same volume. From this result, in Section 4 we treat the problem of generating points inside a pointed order polytope in a uniform random fashion. We finish with the conclusions and open problems.

2. Basic concepts

Consider a finite referential set X of n elements. We will denote subsets of X by capital letters $A, B, \dots$ and by $\mathcal{P}(X)$ the set of subsets of X.

Definition 1. [7] A **capacity** is a function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying

- Monotonicity: $\mu(A) \leq \mu(B)$ if $A \subseteq B$.
- Boundary conditions: $\mu(\emptyset) = 0$, $\mu(X) = 1$.

Let us denote by $\mathcal{FM}(X)$ the set of capacities on X. Note that the set

$$\{\mu(A), A \in \mathcal{P}(X) \setminus \{\emptyset, X\} : \mu \in \mathcal{FM}(X)\}$$

is a convex polytope in $\mathbb{R}^{2^n-2}$. Now, consider the set

$$\mathcal{Q}(X) := \{(A, B) : A \cap B = \emptyset, A, B \subseteq X\}.$$

Definition 2. [18] A **bi-capacity** is a function $\nu : \mathcal{Q}(X) \to [-1, 1]$ satisfying

- Monotonicity: $\nu(A, B) \leq \nu(C, D)$ if $A \subseteq C, B \supseteq D$.
- Boundary conditions: $\nu(X, \emptyset) = 1, \nu(\emptyset, X) = -1, \nu(\emptyset, \emptyset) = 0$.

We will denote by $\mathcal{BCAP}(X)$ the set of all bi-capacities on X. The same as capacities, the set

$$\{\nu(A), A \in \mathcal{Q}(X) \setminus \{(\emptyset, \emptyset), (X, \emptyset), (\emptyset, X)\} : \nu \in \mathcal{BCAP}(X)\}$$

is a convex polytope in $\mathbb{R}^{3^n-3}$.

Let us now introduce order polytopes and pointed order polytopes and establish the relation with $\mathcal{FM}(X)$ and $\mathcal{BCAP}(X)$, respectively. We start with some previous results on posets. See [10] for a general treatment. Let $(P, \preceq)$ (or P for short) be a finite **poset** of p elements, i.e. a set P endowed with a partial relation $\preceq$ that is reflexive, antisymmetric and transitive. Elements of P are denoted $x, y, ...$ Subsets of P are denoted by capital letters $A, B, ...$ or $A_1, A_2, ...$ Posets can be represented as graphs via *Hasse diagrams* (see e.g. Fig. 1 left). We will write $x \lessdot y$ to mean that $x \prec y$ and there is no $z \in P \backslash \{x, y\}$ such that $x \prec z \prec y$. In the Hasse diagram, this translates into there is a line joining x and y. For a poset P, we can define the **dual poset** $P^\partial = (P, \preceq_\partial)$ as $x \preceq_\partial y \Leftrightarrow y \preceq x$.

If $x \in P$ satisfies that $x \not\succeq y, \forall y \in P, y \neq x$, then x is said to be a **minimal element**. The set of minimal elements of P is denoted by $\mathcal{MIN}(P)$. Similarly, if $x \in P$ satisfies that $x \not\preceq y, \forall y \in P, y \neq x$, then x is a **maximal element** and we denote the set of maximal elements of P by $\mathcal{MAX}(P)$.

When any pair of elements in the poset can be compared, the poset is called a **chain**. We will denote by $\boldsymbol{n}$ the chain of n elements (equivalently, a chain of *size* n). A **maximal chain** is a chain in the poset such that it cannot be properly included in another chain. The **height** of P, denoted $h(P)$, is the size of the largest chain in P. For example, poset **1** in Fig. 1 has height 3 corresponding to chain $z - a - y$. Dually, if none of the elements of P are related, we say that P is an **antichain**.

A **filter** or **upset** F is a subset of P such that $x \in F, x \preceq y$ implies $y \in F$. Dually, an **ideal** or **downset** I of P is a subset such that $x \in I, y \preceq x$ implies $y \in I$. We will denote by $\mathcal{F}(P)$ (resp. $\mathcal{I}(P)$) the set of all filters (resp. ideals) of poset P. Note that if $I \in \mathcal{I}(P)$, then $P \setminus I \in \mathcal{F}(P)$ and hence $\mathcal{I}(P) \cong \mathcal{F}(P)^\partial$.

For $a \in P$, we will denote

$$\uparrow a := \{x \in P : a \preceq x\}, \quad \downarrow a := \{x \in P : x \preceq a\}, \quad \updownarrow a := \uparrow a \cup \downarrow a.$$

Now, let us introduce two ways of defining new posets from old that will be used in the paper. Given two posets, $(P, \preceq_P), (Q, \preceq_Q)$, their **direct sum**, denoted $P \oplus Q$, is a poset such that $x \preceq_{P \oplus Q} y$ for every $x \in P$ and $y \in Q$ and preserves the original orders on P and Q. The **disjoint union**, denoted $P \uplus Q$, is a poset $(P \cup Q, \preceq_{P \uplus Q})$ where $x \preceq_{P \uplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$, or $x, y \in Q$ and $x \preceq_Q y$. It can be seen that

$$\mathcal{I}(P_1 \uplus P_2) = \mathcal{I}(P_1) \times \mathcal{I}(P_2). \tag{1}$$

Definition 3. A **linear extension** ϵ of $(P, \preceq)$ is a sorting of the elements of P that is compatible with $\preceq$, i.e. $x \preceq y$ implies that x is before y in the sorting.

For example, (x, z, a, y) is a linear extension of poset **1** of Fig. 1 left. We will denote by $\mathcal{L}(P)$ the set of all linear extensions of poset $(P, \preceq)$ and $e(P) = |\mathcal{L}(P)|$. Next result will be needed in the proofs below.

Lemma 1. *Given two posets P, Q, it follows that*

$$e(P \uplus Q) = \binom{|P| + |Q|}{|P|} e(P)e(Q).$$

We can now introduce the concepts of order polytope and pointed order polytope.

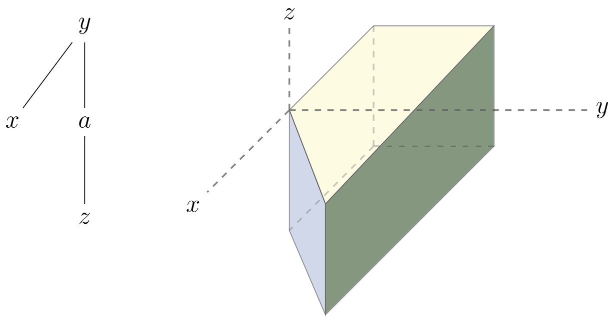

Fig. 1. The poset **1** (left) and its corresponding pointed order polytope $\mathcal{O}(P, a)$ (right).

Definition 4. [33] Let $P = \{x_1, ..., x_p\}$ be a poset. We define the **order polytope** associated to P, denoted $\mathcal{O}(P)$, as the set of points $(f(x_1), ..., f(x_p)) \in \mathbb{R}^P$ satisfying

- $0 \leq f(x) \leq 1, \forall x \in P$.
- $f(x) \leq f(y)$ if $x \preceq y$.

Order polytopes have the advantage that the combinatorial structure of the polytope can be studied in terms of the subjacent poset, and this is usually a simpler problem. For example, vertices of $\mathcal{O}(P)$ are the characteristic functions of filters of P. Note that $\mathcal{FM}(X)$ is the order polytope $\mathcal{O}(\mathcal{P}(X)\backslash\{X, \emptyset\})$, where $A \preceq B$ if and only if $A \subseteq B$ (see [9]). We also denote by $\mathcal{O}_{[a,b]}(P)$ the order polytope defined between a and b instead of between 0 and 1, i.e. $\mathcal{O}(P) = \mathcal{O}_{[0,1]}(P)$. Note that the combinatorial structure of $\mathcal{O}_{[a,b]}(P)$ is the same as the one of $\mathcal{O}(P)$.

Pointed order polytopes appear as an attempt to define a concept similar to order polytopes applying for bi-capacities.

Definition 5. [28] Let P be a poset and take $a \in P$. We define the **pointed order polytope** associated to P and a, denoted $\mathcal{O}(P, a)$, as the set of points $f \in \mathbb{R}^P$ ordered by the elements of P satisfying

- $-1 \leq f(x) \leq 1, \forall x \in P$.
- $f(x) \leq f(y)$, if $x \preceq y$.
- $f(a) = 0$.

Note that $\mathcal{O}(P, a)$ is a polytope of dimension $|P| - 1$, because the value of $f(a)$ is fixed. We will call the pair (P, a) a **pointed poset**.

Example 1. Consider the poset $\mathbf{1} \equiv (P, \preceq)$ where $P := \{x, y, z, a\}$ and whose Hasse diagram is given in Fig. 1 left. Then, $\mathcal{O}(P, a)$ is given by the equations

$$0 \leq f(y) \leq 1, \quad -1 \leq f(x) \leq f(y), \quad f(a) = 0, \quad -1 \leq f(z) \leq 0.$$

As $f(a)$ is fixed, we can draw this polytope in $\mathbb{R}^3$ (Fig. 1 right).

Now, consider the poset $(\mathcal{Q}^*(X), \preceq)$, where

$$\mathcal{Q}^*(X) := \mathcal{Q}(X)\backslash\{(X, \emptyset), (\emptyset, X)\}, \quad (A, B) \preceq (C, D) \Leftrightarrow A \subseteq C, B \supseteq D.$$

Proposition 1. *[28] The polytope $\mathcal{BCAP}(X)$ is the pointed order polytope $\mathcal{O}(\mathcal{Q}^*(X), (\emptyset, \emptyset))$.*

In [28] we have studied the general properties of pointed order polytopes, focusing on differences with the corresponding results for order polytopes. For example, we have characterized the form of the vertices of these polytopes. For this, we have first showed that vertices are the points attaining only values -1, 0 and 1. Hence, a vertex f can be characterized in terms of a partition of P into three sets $\{A_1, A_0, A_{-1}\}$, where A_i is the set of elements $x \in P$ satisfying $f(x) = i,\ i = -1, 0, 1$. We will denote by f_{A_{-1},A_0,A_1} the set function $f_{A_{-1},A_0,A_1} : P \to \{-1, 0, 1\}$ such that $f_{A_{-1},A_0,A_1}(x) = i$ if $x \in A_i$. However, contrary to order polytopes, it can be seen that not every f_{A_{-1},A_0,A_1} defined by a partition $\{A_{-1}, A_0, A_1\}$ of P is a vertex.

Definition 6. [28] Let P be a poset and $a \in P$. We say that a partition $\{A_{-1}, A_0, A_1\}$ of P is a **vertex partition** if it satisfies the following conditions:

1. $A_1 \subseteq P \backslash \{x : x \preceq a\}$ and $A_1 \in \mathcal{F}(P)$.
2. $A_{-1} \subseteq P \backslash \{x : a \preceq x\}$ and $A_{-1} \in \mathcal{I}(P)$.
3. A_0 is a connected subset of P.

Note that in a vertex partition, $a \in A_0$.

Proposition 2. *[28] Let $\mathcal{O}(P, a)$ be a pointed order polytope and consider f_{A_{-1},A_0,A_1} where $\{A_{-1}, A_0, A_1\}$ is a vertex partition of P. Then, f_{A_{-1},A_0,A_1} is a vertex of $\mathcal{O}(P, a)$. Reciprocally, if f_{A_{-1},A_0,A_1} is a vertex of $\mathcal{O}(P, a)$, then $\{A_{-1}, A_0, A_1\}$ is a vertex partition.*

For more results about the geometry of pointed order polytopes, see [28].

Let us finally treat the method of triangulation. For this, we must introduce the concept of simplex. Roughly speaking, a simplex is the counterpart of triangle for more than two dimensions. Thus, a 2-dimensional simplex is a triangle, a 3-dimensional simplex is a triangular pyramid, and so on. Formally,

Definition 7. A **simplex** in $\mathbb{R}^n$ is the convex hull of $n + 1$ points that are affinely independent.

Consequently, for a simplex any subset of vertices defines a face of the simplex. The interest of simplices relies on their good properties as polytopes. In particular, any point in a simplex can be written exactly as a unique convex combination of the vertices. Therefore, generating points in a uniform random fashion in a simplex is very easy (see [12] and Section 4 below). Besides, while computing the volume of a general polytope is a very complex problem, there are formulas to determine the volume of a simplex (see Lemma 7 below).

Definition 8. Consider a polytope $\mathcal{A}$ of dimension m. A **triangulation** of $\mathcal{A}$ is a collection $\Delta = \{\Delta_1, ..., \Delta_r\}$ of m-dimensional simplices satisfying the following properties:

$i)$ Intersection Property: If $\Delta_1, \Delta_2 \in \Delta \Rightarrow \Delta_1 \cap \Delta_2$ is a (possibly empty) common face of Δ_1 and Δ_2.
$ii)$ Union Property:

$$\bigcup_{i=1}^{r} \Delta_i = \mathcal{A}.$$

In other words, a triangulation is a decomposition of a polytope in simplices.[1] Hence, if we succeed in dividing a polytope in simplices (i.e. *triangulate*), the problem of generating points randomly in the polytope becomes much simpler. It just suffices to choose one of the simplices with probability proportional to its volume and then generate a point in the selected simplex in a random fashion. This method of generating points in a polytope is known as the *triangulation method*.

In particular, there is a simple way to triangulate order polytopes based on the order structure of P. This result is given in next theorem.

[1] Some authors define triangulation of a polytope as a decomposition of the polytope into simplices such that the vertices of each simplex are vertices of the original polytope (see e.g. [36]). In this paper we just look for a set of simplices such that their union is the whole polytope and such that the intersection of a pair of simplices is a common face of these simplices. Remark that this relaxation does not affect the basic part of the triangulation method explained below.

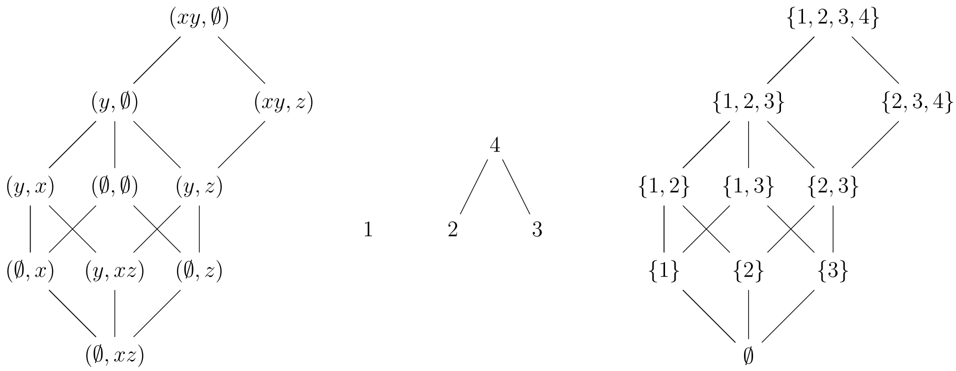

Fig. 2. The poset $(\mathcal{FI}(\mathbf{1}, a), \preceq)$ (left), the hidden poset $\mathbf{1}^*$ (center) and its lattice of ideals (right).

Theorem 1. *[26] Let $(P, \preceq)$ be a poset of p elements.*

1. *If $\preceq$ is a total order on P, then $\mathcal{O}(P)$ is a simplex of volume $\frac{1}{p!}$.*
2. *For any partial ordering $\preceq$ on P, the simplices $\mathcal{O}(P, \leq)$, where $\leq$ is a linear extension of $\preceq$, cover $\mathcal{O}(P)$ and have disjoint interiors. Consequently, $Vol(\mathcal{O}(P)) = \frac{1}{p!} \cdot e(P)$.*

3. A triangulation of $\mathcal{O}(P, a)$

The objective of this section is to construct a triangulation with good properties for $\mathcal{O}(P, a)$. As explained before, the main property usually sought in a triangulation is that all simplices have the same volume, as this simplifies the problem of selecting a simplex in the triangulation method. In this sense, note that for order polytopes we have Theorem 1 that provides a triangulation in these conditions. Thus, we aim to derive a similar result for pointed order polytopes.

Given a linear extension $\epsilon = (x_1, ..., x_p)$ of a poset P, remark that $x_1 \in \mathcal{MIN}(P)$; next, $x_2 \in \mathcal{MIN}(P\backslash\{x_1\})$ and in general, $x_k \in \mathcal{MIN}(P\backslash\{x_1, ..., x_{k-1}\})$. Hence, $\{x_1, ..., x_k\} \in \mathcal{I}(P), \forall k = 1, ..., n$ and linear extensions can be identified to maximal chains $\{I_0, ..., I_{|P|}\}$ in poset $(\mathcal{I}(P), \subseteq)$ via

$$I_0 = \emptyset,\ I_i \setminus I_{i-1} = \epsilon(i) := \{x_i\},\ x_i \in \mathcal{MIN}(P \setminus I_{i-1}). \tag{2}$$

Symmetrically, we can also identify linear extensions to maximal chains in $\mathcal{F}(P)$. Hence, the first aspect that has to be considered for our problem is to find some objects in pointed posets playing the same role as filters and ideals in posets.

With this in mind, we are going to define an important lattice associated to $\mathcal{O}(P, a)$.

Definition 9. Given a pointed poset (P, a), we define the **filter-ideal poset** associated to (P, a) and denoted by $(\mathcal{FI}(P, a), \preceq)$ (or $\mathcal{FI}(P, a)$ for short) as

$$\mathcal{FI}(P, a) := \{(F, I) \in \mathcal{F}(P\backslash \downarrow a) \times \mathcal{I}(P\backslash \uparrow a) \mid F \cap I = \emptyset\}$$

and where $\preceq$ is given by

$$(F_1, I_1) \preceq (F_2, I_2) \Leftrightarrow F_1 \subseteq F_2,\ I_1 \supseteq I_2.$$

Example 2. (Continued Example 1) For the poset $\mathbf{1} \equiv (P, \preceq)$ the corresponding $(\mathcal{FI}(P, a), \preceq)$ is given in Fig. 2 left.

Note that by Definition 6 and Proposition 2, vertices of $\mathcal{O}(P, a)$ can be given as an ideal $I = A_{-1} \subseteq P\backslash \uparrow a$, a filter $F = A_1 \subseteq P\backslash \downarrow a$ and such that $P \setminus (I \cup F) = A_0$ is connected. Hence, any vertex f_{A_{-1}, A_0, A_1} of $\mathcal{O}(P, a)$ can be identified to the pair $(A_1, A_{-1}) \in \mathcal{FI}(P, a)$. However, in general there are elements $(F, I) \in \mathcal{FI}(P, a)$ that are not

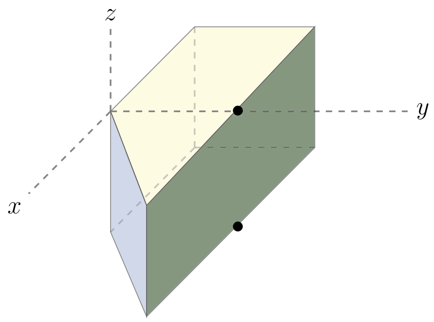

Fig. 3. The pointed order polytope $\mathcal{O}(\mathbf{1}, a)$ with its false vertices.

vertices of $\mathcal{O}(P, a)$ because the corresponding $A_0 = P \setminus (F \cup I)$ is not connected. We will call elements of $\mathcal{FI}(P, a)$ that are not vertices of $\mathcal{O}(P, a)$ as **false vertices.**

Example 3. (Continued Example 1) For $\mathcal{O}(\mathbf{1}, a)$, there are two false vertices, namely $(\{y\}, \{z\})$ and $(\{y\}, \emptyset)$. These vertices correspond to points (0, 1, -1) and (0, 1, 0), respectively. We can see these points in Fig. 3.

Below, we will build a triangulation such that the vertices of the corresponding simplices are elements of $\mathcal{FI}(P, a)$. In next result we study the conditions for $\mathcal{FI}(P, a) = \mathcal{V}(\mathcal{O}(P, a))$, where $\mathcal{V}(\mathcal{O}(P, a))$ denotes the set of vertices of $\mathcal{O}(P, a)$, and hence, the conditions for which this triangulation satisfies the conditions of [36].

Lemma 2. *Let us consider $\mathcal{O}(P, a)$ a pointed polytope and let us denote by $\mathcal{V}(\mathcal{O}(P, a))$ the set of its vertices. Then, $\mathcal{FI}(P, a) = \mathcal{V}(\mathcal{O}(P, a))$ if and only if P can be written as $P = P_1 \oplus a \oplus P_2$.*

Proof. It suffices to show that for any $F \subseteq P \setminus \downarrow a$, $I \subseteq P \setminus \uparrow a$, the set $P \setminus (F \cup I)$ is connected if and only if $P = P_1 \oplus a \oplus P_2$.

If $P = P_1 \oplus a \oplus P_2$, then $F \subseteq P_2$, $I \subseteq P_1$. Hence, for $x \in P_2 \setminus F$, it follows that $x \succeq a$. Similarly, for $y \in P_1 \setminus I$, it follows that $y \preceq a$. Thus, $P \setminus (F \cup I)$ is connected for any (F, I) and the result holds.

Suppose on the other hand that there exists $x \in P$ such that x is not related to a. Then, consider $F = (P \setminus \downarrow a) \setminus \{x\} \in \mathcal{F}(P \setminus \downarrow a)$ and $I = (P \setminus \uparrow a) \setminus \{x\} \in \mathcal{I}(P \setminus \uparrow a)$. Thus, $P \setminus (F \cup I) = \{a, x\}$ that is not connected and hence $f_{F,\{a,x\},I} \notin \mathcal{V}(\mathcal{O}(P, a))$ by Proposition 2. □

For $\mathcal{FI}(P, a)$ the following holds.

Lemma 3. *Let P be a poset and $a \in P$, then $\mathcal{FI}(P, a)$ is a distributive lattice.*

Proof. Let us start showing that $\mathcal{FI}(P, a)$ is a lattice. Indeed, given $(F_1, I_1), (F_2, I_2) \in \mathcal{FI}(P, a)$, we can define

$$(F_1, I_1) \wedge (F_2, I_2) := (F_1 \cap F_2, I_1 \cup I_2), \quad (F_1, I_1) \vee (F_2, I_2) := (F_1 \cup F_2, I_1 \cap I_2).$$

We just need to prove that $(F_1, I_1) \wedge (F_2, I_2) \in \mathcal{FI}(P, a)$ and $(F_1, I_1) \vee (F_2, I_2) \in \mathcal{FI}(P, a)$. For this, observe that since $I_1, I_2 \in \mathcal{I}(P \setminus \uparrow a)$, then $I_1 \cup I_2, I_1 \cap I_2 \in \mathcal{I}(P \setminus \uparrow a)$. Similarly, as $F_1, F_2 \in \mathcal{F}(P \setminus \downarrow a)$ then $F_1 \cup F_2, F_1 \cap F_2 \in \mathcal{F}(P \setminus \downarrow a)$. Moreover,

$$(I_1 \cup I_2) \cap (F_1 \cap F_2) = (F_1 \cap F_2 \cap I_1) \cup (F_1 \cap F_2 \cap I_2) = \emptyset,$$
$$(I_1 \cap I_2) \cap (F_1 \cup F_2) = (I_1 \cap I_2 \cap F_1) \cup (I_1 \cap I_2 \cap F_2) = \emptyset.$$

Therefore, $\wedge$ and $\vee$ are well-defined and $\mathcal{FI}(P, a)$ is a finite lattice. Now, let us show that $\mathcal{FI}(P, a)$ is indeed distributive.

$$\begin{aligned}(F_1, I_1) \vee [(F_2, I_2) \wedge (F_3, I_3)] &= (F_1, I_1) \vee (F_2 \cap F_3, I_2 \cup I_3) \\ &= (F_1 \cup (F_2 \cap F_3), I_1 \cap (I_2 \cup I_3))\end{aligned}$$

$$= ((F_1 \cup F_2) \cap (F_1 \cup F_3), (I_1 \cap I_2) \cup (I_1 \cap I_3))$$
$$= (F_1 \cup F_2, I_1 \cap I_2) \wedge (F_1 \cup F_3, I_1 \cap I_3)$$
$$= [(F_1, I_1) \vee (F_2, I_2)] \wedge [(F_1, I_1) \vee (F_3, I_3)] .$$

Similarly, it can be shown that

$$(F_1, I_1) \wedge [(F_2, I_2) \vee (F_3, I_3)] = [(F_1, I_1) \wedge (F_2, I_2)] \vee [(F_1, I_1) \wedge (F_3, I_3)] .$$

Therefore, the result holds. □

This way, from any pointed poset (P, a), we can get a finite distributive lattice $\mathcal{FI}(P, a)$. On the other hand, by Birkhoff representation theorem [10], we know that every distributive lattice is the ideal lattice of a poset P^*, so $\mathcal{FI}(P, a) = \mathcal{I}(P^*)$. Poset P^* is given by the join-irreducible elements of $\mathcal{FI}(P, a)$, i.e. elements covering exactly one element.

Definition 10. Let P be a poset, $a \in P$ and (P, a) a pointed poset. We define the **hidden poset** P^* of (P, a) as the one that fulfills $\mathcal{FI}(P, a) = \mathcal{I}(P^*)$.

Example 4. (Continued Example 1) In this case, there are four join-irreducible elements in $\mathcal{FI}(P, a)$ that are: $(\emptyset, x), (y, xz), (\emptyset, z), (xy, z)$. For example, $(\emptyset, x)$ just covers $(\emptyset, xz)$. Renaming these elements as 1,2,3,4 respectively, the Hasse diagram of P^* can be seen in Fig. 2 center, and the corresponding lattice of ideals in Fig. 2 right.

This way, we can associate a new poset P^* to any pointed poset (P, a). Thus, it makes sense to think that there is a relationship between the pointed order polytope $\mathcal{O}(P, a)$ and the order polytope $\mathcal{O}(P^*)$. We will exploit this idea in what follows.

Example 5. Note that if $P = P_1 \oplus a \oplus P_2$, we obtain

$$\mathcal{FI}(P, a) = \mathcal{F}(P_2) \times \mathcal{I}(P_1) \cong \mathcal{I}(P_2)^\partial \times \mathcal{I}(P_1) \cong \mathcal{I}(P_2^\partial) \times \mathcal{I}(P_1).$$

Since by Eq. (1) we know that $\mathcal{I}(P_1) \times \mathcal{I}(P_2^\partial) = \mathcal{I}(P_1 \uplus P_2^\partial)$, we conclude that in this case $P^* = P_1 \uplus P_2^\partial$.

Let us study the height of $\mathcal{FI}(P, a)$.

Lemma 4. *Let P be a poset and $a \in P$. Then,*

$$h\left(\mathcal{FI}(P, a)\right) = |P| + |P \setminus \updownarrow a|.$$

Proof. First, note that for any $(F, I) \in \mathcal{FI}(P, a)$, it follows that $I \subseteq P \setminus \uparrow a$ and $F \subseteq P \setminus \downarrow a$. Next, for a maximal chain $(F_1, I_1) \preceq (F_2, I_2) \preceq ... \preceq (F_r, I_r)$, we necessarily have $(F_i, I_i) \lessdot (F_{i+1}, I_{i+1})$. Now observe that $(F_i, I_i) \lessdot (F_{i+1}, I_{i+1})$ if either $|F_{i+1} \setminus F_i| = 1$ or either $|I_i \setminus I_{i+1}| = 1$. Therefore, the maximum possible size of a maximal chain is

$$(|P| - |\uparrow a|) + (|P| - |\downarrow a|) + 1 = |P| + |P \setminus \updownarrow a|.$$

Moreover, consider a maximal chain of ideals $I_1 = P \setminus \uparrow a \supset I_2 \supset ... \supset I_{|P|-|\uparrow a|} = \emptyset$ in $\mathcal{I}(P \setminus \uparrow a)$ and a maximal chain of filters $F_0 = \emptyset \subset F_1 \subset ... \subset F_{|P|-|\downarrow a|} = P \setminus \downarrow a$ in $\mathcal{F}(P \setminus \downarrow a)$ s.t. $|I_i \setminus I_{i+1}| = 1$ and $|F_{i+1} \setminus F_i| = 1$. Note that these chains can always be found, for example, just consider $I_1 = P \setminus \uparrow a$, $I_{i+1} = I_i \setminus \{x_i\}$ s.t. $x_i \in \mathcal{MAX}(I_i)$, and similarly for the maximal chain in $\mathcal{F}(P \setminus \downarrow a)$. Then, we can build the sequence

$$(\emptyset, P \setminus \uparrow a) \lessdot \cdots \lessdot (\emptyset, \emptyset) \lessdot \cdots \lessdot (P \setminus \downarrow a, \emptyset).$$

Hence, the height of $\mathcal{FI}(P, a)$ is exactly $|P| + |P \setminus \updownarrow a|$. □

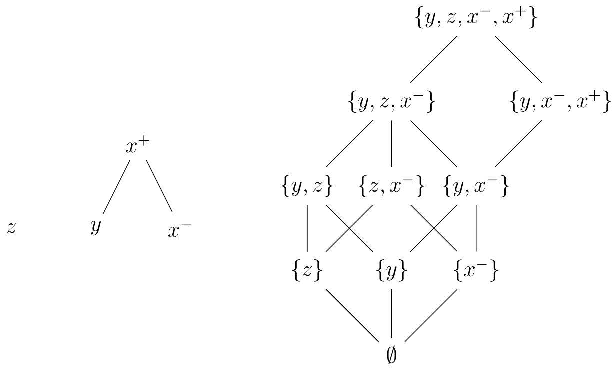

Fig. 4. The hidden poset $\mathbf{1}^*$ (left) and its lattice of ideals (right) in terms of y, z, x^- and x^+.

Example 6. (Continued Example 1). For this pointed order polytope, we can consider the following maximal chain of $\mathcal{FI}(\mathbf{1}, a)$:

$$(\emptyset, xz) - (\emptyset, x) - (\emptyset, \emptyset) - (y, \emptyset) - (xy, \emptyset)$$

Remark that although in the proof of Lemma 4 we have built a chain passing through $(\emptyset, \emptyset)$, other maximal chains can arise without this condition. Consider for example

$$(\emptyset, xz) - (y, xz) - (y, z) - (xy, z) - (xy, \emptyset).$$

Corollary 1. *If $P = P_1 \oplus a \oplus P_2$, then $h(\mathcal{FI}(P, a)) = |P_1| + |P_2| + 1 = |P|$. Reciprocally, if $h(\mathcal{FI}(P, a)) = |P|$, then $P = P_1 \oplus a \oplus P_2$.*

Proof. From the proof of Lemma 4, we see that the height is given by $|P \setminus \uparrow a| + 1 + |P \setminus \downarrow a|$. If $P = P_1 \oplus a \oplus P_2$, then $|P \setminus \uparrow a| = |P_1|, |P \setminus \downarrow a| = P_2$ and the result holds. Reciprocally, if $|P| + |P \setminus \updownarrow a| = h(\mathcal{FI}(P, a)) = |P|$, this implies that $|P \setminus \updownarrow a| = 0$ and hence there is no $x \in P$ such that it is not related to a. Thus, $P = P_1 \oplus a \oplus P_2$. □

Corollary 2. *Consider $\mathcal{FI}(P, a)$ and the corresponding hidden poset P^*. It follows that*

$$|P^*| = |P| + |P \setminus \updownarrow a| - 1.$$

Proof. It suffices to note that maximal chains in $\mathcal{I}(P^*)$ have height $|P^*| + 1$. Hence,

$$|P^*| + 1 = h(\mathcal{I}(P^*)) = h(\mathcal{FI}(P, a)) = |P| + |P \setminus \updownarrow a|. \quad \square$$

Example 7. (Continued Example 1) We have already seen that the elements of P^* are defined in terms of the join-irreducible elements. In this case, we already know that the join-irreducible elements are $(\emptyset, x)$, (y, xz), $(\emptyset, z)$, (xy, z). Consider for example, $(\emptyset, x)$. It follows that it just covers $(\emptyset, xz)$. Now, note that the difference between this pair of elements is that z is not present in the join-irreducible element. Hence, we associate z to $(\emptyset, x)$. Similarly, we associate y to (y, xz) and x to both $(\emptyset, z)$ and (xy, z). Note that x is the only element that cannot be compared to a, so that it can be included in an ideal $I \in \mathcal{I}(P \setminus \uparrow a)$ or in a filter $F \in \mathcal{F}(P \setminus \downarrow a)$. The first case arises for $(\emptyset, z)$ where x is removed from the ideal, and the second one for (xy, z) where x is added to the filter. Hence, elements that cannot be compared to a (so that they are in $P \setminus \updownarrow a$) are related to two elements of P^*. We will name these situations as x^- when x is removed from the ideal and x^+ when x is added to the filter. Poset P^* and $\mathcal{I}(P^*)$ with this notation are given in Fig. 4.

Note that maximal chains in $\mathcal{I}(P^*)$ are related to linear extensions of P^* by Eq. (2). Besides, for a maximal increasing chain in $\mathcal{FI}(P, a)$ and $x \in P \setminus \updownarrow a$, element x is first removed from the ideal and then added to the filter. In other words, x^- appears before x^+ in any $\epsilon \in \mathcal{L}(P^*)$. This translates into $x^- \preceq x^+$ in P^*.

The next problem we have to consider is that if we aim to build a sequence based on maximal chains dividing $\mathcal{O}(P, a)$ in simplices, we need to consider chains of height $|P|$ (one dimension higher than the one of $\mathcal{O}(P, a)$), so that the corresponding simplex has non-null volume in $\mathbb{R}^{p-1}$). Lemma 4 proves that we can find such chains and Corollary 1 also shows that $h(\mathcal{FI}(P, a)) = |P|$ if P can be written as $P = P_1 \oplus a \oplus P_2$. Note at this point the differences with the case of order polytopes, where all maximal chains in $\mathcal{I}(P)$ can be written as linear extensions of P and thus, their height is always $|P|$.

Thus, we have to determine the conditions on chains of length $|P|$ defining a triangulation. A first option is to consider any chain of height $|P|$. However, this does not hold, as next example shows.

Example 8. (Continued Example 1). For this pointed order polytope, let us take the chain of height 4 given by

$$(\emptyset, xz) - (y, xz) - (y, z) - (xy, z)$$

Consider the points of the pointed order polytope with three coordinates (x, y, z). Hence, this chain is related to the following points of the pointed order polytope:

Element	$(\emptyset, xz)$	(y, xz)	(y, z)	(xy, z)
Point	$P_1 := (-1, 0, -1)$	$P_2 := (-1, 1, -1)$	$P_3 := (0, 1, -1)$	$P_4 := (1, 1, -1)$

However, these points do not determine a simplex, as vectors $P_i - P_1,\ i = 2, 3, 4$ are given by $(0, 1, 0)$, $(1, 1, 0)$, $(2, 1, 0)$ and then they are not linearly independent. Note that in this chain element x is first removed in the step $(y, xz) - (y, z)$ and then added in step $(y, z) - (xy, z)$. In other words, x^- (remove x from the ideal) and x^+ (add x to a filter) appear in the corresponding chain $\emptyset - y - x^- - x^+$ of P^*. We will see below that this type of behavior prevents a chain from being a suitable chain.

Definition 11. Let P be a finite poset and $a \in P$. A chain $SC = (F_1, I_1) \prec (F_2, I_2) \prec \dots \prec (F_r, I_r)$ of $\mathcal{FI}(P, a)$ is called a **simplex chain** if:

- The chain has $|P|$ elements, i.e. $r = |P|$.
- If $(F_i, I_i), (F_j, I_j) \in SC$, then $I_i \cap F_j = \emptyset$.

The set of simplex chains of $\mathcal{FI}(P, a)$ is denoted by $\mathcal{SC}(P, a)$. To be consistent with the notation used before in this paper, we also denote $sc(P, a) := |\mathcal{SC}(P, a)|$.

With the first condition, we impose that we obtain a set of $|P|$ points and this is necessary for obtaining a simplex with non-empty volume in $\mathbb{R}^{p-1}$. The second condition avoids chains in the conditions of Example 8, so that an element cannot be added and removed in different parts of the chain. This means that if $x \in P \setminus \updownarrow a$, then either x^- or either x^+ do not appear in the simplex chain.

Example 9. (Continued Example 1) In this case, there are 9 simplex chains in $\mathcal{FI}(P, a)$ that can be obtained from Fig. 2:

$(\emptyset, xz) - (\emptyset, z) - (y, z) - (y, \emptyset)$	$(\emptyset, xz) - (\emptyset, z) - (\emptyset, \emptyset) - (y, \emptyset)$
$(\emptyset, xz) - (y, xz) - (y, z) - (y, \emptyset)$	$(\emptyset, xz) - (y, xz) - (y, x) - (y, \emptyset)$
$(\emptyset, xz) - (\emptyset, x) - (\emptyset, \emptyset) - (y, \emptyset)$	$(\emptyset, xz) - (\emptyset, x) - (y, x) - (y, \emptyset)$
$(\emptyset, z) - (y, z) - (xy, z) - (xy, \emptyset)$	$(\emptyset, z) - (y, z) - (y, \emptyset) - (xy, \emptyset)$
$(\emptyset, z) - (\emptyset, \emptyset) - (y, \emptyset) - (xy, \emptyset)$	

It can be seen that there are 7 other chains of length 4 that do not fulfill the requirements of simplex chains.

A first property of simplex chains is the following:

Lemma 5. *Let P be a finite poset, $a \in P$ and consider a simplex chain $SC = \{(F_1, I_1), ..., (F_p, I_p)\}$ of $\mathcal{FI}(P, a)$ s.t. $(F_1, I_1) \prec (F_2, I_2) \prec ... \prec (F_p, I_p)$. Then,*

$$(F_i, I_i) \lessdot (F_{i+1}, I_{i+1}),\ i = 1, ..., p-1.$$

Consequently, for a simplex chain and any $i = 0, ..., p-1$, either $|F_{i+1} \setminus F_i| = 1$ and $I_i = I_{i+1}$ or either $F_{i+1} = F_i$ and $|I_i \setminus I_{i+1}| = 1$.

Proof. Suppose the result does not hold. Then, there exists i such that (F_{i+1}, I_{i+1}) does not cover (F_i, I_i). But then, $|F_{i+1} \setminus F_i| + |I_i \setminus I_{i+1}| \geq 2$. On the other hand, $|F_{j+1} \setminus F_j| + |I_j \setminus I_{j+1}| \geq 1,\ \forall j \neq i$. As there are p elements in the chain, we conclude that at least an element in $P \setminus \{a\}$ is added and removed in two different parts of the chain, a contradiction with the second condition of Definition 11.

Thus, $|F_{i+1} \setminus F_i| + |I_i \setminus I_{i+1}| = 1$, $\forall i = 1, ..., p-1$. But this means that for any i, either $|F_{i+1} \setminus F_i| = 1$ and $I_i = I_{i+1}$ or either $F_{i+1} = F_i$ and $|I_i \setminus I_{i+1}| = 1$, and hence $(F_i, I_i) \lessdot (F_{i+1}, I_{i+1})$, $\forall i = 1, ..., p-1$. □

Lemma 6. *Consider a simplex chain $(F_1, I_1) \preceq (F_2, I_2) \preceq ... \preceq (F_p, I_p)$. Then, $F_1 = \emptyset$ and $I_p = \emptyset$.*

Proof. It suffices to note that by Lemma 5, either $|F_{i+1} \setminus F_i| = 1$ or either $|I_i \setminus I_{i+1}| = 1$. Hence, if the first term in the chain is (F_1, I_1) with $F_1 \neq \emptyset$, as the height of the chain is $|P|$, then

$$\exists x \in P \setminus \{a\}, i, j \quad \text{s.t.} \quad F_{i+1} \setminus F_i = \{x\}, I_j \setminus I_{j+1} = \{x\},$$

contradicting Definition 11. The proof for ideal $I_p = \emptyset$ is symmetric. □

Remark 1. Note that a simplex chain is a subchain of a maximal chain of $\mathcal{FI}(P, a) \cong \mathcal{I}(P^*)$. By Lemma 5, we know that this subchain satisfies $(F_i, I_i) \lessdot (F_{i+1}, I_{i+1}),\ i = 1, ..., p-1$. On the other hand, for a maximal chain $I'_0 \lessdot I'_1 \lessdot ... \lessdot I'_{|P^*|}$ in $\mathcal{I}(P^*)$, consider the corresponding linear extension ϵ given by Eq. (2), i.e.

$$\epsilon(r) = I'_r \setminus I'_{r-1} = x^*_r \in P^*.$$

Hence, we can associate to a simplex chain the subset of P^* corresponding to the elements of a maximal chain containing the simplex chain. Then, the second condition of Definition 11 states that it cannot happen that x^+, x^- are in this subset for any $x \in P \setminus \updownarrow a$, as conjectured in Example 8.

In the rest of the section, we will associate a simplex to each simplex chain and we will show that these simplices form a triangulation of $\mathcal{O}(P, a)$. To begin with, consider $(F, I) \in \mathcal{FI}(P, a)$. Let us denote

$$\boldsymbol{v}_{(F,I)} := \chi(F) - \chi(I),$$

i.e. the difference of the characteristic functions associated to filter F and ideal I. This is consistent with the notation f_{A_{-1}, A_0, A_1} of vertices of $\mathcal{O}(P, a)$ of Section 2.

Proposition 3. *Let P be a finite poset, $a \in P$ and $SC \in \mathcal{SC}(P, a)$. Then, $Conv(\{\boldsymbol{v}_{(F,I)} : (F, I) \in SC\})$ is a $(|P|-1)$-dimensional simplex contained in $\mathcal{O}(P, a)$.*

Proof. Given $(F, I) \in SC$, note that as $F \in \mathcal{F}(P \setminus \downarrow a)$, $I \in \mathcal{I}(P \setminus \uparrow a)$, then $\boldsymbol{v}_{(F,I)} \in \mathcal{O}(P, a)$. Hence, $\text{Conv}(\{\boldsymbol{v}_{(F,I)} : (F, I) \in SC\}) \subseteq \mathcal{O}(P, a)$.

To prove that $\text{Conv}(\{\boldsymbol{v}_{(F,I)} : (F, I) \in SC\})$ is a simplex, we need to show that $\{\boldsymbol{v}_{(F,I)} : (F, I) \in SC\}$ are affinely independent. Suppose w.l.g.

$$(F_1, I_1) \prec \cdots \prec (F_{|P|}, I_{|P|}).$$

By Lemma 5, $(F_i, I_i) \lessdot (F_{i+1}, I_{i+1})$, so that either $|F_{i+1} \setminus F_i| = 1$ or either $|I_i \setminus I_{i+1}| = 1$. Consequently, $\boldsymbol{v}_{(F_{i+1}, I_{i+1})} - \boldsymbol{v}_{(F_i, I_i)} = \boldsymbol{e}_j$ for some $x_j \in P$.

Since SC is a simplex chain we know that $I_i \cap F_j = \emptyset,\ \forall i, j$. Hence, the new elements x_i appearing at each step are different from each other and thus, vectors $\boldsymbol{v}_{(F_{i+1}, I_{i+1})} - \boldsymbol{v}_{(F_i, I_i)},\ i = 1, ..., p-1$ form the canonical base of $\mathbb{R}^{|P|-1}$. Therefore, the vectors $\{\boldsymbol{v}_{(F_i, I_i)} - \boldsymbol{v}_{(F_{i-1}, I_{i-1})}, i = 1, ..., p-1\}$ are linearly independent and thus $Conv(\{v_{(F,I)} : (F, I) \in SC\})$ is a $(|P|-1)$-dimensional simplex. □

From now on, we will denote the simplex associated to the simplex chain $SC \in \mathcal{SC}(P, a)$ by Δ_{SC}.

Remark 2. As we have seen, simplex chains are subchains of maximal chains of $\mathcal{FI}(P, a)$, or equivalently, of maximal chains of $\mathcal{I}(P^*)$. On the other hand, maximal chains of $\mathcal{I}(P^*)$ can be associated to linear extensions and thus, according to Theorem 1, to simplices triangulating $\mathcal{O}(P^*)$. Note that this means that simplex chains can be seen as some $|P| - 1$ dimensional faces of the simplices triangulating $\mathcal{O}(P^*)$. We derive two consequences of this fact:

- First, as faces of a simplex are simplices, this provides an alternative proof of Proposition 3.
- Second, this means that $\mathcal{O}(P, a)$ is a subpolytope inside $\mathcal{O}_{[-1,1]}(P^*)$.

In next theorem, we establish the main result of this section.

Theorem 2. *Let P a finite poset and $a \in P$. The collection of simplices:*

$$\Delta = \{\Delta_{SC} : SC \in \mathcal{SC}(P, a)\}$$

determines a triangulation of $\mathcal{O}(P, a)$.

Proof. To see that Δ determines a triangulation we need to prove the intersection property and the union property.

Let us start with the intersection property. Consider two simplex chains SC and SC'. By the previous remark, we know that they are faces of two simplices triangulating $\mathcal{O}_{[-1,1]}(P^*)$, say A_1 and A_2. On the other hand, $A_1 \cap A_2$ is a common face of these simplices by Theorem 1. But then, $SC \cap A_1 \cap A_2$ is a face of $A_1 \cap A_2$ and so it is $SC' \cap A_1 \cap A_2$. Hence, $SC \cap SC'$ is a face of SC (resp. SC') and hence, the intersection condition holds.

Let us now show the union property. For this, consider $f \in \mathcal{O}(P, a)$. We may assume w.l.g. that

$$-1 \leq f(x_1) \leq f(x_2) \leq ... \leq f(x_k) \leq f(a) = 0 \leq f(x_{k+1}) \leq ... \leq f(x_{p-1}) \leq 1.$$

We define

$$W_i := -f(x_{k-i+1}), \quad i = 1, ..., k, \quad V_i := 1 - f(x_{p-i}), \quad i = 1, \ldots, p-k-1.$$

Note that $0 \leq W_1 \leq W_2 \leq \ldots \leq W_k \leq 1$ and $0 \leq V_1 \leq V_2 \leq \ldots \leq V_{p-k-1} \leq 1$. Let us rename the set $\{W_1, ..., W_k, V_1, ..., V_{p-k-1}\}$ as $\{Z_1, \ldots, Z_{p-1}\}$ in a way such that

$$Z_1 \leq Z_2 \leq \ldots \leq Z_{p-1}.$$

We also define $Z_p = 1$. Next, define $I_0 := \{x_1, \ldots, x_k\}$, $I_i := \{x_1, \ldots, x_{k-i}\}$, $i = 1, \ldots, k-1$. Similarly, we define $F_0 := \emptyset$, $F_i := \{x_{p-i}, \ldots, x_{p-1}\}$, $i = 1, \ldots, p-k-1$. Note that $I_i \in \mathcal{I}(P \setminus \uparrow a)$ and $F_j \in \mathcal{F}(P \setminus \downarrow a)$, $\forall i, j$. Let us write f as a convex combination of $\boldsymbol{v}_{(F'_i, I'_i)}$, $i = 1, \ldots, p$ for some (F'_i, I'_i) defined as follows.

At step 1, we start considering $(F'_1, I'_1) := (F_0, I_0)$ and assign it the weight $c_1 := Z_1$. Assume we finish step $i-1$ with $(F'_{i-1}, I'_{i-1}) = (F_r, I_s)$, then $(F'_i, I'_i) = (F_{r+1}, I_s)$ if $Z_{i-1} = V_{r+1}$ and $(F'_i, I'_i) = (F_r, I_{s+1})$ if $Z_{i-1} = W_{s+1}$ and assign it weight $c_i := Z_i - Z_{i-1}$.

Thus defined, it follows that

$$f = \sum_{i=1}^{p} c_i \boldsymbol{v}_{(F'_i, I'_i)}.$$

To see this, consider x_i, $i \leq k$. Note that x_i appears in $I'_1, I'_2, \ldots, I'_r$ and it is removed for some I'_{r+1}. Besides, $x_i \notin F'_j$, $\forall j$. Since x_i is removed in step $r+1$, the $k-i$ elements $\{x_{i+1}, \ldots, x_{k-1}, x_k\}$ had already been removed from the ideal in previous steps $1, \ldots, r$. Hence, $Z_r = W_{k-i+1}$, and we get:

$$\sum_{i=1}^{p} c_i \boldsymbol{v}_{(F'_i, I'_i)}(x_i) = -\sum_{i=1}^{r} c_i = -Z_r = -W_{k+1-i} = f(x_i).$$

Now consider element x_{k+i}, $i \geq 1$. Note that x_{k+i} appears in $F'_{r+1}, \ldots, F'_p$ for some $r > 1$ and $x_{k+i} \notin I'_j$, $\forall j$. Since x_{k+i} is added in step r, the $p-k-i-1$ elements $\{x_{k+i+1}, \ldots, x_{p-1}\}$ had already been added to the filter in steps $1, \ldots, r-1$. Hence, $Z_r = V_{p-k-i}$, and we get:

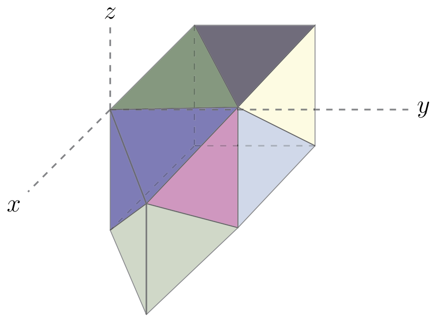

Fig. 5. Decomposition of $\mathcal{O}(\mathbf{1}, a)$ in simplices.

$$\sum_{i=1}^{p} c_i \boldsymbol{v}_{(F'_i, I'_i)}(x_{k+i}) = \sum_{i=r}^{p} c_i = Z_p - Z_r = 1 - V_{p-k-i} = f(x_{k+i}).$$

Moreover, the sequence $c_1, \ldots, c_p$ satisfies $0 \le c_i \le 1,\ \forall i = 1, \ldots, p$ and

$$\sum_{i=1}^{p} c_i = Z_1 + \sum_{i=2}^{p} (Z_i - Z_{i-1}) = Z_p = 1.$$

Remark also that the sequence $(F'_1, I'_1), \ldots, (F'_p, I'_p)$ satisfies the conditions of Definition 11. Hence, the result follows. □

Example 10. (Continued Example 1) For $\mathcal{O}(\mathbf{1}, a)$, we had already seen that there are 9 simplex chains. These simplex chains determine a decomposition of $\mathcal{O}(\mathbf{1}, a)$ in 9 simplices. This decomposition is depicted in Fig. 5. The two first simplex chains lead to simplices that cannot be seen from this angle. For simplex chains 7 and 9, the corresponding triangular pyramid can be seen. For the rest of simplex chains, only a 2-dimensional face (a triangle) can be seen in Fig. 5.

4. On random sampling in $\mathcal{O}(P, a)$

Let us now briefly treat in this section the problem of generating points in $\mathcal{O}(P, a)$ in a uniform random fashion at the light of the results of the previous section. Hence, we are going to apply the triangulation method and consider the triangulation developed before. According to the triangulation method, next step is to select a simplex with probability proportional to its volume. Consequently, we need to compute the volume of each of these simplices. In general, computing the volume of a polytope is a complex problem. However, in the case of simplices, the following result allows to compute the corresponding volume.

Lemma 7. *[19] Let Δ be a k-dimensional simplex with vertices $v_1, \ldots, v_{k+1}$. Then, the k-dimensional volume of Δ is given by:*

$$Vol_k(\Delta) = \sqrt{\frac{|\det(CM_\Delta)|}{2^k (k!)^2}},$$

where $\det(CM_\Delta)$ *is the Cayley-Menger determinant defined as*

$$\det(CM_\Delta) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & d^2_{1,2} & \cdots & d^2_{1,k} & d^2_{1,k+1} \\ 1 & d^2_{2,1} & 0 & \cdots & d^2_{2,k} & d^2_{2,k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & d^2_{k+1,1} & d^2_{k+1,2} & \cdots & d^2_{k+1,k} & 0 \end{vmatrix}$$

being $d^2_{i,j}$ the square of the distance between vertices v_i and v_j.

Now, the following holds:

Proposition 4. *Let P be a finite poset and $a \in P$. Then,*

$$Vol(\Delta_{SC}) = Vol(\Delta_{SC'}), \forall SC, SC' \in \mathcal{SC}(P,a),$$

and this volume is:

$$Vol(\Delta_{SC}) = \frac{1}{(|P|-1)!}.$$

Proof. By Lemma 7, the volume of a simplex depends only on the squared distances between the vertices of the simplex. Consider then $SC \in \mathcal{SC}(P,a)$ given by $(F_1, I_1) \prec \cdots \prec (F_{|P|}, I_{|P|})$. Now, remark that for $i < j$ and the conditions for simplex chain of Definition 11 and Lemma 5, the squared distance between $\boldsymbol{v}_{(F_i,I_i)}$ and $\boldsymbol{v}_{(F_j,I_j)}$ is given by

$$\begin{aligned}\boldsymbol{v}_{(F_j,I_j)} - \boldsymbol{v}_{(F_i,I_i)} &= (\boldsymbol{v}_{(F_j,I_j)} - \boldsymbol{v}_{(F_{j-1},I_{j-1})}) + (\boldsymbol{v}_{(F_{j-1},I_{j-1})} - \boldsymbol{v}_{(F_{j-2},I_{j-2})}) + \ldots + (\boldsymbol{v}_{(F_{i+1},I_{i+1})} - \boldsymbol{v}_{(F_i,I_i)}) \\ &= \boldsymbol{e}_{k_1} + \boldsymbol{e}_{k_2} + \ldots + \boldsymbol{e}_{k_{|j-i|}} \\ &= |j-i|.\end{aligned}$$

On the other hand, we note that by Theorem 1, if we have a chain $Q = \{x_1, \ldots, x_q\}$, with $x_1 \prec x_2 \prec \ldots \prec x_q$, the corresponding order polytope $\mathcal{O}(Q)$ in $\mathbb{R}^q$ is a simplex whose volume is $\frac{1}{|Q|!}$. The vertices of this polytope are $\boldsymbol{w}_i := \chi_{\{x_i,\ldots,x_q\}}$ for $i = 1, \ldots, q$ and vector $\boldsymbol{w}_0 := \boldsymbol{0}$. Hence, the squared distance between $\boldsymbol{w}_i$ and $\boldsymbol{w}_j$ is again $|i-j|$. Consequently, the distances between vertices of the simplex chain SC are the same as the distances between vertices in the simplex $\mathcal{O}(Q)$ where Q is a chain such that $|Q| = |P| - 1$. Applying Lemma 7, the result follows. □

Corollary 3. *The volume of $\mathcal{O}(P,a)$ is given by:*

$$Vol(\mathcal{O}(P,a)) = \frac{sc(P,a)}{(|P|-1)!}.$$

Putting all these results together, we obtain a generalization of Theorem 1 for pointed order polytopes.

Theorem 3. *Let $(P, \preceq)$ be a poset of p elements and consider $a \in P$.*

1. *If $SC \in \mathcal{SC}(P,a)$, then the corresponding simplex Δ_{SC} has volume $\frac{1}{(|P|-1)!}$.*
2. *For any partial ordering $\preceq$ on P, the set of simplices defined via simplex chains cover $\mathcal{O}(P,a)$ and have disjoint interiors. Consequently,*

$$vol(\mathcal{O}(P,a)) = \frac{1}{(|P|-1)!} \cdot sc(P,a).$$

Proof. *1)* is Proposition 4. *2)* follows from Proposition 4 and Theorem 2. □

Consequently, a way to select points in a uniform random fashion for a pointed order polytope is given in Algorithm 1.

Algorithm 1 RANDOM SAMPLING IN $\mathcal{O}(P,a)$.

Step 1: Choose a simplex chain SC uniformly at random among all simplex chains for (P,a).

Step 2: Sample a point uniformly at random in the simplex Δ_{SC}.

For Step 2, given a simplex chain $SC \in \mathcal{SC}(P,a)$, it is easy to sample a uniform point inside the simplex Δ_{SC} [32]. Indeed, suppose that SC is given by the sets $(F_1, I_1), \cdots, (F_p, I_p)$. First, we generate $p-1$ identically distributed and independent random variables $U_1, \ldots, U_{p-1}$ with uniform distribution $U(0,1)$. Next, we consider the order statistics for this sample $U_{(1)}, \ldots, U_{(p-1)}$ and define $\alpha_i := U_{(i)} - U_{(i-1)}$, $1 \le i \le p$, where $U_{(0)} := 0$, and $U_{(p)} := 1$. Observe that $\sum_{i=1}^{p} \alpha_i = 1$. Finally, point $\boldsymbol{v} = \sum_{i=1}^{p} \alpha_i \boldsymbol{v}_{(F_i, I_i)}$ is generated with uniform distribution in the simplex.

Hence, the main problem of Algorithm 1 is to generate a simplex chain in a uniform random fashion (Step 1). This is not surprising, as we have seen that the concept of simplex chain is related to the concept of linear extension and it is well-known [5] that determining the number of linear extensions or generating a linear extension randomly for general posets is a ♯P-complete problem and only partial results are known.

The same as with linear extensions, a reasonable way to choose uniformly a random simplex chain would be to use Markov chains (see e.g. [2] for an example of application of Markov chains for generating linear extensions in the field of Capacities). Many authors use Metropolis-Hastings style chains to get asymptotically uniform samplers (see [21,20]). To do this, we can just give a definition of adjacent simplex chains (see [21]) such that the graph of simplex chains $SC \in \mathcal{SC}(P,a)$ is connected under this definition of adjacency. Then, we can choose a starting simplex chain and move to an adjacent simplex chain in a way so that all simplex chains have the same probability of being selected (or remaining in the same simplex chain with some probability). This kind of algorithms gives uniform samples when the number of movements is large enough.

Remark however that the previous method relies on the knowledge of all simplex chains or at least, a method to derive an adjacent simplex chain from another in a suitable way. The same as for linear extensions, the number of simplex chains grows very fast with $|P|$. To see this, note that simplex chains are subchains of maximal chains in lattice $\mathcal{FI}(P,a)$ and in general, the number of maximal chains in a distributive lattice grows very fast.

To shed more light on this fact, let us see a couple of partial results about the number of simplex chains and its random generation.

Lemma 8. *If $P = P_1 \oplus a \oplus P_2$, then $sc(P,a) = e(P_1 \uplus P_2)$.*

Proof. If $P = P_1 \oplus a \oplus P_2$, then $\mathcal{O}(P,a) \cong \mathcal{O}_{[-1,0]}(P_1) \times \mathcal{O}_{[0,1]}(P_2)$. Thus,

$$\frac{sc(P,a)}{(|P_1|+|P_2|)!} = \mathrm{Vol}\left(\mathcal{O}(P,a)\right) = \mathrm{Vol}\left(\mathcal{O}_{[-1,0]}(P_1)\right) \cdot \mathrm{Vol}\left(\mathcal{O}_{[0,1]}(P_2)\right) = \frac{e(P_1)}{|P_1|!} \cdot \frac{e(P_2)}{|P_2|!} \Rightarrow$$
$$sc(P,a) = \binom{|P_1|+|P_2|}{|P_1|} e(P_1)e(P_2) = e(P_1 \uplus P_2). \quad \square$$

Proposition 5. *Let (P,a) be a finite pointed poset and P^* its hidden poset. Then*

$$sc(P,a) \le (|P \setminus \updownarrow a| + 1)\, e(P^*).$$

Proof. By construction, there are $e(P^*)$ maximal chains in $\mathcal{FI}(P,a)$. Every simplex chain is a subchain of some of these maximal chains. Moreover, these chains should have consecutive elements. As $h(\mathcal{IF}(P,a)) = |P| + |P \setminus \updownarrow a|$ by Lemma 4, each maximal chain has at most $|P \setminus \updownarrow a| + 1$ simplex chains inside. Note that not necessarily all of these subchains will be simplex chains. Besides, it could happen that the same simplex chain could arise from several maximal chains in P^*. Hence, the result holds. □

At this point, note that the all maximal chains in a distributive lattice share the same length. Besides, maximal chains can be obtained via a path upwards in the Hasse diagram of the lattice. Hence, we can think on generating a maximal chain in a random fashion and once this maximal chain is derived, select a simplex chain inside. Note that this procedure is not valid even if we could generate randomly a maximal chain because the number of simplex chains inside a maximal chain could be in general different for different maximal chains, as next example shows.

Example 11. (Continued Example 1) For $\mathcal{FI}(\mathbf{1},a)$, we have the maximal chain

$$(\emptyset, xz) - (\emptyset, x) - (y, x) - (y, \emptyset) - (yx, \emptyset)$$

that has one simplex chain, namely

$$(\emptyset, xz) - (\emptyset, x) - (y, x) - (y, \emptyset),$$

and the maximal chain

$$(\emptyset, xz) - (\emptyset, z) - (\emptyset, \emptyset) - (y, \emptyset) - (yx, \emptyset),$$

that has two simplex chains, namely

$$(\emptyset, xz) - (\emptyset, z) - (\emptyset, \emptyset) - (y, \emptyset), \quad (\emptyset, z) - (\emptyset, \emptyset) - (y, \emptyset) - (yx, \emptyset).$$

Let us now treat the problem of generating simplex chains in more detail. For a simplex chain $(F_1, I_1) \preceq (F_2, I_2) \preceq ... \preceq (F_p, I_p)$, let us define

$$F := \bigcup_{i=1}^{p} F_i, \; I := \bigcup_{i=1}^{p} I_i.$$

As $F_i \subseteq P \setminus \downarrow a$, we conclude that $F \in \mathcal{F}(P \setminus \downarrow a)$ because it is a union of filters. Similarly, $I \in \mathcal{I}(P \setminus \downarrow a)$. Moreover, as $F_i \subseteq F_{i+1}, I_i \supseteq I_{i+1}$, then,

$$F = F_p, \quad I = I_1.$$

Lemma 9. *Consider a simplex chain $(F_1, I_1) \preceq (F_2, I_2) \preceq ... \preceq (F_p, I_p)$. Then, $\{F, I\}$ determines a partition on $P \setminus \{a\}$.*

Proof. We know from Definition 11 that $F_i \cap I_j = \emptyset, \forall i, j$. Therefore, an element $x \in P \setminus \{a\}$ satisfies that if $x \in I_i$, then $x \notin F_j, \forall j$ and similarly if $x \in F_i$, then $x \notin I_j, \forall j$. Hence, $F \cap I = \emptyset$. On the other hand, as a simplex chain has length $|P| - 1$, we conclude that for any $x \in P \setminus \{a\}$, either $x \in F_i$ for some i or either $x \in I_j$ for some j. Consequently, a simplex chain defines a partition $\{F, I\}$ on $P \setminus \{a\}$. □

Note however that there is not a bijection between simplex chains and partitions as different simplex chains may define the same partition.

On the other hand, by Lemma 5, when building a simplex chain starting at $(\emptyset, I)$, at step i with (F_i, I_i), we either add an element to F_i or remove an element from I_i. And this element should be chosen so that $(F_{i+1}, I_{i+1}) \in \mathcal{FI}(P, a)$.

Putting all these facts together, any simplex chain can be built as explained in Algorithm 2.

Algorithm 2 GENERATION OF SIMPLEX CHAINS.

Step 1: Select an ideal I' in $P \setminus \updownarrow a$.

Step 2: Define $I := I' \cup (\downarrow a \setminus a)$, and $F = P \setminus (I \cup \{a\})$.

Step 3: Generate a random linear extension $x_1 \prec x_2 \prec \ldots \prec x_{|I|}$ of I and a random linear extension $y_1 \prec y_2 \prec \ldots \prec y_{|F|}$ of F.

Step 4: Define $I_j := \{x_1, \ldots, x_{|I|-j+1}\}, \; j = 1, \ldots, |I|$, and $F_j := \{y_{|F|-j+1}, \ldots, y_{|F|}\}, \; j = 1, \ldots, |F|$.

Step 5: Select $|I|$ positions $i_1 < \ldots < i_{|I|}$ on $\{1, \ldots, |P| - 1\}$.

Step 6: The simplex chain is built as follows: Start with $(F_0 = \emptyset, I_0 = I)$. Assume at step $i - 1$ we have the element (F_s, I_r), then at step i we consider (F_{s+1}, I_r) if $i \notin \{i_1, \ldots, i_{|I|}\}$ and (F_s, I_{r+1}) if $i \in \{i_1, \ldots, i_{|I|}\}$.

Lemma 10. *Let $SC \in \mathcal{SC}(P, a)$. Then, SC can be generated by Algorithm 2.*

Proof. Let us suppose that SC is given by $(F_1, I_1) \prec (F_2, I_2) \prec \cdots \prec (F_p, I_p)$. Then, we consider

$$I = I_1 := \{x_1, x_2, \ldots, x_{|I|}\}, \; F = F_p := \{y_1, y_2, \ldots, y_{|F|}\}.$$

By Lemma 5, either $|F_{i+1} \setminus F_i| = 1$ or either $|I_i \setminus I_{i+1}| = 1$. Assume w.l.g. that $|F_2 \setminus F_1| = 1$, thus $F_2 \setminus F_1 = \{y_1\}$. Hence, as F_2 is a filter, $y_1 \not\succeq y_j, \ \forall j < |F|$. Proceeding this way, this allows us to build a linear extension $y_1 \prec y_2 \prec \ldots \prec y_{|F|}$ on F to apply Step 3. Similarly, we can build another linear extension $x_1 \prec x_2 \prec \ldots \prec x_{|I|}$ on I. Finally, it suffices to select $i_1, ..., i_{|I|}$ as the positions in the chain where $|I_i \setminus I_{i+1}| = 1$ for Step 5 to recover SC. □

Besides, if either I at Step 1 varies, or either the linear extensions of Step 3 change, or either the selection of positions at Step 5 varies, then the simplex chain obtained in the procedure varies. In other words, any simplex chain can be obtained in just one form following Algorithm 2.

Let us now comment about the applicability of Algorithm 2 for generating random simplex chains. The delicate points are Steps 1 and 3. For Step 1, the selection of ideal I' determines the whole procedure. This ideal I' should be selected in a way so that each ideal is selected with probability proportional to the number of simplex chains derived from I'. However, this would imply the knowledge of all simplex chains in order to compute these probabilities and this is possible only for very special cases as e.g. Examples 12 and 13. Hence, we propose for the general case to select an ideal via a pseudo-random algorithm and proceed with Algorithm 2. Besides, Step 3 assumes that we can generate a linear extension for I and F uniformly at random, a problem that we know that is ♯P-complete, too. The same as Step 1, we propose to apply a method to generate linear extensions (e.g. based on Markov chains) and proceed with the Algorithm.

Let us see some examples of how we can apply this algorithm in some special cases.

Example 12. Suppose P is a chain. Hence, $P = \boldsymbol{n} \oplus a \oplus \boldsymbol{m}$. Consequently, in this case, we have only one option for I and F: $I = \boldsymbol{n}, F = \boldsymbol{m}$. Hence, it follows that in Step 3 we just have a linear extension for I and F, so that $I_j = \boldsymbol{n-j}, j = 1, ..., n$ and $F_j = \boldsymbol{j}, j = 1, ..., m$. Thus, it just suffices to determine the positions of each ideal I_j in the chain. By Step 5 we have

$$\binom{n+m}{n}$$

different simplex chains and the volume of $\mathcal{O}(P, a)$ is

$$Vol(\mathcal{O}(P, a)) = \frac{\binom{n+m}{n}}{(n+m)!} = \frac{1}{n!m!}.$$

Example 13. Suppose now P is an antichain. In this case, $\mathcal{O}(P, a) = [-1, 1]^{|P|-1}$. Hence, filter I can be any subset of $P \setminus \{a\}$, so that we have $2^{|P|-1}$ possible choices. Besides, at Step 3, we have to derive a linear extension in I, and in this case we have $|I|!$ possibilities. Similarly, we have $(|P| - 1 - |I|)!$ linear extensions for F. Finally, Step 5 states that we have to decide the positions of each element of the linear extension. Hence, for fixed I such that $|I| = r$, there are $\binom{|P|-1}{r}$ simplex chains for fixed linear extensions of I and F. Consequently, for fixed I s.t. $|I| = r$ we can derive

$$r!(|P| - 1 - r)!\binom{|P| - 1}{r} = (|P| - 1)!$$

simplex chains. This means that the number of simplex chains is the same for any I. As there are $2^{|P|-1}$ possible choices for I, we conclude that for P being an antichain there are $2^{|P|-1}(|P| - 1)!$ simplex chains and

$$Vol(\mathcal{O}(P, a)) = \frac{2^{|P|-1}(|P| - 1)!}{(|P| - 1)!} = 2^{|P|-1}.$$

Moreover, as explained before, note that Algorithm 2 provides a uniformly at random simplex chain if I' can be selected with probability proportional to the number of simplex chains that can be generated and linear extensions can be generated at random. In this case, there is no problem to generate linear extensions for fixed I' and, as the number of simplex chains is the same for any I', the procedure generates a simplex chain at random if I' in Step 1 is chosen randomly. And this can be done including $x \in P \setminus \{a\}$ in I' with probability 0.5 for all $x \in P \setminus \{a\}$.

Example 14. Let us finally treat the case of $P = P_1 \oplus a \oplus P_2$. In this case, the same as in Example 12, it follows that there is only one option for I, namely $I = P_1$. Similarly, $F = P_2$. Now, consider according Step 3 a linear extension of I and a linear extension of F. For such pair of linear extensions, we can derive applying Step 5 $\binom{|P_1|+|P_2|}{|P_1|}$ different simplex chains. Hence, in order to generate a random simplex chain, it just suffices to generate randomly two linear extensions of P_1 and P_2. Note the similarities with the corresponding result for order polytopes, where it just suffices to generate a linear extension of the subjacent poset. Besides,

$$Vol(\mathcal{O}(P,a)) = \frac{e(P_1)e(P_2)\binom{|P_1|+|P_2|}{|P_1|}}{(|P_1|+|P_2|)!} = \frac{e(P_1 \uplus P_2)}{(|P_1|+|P_2|)!} = Vol(\mathcal{O}(P_1 \uplus P_2)).$$

On the other hand, note that for fixed linear extensions of F and I, Step 5 always provides a random simplex chain.

5. Conclusions and open problems

In this paper we have introduced a procedure to triangulate pointed order polytopes. Besides, all the simplices in this triangulation share the same volume. This makes this triangulation very appealing for solving the problem of generating points in a uniform random fashion inside a pointed order polytope.

The results obtained in the paper generalize the corresponding results for order polytopes. This is not surprising, as pointed order polytopes are an extension of order polytopes. If $P = P_1 \oplus a \oplus P_2$, the results in the paper extend in a very intuitive way the corresponding results for order polytopes (see Lemma 2, Example 5, Corollary 1, Example 14). The main difficulty, as it has been shown in Section 3, arises when there exist elements in P that cannot be compared to a. If this is the case, this extension is not straightforward and several differences arise for the case of pointed order polytopes. First, for order polytopes, the triangulation is related to linear extensions of the subjacent poset, i.e. maximal chains in $\mathcal{I}(P)$. For pointed order polytopes, the triangulation is linked to maximal chains in the lattice $\mathcal{FI}(P,a)$ (or linear extensions of P^*), but it also needs to consider subchains of these maximal chains. Moreover, in the case of order polytopes, a triangulation can be easily derived such that all vertices in any simplex are vertices of the order polytope, while for pointed order polytopes the triangulation obtained does not satisfy this property in general (see Lemma 2).

Although we have solved the problem of deriving a triangulation with good properties for pointed order polytopes, there are several problems that remain unsolved for the problem of random generation. For this, it is necessary to solve the problem of selecting a simplex chain in a uniform random fashion as it has been shown in Section 4. As it can be seen in that section, this problem seems to be very difficult as the corresponding problem for order polytopes, i.e. generate linear extensions randomly, is a ♯P-complete problem. Similarly, the problem of finding the volume of a pointed order polytope relies on finding the number of simplex chains.

Hence, it is necessary to study different cases in a separated way. The most relevant example of pointed order polytope is perhaps the set of all bi-capacities over a finite referential set and some other subfamilies of bi-capacities being pointed order polytopes. In these cases, the number of simplex chains grows very fast with the cardinality of n. For the case of bi-capacities, Algorithm 2 developed in the paper implies that we just need to select which elements of $\{(A,B) : A \neq \emptyset, B \neq \emptyset\}$ are included in I'. And as $I = I' \cup \{(\emptyset, B) : B \neq \emptyset\}$ is an ideal, these elements should constitute an ideal of $\{(A,B) : A \neq \emptyset, B \neq \emptyset\}$. Finally, starting with $(\emptyset, I)$, at first step we either add a maximal element of F or either we remove a maximal element of I. Note however that in this case the limitations of Algorithm 2 stated at the end of Section 4 arise.

On the other hand, we feel that these algorithms could be a good approximation for generating points of a pointed order polytope in a random fashion. Moreover, a promising research line is to combine this way to triangulate the polytope with random procedures based on Markov chains, in a way similar to the procedure proposed in [2] for k-interactive capacities. This is the problem that we aim to study in the future.

Another research line is to derive algorithms being pseudo-random but with a large range of applicability. In this line, Algorithm 2 combined with pseudo-random algorithms for deriving linear extensions (see e.g. [8]) could be an appealing option.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This paper has been supported by the Ministry of Economy and Competitiveness of Spain under Grant PID2021-124933NB-100.

References

[1] J. Abbas, The bipolar Choquet integral based on ternary-element sets, J. Artif. Intell. Soft Comput. Res. 6 (1) (2016) 13–21.
[2] G. Beliakov, F.J. Cabrerizo, E. Herrera-Viedma, J.Z. Wu, Random generation of k-interactive capacities, Fuzzy Sets Syst. 430 (2022) 48–55.
[3] G. Beliakov, J.Z. Wu, Learning fuzzy measures from data: simplifications and optimisation strategies, Inf. Sci. 494 (2019) 100–113.
[4] G. Beliakov, J.Z. Wu, Random generation of capacities and its application in comprehensive decision aiding, Inf. Sci. 577 (2021) 424–435.
[5] G. Brightwell, P. Winkler, Counting linear extensions, Order 8 (1991) 225–242.
[6] D. Candeloro, R. Mesiar, A.R. Sambucini, A special class of fuzzy measures. Choquet integral and applications, Fuzzy Sets Syst. 355 (2019) 83–99.
[7] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[8] E.F. Combarro, J. Hurtado de Saracho, I. Díaz, Minimals plus: an improved algorithm for the random generation of linear extensions of partially ordered sets, Inf. Sci. 501 (2019) 50–67.
[9] E.F. Combarro, P. Miranda, Adjacency on the order polytope with applications to the theory of fuzzy measures, Fuzzy Sets Syst. 180 (2010) 384–398.
[10] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002.
[11] D. Denneberg, Non-additive Measures and Integral, Kluwer Academic, Dordrecht (The Netherlands), 1994.
[12] L. Devroye, Non-Uniform Random Variate Generation, Springer-Verlag, New York, 1986.
[13] P. García-Segador, P. Miranda, Applying Young diagrams to 2-symmetric fuzzy measures with an application to general fuzzy measures, Fuzzy Sets Syst. 379 (2020) 20–36.
[14] M. Grabisch, Alternative representations of discrete fuzzy measures for decision making, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 5 (1997) 587–607.
[15] M. Grabisch, k-order additive discrete fuzzy measures and their representation, Fuzzy Sets Syst. 92 (1997) 167–189.
[16] M. Grabisch, Set Functions, Games and Capacities in Decision Making, Theory and Decision Library, vol. 46, Springer, 2016.
[17] M. Grabisch, C. Labreuche, Bi-capacities. I. The Choquet integral, Fuzzy Sets Syst. 151 (2) (2005) 236–259.
[18] M. Grabisch, C. Labreuche, Bi-capacities. II. Definition, Möbius transform and interaction, Fuzzy Sets Syst. 151 (2) (2005) 211–236.
[19] P. Gritzmann, V. Klee, On the complexity of some basic problems in computational convexity, in: Polytopes: Abstract, Convex and Computational, Kluwer Academic, Dordrecht, 1994, pp. 374–466.
[20] M. Huber, Fast perfect sampling from linear extensions, Discrete Math. 306 (4) (2006) 420–428.
[21] A. Karzanov, L. Khachiyan, On the conductance of order Markov chains, Order 8 (1) (1995) 7–15.
[22] I. Kojadinovic, J.L. Marichal, Entropy of bi-capacities, Eur. J. Oper. Res. 178 (1) (2007) 168–184.
[23] F. Lange, M. Grabisch, New axiomatizations of Shapley interaction index for bi-capacities, Fuzzy Sets Syst. 176 (2011) 64–75.
[24] J. Leydold, W. Hörmann, A sweep-plane algorithm for generating random tuples in simple polytopes, J. Math. Comput. 67 (1998) 1617–1635.
[25] J. Li, X. Yao, X. Sun, D. Wu, Determining the fuzzy measure in multiple criteria decision aiding from the tolerance perspective. Representation of associative functions, Eur. J. Oper. Res. 264 (2018) 428–439.
[26] J. Matousek, Lectures on Discrete Geometry, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.
[27] P. Miranda, P. García-Segador, Order cones: a tool for deriving k-dimensional faces of cones of subfamilies of monotone games, Ann. Oper. Res. 295 (1) (2020) 117–137.
[28] P. Miranda, P. García-Segador, Pointed order polytopes: studying geometrical aspects of the polytope of bi-capacities, Fuzzy Sets Syst. 444 (2022) 182–205.
[29] Y. Narukawa, V. Torra, M. Sugeno, Choquet integral with respect to a symmetric fuzzy measure of a function on the real line, Ann. Oper. Res. 244 (2016) 571–581.
[30] C.E. Osgood, G.J. Suci, P.H. Tannenbaum, The Measurement of Meaning, University of Illinois Press, Urbana (Il.), 1957.
[31] A. Rolland, J. Ah-Pine, B. Mayag, Elicitation of 2-additive bi-capacity parameters, EURO J. Dec. Processes 3 (1) (2015) 5–28.
[32] R.Y. Rubinstein, Generating random vectors uniformly distributed inside and on the surface of different regions, Eur. J. Oper. Res. 10 (1982) 205–209.
[33] R. Stanley, Two poset polytopes, Discrete Comput. Geom. 1 (1) (1986) 9–23.
[34] M. Sugeno, Theory of fuzzy integrals and its applications, PhD thesis, Tokyo Institute of Technology, 1974.
[35] L. Xie, M. Grabisch, The core of bicapacities and bipolar games, Fuzzy Sets Syst. 158 (9) (2007) 1000–1012.
[36] G. Ziegler, Lectures on Polytopes, Springer-Verlag, 1995.