UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

Departamento de Análisis Matemático

TESIS DOCTORAL

Limiting interpolation methods

(Métodos límite de interpolación)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Alba Segurado López

Director

Fernando Cobos Díaz

Madrid, 2015

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Memoria para optar al grado de doctor
con mención de *Doctorado Europeo*
presentada por

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bajo la dirección del doctor

Fernando Cobos Díaz

MADRID, 2015

Persevera, per severa, per se vera.

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Sobre esta tesis

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Además, participé entre enero de 2012 y diciembre de 2014 como miembro del equipo investigador del proyecto de título *Interpolación, Espacios de Funciones y Aplicaciones* de referencia MTM2010-15814 financiado por el Ministerio de Ciencia e Innovación, y desde enero de 2015 como miembro del equipo de trabajo del proyecto de título *Interpolación, Aproximación, Entropía y Espacios de Funciones* de referencia MTM2013-42220-P financiado por el Ministerio de Economía y Competitividad.

A lo largo de estos cuatro años, he podido realizar una estancia de investigación en la universidad Friedrich-Schiller-Universität de Jena (Alemania) con la doctora Dorothee D. Haroske como profesora responsable. Durante dicha estancia, pude ampliar mis conocimientos y trabajar junto con un grupo de referencia internacional como el grupo "Funktionenräume".

Fruto del trabajo de estos cuatro años son los artículos

- F. Cobos, A. Segurado. Limiting real interpolation methods for arbitrary Banach couples. *Studia Math.* 213 (2012), 243–273.

- F. Cobos, A. Segurado. Bilinear operators and limiting real methods. In *Function Spaces X*. Vol. 102 of *Banach Center Publ.* (2014), 57–70.

- F. Cobos, A. Segurado. Some reiteration formulae for limiting real methods. *J. Math. Anal. Appl.* 411 (2014), 405–421.

- F. Cobos, A. Segurado. Description of logarithmic interpolation spaces by means of the J-functional and applications. *J. Funct. Anal.* 268 (2015), 2906–2945.

Contents

Resumen

El marco de esta memoria es la Teoría de Interpolación y, más concretamente, los métodos límite de interpolación.

La Teoría de Interpolación es una rama del Análisis Funcional con importantes aplicaciones en el Análisis Armónico, la Teoría de Aproximación, las Ecuaciones en Derivadas Parciales, la Teoría de Operadores y otras áreas de las matemáticas. Se pueden consultar, por ejemplo, los libros de Butzer y Berens [9], Bergh y Löfström [5], Triebel [80, 81], König [63], Bennett y Sharpley [4], Brudnyĭ y Kruglyak [8] o Connes [39]. Dado un par (compatible) de espacios de Banach (A_0, A_1) y usando las construcciones de la teoría de interpolación, uno puede producir, entre otras cosas, una familia de espacios cuyas propiedades, en cierto modo, mezclan las de A_0 y A_1. Esto es muy útil en muchos contextos.

Los orígenes de la Teoría de Interpolación se remontan a la primera mitad del siglo XX con el teorema de Riesz (1927), la prueba de Thorin (1938) para escalares complejos y el teorema de Marcinkiewicz (1939). Estos resultados aparecieron como herramientas para resolver ciertos problemas en el Análisis Armónico, como por ejemplo el teorema de Hausdorff-Young. La versión más sencilla del teorema de Riesz-Thorin afirma que si T es un operador lineal y continuo de L_{p_0} en L_{p_0} y de L_{p_1} en L_{p_1}, donde $1 \le p_0 \le p_1 \le \infty$, entonces también es acotado de L_p en L_p para $p_0 < p < p_1$. Por otro lado, el teorema de Marcinkiewicz es el resultado correspondiente cuando uno sustituye los espacios de llegada por espacios L_p-débil. Así, el teorema de Marcinkiewicz puede emplearse en algunos casos donde falla el teorema de Riesz-Thorin. Estos resultados en sí tienen diversas aplicaciones en el Análisis Matemático (ver, por ejemplo, [86, Capítulo 12]).

En la década de los 60, autores como Lions, Peetre, Aronszajn, Gagliardo, Calderón y Krein iniciaron lo que ahora se conoce como la teoría abstracta de interpolación. Su principal motivación era el estudio de ciertos problemas sobre ecuaciones en derivadas parciales en el marco de la escala de espacios de Sobolev $H^s(\Omega)$. Su enfoque era functorial, esto es, su interés se centraba en obtener construcciones generales (functores o métodos de interpolación) que a cada par compatible de espacios de Banach (A_0, A_1) le hacen corresponder un espacio de interpolación $A = \mathcal{F}(A_0, A_1)$.

Los métodos que más interés han despertado son el método complejo y el método real. El método complejo se presentó en el trabajo [10] de Calderón; su construcción se basa en las ideas de la prueba de Thorin del teorema de Riesz. Por otro lado, el método real está conectado con el teorema de Marcinkiewicz y se introdujo en el artículo de Lions y Peetre [67]. En la actualidad, la presentación usual del método real es mediante el K-funcional de Peetre. Recordemos que, dados un par compatible de espacios de Banach $\bar{A} = (A_0, A_1)$ y $t > 0$, el K-funcional se define como

$$K(t,a) = K(t,a;\bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1.$$

Para $1 \leq q \leq \infty$ y $0 < \theta < 1$, el espacio de interpolación real $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ se define como la colección de vectores $a \in A_0 + A_1$ para los que la norma

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}$$

es finita.

Una de las ventajas del método real es que el K-funcional se puede obtener de manera explícita en ciertas situaciones y que está relacionado con otras nociones importantes del Análisis Matemático. Por ejemplo, en el marco de la Teoría de Aproximación, algunos módulos de suavidad se pueden interpretar como K-funcionales sobre pares de espacios adecuados. Otra gran ventaja de este método es su flexibilidad. Se puede extender a pares de espacios cuasi-Banach y también a grupos Abelianos normados (ver [5]).

Aplicando el método real al par (L_1, L_∞), resultan espacios de Lebesgue y de Lorentz

$$(L_1, L_\infty)_{\theta,q} = L_{p,q} \quad \text{si } 1/p = 1 - \theta$$

(ver [5, 80, 4]). Para obtener espacios de Lorentz-Zygmund $L_{p,q}(\log L)_\gamma$, tenemos que reemplazar en la definición del método real t^θ por una función más general $f(t)$ (ver el trabajo de Gustavsson [55]). El caso en que $f(t) = t^\theta g(t)$ es de especial interés. Aquí, g es una potencia de $1 + |\log t|$ o, en general, una función de variación lenta; estos casos se estudian en los trabajos de Doktorskii [43], Evans y Opic [46], Evans, Opic y Pick [47], Gogatishvili, Opic y Trebels [52] y Ahmed, Edmunds, Evans y Karadzhov [1].

Con esta definición, θ puede tomar los valores 1 y 0, pero, en estos casos límite, la función extra $g(t)$ es esencial para que la definición tenga sentido y no quede el espacio sólo en $\{0\}$. No obstante, si los espacios de Banach están relacionados mediante una inclusión continua, por ejemplo $A_0 \hookrightarrow A_1$, entonces se pueden definir los espacios límite $(A_0, A_1)_{0,q;J}$ y $(A_0, A_1)_{1,q;K}$ sin la ayuda de una función auxiliar, simplemente haciendo una modificación natural en la definición del método real. Estos métodos límite han sido estudiados en los trabajos de Gomez y Milman [54], Cobos, Fernández-Cabrera, Kühn y Ullrich [19], Cobos, Fernández-Cabrera y Mastyło [24], Cobos y Kühn [29] y Cobos, Fernández-Cabrera y Martínez [22], donde se aplican para trabajar con integrales singulares [54], aproximación de integrales estocásticas [29] y caracterizar los espacios de sucesiones de Cèsaro por interpolación [24], entre otras cosas. El espacio $(A_0, A_1)_{0,q;J}$ es muy próximo a A_0 y $(A_0, A_1)_{1,q;K}$ es cercano a A_1; este hecho es importante en las aplicaciones.

Trabajar en el caso ordenado $A_0 \hookrightarrow A_1$ es básico para los argumentos de estos artículos, pero, desde el punto de vista de la Teoría de Interpolación, esto es sólo una restricción. Por ello, es natural estudiar la extensión de estos métodos límite a pares arbitrarios, no necesariamente ordenados. Esta cuestión fue considerada por Cobos, Fernández-Cabrera y Silvestre en [25, 26], siendo su principal objetivo el describir los espacios que surgen al interpolar la 4-upla $\{A_0, A_1, A_1, A_0\}$ con los métodos asociados al cuadrado unidad. Presentaron varios K- y J-métodos de modo que, a lo largo de las diagonales del cuadrado, los espacios de interpolación son sumas (en el caso K) o intersecciones (en el caso J) de espacios límite y espacios de interpolación real.

El objetivo de una buena parte de esta memoria es desarrollar una teoría lo más completa posible sobre métodos límite para pares arbitrarios. Así, en los Capítulos 3, 4 y 5, presentamos una familia de K-métodos y una familia de J-métodos que están relacionadas por dualidad, que extienden las definiciones de Gomez y Milman y de Cobos, Fernández-Cabrera, Kühn y Ullrich a pares arbitrarios y que producen una teoría lo suficientemente rica.

La definición precisa dada en el Capítulo 3 de los K- y los J-espacios límite es como sigue:

Definición 1. Sea $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach y sea $1 \leq q \leq \infty$. El espacio $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ está formado por todos aquellos $a \in A_0 + A_1$ para los que la siguiente norma es finita:

$$\|a\|_{\bar{A}_{q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.$$

Definición 2. Sea $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach y sea $1 \leq q \leq \infty$. El espacio $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ está formado por todos aquellos $a \in A_0 + A_1$ para los que existe una función fuertemente medible $u(t)$ con valores en $A_0 \cap A_1$ que representa a a como sigue

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergencia en } A_0 + A_1\text{)} \tag{1}$$

y tal que

$$\left(\int_0^1 \left[t^{-1} J(t, u(t))\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{2}$$

La norma $\|a\|_{\bar{A}_{q;J}}$ en $\bar{A}_{q;J}$ se define como el ínfimo en (2) sobre todas las posibles representaciones de a como en (1) de modo que también se tiene (2).

En ese capítulo, mostramos la relación entre estos métodos y otros métodos límite, y también con el método real clásico $\bar{A}_{\theta,q}$. En concreto, comprobamos que estas definiciones generalizan a pares arbitrarios las dadas por Gomez y Milman y por Cobos, Fernández-Cabrera, Kühn y Ullrich. Además, probamos que estos métodos son límite en el siguiente sentido:

Teorema 1. *Sea $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach. Sean $1 \leq p, q, r \leq \infty$ y $0 < \theta < 1$. Entonces, se tiene que*

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{p;J} \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{r;K} \hookrightarrow A_0 + A_1.$$

De hecho, los J-espacios límite son muy próximos a la intersección $A_0 \cap A_1$, y los K-espacios son próximos a $A_0 + A_1$. Tanto es así, que las estimaciones para las normas de los operadores interpolados por estos métodos son peores que en los casos límite ordenados.

Este mal comportamiento se va a ver reflejado en la interpolación de operadores compactos. Recordemos que, dados dos pares compatibles de espacios de Banach $\bar{A} = (A_0, A_1)$ y $\bar{B} = (B_0, B_1)$ y un operador lineal $T \in \mathcal{L}(\bar{A}, \bar{B})$ tal que cualquiera de las dos restricciones $T : A_j \longrightarrow B_j$ es compacta, entonces también es compacto $T : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}$ para todo $1 \leq q \leq \infty$ y todo $0 < \theta < 1$ (ver [40] y [30]). En el caso ordenado, donde $A_0 \hookrightarrow A_1$ y $B_0 \hookrightarrow B_1$, para garantizar la compacidad del operador interpolado por el J- o el K-método límite, también es suficiente que una de las dos restricciones sea compacta, pero no una cualquiera: la compacidad de $T : A_0 \longrightarrow B_0$ es suficiente para garantizar que el operador $T : \bar{A}_{0,q;J} \longrightarrow \bar{B}_{0,q;J}$ es compacto, mientras que para tener la compacidad de $T : \bar{A}_{1,q;K} \longrightarrow \bar{B}_{1,q;K}$, necesitamos que la otra restricción, $T : A_1 \longrightarrow B_1$, sea compacta (ver [19]). En el Capítulo 3, mostramos que en el caso límite general no basta con que una sola restricción, cualquiera que sea, sea compacta, pero, si ambas restricciones son compactas, entonces el operador interpolado por el K- y por el J-método sí es compacto.

También en el Capítulo 3 mostramos cómo uno puede describir los K-espacios límite usando el J-funcional y algunas consecuencias de dicha descripción: primero, damos la siguiente definición.

Definición 3. Sea $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach y sea $1 \leq q \leq \infty$. Pongamos $\rho(t) = 1 + |\log t|$ y $\mu(t) = t^{-1}(1 + |\log t|)$. El espacio $\bar{A}_{\{\rho,\mu\},q;J}$ consiste en todos aquellos elementos $a \in A_0 + A_1$ para los que existe una representación

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergencia en } A_0 + A_1\text{)}, \tag{3}$$

donde $u(t)$ es una función fuertemente medible con valores en $A_0 \cap A_1$ y tal que

$$\left(\int_0^1 [\rho(t) J(t, u(t))]^q \frac{dt}{t} \right)^{1/q} + \left(\int_1^\infty [\mu(t) J(t, u(t))]^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{4}$$

La norma en $\bar{A}_{\{\rho,\mu\},q;J}$ se define como el ínfimo de los valores (4) sobre todas las posibles representaciones u de a que satisfacen (3) y (4).

Seguidamente, mostramos que los espacios $\bar{A}_{\{\rho,\mu\},q;J}$ coinciden con los K-espacios límite:

Teorema 2. *Sea $\bar{A} = (A_0, A_1)$ un par compatible de espacios de Banach y sea $1 \leq q < \infty$. Entonces, se tiene con equivalencia de normas $(A_0, A_1)_{q;K} = (A_0, A_1)_{\{\rho,\mu\},q;J}$.*

El teorema de equivalencia anterior no es cierto para $q = \infty$, como probamos con un contraejemplo.

Asimismo, tratamos la dualidad entre K- y J-espacios límite, y, al final del capítulo, damos algunos ejemplos de espacios obtenidos por los métodos límite. Primero, trabajando con cualquier espacio de medida σ-finito, caracterizamos los espacios límite generados por el par (L_∞, L_1). Luego, consideramos un par formado por dos espacios L_q con pesos y, como aplicación, determinamos los espacios generados por el par de espacios de Sobolev (H^{s_0}, H^{s_1}). También consideramos el caso del par de espacios de Besov $(B^{s_0}_{p,q}, B^{s_1}_{p,q})$. Por último, empleamos los métodos límite para obtener un resultado de tipo Hausdorff-Young para el espacio de Zygmund $L_2(\log L)_{-1/2}([0, 2\pi])$. Todo el contenido del Capítulo 3 aparece en el artículo [35].

En el Capítulo 4, consideramos la interpolación de operadores bilineales mediante estos métodos límite. El problema del comportamiento por interpolación de los operadores bilineales es una cuestión clásica que ya estudiaron Lions y Peetre [67] y Calderón [10] en sus trabajos sobre el método real y el método complejo, respectivamente. Los resultados en este campo han tenido una gran cantidad de aplicaciones interesantes en el Análisis, como la continuidad de ciertos operadores de convolución, la interpolación entre un espacio de Banach y su dual, la estabilidad de álgebras de Banach bajo interpolación y la interpolación de espacios de operadores lineales y acotados (ver los trabajos de Peetre [75], Mastyło [69], Cobos y Fernández-Cabrera [17, 18] y la bibliografía que en ellos aparece). Comenzamos el capítulo mostrando el siguiente resultado:

Teorema 3. *Sean* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *y* $\bar{C} = (C_0, C_1)$ *pares compatibles de espacios de Banach y sean* $1 \le p, q, r \le \infty$ *con* $1/p + 1/q = 1 + 1/r$*. Supongamos que*

$$R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$$

es un operador bilineal y acotado cuyas restricciones a $A_j \times B_j$ *definen operadores acotados*

$$R : A_j \times B_j \to C_j$$

de norma M_j *(*$j = 0, 1$*). Entonces, las restricciones*

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;J} \to (C_0, C_1)_{r;J}$$

y

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;K} \to (C_0, C_1)_{r;K}$$

también son acotadas, con norma $M \le \text{máx}\,(M_0, M_1)$*.*

Asimismo, probamos que los resultados correspondientes de tipo $K \times J \to J$ y $K \times K \to K$ no son ciertos. Como aplicación, establecemos una fórmula de interpolación para espacios de operadores lineales y acotados.

Seguidamente, mostramos que estos métodos no preservan la estructura de álgebra de Banach. Los resultados se recogen en el siguiente teorema:

Teorema 4. *Los espacios* $(\ell_1, \ell_1(2^{-m}))_{q;K}$ *(para* $1 \le q < \infty$*) y* $(\ell_1, \ell_1(2^{-m}))_{q;J}$ *(para* $1 < q < \infty$*), con la convolución definida como multiplicación, no son álgebras de Banach.*

Finalizamos el capítulo comparando las estimaciones para las normas de los operadores bilineales con las de los operadores lineales interpolados por los métodos límite. Además, establecemos un resultado relacionado con la norma del operador lineal interpolado. Este teorema complementa lo mostrado en el Capítulo 3 sobre este tema:

Teorema 5. *Sea* $1 \le q \le \infty$*. Entonces,*

$$\sup\left\{ \|T\|_{\bar{A}_{q;K}, \bar{B}_{q;K}} : \|T\|_{A_0,B_0} \le s, \|T\|_{A_1,B_1} \le t \right\} \sim \text{máx}(s, t),$$

donde el supremo se toma sobre todos los posibles pares compatibles de espacios de Banach $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *y todos los operadores* $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ *que satisfacen las condiciones que hemos mencionado.*

Además, si $q = \infty$*, el supremo se alcanza y la equivalencia anterior es de hecho una igualdad.*

Los resultados más importantes del Capítulo 4 forman el artículo [36].

El Capítulo 5 describe el contenido del artículo [37] y se refiere a fórmulas de reiteración, es decir, estabilidad, para los métodos límite. La reiteración es una cuestión central en el estudio de cualquier método de interpolación. Las fórmulas de reiteración permiten determinar gran número de espacios de interpolación y tienen diversas aplicaciones en Análisis. Por ejemplo, en el caso del método real, la reiteración permite deducir estimaciones fuertes (esto es, $L_p \to L_q$) de estimaciones débiles (ver los libros de Bergh y Löfström [5], Triebel [80], Bennett y Sharpley [4] y Brudnyĭ y Krugljak [8]).

Una forma de establecer el teorema de reiteración para el método real $(A_0, A_1)_{\theta,q}$ es a través de la fórmula de Holmstedt [57], que proporciona el K-funcional del par de espacios de interpolación real $((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1})$ en términos del K-funcional de (A_0, A_1). Recordemos que, en este caso, $0 < \theta < 1$.

Varios autores han seguido esta línea de investigación. Así se hace, por ejemplo, en los artículos de Asekritova [2], Evans y Opic [46], Evans, Opic y Pick [47], Gogatishvili, Opic y Trebels [52] o Ahmed, Edmunds, Evans y Karadzhov [1]. Los últimos cuatro artículos mencionados versan sobre la extensión del método real que se obtiene al sustituir en la definición t^θ por $t^\theta g(t)$, donde g es una función logarítmica quebrada o, más en general, una función de variación lenta. En estos trabajos, se obtuvieron fórmulas de tipo Holmstedt y resultados de reiteración que involucran a la función g.

La extensión del resultado de Holmstedt a los K-espacios límite para pares ordenados se hace en el artículo de Gomez y Milman [54]. Para el caso de los J-espacios límite para pares ordenados, se puede ver una fórmula de reiteración en el artículo de Cobos, Fernández-Cabrera, Kühn y Ullrich [19].

Nuestro objetivo en el Capítulo 5 es establecer fórmulas de reiteración para los K- y J-espacios límite actuando entre pares arbitrarios. Este caso no lo cubre ninguno de los artículos citados anteriormente. Mostramos estimaciones que están adaptadas al tipo de espacios con los que trabajamos y que permiten determinar explícitamente los espacios resultantes, pues muestran los pesos que aparecen con el K-funcional.

Comenzamos el capítulo obteniendo fórmulas de tipo Holmstedt para el K-funcional de pares formados por un espacio interpolado límite y un espacio del par original. A partir de esas fórmulas, obtenemos algunos resultados de reiteración. Los espacios que obtenemos al interpolar un método límite y un espacio del par original se pueden expresar como una intersección $V \cap W$, donde

$$\begin{cases} V = \{a \in A_0 + A_1 : K(s,a)/v(s) \in L_q((0,1), ds/s)\}, \\ W = \{a \in A_0 + A_1 : K(s,a)/w(s) \in L_q((1,\infty), ds/s)\}. \end{cases} \tag{5}$$

Aquí, v y w son funciones de la forma $s^i b(s)$, siendo b una cierta función de variación lenta e $i = 0$ ó 1. Si los dos espacios involucrados son de interpolación clásica, el espacio resultante al aplicarles un método límite también tendrá la forma (5), pero, en este caso, las funciones v y w son de la forma $s^\theta h(s)$, donde $0 < \theta < 1$ y h es una función logarítmica.

Por último, empleamos estos resultados para determinar los espacios generados por ciertos pares de espacios de funciones y de espacios de operadores. Algunos de estos resultados se engloban en el siguiente teorema:

Teorema 6. *Sea* (Ω, μ) *un espacio de medida* σ*-finito y resonante y sean* $1 < p_0, p_1 < \infty, 1 < q \leq \infty$ *y* $1/q + 1/q' = 1$*. Entonces,*

$$(L_{p_0,q}, L_{p_1,q})_{q;J} = L_{p_0,q}(\log L)_{-1/q'} \cap L_{p_1,q}(\log L)_{-1/q'} \quad y$$
$$(L_{p_0,q}, L_{p_1,q})_{q;K} = L_{p_0,q}(\log L)_{1/q} + L_{p_1,q}(\log L)_{1/q}$$

con equivalencia de normas.

El último capítulo de la memoria se refiere a cuestiones relativas a los métodos logarítmicos de interpolación, es decir, a los espacios $(A_0, A_1)_{\theta,q,\mathbb{A}}$, cuya norma viene dada por

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \left[t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q},$$

donde tomamos $1 \leq q \leq \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell(t) = 1 + |\log t|$,

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{si } 0 < t \leq 1, \\ \ell^{\alpha_\infty}(t) & \text{si } 1 < t < \infty, \end{cases}$$

y ahora no sólo $0 < \theta < 1$, sino que también θ puede tomar los valores 0 y 1. De hecho, son estos valores extremos en los que estamos interesados, pues, como se puede ver en [45, Proposición 2.1], en el caso ordenado, los métodos $(A_0, A_1)_{0,q,\mathbb{A}}$ y $(A_0, A_1)_{1,q,\mathbb{A}}$ están relacionados con los métodos de interpolación límite.

Los espacios $(A_0, A_1)_{\theta,q,\mathbb{A}}$ se estudian en los artículos de Gustavsson [55], Doktorskii [43], Evans y Opic [46], Evans, Opic y Pick [47] y las referencias allí citadas.

Si $0 < \theta < 1$, entonces $t^{-\theta}\ell^{\mathbb{A}}(t)$ satisface las hipótesis de [55], y así $(A_0, A_1)_{\theta,q,\mathbb{A}}$ es sólo un caso especial del método real con un parámetro funcional, cuya teoría está bien establecida (ver [55, 60, 77]). Sin embargo, si $\theta = 0$ ó 1, hay varias cuestiones naturales que todavía no se habían estudiado y que se tratan en el Capítulo 6. Comenzamos dando la descripción de $(A_0, A_1)_{0,q,\mathbb{A}}$ y $(A_0, A_1)_{1,q,\mathbb{A}}$ por medio del J-funcional y después usamos esa descripción para mostrar las propiedades de interpolación por esos métodos de los operadores compactos y de los operadores débilmente compactos, y también para determinar el dual de $(A_0, A_1)_{0,q,\mathbb{A}}$ y $(A_0, A_1)_{1,q,\mathbb{A}}$.

Mostramos que, contrariamente al caso $0 < \theta < 1$, cuando $\theta = 0$ ó 1, la J-descripción depende de la relación entre $\mathbb{A}$ y q: en ocasiones, se debe añadir una unidad a la potencia del logaritmo, en otras hay que insertar además un logaritmo iterado, y en otras, la J-descripción ni siquiera existe.

La interpolación de operadores compactos tiene sus raíces en la versión reforzada del teorema de Riesz-Thorin probado por Krasnosel'skiĭ [64]. Recientemente, Edmunds y Opic [45] establecieron una variante límite del teorema de Krasnosel'skiĭ para espacios de medida finita: si $T : L_{p_0} \longrightarrow L_{q_0}$ es compacto y $T : L_{p_1} \longrightarrow L_{q_1}$ es acotado, entonces T también es compacto actuando entre espacios de Lorentz-Zygmund que son muy próximos a L_{p_0} y L_{q_0}. Las técnicas usadas en [45] aprovechan el hecho de trabajar con espacios de Lebesgue.

Más tarde, Cobos, Fernández-Cabrera y Martínez [23] obtuvieron versiones abstractas de los resultados de [45] que funcionan para pares compatibles de espacios de Banach. No obstante, asumían que el segundo par está ordenado por inclusión, esto es, $B_1 \hookrightarrow B_0$ ó $B_0 \hookrightarrow B_1$. La hipótesis del orden se corresponde con la hipótesis de la medida finita de los espacios de Lebesgue en [45]. Usando las J-representaciones y una estrategia distinta a la de [23], mostramos aquí que se puede eliminar la restricción de que el segundo par sea ordenado. En concreto, mostramos los siguientes resultados de compacidad:

Teorema 7. *Sean* $\bar{A} = (A_0, A_1)$ *y* $\bar{B} = (B_0, B_1)$ *dos pares compatibles de espacios de Banach. Consideremos un operador lineal* $T \in \mathcal{L}(\bar{A}, \bar{B})$ *tal que la restricción* $T : A_0 \longrightarrow B_0$ *es compacta. Tomemos también* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *y* $1 \le q \le \infty$ *tales que*

$$\alpha_\infty + 1/q < 0 \text{ si } q < \infty \quad \text{ó} \quad \alpha_\infty \le 0 \text{ si } q = \infty.$$

Entonces, también es compacta la restricción

$$T : (A_0, A_1)_{0,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{0,q,\mathbb{A}}.$$

Teorema 8. *Sean* $\bar{A} = (A_0, A_1)$ *y* $\bar{B} = (B_0, B_1)$ *dos pares compatibles de espacios de Banach. Consideremos un operador lineal* $T \in \mathcal{L}(\bar{A}, \bar{B})$ *tal que la restricción* $T : A_1 \longrightarrow B_1$ *es compacta. Tomemos también* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *y* $1 \le q \le \infty$ *tales que*

$$\alpha_0 + 1/q < 0 \text{ si } q < \infty \quad \text{ó} \quad \alpha_0 \le 0 \text{ si } q = \infty.$$

Entonces, también es compacta la restricción

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}.$$

Estos teoremas permiten deducir resultados sobre interpolación de operadores compactos entre espacios de Lorentz-Zygmund generalizados $L_{p,q}(\log L)_{\mathbb{A}}(\Omega)$. Aquí (Ω, μ) es un espacio de medida σ-finita, $1 < p < \infty$, $1 \le q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ y la norma en el espacio de funciones está dada por

$$\|f\|_{L_{p,q}(\log L)_{\mathbb{A}}(\Omega)} = \left(\int_0^\infty \left[t^{1/p} \ell^{\mathbb{A}}(t) f^{**}(t) \right]^q \frac{dt}{t} \right)^{1/q}$$

donde $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$ y f^* es la reordenada no creciente de f. Se tienen las siguientes variantes para espacios de medida σ-finita (no necesariamente finita) del teorema de Edmunds y Opic que aparece en [45]:

Corolario 1. *Sean* (Ω, μ) *y* (Θ, ν) *espacios de medida σ-finita. Tomemos* $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ *y* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *con* $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. *Sea* T *un operador lineal tal que*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \text{ es compacto y } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \text{ es acotado.}$$

Entonces,

$$T : L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_0,q)}}(\Omega) \longrightarrow L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_0,q)}}(\Theta)$$

también es compacto.

Corolario 2. *Sean* (Ω,μ) *y* (Θ,ν) *espacios de medida* σ*-finita. Tomemos* $1 \le p_0 < p_1 < \infty$, $1 \le q_0 < q_1 < \infty$, $1 \le q < \infty$ *y* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *con* $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. *Sea* T *un operador lineal tal que*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \text{ es acotado y } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \text{ es compacto.}$$

Entonces,

$$T : L_{p_1,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_1,q)}}(\Omega) \longrightarrow L_{q_1,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_1,q)}}(\Theta)$$

también es compacto.

Asimismo, empleamos las J-representaciones para caracterizar el comportamiento de los operadores débilmente compactos bajo interpolación cuando $\theta = 0$ ó 1. En particular, mostramos el siguiente resultado:

Corolario 3. *Sea* $\bar{A} = (A_0, A_1)$ *un par compatible de espacios de Banach. Tomemos* $1 < q < \infty$ *y sea* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$.

(a) Si $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$, *entonces el espacio* $(A_0, A_1)_{1,q,\mathbb{A}}$ *es reflexivo si y sólo si la inclusión* $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ *es débil-compacta.*

(b) Si $\alpha_0 + 1/q < 0$ *y* $\alpha_\infty + 1/q < 0$, *entonces el espacio* $(A_0, A_1)_{1,q,\mathbb{A}}$ *es reflexivo si y sólo si la inclusión* $A_1 \hookrightarrow A_0 + A_1$ *es débil-compacta.*

Por último, obtenemos los espacios duales de $(A_0, A_1)_{1,q,\mathbb{A}}$ y $(A_0, A_1)_{0,q,\mathbb{A}}$ en términos del K-funcional. A diferencia de la teoría clásica, mostramos, con la ayuda de las J-representaciones, que el dual de $(A_0, A_1)_{1,q,\mathbb{A}}$ (respectivamente, $(A_0, A_1)_{0,q,\mathbb{A}}$) depende de la relación entre q y $\mathbb{A}$.

Los resultados de este último capítulo forman el artículo [38].

Chapter 1

Introduction

The main topic of this thesis is interpolation theory and, more specifically, limiting interpolation methods.

Interpolation theory is a branch of functional analysis with important applications in harmonic analysis, approximation theory, partial differential equations, operator theory and some other areas of mathematics. See, for instance, the monographs by Butzer and Berens [9], Bergh and Löfström [5], Triebel [80, 81], König [63], Bennett and Sharpley [4], Brudnyĭ and Kruglyak [8] or Connes [39]. Among other things, given two (compatible) Banach spaces A_0 and A_1, and using the constructions of interpolation theory, one can produce a family of spaces whose properties are intuitively a mixture of those of A_0 and A_1. This is very useful in many contexts.

The origins of interpolation theory go back to the first half of the 20th century with Riesz's theorem (1927), Thorin's proof (1938) for complex scalars and Marcinkiewicz's theorem (1939). These results appeared as a tool for solving certain problems in harmonic analysis, like the Hausdorff-Young theorem. The simplest version of the Riesz-Thorin theorem states that if T is a linear operator that maps continuously L_{p_0} into L_{p_0} and L_{p_1} into L_{p_1}, where $1 \le p_0 \le p_1 \le \infty$, then it also maps L_p into L_p for $p_0 < p < p_1$. On the other hand, Marcinkiewicz's theorem is the corresponding result when one replaces the target spaces by (the larger) weak-L_p spaces. Therefore the Marcinkiewicz theorem can be used in cases where the Riesz-Thorin theorem fails. These results themselves have found a variety of applications in analysis (see, for instance, [86, Chapter 12]).

In the 1960's, authors like Lions, Peetre, Aronszajn, Gagliardo, Calderón and Krein started what is now known as abstract interpolation theory. Their main motivation was the study of certain problems in partial differential equations that dealt with the scale of Sobolev spaces $H^s(\Omega)$. Their approach was functorial, that is, they were interested in obtaining general constructions (interpolation methods) that produce an interpolation space $A = \mathcal{F}(A_0, A_1)$ for each pair of spaces (A_0, A_1).

There are several procedures for generating interpolation spaces, among which are the complex method and the real method. The complex method was presented in Calderón's seminal paper [10] and its construction is based on the ideas in Thorin's proof of Riesz's theorem. The real method, on the other hand, is connected to Marcinkiewicz's theorem; it was introduced in Lions and Peetre's work [67]. Currently, the most usual way to present the real method is by means of Peetre's K-functional. Recall that, given a Banach couple $\bar{A} = (A_0, A_1)$ and $t > 0$, the K-functional is defined as

$$K(t, a) = K(t, a; \bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1.$$

For $1 \leq q \leq \infty$ and $0 < \theta < 1$, the real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ is defined as the collection of vectors $a \in A_0 + A_1$ for which the following norm is finite

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty [t^{-\theta} K(t, a)]^q \frac{dt}{t}\right)^{1/q}.$$

An advantage of the real method is its flexibility. In fact, it can be easily extended to quasi-Banach spaces and also to normed Abelian groups (see [5]). Furthermore, the K-functional can be in certain situations explicitly obtained and is related to other important concepts in analysis. For instance, in the context of approximation theory, some moduli of smoothness can be interpreted as K-functionals on suitable couples of spaces.

Working with the couple of Lebesgue spaces (L_1, L_∞), the real method produces Lebesgue and Lorentz spaces

$$(L_1, L_\infty)_{\theta,q} = L_{p,q} \quad \text{if } 1/p = 1 - \theta$$

(see [5, 80, 4]). In order to obtain Lorentz-Zygmund spaces $L_{p,q}(\log L)_\gamma$, we need to replace t^θ by a more general function $f(t)$ in the definition of the real method (see the article by Gustavsson [55]). The case where $f(t) = t^\theta g(t)$ is of special interest. Here g is a power of $1 + |\log t|$ or, more generally, a slowly varying function; these cases are studied in the papers by Doktorskii [43], Evans and Opic [46], Evans, Opic and Pick [47], Gogatishvili, Opic and Trebels [52] and Ahmed, Edmunds, Evans and Karadzhov [1].

With this definition θ can take the values 1 and 0, but in these limit cases the extra function $g(t)$ is essential to get a meaningful definition and to obtain a space that is not just $\{0\}$. However, if the Banach spaces are related by a continuous embedding, say $A_0 \hookrightarrow A_1$, then the limiting spaces $(A_0, A_1)_{0,q;J}$ and $(A_0, A_1)_{1,q;K}$ can be defined without the help of an auxiliary function, just by making a natural modification in the definition of the real interpolation method. These limiting methods have been studied in the papers by Gomez and Milman [54], Cobos, Fernández-Cabrera, Kühn and Ullrich [19], Cobos, Fernández-Cabrera and Mastyło [24], Cobos and Kühn [29] and Cobos, Fernández-Cabrera and Martínez [22], where they are applied to work with singular integrals [54], approximation of stochastic integrals [29] and to characterise Cèsaro sequence spaces by interpolation [24], among other things. The space $(A_0, A_1)_{0,q;J}$ is very close to A_0 and $(A_0, A_1)_{1,q;K}$ is near A_1; this fact is important in applications.

To be in the ordered case $A_0 \hookrightarrow A_1$ is basic for the arguments of those papers, but it is only a restriction from the point of view of interpolation theory. For this reason, it is natural to study the extension of these limiting methods to arbitrary, not necessarily ordered, couples of Banach spaces. This question has been considered by Cobos, Fernández-Cabrera and Silvestre in [25, 26], their main target being to describe the spaces that arise when interpolating the 4-tuple $\{A_0, A_1, A_1, A_0\}$ by the methods associated to the unit square. Several K- and J-methods were introduced to the effect that along the diagonals of the square the interpolated spaces are sums (in the K-case) or intersections (in the J-case) of limiting methods and real interpolation spaces.

Our goal in a large part of this thesis is to develop a comprehensive theory of limiting methods for arbitrary couples. In Chapters 3, 4 and 5 we present a family of K-methods and a family of J-methods that are related by duality, that extend to arbitrary couples the definitions by Gomez and Milman and by Cobos, Fernández-Cabrera, Kühn and Ullrich and that produce a sufficiently rich theory.

The concrete definition of the limiting K- and J-spaces given in Chapter 3 is as follows.

Definition 1.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. We define the space $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ as the collection of all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\bar{A}_{q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.$$

Definition 1.2. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is formed by all those $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t)\frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1\text{)} \tag{1.1}$$

and

$$\left(\int_0^1 \left[t^{-1}J(t,u(t))\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty J(t,u(t))^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{1.2}$$

The norm $\|a\|_{\bar{A}_{q;J}}$ in $\bar{A}_{q;J}$ is the infimum in (1.2) over all representations that satisfy (1.1) and (1.2).

We study in that chapter the relationship between these methods and other limiting methods and also with the classical real method $\bar{A}_{\theta,q}$. In concrete terms we prove that these definitions generalise to arbitrary couples those given by Gomez and Milman and by Cobos, Fernández-Cabrera, Kühn and Ullrich. We also show that these methods are limiting in the following sense.

Theorem 1.1. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple. Let* $1 \leq p, q, r \leq \infty$ *and* $0 < \theta < 1$. *Then*

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{p;J} \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{r;K} \hookrightarrow A_0 + A_1.$$

The limiting J-spaces are very close to the intersection $A_0 \cap A_1$ and the K-spaces are near $A_0 + A_1$, so much so that the estimates for the norms of the operators interpolated by these methods are worse than in the limiting ordered case.

This bad behaviour is reflected in the interpolation properties of compact operators. Recall that given two Banach couples $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ and a linear operator $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that any of the restrictions $T : A_j \longrightarrow B_j$ is compact, then $T : \bar{A}_{\theta,q} \longrightarrow \bar{B}_{\theta,q}$ is also compact for any $1 \leq q \leq \infty$ and any $0 < \theta < 1$ (see [40] and [30]). This no longer holds when one works with limiting methods. Indeed, in the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, we also need one of the restrictions, but not just any one of them, to be compact so as to guarantee the compactness of the interpolated operator by the limiting J- or K-method. More precisely, the compactness of the restriction $T : A_0 \longrightarrow B_0$ is sufficient to ensure that the operator $T : \bar{A}_{0,q;J} \longrightarrow \bar{B}_{0,q;J}$ is compact, whereas in order to have the compactness of $T : \bar{A}_{1,q;K} \longrightarrow \bar{B}_{1,q;K}$, we need the other restriction, $T : A_1 \longrightarrow B_1$, to be compact (see [19]). In Chapter 3 we show that in the general limiting case the fact that one restriction, whichever one, is compact is not enough, but, if both are compact, then the interpolated operator by the K- and the J-method is compact.

Moreover, we study in Chapter 3 how one can describe limiting K-spaces by means of the J-functional and we give some consequences of this description: First we give the following definition.

Definition 1.3. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Write $\rho(t) = 1 + |\log t|$ and $\mu(t) = t^{-1}(1 + |\log t|)$. The space $\bar{A}_{\{\rho,\mu\},q;J}$ is formed by all those elements $a \in A_0 + A_1$ for which there is a representation

$$a = \int_0^\infty u(t)\frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1\text{)} \tag{1.3}$$

where $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ and such that

$$\left(\int_0^1 [\rho(t)J(t, u(t))]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [\mu(t)J(t, u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{1.4}$$

The norm in $\bar{A}_{\{\rho,\mu\},q;J}$ is given by taking the infimum of the values (1.4) over all possible representations u of a satisfying (1.3) and (1.4).

Then we show that the spaces $\bar{A}_{\{\rho,\mu\},q;J}$ coincide with the limiting K-spaces

Theorem 1.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q < \infty$. Then we have with equivalent norms $(A_0, A_1)_{q;K} = (A_0, A_1)_{\{\rho,\mu\},q;J}$.*

The equivalence theorem is not true when $q = \infty$, as we show with a counterexample.

Furthermore, we establish the duality relationship between limiting K- and J-methods, and at the end of the chapter we give some examples of spaces obtained by these limiting methods. First, working with any σ-finite measure space, we characterise the limiting spaces generated by the couple (L_∞, L_1). Then we consider a couple formed by two weighted L_q-spaces and, as an application, we determine the spaces generated by the Sobolev couple (H^{s_0}, H^{s_1}). We also consider the case of the couple $(B^{s_0}_{p,q}, B^{s_1}_{p,q})$ of Besov spaces. Finally, we apply the limiting methods to obtain a Hausdorff-Young type result for the Zygmund space $L_2(\log L)_{-1/2}([0, 2\pi])$. The contents of Chapter 3 appear in the paper [35].

In Chapter 4 we consider the interpolation of bilinear operators under these limiting methods. The problem of the behaviour of bilinear operators under interpolation is a classical question that was already studied by Lions and Peetre [67] and Calderón [10] in their seminal papers on the real and the complex interpolation methods, respectively. The results in this field have found a variety of interesting applications in analysis including boundedness of convolution operators, interpolation between a Banach space and its dual, stability of Banach algebras under interpolation or interpolation of spaces of bounded linear operators (see the articles by Peetre [75], Mastyło [69], Cobos and Fernández-Cabrera [17, 18] and the references given there). We start the chapter with the following result.

Theorem 1.3. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and* $\bar{C} = (C_0, C_1)$ *be Banach couples and let* $1 \le p, q, r \le \infty$ *with* $1/p + 1/q = 1 + 1/r$. *Suppose that*

$$R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$$

is a bounded bilinear operator whose restrictions $A_j \times B_j$ *define bounded operators*

$$R : A_j \times B_j \to C_j$$

with norms M_j *(*$j = 0, 1$*). Then the restrictions*

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;J} \to (C_0, C_1)_{r;J}$$

and

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;K} \to (C_0, C_1)_{r;K}$$

are also bounded with norm $M \le \text{máx}\,(M_0, M_1)$.

Moreover, we show that the corresponding results of the type $K \times J \to J$ and $K \times K \to K$ do not hold. As an application we establish an interpolation formula for spaces of bounded linear operators.

Then we check that these methods do not preserve the Banach-algebra structure. The results are collected in the following theorem.

Theorem 1.4. *The spaces* $(\ell_1, \ell_1(2^{-m}))_{q;K}$ *(for* $1 \le q < \infty$*) and* $(\ell_1, \ell_1(2^{-m}))_{q;J}$ *(for* $1 < q < \infty$*) are not Banach algebras if multiplication is defined as convolution.*

We end the chapter comparing the estimates for the norms of bilinear operators with those of linear operators interpolated under limiting methods. We also establish a result that is related to the norm of interpolated linear operators. This theorem complements what is shown in Chapter 3 regarding this matter.

Theorem 1.5. *Let* $1 \le q \le \infty$. *Then*

$$\sup\left\{\|T\|_{\bar{A}_{q;K}, \bar{B}_{q;K}} : \|T\|_{A_0, B_0} \le s, \|T\|_{A_1, B_1} \le t\right\} \sim \max(s, t),$$

where the supremum is taken over all Banach pairs $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and all* $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ *satisfying the stated conditions.*

In addition, if $q = \infty$, *the supremum is attained and the previous equivalence is actually an equality.*

The most important results in Chapter 4 form the article [36].

Chapter 5 describes the contents of the paper [37] and refers to reiteration, that is, stability, formulae for limiting methods. Reiteration is a central question in the study of any interpolation method. Reiteration formulae allow to determine many interpolation spaces and have found interesting applications in analysis. For example, in the case of the real method, reiteration allows to derive strong (i.e. $L_p \to L_q$) estimates for operators from weak type estimates (see the books by Bergh and Löfström [5], Triebel [80], Bennett and Sharpley [4] and Brudnyĭ and Krugljak [8]).

One way to establish the reiteration theorem for the real method $(A_0, A_1)_{\theta,q}$ is by means of Holmstedt's formula [57], which gives the K-functional of the couple of real interpolation spaces $((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1})$ in terms of the K-functional of the original couple (A_0, A_1). Recall that in this case $0 < \theta < 1$.

Several authors have followed this line of research. See, for example, the papers by Asekritova [2], Evans and Opic [46], Evans, Opic and Pick [47], Gogatishvili, Opic and Trebels [52] or Ahmed, Edmunds, Evans and Karadzhov [1]. The last four mentioned papers deal with the extension of the real method which is obtained by replacing in the definition t^θ by $t^\theta g(t)$, where g is a broken logarithmic function or, more generally, a slowly varying function. In these articles the authors obtained Holsmtedt-type formulae and reiteration results where the function g is involved.

The extension of Holmstedt's result to limiting K-spaces for ordered couples is done in the paper by Gomez and Milman [54]. For the case of limiting J-spaces for ordered couples, a reiteration formula can be found in the article by Cobos, Fernández-Cabrera, Kühn and Ullrich [19].

Our aim in Chapter 5 is to establish reiteration formulae for limiting K- and J-methods acting on arbitrary couples. This case is not covered in any of the papers that we have mentioned. We obtain estimates that are adapted to the kinds of spaces that we consider and that allow us to explicitly determine the resulting spaces, since they show the weights that appear with the K-functional.

We start the chapter by deriving Holmstedt-type formulae for the K-functional of couples formed by a limiting interpolation space and a space of the original couple. From these formulae we derive some reiteration results. The spaces that we obtain by interpolating a limiting method and a space that belongs to the original couple can be expressed as an intersection $V \cap W$, where

$$\left\{ \begin{array}{l} V = \{a \in A_0 + A_1 : K(s,a)/v(s) \in L_q((0,1), ds/s)\}, \\ W = \{a \in A_0 + A_1 : K(s,a)/w(s) \in L_q((1,\infty), ds/s)\}. \end{array} \right. \tag{1.5}$$

Here, v and w are functions of the form $s^i b(s)$, b being a certain slowly varying function and $i = 0$ or 1. A limiting method applied to a couple of real interpolation spaces is also of the form (1.5), but in this case the functions v and w have the form $s^\theta h(s)$, where $0 < \theta < 1$ and h is a logarithmic function.

Finally we apply these results to determine the spaces generated by some couples of function spaces and couples of spaces of operators. Some of these results are included in the following theorem.

Theorem 1.6. *Let* (Ω, μ) *be a resonant, σ-finite measure space and let* $1 < p_0, p_1 < \infty, 1 < q \le \infty$ *and* $1/q + 1/q' = 1$. *Then*

$$(L_{p_0,q}, L_{p_1,q})_{q;J} = L_{p_0,q}(\log L)_{-1/q'} \cap L_{p_1,q}(\log L)_{-1/q'} \quad \textit{and}$$
$$(L_{p_0,q}, L_{p_1,q})_{q;K} = L_{p_0,q}(\log L)_{1/q} + L_{p_1,q}(\log L)_{1/q}$$

with equivalence of norms.

The last chapter of the thesis refers to questions related to logarithmic interpolation spaces, that is, to the spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ whose norm is given by

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \left[t^{-\theta}\ell^{\mathbb{A}}(t)K(t,a)\right]^q \frac{dt}{t}\right)^{1/q}.$$

Here $1 \le q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell(t) = 1 + |\log t|$,

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \le 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty, \end{cases}$$

and now not only $0 < \theta < 1$ but also θ can take the values 0 and 1. In fact it is these two extreme values which we are interested in, since, as can be seen in [45, Proposition 2.1], in the ordered case, the methods $(A_0, A_1)_{0,q,\mathbb{A}}$ and $(A_0, A_1)_{1,q,\mathbb{A}}$ are related to limiting interpolation methods.

The spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ are studied in the papers by Gustavsson [55], Doktorskii [43], Evans and Opic [46], Evans, Opic and Pick [47] and the references given there.

If $0 < \theta < 1$ then $t^{-\theta}\ell^{\mathbb{A}}(t)$ satisfies the assumptions used in [55], so $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is just a special case of the real method with a function parameter whose theory is well-established (see [55, 60, 77]). However, if $\theta = 0$ or 1, there are certain natural questions that have not been studied yet and that are dealt with in Chapter 6. We start by giving the description of $(A_0, A_1)_{0,q,\mathbb{A}}$ and $(A_0, A_1)_{1,q,\mathbb{A}}$ by means of the J-functional and then we use this description to show the interpolation properties by these methods of compact and weakly compact operators, and also to determine the dual of $(A_0, A_1)_{0,q,\mathbb{A}}$ and $(A_0, A_1)_{1,q,\mathbb{A}}$.

We show that, on the contrary to the case $0 < \theta < 1$, when $\theta = 0$ or 1, the J-description depends on the relationship between $\mathbb{A}$ and q: Sometimes one should add one unit to the power of the logarithm, some other times an iterated logarithm should be inserted in addition, and some other times the J-description does not exist at all.

The problem of how compact operators behave under interpolation has its root in the reinforced version of the Riesz-Thorin theorem given by Krasnosel'skiĭ [64]. Recently Edmunds and Opic [45] established a limiting variant of Krasnosel'skiĭ's theorem for finite measure spaces to the effect that if $T : L_{p_0} \longrightarrow L_{q_0}$ compactly and $T : L_{p_1} \longrightarrow L_{q_1}$ boundedly, then T is also compact when acting between Lorentz-Zygmund spaces which are very close to L_{p_0} and L_{q_0}. The techniques used in [45] take advantage of dealing with Lebesgue spaces.

Very recently, Cobos, Fernández-Cabrera and Martínez [23] obtained abstract versions of the results of [45] which work for more general Banach couples. However, they assumed that the second couple is ordered by inclusion, that is, $B_1 \hookrightarrow B_0$ or $B_0 \hookrightarrow B_1$. This embedding hypothesis corresponds to the finiteness of the measure spaces in [45]. Using J-representations and a different approach to [23], we show here that the embedding restrictions can be removed. In concrete terms we show the following compactness results.

Theorem 1.7. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach couples. Consider a linear operator $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that the restriction $T : A_0 \longrightarrow B_0$ is compact. For any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$ such that*

$$\alpha_\infty + 1/q < 0 \textit{ if } q < \infty \quad \textit{or} \quad \alpha_\infty \le 0 \textit{ if } q = \infty,$$

we have that

$$T : (A_0, A_1)_{0,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{0,q,\mathbb{A}}$$

is also compact.

Theorem 1.8. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach couples. Consider a linear operator $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that the restriction $T : A_1 \longrightarrow B_1$ is compact. For any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$ such that*

$$\alpha_0 + 1/q < 0 \textit{ if } q < \infty \quad \textit{or} \quad \alpha_0 \le 0 \textit{ if } q = \infty,$$

we have that

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$$

is also compact.

As a consequence of these theorems we derive results on interpolation properties of compact operators acting between generalised Lorentz-Zygmund spaces $L_{p,q}(\log L)_{\mathbb{A}}(\Omega)$. Here (Ω, μ) is a σ-finite measure space, $1 < p < \infty$, $1 \le q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and the norm in the function space is given by

$$\|f\|_{L_{p,q}(\log L)_{\mathbb{A}}(\Omega)} = \left(\int_0^\infty \left[t^{1/p} \ell^{\mathbb{A}}(t) f^{**}(t) \right]^q \frac{dt}{t} \right)^{1/q} \tag{1.6}$$

where $f^{**}(t) = t^{-1} \int_0^t f^*(s)ds$ and f^* is the non-increasing rearrangement of f. We show the following versions of Edmunds and Opic's theorem in [45] for σ-finite (not necessarily finite) measure spaces.

Corollary 1.9. *Let (Ω, μ), (Θ, ν) be σ-finite measure spaces. Take $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \textit{ compactly and } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \textit{ boundedly.}$$

Then

$$T : L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_0,q)}}(\Omega) \longrightarrow L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_0,q)}}(\Theta)$$

is also compact.

Corollary 1.10. *Let (Ω, μ), (Θ, ν) be σ-finite measure spaces. Take $1 \le p_0 < p_1 < \infty$, $1 \le q_0 < q_1 < \infty$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Let T be a linear operator such that*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \textit{ boundedly and } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \textit{ compactly.}$$

Then

$$T : L_{p_1,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_1,q)}}(\Omega) \longrightarrow L_{q_1,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_1,q)}}(\Theta)$$

is also compact.

We also use J-representations to characterise the behaviour of weakly compact operators under interpolation when $\theta = 0$ or 1. In particular we show the following result.

Corollary 1.11. *Assume that* $\bar{A} = (A_0, A_1)$ *is a Banach couple and let* $1 < q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$.

(a) Suppose that $\alpha_0 + 1/q < 0 \leq \alpha_\infty + 1/q$. *Then* $(A_0, A_1)_{1,q,\mathbb{A}}$ *is reflexive if and only if the embedding* $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ *is weakly compact.*

(b) If $\alpha_0 + 1/q < 0$ *and* $\alpha_\infty + 1/q < 0$*, then* $(A_0, A_1)_{1,q,\mathbb{A}}$ *is reflexive if and only if the embedding* $A_1 \hookrightarrow A_0 + A_1$ *is weakly compact.*

Finally we determine the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ and $(A_0, A_1)_{0,q,\mathbb{A}}$ in terms of the K-functional. We show with the help of J-representations that in contrast to the classical theory, the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ (respectively, $(A_0, A_1)_{0,q,\mathbb{A}}$) depends on the relationship between q and $\mathbb{A}$.

The results in this chapter form the paper [38].

Chapter 2

Preliminaries

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in a common Hausdorff topological vector space $\mathcal{A}$, $A_0, A_1 \hookrightarrow \mathcal{A}$. Then it makes sense to consider the vector spaces $A_0 \cap A_1$ and

$$A_0 + A_1 = \{a \in \mathcal{A} : \exists a_0 \in A_0, a_1 \in A_1 \text{ with } a = a_0 + a_1\}$$

endowed with the natural norms

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

and

$$\|a\|_{A_0+A_1} = \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\right\},$$

respectively. Clearly $A_0 \cap A_1 \hookrightarrow A_0, A_1 \hookrightarrow A_0 + A_1$, so, once constructed $A_0 \cap A_1$ and $A_0 + A_1$, one can disregard $\mathcal{A}$ and consider $A_0 + A_1$ as the ambient space.

Given a Banach couple $\bar{A} = (A_0, A_1)$, a normed space $A \hookrightarrow \mathcal{A}$ is said to be an *intermediate space* with respect to $\bar{A}$ if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. An *interpolation space* between A_0 and A_1 is any intermediate space A with respect to the couple $\bar{A}$ such that for every $T \in \mathcal{L}(A_0 + A_1, A_0 + A_1)$ whose restriction to A_0 belongs to $\mathcal{L}(A_0, A_0)$ and whose restriction to A_1 belongs to $\mathcal{L}(A_1, A_1)$, the restriction of T to A belongs to $\mathcal{L}(A, A)$.

As we stated before, the complex interpolation method is based on the ideas in Thorin's proof of Riesz's theorem. The Riesz-Thorin theorem will be mentioned in Chapter 6 and reads as follows.

Theorem 2.1. [Riesz-Thorin theorem] *Let (Ω, μ) and (Θ, ν) be σ-finite measure spaces. Take any values $1 \le p_0, p_1, q_0, q_1 \le \infty$ and let T be a linear operator such that*

$$\begin{aligned} T : L_{p_0}(\Omega, \mu) &\longrightarrow L_{q_0}(\Theta, \nu) \quad \textit{with norm } M_0 \textit{ and} \\ T : L_{p_1}(\Omega, \mu) &\longrightarrow L_{q_1}(\Theta, \nu) \quad \textit{with norm } M_1. \end{aligned}$$

Take $0 < \theta < 1$ *and*
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$
Then the restriction of T *to* $L_p(\Omega,\mu)$ *is a bounded operator,*
$$T : L_p(\Omega,\mu) \longrightarrow L_q(\Theta,\nu),$$
with norm $M \leq M_0^{1-\theta} M_1^{\theta}$.

On the other hand, the root of the real interpolation method $(A_0, A_1)_{\theta,q}$ is the celebrated Marcinkiewicz interpolation theorem, which states the following.

Theorem 2.2. [Marcinkiewicz's interpolation theorem] *Let* (Ω,μ) *and* (Θ,ν) *be* σ*-finite measure spaces. Take any values* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *with* $q_0 \neq q_1$ *and let* T *be a linear operator such that*
$$T : L_{p_0}(\Omega,\mu) \to L_{q_0,\infty}(\Theta,\nu) \text{ with norm } M_0 \quad \text{and} \quad T : L_{p_1}(\Omega,\mu) \to L_{q_1,\infty}(\Theta,\nu) \text{ with norm } M_1.$$
Let $0 < \theta < 1$ *and put*
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$
Then, if $p \leq q$, *the restriction* $T : L_p(\Omega,\mu) \to L_q(\Theta,\nu)$ *is also bounded with norm* $M \leq C M_0^{1-\theta} M_1^{\theta}$, *where* C *does not depend on* T.

2.1 The real interpolation method

The real method can be defined in several equivalent ways, but the most common are those given by Peetre's K- and J-functionals. For $t > 0$, *Peetre's* K- *and* J*-functionals* are the norms on $A_0 + A_1$ and $A_0 \cap A_1$, respectively, defined by
$$K(t,a) = K(t,a;\bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in A_0 + A_1,$$
and
$$J(t,a) = J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \; a \in A_0 \cap A_1 .$$
Notice that $K(1,\cdot) = \|\cdot\|_{A_0+A_1}$ and $J(1,\cdot) = \|\cdot\|_{A_0\cap A_1}$. Moreover, for each $t > 0$, $K(t,\cdot)$ is equivalent to $\|\cdot\|_{A_0+A_1}$ and $J(t,\cdot)$ is equivalent to $\|\cdot\|_{A_0\cap A_1}$.

With the help of these functionals we can define the (classical) real interpolation spaces. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. The *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$, viewed as a K-space, consists of all $a \in A_0 + A_1$ for which the norm
$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty [t^{-\theta} K(t,a)]^q \frac{dt}{t}\right)^{1/q} \tag{2.1}$$
is finite (when $q = \infty$ the integral should be replaced by a supremum). See [5, 4, 8, 80]. It follows from the equivalence theorem that $\bar{A}_{\theta,q}$ coincides with the collection of all those $a \in A_0 + A_1$ for

which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ that represents a as follows

$$a = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1\text{)} \tag{2.2}$$

and such that

$$\left(\int_0^\infty [t^{-\theta} J(t, u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty\,. \tag{2.3}$$

We refer to [84] for details on the Bochner integral. Moreover,

$$\|a\|_{\bar{A}_{\theta,q;J}} = \inf\left\{\left(\int_0^\infty [t^{-\theta} J(t, u(t))]^q \frac{dt}{t}\right)^{1/q} : u(t) \text{ satisfies (2.2) and (2.3)}\right\} \tag{2.4}$$

is an equivalent norm to $\|\cdot\|_{\bar{A}_{\theta,q}}$.

As we mentioned before, the real method produces interpolation spaces. The following theorem shows this fact and also generalises Marcinkiewicz's theorem to arbitrary Banach spaces. Given two Banach couples $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$, we write $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ if T is a linear operator, $T : A_0 + A_1 \longrightarrow B_0 + B_1$, for which restrictions $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ are bounded. In addition, we write $M_j = \|T\|_{A_j, B_j}$.

Theorem 2.3. [Interpolation Theorem] *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach couples and let $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$. Then, for $0 < \theta < 1$ and $1 \le q \le \infty$, the restriction of T to $(A_0, A_1)_{\theta,q}$ is a bounded operator,*

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q},$$

and its norm is $M \le M_0^{1-\theta} M_1^{\theta}$.

We end this section by giving an example. Let (Ω, μ) be a σ-finite measure space. Given any measurable function f which is finite almost everywhere, the *non-increasing rearrangement of* f is defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : |f(x)| > s\}) \le t\}\,. \tag{2.5}$$

Let $1 \le p, q \le \infty$. We define the *Lorentz space* $L_{p,q}(\Omega)$ as the set of all equivalence classes of measurable functions f for which the following functional is finite

$$\|f\|_{p,q} = \left(\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q}.$$

Note that if $p = q$ then $L_{p,p}(\Omega)$ coincides with the Lebesgue space $L_p(\Omega)$. So, the scale of Lorentz spaces is a refinement of the scale of Lebesgue spaces.

The couple of Lebesgue spaces $(L_\infty(\Omega), L_1(\Omega))$ is a Banach couple. It turns out that if we apply the real method to this couple, we obtain a Lorentz space. Namely, if $1 \le q \le \infty$ and $0 < \theta < 1$, we have that

$$(L_\infty(\Omega), L_1(\Omega))_{\theta,q} = L_{1/\theta,q}(\Omega).$$

We will mention more results on the real method throughout the thesis. All of them, and examples that deal with other spaces, appear in [80, 5, 4, 8].

2.2 Extensions of the real method

We stated before that the real method is very flexible and can be easily extended, and we mentioned how one could generalise the definition to other *kinds* of couples of spaces (for instance, quasi-Banach spaces or even normed Abelian groups).

Another possibility is to change the *norm* in the definition. If one replaces the usual weighted L_q norm by a more general lattice norm Γ, one obtains the so-called *general real method*, introduced by Peetre in [74]. This method plays an important role, as can be seen in the book by Brudnyĭ and Krugljak [8] and the articles by Cwikel and Peetre [41] and by Nilsson [71, 72]. Among other applications, it turns out that one can describe all interpolation spaces with respect to the couple (L_∞, L_1) by means of this general real method (see [8] or [72]). It will appear in Chapter 4.

A special case of the general real method consists in replacing in the definition of $(A_0, A_1)_{\theta,q}$ the function t^θ by a more general *function* $f(t)$ (see the paper by Gustavsson [55]). The case where $f(t) = t^\theta g(t)$ is of special interest; the definition of these methods is as follows. The interpolation space $\bar{A}_{\theta,g,q} = (A_0, A_1)_{\theta,g,q}$ consists of all $a \in A_0 + A_1$ for which the norm

$$\|a\|_{\bar{A}_{\theta,g,q}} = \left(\int_0^\infty [t^{-\theta} g(t) K(t,a)]^q \frac{dt}{t} \right)^{1/q} \tag{2.6}$$

is finite. Here, g is a power of $1 + |\log t|$ or, more generally, a slowly varying function.

In order to illustrate the importance of these methods we give the following example. Working with the couple of Lebesgue spaces (L_{p_0}, L_{p_1}), the real method produces Lebesgue and Lorentz spaces (see [5, 80, 4]). The Lorentz-Zygmund space $L_{p,q}(\log L)_b$ can be obtained from the couple (L_{p_0}, L_{p_1}) by using this extension of the real method. In fact, we have that

$$(L_\infty, L_1)_{1/p,\rho_b,q} = L_{p,q}(\log L)_b, \text{ where } \rho_b = (1 + |\log t|)^b.$$

Recall that if (Ω, μ) is a σ-finite measure space, $1 \le p, q \le \infty$ and $b \in \mathbb{R}$, the *Lorentz-Zygmund function space* $L_{p,q}(\log L)_b(\Omega)$ consists of all (equivalence classes of) measurable functions f on Ω such that the functional

$$\|f\|_{L_{p,q}(\log L)_b(\Omega)} = \left(\int_0^\infty \left[t^{1/p} (1 + |\log t|)^b f^*(t) \right]^q \frac{dt}{t} \right)^{1/q}$$

is finite. Here f^* is the *non-increasing rearrangement of* f defined above. We refer to [3, 4, 44] for properties of these spaces. Note that if $q = p$ then $L_{p,p}(\log L)_b(\Omega)$ is the Zygmund space $L_p(\log L)_b(\Omega)$. If in addition $b = 0$, then $L_{p,p}(\log L)_0(\Omega)$ is the Lebesgue space $L_p(\Omega)$.

Several authors like Gustavsson [55], Doktorskii [43], Evans and Opic [46], Evans, Opic and Pick [47] have focused on the special case where the function g in (2.6) is a broken logarithmic function. We denote the resulting space by $(A_0, A_1)_{\theta,q,\mathbb{A}}$, which is normed by

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \left[t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q}. \tag{2.7}$$

Here $1 \le q \le \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell(t) = 1 + |\log t|$,

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \le 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty, \end{cases}$$

and now not only $0 < \theta < 1$ but also θ can take the values 0 and 1. We will work with these limiting methods in Chapter 6.

Before presenting another extension that we shall consider, we need to establish the following notation. If X, Y are non-negative quantities depending on certain parameters, we put $X \lesssim Y$ if there is a constant $c > 0$ independent of the parameters involved in X and Y such that $X \le cY$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

When we defined the real method, we asked for θ to be strictly between 0 and 1. A natural question, thus, is the following: Can we take $\theta = 0$ or $\theta = 1$? This was already considered by Butzer and Berens in [9], where they showed that if we take $\theta = 0$ or 1 and $q = \infty$ in (2.1) or $\theta = 0$ or 1 and $q = 1$ in (2.4), then the resulting spaces are also interpolation spaces. However, for any other values of q, the spaces with $\theta = 0, 1$ are meaningless, that is, they are just the trivial space $\{0\}$, which need not be even an intermediate space. Indeed, in order to simplify, suppose that $A_0 \hookrightarrow A_1$. Take $a \in A_0 + A_1 = A_1$. Then clearly $K(t, a; A_0, A_1) \le t \|a\|_{A_1}$. Conversely, if $a \in A_1$ and $a = a_0 + a_1$ is any representation of a with $a_j \in A_j$ and $0 < t < 1$, then

$$t\|a\|_{A_1} \le t\|a_0\|_{A_1} + t\|a_1\|_{A_1} \lesssim t\|a_0\|_{A_0} + t\|a_1\|_{A_1} \le \|a_0\|_{A_0} + t\|a_1\|_{A_1},$$

so, taking the infimum over all possible representations, we obtain $t\|a\|_{A_1} \lesssim K(t, a; A_0, A_1)$. This implies that

$$\text{if } A_0 \hookrightarrow A_1 \text{ and } 0 < t < 1, \text{ then } K(t, a; A_0, A_1) \sim t\|a\|_{A_1}. \tag{2.8}$$

Now, if we take $\theta = 1$ in (2.1), we obtain

$$\left(\int_0^1 \left[t^{-1}K(t,a)\right]^q \frac{dt}{t} + \int_1^\infty \left[t^{-1}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q},$$

and, by (2.8),

$$\left(\int_0^1 \left[t^{-1}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} \sim \|a\|_{A_1} \left(\int_0^1 t^{-q} \frac{dt}{t}\right)^{1/q},$$

which is divergent unless $q = \infty$. The proof of the general K- and J-cases can be seen in [9, Propositions 3.2.5 and 3.2.7].

The extension that we are about to describe corresponds to taking the limiting values $\theta = 0$ and $\theta = 1$. This can be done in the logarithmic case (2.7), but in these limit cases the extra function $\ell^{\mathbb{A}}(t)$ is essential to get a meaningful definition. For this extension, instead of replacing t^θ by a more general function $t^\theta g(t)$, the original definition is modified in the most natural way, without the help of auxiliary functions. The following result motivates the definitions of these limiting methods. Suppose that the Banach spaces are related by a continuous embedding, say, for instance, $A_0 \hookrightarrow A_1$.

Proposition 2.4. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple with $A_0 \hookrightarrow A_1$ and let $0 < \theta < 1$ and $1 \le q \le \infty$.*

(i) *The space* $\bar{A}_{\theta,q}$*, seen as a* K*-space, coincides with the collection* $\bar{A}_{\theta,q;K}$ *of all those* $a \in A_1$ *for which the following norm is finite*

$$\|a\|_{\bar{A}_{\theta,q;K}} = \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q} \tag{2.9}$$

with equivalent norms.

(ii) *The space* $\bar{A}_{\theta,q}$*, seen as a* J*-space, coincides with the collection* $\bar{A}_{\theta,q;J}$ *of all those* $a \in A_1$ *for which there is a strongly measurable function* $u(t)$ *with values in* A_0 *that represents* a *as follows*

$$a = \int_1^\infty u(t)\frac{dt}{t} \quad (\textit{convergence in } A_1) \tag{2.10}$$

and such that

$$\left(\int_1^\infty [t^{-\theta}J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty\,. \tag{2.11}$$

Moreover,

$$\|a\|_{\bar{A}_{\theta,q;J}} = \inf\left\{\left(\int_1^\infty [t^{-\theta}J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} : u(t) \textit{ satisfies } (2.10) \textit{ and } (2.11)\right\} \tag{2.12}$$

is an equivalent norm to $\|\cdot\|_{\bar{A}_{\theta,q}}$.

Proof. We have by (2.8) that

$$\begin{aligned}\|a\|_{\bar{A}_{\theta,q}} &\sim \left(\int_0^1 [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \|a\|_{A_1}\left(\int_0^1 t^{(1-\theta)q} \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since $0 < \theta < 1$, the first integral in the second line is a constant. Moreover, $K(t,\cdot)$ is non-decreasing with t, so

$$\begin{aligned}\|a\|_{\bar{A}_{\theta,q}} &\sim \|a\|_{A_1} + \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q} \sim K(1,a)\left(\int_1^\infty t^{-\theta q} \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}\\ &\le 2\left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since we also have that

$$\left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q} \le \|a\|_{\bar{A}_{\theta,q}},$$

we derive that, if $A_0 \hookrightarrow A_1$, $0 < \theta < 1$ and $1 \le q \le \infty$, then

$$\|a\|_{\bar{A}_{\theta,q}} \sim \left(\int_1^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.$$

Next we prove (ii). Let $a \in \bar{A}_{\theta,q;J}$ and let $u(t)$ be such that $\int_1^\infty u(t)dt/t = a$, and put

$$v(t) = \begin{cases} 0 & \text{if } 0 < t \le 1,\\ u(t) & \text{if } 1 < t < \infty.\end{cases}$$

Then clearly $a = \int_0^\infty v(t)\,dt/t$ and

$$\left(\int_0^\infty \left[t^{-\theta}J(t,v(t))\right]^q \frac{dt}{t}\right)^{1/q} \le \left(\int_1^\infty \left[t^{-\theta}J(t,u(t))\right]^q \frac{dt}{t}\right)^{1/q},$$

so $\|a\|_{\bar{A}_{\theta,q}} \le \|a\|_{\bar{A}_{\theta,q;J}}$.

Conversely, let $u(t)$ be such that $\int_0^\infty u(t)\frac{dt}{t} = a$ and (2.3) is satisfied. Then $\int_0^1 u(s)\frac{ds}{s}$ belongs to A_0 because for $1/q + 1/q' = 1$ we obtain

$$\int_0^1 \|u(s)\|_{A_0} \frac{ds}{s} \le \int_0^1 J(s,u(s))\frac{ds}{s} \le \left(\int_0^1 s^{\theta q'}\frac{ds}{s}\right)^{1/q'} \left(\int_0^1 \left[s^{-\theta}J(s,u(s))\right]^q \frac{ds}{s}\right)^{1/q} < \infty. \quad (2.13)$$

Put

$$v(t) = \begin{cases} u(t) + \dfrac{1}{\log 2}\int_0^1 u(s)\frac{ds}{s} & \text{if } 1 < t \le 2, \\ u(t) & \text{if } t > 2. \end{cases}$$

Then, we have that

$$\int_1^\infty v(t)\frac{dt}{t} = \int_0^1 u(t)\frac{dt}{t} + \int_1^2 u(t)\frac{dt}{t} + \int_2^\infty u(t)\frac{dt}{t} = a,$$

and

$$\int_1^\infty \left[t^{-\theta}J(t,v(t))\right]^q \frac{dt}{t} \lesssim \int_1^2 \left[t^{-\theta}J\Big(t,\int_0^1 u(s)\frac{ds}{s}\Big)\right]^q \frac{dt}{t} + \int_1^2 \left[t^{-\theta}J(t,u(t))\right]^q \frac{dt}{t} + \int_2^\infty \left[t^{-\theta}J(t,u(t))\right]^q \frac{dt}{t}.$$

For $t > 1$ we have that $J(t,u(s)) \lesssim t\,\|u(s)\|_{A_0}$. Indeed, since $A_0 \hookrightarrow A_1$,

$$J(t,u(s)) \le \|u(s)\|_{A_0} + t\,\|u(s)\|_{A_1} \lesssim \|u(s)\|_{A_0} + t\,\|u(s)\|_{A_0} \lesssim t\,\|u(s)\|_{A_0}.$$

Moreover,

$$J\Big(t,\int_0^1 u(s)\frac{ds}{s}\Big) \le \int_0^1 J(t,u(s))\frac{ds}{s}.$$

This gives that

$$\begin{aligned}\int_1^2 \left[t^{-\theta}J\Big(t,\int_0^1 u(s)\frac{ds}{s}\Big)\right]^q \frac{dt}{t} &\le \int_1^2 \left[t^{-\theta}\int_0^1 J(t,u(s))\frac{ds}{s}\right]^q \frac{dt}{t} \\ &\le \int_1^2 \left[t^{1-\theta}\int_0^1 \|u(s)\|_{A_0}\frac{ds}{s}\right]^q \frac{dt}{t} \\ &\lesssim \int_0^1 \left[s^{-\theta}J(s,u(s))\right]^q \frac{ds}{s},\end{aligned}$$

where we have used (2.13) in the last inequality. This ends the proof. □

Note that the difference between the equivalent definitions given in Proposition 2.4 and the original ones is that the integrals are not on $(0,\infty)$ but only on $(1,\infty)$.

In 1986, Gomez and Milman ([54]) realised that one can take $\theta = 1$ in (2.9), obtaining spaces that are not only intermediate, but also interpolation spaces. If $A_0 \hookrightarrow A_1$, the *limiting* K*-spaces for ordered couples* $\bar{A}_{1,q;K}$ are thus defined as the collection of all those $a \in A_1$ for which the following norm is finite

$$\|a\|_{\bar{A}_{1,q;K}} \sim \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q}. \tag{2.14}$$

Later on, Cobos, Fernández-Cabrera, Kühn and Ullrich ([19]) noticed that one can also take $\theta = 0$ in the equivalent definition by means of the J-functional (2.12), obtaining also interpolation spaces. If $A_0 \hookrightarrow A_1$, the *limiting* J*-spaces for ordered couples* $\bar{A}_{0,q;J}$ are thus defined as the collection of all those $a \in A_1$ for which there is a strongly measurable function $u(t)$ with values in A_0 that represents a as follows

$$a = \int_1^\infty u(t)\frac{dt}{t} \quad (\text{convergence in } A_1) \tag{2.15}$$

and such that

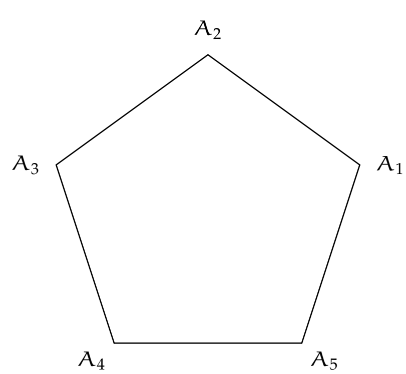
$$\left(\int_1^\infty J(t,u(t))^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{2.16}$$

They defined the norm on this space as

$$\|a\|_{\bar{A}_{0,q;J}} = \inf\left\{\left(\int_1^\infty J(t,u(t))^q \frac{dt}{t}\right)^{1/q} : u(t) \text{ satisfies (2.15) and (2.16)}\right\}. \tag{2.17}$$

The spaces $\bar{A}_{1,q;K}$ and $\bar{A}_{0,q;J}$ arise when interpolating the 4-tuple $\{A_0, A_1, A_1, A_0\}$ by the methods associated to the unit square. Let us recall the definition of the K- and J-methods associated to a convex polygon in the plane.

Motivated by certain problems in the theory of function spaces, authors like Foiaş and Lions [51], Sparr [79] and Fernandez [48] among others have studied interpolation methods for finite families (N-tuples) of Banach spaces. In 1991, Cobos and Peetre [33] introduced a K- and a J-method for N-tuples of Banach spaces that are associated to a convex polygon Π in the plane and a point (α,β) in the interior of Π. The construction is as follows: Consider a Banach N-tuple $\bar{A} = (A_1, A_2, \ldots, A_N)$, that is, N Banach spaces $A_1, \ldots, A_N$ that are linearly and continuously embedded in a Hausdorff topological vector space $\mathcal{A}$. We designate by $\Delta(\bar{A})$ the intersection $A_1 \cap A_2 \cap \cdots \cap A_N$, and $\Sigma(\bar{A})$ stands for $A_1 + A_2 + \cdots + A_N$. Imagine each Banach space A_j sitting on the vertex P_j of a convex polygon $\Pi = \overline{P_1, P_2, \ldots, P_N}$ in $\mathbb{R}^2$.



Let the coordinates of P_j be $\left(x_j, y_j\right)$ and put

$$K(t,s;a) = \inf\left\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \left\|a_j\right\|_{A_j} : a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\right\},\ t,s>0,\ a \in \Sigma\left(\bar{A}\right)$$

and

$$J(t,s;a) = \max_{1\le j\le N}\left\{t^{x_j} s^{y_j}\left\|a_j\right\|_{A_j}\right\},\ t,s>0,\ a\in\Delta\left(\bar{A}\right).$$

Choose any (α,β) in the interior of Π and any $1\le q\le\infty$. Then we can define the space $\bar{A}_{(\alpha,\beta),q;K}$ as the set of all $a\in\Sigma\left(\bar{A}\right)$ for which the following norm is finite.

$$\|a\|_{\bar{A}_{(\alpha,\beta),q;K}} = \left(\int_0^\infty\int_0^\infty \left[t^{-\alpha}s^{-\beta}K(t,s;a)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q}. \tag{2.18}$$

In addition, we define the space $\bar{A}_{(\alpha,\beta),q;J}$ as the one consisting of all elements $a\in\Sigma\left(\bar{A}\right)$ which can be represented as

$$a = \int_0^\infty\int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s}\ \text{(convergence in } \Sigma\left(\bar{A}\right)),$$

where $u(t,s)$ is a strongly measurable funtion with values in $\Delta\left(\bar{A}\right)$ and

$$\left(\int_0^\infty\int_0^\infty\left[t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}<\infty.$$

We define the norm in $\bar{A}_{(\alpha,\beta),q;J}$ as follows.

$$\|a\|_{\bar{A}_{(\alpha,\beta),q;J}} = \inf\left\{\left(\int_0^\infty\int_0^\infty\left[t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\right\} \tag{2.19}$$

where the infimum is taken over all representations $u(t,s)$ of a as above.

If we take $\Pi = \{(0,0),(1,0),(0,1),(1,1)\}$ (that is, the unit square), we obtain the spaces studied by Fernandez (see [48]) for 4-tuples. If $\Pi = \{(0,0),(1,0),(0,1)\}$, we recover the Sparr spaces for triples (see [79]).

Working with the methods associated to polygons, K- and J- spaces are different in general, since the fundamental lemma ([5, Lemma 3.3.2]) does not extend to the context of N-tuples. However, we have the following continuous embedding

$$\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K},$$

see [33, Theorem 1.3].

In the theory of K- and J-spaces defined by a polygon, there is a case which is harder. Namely, when the point (α,β) is in any diagonal of Π. It turns out that if $A_0\hookrightarrow A_1$ and we interpolate the 4-tuple $\{A_0,A_1,A_1,A_0\}$ using the unit square, then when (α,β) is on the diagonal $\beta = 1-\alpha$, K-spaces coincide with $(A_0, A_1)_{1,q;K}$ (see [20, Theorem 3.5]). For J-spaces, we have that along the diagonal $\alpha = \beta$ they coincide with $(A_0, A_1)_{0,q;J}$ (see [19, Theorem 5.1]).

The case of a 4-tuple $\{A_0, A_1, A_1, A_0\}$ when there is no relationship between A_0 and A_1 has been studied by Cobos, Fernández-Cabrera and Silvestre in [25, 26]. For this aim they introduced several limiting K- and J-methods to the effect that along the diagonals of the square, the interpolated spaces are sums of limiting methods and real interpolation spaces in the K-case, while they are intersections of limiting methods and real interpolation spaces in the J-case.

The K-spaces $\tilde{A}_{1,q;K}$ and $\tilde{A}_{0,q;K}$ that they considered are defined as the collections of all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\tilde{A}_{1,q;K}} = \sup_{0<t\leq 1} t^{-1}K(t,a) + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t} \right)^{1/q} \tag{2.20}$$

and

$$\|a\|_{\tilde{A}_{0,q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t} \right)^{1/q} + \sup_{1\leq t<\infty} K(t,a), \tag{2.21}$$

respectively. On the other hand, the J-spaces $\tilde{A}_{0,q;J}$ and $\tilde{A}_{1,q;J}$ are defined as the collection of all $a \in A_0 + A_1$ which can be represented as

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad \text{(convergence in } A_0 + A_1), \tag{2.22}$$

where $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ such that

$$\int_0^1 J(t,u(t)) \frac{dt}{t} + \left(\int_1^\infty J(t,u(t))^q \frac{dt}{t} \right)^{1/q} < \infty \quad (\text{for } \tilde{A}_{0,q;J}), \quad \text{or} \tag{2.23}$$

$$\left(\int_0^1 \left[t^{-1}J(t,u(t))\right]^q \frac{dt}{t} \right)^{1/q} + \int_1^\infty t^{-1}J(t,u(t)) \frac{dt}{t} < \infty \quad (\text{for } \tilde{A}_{1,q;J}). \tag{2.24}$$

The norm in $\tilde{A}_{0,q;J}$ (respectively, in $\tilde{A}_{1,q;J}$) is given by the infimum in (2.23) (respectively, (2.24)) over all representations (2.22), (2.23) (respectively, (2.22), (2.24)). We refer to [25, 26] for details on these K- and J-spaces.

Chapter 3

Limiting real interpolation methods for arbitrary Banach couples

In the following three chapters we shall develop a comprehensive theory of limiting methods for arbitrary couples. We will present a family of K-methods and a family of J-methods which extend in a natural way definitions (2.14) and (2.17), given in [54] and [19] for ordered couples, to arbitrary couples. It turns out that these K- and J-methods are related by duality and that they allow to produce a sufficiently rich theory. In terms of the interpolation of the 4-tuple $\{A_0, A_1, A_1, A_0\}$ by the methods associated to the unit square (see [25, 26]), the choice we make corresponds to the methods that arise using the centre of the square.

This chapter is organised as follows. In Sections 3.1 and 3.2, we introduce the limiting K- and J-methods. We also establish there their basic properties and we study their connection with the methods developed for the ordered case (definitions (2.14) and (2.17)) and with the methods considered in [25] and [26]. There is a price to be paid for having methods for general couples: They satisfy worse norm estimates for interpolated operators than in the ordered case and, as a consequence, interpolation properties of compact operators are also worse than in the ordered case. Interpolation of compact operators is discussed in Section 3.3. As we show there, given $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$, a sufficient condition for the interpolated operator by limiting methods to be compact is that both restrictions $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ are compact.

Section 3.4 is devoted to the description of the limiting K-spaces using the J-functional. This can be done provided that $1 \le q < \infty$. Some consequences of that description are also shown there. Duality between limiting K- and J-spaces is discussed in Section 3.5, while Section 3.6 contains some examples of limiting spaces obtained by these methods. Namely, we work with the couple $(L_\infty(\Omega), L_1(\Omega))$ where Ω is a σ-finite measure space, and also with couples of weighted L_q-spaces and of Besov spaces. We also apply the limiting methods to obtain a Hausdorff-Young type result for the Zygmund space $L_2(\log L)_{-1/2}([0, 2\pi])$. The results in this chapter form the paper [35].

Subsequently, for $1 \le q \le \infty$, we let ℓ_q be the usual space of q-summable scalar sequences and c_0 is the space of null sequences. Given any sequence (λ_m) of positive numbers and any sequence (W_m) of Banach spaces, we write $\ell_q(\lambda_m W_m)$ for the space of all vector-valued sequences $w = (w_m)$ with $w_m \in W_m$ and such that

$$\|w\|_{\ell_q(\lambda_m W_m)} = \left(\sum_m [\lambda_m \|w_m\|_{W_m}]^q\right)^{1/q} < \infty\,.$$

If for each m the space W_m is equal to the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we simply write $\ell_q(\lambda_m)$. The space $c_0(\lambda_m W_m)$ is defined similarly.

3.1 Limiting K-spaces

We start by introducing the limiting K-spaces that we will consider in the following.

Definition 3.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. We define the space $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ as the collection of all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\bar{A}_{q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q}.$$

Since

$$K(t,a;A_0,A_1) = tK(t^{-1},a;A_1,A_0)\,,$$

we have that

$$(A_0,A_1)_{q;K} = (A_1,A_0)_{q;K}\,. \tag{3.1}$$

Indeed, let $a \in A_0 + A_1$. By a change of variable, it follows that

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{q;K}} &= \left(\int_0^1 K(t,a;A_0,A_1)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,a;A_0,A_1)]^q \frac{dt}{t}\right)^{1/q}\\
&= \left(\int_0^1 [tK(t^{-1},a;A_1,A_0)]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty K(t^{-1},a;A_1,A_0)^q \frac{dt}{t}\right)^{1/q}\\
&= \left(\int_1^\infty [s^{-1}K(s,a;A_1,A_0)]^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^1 K(s,a;A_1,A_0)^q \frac{ds}{s}\right)^{1/q} = \|a\|_{(A_1,A_0)_{q;K}}.
\end{aligned}$$

The following lemma shows that $\bar{A}_{q;K}$ is an intermediate space between A_0 and A_1, and that it is larger than any real interpolation space.

Lemma 3.1. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \theta < 1$ and $1 \le q, r \le \infty$. Then we have*

$$A_0 \cap A_1 \hookrightarrow (A_0,A_1)_{\theta,r} \hookrightarrow (A_0,A_1)_{q;K} \hookrightarrow A_0 + A_1\,.$$

Moreover, $(A_0,A_1)_{\infty;K} = A_0 + A_1$ with equivalent norms.

Proof. It is well-known that $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,r} \hookrightarrow (A_0, A_1)_{\theta,\infty}$ (see [5] or [80]). Take any vector $a \in (A_0, A_1)_{\theta,\infty}$. We have

$$\left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} \le \left(\int_0^1 t^{\theta q} \frac{dt}{t}\right)^{1/q} \|a\|_{\bar{A}_{\theta,\infty}} = c_1 \|a\|_{\bar{A}_{\theta,\infty}}$$

and

$$\left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q} \le \left(\int_1^\infty t^{(\theta-1)q} \frac{dt}{t}\right)^{1/q} \|a\|_{\bar{A}_{\theta,\infty}} = c_2 \|a\|_{\bar{A}_{\theta,\infty}}.$$

Whence, $(A_0, A_1)_{\theta,\infty} \hookrightarrow (A_0, A_1)_{q;K}$.

Assume now that $a \in (A_0, A_1)_{q;K}$. Using that $K(t,a)$ is a non-decreasing function of t, we derive with $c_3 = (\int_1^\infty t^{-q} dt/t)^{-1/q}$ that

$$\|a\|_{A_0+A_1} = c_3 \left(\int_1^\infty t^{-q} \frac{dt}{t}\right)^{1/q} K(1,a) \le c_3 \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q} \le c_3 \|a\|_{\bar{A}_{q;K}}.$$

Finally, if $q = \infty$ we have

$$\|a\|_{\bar{A}_{\infty;K}} = \sup_{0<t\le 1} K(t,a) + \sup_{1<t<\infty} t^{-1}K(t,a) = \|a\|_{A_0+A_1},$$

as desired. □

Remark 3.1. In the ordered case where $A_0 \hookrightarrow A_1$, if we disregard the term with the integral over $(0,1)$ in Definition 3.1, then we recover the spaces $\bar{A}_{1,q;K}$ introduced in (2.14), in the previous chapter. Notice that $\bar{A}_{q;K}$ extends $\bar{A}_{1,q;K}$ to arbitrary couples because if $A_0 \hookrightarrow A_1$ we have by (2.8) that

$$\left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^1 t^q \frac{dt}{t}\right)^{1/q} \|a\|_{A_1} \le c\|a\|_{\bar{A}_{1,q;K}}.$$

So, $\bar{A}_{q;K} = \bar{A}_{1,q;K}$ with equivalence of norms.

In the following proposition we show that $\bar{A}_{q;K}$ is complete.

Proposition 3.2. *If A_0 and A_1 are complete, then so is $\bar{A}_{q;K}$ for any $1 \le q \le \infty$.*

Proof. The proof follows the same lines as the proof of [5, Theorem 3.4.2 (a)]. Suppose that we have $\sum_{j\in\mathbb{N}} \|a_j\|_{\bar{A}_{q;K}} < \infty$. Then, by Lemma 3.1, $\bar{A}_{q;K} \hookrightarrow A_0 + A_1$, so we also have $\sum_{j\in\mathbb{N}} \|a_j\|_{A_0+A_1} < \infty$. Since $A_0 + A_1$ is complete, $\sum_{j\in\mathbb{N}} a_j$ converges in $A_0 + A_1$ to an element a. Moreover

$$\begin{aligned}
\Big\|\sum_{j>N} a_j\Big\|_{\bar{A}_{q;K}} &= \left(\int_0^1 K(t, \sum_{j>N} a_j)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t, \sum_{j>N} a_j)]^q \frac{dt}{t}\right)^{1/q} \\
&\le \sum_{j>N} \left(\int_0^1 K(t,a_j)^q \frac{dt}{t}\right)^{1/q} + \sum_{j>N} \left(\int_1^\infty [t^{-1}K(t,a_j)]^q \frac{dt}{t}\right)^{1/q} \\
&= \sum_{j>N} \|a_j\|_{\bar{A}_{q;K}} \xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

It follows that $a \in \bar{A}_{q;K}$ and $\sum_{j\in\mathbb{N}} a_j$ converges to a in $\bar{A}_{q;K}$. □

Next we show the connection between $\bar{A}_{q;K}$ and the limiting spaces $\tilde{A}_{1,q;K}$ and $\tilde{A}_{0,q;K}$ introduced in [25]. Recall that their definition is given in equations (2.20) and (2.21).

Proposition 3.3. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then we have with equivalent norms*

$$\bar{A}_{q;K} = \tilde{A}_{0,q;K} + \tilde{A}_{1,q;K}.$$

Proof. Let $a = x_0 + x_1$ with $x_0 \in \tilde{A}_{0,q;K}$ and $x_1 \in \tilde{A}_{1,q;K}$. Then

$$\begin{aligned}\left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q} &\le \left(\int_1^\infty [t^{-1}K(t,x_0)]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,x_1)]^q \frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_1^\infty t^{-q} \frac{dt}{t}\right)^{1/q} \|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}} \le c_1(\|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}}).\end{aligned}$$

Similarly

$$\begin{aligned}\left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} &\le \left(\int_0^1 K(t,x_0)^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 K(t,x_1)^q \frac{dt}{t}\right)^{1/q} \\ &\le \|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}} \left(\int_0^1 t^q \frac{dt}{t}\right)^{1/q} \le c_2(\|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}}).\end{aligned}$$

This yields the continuous embedding $\tilde{A}_{0,q;K} + \tilde{A}_{1,q;K} \hookrightarrow \bar{A}_{q;K}$.

Conversely, let $a \in \bar{A}_{q;K}$ and take any representation $a = x_0 + x_1$ with $x_j \in A_j$ $(j = 0,1)$ and $\|x_0\|_{A_0} + \|x_1\|_{A_1} \le 2K(1,a) = 2\|a\|_{A_0+A_1}$. We claim that $x_j \in \tilde{A}_{j,q;K}$ for $j = 0,1$. Indeed,

$$\begin{aligned}\|x_0\|_{\tilde{A}_{0,q;K}} &= \left(\int_0^1 K(t,x_0)^q \frac{dt}{t}\right)^{1/q} + \sup_{1\le t<\infty} K(t,x_0) \\ &\le \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 K(t,x_1)^q \frac{dt}{t}\right)^{1/q} + \|x_0\|_{A_0} \\ &\le \|a\|_{\bar{A}_{q;K}} + \left(\int_0^1 t^q \frac{dt}{t}\right)^{1/q} \|x_1\|_{A_1} + \|x_0\|_{A_0} \\ &\le \|a\|_{\bar{A}_{q;K}} + c_1\|a\|_{A_0+A_1} \le c_2\|a\|_{\bar{A}_{q;K}},\end{aligned}$$

where we have used Lemma 3.1 in the last inequality. For x_1 we obtain

$$\begin{aligned}\|x_1\|_{\tilde{A}_{1,q;K}} &\le \|x_1\|_{A_1} + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,x_0)]^q \frac{dt}{t}\right)^{1/q} \\ &\le \|x_1\|_{A_1} + \|a\|_{\bar{A}_{q;K}} + \left(\int_1^\infty t^{-q} \frac{dt}{t}\right)^{1/q} \|x_0\|_{A_0} \le c_3\|a\|_{\bar{A}_{q;K}}.\end{aligned}$$

Whence, $a \in \tilde{A}_{0,q;K} + \tilde{A}_{1,q;K}$ and $\|a\|_{\tilde{A}_{0,q;K}+\tilde{A}_{1,q;K}} \le (c_2 + c_3)\|a\|_{\bar{A}_{q;K}}$. This completes the proof. □

As a direct consequence we can show the relationship of these limiting methods with the methods associated to the unit square (see Section 2.2). According to [25, Theorem 4.1], we have that

$(A_0, A_1, A_1, A_0)_{(1/2,1/2),q;K} = \tilde{A}_{0,q;K} + \tilde{A}_{1,q;K}$, where $(\cdot,\cdot,\cdot,\cdot)_{(\alpha,\beta),q;K}$ stands for the K-method associated to the unit square (see the definition in (2.18)). It follows, thus, by Proposition 3.3 that

$$\bar{A}_{q;K} = (A_0, A_1, A_1, A_0)_{(1/2,1/2),q;K}. \tag{3.2}$$

Besides the relation described in Remark 3.1, the following lemma shows another interesting connection between $\bar{A}_{q;K}$ and the space $\bar{A}_{1,q;K}$ defined for ordered couples (2.14).

Lemma 3.4. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then we have with equivalence of norms*

$$(A_0, A_1)_{q;K} = (A_0 \cap A_1, A_0 + A_1)_{1,q;K}.$$

Proof. Let $\bar{K}(t,a) = K(t,a; A_0 \cap A_1, A_0 + A_1)$ and $K(t,a) = K(t,a; A_0, A_1)$. According to [68, Theorem 3], for $1 < t < \infty$ and $a \in A_0 + A_1$, we have that $\bar{K}(t,a) \sim tK(t^{-1},a) + K(t,a)$. Consequently,

$$\begin{aligned}\|a\|_{(A_0\cap A_1, A_0+A_1)_{1,q;K}} &= \left(\int_1^\infty [t^{-1}\bar{K}(t,a)]^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [t^{-1}K(t,a)]^q \frac{dt}{t}\right)^{1/q}\\ &= \|a\|_{(A_0,A_1)_{q;K}}.\end{aligned}$$ □

The next lemma shows a discrete norm which is equivalent to $\|\cdot\|_{\bar{A}_{q;K}}$.

Lemma 3.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then the space $\bar{A}_{q;K}$ coincides with the collection of all those $a \in A_0 + A_1$ for which the norm*

$$\|a\|_{q;K} = \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{-m})\, K(2^m, a)]^q\right)^{1/q}$$

is finite. Moreover, $\|\cdot\|_{\bar{A}_{q;K}} \sim \|\cdot\|_{q;K}$.

Proof. Clearly $\|\cdot\|_{q;K}$ is a norm. In addition, we have that

$$\|a\|_{\bar{A}_{q;K}} \sim \left(\int_0^\infty [\min(1,t^{-1})K(t,a)]^q \frac{dt}{t}\right)^{1/q} = \left(\sum_{m=-\infty}^{\infty} \int_{2^{m-1}}^{2^m} [\min(1,t^{-1})K(t,a)]^q \frac{dt}{t}\right)^{1/q}.$$

Pick $t \in [2^{m-1}, 2^m)$. Then, if $q < \infty$, it is easy to see that

$$\begin{aligned}\int_{2^{m-1}}^{2^m} [\min(1,2^{-m})K(2^{m-1},a)]^q \frac{dt}{t} &\le \int_{2^{m-1}}^{2^m} [\min(1,t^{-1})K(t,a)]^q \frac{dt}{t}\\ &\le \int_{2^{m-1}}^{2^m} [\min(1,2^{1-m})K(2^m,a)]^q \frac{dt}{t},\end{aligned}$$

which implies

$$\begin{aligned}2^{-q}\log 2\left[\min(1,2^{1-m})K(2^{m-1},a)\right]^q &\le \int_{2^{m-1}}^{2^m} [\min(1,t^{-1})K(t,a)]^q \frac{dt}{t}\\ &\le 2^q \log 2\,[\min(1,2^{-m})K(2^m,a)]^q\end{aligned}$$

and, thus, the desired equivalence. On the other hand, if $q = \infty$, what we get is

$$\frac{1}{2}\min(1,2^{1-m})K(2^{m-1},a) \le \sup_{t\in[2^{m-1},2^m)} \min(1,t^{-1})K(t,a) \le 2\min(1,2^{-m})K(2^m,a).$$

This ends the proof. □

Note that a trivial consequence of this lemma is that K-spaces are increasing with q, that is, if $p \le q$ then $\bar{A}_{p;K} \hookrightarrow \bar{A}_{q;K}$.

Let $\bar{B} = (B_0, B_1)$ be another Banach couple. Recall that by $T \in \mathcal{L}(\bar{A}, \bar{B})$ we mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restrictions $T : A_j \longrightarrow B_j$ are bounded for $j = 0, 1$. It is not hard to check that for any $1 \le q \le \infty$, the restriction $T : \bar{A}_{q;K} \longrightarrow \bar{B}_{q;K}$ is also bounded with

$$\|T\|_{\bar{A}_{q;K}, \bar{B}_{q;K}} \le \max(\|T\|_{A_0, B_0}, \|T\|_{A_1, B_1}) .$$

Indeed, let $a \in \bar{A}_{q;K}$ and pick any decomposition $a = a_0 + a_1$ with $a_j \in A_j$. Since T is linear, $Ta = Ta_0 + Ta_1$ and by hypothesis $Ta_j \in B_j$. Therefore, for any $t > 0$,

$$\begin{aligned} K(t, Ta; B_0, B_1) &\le \|Ta_0\|_{B_0} + t\,\|Ta_1\|_{B_1} \le \|T\|_{A_0, B_0}\|a_0\|_{A_0} + t\|T\|_{A_1, B_1}\|a_1\|_{A_1} \\ &\le \max\left(\|T\|_{A_0, B_0}, \|T\|_{A_1, B_1}\right)\left(\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right) . \end{aligned}$$

Taking the infimum over all possible decompositions of a, we derive that

$$K(t, Ta; B_0, B_1) \le \max\left(\|T\|_{A_0, B_0}, \|T\|_{A_1, B_1}\right) K\left(t, a; A_0, A_1\right)$$

which implies the result.

The estimate $\|T\|_{\mathcal{F}(\bar{A}), \mathcal{F}(\bar{B})} \lesssim \max(\|T\|_{A_0, B_0}, \|T\|_{A_1, B_1})$ is actually true for any interpolation method $\mathcal{F}$ provided that the couples of spaces are Banach couples (see [5, Theorem 2.4.2]). However, it may be improved for certain interpolation methods. Indeed, for the real method it is well-known that if $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ then

$$\|T\|_{\bar{A}_{\theta,q}, \bar{B}_{\theta,q}} \le \|T\|^{1-\theta}_{A_0, B_0} \|T\|^{\theta}_{A_1, B_1} \tag{3.3}$$

(see, for example, [5, Theorem 3.1.2]). For limiting real methods, estimate (3.3) is no longer true. In the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, it is shown in [19, Theorem 7.9] that if $1 \le q < \infty$ then

$$\|T\|_{\bar{A}_{1,q;K}, \bar{B}_{1,q;K}} \le c\|T\|_{A_1, B_1}\left(1 + \max\left\{0, \log\frac{\|T\|_{A_0, B_0}}{\|T\|_{A_1, B_1}}\right\}\right), \tag{3.4}$$

where M does not depend on T, $\bar{A}$ or $\bar{B}$. However, for general couples, even this weaker estimate fails, as the following example shows.

Counterexample 3.1. Let $1 \le q < \infty$. Consider the couples $(\ell_q(e^{-n}), \ell_q)$ and $(\mathbb{K}, \mathbb{K})$, where sequences have indices on $\mathbb{N}$. For $k \in \mathbb{N}$, let T_k be the linear operator defined by $T_k\xi = e^{-k}\xi_k$. Clearly $T_k \in \mathcal{L}((\ell_q(e^{-n}), \ell_q), (\mathbb{K}, \mathbb{K}))$ with $\|T\|_{\ell_q(e^{-n}), \mathbb{K}} = 1$ and $\|T\|_{\ell_q, \mathbb{K}} = e^{-k}$. According to Lemma 3.4 and [19, Lemma 7.2 and Remark 7.3], we have that

$$(\ell_q(e^{-n}), \ell_q)_{q;K} = (\ell_q, \ell_q(e^{-n}))_{1,q;K} = \ell_q(n^{1/q}e^{-n}) .$$

Moreover, $(\mathbb{K}, \mathbb{K})_{q;K} = \mathbb{K}$ with equivalence of norms. Hence,

$$\|T_k\|_{(\ell_q(e^{-n}), \ell_q)_{q;K}, (\mathbb{K}, \mathbb{K})_{q;K}} \sim k^{-1/q} .$$

Since there is no $c > 0$ such that $k^{-1/q} \le cke^{-k}$ for all $k \in \mathbb{N}$, it follows that (3.4) does not hold in general outside of the ordered case.

3.2 Limiting J-spaces

Now we turn our attention to J-spaces.

Definition 3.2. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is formed by all those $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1) \tag{3.5}$$

and

$$\left(\int_0^1 \left[t^{-1} J(t, u(t))\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{3.6}$$

The norm $\|a\|_{\bar{A}_{q;J}}$ in $\bar{A}_{q;J}$ is the infimum in (3.6) over all representations that satisfy (3.5) and (3.6).

These spaces were introduced in [26] under the notation $\bar{A}_{\{1,0\},q;J}$. Next we show that they are intermediate spaces with respect to the couple $\bar{A}$ and that they are smaller than any space $\bar{A}_{\theta,r}$.

Lemma 3.6. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \theta < 1$ and $1 \leq q, r \leq \infty$. Then we have $A_0 \cap A_1 \hookrightarrow \bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1$. Moreover, $\bar{A}_{1;J} = A_0 \cap A_1$ with equivalent norms.*

Proof. Let $a \in A_0 \cap A_1$. Take $u(t) = a\chi_{(1,e)}(t)$, where $\chi_I(t)$ is the characteristic function on the interval I. Then $a = \int_0^\infty u(t) \frac{dt}{t}$ and we obtain

$$\|a\|_{\bar{A}_{q;J}} \leq \left(\int_1^e J(t, a)^q \frac{dt}{t}\right)^{1/q} \leq c \|a\|_{A_0 \cap A_1}.$$

Suppose now that $a \in \bar{A}_{q;J}$ and let $a = \int_0^\infty u(t) \frac{dt}{t}$ be a representation of a satisfying (3.6). Then, it is also a representation of a in $\bar{A}_{\theta,1}$ because, using Hölder's inequality, we have

$$\int_0^1 t^{-\theta} J(t, u(t)) \frac{dt}{t} \leq \left(\int_0^1 \left[t^{-1} J(t, u(t))\right]^q \frac{dt}{t}\right)^{1/q} \left(\int_0^1 t^{(1-\theta)q'} \frac{dt}{t}\right)^{1/q'}$$

and

$$\int_1^\infty t^{-\theta} J(t, u(t)) \frac{dt}{t} \leq \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t}\right)^{1/q} \left(\int_1^\infty t^{-\theta q'} \frac{dt}{t}\right)^{1/q'}.$$

Therefore, $\bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,1}$. Since $\bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1$ (see [5] or [80]), it follows that

$$A_0 \cap A_1 \hookrightarrow \bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1.$$

Finally, let $q = 1$ and $a \in \bar{A}_{1;J}$. Take any representation $a = \int_0^\infty u(t) \frac{dt}{t}$ of a in $\bar{A}_{1;J}$. Then the integral is absolutely convergent in $A_0 \cap A_1$ because, since $J(t, v)$ is a non-decreasing function of t and $t^{-1}J(t, v)$ is non-increasing, we get

$$\begin{aligned}\int_0^\infty \|u(t)\|_{A_0 \cap A_1} \frac{dt}{t} &= \int_0^1 J(1, u(t)) \frac{dt}{t} + \int_1^\infty J(1, u(t)) \frac{dt}{t} \\ &\leq \int_0^1 t^{-1} J(t, u(t)) \frac{dt}{t} + \int_1^\infty J(t, u(t)) \frac{dt}{t}.\end{aligned}$$

Consequently, $a \in A_0 \cap A_1$ and $\|a\|_{A_0 \cap A_1} \leq \|a\|_{\bar{A}_{1;J}}$. The proof is completed. □

It is shown in [26, Theorem 4.1] that

$$\bar{A}_{q;J} = (A_0, A_1, A_1, A_0)_{(1/2,1/2),q;J}\,, \tag{3.7}$$

where $(\cdot,\cdot,\cdot,\cdot)_{(\alpha,\beta),q;J}$ is the J-method defined by the unit square (see the definition in equation (2.19) and in [33]). In particular if $A_0 \hookrightarrow A_1$ and $\bar{A}_{0,q;J}$ is the space introduced in (2.17), we have that $\bar{A}_{q;J} = \bar{A}_{0,q;J}$.

For $1 < q \le \infty$, the spaces $\bar{A}_{q;J}$ can be also described using the K-functional. Recall that in the previous chapter we mentioned the equivalence theorem for the classical real method; in that case the integral expressions in the definitions by the K- and J- methods were very similar, the only difference being that one replaces the K- by the J- functional and a by $u(t)$ (see (2.1) and (2.4)). This time we will need a correction factor in order to have equivalence. In fact, according to [26, Theorem 3.10], we have with equivalence of norms

$$\bar{A}_{q;J} = \bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}, \tag{3.8}$$

where $\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}$ is formed by all those $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}} = \left(\int_0^1 \left[\frac{K(t,a)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Note that if we now suppose that $A_0 \hookrightarrow A_1$ then, by (2.8) and the facts that the K-functional is non-decreasing and that $q > 1$, it follows that

$$\begin{aligned}\|a\|_{\bar{A}_{q;J}} &\sim \|a\|_{A_1}\left(\int_0^1 (1-\log t)^{-q}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}\\ &\sim K(1,a)\left(\int_1^\infty (1+\log t)^{-q}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since we also have that

$$\left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \|a\|_{\bar{A}_{q;J}}\,,$$

we derive that, if $A_0 \hookrightarrow A_1$, then

$$\|a\|_{\bar{A}_{q;J}} \sim \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q},$$

recovering the equivalence formula given in [19, Theorem 4.2].

Equality (3.8) is not true if $q = 1$. Indeed, let $\{0\} \neq A_0 \hookrightarrow A_1$, being the embedding of norm less than or equal to 1. Then $K(t,a) = t\|a\|_{A_1}$ if $0 < t \le 1$ (see the argument that leads to (2.8)). By Lemma 3.6, $\bar{A}_{1;J} = A_0$. However, $\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},1;K} = \{0\}$ because for any $a \neq 0$ we obtain

$$\int_0^1 \frac{K(t,a)}{t(1-\log t)}\frac{dt}{t} = \|a\|_{A_1}\int_0^1 \frac{1}{1-\log t}\frac{dt}{t} = \infty.$$

In fact, the $(1;J)$-method cannot be described using the K-functional. Recall that for any Banach couple (A_0, A_1), one has $K(t, a; A_0, A_1) = K\left(t, a; \widetilde{A_0}, \widetilde{A_1}\right)$, where $\widetilde{A_j}$ is the Gagliardo completion of A_j in $A_0 + A_1$ (see [4, Theorem 5.1.5]). Whence, if the $(1;J)$-method could be described using the K-functional, we would have for any Banach couple

$$A_0 \cap A_1 = (A_0, A_1)_{1;J} = \left(\widetilde{A_0}, \widetilde{A_1}\right)_{1;J} = \widetilde{A_0} \cap \widetilde{A_1}.$$

However, if we take $A_0 = c_0$ and $A_1 = \ell_\infty(2^{-n})$, then $\widetilde{A_0} = \ell_\infty$ and $A_0 \cap A_1 = c_0 \neq \ell_\infty = \widetilde{A_0} \cap \widetilde{A_1}$.

The following result is based on the K-description of $\bar{A}_{q;J}$.

Lemma 3.7. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and take any $1 \leq q \leq \infty$. Then we have that $(A_0, A_1)_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{0,q;J}$ with equivalence of norms. In particular, the J-method is symmetric, that is, $(A_0, A_1)_{q;J} = (A_1, A_0)_{q;J}$.*

Proof. If $q = 1$, we have $(A_0, A_1)_{1;J} = A_0 \cap A_1 = (A_0 \cap A_1, A_0 + A_1)_{0,1;J}$. Otherwise, if $q \neq 1$, set $\bar{K}(t, a) = K(t, a; A_0 \cap A_1, A_0 + A_1)$. By [19, Theorem 4.2], we obtain

$$\|a\|_{(A_0 \cap A_1, A_0 + A_1)_{0,q;J}} \sim \left(\int_1^\infty \left[\frac{\bar{K}(t,a)}{1 + \log t}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Using [68, Theorem 3], a change of variable and (3.8), we derive that

$$\left(\int_1^\infty \left[\frac{\bar{K}(t,a)}{1 + \log t}\right]^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^1 \left[\frac{K(s,a)}{s(1 - \log s)}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1 + \log t}\right]^q \frac{dt}{t}\right)^{1/q} \sim \|a\|_{(A_0, A_1)_{q;J}}.$$

Consequently, $(A_0, A_1)_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{0,q;J}$. This equality gives the symmetry relationship $(A_0, A_1)_{q;J} = (A_1, A_0)_{q;J}$. □

The following lemma shows an equivalent discrete definition for J-spaces.

Lemma 3.8. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. Then the space $\bar{A}_{q;J}$ can be also defined as the collection of all those $a \in A_0 + A_1$ for which there exists a sequence $(u_m)_{m \in \mathbb{Z}} \subset A_0 \cap A_1$ such that*

$$a = \sum_{m=-\infty}^{\infty} u_m \qquad \textit{(convergence in } A_0 + A_1\textit{)} \tag{3.9}$$

and

$$\left(\sum_{m=-\infty}^{\infty} \left[\max(1, 2^{-m}) J(2^m, u_m)\right]^q\right)^{1/q} < \infty. \tag{3.10}$$

Moreover, the norm $\|a\|_{q;J}$ given by the infimum in (3.10) *over all representations that satisfy* (3.9) *and* (3.10) *is equivalent to $\|a\|_{\bar{A}_{q;J}}$.*

Proof. Let $a \in A_0 + A_1$ and take any sequence $(u_m)_{m\in\mathbb{Z}} \subset A_0 \cap A_1$ that satisfies (3.9) and (3.10) and such that

$$\left(\sum_{m=-\infty}^{\infty} \left[\max\left(1,2^{-m}\right) J\left(2^m,u_m\right)\right]^q\right)^{1/q} \le 2\left\|a\right\|_{q;J}.$$

Put $u(t) = \sum_{m=-\infty}^{\infty} \frac{u_m}{\log 2}\chi_{[2^m,2^{m+1})}(t)$. Then clearly $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ and

$$\int_0^\infty u(t)\frac{dt}{t} = \sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}} \frac{u_m}{\log 2}\frac{dt}{t} = \sum_{m=-\infty}^{\infty} u_m \quad \text{(convergence in } A_0 + A_1\text{)}.$$

Moreover, since $J(t,\cdot)$ is non-decreasing and $t^{-1}J(t,\cdot)$ is non-increasing,

$$\begin{aligned}
\left(\int_0^1 \left[t^{-1}J\left(t,u(t)\right)\right]^q \frac{dt}{t}\right)^{1/q} &+ \left(\int_1^\infty J\left(t,u(t)\right)^q \frac{dt}{t}\right)^{1/q}\\
&\sim \left(\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\left[\frac{\max(1,t^{-1})}{\log 2}J(t,u_m)\right]^q\frac{dt}{t}\right)^{1/q}\\
&\lesssim \left(\sum_{m=-\infty}^{\infty}\int_{2^m}^{2^{m+1}}\left[\frac{\max(1,2^{-m})}{\log 2}J(2^m,u_m)\right]^q\frac{dt}{t}\right)^{1/q}\\
&= \left(\sum_{m=-\infty}^{\infty}\left[\max\left(1,2^{-m}\right) J\left(2^m,u_m\right)\right]^q\right)^{1/q} \lesssim \left\|a\right\|_{q;J}.
\end{aligned}$$

Conversely, let $a \in \bar{A}_{q;J}$ and let $u(t)$ be a strongly measurable function with values in $A_0 \cap A_1$ and such that $a = \int_0^\infty u(t)\frac{dt}{t}$ and

$$\left(\int_0^1 \left[t^{-1}J\left(t,u(t)\right)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty J\left(t,u(t)\right)^q \frac{dt}{t}\right)^{1/q} \le 2\left\|a\right\|_{\bar{A}_{q;J}}.$$

Since

$$\begin{aligned}
\max\left(\left\|\int_{2^m}^{2^{m+1}} u(s)\frac{ds}{s}\right\|_{A_0}, t\left\|\int_{2^m}^{2^{m+1}} u(s)\frac{ds}{s}\right\|_{A_1}\right) &\lesssim \int_{2^m}^{2^{m+1}} \max\left(\|u(s)\|_{A_0}, t\,\|u(s)\|_{A_1}\right)\frac{ds}{s}\\
&= \int_{2^m}^{2^{m+1}} J(t,u(s))\frac{ds}{s} \le \int_{2^m}^{2^{m+1}} \max\left(1,\frac{t}{s}\right)J(s,u(s))\frac{ds}{s} < \infty,
\end{aligned} \tag{3.11}$$

we have that $u_m = \int_{2^m}^{2^{m+1}} u(t)\frac{dt}{t}$ belongs to $A_0 \cap A_1$ with

$$J(t,u_m) \le \int_{2^m}^{2^{m+1}} \max\left(1,\frac{t}{s}\right)J(s,u(s))\frac{ds}{s}.$$

So, $J\left(2^m,u_m\right) \le \int_{2^m}^{2^{m+1}} J(t,u(t))\frac{dt}{t}$. It is clear that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$). Therefore, by Hölder's inequality,

$$\begin{aligned}
\left(\sum_{m=-\infty}^{\infty} \left[\max\left(1,2^{-m}\right) J\left(2^m,u_m\right)\right]^q\right)^{1/q} &\le \left(\sum_{m=-\infty}^{\infty}\left[\max\left(1,2^{-m}\right)\int_{2^m}^{2^{m+1}} J(t,u(t))\frac{dt}{t}\right]^q\right)^{1/q}\\
&\lesssim \left(\int_0^\infty \left[\max(1,t^{-1})J\left(t,u(t)\right)\right]^q\frac{dt}{t}\right)^{1/q} \lesssim \left\|a\right\|_{\bar{A}_{q;J}}.
\end{aligned}$$

This ends the proof. □

Just as with the K-method, Lemma 3.8 shows that J-spaces are also ordered in the following way: $\bar{A}_{p;J} \hookrightarrow \bar{A}_{q;J}$ if $p \leq q$.

With the help of this lemma we can show that spaces $\bar{A}_{q;J}$ are complete.

Proposition 3.9. *If A_0 and A_1 are complete, then so is $\bar{A}_{q;J}$ for any $1 \leq q \leq \infty$.*

Proof. For $m \in \mathbb{Z}$ let G_m be the space $A_0 \cap A_1$ normed by $\max(1, 2^{-m})J(2^m, \cdot; A_0, A_1)$. Clearly G_m is complete and therefore $\ell_q(G_m)$ is also complete for $1 \leq q \leq \infty$. Put $W = \ell_q(G_m)$ and

$$V = \left\{ (w_m) \in W : \sum_{m=-\infty}^{\infty} w_m = 0 \text{ (convergence in } A_0 + A_1) \right\}.$$

We will show that V is closed in W. Let $(v^n) \subset V$ be a Cauchy sequence, $v^n = (w_m^n)_{m=-\infty}^{\infty}$. Since W is complete, there exists a certain $v = (w_m)_{m=-\infty}^{\infty} \in W$ such that $(v^n) \to v$ in W. In order to prove that $v \in V$, all we need to do is to show that

$$\left\| \sum_{|m| \geq M} w_m \right\|_{A_0 + A_1} \xrightarrow[M \to \infty]{} 0.$$

Let $\varepsilon > 0$. Since $(v^n) \to v$ in W, there exists $n \in \mathbb{N}$ such that $\|v^n - v\|_W < \varepsilon/6$. On the other hand, for such an n, there exists a certain $M_1 \in \mathbb{N}$ such that for all $M \geq M_1$ we have $\left\| \sum_{|m| \geq M} w_m^n \right\|_{A_0 + A_1} < \varepsilon/2$. Therefore, for each $M \geq M_1$ and the chosen n we have that

$$\begin{aligned} \left\| \sum_{|m| \geq M} w_m \right\|_{A_0 + A_1} &\leq \left\| \sum_{|m| \geq M} w_m - w_m^n \right\|_{A_0 + A_1} + \left\| \sum_{|m| \geq M} w_m^n \right\|_{A_0 + A_1} \\ &< K(1, \sum_{|m| \geq M} w_m - w_m^n) + \varepsilon/2 \leq \sum_{|m| \geq M} K(1, w_m - w_m^n) + \varepsilon/2. \end{aligned}$$

According to [5, Lemma 3.2.1 (2)], $K(t, a) \leq \min(1, t/s)J(s, a)$ for any vector $a \in A_0 \cap A_1$. Therefore,

$$\begin{aligned} K(1, w_m - w_m^n) &\leq \min(1, 2^{-m})J(2^m, w_m - w_m^n) = \frac{\min(1, 2^{-m})}{\max(1, 2^{-m})} \|w_m - w_m^n\|_{G_m} \\ &= \min(2^m, 2^{-m}) \|w_m - w_m^n\|_{G_m}, \end{aligned}$$

so, by Hölder's inequality,

$$\begin{aligned} \left\| \sum_{|m| \geq M} w_m \right\|_{A_0 + A_1} &\leq \sum_{m=-\infty}^{\infty} \min(2^m, 2^{-m}) \|w_m - w_m^n\|_{G_m} + \varepsilon/2 \\ &\leq \left(\sum_{m=-\infty}^{\infty} \|w_m - w_m^n\|_{G_m}^q \right)^{1/q} \left(\sum_{m=-\infty}^{\infty} \min(2^m, 2^{-m})^{q'} \right)^{1/q'} + \varepsilon/2 \\ &\leq 3 \|v - v^n\|_W + \varepsilon/2 < \varepsilon. \end{aligned}$$

This gives that $v \in V$, that is, V is closed in W.

Finally, since $\bar{A}_{q;J} = W/V$, W is complete and V is closed in W, we derive that $\bar{A}_{q;J}$ is complete, as desired. □

It is easy to check that if $T \in \mathcal{L}(\bar{A}, \bar{B})$, then the interpolated operator $T : \bar{A}_{q;J} \longrightarrow \bar{B}_{q;J}$ is also bounded with $\|T\|_{\bar{A}_{q;J}, \bar{B}_{q;J}} \leq \max\left\{ \|T\|_{A_0, B_0}, \|T\|_{A_1, B_1} \right\}$.

3.3 Compact operators

Interpolation of compact operators is a classical question that has attracted the attention of many authors (see [12] and the references given there). In 1992, those efforts culminated in Cwikel [40] and Cobos, Kühn and Schonbek [30] proving that if $T \in \mathcal{L}(\bar{A}, \bar{B})$ and any one of the restrictions $T : A_j \longrightarrow B_j$ $(j = 0, 1)$ is compact, then the interpolated operator by the classical real method $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ is also compact.

In this section we study how compact operators behave under limiting K- and J-methods for arbitrary couples. First we consider the K-methods defined in Section 3.1.

For limiting methods in the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, it was proved by Cobos, Fernández-Cabrera, Kühn and Ullrich [19] that the compactness of $T : A_1 \longrightarrow B_1$ implies that $T : \bar{A}_{1,q;K} \longrightarrow \bar{B}_{1,q;K}$ is also compact, whereas the compactness of $T : A_0 \longrightarrow B_0$ is not enough (see [19, Counterexample 7.11 and Theorem 7.14].

In the general case, we have already pointed out the bad behaviour of the $(q; K)$-method concerning estimates for the norm of the interpolated operators (see Counterexample 3.1). This suggests poorer properties with respect to interpolation of compact operators. Next, we show with an example based on [19, Counterexample 7.11] that in contrast to the ordered case, if $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_1 \longrightarrow B_1$ is compact, it might happen that $T : \bar{A}_{q;K} \longrightarrow \bar{B}_{q;K}$ fails to be compact.

Counterexample 3.2. Let $1 \le q < \infty$ and consider the Banach couples $\bar{A} = (\ell_q(3^{-n}), \ell_q)$ and $\bar{B} = (\ell_q(2^{-n}), \ell_q)$. Let D be the diagonal operator defined by $D(\xi_n) = ((2/3)^n \xi_n)$. Then clearly $D : \ell_q(3^{-n}) \longrightarrow \ell_q(2^{-n})$ is bounded and $D : \ell_q \longrightarrow \ell_q$ is compact, because it is a diagonal operator whose associated sequence has null limit. However, according to Lemma 3.4 and [19, Lemma 7.2 and Remark 7.3], we have that

$$(\ell_q(3^{-n}), \ell_q)_{q;K} = \ell_q\left(n^{1/q}3^{-n}\right) \quad \text{and} \quad (\ell_q(2^{-n}), \ell_q)_{q;K} = \ell_q\left(n^{1/q}2^{-n}\right),$$

and $D : \ell_q\left(n^{1/q}3^{-n}\right) \longrightarrow \ell_q\left(n^{1/q}2^{-n}\right)$ fails to be compact. Indeed, for each $n \in \mathbb{N}$ consider the vector $u_n = (n^{-1/q}3^n e_n)$, where e_n has all of its coordinates equal to 0 except for the n^{th} one, which is equal to 1. Then (u_n) is a bounded sequence in $\ell_q(3^{-n}n^{1/q})$. Since $Du_n = 2^n n^{-1/q} e_n$, $\|Du_n - Du_m\|_{\ell_q(n^{1/q}2^{-n})} = \|e_n + e_m\|_{\ell_q} = 2^{1/q}$ if $m \ne n$. This implies that (Du_n) cannot have a convergent subsequence in $\ell_q\left(n^{1/q}2^{-n}\right)$.

Nevertheless, if the first couple reduces to a single Banach space, then the behaviour of the $(q; K)$-method improves.

Proposition 3.10. *Let* A *be a Banach space, let* $\bar{B} = (B_0, B_1)$ *be a Banach couple and let* $1 \le q \le \infty$. *If* T *is a linear operator such that* $T : A \longrightarrow B_j$ *is bounded for* $j = 0, 1$ *and one of these restrictions is compact, then* $T : A \longrightarrow \bar{B}_{q;K}$ *is also compact.*

Proof. Clearly, $T : A \longrightarrow B_0 + B_1$ compactly and $T : A \longrightarrow B_0 \cap B_1$ boundedly. If $1 \le q < \infty$, we derive that $T : A \longrightarrow (B_0 \cap B_1, B_0 + B_1)_{1,q;K} = (B_0, B_1)_{q;K}$ is compact by Lemma 3.4 and [19, Theorem 7.14]. If $q = \infty$, the result follows from the last part of Lemma 3.1. □

This result has been recently improved by Fernández-Cabrera and Martínez [50, Corollary 3.10] who showed that the compactness of $T : A \to B_0 + B_1$ is sufficient to guarantee the compactness of $T : A \to \bar{B}_{q;K}$.

In order to establish the compactness result in the general case, given any Banach couple (A_0, A_1), we write $\left(A_0^\circ, A_1^\circ\right)$ for the Banach couple formed by the closures of $A_0 \cap A_1$ in A_j for $j = 0, 1$.

Theorem 3.11. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, let $1 \le q \le \infty$ and let $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$. If $T : A_j \longrightarrow B_j$ is compact for $j = 0, 1$, then $T : \left(A_0^\circ, A_1^\circ\right)_{q;K} \longrightarrow \left(B_0^\circ, B_1^\circ\right)_{q;K}$ is compact as well.*

Proof. According to [49, Corollary 4.4], if $T : A_j \longrightarrow B_j$ compactly for $j = 0, 1$ then

$$T : (A_0^\circ, A_1^\circ, A_1^\circ, A_0^\circ)_{(1/2,1/2),q;K} \longrightarrow (B_0^\circ, B_1^\circ, B_1^\circ, B_0^\circ)_{(1/2,1/2),q;K}$$

is also compact. The result follows from (3.2). □

Remark 3.2. We will show at the end of Section 3.5 that $\left(A_0^\circ, A_1^\circ\right)_{q;K} = (A_0, A_1)_{q;K}$ whenever $q < \infty$.

Next we turn our attention to the $(q; J)$-method. In the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, if $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ and $T : A_0 \longrightarrow B_0$ compactly, then $T : \bar{A}_{0,q;J} \longrightarrow \bar{B}_{0,q;J}$ is compact (see [19, Theorem 6.4]). However, in the general case, compactness of $T : A_0 \longrightarrow B_0$ is not enough to imply that $T : \bar{A}_{q;J} \longrightarrow \bar{B}_{q;J}$ is compact. An example can be given by reversing the order of the couples in [19, Counterexample 6.2] and using Lemma 3.7.

Counterexample 3.3. Let $1 \le q \le \infty$ and consider the couples of sequence spaces $\bar{A} = (\ell_q(2^{-n}), \ell_q)$ and $\bar{B} = (\ell_q(3^{-n}), \ell_q)$. Let I be the identity operator. Then $I : \ell_q(2^{-n}) \longrightarrow \ell_q(3^{-n})$ is compact because it is the limit of the sequence of finite rank operators given by

$$P_m(\xi_n) = (\xi_1, \xi_2, \dots, \xi_m, 0, 0, \dots).$$

Moreover, $I : \ell_q \longrightarrow \ell_q$ is bounded. However, by Lemma 3.7 and [19, Corollary 3.6], we have that

$$\begin{aligned}(\ell_q(2^{-n}), \ell_q)_{q;J} &= (\ell_q, \ell_q(2^{-n}))_{0,q;J} = \ell_q(n^{-1/q'}) \text{ and}\\ (\ell_q(3^{-n}), \ell_q)_{q;J} &= (\ell_q, \ell_q(3^{-n}))_{0,q;J} = \ell_q(n^{-1/q'}),\end{aligned}$$

where $1/q + 1/q' = 1$. And it is clear that $I : \ell_q(n^{-1/q'}) \longrightarrow \ell_q(n^{-1/q'})$ is not compact.

The following result shows sufficient conditions for interpolation of compact operators for the $(q; J)$-method.

Proposition 3.12. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let B be a Banach space and let $1 \le q \le \infty$. If T is a linear operator such that $T : A_j \longrightarrow B$ is bounded for $j = 0, 1$ and any of these two restrictions is compact, then $T : \bar{A}_{q;J} \longrightarrow B$ is also compact.*

Proof. It is clear that $T : A_0 \cap A_1 \longrightarrow B$ is compact. If $q = 1$, the result follows using that $\bar{A}_{1;J} = A_0 \cap A_1$. Assume now that $1 < q \leq \infty$. We have that $T : A_0 + A_1 \longrightarrow B$ is bounded because

$$\begin{aligned}\|T(a_0 + a_1)\|_B &\leq \|Ta_0\|_B + \|Ta_1\|_B \\ &\leq \max\left\{\|T\|_{A_0}, \|T\|_{A_1}\right\}\left(\|a_0\|_{A_0} + \|a_1\|_{A_1}\right).\end{aligned}$$

Hence, applying [19, Theorem 6.4] to the couples $(A_0 \cap A_1, A_0 + A_1), (B, B)$ and using Lemma 3.7, we conclude that

$$T : \bar{A}_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{0,q;J} \longrightarrow B$$

is also compact. □

Fernández-Cabrera and Martínez also improved this result in [50, Corollary 3.9]: They showed that the compactness of $T : A_0 \cap A_1 \to B$ is sufficient to have that $T : \bar{A}_{q;J} \to B$ is compact.

We finish this section with a consequence of (3.7) and [33, Theorem 6.1].

Theorem 3.13. *Let* $\bar{A} = (A_0, A_1)$ *and* $\bar{B} = (B_0, B_1)$ *be Banach couples, let* $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ *and* $1 \leq q \leq \infty$. *If* $T : A_j \longrightarrow B_j$ *is compact for* $j = 0, 1$, *then* $T : \bar{A}_{q;J} \longrightarrow \bar{B}_{q;J}$ *is also compact.*

3.4 Description of K-spaces using the J-functional

In (3.8) we have pointed out that limiting J-spaces can be described by using the K-functional provided that $1 < q \leq \infty$. In this section we study the description of limiting K-spaces using the J-functional.

Recall that in the ordered case where $A_0 \hookrightarrow A_1$ it was shown in [19, Theorem 7.6] that if $1 \leq q < \infty$ the limiting K-space $\bar{A}_{1,q;K}$ can be also realised as the collection of vectors $a \in A_1$ for which there is a representation

$$a = \int_1^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } A_1), \tag{3.12}$$

where $u(t)$ is a strongly measurable function with values in A_0 and such that

$$\left(\int_1^\infty \left[t^{-1}(1 + \log t)J(t, u(t))\right]^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{3.13}$$

The norm is defined as the infimum over all possible representations u of a satisfying (3.12) and (3.13) of the values (3.13).

Definition 3.3. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. Write $\rho(t) = 1 + |\log t|$ and $\mu(t) = t^{-1}(1 + |\log t|)$. The space $\bar{A}_{\{\rho,\mu\},q;J}$ is formed by all those elements $a \in A_0 + A_1$ for which there is a representation

$$a = \int_0^\infty u(t)\frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1) \tag{3.14}$$

where $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ and such that

$$\left(\int_0^1 [\rho(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [\mu(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{3.15}$$

The norm in $\bar{A}_{\{\rho,\mu\},q;J}$ is given by taking the infimum of the values (3.15) over all possible representations u of a satisfying (3.14) and (3.15).

Note that if in this definition we suppose that $A_0 \hookrightarrow A_1$, then we recover the equivalent definition for $\bar{A}_{1,q;K}$ given in [19]. In order to establish this, one only has to slightly modify the argument in the proof of Proposition 2.4, (ii).

The following result shows the relationship between the spaces introduced in Definition 3.3 and limiting K-spaces.

Theorem 3.14. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and $1 \le q < \infty$. Then we have with equivalence of norms $(A_0,A_1)_{q;K} = (A_0,A_1)_{\{\rho,\mu\},q;J}$.*

Proof. Let $a \in (A_0,A_1)_{\{\rho,\mu\},q;J}$ and choose a representation $a = \int_0^\infty u(s)\frac{ds}{s}$ of a such that

$$\left(\int_0^1 [\rho(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty [\mu(t)J(t,u(t))]^q \frac{dt}{t}\right)^{1/q} \le 2\,\|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.$$

For any $0 < t < \infty$, we have that

$$\begin{aligned} K(t,a) &\le \int_0^\infty K(t,u(s))\frac{ds}{s} \le \int_0^\infty \min(1,t/s)\,J(s,u(s))\frac{ds}{s} \\ &= \int_0^t J(s,u(s))\frac{ds}{s} + \int_t^\infty \frac{t}{s}J(s,u(s))\frac{ds}{s}. \end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned} \|a\|_{\bar{A}_{q;K}} &= \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{-1}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_0^1 \left[\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[\int_t^\infty \frac{t}{s}J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_1^\infty \left[\frac{1}{t}\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\int_t^\infty \frac{1}{s}J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We shall estimate each of these terms separately. Let $h \in L_{q'}((0,1),dt/t)$ with $\|h\|_{L_{q'}} = 1$ and such that

$$I_1 = \left(\int_0^1 \left[\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} = \int_0^1 h(t)\int_0^t J(s,u(s))\frac{ds}{s}\frac{dt}{t}.$$

Using Fubini's theorem, Hölder's inequality, changing variables and applying Hardy's inequality (see [65]), we obtain

$$I_1 = \int_0^1\int_s^1 h(t)J(s,u(s))\frac{dt}{t}\frac{ds}{s} = \int_0^1 J(s,u(s))\,\rho(s)\frac{1}{\rho(s)}\int_s^1 h(t)\frac{dt}{t}\frac{ds}{s}$$

$$
\begin{aligned}
&\leq \left(\int_0^1 [\rho(s)J(s,u(s))]^q \frac{ds}{s}\right)^{1/q} \left(\int_0^1 \left[\frac{1}{1-\log s}\int_s^1 h(t)\frac{dt}{t}\right]^{q'} \frac{ds}{s}\right)^{1/q'}\\
&\lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \left(\int_0^\infty \left[\frac{1}{1+x}\int_{e^{-x}}^1 h(t)\frac{dt}{t}\right]^{q'} dx\right)^{1/q'}\\
&\leq \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \left(\int_0^\infty \left[\frac{1}{x}\int_0^x h(e^{-s})ds\right]^{q'} dx\right)^{1/q'}\\
&\lesssim \left(\int_0^\infty h(e^{-s})^{q'} ds\right)^{1/q'} \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} = \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.
\end{aligned}
$$

As for I_2, by Hölder's inequality and the fact that $s(1-\log s)$ is increasing in $(0,1)$, we get

$$
\begin{aligned}
I_2 &\leq \left(\int_0^1 \left[t\int_t^1 \frac{1-\log s}{s(1-\log s)} J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[t\int_1^\infty \frac{1}{s}J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&\leq \left(\int_0^1 \left[\frac{1}{(1-\log t)}\int_t^1 (1-\log s)J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&\quad + \left(\int_0^1 t^q \left(\int_1^\infty \left[\frac{1+\log s}{s}J(s,u(s))\right]^q \frac{ds}{s}\right)\left[\int_1^\infty (1+\log s)^{-q'}\frac{ds}{s}\right]^{q/q'} \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

The last integral is finite because $q' = (1-1/q)^{-1}$ is bigger than 1. Changing variables and using Hardy's inequality, we derive

$$
\begin{aligned}
I_2 &\lesssim \left(\int_0^\infty \left[\frac{1}{1+v}\int_0^v (1+x)J(e^{-x},u(e^{-x}))\,dx\right]^q dv\right)^{1/q} + \left(\int_0^1 t^q \int_1^\infty [\mu(s)J(s,u(s))]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q}\\
&\lesssim \left(\int_0^\infty [(1+x)J(e^{-x},u(e^{-x}))]^q\,dx\right)^{1/q} + \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.
\end{aligned}
$$

As for I_3, using Hölder's inequality and also applying Hardy's inequality to the function $s^{-1}J(s,u(s))\chi_{(1,\infty)}(s)$, we have that

$$
\begin{aligned}
I_3 &\leq \left(\int_1^\infty \left[t^{-1}\int_0^1 J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{-1}\int_1^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&\leq \left(\int_1^\infty t^{-q}\left(\int_0^1 [\rho(s)J(s,u(s))]^q \frac{ds}{s}\right)\left(\int_0^1 \left[\frac{1}{1-\log s}\right]^{q'}\frac{ds}{s}\right)^{q/q'} \frac{dt}{t}\right)^{1/q}\\
&\quad + \left(\int_1^\infty [s^{-1}J(s,u(s))]^q \frac{ds}{s}\right)^{1/q}\\
&\lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} + \left(\int_1^\infty \left[\frac{1+\log s}{s}J(s,u(s))\right]^q (1+\log s)^{-q}\frac{ds}{s}\right)^{1/q}\\
&\leq \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} + \sup_{1\leq s<\infty}(1+\log s)^{-q}\,\|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.
\end{aligned}
$$

In order to estimate the last term I_4, we proceed as in the case of I_1. Namely, choose a function $h \in L_{q'}((1,\infty), dt/t)$ with $\|h\|_{L_{q'}} = 1$ and such that

$$I_4 = \int_1^\infty h(t) \int_t^\infty s^{-1} J(s, u(s)) \frac{ds}{s} \frac{dt}{t}.$$

We obtain

$$\begin{aligned} I_4 &= \int_1^\infty s^{-1} J(s, u(s)) \int_1^s h(t) \frac{dt}{t} \frac{ds}{s} \\ &\le \left(\int_1^\infty [\mu(s) J(s, u(s))]^q \frac{ds}{s} \right)^{1/q} \left(\int_1^\infty \left[\frac{1}{1+\log s} \int_1^s h(t) \frac{dt}{t} \right]^{q'} \frac{ds}{s} \right)^{1/q'} \\ &\le \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \left(\int_0^\infty \left(\frac{1}{1+x} \int_0^x h(e^y) dy \right)^{q'} dx \right)^{1/q'} \\ &\lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \left(\int_0^\infty h(e^x)^{q'} dx \right)^{1/q'} = \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}. \end{aligned}$$

Consequently, $(A_0, A_1)_{\{\rho,\mu\},q;J} \hookrightarrow (A_0, A_1)_{q;K}$.

Conversely, take any $a \in (A_0, A_1)_{q;K}$. Then

$$\int_0^1 K(t,a)^q \frac{dt}{t} + \int_1^\infty \left[t^{-1} K(t,a) \right]^q \frac{dt}{t} < \infty. \tag{3.17}$$

Since $K(t,a)$ (respectively, $t^{-1}K(t,a)$) is non-decreasing (respectively, non-increasing) in t, it follows from (3.17) that

$$K(t,a) \longrightarrow 0 \text{ as } t \to 0 \text{ and } \frac{K(t,a)}{t} \longrightarrow 0 \text{ as } t \to \infty. \tag{3.18}$$

For $\nu \in \mathbb{Z}$, put

$$\eta_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

We can find decompositions $a = a_{0,\nu} + a_{1,\nu}$, with $a_{j,\nu} \in A_j$, $j = 0,1$ such that

$$\begin{aligned} \|a_{0,\nu}\|_{A_0} + \eta_{\nu+1} \|a_{1,\nu}\|_{A_1} &\le 2K(\eta_{\nu+1}, a) && \text{if } \nu \le 1 \text{, and} \\ \eta_{\nu-1}^{-1} \|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} &\le 2\widetilde{K}\left(\eta_{\nu-1}^{-1}, a\right) && \text{if } \nu > 1, \end{aligned}$$

where $\widetilde{K}(t,a) = K(t,a;A_1,A_0)$.

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$, $\nu \in \mathbb{Z}$. Given any $N, M \in \mathbb{N}$, we have

$$\left\| a - \sum_{\nu=-N}^{M} u_\nu \right\|_{A_0+A_1} = \|a - a_{0,M} + a_{0,-N-1}\|_{A_0+A_1} \le \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1}.$$

By (3.18), the last two terms go to 0 as $N, M \to \infty$. Hence, $a = \sum_{\nu\in\mathbb{Z}} u_\nu$ in $A_0 + A_1$.

Let $L_\nu = (\eta_{\nu-1}, \eta_\nu]$, $\nu \in \mathbb{Z}$. It follows that

$$\int_{L_\nu} \frac{dt}{t} = \begin{cases} 2^{-\nu-1} \log 2 & \text{if } \nu < 0, \\ \log 2 & \text{if } \nu = 0, 1, \\ 2^{\nu-2} \log 2 & \text{if } \nu > 1. \end{cases}$$

Let

$$v(t) = \begin{cases} \dfrac{u_\nu}{2^{-\nu-1} \log 2} & \text{if } t \in L_\nu \text{ and } \nu < 0, \\ \dfrac{u_\nu}{\log 2} & \text{if } t \in L_\nu \text{ and } \nu = 0, 1, \\ \dfrac{u_\nu}{2^{\nu-2} \log 2} & \text{if } t \in L_\nu \text{ and } \nu > 1. \end{cases}$$

Then $a = \int_0^\infty v(t) \frac{dt}{t}$ (convergence in $A_0 + A_1$). Next we show that this is a suitable representation of a in the J-space.

If $\nu < 0$ and $t \in L_\nu$, we have that

$$\begin{aligned} J(t, v(t)) &= \frac{J(t, u_\nu)}{2^{-\nu-1} \log 2} \lesssim 2^{\nu+1} J(\eta_\nu, u_\nu) \\ &\le 2^{\nu+1} \left(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0} + \eta_\nu \|a_{1,\nu-1}\|_{A_1} + \eta_\nu \|a_{1,\nu}\|_{A_1} \right) \lesssim 2^{\nu+1} K(\eta_{\nu+1}, a). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{L_\nu} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} &\lesssim \left[2^{\nu+1} K(\eta_{\nu+1}, a)\right]^q \int_{2^{-2^{-\nu}}}^{2^{-2^{-\nu-1}}} (1 - \log t)^q \frac{dt}{t} \\ &\le \left[2^{\nu+1} K(\eta_{\nu+1}, a)\right]^q (1 + 2^{-\nu} \log 2)^q \, 2^{-\nu-1} \log 2 \lesssim 2^{-\nu-1} K(\eta_{\nu+1}, a)^q. \end{aligned}$$

Now we distinguish three subcases. If $\nu < -2$, we derive

$$\int_{L_\nu} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_{\nu+1}, a)^q \int_{L_{\nu+2}} \frac{dt}{t} \le \int_{L_{\nu+2}} K(t, a)^q \frac{dt}{t}.$$

If $\nu = -2$, we get

$$\int_{L_{-2}} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_{-1}, a)^q \int_{L_0} \frac{dt}{t} \le \int_{L_0} K(t, a)^q \frac{dt}{t}.$$

In the remaining case $\nu = -1$, we obtain

$$\int_{L_{-1}} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_0, a)^q \lesssim \int_{L_1} \left[\frac{K(t, a)}{t}\right]^q \frac{dt}{t}.$$

Suppose now $\nu > 1$. A change of variables yields that

$$\begin{aligned} \int_{L_\nu} \left[\frac{1 + \log t}{t} J(t, v(t))\right]^q \frac{dt}{t} &= \int_{\eta_\nu^{-1}}^{\eta_{\nu-1}^{-1}} [(1 - \log s) s J(1/s, v(1/s))]^q \frac{ds}{s} \\ &= \int_{\eta_\nu^{-1}}^{\eta_{\nu-1}^{-1}} \left[(1 - \log s) \widetilde{J}(s, v(1/s))\right]^q \frac{ds}{s}, \end{aligned}$$

where $\widetilde{J}(s,w) = J(s,w;A_1,A_0)$. If $s \in \left(\eta_\nu^{-1}, \eta_{\nu-1}^{-1}\right]$, then $1/s \in L_\nu$, and we get

$$\begin{aligned}\widetilde{J}(s,v(1/s)) &= \frac{\widetilde{J}(s,u_\nu)}{2^{\nu-2}\log 2} \lesssim \frac{\widetilde{J}\left(\eta_{\nu-1}^{-1},u_\nu\right)}{2^{\nu-2}} \le 2^{2-\nu}\left[\eta_{\nu-1}^{-1}\left(\|a_{0,\nu}\|_{A_0}+\|a_{0,\nu-1}\|_{A_0}\right)+\|a_{1,\nu-1}\|_{A_1}+\|a_{1,\nu}\|_{A_1}\right]\\ &\lesssim 2^{2-\nu}\widetilde{K}\left(\eta_{\nu-2}^{-1},a\right).\end{aligned}$$

This implies that

$$\int_{L_\nu}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t} \lesssim \left[2^{2-\nu}\widetilde{K}\left(\eta_{\nu-2}^{-1},a\right)\right]^q(1+\log\eta_\nu)^q\int_{L_\nu}\frac{dt}{t} \lesssim \left[\frac{K(\eta_{\nu-2},a)}{\eta_{\nu-2}}\right]^q\int_{L_\nu}\frac{dt}{t}.$$

Now, if $\nu > 2$, we derive

$$\begin{aligned}\int_{L_\nu}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t} &\lesssim \left[\frac{K(\eta_{\nu-2},a)}{\eta_{\nu-2}}\right]^q 2^{\nu-2} \lesssim \left[\frac{K(\eta_{\nu-2},a)}{\eta_{\nu-2}}\right]^q\int_{L_{\nu-2}}\frac{dt}{t}\\ &\le \int_{L_{\nu-2}}\left[\frac{K(t,a)}{t}\right]^q\frac{dt}{t}.\end{aligned}$$

If $\nu = 2$, we have

$$\int_{L_2}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t} \lesssim \left[\frac{K(\eta_0,a)}{\eta_0}\right]^q\int_{L_2}\frac{dt}{t} \lesssim K(1/2,a)^q\int_{L_0}\frac{dt}{t} \lesssim \int_{L_0}K(t,a)^q\frac{dt}{t}.$$

Finally, we focus on the two remaining cases: $\nu = 0,1$. If $\nu = 0$ and $t \in L_0$, then

$$J(t,v(t)) = \frac{J(t,u_0)}{\log 2} \lesssim \|a_{0,0}\|_{A_0}+\|a_{0,-1}\|_{A_0}+\|a_{1,-1}\|_{A_1}+\|a_{1,0}\|_{A_1} \lesssim K(2,a).$$

Hence,

$$\int_{L_0}[(1-\log t)J(t,v(t))]^q\frac{dt}{t} \lesssim K(2,a)^q \lesssim \int_{L_1}\left[t^{-1}K(t,a)\right]^q\frac{dt}{t}.$$

If $\nu = 1$ and $t \in L_1$, then $J(t,v(t)) \lesssim K(4,a)$, and so

$$\int_{L_1}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t} \lesssim \left[\frac{K(4,a)}{4}\right]^q\int_{L_2}\frac{dt}{t} \lesssim \int_{L_2}\left[t^{-1}K(t,a)\right]^q\frac{dt}{t}.$$

With all these estimates, we have that

$$\begin{aligned}&\left(\int_0^1[(1-\log t)J(t,v(t))]^q\frac{dt}{t}\right)^{1/q}+\left(\int_1^\infty\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t}\right)^{1/q}\\ &\quad=\left(\sum_{\nu=-\infty}^{-1}\int_{L_\nu}[(1-\log t)J(t,v(t))]^q\frac{dt}{t}+\int_{L_0}[(1-\log t)J(t,v(t))]^q\frac{dt}{t}\right)^{1/q}\\ &\quad\quad+\left(\sum_{\nu=2}^{\infty}\int_{L_\nu}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t}+\int_{L_1}\left[\frac{1+\log t}{t}J(t,v(t))\right]^q\frac{dt}{t}\right)^{1/q}\\ &\quad\lesssim\left(\sum_{\nu=-\infty}^{-2}\int_{L_{\nu+2}}K(t,a)^q\frac{dt}{t}+\int_{L_1}\left[t^{-1}K(t,a)\right]^q\frac{dt}{t}\right)^{1/q}\end{aligned}$$

$$+\left(\int_{L_0} K(t,a)^q \frac{dt}{t} + \sum_{\nu=3}^{\infty} \int_{L_{\nu-2}} \left[t^{-1}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^{\infty} \left[t^{-1}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q}.$$

This shows that $(A_0,A_1)_{q;K} \hookrightarrow (A_0,A_1)_{\{\rho,\mu\},q;J}$ and completes the proof. □

Remark 3.3. In the proof of Theorem 3.14, the assumption $q \neq \infty$ has allowed us to use Hardy's inequality, as well as to guarantee the convergence of certain integrals. So, it is essential for the arguments. In fact, equality $\bar{A}_{\infty;K} = \bar{A}_{\{\rho,\mu\},\infty;J}$ does not hold in general: Assume that $A_0 \hookrightarrow A_1$ with the closure of A_0 in A_1, A_0^o, being different from A_1 (take for instance $A_0 = \ell_1$ and $A_1 = \ell_\infty$). By Lemma 3.1, we have $\bar{A}_{\infty;K} = A_0 + A_1 = A_1$. However, $\bar{A}_{\{\rho,\mu\},\infty;J} \subset A_0^o \neq A_1$. Indeed, take any $a \in \bar{A}_{\{\rho,\mu\},\infty;J}$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a J-representation of a with

$$\max\left\{\sup_{0<t<1}(1-\log t)J(t,u(t)), \sup_{1\le t<\infty}\frac{1+\log t}{t}J(t,u(t))\right\} \le 2\|a\|_{\bar{A}_{\{\rho,\mu\},\infty;J}}.$$

Then $\lim_{N\to\infty}\left\|a - \int_{1/N}^N u(t)\frac{dt}{t}\right\|_{A_1} = 0$ and $\int_{1/N}^N u(t)\frac{dt}{t}$ belongs to A_0 because

$$\int_{1/N}^N \|u(t)\|_{A_0}\frac{dt}{t} \le \int_{1/N}^1 \frac{J(t,u(t))}{1-\log t}(1-\log t)\frac{dt}{t} + \int_1^N \frac{t}{1+\log t}\frac{1+\log t}{t}J(t,u(t))\frac{dt}{t} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},\infty;J}}.$$

Remark 3.4. In a more general way, the $(\infty;K)$-method does not admit a description as a J-space. Indeed, given any Banach couple $\bar{A} = (A_0,A_1)$, using Hölder's inequality, it is not hard to check that if $u(t)$ satisfies condition (3.15), then the integral $\int_0^\infty u(t)\frac{dt}{t}$ is convergent in A_0+A_1. Besides, if $t>0$ and $w \in A_0\cap A_1$ then $J(t,w;A_0,A_1) = J\left(t,w;A_0^o,A_1^o\right)$, because $A_0\cap A_1 = A_0^o \cap A_1^o$ and the norms of A_j and A_j^o coincide for $j=0,1$. These two facts imply that

$$(A_0,A_1)_{\{\rho,\mu\},q;J} = (A_0^o,A_1^o)_{\{\rho,\mu\},q;J}. \tag{3.19}$$

Equality (3.19) holds for any general J-method as considered in [8] because our assumptions on $J(t,u(t))$ still imply the convergence of $\int_0^\infty u(t)\frac{dt}{t}$ in A_0+A_1 (see [8, page 362]). Since for the couple (ℓ_1,ℓ_∞) we have

$$(\ell_1,\ell_\infty)_{\infty;K} = \ell_\infty \neq c_0 = (\ell_1,c_0)_{\infty;K} = (\ell_1^o,\ell_\infty^o)_{\infty;K},$$

we conclude that the $(\infty;K)$-method does not admit a description by means of the J-functional.

Corollary 3.15. *Let $\bar{A} = (A_0,A_1)$ be a Banach couple and let $1 \le q < \infty$. Then $A_0\cap A_1$ is dense in $\bar{A}_{q;K}$.*

Proof. By Theorem 3.14, we can work with the norm $\|\cdot\|_{\bar{A}_{\{\rho,\mu\},q;J}}$. Let $a \in \bar{A}_{q;K}$ and take any $\varepsilon > 0$. We can find a J-representation $a = \int_0^\infty u(t)\frac{dt}{t}$ of a satisfying (3.15). Let $N \in \mathbb{N}$ such that

$$\left(\int_0^{1/N}\left[\rho(t)J(t,u(t))\right]^q\frac{dt}{t}\right)^{1/q} + \left(\int_N^\infty\left[\mu(t)J(t,u(t))\right]^q\frac{dt}{t}\right)^{1/q} < \varepsilon.$$

Using Hölder's inequality and the continuity of the function $t^{-1}(1-\log t)^{-1}$ on $[1,1/N]$ and of $t(1+\log t)^{-1}$ on $[1,N]$, we get

$$\begin{aligned}\int_{1/N}^{N}\|u(t)\|_{A_0\cap A_1}\frac{dt}{t} &\le \int_{1/N}^{1} t^{-1}J(t,u(t))\frac{dt}{t}+\int_{1}^{N}J(t,u(t))\frac{dt}{t}\\ &\le \left(\int_{1/N}^{1}[\rho(t)J(t,u(t))]^q\frac{dt}{t}\right)^{1/q}\left(\int_{1/N}^{1}[t\rho(t)]^{-q'}\frac{dt}{t}\right)^{1/q'}\\ &\quad+\left(\int_{1}^{N}[\mu(t)J(t,u(t))]^q\frac{dt}{t}\right)^{1/q}\left(\int_{1}^{N}\mu(t)^{-q'}\frac{dt}{t}\right)^{1/q'}<\infty.\end{aligned}$$

Therefore, $w=\int_{1/N}^{N}u(t)\frac{dt}{t}$ belongs to $A_0\cap A_1$. Since $a-w=\int_0^{1/N}u(t)\frac{dt}{t}+\int_N^\infty u(t)\frac{dt}{t}$, we obtain that

$$\|a-w\|_{\bar{A}_{\{\rho,\mu\},q;J}}\le\left(\int_0^{1/N}[\rho(t)J(t,u(t))]^q\frac{dt}{t}\right)^{1/q}+\left(\int_N^\infty[\mu(t)J(t,u(t))]^q\frac{dt}{t}\right)^{1/q}<\varepsilon.$$

This shows the density of $A_0\cap A_1$ in $\bar{A}_{q;K}$. □

It follows from (3.19) and Theorem 3.14 that $\left(A_0^{o},A_1^{o}\right)_{q;K}=(A_0,A_1)_{q;K}$ if $1\le q<\infty$. Hence, as a direct consequence of Theorem 3.11, we derive the following.

Corollary 3.16. *Let* $\bar{A}=(A_0,A_1)$ *and* $\bar{B}=(B_0,B_1)$ *be Banach couples, let* $1\le q<\infty$, *and let* $T\in\mathcal{L}\left(\bar{A},\bar{B}\right)$. *If* $T:A_j\longrightarrow B_j$ *is compact for* $j=0,1$, *then* $T:\bar{A}_{q;K}\longrightarrow\bar{B}_{q;K}$ *is also compact.*

3.5 Duality

This section is devoted to the study of the dual spaces $(A_0,A_1)^*_{q;K}$ and $(A_0,A_1)^*_{q;J}$ of the limiting K- and J-spaces. Duality is a classical question in interpolation theory that for the case of the real method $(A_0,A_1)_{\theta,q}$ has its roots in the papers by Lions [66] and Lions and Peetre [67].

Recall that a pair of normed spaces (A_0,A_1) is said to be *regular* if $A_0\cap A_1$ is dense in A_0 and A_1. Given a regular Banach couple, the mappings

$$\begin{aligned}\phi_j : A_j^* &\longrightarrow (A_0\cap A_1)^*\\ f &\longmapsto f|_{A_0\cap A_1}\end{aligned}$$

are linear embeddings for $j=0,1$. Thus, if (A_0,A_1) is regular then $A_0^*,A_1^*\hookrightarrow(A_0\cap A_1)^*$ by means of ϕ_0 and ϕ_1.

Let

$$A_j'=\left\{f|_{A_0\cap A_1}: f\in A_j^* \text{ and } \|f\|_{A_j'}=\|f\|_{A_j^*}\right\},\qquad j=0,1.$$

Then A_j' is clearly isometric to A_j^*. Moreover, since $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ whenever $1 \le q < \infty$, the space

$$(A_0, A_1)'_{\theta,q} = \left\{ f|_{A_0 \cap A_1} : f \in (A_0, A_1)^*_{\theta,q} \text{ and } \|f\|_{(A_0,A_1)'_{\theta,q}} = \|f\|_{(A_0,A_1)^*_{\theta,q}} \right\}$$

is meaningful.

In the classical real case, we have the following duality relationship: Whenever $1 \le q < \infty$ and $0 < \theta < 1$, $\left((A_0, A_1)_{\theta,q}\right)' = \left(A_0', A_1'\right)_{\theta,q'}$. In the proof of this duality result (see, for instance, [5, Theorem 3.7.1] or [80, Theorem 1.11.2]), one actually shows relationships between the dual of the K-space and the J-space and viceversa. Namely, in [5, Theorem 3.7.1], it is shown that $(A_0, A_1)'_{\theta,q;J} \hookrightarrow (A_0', A_1')_{\theta,q';K}$ and $(A_0', A_1')_{\theta,q';J} \hookrightarrow (A_0, A_1)'_{\theta,q;K}$, and the result is obtained using the equivalence theorem.

In the ordered case where $A_0 \hookrightarrow A_1$, it was shown in [19, Theorems 8.1 and 8.2] that if A_0 is dense in A_1 and $1 < q < \infty$, $(A_0, A_1)'_{0,q;J} = \left(A_1', A_0'\right)_{1,q';K}$ and $(A_0, A_1)'_{1,q;K} = \left(A_1', A_0'\right)_{0,q';J}$, where q' is the conjugate exponent of q.

The following theorems show the duality relationships for the limiting methods in the general case.

Theorem 3.17. *Let* $1 \le q \le \infty$, $1/q + 1/q' = 1$ *and let* $\bar{A} = (A_0, A_1)$ *be a regular Banach couple. Then, we have with equivalent norms* $(A_0, A_1)'_{q;K} = \left(A_0', A_1'\right)_{q';J}$.

Proof. By [5, Theorem 2.7.1], we know that $(A_0 + A_1)' = A_0' \cap A_1'$ and $(A_0 \cap A_1)' = A_0' + A_1'$. Whence, using Lemmata 3.6 and 3.1, we obtain

$$(A_0, A_1)'_{\infty;K} = (A_0 + A_1)' = A_0' \cap A_1' = \left(A_0', A_1'\right)_{1;J}.$$

If $1 < q < \infty$, we derive from Lemmata 3.7, 3.4 and [19, Theorem 8.2] that

$$\begin{aligned}(A_0, A_1)'_{q;K} &= (A_0 \cap A_1, A_0 + A_1)'_{1,q;K} = \left(A_0' \cap A_1', A_0' + A_1'\right)_{0,q';J} \\ &= \left(A_0', A_1'\right)_{q';J}.\end{aligned}$$

The remaining case $q = 1$ can be treated as when $1 < q < \infty$ because the arguments in [19, Theorem 8.2] also work for $q = 1$. □

Theorem 3.18. *Let* $1 \le q < \infty$, $1/q + 1/q' = 1$ *and let* $\bar{A} = (A_0, A_1)$ *be a regular Banach couple. Then we have with equivalent norms* $(A_0, A_1)'_{q;J} = \left(A_0', A_1'\right)_{q';K}$.

Proof. The case $q = 1$ follows again by Lemmata 3.6 and 3.1 and [5, Theorem 2.7.1]. Namely

$$(A_0, A_1)'_{1;J} = (A_0 \cap A_1)' = A_0' + A_1' = \left(A_0', A_1'\right)_{\infty;K}.$$

For $1 < q < \infty$, by Lemmata 3.7 and 3.4 and [19, Theorem 8.1], we derive

$$\begin{aligned}(A_0, A_1)'_{q;J} &= (A_0 \cap A_1, A_0 + A_1)'_{0,q;J} = \left(A_0' \cap A_1', A_0' + A_1'\right)_{1,q';K} \\ &= \left(A_0', A_1'\right)_{q';K}.\end{aligned}$$

□

In order to study the dual of the J-space when $q = \infty$, define $(A_0, A_1)_{c_0;J}$ as the collection of all $a \in A_0 + A_1$ for which there exists a sequence $(u_m)_{m\in\mathbb{Z}} \subset A_0 \cap A_1$ such that $a = \sum_{m\in\mathbb{Z}} u_m$ (convergence in $A_0 + A_1$) and

$$\max(1, 2^{-m}) J(2^m, u_m) \xrightarrow[m\to\pm\infty]{} 0. \tag{3.20}$$

We put

$$\|a\|_{\bar{A}_{c_0;J}} = \inf_{a=\sum u_m} \left[\sup_{m\in\mathbb{Z}} \max(1, 2^{-m}) J(2^m, u_m) \right].$$

Lemma 3.19. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $(A_0, A_1)^{o}_{\infty;J}$ be the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{\infty;J}$. Then we have with equivalence of norms $(A_0, A_1)_{c_0;J} = (A_0, A_1)^{o}_{\infty;J}$.*

Proof. Let $a \in (A_0, A_1)_{c_0;J}$. Choose $(u_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m\in\mathbb{Z}} u_m$ and satisfying (3.20). Given any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that if $|m| \geq M$, then $\max(1, 2^{-m}) J(2^m, u_m) < \varepsilon/2$. Consider the vector $w = \sum_{|m|\leq M} u_m \in A_0 \cap A_1$. Since $a - w$ can be represented in $\bar{A}_{\infty;J}$ by means of the sequence

$$v_m = \begin{cases} u_m & \text{if } |m| > M, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\|a - w\|_{\bar{A}_{\infty;J}} \lesssim \sup_{m>M} J(2^m, u_m) + \sup_{m<-M} 2^{-m} J(2^m, u_m) \leq \varepsilon.$$

This implies that $a \in (A_0, A_1)^{o}_{\infty;J}$.

In order to show the converse embedding, we shall prove that

$$(A_0, A_1)^{o}_{\infty;J} \hookrightarrow X \hookrightarrow (A_0, A_1)_{c_0;J},$$

where X is the set of all vectors $a \in A_0 + A_1$ such that

$$\max(1, t^{-1})(1 + |\log t|)^{-1} K(t, a) \longrightarrow 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

normed by

$$\|a\|_X = \sup_{0<t<\infty} \max(1, t^{-1})(1 + |\log t|)^{-1} K(t, a).$$

Let $a \in (A_0, A_1)^{o}_{\infty;J}$. Then, given any $\varepsilon > 0$, there is $w \in A_0 \cap A_1$ such that $\|a - w\|_{\bar{A}_{\infty;J}} \leq \varepsilon/2$. Find $M > 1$ such that

$$(1 + |\log t|)^{-1} \|w\|_{A_0\cap A_1} \leq \varepsilon/2 \text{ if } t > M.$$

Then we also have that

$$(1 + |\log t|)^{-1} \|w\|_{A_0\cap A_1} \leq \varepsilon/2 \text{ if } 0 < t < 1/M.$$

Besides, by (3.8),

$$K(t, a) \leq K(t, a - w) + K(t, w) \leq \frac{(1 + |\log t|)}{\max(1, t^{-1})} \|a - w\|_{\infty;J} + \min(1, t) \|w\|_{A_0\cap A_1}.$$

Consequently, if $0 < t < 1/M$ or $t > M$, we derive that

$$\max(1, t^{-1})(1 + |\log t|)^{-1} K(t, a) \le \|a - w\|_{\infty;J} + (1 + |\log t|)^{-1} \|w\|_{A_0 \cap A_1} \le \varepsilon.$$

So $a \in X$. Now we show the continuous embedding. Take a representation of a in the J-space, $a = \int_0^\infty u(s) ds/s$, such that

$$\sup_{0<s<\infty} \max(1, s^{-1}) J(s, u(s)) \le 2 \|a\|_{\bar{A}_{\infty;J}}.$$

Then, by (3.16),

$$\begin{aligned}
\|a\|_X &\le \sup_{0<t<\infty} \max(1, t^{-1})(1 + |\log t|)^{-1} \int_0^t J(s, u(s)) \frac{ds}{s} \\
&\quad + \sup_{0<t<\infty} \max(1, t^{-1})(1 + |\log t|)^{-1} \int_t^\infty \frac{t}{s} J(s, u(s)) \frac{ds}{s} \\
&\le \sup_{0<t<1} t^{-1}(1 + |\log t|)^{-1} \int_0^t J(s, u(s)) \frac{ds}{s} + \sup_{1\le t<\infty} (1 + |\log t|)^{-1} \int_0^1 J(s, u(s)) \frac{ds}{s} \\
&\quad + \sup_{1\le t<\infty} (1 + |\log t|)^{-1} \int_1^t J(s, u(s)) \frac{ds}{s} + \sup_{0<t<1} (1 + |\log t|)^{-1} \int_t^1 \frac{J(s, u(s))}{s} \frac{ds}{s} \\
&\quad + \sup_{0<t<1} (1 + |\log t|)^{-1} \int_1^\infty \frac{J(s, u(s))}{s} \frac{ds}{s} + \sup_{1\le t<\infty} t(1 + |\log t|)^{-1} \int_t^\infty \frac{J(s, u(s))}{s} \frac{ds}{s} = \sum_{j=1}^6 S_j.
\end{aligned}$$

We have that

$$S_1 \le \sup_{0<t<1} t^{-1}(1 + |\log t|)^{-1} \int_0^t s \frac{ds}{s} \sup_{0<s<t} \frac{J(s, u(s))}{s} \le \sup_{0<s<1} \frac{J(s, u(s))}{s} \lesssim \|a\|_{\bar{A}_{\infty;J}}.$$

Similarly,

$$S_6 \le \sup_{1\le t<\infty} t(1 + |\log t|)^{-1} \int_t^\infty s^{-1} \frac{ds}{s} \sup_{t<s<\infty} J(s, u(s)) \le \sup_{1<s<\infty} J(s, u(s)) \lesssim \|a\|_{\bar{A}_{\infty;J}}.$$

On the other hand,

$$S_2 = \int_0^1 J(s, u(s)) \frac{ds}{s} \le \sup_{0<s<1} \frac{J(s, u(s))}{s} \int_0^1 ds \lesssim \|a\|_{\bar{A}_{\infty;J}}$$

and similarly

$$S_5 = \int_1^\infty \frac{J(s, u(s))}{s} \frac{ds}{s} \lesssim \|a\|_{\bar{A}_{\infty;J}}.$$

Finally,

$$\begin{aligned}
S_3 &\le \sup_{1\le t<\infty} (1 + |\log t|)^{-1} \int_1^t \frac{ds}{s} \sup_{1\le s<t} J(s, u(s)) \le \sup_{1\le s<\infty} J(s, u(s)) \sup_{1\le t<\infty} (1 + |\log t|)^{-1}(1 + |\log t|) \\
&\lesssim \|a\|_{\bar{A}_{\infty;J}}
\end{aligned}$$

and also

$$S_4 \le \sup_{0<t<1} (1+|\log t|)^{-1} \int_t^1 \frac{ds}{s} \sup_{t<s<1} \frac{J(s,u(s))}{s} \lesssim \|a\|_{\bar{A}_{\infty;J}} .$$

This shows that $(A_0, A_1)^{o}_{\infty;J} \hookrightarrow X$.

Next we prove the second embedding. Let $a \in X$. Then

$$t^{-1}(1+|\log t|)^{-1}K(t,a) \longrightarrow 0 \text{ as } t \to 0 \quad \text{and} \quad (1+|\log t|)^{-1}K(t,a) \longrightarrow 0 \text{ as } t \to \infty.$$

Therefore,

$$K(t,a) = t^{-1}(1+|\log t|)^{-1}K(t,a)t(1+|\log t|) \longrightarrow 0 \text{ as } t \to 0 \tag{3.21}$$

and

$$t^{-1}K(t,a) = t^{-1}(1+|\log t|)(1+|\log t|)^{-1}K(t,a) \longrightarrow 0 \text{ as } t \to \infty. \tag{3.22}$$

For $\nu \in \mathbb{Z}$, put

$$\eta_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

We can find decompositions $a = a_{0,\nu} + a_{1,\nu}$, with $a_{j,\nu} \in A_j$, $j = 0,1$, such that

$$\begin{aligned} \|a_{0,\nu}\|_{A_0} + \eta_{\nu+1} \|a_{1,\nu}\|_{A_1} &\le 2K(\eta_{\nu+1}, a) && \text{if } \nu > 0 \text{, and} \\ \eta_{\nu-1}^{-1} \|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} &\le 2\widetilde{K}\left(\eta_{\nu-1}^{-1}, a\right) && \text{if } \nu \le 0, \end{aligned}$$

where $\widetilde{K}(t,a) = K(t,a;A_1,A_0)$.

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$, $\nu \in \mathbb{Z}$. Given any $N, M \in \mathbb{N}$, we have

$$\left\| a - \sum_{\nu=-N}^{M} u_\nu \right\|_{A_0+A_1} = \|a - a_{0,M} + a_{0,-N-1}\|_{A_0+A_1} \le \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} .$$

By (3.21) and (3.22), the last two terms go to 0 as $N, M \to \infty$. Hence, we have that $a = \sum_{\nu\in\mathbb{Z}} u_\nu$ in $A_0 + A_1$.

Consider now the sequence of vectors given by

$$v_m = \begin{cases} \dfrac{u_\nu}{2^{-\nu-1}} & \text{if } \eta_{\nu-1} \le 2^m < \eta_\nu \text{ and } \nu < 0, \\ u_\nu & \text{if } \eta_{\nu-1} \le 2^m < \eta_\nu \text{ and } \nu = 0,1, \\ \dfrac{u_\nu}{2^{\nu-2}} & \text{if } \eta_{\nu-1} \le 2^m < \eta_\nu \text{ and } \nu > 1. \end{cases}$$

Then

$$\sum_{m=-\infty}^{\infty} v_m = \sum_{\nu=-\infty}^{\infty} \sum_{\eta_{\nu-1} \le 2^m < \eta_\nu} v_m = \sum_{\nu=-\infty}^{\infty} u_\nu = a \quad (\text{convergence in } A_0 + A_1).$$

Next we show that this is a suitable representation of a in the J-space.

If $\nu > 1$ and $\eta_{\nu-1} \le 2^m < \eta_\nu$, we have that

$$\begin{aligned} J(2^m, v_m) = J\left(2^m, \frac{u_\nu}{2^{\nu-2}}\right) &\lesssim 2^{-\nu} J(\eta_\nu, u_\nu) \le 2^{-\nu}\left(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0} + \eta_\nu \|a_{1,\nu}\|_{A_1} + \eta_\nu \|a_{1,\nu-1}\|_{A_1}\right) \\ &\lesssim 2^{-\nu}\left[K(\eta_{\nu+1}, a) + K(\eta_\nu, a)\right] \lesssim 2^{-\nu} K(\eta_{\nu+1}, a) \sim (1 + |\log \eta_{\nu+1}|)^{-1} K(\eta_{\nu+1}, a). \end{aligned} \tag{3.23}$$

Suppose now $\nu < 0$. Then, if $\eta_{\nu-1} \le 2^m < \eta_\nu$,

$$\begin{aligned} 2^{-m} J(2^m, v_m) &\le \eta_{\nu-1}^{-1} J(\eta_{\nu-1}, v_m) \sim \eta_{\nu-1}^{-1} 2^\nu J(\eta_{\nu-1}, u_\nu) = 2^\nu \widetilde{J}(\eta_{\nu-1}^{-1}, u_\nu) \\ &\le 2^\nu \left(\eta_{\nu-1}^{-1} \|a_{0,\nu}\|_{A_0} + \eta_{\nu-1}^{-1} \|a_{0,\nu-1}\|_{A_0} + \|a_{1,\nu}\|_{A_1} + \|a_{1,\nu-1}\|_{A_1}\right) \\ &\lesssim 2^\nu \left[\widetilde{K}(\eta_{\nu-1}^{-1}, a) + \widetilde{K}(\eta_{\nu-2}^{-1}, a)\right] \lesssim 2^\nu \widetilde{K}(\eta_{\nu-2}^{-1}, a) \\ &\sim (1 + |\log \eta_{\nu-2}|)^{-1} \eta_{\nu-2}^{-1} K(\eta_{\nu-2}, a). \end{aligned} \tag{3.24}$$

Finally, we also have that

$$2J(2^{-1}, v_{-1}) \lesssim K(1, a) \qquad \text{and} \qquad J(1, v_0) \lesssim K(1, a). \tag{3.25}$$

Equations (3.23), (3.24) and (3.25) imply that

$$\max(1, 2^{-m}) J(2^m, v_m) \longrightarrow 0 \text{ as } m \to \pm\infty$$

and also that

$$\begin{aligned} \sup_{m \in \mathbb{Z}} \max(1, 2^{-m}) J(2^m, v_m) &\lesssim \sup_{\nu \in \mathbb{Z}} \max(1, \eta_\nu^{-1}) K(\eta_\nu, a)(1 + |\log \eta_\nu|)^{-1} \\ &\le \sup_{0 < t < \infty} \max(1, t^{-1}) K(t, a)(1 + |\log t|)^{-1} = \|a\|_X . \end{aligned}$$

This ends the proof. □

Theorem 3.20. *We have* $\left((A_0, A_1)^{o}_{\infty;J}\right)' = \left(A_0', A_1'\right)_{1;K}$ *with equivalence of norms.*

Proof. With the help of Lemma 3.19, we can proceed similarly to [19, Theorem 8.1]. In other words, put

$$G_m = \begin{cases} A_0 \cap A_1 \text{ normed by } J(2^m, \cdot) & \text{if } m \in \mathbb{N}, \\ A_0 \cap A_1 \text{ normed by } \|\cdot\|_{A_0 \cap A_1} & \text{if } m = 0, \\ A_0 \cap A_1 \text{ normed by } 2^{-m} J(2^m, \cdot) & \text{if } -m \in \mathbb{N}. \end{cases}$$

Let $W = c_0(G_m)_{m \in \mathbb{Z}}$ and put

$$M = \left\{ (w_m) \in W : \sum_{m \in \mathbb{Z}} w_m = 0 \text{ (convergence in } A_0 + A_1) \right\}.$$

As usual, let

$$M^\perp = \left\{ \tilde{f} \in W^* : \tilde{f}(w_m) = 0 \text{ for each } (w_m) \in M \right\}.$$

The space $(A_0, A_1)^o_{\infty;J} = (A_0, A_1)_{c_0;J}$ coincides with W/M with equivalent norms. Therefore,

$$\left((A_0, A_1)^o_{\infty;J} \right)^* = (W/M)^* = M^\perp.$$

Let us identify $M^\perp$.

Put

$$F_m = \begin{cases} A_0' + A_1' \text{ normed by } 2^{-m} K\left(2^m, \cdot; A_0', A_1'\right) & \text{if } m \in \mathbb{N}, \\ A_0' + A_1' \text{ normed by } \|\cdot\|_{A_0'+A_1'} & \text{if } m = 0, \\ A_0' + A_1' \text{ normed by } K\left(2^m, \cdot; A_0', A_1'\right) & \text{if } -m \in \mathbb{N}. \end{cases}$$

For each $m \in \mathbb{Z}$, we have that $G_m' = F_{-m}$ with equal norms. Hence $W^* = \ell_1(F_{-m})$. This means that functionals $\tilde{f} \in W^*$ are given by sequences $(f_{-m}) \in \ell_1(F_{-m})$ with

$$\tilde{f}(w_m) = \sum_{m \in \mathbb{Z}} f_{-m}(w_m) \qquad \text{and} \qquad \left\|\tilde{f}\right\|_{W^*} = \sum_{m \in \mathbb{Z}} \|f_{-m}\|_{F_{-m}}.$$

We claim that if $\tilde{f} \in M^\perp$ then $f_n = f_m$ for all $n, m \in \mathbb{Z}$. Indeed, if there is $a \in A_0 \cap A_1$ such that $f_n(a) \neq f_m(a)$, then for the sequence $w = (w_k) \in W$ defined by $w_k = a$ if $k = -n$, $w_k = -a$ if $k = -m$ and $w_k = 0$ for the rest of $k \in \mathbb{Z}$, we have that $\tilde{f}(w_k) = f_n(a) - f_m(a) \neq 0$, but $w \in M$.

Conversely, let $f \in (A_0 \cap A_1)'$ with $(\dots, f, f, f, \dots) \in W^*$. We claim that the functional $\tilde{f}$ defined by this constant sequence belongs to $M^\perp$. Indeed, take any $(w_m) \in M$. Let us show that actually $\tilde{f}(w_m) = \sum_{m \in \mathbb{Z}} f(w_m) = 0$. Since $(\dots, f, f, f, \dots) \in W^* = \ell_1(F_{-m})$, we derive that $(\dots, f, f, f, \dots)$ belongs to $\left(A_0', A_1'\right)_{1;K}$. Using the J-representation of this space given by Theorem 3.14, we can find $\left(g_j\right) \subset A_0' \cap A_1'$ such that $f = \sum_{j \in \mathbb{Z}} g_j$ (convergence in $A_0' + A_1'$) and

$$\left\| f - \sum_{j=-N}^{M} g_j \right\|_{(A_1', A_0')_{1;K}} \longrightarrow 0 \quad \text{as } M, N \to \infty.$$

Hence, given any $\varepsilon > 0$, there is $L \in \mathbb{N}$ such that

$$\begin{aligned} \left\| f - \sum_{|j| \le L} g_j \right\|_{(A_0', A_1')_{1;K}} &= \left\| \left(\dots, f - \sum_{|j| \le L} g_j, f - \sum_{|j| \le L} g_j, f - \sum_{|j| \le L} g_j, \dots \right) \right\|_{W^*} \\ &< \frac{\varepsilon}{2\, \|(w_m)\|_W}. \end{aligned}$$

Let $g = \sum_{|j| \le L} g_j$. Then $g \in A_0' \cap A_1' = (A_0 + A_1)'$. Since $\sum_{m \in \mathbb{Z}} w_m = 0$ in $A_0 + A_1$, we can find $N \in \mathbb{N}$ such that for any $m \ge N$ we have

$$\left| g\left(\sum_{k=-m}^{m} w_k \right) \right| < \frac{\varepsilon}{2}.$$

Therefore, for each $m \geq N$, we derive that

$$\begin{aligned}\left|\sum_{k=-m}^{m} f(w_k)\right| &= \left|\sum_{k=-m}^{m} f(w_k) - g\left(\sum_{k=-m}^{m} w_k\right) + g\left(\sum_{k=-m}^{m} w_k\right)\right| \\ &\leq \|(\dots, f-g, f-g, f-g, \dots)\|_{W^*} \|(w_n)\|_W + \left|g\left(\sum_{k=-m}^{m} w_k\right)\right| \\ &< \frac{\varepsilon}{2\|(w_m)\|_W}\|(w_m)\|_W + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This yields that $\tilde{f} \in M^\perp$.

Consequently, $\left((A_0, A_1)^{o}_{\infty;J}\right)'$ consists of all $f \in (A_0 \cap A_1)' = A_0' + A_1'$ for which the sequence $\left(\min(1, 2^{-m}) K\left(2^m, f; A_0', A_1'\right)\right) \in \ell_1$. This establishes that $\left((A_0, A_1)^{o}_{\infty;J}\right)' = \left(A_0', A_1'\right)_{1;K}$ and ends the proof. □

3.6 Examples

Let (Ω, μ) be a σ-finite measure space. In order to determine the spaces generated by limiting interpolation from the couple $(L_\infty(\Omega), L_1(\Omega))$, we recall that for $1 \leq p, q \leq \infty$ and $b \in \mathbb{R}$, the Lorentz-Zygmund space $L_{p,q}(\log L)_b(\Omega)$ is defined to be the collection of all (equivalence classes of) measurable functions f on Ω such that the functional

$$\|f\|_{L_{p,q}(\log L)_b(\Omega)} = \left(\int_0^\infty \left(t^{1/p}(1+|\log t|)^b f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}$$

is finite. The space $L_{(p,q)}(\log L)_b(\Omega)$ is defined similarly but replacing f^* (defined in (2.5)) by $f^{**}(t) = t^{-1}\int_0^t f^*(s)ds$. According to [44, Lemma 3.4.39], $L_{p,q}(\log L)_b(\Omega) = L_{(p,q)}(\log L)_b(\Omega)$ provided that $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $b \in \mathbb{R}$.

Working with limiting ordered methods, it was shown in [19, Corollary 4.3] that if Ω is a *finite* measure space then $(L_\infty(\Omega), L_1(\Omega))_{0,q;J} = L_{\infty,q}(\log L)_{-1}(\Omega)$ with equivalent norms. One obviously needs Ω to be of finite measure in order to make sure that the couple $(L_\infty(\Omega), L_1(\Omega))$ is ordered. In the limiting general case, we can recover the hypothesis of Ω being a σ-finite measure space, since no order relationship is needed.

Theorem 3.21. *Let (Ω, μ) be a σ-finite measure space.*

(i) *If $1 < q \leq \infty$ then*

$$(L_\infty(\Omega), L_1(\Omega))_{q;J} = L_{(\infty,q)}(\log L)_{-1}(\Omega) \cap L_{(1,q)}(\log L)_{-1}(\Omega).$$

(ii) *If $1 \leq q < \infty$ then*

$$(L_\infty(\Omega), L_1(\Omega))_{q;K} = \left\{f : \|f\| = \left(\int_0^\infty [\min(1,t) f^{**}(t)]^q \frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

Proof. It is well-known (see [5] or [80]) that

$$K(t, f; L_\infty(\Omega), L_1(\Omega)) = f^{**}(1/t). \tag{3.26}$$

According to (3.8), we obtain

$$\begin{aligned}\|f\|_{(L_\infty,L_1)_{q;J}} &\sim \left(\int_0^1 \left[\frac{f^{**}(1/t)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{f^{**}(1/t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}\\ &= \left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Now we study each of these two terms. Using that $f^{**}(t)$ is non-increasing, we get

$$\begin{aligned}\left(\int_1^\infty \left[\frac{f^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} &\le f^{**}(1)\left(\int_1^\infty (1+\log t)^{-q}\frac{dt}{t}\right)^{1/q} \sim f^{**}(1)\left(\int_0^1 (1-\log t)^{-q}\frac{dt}{t}\right)^{1/q}\\ &\le \left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Hence,

$$\left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^\infty \left[\frac{f^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} = \|f\|_{L_{(\infty,q)}(\log L)_{-1}}. \tag{3.27}$$

For the second term, we observe that

$$\begin{aligned}\left(\int_0^1 \left[\frac{tf^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} &\le \left(\int_0^1 f^*(s)ds\right)\left(\int_0^1 (1-\log t)^{-q}\frac{dt}{t}\right)^{1/q}\\ &\lesssim \left(\int_0^1 f^*(s)ds\right)\left(\int_1^\infty (1+\log t)^{-q}\frac{dt}{t}\right)^{1/q} \le \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

So,

$$\left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_{L_{(1,q)}(\log L)_{-1}}.$$

This yields (i). Formula (ii) follows by inserting (3.26) in the interpolation norm. Namely,

$$\begin{aligned}\|f\|_{(L_\infty,L_1)_{q;K}} &\sim \left(\int_0^1 f^{**}(1/t)^q\frac{dt}{t} + \int_1^\infty \left[t^{-1}f^{**}(1/t)\right]^q \frac{dt}{t}\right)^{1/q}\\ &\sim \left(\int_0^1 [tf^{**}(t)]^q\frac{dt}{t} + \int_1^\infty f^{**}(t)^q \frac{dt}{t}\right)^{1/q}\\ &= \left(\int_0^\infty [\min(1,t)f^{**}(t)]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

This ends the proof. □

Let now ω be a weight on Ω, that is, a positive measurable function on Ω. As usual, we put

$$L_q(\omega) = \left\{ f : \|f\|_{L_q(\omega)} = \|\omega f\|_{L_q} < \infty \right\}.$$

The following theorem deals with the Banach couple $(L_q(\omega_0), L_q(\omega_1))$ where ω_0 and ω_1 are weights on Ω. If one applies the classical real method to this couple with $1 \le q \le \infty$, it is known that $(L_q(\omega_0), L_q(\omega_1))_{\theta,q} = L_q(\omega)$, where $\omega(x) = \omega_0^{1-\theta}(x)\omega_1^{\theta}(x)$ (see, for instance, [5, Theorem 5.4.1]). Regarding the limiting ordered cases, it is shown in [19, Theorems 4.8 and 7.4] that

$$(L_q(\omega_0), L_q(\omega_1))_{0,q;J} = L_q(\omega_J), \text{ where } \omega_J(x) = \omega_0(x)\left(1 + \log\frac{\omega_0(x)}{\omega_1(x)}\right)^{-1/q'}, \text{ and that}$$

$$(L_q(\omega_0), L_q(\omega_1))_{1,q;K} = L_q(\omega_K), \text{where } \omega_K(x) = \omega_1(x)\left(1 + \log\frac{\omega_0(x)}{\omega_1(x)}\right)^{1/q}.$$

One obviously needs to have that $\omega_0(x) \ge \omega_1(x)$ μ-almost everywhere in order to make sure that the Banach couple $(L_q(\omega_0), L_q(\omega_1))$ is ordered. In the limiting general case, one no longer needs this order relationship between the weights; the result turns out to be as follows.

Theorem 3.22. *Let (Ω,μ) be a σ-finite measure space, let $1 \le q \le \infty$, $1/q + 1/q' = 1$ and let ω_0, ω_1 be weights on Ω.*

(i) *We have with equivalence of norms*

$$(L_q(\omega_0), L_q(\omega_1))_{q;K} = L_q(\omega_K),$$

where

$$\omega_K(x) = \min(\omega_0(x), \omega_1(x))\left(1 + \left|\log\frac{\omega_0(x)}{\omega_1(x)}\right|\right)^{1/q}.$$

(ii) *For the $(q;J)$-method, we have with equivalence of norms,*

$$(L_q(\omega_0), L_q(\omega_1))_{q;J} = L_q(\omega_J),$$

where

$$\omega_J(x) = \max(\omega_0(x), \omega_1(x))\left(1 + \left|\log\frac{\omega_0(x)}{\omega_1(x)}\right|\right)^{-1/q'}.$$

Proof. It is easy to check that

$$L_q(\omega_0) \cap L_q(\omega_1) = L_q\left(\max\left(\omega_0, \omega_1\right)\right) \quad \text{and} \quad L_q(\omega_0) + L_q(\omega_1) = L_q\left(\min\left(\omega_0, \omega_1\right)\right).$$

Whence, by Lemma 3.4,

$$(L_q(\omega_0), L_q(\omega_1))_{q;K} = \left(L_q\left(\max\left(\omega_0, \omega_1\right)\right), L_q\left(\min\left(\omega_0, \omega_1\right)\right)\right)_{q;K}.$$

Now (i) follows from the corresponding result for the ordered case (see [19, Theorem 7.4]). The proof of (ii) is similar but using now Lemma 3.7 and [19, Theorem 4.8]. □

Next we show a consequence of this result on interpolation of a certain class of Sobolev spaces. We put $\mathcal{S}\left(\mathbb{R}^d\right)$ and $\mathcal{S}'\left(\mathbb{R}^d\right)$ for the Schwarz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. The symbol $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for the inverse Fourier transform. For $s \in \mathbb{R}$, we denote by $H^s = H_2^s\left(\mathbb{R}^d\right)$ the set of all $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that

$$\|f\|_{H^s} = \left\|\left(1 + \|x\|_{\mathbb{R}^d}^2\right)^{s/2} \mathcal{F}f\right\|_{L_2(\mathbb{R}^d)} < \infty.$$

In a more general way, we put

$$H^\varphi = \left\{f \in \mathcal{S}'\left(\mathbb{R}^d\right) : \|f\|_{H^\varphi} = \|\varphi(x)\mathcal{F}f\|_{L_2(\mathbb{R}^d)} < \infty\right\}$$

(see [59, 70]), where φ is a temperate weight function in the sense of [59, Definition 10.1.1]. Recall that a function φ defined on $\mathbb{R}^d$ is said to be a *temperate weight* if there exist two constants $C, N > 0$ such that

$$\varphi\left(\xi + \eta\right) \le \left(1 + C\|\xi\|_{\mathbb{R}^d}\right)^N \varphi(\eta) \qquad \forall \xi, \eta \in \mathbb{R}^d.$$

As a direct consequence of Theorem 3.22 and the interpolation property of the $(q; K)$- and $(q; J)$-methods, we obtain the following.

Corollary 3.23. *Let* $-\infty < s_1 < s_0 < \infty$. *Put*

$$\varphi_j(x) = \left(1 + \|x\|_{\mathbb{R}^d}^2\right)^{s_j/2} \left(1 + \frac{1}{2}(s_0 - s_1)\log\left(1 + \|x\|_{\mathbb{R}^d}^2\right)\right)^{(-1)^{(j+1)}/2}$$

where $j = 0, 1$. *Then we have with equivalence of norms*

$$(H^{s_0}, H^{s_1})_{2;K} = H^{\varphi_1} \quad \textit{and} \quad (H^{s_0}, H^{s_1})_{2;J} = H^{\varphi_0}.$$

Proof. Put $w_s(x) := \left(1 + \|x\|_{\mathbb{R}^d}^2\right)^{s/2}$. Since the Fourier transform is an isometry in L_2, we have that

$$\mathcal{F} : H^s \longrightarrow L_2\left(w_s\right)$$

is also an isometry; it is actually an isometric isomorphism. Therefore, interpolating $\mathcal{F}$ and $\mathcal{F}^{-1}$, we get that

$$\mathcal{F} : (H^{s_0}, H^{s_1})_{2,J} \longrightarrow \left(L_2\left(w_{s_0}\right), L_2\left(w_{s_1}\right)\right)_{2,J} \qquad \text{and}$$
$$\mathcal{F}^{-1} : \left(L_2\left(w_{s_0}\right), L_2\left(w_{s_1}\right)\right)_{2,J} \longrightarrow (H^{s_0}, H^{s_1})_{2,J}$$

are continuous. Therefore,

$$f \in (H^{s_0}, H^{s_1})_{2,J} \iff \mathcal{F}f \in \left(L_2\left(w_{s_0}\right), L_2\left(w_{s_1}\right)\right)_{2,J}.$$

Applying Theorem 3.22 we obtain that $\left(L_2\left(w_{s_0}\right), L_2\left(w_{s_1}\right)\right)_{2,J} = L_2\left(w_J\right)$, where

$$w_J(x) = w_{s_0}(x)\left(1 + \log\frac{w_{s_0}}{w_{s_1}}\right)^{-1/2} = \left(1 + \|x\|_{\mathbb{R}^d}^2\right)^{s_0/2}\left(1 + \frac{1}{2}\left(s_0 - s_1\right)\log\left(1 + \|x\|_{\mathbb{R}^d}^2\right)\right)^{-1/2},$$

and thus

$$f \in (H^{s_0}, H^{s_1})_{2,J} \iff \mathcal{F}f \in L_2\left(w_J\right) \iff \|w_J\mathcal{F}f\|_{L_2} < \infty \iff f \in H^{\varphi_0}.$$

The other formula follows similarly. $\square$

Remark 3.5. The functions φ_0 and φ_1 in Corollary 3.23 are temperate weights in the sense of Hörmander (see [59]). Let us see why.

In [59, Example 10.1.2] it is shown that the function $\left(1+\|\xi\|_{\mathbb{R}^d}^2\right)^{s/2}$ is a temperate weight for any $s\in\mathbb{R}$, and with $N=2$ and $C=1$, that is,

$$\frac{\left(1+\|\xi\|_{\mathbb{R}^d}^2\right)^{s/2}}{\left(1+\|\eta\|_{\mathbb{R}^d}^2\right)^{s/2}} \le (1+\|\xi-\eta\|_{\mathbb{R}^d})^2 \quad \text{for } \xi,\eta\in\mathbb{R}^d \text{ with } \|\xi\|_{R^d}\ge\|\eta\|_{R^d}. \tag{3.28}$$

Moreover, it follows from [59, Theorem 10.1.4] that if k_1 and k_2 are temperate weights and $s\in\mathbb{R}$ then k_1k_2 and k_1^s are also temperate weights. Therefore, all we need to do is to show that

$$\frac{w(\xi)}{w(\eta)} \le (1+C\,\|\xi-\eta\|_{\mathbb{R}^d})^N$$

for $\|\xi\|_{\mathbb{R}^d}\ge\|\eta\|_{\mathbb{R}^d}$, certain $C,N>0$, $w(\xi)=1+r\log\left(1+\|\xi\|_{\mathbb{R}^d}^2\right)$ and $r>0$. Using the fact that

$$\frac{1+\|\xi\|_{\mathbb{R}^d}^2}{1+\|\eta\|_{\mathbb{R}^d}^2}\ge 1$$

and (3.28), we derive that

$$\begin{aligned}\frac{w(\xi)}{w(\eta)} &= 1+\frac{r\left(\log\left(1+\|\xi\|_{\mathbb{R}^d}^2\right)-\log\left(1+\|\eta\|_{\mathbb{R}^d}^2\right)\right)}{1+r\log\left(1+\|\eta\|_{\mathbb{R}^d}^2\right)} \le 1+r\log\frac{1+\|\xi\|_{\mathbb{R}^d}^2}{1+\|\eta\|_{\mathbb{R}^d}^2} \le 1+r\log\left(1+\|\xi-\eta\|_{\mathbb{R}^d}\right)^2\\ &= 1+2r\log\left(1+\|\xi-\eta\|_{\mathbb{R}^d}\right) \le C\left(1+\|\xi-\eta\|_{\mathbb{R}^d}\right),\end{aligned}$$

where the last inequality is due to the fact that

$$\lim_{x\to\infty}\frac{1+r\log(1+x)}{1+x}=0.$$

Next consider a *dyadic resolution of unity* in $\mathbb{R}^d$, that is, a family $(\phi_n)_{n=0}^\infty\subset\mathcal{S}\left(\mathbb{R}^d\right)$ such that

- $\operatorname{supp}\phi_0\subset\left\{x\in\mathbb{R}^d:\|x\|_{\mathbb{R}^d}\le 2\right\}$,
- $\operatorname{supp}\phi_n\subset\left\{x\in\mathbb{R}^d:2^{n-1}\le\|x\|_{\mathbb{R}^d}\le 2^{n+1}\right\}$, $n\in\mathbb{N}$,
- $\sup_{x\in\mathbb{R}^d}|D^\alpha\phi_n(x)|\le c_\alpha 2^{-n|\alpha|}$, $n\in\mathbb{N}\cup\{0\}$, $\alpha\in(\mathbb{N}\cup\{0\})^d$,
- $\sum_{n=0}^\infty\phi_n(x)=1$, $x\in\mathbb{R}^d$.

For $s\in\mathbb{R}$ and $1\le p,q\le\infty$, the Besov space $B_{p,q}^s$ consists of all those $f\in\mathcal{S}'\left(\mathbb{R}^d\right)$ such that

$$\|f\|_{B_{p,q}^s}=\left(\sum_{n=0}^\infty\left(2^{sn}\left\|\mathcal{F}^{-1}\left(\phi_n\mathcal{F}f\right)\right\|_{L_p(\mathbb{R}^d)}\right)^q\right)^{1/q}<\infty.$$

Spaces $B^{s,b}_{p,q}$, where $b \in \mathbb{R}$, are defined similarly but replacing the role of t^s by $t^s(1+|\log t|)^b$ in the above definition. That is,

$$\|f\|_{B^{s,b}_{p,q}} = \left(\sum_{n=0}^{\infty}\left(2^{sn}(1+n)^b\left\|\mathcal{F}^{-1}(\phi_n\mathcal{F}f)\right\|_{L_p(\mathbb{R}^d)}\right)^q\right)^{1/q} < \infty.$$

Spaces $B^{s,b}_{p,q}$ are a special case of Besov spaces of generalised smoothness, which were considered in [16, 28] among other papers. They are of interest in fractal analysis and the related spectral theory (see [82, 83] and the references given there).

Theorem 3.24. *Let* $-\infty < s_1 < s_0 < \infty$, $1 \le p, q \le \infty$ *and* $1/q + 1/q' = 1$. *Then we have with equivalence of norms*

$$\left(B^{s_0}_{p,q}, B^{s_1}_{p,q}\right)_{q;K} = B^{s_1,1/q}_{p,q} \qquad \text{and} \qquad \left(B^{s_0}_{p,q}, B^{s_1}_{p,q}\right)_{q;J} = B^{s_0,-1/q'}_{p,q}.$$

Proof. It is shown in [5, Theorem 6.4.3] and [80, Theorem 2.3.2 a)] that $B^{s_j}_{p,q}$ is a retract of $\ell_q\left(2^{ns_j}L_p\right)$ for $j = 0, 1$. Besides, by Remark 3.1 and [19, Remark 7.3], we derive that

$$\left(\ell_q\left(2^{ns_0}L_p\right), \ell_q\left(2^{ns_1}L_p\right)\right)_{q;K} = \ell_q\left((1+n)^{1/q}2^{ns_1}L_p\right).$$

These two results yield the formula for the limiting K-method. The proof for the J-case has the same structure, but using now [19, Corollary 3.6]. □

We finish this chapter with an application of limiting methods to Fourier coefficients. Let $\Omega = [0, 2\pi]$ with the Lebesgue measure and, given $f \in L_1([0, 2\pi])$, write (c_m) for the sequence of its Fourier coefficients, defined by

$$c_m = \hat{f}(m) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx, \qquad m \in \mathbb{Z}.$$

We designate by (c^*_m) the decreasing rearrangement of the sequence $(|c_m|)$ given by

$$c^*_1 = \max\{|c_m| : m \in \mathbb{Z}\} = |c_{m_1}|, \quad c^*_2 = \max\{|c_m| : m \in \mathbb{Z} \setminus \{m_1\}\} = |c_{m_2}|,$$

and so on.

Theorem 3.25. *If* $f \in L_2(\log L)_{-1/2}$, *then* $\sum_{n=1}^{\infty}(1+\log n)^{-1}\left(c^*_n\right)^2 < \infty$.

Proof. Let $F(f) = \left(\hat{f}(m)\right)$ be the operator assigning to each function f the sequence of its Fourier coefficients. As is well-known, both the restrictions $F : L_2([0, 2\pi]) \longrightarrow \ell_2$ and $F : L_1([0, 2\pi]) \longrightarrow \ell_\infty$ are bounded. Whence, interpolating by the $(2; J)$-method, we obtain that

$$F : \left(L_2([0, 2\pi]), L_1([0, 2\pi])\right)_{2;J} \longrightarrow (\ell_2, \ell_\infty)_{2;J}$$

is also bounded. Now we proceed to identify these spaces. Since

$$L_2([0, 2\pi]) = \left(L_\infty([0, 2\pi]), L_1([0, 2\pi])\right)_{1/2,2},$$

it follows from [19, Theorem 4.6] and (3.26) that

$$\|f\|_{(L_2,L_1)_{2;J}} \sim \left(\int_1^\infty \left[t^{-1/2}(1+\log t)^{-1/2} f^{**}(1/t)\right]^2 \frac{dt}{t}\right)^{1/2} \sim \left(\int_0^{2\pi} \left[t^{1/2}(1+|\log t|)^{-1/2} f^{**}(t)\right]^2 \frac{dt}{t}\right)^{1/2}$$
$$\sim \|f\|_{L_2(\log L)_{-1/2}},$$

where we have used [44, Lemma 3.4.39] in the last equivalence. As for the sequence space, since $K(t,\xi;\ell_1,\ell_\infty) \sim \sum_{j=1}^{[t]} \xi_j^*$ (see [80, page 126]), where $[t]$ is the largest integer less than or equal to t, using again [19, Theorem 4.6], we obtain

$$\|\xi\|_{(\ell_2,\ell_\infty)_{2;J}} \sim \left(\sum_{n=1}^{\infty} \int_n^{n+1} \left[t^{-1/2}(1+\log t)^{-1/2} \sum_{j=1}^{[t]} \xi_j^*\right]^2 \frac{dt}{t}\right)^{1/2} \sim \left(\sum_{n=1}^{\infty} \left[n^{-1/2}(1+\log n)^{-1/2} \sum_{j=1}^{n} \xi_j^*\right]^2 n^{-1}\right)^{1/2}$$
$$\geq \left(\sum_{n=1}^{\infty} (1+\log n)^{-1} (\xi_n^*)^2\right)^{1/2}.$$

This yields the result. □

Other results on Fourier coefficients can be found in [3, 54].

Theorem 3.25 has been extended in [14, Theorem 5.3] to functions f in $L_{2,q}(\log L)_{\gamma+1/\min(2,q)}$ for $0 < q \leq \infty$ and $\gamma < -1/q$. In that case $(\hat{f}(m))$ belongs to the Lorentz-Zygmund sequence space $\ell_{2,q}(\log \ell)_{\gamma+1/\max(2,q)}$.

Chapter 4

Bilinear operators and limiting real methods

In this chapter we study the behaviour of bilinear operators under limiting real methods. Let us state the problem. Take the Banach couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and $\bar{C} = (C_0, C_1)$ and consider a bilinear and continuous operator $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow C_0 + C_1$ whose restrictions $T : A_j \times B_j \longrightarrow C_j$ are bounded with norms M_j for $j = 0, 1$. The question is, if we can interpolate this bilinear operator as we did with linear operators.

In the classical setting, if $1 \le p, q, r \le \infty$ and $1/r + 1 = 1/p + 1/q$, then for every $0 < \theta < 1$ the restriction

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (C_0, C_1)_{\theta,r}$$

is also continuous. This was proved by Lions and Peetre in [67]. By the equivalence theorem, one can interpret these classical real interpolation spaces as K- or J-spaces. In Section 4.1, we show that the bilinear interpolation theorems $J \times J \to J$ and $J \times K \to K$ hold, and that there are no similar results of the type $K \times J \to J$ and $K \times K \to K$. As an application, we establish an interpolation formula for spaces of bounded linear operators. Then, in Section 4.2, we check if the limiting methods preserve the Banach-algebra structure. Finally, in Section 4.3, we compare norm estimates for bilinear operators with norm estimates for linear operators. We establish two results which complement those shown in Chapter 3. The main results of this chapter have appeared in the article [36].

4.1 Interpolation of bilinear operators

It will be useful to work with the following discrete norm in the K-space

$$\|a\|_{q;K} = \left(\sum_{m=-\infty}^{\infty} [\min(1, 2^{-m}) K(2^m, a)]^q \right)^{1/q},$$

which is equivalent to $\|\cdot\|_{\bar{A}_{q;K}}$ by Lemma 3.5. As we mentioned in the previous chapter, a consequence of this discrete representation of $\bar{A}_{q;K}$ is that

$$\bar{A}_{1;K} \hookrightarrow \bar{A}_{q;K} \quad , \quad 1 \leq q \leq \infty \,. \tag{4.1}$$

For the J-space, we work with

$$\|a\|_{q;J} = \inf\left\{\left(\sum_{m=-\infty}^{\infty} [\max(1,2^{-m})\, J(2^m, u_m)]^q\right)^{1/q}\right\},$$

where the infimum is taken over all possible representations $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) with $(u_m) \subset A_0 \cap A_1$ satisfying that

$$\left(\sum_{m=-\infty}^{\infty} [\max(1,2^{-m})\, J(2^m, u_m)]^q\right)^{1/q} < \infty, \tag{4.2}$$

see Lemma 3.8.

Remark 4.1. Note that if $(u_m) \subset A_0 \cap A_1$ satisfies (4.2), then the series is absolutely convergent in $A_0 + A_1$ because

$$\begin{aligned}\sum_{m=-\infty}^{\infty} K(1, u_m) &\leq \sum_{m=-\infty}^{\infty} \min(1,2^{-m})\, J(2^m, u_m)\\ &\leq \left(\sum_{m=-\infty}^{\infty} [\max(1,2^{-m})\, J(2^m, u_m)]^q\right)^{1/q} \times \left(\sum_{m=-\infty}^{\infty} \left[\frac{\min(1,2^{-m})}{\max(1,2^{-m})}\right]^{q'}\right)^{1/q'} < \infty.\end{aligned}$$

As usual $1/q + 1/q' = 1$.

The following two theorems are a consequence of the results of [13] and connections (3.2) and (3.7) between limiting methods and interpolation methods associated to the unit square (see [25, 26]). However, we give here more simple direct proofs.

Theorem 4.1. *Let* $\bar{A} = (A_0, A_1)\,, \bar{B} = (B_0, B_1)$ *and* $\bar{C} = (C_0, C_1)$ *be Banach couples and let* $1 \leq p, q, r \leq \infty$ *with* $1/p + 1/q = 1 + 1/r$. *Suppose that*

$$R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$$

is a bounded bilinear operator whose restrictions to $A_j \times B_j$ *define bounded operators*

$$R : A_j \times B_j \to C_j$$

with norms M_j *(*$j = 0, 1$*). Then the restriction*

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;J} \to (C_0, C_1)_{r;J}$$

is also bounded with norm $M \leq \max(M_0, M_1)$.

Proof. Take any $a \in (A_0, A_1)_{p;J}$ and $b \in (B_0, B_1)_{q;J}$, and consider any arbitrary J-representations $a = \sum_{m=-\infty}^{\infty} a_m$, $b = \sum_{m=-\infty}^{\infty} b_m$. For each $k \in \mathbb{Z}$, put

$$c_k = \sum_{m=-\infty}^{\infty} R(a_m, b_{k-m}).$$

Then $c_k \in C_0 \cap C_1$ because

$$\begin{aligned}\sum_{m=-\infty}^{\infty} J\left(2^k, R(a_m, b_{k-m})\right) &\le \sum_{m=-\infty}^{\infty} \max\left(M_0 \|a_m\|_{A_0} \|b_{k-m}\|_{B_0}, M_1 2^m \|a_m\|_{A_1} 2^{k-m} \|b_{k-m}\|_{B_1}\right) \\ &\le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J\left(2^{k-m}, b_{k-m}\right)\end{aligned}$$

and the last sum is finite as we will show in the course of the next paragraph. Hence, the sequence $(c_k)_{k=-\infty}^{\infty} \subset C_0 \cap C_1$ with

$$J\left(2^k, c_k\right) \le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J\left(2^{k-m}, b_{k-m}\right).$$

Next we show that the series $\sum_{k=-\infty}^{\infty} c_k$ is absolutely convergent in $C_0 + C_1$. According to Remark 4.1, this holds if (c_k) satisfies (4.2). We check this last fact by using Young's inequality. We have

$$\begin{aligned}&\left(\sum_{k=-\infty}^{\infty} \left[\max\left(1, 2^{-k}\right) J\left(2^k, c_k\right)\right]^r\right)^{1/r} \\ &\le \max(M_0, M_1) \left(\sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \max(1, 2^{-m}) J(2^m, a_m) \max\left(1, 2^{-(k-m)}\right) J\left(2^{k-m}, b_{k-m}\right)\right]^r\right)^{1/r} \\ &\le \max(M_0, M_1) \left(\sum_{m=-\infty}^{\infty} \left[\max(1, 2^{-m}) J(2^m, a_m)\right]^p\right)^{1/p} \left(\sum_{k=-\infty}^{\infty} \left[\max\left(1, 2^{-k}\right) J\left(2^k, b_k\right)\right]^q\right)^{1/q} < \infty.\end{aligned} \tag{4.3}$$

These arguments allow also to show that

$$\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K(1, R(a_m, b_{k-m})) < \infty.$$

Indeed, since

$$\begin{aligned}K(1, R(a_m, b_{k-m})) &\le \min(M_0 \|a_m\|_{A_0} \|b_{k-m}\|_{B_0}, 2^{-k} M_1 2^m \|a_m\|_{A_1} 2^{k-m} \|b_{k-m}\|_{B_1}) \\ &\le \max(M_0, M_1) \min(1, 2^{-k}) J(2^m, a_m) J(2^{k-m}, b_{k-m}),\end{aligned}$$

proceeding as in Remark 4.1, we obtain with

$$L = \left(\sum_{k=-\infty}^{\infty} \left[\frac{\min\left(1, 2^{-k}\right)}{\max\left(1, 2^{-k}\right)}\right]^{r'}\right)^{1/r'}$$

that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K(1, R(a_m, b_{k-m})) \\
&\quad \le \max(M_0, M_1) \sum_{k=-\infty}^{\infty} \min(1, 2^{-k}) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}) \\
&\quad \le L \max(M_0, M_1) \left(\sum_{k=-\infty}^{\infty} \left[\max(1, 2^{-k}) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}) \right]^r \right)^{1/r} \\
&\quad \le L \max(M_0, M_1) \left(\sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \max(1, 2^{-m}) J(2^m, a_m) \max\left(1, 2^{-(k-m)}\right) J\left(2^{k-m}, b_{k-m}\right) \right]^r \right)^{1/r}.
\end{aligned}$$

Using now Young's inequality, we get that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K(1, R((a_m, b_{k-m})) &\le L \max(M_0, M_1) \left(\sum_{m=-\infty}^{\infty} [\max(1, 2^{-m}) J(2^m, a_m)]^p \right)^{1/p} \\
&\quad \times \left(\sum_{k=-\infty}^{\infty} \left[\max\left(1, 2^{-k}\right) J\left(2^k, b_k\right)\right]^q \right)^{1/q} < \infty .
\end{aligned}$$

A change in the order of summation in the double series yields that

$$R(a, b) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R(a_m, b_{k-m}) = \sum_{k=-\infty}^{\infty} c_k .$$

Consequently, by (4.3), we derive

$$\begin{aligned}
\|R(a, b)\|_{r;J} &\le \left(\sum_{k=-\infty}^{\infty} \left[\max\left(1, 2^{-k}\right) J\left(2^k, c_k\right)\right]^r \right)^{1/r} \\
&\le \max(M_0, M_1) \left(\sum_{m=-\infty}^{\infty} [\max(1, 2^{-m}) J(2^m, a_m)]^p \right)^{1/p} \\
&\quad \times \left(\sum_{k=-\infty}^{\infty} \left[\max\left(1, 2^{-k}\right) J\left(2^k, b_k\right)\right]^q \right)^{1/q} .
\end{aligned}$$

Now the result follows by taking the infimum over all possible J-representations of a and b. □

Theorem 4.2. *Let* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and* $\bar{C} = (C_0, C_1)$ *be Banach couples and let* $1 \le p, q, r \le \infty$ *with* $1/p + 1/q = 1 + 1/r$. *Assume that*

$$R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$$

is a bounded bilinear operator whose restrictions to $A_j \times B_j$ *define bounded operators*

$$R : A_j \times B_j \to C_j$$

with norms M_j $(j = 0, 1)$. *Then the restriction*

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;K} \to (C_0, C_1)_{r;K}$$

is also bounded with norm $M \le \max(M_0, M_1)$.

Proof. Take any $a \in (A_0, A_1)_{p;J}$ and $b \in (B_0, B_1)_{q;K}$. Let $(\lambda_m)_{m=-\infty}^{\infty}$ be a sequence of positive numbers such that

$$\sum_{m=-\infty}^{\infty} \min(1, 2^{-m})^q \lambda_m^q = 1,$$

and let $\varepsilon > 0$. For each $m \in \mathbb{Z}$ choose a representation $b = b_0^{(m)} + b_1^{(m)}$ of b in $B_0 + B_1$ such that

$$\left\| b_0^{(m)} \right\|_{B_0} + 2^m \left\| b_1^{(m)} \right\|_{B_1} \le K(2^m, b) + \varepsilon \lambda_m.$$

Pick any J-representation $a = \sum_{m=-\infty}^{\infty} a_m$ of a. Then, for each $k \in \mathbb{Z}$, we have that

$$\begin{aligned}
K\left(2^k, R(a,b)\right) &\le \sum_{m=-\infty}^{\infty} K\left(2^k, R(a_m, b)\right) \\
&\le \sum_{m=-\infty}^{\infty} \left[K\left(2^k, R(a_m, b_0^{(k-m)})\right) + K\left(2^k, R(a_m, b_1^{(k-m)})\right) \right] \\
&\le \sum_{m=-\infty}^{\infty} \left[M_0 \|a_m\|_{A_0} \left\| b_0^{(k-m)} \right\|_{B_0} + 2^k M_1 \|a_m\|_{A_1} \left\| b_1^{(k-m)} \right\|_{B_1} \right] \\
&\le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) \left[\left\| b_0^{(k-m)} \right\|_{B_0} + 2^{k-m} \left\| b_1^{(k-m)} \right\|_{B_1} \right] \\
&\le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) \left(K\left(2^{k-m}, b\right) + \varepsilon \lambda_{k-m} \right).
\end{aligned}$$

Therefore, by Young's inequality, we derive

$$\begin{aligned}
\|R(a,b)\|_{r;K} &= \left(\sum_{k=-\infty}^{\infty} \left[\min\left(1, 2^{-k}\right) K\left(2^k, R(a,b)\right) \right]^r \right)^{1/r} \\
&\le \max(M_0, M_1) \left(\sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \max(1, 2^{-m}) J(2^m, a_m) \right.\right. \\
&\qquad\qquad \left.\left. \times \min\left(1, 2^{-(k-m)}\right) \left(K\left(2^{k-m}, b\right) + \varepsilon \lambda_{k-m} \right) \right]^r \right)^{1/r} \\
&\le \max(M_0, M_1) \left(\sum_{m=-\infty}^{\infty} \left[\max(1, 2^{-m}) J(2^m, a_m) \right]^p \right)^{1/p} \\
&\quad \times \left(\sum_{m=-\infty}^{\infty} \left[\min(1, 2^{-m}) \left(K(2^m, b) + \varepsilon \lambda_m \right) \right]^q \right)^{1/q} \\
&\le \max(M_0, M_1) \left(\sum_{m=-\infty}^{\infty} \left[\max(1, 2^{-m}) J(2^m, a_m) \right]^p \right)^{1/p} \left(\|b\|_{q;K} + \varepsilon \right).
\end{aligned}$$

Taking the infimum over all J-representations of a and letting ε go to 0, we get that

$$\|R(a,b)\|_{r;K} \le \max(M_0, M_1) \|a\|_{p;J} \|b\|_{q;K},$$

as desired. □

Remark 4.2. In applications, there are times when one is only given a continuous bilinear operator $R : (A_0 + A_1) \times (B_0 \cap B_1) \to C_0 + C_1$ whose restrictions $R : A_j \times \left(B_0 \cap B_1, \|\cdot\|_{B_j}\right) \to C_j$ are bounded for $j = 0, 1$, and where the couple $\bar{B}$ satisfies that $B_0 \cap B_1$ is dense in B_j for $j = 0, 1$. The question is to show that R has a bounded extension to the interpolation spaces. This means, for the case of Theorem 4.2, an extension from $\bar{A}_{p;J} \times \bar{B}_{q;K}$ into $\bar{C}_{r;K}$.

This problem has a positive answer provided that $q < \infty$. Namely, if $b \in B_0 \cap B_1$, the argument in the proof of Theorem 4.2 gives that

$$\|R(a, b)\|_{r;K} \leq \max(M_0, M_1) \|a\|_{p;J} \|b\|_{q;K}.$$

Since $B_0 \cap B_1$ is dense in $\bar{B}_{q;K}$ when $q < \infty$ (see Corollary 3.15), the bounded extension is possible.

Next we show an application of this remark to interpolation of operator spaces.

Theorem 4.3. *Let* $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ *be Banach couples with* $A_0 \cap A_1$ *dense in* A_j *for* $j = 0, 1$. *Assume that* $1 \leq p, q, r \leq \infty$ *with* $q < \infty$ *and* $1/p + 1/q = 1 + 1/r$. *Then,*

$$(\mathcal{L}(A_0, B_0), \mathcal{L}(A_1, B_1))_{p;J} \subset \mathcal{L}\left(\bar{A}_{q;K}, \bar{B}_{r;K}\right).$$

Proof. Let $R : \mathcal{L}(A_0 \cap A_1, B_0 + B_1) \times (A_0 \cap A_1) \to B_0 + B_1$ be the bounded bilinear operator defined by $R(T, a) = Ta$. It is clear that $R : \mathcal{L}(A_j, B_j) \times \left(A_0 \cap A_1, \|\cdot\|_{A_j}\right) \to B_j$ is also bounded for $j = 0, 1$. Whence, by Remark 4.2, we obtain that R has a bounded extension

$$R : (\mathcal{L}(A_0, B_0), \mathcal{L}(A_1, B_1))_{p;J} \times (A_0, A_1)_{q;K} \to (B_0, B_1)_{r;K}.$$

Therefore, the wanted inclusion follows. □

If we exchange the role of J- and K-methods in Theorem 4.2, then the corresponding statement does not hold as the next example shows.

Counterexample 4.1. Let (A_0, A_1) be a Banach couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. Put $R : (A_0 + A_1) \times (\mathbb{K} + \mathbb{K}) \to A_0 + A_1$ for the bounded bilinear operator defined by $R(a, \lambda) = \lambda a$. It is clear that restrictions $R : A_j \times \mathbb{K} \to A_j$ are bounded for $j = 0, 1$. If the bilinear theorem $K \times J \to J$ were true, then for any $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$ we would have that the restriction $R : (A_0, A_1)_{p;K} \times (\mathbb{K}, \mathbb{K})_{q;J} \to (A_0, A_1)_{r;J}$ is bounded. This yields that $(A_0, A_1)_{p;K} \hookrightarrow (A_0, A_1)_{r;J}$. However, take any $0 < \theta < 1$ and $1 \leq s \leq \infty$. By Lemmata 3.1 and 3.6 we have the inclusions $(A_0, A_1)_{r;J} \hookrightarrow (A_0, A_1)_{\theta,s} \hookrightarrow (A_0, A_1)_{p;K}$. Therefore, we conclude that $(A_0, A_1)_{\theta,s} = (A_0, A_1)_{\mu,s}$ for any $0 < \theta \neq \mu < 1$, which is impossible (see [61, Theorem 3.1]).

Concerning Theorem 4.1, there is no similar result for K-spaces. In order to show this, we establish first an auxiliary result. For $n \in \mathbb{N}$, let ℓ_q^n be the space $\mathbb{K}^n$ with the ℓ_q-norm, and if $\left(\omega_j\right)_{j=1}^n$ is a positive n-tuple, write $\ell_q^n\left(\omega_j\right)$ for the corresponding weighted ℓ_q^n-space. We put $\ell_q^n\left(n^{1/q}\right)$ for the space $\ell_q^n\left(\omega_j\right)$ if $\omega_j = n^{1/q}$ for $1 \leq j \leq n$.

Lemma 4.4. *Let* $n \in \mathbb{N}$ *and* $1 \leq q \leq \infty$. *Then*

$$\ell_1^n\left(j2^{-j}\right) \hookrightarrow \left(\ell_1^n, \ell_1^n\left(2^{-j}\right)\right)_{q;K} \quad \textit{and} \quad \ell_\infty^n\left(n^{1/q}\right) \hookrightarrow \left(\ell_\infty^n, \ell_\infty^n\left(2^j\right)\right)_{q;K},$$

and the norms of the embeddings can be bounded from above with constants independent of n.

Proof. By Remark 3.1 and [19, Lemma 7.2], we have $\left(\ell_1^n, \ell_1^n\left(2^{-j}\right)\right)_{1;K} = \ell_1^n\left(j2^{-j}\right)$ with equivalence of norms, where the constants of equivalence do not depend on n. Hence equation (4.1) implies that $\ell_1^n\left(j2^{-j}\right) \hookrightarrow \left(\ell_1^n, \ell_1^n\left(2^{-j}\right)\right)_{q;K}$.

To prove the second embedding of the statement, note that by (3.1)

$$\left(\ell_\infty^n, \ell_\infty^n\left(2^j\right)\right)_{q;K} = \left(\ell_\infty^n\left(2^j\right), \ell_\infty^n\right)_{q;K}$$

and that $K(t, \xi; \ell_\infty^n(2^j), \ell_\infty^n) = \max_{1\le j\le n} \min(2^j, t)|\xi_j|$. Hence, using again Remark 3.1, we obtain

$$\begin{aligned}\|\xi\|^q_{(\ell_\infty^n, \ell_\infty^n(2^j))_{q;K}} \sim \sum_{m=1}^{\infty} 2^{-mq} \max_{1\le j\le n} \min(2^{jq}, 2^{mq})|\xi_j|^q &= \sum_{m=1}^{n} \max_{1\le j\le n} \min(2^{(j-m)q}, 1)|\xi_j|^q \\ &+ \sum_{m=n+1}^{\infty} 2^{-mq} \max_{1\le j\le n} \min(2^{jq}, 2^{mq})|\xi_j|^q = S_1 + S_2,\end{aligned}$$

where the constants in the equivalence do not depend on n.

Next we estimate S_2. Let $k \le n$; we obtain

$$\begin{aligned}S_2 &= \sum_{m=n+1}^{\infty} 2^{-mq} \max_{1\le j\le n} 2^{jq}|\xi_j|^q = \frac{2^{-(n+1)q}}{1-2^{-q}} \max_{1\le j\le n} 2^{jq}|\xi_j|^q \sim \max_{1\le j\le n} 2^{(j-n)q}|\xi_j|^q \\ &\le \max_{1\le j\le n} \min(1, 2^{(j-k)q})|\xi_j|^q \le S_1.\end{aligned}$$

Consequently,

$$\|\xi\|^q_{(\ell_\infty^n, \ell_\infty^n(2^j))_{q;K}} \sim \sum_{m=1}^{n} \max_{1\le j\le n} \min(2^{(j-m)q}, 1)|\xi_j|^q \le \sum_{m=1}^{n} \max_{1\le j\le n} |\xi_j|^q = \max_{1\le j\le n} n|\xi_j|^q = \|\xi\|^q_{\ell_\infty(n^{1/q})}. \qquad \square$$

Counterexample 4.2. Take any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$, consider the couples $\bar{A} = \left(\ell_1^n, \ell_1^n\left(2^{-j}\right)\right)$, $\bar{B} = \left(\ell_\infty^n, \ell_\infty^n\left(2^j\right)\right)$, $\bar{C} = (\mathbb{K}, \mathbb{K})$, and let R be the bilinear operator defined by $R\left(\left(\xi_j\right), \left(\eta_j\right)\right) = \sum_{j=1}^n \xi_j\eta_j$. It is easy to check that $R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$ is bounded, and the restrictions $R : A_j \times B_j \to C_j$ are also bounded, with norm 1 for $j = 0, 1$. If the bilinear theorem $K \times K \to K$ were true, using Lemma 4.4 there would be some $M < \infty$ such that

$$\|R : \ell_1^n(j2^{-j}) \times \ell_\infty^n(n^{1/q}) \to \mathbb{K}\| \le M$$

for every $n \in \mathbb{N}$. Take $\xi = (0, \ldots, 0, 2^n/n)$ and $\eta = (0, \ldots, 0, n^{-1/q})$. Then we have $\|\xi\|_{\ell_1^n(j2^{-j})} = 1$, $\|\eta\|_{\ell_\infty^n(n^{1/q})} = 1$ and $R(\xi, \eta) = 2^n/n^{1+1/q}$. It follows that $2^n/n^{1+1/q} \le M$ for every $n \in \mathbb{N}$ which is impossible.

4.2 Interpolation of Banach algebras

Interpolation of Banach algebras was first considered in 1963 by Bishop [6] to study questions of analytic continuation. One year later Calderón published his seminal paper on the complex method [10]; among many other results, he showed that the Banach-algebra structure is stable under complex interpolation. As can be seen in the articles by Zafran [85] and Kaijser [62], the more general methods $(\cdot,\cdot)_{\theta,g,1}$ (defined in Chapter 2) also interpolate Banach algebras. Later on, Blanco, Kaijser and Ransford [7] showed another class of real interpolation methods that also preserve this structure, but the question of whether the classical real method $(\cdot,\cdot)_{\theta,q}$ interpolates Banach algebras for $q > 1$ still remained open. However, in 2006, Cobos, Fernández-Cabrera and Martínez [21], working with the general real method $(\cdot,\cdot)_\Gamma$, showed a necessary and sufficient condition on the lattice norm Γ for the general real method $(\cdot,\cdot)_\Gamma$ to interpolate Banach algebras. As a consequence of the result on the method $(\cdot,\cdot)_\Gamma$, they derived that the real method $(\cdot,\cdot)_{\theta,q}$ respects the Banach-algebra structure only if $q = 1$.

First we give some definitions and results that we shall need. They all appear in [21] and [18]. A *Banach algebra* A is an algebra which is also a Banach space and for which there exists a constant $c_A > 0$ such that for all $a, b \in A$ we have

$$\|ab\|_A \le c_A \|a\|_A \|b\|_A .$$

On the other hand, a *couple of Banach algebras* $\bar{A} = (A_0, A_1)$ is a Banach couple consisting of two Banach algebras A_0, A_1 such that the two multiplications agree on $A_0 \cap A_1$.

Now, let Γ be a Banach space of real-valued sequences with $\mathbb{Z}$ as its index space. Assume that Γ is a *lattice*, that is, whenever $|\xi_m| \le |\mu_m|$ for each $m \in \mathbb{Z}$ and $(\mu_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and also $\|(\xi_m)\|_\Gamma \le \|(\mu_m)\|_\Gamma$. Suppose further that Γ contains all sequences with only finitely many non-zero coordinates.

We say that Γ is K-*non-trivial* if $(\min(1, 2^m)) \in \Gamma$ and we say that it is J-*non-trivial* if

$$\sup\left\{\sum_{m=-\infty}^{\infty} \min(1, 2^{-m})\,|\xi_m| : \|(\xi_m)\|_\Gamma \le 1\right\} < \infty.$$

If Γ is a K-non-trivial sequence space, then given any Banach couple $\bar{A}$, we can define the *abstract K-space* $\bar{A}_{\Gamma,K}$ as the space consisting of all $a \in A_0 + A_1$ such that $(K(2^m, a)) \in \Gamma$ endowed with the norm $\|a\|_{\Gamma,K} := \|(K(2^m, a))\|_\Gamma$.

Similarly, if Γ is J-non-trivial, then we can define the *abstract* J-*space* $\bar{A}_{\Gamma,J}$ as the one consisting of all $a \in A_0 + A_1$ that may be written as $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), with $(u_m) \subset A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. We set the following norm on this space

$$\|a\|_{\Gamma,J} := \inf\left\{\|(J(2^m, u_m))\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m\right\}.$$

It is shown in [71] that if Γ is K-non-trivial then $(\ell_1, \ell_1(2^{-m}))_{\Gamma,K} \hookrightarrow \Gamma$ and that, if it is J-non-trivial, then $\Gamma \hookrightarrow (\ell_1, \ell_1(2^{-m}))_{\Gamma,J}$.

Let $\mathcal{F}$ be an interpolation method and suppose that $A_0 \cap A_1$ is dense in $\mathcal{F}(A_0, A_1)$ for any Banach couple (A_0, A_1). We say that the interpolation method $\mathcal{F}$ *preserves the Banach-algebra structure* if given any couple of Banach algebras $\bar{A}$ there exists a constant $c_{\mathcal{F}(\bar{A})} > 0$ such that

$$\|ab\|_{\mathcal{F}(\bar{A})} \le c_{\mathcal{F}(\bar{A})} \|a\|_{\mathcal{F}(\bar{A})} \|b\|_{\mathcal{F}(\bar{A})}$$

for all $a, b \in A_0 \cap A_1$. Since the intersection $A_0 \cap A_1$ is dense in $\mathcal{F}(A_0, A_1)$, we can extend multiplication by continuity to the whole space $\mathcal{F}(A_0, A_1)$, making of this space a Banach algebra.

The limiting K- and J-methods we introduced in Chapter 3 correspond to the abstract K-method with $\Gamma = \ell_q(\min(1, 2^{-m}))$ and to the abstract J-method with $\Gamma = \ell_q(\max(1, 2^m))$. They do not satisfy the hypotheses in the article by Cobos, Fernández-Cabrera and Martínez [21] or those in [18]. In these papers the authors suppose that the Calderón transform

$$\Omega(\xi_m) = \left(\sum_{k=-\infty}^{\infty} \min(1, 2^{m-k}) \, |\xi_k|\right)_{m \in \mathbb{Z}}$$

is bounded in Γ, where Γ is the lattice associated to the limiting methods. By [71, Lemma 2.5], this implies that $\bar{A}_{\Gamma;J} = \bar{A}_{\Gamma;K}$, which, as we already know, is not our case.

In order to study the interpolation of Banach algebras by limiting K- and J-methods, we will work with the couple of Banach algebras $(\ell_1, \ell_1(2^{-m}))$, where the spaces are indexed by $\mathbb{Z}$ and multiplication is defined as convolution. First we show the following lemma.

Lemma 4.5. *Let $(F_m)_{m \in \mathbb{Z}}$ be a sequence of spaces, let $\lambda > 0$, $\lambda \neq 1$ and let $1 \leq q \leq \infty$. Then*

$$\left(\ell_q(F_m), \ell_q(\lambda^{-m} F_m)\right)_{q;K} = \ell_q\left((1 + |m|)^{1/q} \min(1, \lambda^{-m}) \, F_m\right)$$

with equivalent norms.

Proof. Note that it is sufficient to prove the result for $\lambda > 1$, since if $\lambda < 1$ we can take $G_m = \lambda^{-m} F_m$ and obtain

$$\left(\ell_q(F_m), \ell_q(\lambda^{-m} F_m)\right)_{q;K} = \left(\ell_q\left((\lambda^{-1})^{-m} G_m\right), \ell_q(G_m)\right)_{q;K} = \left(\ell_q(G_m), \ell_q\left((\lambda^{-1})^{-m} G_m\right)\right)_{q;K}.$$

Suppose first that $q < \infty$. Let $t > 0$ and choose any $a = (a_n) \in \ell_q(F_m) + \ell_q(\lambda^{-m} F_m)$. Pick the largest $n \in \mathbb{Z}$ such that $t\lambda^{-n} > 1$, and consider the decomposition $a = \bar{a}_0 + \bar{a}_1$, where we have taken $\bar{a}_0 = (\ldots, a_{n-1}, a_n, 0, 0, \ldots)$ and $\bar{a}_1 = (\ldots, 0, 0, a_{n+1}, a_{n+2}, \ldots)$. Then $\bar{a}_0 \in \ell_q(F_m)$ and $\bar{a}_1 \in \ell_q(\lambda^{-m} F_m)$, and

$$K(t, a) \lesssim \left(\sum_{m=-\infty}^{n} \|a_m\|_{F_m}^q + \sum_{m=n+1}^{\infty} t^q \lambda^{-mq} \|a_m\|_{F_m}^q\right)^{1/q} = \left(\sum_{m=-\infty}^{\infty} \min(1, t\lambda^{-m})^q \, \|a_m\|_{F_m}^q\right)^{1/q}.$$

On the other hand, given any decomposition $a = \bar{a}_0 + \bar{a}_1$, where $\bar{a}_0 = (a_m^0)$ and $\bar{a}_1 = (a_m^1)$, we have that

$$\left(\sum_{m=-\infty}^{\infty} \min(1, t\lambda^{-m})^q \, \|a_m\|_{F_m}^q\right)^{1/q} \leq \left(\sum_{m=-\infty}^{\infty} \|a_m^0\|_{F_m}^q\right)^{1/q} + t\left(\sum_{m=-\infty}^{\infty} \lambda^{-mq} \|a_m^1\|_{F_m}^q\right)^{1/q},$$

so

$$K(t, a; \ell_q(F_m), \ell_q(\lambda^{-m} F_m)) \sim \left(\sum_{m=-\infty}^{\infty} \min(1, t\lambda^{-m})^q \, \|a_m\|_{F_m}^q\right)^{1/q}. \tag{4.4}$$

This implies that

$$\begin{aligned}
\|a\|_{(\ell_q(F_m),\ell_q(\lambda^{-m}F_m))_{q;K}} &\sim \left(\sum_{m=-\infty}^{\infty} \min(1,\lambda^{-mq}) \sum_{n=-\infty}^{\infty} \min(1,\lambda^{m-n})^q \|a_n\|_{F_n}^q\right)^{1/q} \\
&= \left(\sum_{n=-\infty}^{0} \|a_n\|_{F_n}^q \left[\sum_{m=-\infty}^{n} \lambda^{(m-n)q} + \sum_{m=n+1}^{0} 1 + \sum_{m=1}^{\infty} \lambda^{-mq}\right]\right. \\
&\quad \left. + \sum_{n=1}^{\infty} \|a_n\|_{F_n}^q \left[\sum_{m=-\infty}^{0} \lambda^{(m-n)q} + \sum_{m=1}^{n} \lambda^{-nq} + \sum_{m=n+1}^{\infty} \lambda^{-mq}\right]\right)^{1/q} \\
&\sim \left(\sum_{n=-\infty}^{0} \|a_n\|_{F_n}^q (1+|n|) + \sum_{n=1}^{\infty} \lambda^{-nq} \|a_n\|_{F_n}^q (1+|n|)\right)^{1/q} \\
&= \|a\|_{\ell_q\left((1+|m|)^{1/q}\min(1,\lambda^{-m})F_m\right)},
\end{aligned}$$

as desired. If $q = \infty$, the result follows easily. □

The following proposition shows that the K-method does not preserve the Banach-algebra structure. We take $q < \infty$ in order to make sure that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{q;K}$

Proposition 4.6. *Let $1 \le q < \infty$. Then $(\ell_1, \ell_1(2^{-m}))_{q;K}$ is not a Banach algebra if multiplication is defined as convolution.*

Proof. Let $1 < q < \infty$. Then $(\ell_1, \ell_1(2^{-m}))_{q;K} = (\ell_1, \ell_1(2^{-m}))_{\Gamma,K} = (\ell_1, \ell_1(2^{-m}))_{\widetilde{\Gamma},J}$, where the lattices Γ and $\widetilde{\Gamma}$ are given by $\Gamma = \ell_q(\min(1, 2^{-m}))$ and $\widetilde{\Gamma} = \ell_q(\min(1, 2^{-m})(1+|m|))$. In addition, we have that Γ is K-non trivial and $\widetilde{\Gamma}$ is J-non trivial. Therefore,

$$\ell_q(\min(1,2^{-m})(1+|m|)) \hookrightarrow \left(\ell_1, \ell_1(2^{-m})\right)_{q;K} \hookrightarrow \ell_q(\min(1,2^{-m})).$$

In order to show that the space $(\ell_1, \ell_1(2^{-m}))_{q;K}$ is not a Banach algebra we will find two vectors $(a_m), (b_m) \in \ell_q(\min(1,2^{-m})(1+|m|))$ such that $(a_m) * (b_m) \notin \ell_q(\min(1,2^{-m}))$.

Take

$$a_m = \begin{cases} (1+|m|)^{\alpha-2} & \text{if } m < 0, \\ 0 & \text{if } m \ge 0, \end{cases} \quad \text{and} \quad b_m = \begin{cases} 0 & \text{if } m < 0, \\ 2^m(1+|m|)^{\alpha-2} & \text{if } m \ge 0, \end{cases}$$

with $0 < \alpha < 1/q'$. Then, since $(1-\alpha)q > (1-1/q')q = 1$, we have

$$\|(a_m)\|_{\widetilde{\Gamma}} = \left(\sum_{m=-\infty}^{-1} \frac{1}{(1+|m|)^{(1-\alpha)q}}\right)^{1/q} < \infty.$$

Moreover, it is clear that

$$\|(b_m)\|_{\widetilde{\Gamma}} = \left(\sum_{m=0}^{\infty} \frac{2^{-mq}2^{mq}}{(1+|m|)^{(1-\alpha)q}}\right)^{1/q} < \infty,$$

so $(a_m), (b_m) \in \widetilde{\Gamma}$. However,

$$\sum_{k=-\infty}^{\infty} a_k b_{m-k} = \sum_{k=-\infty}^{-1} (1+|k|)^{\alpha-2} b_{m-k} = \sum_{k=-\infty}^{\min(-1,m)} (1+|k|)^{\alpha-2} 2^{m-k} (1+|m-k|)^{\alpha-2},$$

and

$$\left\| \left(\sum_{k=-\infty}^{\infty} a_k b_{m-k} \right)_m \right\|_{\Gamma} \geq \left(\sum_{m=-\infty}^{-1} \left(\sum_{k=-\infty}^{m} 2^{m-k} (1+|k|)^{\alpha-2} (1+|m-k|)^{\alpha-2} \right)^q \right)^{1/q} = \infty.$$

Next, we study the case $q = 1$. By Lemma 4.5, we have that

$$(\ell_1, \ell_1(2^{-m}))_{1;K} = \ell_1 \left(\min(1, 2^{-m})(1+|m|)\right).$$

Now, this space is a Banach algebra if and only if the following supremum is finite

$$\sup_{m,n\in\mathbb{Z}} \frac{\min(1,2^{-m})(|m|+1)}{\min(1,2^{-n})(1+|n|)\min(1,2^{n-m})(1+|n-m|)}$$

(see [7, Proposition 2.3]). But, if $n < 0 < m$, the quotient is

$$\frac{2^{-m}(|m|+1)}{(|n|+1)2^{n-m}(|m-n|+1)} \xrightarrow[n\to-\infty]{} \infty.$$

This ends the proof. $\square$

Next we show that the limiting J-method does not preserve the Banach-algebra structure either. This time the argument will be slightly different to the one that we have used for the K-method. For $1 \leq q \leq \infty$, we have that $(\ell_1, \ell_1(2^{-m}))_{q;J} = (\ell_1, \ell_1(2^{-m}))_{\Delta;J}$, where $\Delta = \ell_q(\max(1, 2^{-m}))$. In addition, Δ is J-non trivial. This implies that

$$\ell_q(\max(1,2^{-m})) \hookrightarrow (\ell_1, \ell_1(2^{-m}))_{q;J}.$$

Instead of finding a larger space, we shall characterise the limiting J-space. Once again, we will take $q < \infty$ to make sure that $A_0 \cap A_1$ is dense in $\bar{A}_{q;J}$.

Applying (4.4) to the Banach couple $(\ell_1, \ell_1(2^{-m}))$ we get that

$$K\left(2^m, a; \ell_1, \ell_1(2^{-k})\right) \sim \sum_{k=-\infty}^{\infty} \min\left(1, 2^{m-k}\right) |a_k|.$$

This formula and (3.8) give that

$$(\ell_1, \ell_1(2^{-m}))_{q;J} = \left\{ (a_m) \in \ell_1 + \ell_1(2^{-m}) : \left(\sum_{k=-\infty}^{\infty} \min\left(1, 2^{m-k}\right) |a_k| \right) \in \ell_q \left(\max\left(1, 2^{-m}\right) (1+|m|)^{-1} \right) \right\} \tag{4.5}$$

for $q > 1$.

Proposition 4.7. *Let $1 < q < \infty$. Then $(\ell_1, \ell_1(2^{-m}))_{q;J}$ is not a Banach algebra if multiplication is defined as convolution.*

Proof. Since $\ell_q(\max(1,2^{-m})) \hookrightarrow (\ell_1,\ell_1(2^{-m}))_{q;J}$, we will have the result if we find two vectors $(a_m),(b_m) \in \ell_q(\max(1,2^{-m}))$ such that $(a_m)*(b_m) \notin (\ell_1,\ell_1(2^{-m}))_{q;J}$.

Take $(a_m) = (b_m) = (\min(1,2^m)(1+|m|)^{-\alpha})$, where $\frac{1}{q} < \alpha < \frac{1}{2}\left(1+\frac{1}{q}\right)$. Note that such α exists because $q > 1$. Then, since $\alpha q > 1$, we have that

$$\|(a_m)\|_{\ell_q(\max(1,2^{-m}))} = \left(\sum_{m=-\infty}^{\infty}(1+|m|)^{-\alpha q}\right)^{1/q} < \infty.$$

On the other hand, if $m > 0$, then

$$\begin{aligned}\sum_{k=-\infty}^{\infty} a_k a_{m-k} &= \sum_{k=-\infty}^{\infty}\min(1,2^k)(1+|k|)^{-\alpha}\min(1,2^{m-k})(1+|m-k|)^{-\alpha}\\ &\geq \sum_{k=0}^{m}(1+k)^{-\alpha}(1+m-k)^{-\alpha} \geq (1+m)^{-\alpha}(1+m)^{-\alpha}(1+m) = (1+m)^{1-2\alpha},\end{aligned}$$

and, similarly, if $m \leq 0$,

$$\sum_{k=-\infty}^{\infty} a_k a_{m-k} \geq \sum_{k=m}^{0} 2^m(1+|k|)^{-\alpha}(1+|m-k|)^{-\alpha} = 2^m\sum_{j=0}^{|m|}(1+j)^{-\alpha}(1+|m|-j)^{-\alpha} \geq 2^m(1+|m|)^{1-2\alpha}.$$

Thus $(a*a)_m \geq \min(1,2^m)(1+|m|)^{1-2\alpha}$. Next we show that $a*a$ does not satisfy condition (4.5). If $m > 0$, we have that

$$\sum_{k=-\infty}^{\infty}\min\left(1,2^{m-k}\right)\min(1,2^k)(1+|k|)^{1-2\alpha} \geq \sum_{k=\left[\frac{m}{2}\right]}^{m}(1+|k|)^{1-2\alpha} \gtrsim (1+m)^{2-2\alpha}$$

and similarly, if $m \leq 0$,

$$\sum_{k=-\infty}^{\infty}\min\left(1,2^{m-k}\right)\min(1,2^k)(1+|k|)^{1-2\alpha} \geq \sum_{k=m}^{\left[\frac{m}{2}\right]} 2^m(1+|k|)^{1-2\alpha} \gtrsim 2^m(1+|m|)^{2-2\alpha}.$$

Therefore,

$$\sum_{k=-\infty}^{\infty}\min\left(1,2^{m-k}\right)|(a*a)_k| \geq \min(1,2^m)(1+|m|)^{2-2\alpha},$$

and so

$$\left\|\sum_{k=-\infty}^{\infty}\min\left(1,2^{m-k}\right)|(a*a)_k|\right\|_{\ell_q(\max(1,2^{-m})(1+|m|)^{-1})} \geq \left(\sum_{m=-\infty}^{\infty}(1+|m|)^{(1-2\alpha)q}\right)^{1/q},$$

which is divergent since $(1-2\alpha)q > -1$. This ends the proof. □

4.3 Norm estimates

In this final section we compare norm estimates for bilinear operators with the norms of linear operators interpolated by the limiting methods. We start with an auxiliary result.

Lemma 4.8. *Let* $\bar{E} = (\mathbb{K}, \mathbb{K})$. *Then,* $\bar{E}_{1;J} = \mathbb{K}$ *and* $\|\cdot\|_{1;J}$ *coincides with* $|\cdot|$.

Proof. If $\lambda \in \mathbb{K}$, we can take the representation $\lambda = \sum_{m=-\infty}^{\infty} \nu_m$ with $\nu_m = 0$ for $m \neq 0$ and $\nu_0 = \lambda$. It follows that $\|\lambda\|_{1;J} \leq |\lambda|$. Conversely, given any J-representation $\lambda = \sum_{m=-\infty}^{\infty} \lambda_m$ of λ, we have

$$|\lambda| \leq \sum_{m=-\infty}^{\infty} |\lambda_m| \leq \sum_{m=-\infty}^{\infty} \max\left(1, 2^{-m}\right) J\left(2^m, \lambda_m\right).$$

Hence, $|\lambda| \leq \|\lambda\|_{1;J}$. □

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$. Put $\bar{E} = (\mathbb{K}, \mathbb{K})$, and define the bilinear operator R by $R(\lambda, a) = \lambda T a$. The operator R is bounded from $(\mathbb{K} + \mathbb{K}) \times (A_0 + A_1)$ into $B_0 + B_1$, and the restrictions $R : \mathbb{K} \times A_j \to B_j$ are also bounded. It follows from Lemma 4.8 that for any $1 \leq q \leq \infty$ we have that

$$\|T\|_{\mathcal{L}\left(\bar{A}_{q;K}, \bar{B}_{q;K}\right)} = \left\|R : \bar{E}_{1;J} \times \bar{A}_{q;K} \to \bar{B}_{q;K}\right\|.$$

Whence, norm estimates for interpolated bilinear operators cannot be better than the corresponding estimates for interpolated linear operators.

Recall that we showed in Counterexample 3.1 that even the weaker estimate

$$\|T\|_{\bar{A}_{q;K}, \bar{B}_{q;K}} \lesssim \|T\|_{A_1, B_1} \left[1 + \max\left\{0, \log \frac{\|T\|_{A_1, B_1}}{\|T\|_{A_0, B_0}}\right\}\right],$$

shown in [19, Theorem 7.9] fails for general couples. Next we establish two results which complement those that appear in Chapter 3 and illustrate the poor norm estimates that are fulfilled for the limiting methods. Subsequently, we work with the continuous norm $\|\cdot\|_{\bar{A}_{q;K}}$ of the limiting K-space.

Proposition 4.9. *For any* $s, t > 0$, *there exist Banach couples* $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and an operator* $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ *such that* $\|T\|_{A_0, B_0} = s$, $\|T\|_{A_1, B_1} = t$ *and*

$$\|T\|_{\bar{A}_{\infty;K}, \bar{B}_{\infty;K}} = \max(s, t).$$

Proof. Let $B_0 = B_1 = \mathbb{K}$ with the usual norm $|\cdot|$. Take $A_0 = A_1 = \mathbb{K}$ normed with $\|\lambda\|_{A_0} = s^{-1}|\lambda|$ and $\|\lambda\|_{A_1} = t^{-1}|\lambda|$, respectively, and put $T\lambda = \lambda$. It is clear that $\|T\|_{A_0, B_0} = s$ and $\|T\|_{A_1, B_1} = t$. Since

$$\|\lambda\|_{\bar{B}_{\infty;K}} = 2K\left(1, \lambda; B_0, B_1\right) = 2|\lambda|$$

and

$$\|\lambda\|_{\bar{A}_{\infty;K}} = 2K\left(1, \lambda; A_0, A_1\right) = 2\min\left(s^{-1}, t^{-1}\right)|\lambda|,$$

we derive that

$$\|T\|_{\bar{A}_{\infty;K}, \bar{B}_{\infty;K}} = \max(s, t),$$

as desired. □

We close the chapter with the case $q < \infty$.

Theorem 4.10. *Let* $1 \le q < \infty$. *Then*

$$\sup\left\{\|T\|_{\bar{A}_{q;K},\bar{B}_{q;K}} : \|T\|_{A_0,B_0} \le s, \|T\|_{A_1,B_1} \le t\right\} \sim \max(s,t),$$

where the supremum is taken over all Banach pairs $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ *and all* $T \in \mathcal{L}\left(\bar{A}, \bar{B}\right)$ *satisfying the stated conditions.*

Proof. According to [27, Corollary 1.7],

$$\sup\left\{\|T\|_{\bar{A}_{q;K},\bar{B}_{q;K}} : \|T\|_{A_0,B_0} \le s, \|T\|_{A_1,B_1} \le t\right\} \sim sg(t/s),$$

where

$$g(\tau) = \sup_{\alpha\in(0,\infty)} \frac{\left\|\frac{\min(1,\alpha\tau\cdot)}{\max(1,\cdot)}\right\|_{L_q((0,\infty),dt/t)}}{\left\|\frac{\min(1,\alpha\cdot)}{\max(1,\cdot)}\right\|_{L_q((0,\infty),dt/t)}} = \sup_{\alpha\in(0,\infty)} C_{\alpha,\tau}.$$

Let us compute g. We start with the case $1/\tau < 1$. We have that

$$\sup_{\alpha\in(0,\infty)} C_{\alpha,\tau} = \max\left(\sup_{0<\alpha<1/\tau} C_{\alpha,\tau}, \sup_{1/\tau\le\alpha<1} C_{\alpha,\tau}, \sup_{\alpha\ge1} C_{\alpha,\tau}\right).$$

Let $0 < \alpha < 1/\tau$. Then

$$\begin{aligned}\left(\int_0^\infty \left[\frac{\min(1,\alpha\tau t)}{\max(1,t)}\right]^q \frac{dt}{t}\right)^{1/q} &= \left(\int_0^1 (\alpha\tau t)^q \frac{dt}{t} + \int_1^{1/\alpha\tau} (\alpha\tau)^q \frac{dt}{t} + \int_{1/\alpha\tau}^\infty t^{-q}\frac{dt}{t}\right)^{1/q}\\ &= (2/q - \log(\alpha\tau))^{1/q}\,\alpha\tau,\end{aligned}$$

and

$$\left(\int_0^\infty \left[\frac{\min(1,\alpha t)}{\max(1,t)}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^1 (\alpha t)^q \frac{dt}{t} + \int_1^{1/\alpha} \alpha^q \frac{dt}{t} + \int_{1/\alpha}^\infty t^{-q}\frac{dt}{t}\right)^{1/q} = (2/q - \log(\alpha))^{1/q}\,\alpha,$$

so

$$\begin{aligned}\sup_{0<\alpha<1/\tau} C_{\alpha,\tau} &= \sup_{0<\alpha<1/\tau} \tau\left[\frac{2/q - \log(\alpha\tau)}{2/q - \log\alpha}\right]^{1/q} = \sup_{0<\alpha<1/\tau} \tau\left[\frac{2/q - \log\alpha - \log\tau}{2/q - \log\alpha}\right]^{1/q}\\ &= \sup_{0<\alpha<1/\tau} \tau\left[1 - \frac{\log\tau}{2/q - \log\alpha}\right]^{1/q} = \tau.\end{aligned}$$

Now, let $1/\tau \le \alpha < 1$. Then

$$\left(\int_0^\infty \left[\frac{\min(1,\alpha\tau t)}{\max(1,t)}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^{1/\alpha\tau} (\alpha\tau t)^q \frac{dt}{t} + \int_{1/\alpha\tau}^1 \frac{dt}{t} + \int_1^\infty t^{-q}\frac{dt}{t}\right)^{1/q} = (2/q + \log(\alpha\tau))^{1/q},$$

so in this case

$$C_{\alpha,\tau} = \left[\frac{2/q + \log\alpha + \log\tau}{\alpha^q(2/q - \log\alpha)}\right]^{1/q}.$$

We have that

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(\alpha,\tau) = 0 \iff \log\alpha = \frac{-q\log\tau \pm \sqrt{q^2\log^2\tau + 4q\log\tau}}{2q}.$$

Since $\log\tau > 0$, one of the roots is positive, and the other one is less than or equal to

$$\frac{-q\log\tau - q\log\tau}{2q} = \log(1/\tau).$$

This implies that the derivative does not change its sign on the interval $1/\tau \le \alpha \le 1$. Since

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(1,\tau) = \frac{1^{q-1}}{(1^q(2/q - \log 1))^2}\log 1/\tau < 0,$$

we derive that $C_{\alpha,\tau}$ is decreasing on $[1/\tau, 1]$, and therefore,

$$\sup_{1/\tau\le\alpha<1} C_{\alpha,\tau} = \tau\left[\frac{2/q - \log\tau + \log\tau}{2/q + \log\tau}\right]^{1/q} = \tau\left[1 - \frac{\log\tau}{2/q + \log\tau}\right]^{1/q}.$$

In the case $\alpha \ge 1$, we have that

$$\left(\int_0^\infty \left[\frac{\min(1,\alpha t)}{\max(1,t)}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^{1/\alpha} (\alpha t)^q \frac{dt}{t} + \int_{1/\alpha}^1 \frac{dt}{t} + \int_1^\infty t^{-q}\frac{dt}{t}\right)^{1/q} = (2/q + \log\alpha)^{1/q},$$

so

$$\sup_{\alpha\ge1} C_{\alpha,\tau} = \sup_{\alpha\ge1}\left[\frac{2/q + \log(\alpha\tau)}{2/q + \log\alpha}\right]^{1/q} = \sup_{\alpha\ge1}\left[1 + \frac{\log\tau}{2/q + \log\alpha}\right]^{1/q} = \left[1 + \frac{q\log\tau}{2}\right]^{1/q}.$$

Therefore,

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\left(\tau, \tau\left[1 - \frac{\log\tau}{2/q + \log\tau}\right]^{1/q}, \left[1 + \frac{q\log\tau}{2}\right]^{1/q}\right).$$

It is easy to check that the second value in the maximum is less than or equal to τ. To compare the last term, put $f(\tau) = 2\tau^q - 2 - q\log\tau = 2\tau^q - 2 - \log\tau^q$ for $\tau \ge 1$. We have that $f(1) = 0$ and that $f'(\tau) = q\tau^{-1}(2\tau^q - 1) > 0$, so $f(\tau) \ge 0$, and therefore $\tau \ge \left[1 + \frac{q\log\tau}{2}\right]^{1/q}$, that is, if $1/\tau < 1$, we have that $g(\tau) = \tau$.

Next consider the case $1/\tau \ge 1$. Then

$$\sup_{\alpha\in(0,\infty)} C_{\alpha,\tau} = \max\left(\sup_{0<\alpha<1} C_{\alpha,\tau}, \sup_{1\le\alpha\le1/\tau} C_{\alpha,\tau}, \sup_{\alpha>1/\tau} C_{\alpha,\tau}\right).$$

If $0 < \alpha < 1$, using what we already have, we get that

$$\sup_{0<\alpha<1} C_{\alpha,\tau} = \sup_{0<\alpha<1} \tau\left[\frac{2/q - \log(\alpha\tau)}{2/q - \log\alpha}\right]^{1/q} = \sup_{0<\alpha<1} \tau\left[1 - \frac{\log\tau}{2/q - \log\alpha}\right]^{1/q} = \tau\left[1 - \frac{q\log\tau}{2}\right]^{1/q}.$$

If $1 \le \alpha \le 1/\tau$, we have that

$$\sup_{1\le\alpha\le 1/\tau} C_{\alpha,\tau} = \sup_{1\le\alpha\le 1/\tau} \alpha\tau \left[\frac{2/q - \log(\alpha\tau)}{2/q + \log\alpha}\right]^{1/q},$$

and since

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(\alpha,\tau) = \frac{(\tau\alpha)^{q-1}\tau}{(2/q+\log\alpha)^2}\left(-q\log\alpha\left(\log\alpha - \log(1/\tau)\right) + \log(1/\tau)\right) > 0,$$

we obtain that

$$\sup_{1\le\alpha\le 1/\tau} C_{\alpha,\tau} = \left[\frac{2/q}{2/q+\log(1/\tau)}\right]^{1/q} = \left[1 + \frac{q\log\tau}{2-q\log\tau}\right]^{1/q}.$$

Finally, if $\alpha > 1/\tau$, it follows that

$$\sup_{\alpha>1/\tau} C_{\alpha,\tau} = \sup_{\alpha>1/\tau}\left[\frac{2/q+\log(\alpha\tau)}{2/q+\log\alpha}\right]^{1/q} = \sup_{\alpha>1/\tau}\left[1+\frac{\log\tau}{2/q+\log\alpha}\right]^{1/q} = 1,$$

and therefore

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\left(\tau\left[1-\frac{q\log\tau}{2}\right]^{1/q}, \left[1+\frac{q\log\tau}{2-q\log\tau}\right]^{1/q}, 1\right).$$

Clearly the second term is less than equal to 1. In order to compare the first one, we consider $h(\tau) = \tau^q\left(1 - \frac{q}{2}\log\tau\right) - 1$ for $0 < \tau \le 1$. Then $h(1) = 0$ and $h'(\tau) = \frac{q}{2}\tau^{q-1}(1 - q\log\tau) > 0$, so $h(\tau) \le 0$ whenever $0 < \tau \le 1$. This yields that

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\left(\tau\left[1-\frac{q\log\tau}{2}\right]^{1/q}, \left[1+\frac{q\log\tau}{2-q\log\tau}\right]^{1/q}, 1\right) = 1.$$

Consequently, $g(\tau) = \max(1,\tau)$, which completes the proof. □

Chapter 5

Some reiteration formulae for limiting real methods

Reiteration is a very important question when one studies any interpolation method. The classical reiteration theorem reads as follows. Let $0 < \theta_0, \theta_1, \theta < 1$ and $1 \le q_0, q_1, q \le \infty$. Then

$$\big((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1}\big)_{\theta,q} = (A_0, A_1)_{\eta,q}\,,$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$. From this formula we derive that reiteration is blind to the parameters q_0 and q_1 in the classical setting. Another consequence is that the classical real method is stable under interpolation, since the resulting space is in the same scale of interpolation spaces. This will no longer be true in the limiting case.

One way to prove this theorem for the real method $(A_0, A_1)_{\theta,q}$ $(0 < \theta < 1)$ is by expressing the K-functional of the couple of real interpolation spaces $((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1})$ in terms of the K-functional of (A_0, A_1) (Holmstedt's formula, [57]).

Gomez and Milman [54, Theorem 3.6] extended Holmstedt's result to limiting real spaces for couples that are ordered by inclusion. They deal with K-spaces that correspond to the choice $\theta = 1$ in the construction of the real method defined in (2.14). Later, Cobos, Fernández-Cabrera, Kühn and Ullrich [19, Theorem 4.6] considered the case of limiting ordered J-spaces, which fits to the choice $\theta = 0$ (see the definition in (2.17)). Our aim in this chapter is to continue the research on limiting real methods by studying their reiteration properties in the case of arbitrary Banach couples, not necessarily ordered. The results in this chapter form the article [37].

We work with the limiting K-spaces $(A_0, A_1)_{q;K}$ and the limiting J-spaces $(A_0, A_1)_{q;J}$ studied in the previous two chapters. We shall use the fact that elements of $(A_0, A_1)_{q;K}$ are characterised by the condition $\min(1, 1/t)K(t, a) \in L_q((0,\infty), dt/t)$ and those of $(A_0, A_1)_{q;J}$ by

$$\max(1, 1/t)(1 + |\log t|)^{-1}K(t, a) \in L_q((0,\infty), dt/t)$$

for $q > 1$, see (3.8). Since $1/t$ appears only in part of the interval $(0, \infty)$, the results of [46, 47, 52, 1] do not cover the cases that we study here. The estimates that we obtain allow to determine explicitly the resulting spaces because they show the weights that appear with the K-functional.

We start by deriving Holmstedt type formulae for the K-functional of couples formed by a limiting space and a space of the original couple. This is done in Section 5.1. Then, in Section 5.2 we derive some reiteration results for limiting methods and finally, in Section 5.3, we apply the results to determine the spaces generated by some couples of function spaces and couples of spaces of operators.

The following formulae for the real method were established by Holmstedt [57] (see also [5, §3.6]): Let $0 < \theta_0 < \theta_1 < 1, \lambda = \theta_1 - \theta_0, 1 \le q_0, q_1 \le \infty$ and put $\bar{X} = \left(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1}\right)$. Then

$$K\left(t, a; \bar{X}\right) \sim \left(\int_0^{t^{1/\lambda}} \left[\frac{K(s, a; \bar{A})}{s^{\theta_0}}\right]^{q_0} \frac{ds}{s}\right)^{1/q_0} + t\left(\int_{t^{1/\lambda}}^{\infty} \left[\frac{K(s, a; \bar{A})}{s^{\theta_1}}\right]^{q_1} \frac{ds}{s}\right)^{1/q_1}. \tag{5.1}$$

Moreover, if $0 < \theta < 1$ and $1 \le q \le \infty$,

$$K\left(t, a; \bar{A}_{\theta,q}, A_1\right) \sim \left(\int_0^{t^{1/1-\theta}} \left[\frac{K(s, a; \bar{A})}{s^{\theta}}\right]^{q} \frac{ds}{s}\right)^{1/q} \tag{5.2}$$

and

$$K\left(t, a; A_0, \bar{A}_{\theta,q}\right) \sim t\left(\int_{t^{1/\theta}}^{\infty} \left[\frac{K(s, a; \bar{A})}{s^{\theta}}\right]^{q} \frac{ds}{s}\right)^{1/q}. \tag{5.3}$$

5.1 Limiting estimates for the K-functional

In this section we extend (5.2) and (5.3) to limiting real spaces. Subsequently, $K(t, a)$ stands for the K-functional of $\bar{A} = (A_0, A_1)$. We write $K\left(t, a; \bar{X}\right) = K\left(t, a; X_0, X_1\right)$ for the K-functional of a couple $\bar{X} = (X_0, X_1)$ different from $\bar{A}$.

We start with the J-spaces and we distinguish the cases $0 < t < 1$ and $1 \le t < \infty$. We shall use that if $0 < \lambda < 1$ then

$$\left(\int_\lambda^1 \left[\frac{1}{s(1 - \log s)}\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \frac{1}{\lambda(1 - \log \lambda)}. \tag{5.4}$$

Indeed, if $1 \le q < \infty$, we have that

$$\begin{aligned}\int_\lambda^1 s^{-q}(1 - \log s)^{-q} \frac{ds}{s} &= \int_\lambda^1 s^{-q+1/2}(1 - \log s)^{-q} s^{-1/2} \frac{ds}{s} \lesssim \lambda^{-q+1/2}(1 - \log \lambda)^{-q} \int_\lambda^1 s^{-1/2} \frac{ds}{s} \\ &\sim \lambda^{-q+1/2}(1 - \log \lambda)^{-q}(\lambda^{-1/2} - 1) \le \lambda^{-q}(1 - \log \lambda)^{-q}.\end{aligned}$$

The case $q = \infty$ is trivial.

Subsequently, the proofs are given for $q < \infty$. The case $q = \infty$ can be carried out similarly.

Lemma 5.1. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple, let* $1 < q \leq \infty$, $1/q + 1/q' = 1$ *and* $0 < t < 1$. *Put* $\bar{X} = \left(\bar{A}_{q;J}, A_1\right)$. *Then, for any* $a \in \bar{A}_{q;J} + A_1$, *we have that*

$$K\left(t, a; \bar{X}\right) \sim \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Proof. First we note that

$$\left(\int_0^{e^{-t^{-q'}}} (1-\log s)^{-q} \frac{ds}{s}\right)^{1/q} \sim (1+t^{-q'})^{-1/q'} \sim [\max(1, t^{-q'})]^{-1/q'} = t. \tag{5.5}$$

Take any decomposition $a = x_0 + a_1$ with $x_0 \in \bar{A}_{q;J}$ and $a_1 \in A_1$. Then, by (5.5),

$$\begin{aligned}
&\left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\qquad \leq \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,x_0)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a_1)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\qquad \leq \|x_0\|_{\bar{A}_{q;J}} + \left(\int_0^{e^{-t^{-q'}}} (1-\log s)^{-q} \frac{ds}{s}\right)^{1/q} \|a_1\|_{A_1} \\
&\qquad \lesssim \|x_0\|_{\bar{A}_{q;J}} + t\,\|a_1\|_{A_1}.
\end{aligned}$$

Taking the infimum over all possible decompositions of a, we derive that

$$K\left(t, a; \bar{X}\right) \gtrsim \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Conversely, according to the definition of $K(t,a)$, we may decompose $a = a_0(t) + a_1(t)$ in a way such that $a_j(t) \in A_j$ and $\|a_0(t)\|_{A_0} + t\,\|a_1(t)\|_{A_1} \leq 2K(t,a)$. We claim that $a_0(e^{-t^{-q'}}) \in \bar{A}_{q;J}$. Indeed,

$$\begin{aligned}
\|a_0(e^{-t^{-q'}})\|_{\bar{A}_{q;J}} &\leq \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a_0(e^{-t^{-q'}}))}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_{e^{-t^{-q'}}}^1 \left[\frac{K(s,a_0(e^{-t^{-q'}}))}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[\frac{K(s,a_0(e^{-t^{-q'}}))}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} = I_1 + I_2 + I_3.
\end{aligned}$$

Let us estimate each term. We have

$$I_1 \leq \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a_1(e^{-t^{-q'}}))}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q}$$

and

$$\left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a_1(e^{-t^{-q'}}))}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}\leq\left(\int_0^{e^{-t^{-q'}}}\left[\frac{\|a_1(e^{-t^{-q'}})\|_{A_1}}{1-\log s}\right]^q\frac{ds}{s}\right)^{1/q}$$
$$\lesssim\frac{K\left(e^{-t^{-q'}},a\right)}{e^{-t^{-q'}}}\left(\int_0^{e^{-t^{-q'}}}(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q}$$
$$\leq\left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a)}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q},$$

where in the last inequality we have used that $K(s,a)/s$ is a decreasing function.

As for I_2, by (5.4) and (5.5), we obtain

$$I_2\lesssim\left\|a_0(e^{-t^{-q'}})\right\|_{A_0}\left(\int_{e^{-t^{-q'}}}^1\left[\frac{1}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}\lesssim K(e^{-t^{-q'}},a)\left[e^{-t^{-q'}}(1+t^{-q'})\right]^{-1}$$
$$\leq\frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}(1+t^{-q'})^{-1/q'}\sim\frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}\left(\int_0^{e^{-t^{-q'}}}(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q}$$
$$\leq\left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a)}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}.$$

In order to estimate the third term, recall that $e^x\geq 1+x$ for $x\geq 0$. This yields that $e^{q't^{-q'}}\geq 1+t^{-q'}$ and so

$$e^{-t^{-q'}}\leq(1+t^{-q'})^{-1/q'}\sim\left(\int_0^{e^{-t^{-q'}}}(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q}.$$

It follows that

$$I_3\lesssim K(e^{-t^{-q'}},a)\left(\int_1^\infty(1+\log s)^{-q}\frac{ds}{s}\right)^{1/q}\sim K(e^{-t^{-q'}},a)=\frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}e^{-t^{-q'}}$$
$$\lesssim\frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}\left(\int_0^{e^{-t^{-q'}}}(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q}\leq\left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a)}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}.$$

Summing the three estimates we conclude that $a_0(e^{-t^{-q'}})$ belongs to $\bar{A}_{q;J}$ with

$$\|a_0(e^{-t^{-q'}})\|_{\bar{A}_{q;J}}\lesssim\left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a)}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}.$$

On the other hand, using (5.5), we get

$$t\|a_1(e^{-t^{-q'}})\|_{A_1} \lesssim \frac{t}{e^{-t^{-q'}}} K(e^{-t^{-q'}}, a) \lesssim \frac{K(e^{-t^{-q'}}, a)}{e^{-t^{-q'}}} \left(\int_0^{e^{-t^{-q'}}} (1-\log s)^{-q} \frac{ds}{s} \right)^{1/q}$$
$$\le \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)} \right]^q \frac{ds}{s} \right)^{1/q}.$$

Consequently,

$$K\left(t, a; \bar{X}\right) \le \|a_0(e^{-t^{-q'}})\|_{\bar{A}_{q;J}} + t\|a_1(e^{-t^{-q'}})\|_{A_1} \lesssim \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)} \right]^q \frac{ds}{s} \right)^{1/q}.$$

This completes the proof. □

In the case $1 \le t < \infty$, the estimate requires the increasing function $\Psi : (0,1] \to (0,1]$ defined by $\Psi(s) = s(1+|\log s|)^{1/q'}$.

Lemma 5.2. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple, let* $1 < q \le \infty, 1/q + 1/q' = 1$ *and* $1 \le t < \infty$. *Put* $\bar{X} = \left(\bar{A}_{q;J}, A_1\right)$. *Then, for any* $a \in \bar{A}_{q;J} + A_1$, *we have that*

$$K\left(t, a; \bar{X}\right) \sim \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)} \right]^q \frac{ds}{s} \right)^{1/q} + \left(\int_1^{\infty} \left[\frac{K\left(\min(1/\Psi^{-1}(t^{-1}), s), a\right)}{1+\log s} \right]^q \frac{ds}{s} \right)^{1/q},$$

where $\Psi(s) = s(1+|\log s|)^{1/q'}$.

Proof. Let

$$Q(t,a) = \left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)} \right]^q \frac{ds}{s} \right)^{1/q} + \left(\int_1^{\infty} \left[\frac{K\left(\min(1/\Psi^{-1}(t^{-1}), s), a\right)}{1+\log s} \right]^q \frac{ds}{s} \right)^{1/q}$$
$$= Q_1(t,a) + Q_2(t,a).$$

Given any decomposition $a = x_0 + a_1$ with $x_0 \in \bar{A}_{q;J}$ and $a_1 \in A_1$, we have

$$Q_2(t,a) \le Q_2(t,x_0) + Q_2(t,a_1) \le \|x_0\|_{\bar{A}_{q;J}} + \left(\int_1^{\infty} \left[\frac{\min(1/\Psi^{-1}(t^{-1}), s)}{1+\log s} \right]^q \frac{ds}{s} \right)^{1/q} \|a_1\|_{A_1}$$
$$= \|x_0\|_{\bar{A}_{q;J}} + L_1 \|a_1\|_{A_1}.$$

According to (5.4), we obtain

$$L_1 \le \left(\int_1^{1/\Psi^{-1}(t^{-1})} \left[\frac{s}{1+\log s} \right]^q \frac{ds}{s} \right)^{1/q} + \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} \left[\frac{1/\Psi^{-1}(t^{-1})}{1+\log s} \right]^q \frac{ds}{s} \right)^{1/q}$$

$$\sim \left(\int_{\Psi^{-1}(t^{-1})}^{1} [s(1-\log s)]^{-q}\,\frac{ds}{s}\right)^{1/q} + \frac{1}{\Psi^{-1}(t^{-1})}\,\frac{1}{(1-\log \Psi^{-1}(t^{-1}))^{1/q'}}$$
$$\lesssim \frac{1}{\Psi^{-1}(t^{-1})(1-\log \Psi^{-1}(t^{-1}))^{1/q'}} = \frac{1}{\Psi(\Psi^{-1}(t^{-1}))} = t.$$

Whence $Q_2(t,a) \lesssim \|x_0\|_{\bar{A}_{q;J}} + t\,\|a_1\|_{A_1}$.

As for $Q_1(t,a)$, we derive

$$Q_1(t,a) \le Q_1(t,x_0) + Q_1(t,a_1) \le \|x_0\|_{\bar{A}_{q;J}} + \left(\int_0^{e^{-t^{-q'}}} (1-\log s)^{-q}\,\frac{ds}{s}\right)^{1/q} \|a_1\|_{A_1} \lesssim \|x_0\|_{\bar{A}_{q;J}} + t\,\|a\|_{A_1}.$$

Therefore, $Q(t,a) = Q_1(t,a) + Q_2(t,a) \lesssim \|x_0\|_{\bar{A}_{q;J}} + t\,\|a_1\|_{A_1}$. Taking the infimum over all possible representations of a, we conclude that $Q(t,a) \lesssim K\left(t,a;\bar{X}\right)$.

To check the converse inequality, we choose two vectors $a_0(t) \in A_0$ and $a_1(t) \in A_1$ such that $a = a_0(t) + a_1(t)$ and $\|a_0(t)\|_{A_0} + t\,\|a_1(t)\|_{A_1} \le 2K(t,a)$. First we show that $a_0(1/\Psi^{-1}(t^{-1}))$ belongs to $\bar{A}_{q;J}$. We have

$$\begin{aligned}\left\|a_0(1/\Psi^{-1}(t^{-1}))\right\|_{\bar{A}_{q;J}} &\le \left(\int_0^1 \left[\frac{K\left(s,a_0(1/\Psi^{-1}(t^{-1}))\right)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} \\ &\quad + \left(\int_1^{1/\Psi^{-1}(t^{-1})} \left[\frac{K\left(s,a_0(1/\Psi^{-1}(t^{-1}))\right)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} \\ &\quad + \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} \left[\frac{K\left(s,a_0(1/\Psi^{-1}(t^{-1}))\right)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

In order to estimate I_1, we use that $a_0(1/\Psi^{-1}(t^{-1})) = a - a_1(1/\Psi^{-1}(t^{-1}))$. We get

$$\begin{aligned}I_1 \le &\left(\int_0^{e^{-t^{-q'}}} \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_{e^{-t^{-q'}}}^1 \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} \\ &+ \left(\int_0^1 \left[\frac{K\left(s,a_1(1/\Psi^{-1}(t^{-1}))\right)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q} = Q_1(t,a) + J_1 + J_2.\end{aligned}$$

Moreover,

$$J_1 \le \frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}} \left(\int_{e^{-t^{-q'}}}^1 (1-\log s)^{-q}\,\frac{ds}{s}\right)^{1/q} \lesssim \frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}.$$

Since

$$\left(\int_0^{e^{-t^{-q'}}} (1-\log s)^{-q}\,\frac{ds}{s}\right)^{1/q} \sim (1+t^{-q'})^{-1/q'} \ge 2^{-1/q'}$$

and $K(s,a)/s$ is a decreasing function, we obtain

$$J_1 \lesssim \frac{K(e^{-t^{-q'}},a)}{e^{-t^{-q'}}}\left(\int_0^{e^{-t^{-q'}}}(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q} \le \left(\int_0^{e^{-t^{-q'}}}\left[\frac{K(s,a)}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q} = Q_1(t,a).$$

To estimate J_2, note that

$$\begin{aligned} t\Psi^{-1}(t^{-1}) &= \frac{\Psi^{-1}(t^{-1})}{\Psi(\Psi^{-1}(t^{-1}))} = \frac{\Psi^{-1}(t^{-1})}{\Psi^{-1}(t^{-1})\,(1-\log\Psi^{-1}(t^{-1}))^{1/q'}} \\ &= \left(1-\log\Psi^{-1}(t^{-1})\right)^{-1/q'} \sim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty}(1+\log s)^{-q}\frac{ds}{s}\right)^{1/q}. \end{aligned} \tag{5.6}$$

We derive

$$\begin{aligned} J_2 &\le \left(\int_0^1(1-\log s)^{-q}\frac{ds}{s}\right)^{1/q}\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \sim \left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \\ &\lesssim \frac{K\left(1/\Psi^{-1}(t^{-1}),a\right)}{1/\Psi^{-1}(t^{-1})} \le t\Psi^{-1}(t^{-1})K\left(1/\Psi^{-1}(t^{-1}),a\right) \\ &\sim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty}(1+\log s)^{-q}\frac{ds}{s}\right)^{1/q}K\left(1/\Psi^{-1}(t^{-1}),a\right) \\ &= \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty}\left[\frac{K\left(\min(1/\Psi^{-1}(t^{-1}),s),a\right)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q} \le Q_2(t,a). \end{aligned}$$

Consequently, $I_1 \lesssim Q_1(t,a)+Q_2(t,a) = Q(t,a)$.

We proceed now to estimate I_2. We have

$$\begin{aligned} I_2 &\le \left(\int_1^{1/\Psi^{-1}(t^{-1})}\left[\frac{K(s,a)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q} + \left(\int_1^{1/\Psi^{-1}(t^{-1})}\left[\frac{K\left(s,a_1(1/\Psi^{-1}(t^{-1}))\right)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q} \\ &\le Q_2(t,a) + \left(\int_1^{1/\Psi^{-1}(t^{-1})}\left[\frac{s}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q}\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} = Q_2(t,a)+J_3. \end{aligned}$$

According to (5.4) and (5.6), we get

$$\begin{aligned} J_3 &= \left(\int_{\Psi^{-1}(t^{-1})}^{1}\left[\frac{1}{s(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \\ &\lesssim \frac{1}{\Psi^{-1}(t^{-1})(1-\log\Psi^{-1}(t^{-1}))}\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \\ &\le t\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \lesssim t\frac{K\left(1/\Psi^{-1}(t^{-1}),a\right)}{1/\Psi^{-1}(t^{-1})} \\ &\sim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty}(1+\log s)^{-q}\frac{ds}{s}\right)^{1/q}K\left(1/\Psi^{-1}(t^{-1}),a\right) \\ &= \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty}\left[\frac{K\left(1/\Psi^{-1}(t^{-1}),a\right)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q} \le Q_2(t,a). \end{aligned}$$

As for I_3, we derive

$$\begin{aligned} I_3 &\lesssim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} (1+\log s)^{-q} \frac{ds}{s}\right)^{1/q} \left\|a_0(1/\Psi^{-1}(t^{-1}))\right\|_{A_0} \\ &\lesssim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} (1+\log s)^{-q} \frac{ds}{s}\right)^{1/q} K(1/\Psi^{-1}(t^{-1}), a) \\ &= \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} \left[\frac{K(1/\Psi^{-1}(t^{-1}), a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} \le Q_2(t,a). \end{aligned}$$

Collecting the estimates for I_1, I_2 and I_3, we conclude that $a_0(1/\Psi^{-1}(t^{-1}))$ belongs to $\bar{A}_{q;J}$ with $\left\|a_0(1/\Psi^{-1}(t^{-1}))\right\|_{\bar{A}_{q;J}} \lesssim Q(t,a)$.

It is clear that

$$K\left(t,a;\bar{X}\right) \le \left\|a_0(1/\Psi^{-1}(t^{-1}))\right\|_{\bar{A}_{q;J}} + t\left\|a_1(1/\Psi^{-1}(t^{-1}))\right\|_{A_1} \lesssim Q(t,a) + t\Psi^{-1}(t^{-1})K\left(1/\Psi^{-1}(t^{-1}), a\right),$$

and, by (5.6), it follows that

$$t\Psi^{-1}(t^{-1})K\left(1/\Psi^{-1}(t^{-1}), a\right) \sim \left(\int_{1/\Psi^{-1}(t^{-1})}^{\infty} (1+\log s)^{-q} \frac{ds}{s}\right)^{1/q} K\left(1/\Psi^{-1}(t^{-1}), a\right) \lesssim Q_2(t,a) \lesssim Q(t,a).$$

This implies that $K\left(t,a;\bar{X}\right) \lesssim Q(t,a)$ and finishes the proof. □

Next we establish the limiting version of (5.3) for J-spaces.

Lemma 5.3. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple, let* $1 < q \le \infty$ *and* $1/q + 1/q' = 1$. *Put* $\bar{Y} = \left(A_0, \bar{A}_{q;J}\right)$.

If $0 < t < 1$, *then*

$$K\left(t,a;\bar{Y}\right) \sim t\left(\int_{e^{t^{q'}}}^{\infty} \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} + t\left(\int_0^1 \left[\frac{K\left(\max(s,\Psi^{-1}(t)), a\right)}{\max(s,\Psi^{-1}(t))(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q},$$

where $\Psi(s) = s(1+|\log s|)^{1/q'}$.

If $1 \le t < \infty$, *then*

$$K\left(t,a;\bar{Y}\right) \sim t\left(\int_{e^{t^{q'}}}^{\infty} \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Proof. By the symmetry property of the limiting J-method shown in Lemma 3.7, we obtain

$$K\left(t,a;\bar{Y}\right) = K\left(t,a;A_0,(A_1,A_0)_{q;J}\right) = tK\left(t^{-1},a;(A_1,A_0)_{q;J},A_0\right).$$

Therefore, for $0 < t < 1$, using Lemma 5.2, we derive

$$K\left(t,a;\bar{Y}\right) \sim t\left(\int_0^{e^{-t^{q'}}} \left[\frac{K\left(s,a;A_1,A_0\right)}{s(1-\log s)}\right]^q \frac{ds}{s}\right)^{1/q}$$

$$
\begin{aligned}
&+ t\left(\int_1^{\infty}\left[\frac{K\left(\min(1/\Psi^{-1}(t),s),a;A_1,A_0\right)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q}\\
&= t\left(\int_0^{e^{-t^{q'}}}\left[\frac{K(s^{-1},a)}{1-\log s}\right]^q\frac{ds}{s}\right)^{1/q}\\
&\quad + t\left(\int_1^{\infty}\left[\frac{\min(1/\Psi^{-1}(t),s)K\left(\min(1/\Psi^{-1}(t),s)^{-1},a\right)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q}\\
&= t\left(\int_{e^{t^{q'}}}^{\infty}\left[\frac{K(s,a)}{1+\log s}\right]^q\frac{ds}{s}\right)^{1/q} + t\left(\int_0^1\left[\frac{K\left(\max(s,\Psi^{-1}(t)),a\right)}{\max(s,\Psi^{-1}(t))(1-\log s)}\right]^q\frac{ds}{s}\right)^{1/q}.
\end{aligned}
$$

The proof in the case $1 \le t < \infty$ follows the same line, this time using Lemma 5.1. □

As we show next, the arguments used in the proofs of Lemmata 5.1, 5.2 and 5.3 may be modified to establish the corresponding results on K-spaces. We deal first with the case when $0 < t \le 1$. In this case the estimate requires the increasing function $\Phi : (0,1] \longrightarrow (0,1]$ given by $\Phi(t) = t(1+|\log t|)^{1/q}$.

Lemma 5.4. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple, let* $1 \le q \le \infty$ *and* $0 < t \le 1$. *Put* $\bar{X} = (\bar{A}_{q;K}, A_1)$. *Then, for any* $a \in \bar{A}_{q;K} + A_1$, *we have that*

$$
K\left(t,a;\bar{X}\right) \sim \left(\int_0^1 K\left(\min(\Phi^{-1}(t),s),a\right)^q\frac{ds}{s}\right)^{1/q},
$$

where $\Phi(t) = t(1+|\log t|)^{1/q}$.

Proof. Let $a \in \bar{A}_{q;K} + A_1$ and pick any decomposition of a, $a = a_0 + a_1$, such that $a_0 \in \bar{A}_{q;K}$ and $a_1 \in A_1$. Then

$$
\begin{aligned}
\left(\int_0^1 K\left(\min\left(\Phi^{-1}(t),s\right),a\right)^q\frac{ds}{s}\right)^{1/q} &\le \left(\int_0^1 K\left(\min\left(\Phi^{-1}(t),s\right),a_0\right)^q\frac{ds}{s}\right)^{1/q}\\
&\quad + \left(\int_0^1 K\left(\min\left(\Phi^{-1}(t),s\right),a_1\right)^q\frac{ds}{s}\right)^{1/q}\\
&\le \|a_0\|_{\bar{A}_{q;K}} + I_2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_2 &\le \|a_1\|_{A_1}\left(\int_0^1 \min\left(\Phi^{-1}(t),s\right)^q\frac{ds}{s}\right)^{1/q} \le \|a_1\|_{A_1}\left[\left(\int_0^{\Phi^{-1}(t)} s^q\frac{ds}{s}\right)^{1/q} + \left(\int_{\Phi^{-1}(t)}^1 \Phi^{-1}(t)^q\frac{ds}{s}\right)^{1/q}\right]\\
&\sim \|a_1\|_{A_1}\left[\Phi^{-1}(t) + \Phi^{-1}(t)\left(-\log\Phi^{-1}(t)\right)^{1/q}\right] = \|a_1\|_{A_1}\Phi^{-1}(t)(1+\left|\log\Phi^{-1}(t)\right|^{1/q})\\
&\sim \|a_1\|_{A_1}\Phi^{-1}(t)(1+\left|\log\Phi^{-1}(t)\right|)^{1/q} = \Phi(\Phi^{-1}(t))\,\|a_1\|_{A_1} = t\,\|a_1\|_{A_1}.
\end{aligned}
$$

Therefore,

$$\left(\int_0^1 K\left(\min\left(\Phi^{-1}(t),s\right),a\right)^q \frac{ds}{s}\right)^{1/q} \lesssim \|a_0\|_{\bar{A}_{q;K}} + t\,\|a_1\|_{A_1}.$$

Taking the infimum over all possible decompositions of a, we obtain that

$$\left(\int_0^1 K\left(\min\left(\Phi^{-1}(t),s\right),a\right)^q \frac{ds}{s}\right)^{1/q} \lesssim K\left(t,a;\bar{X}\right).$$

Conversely, pick any representation of a, $a = a_0(t) + a_1(t)$ with $a_j(t) \in A_j$, $j = 0,1$, such that $\|a_0(t)\|_{A_0} + t\,\|a_1(t)\|_{A_1} \le 2K\left(t,a;\bar{A}\right)$. First we show that $a_0\left(\Phi^{-1}(t)\right)$ belongs to $\bar{A}_{q;K}$. We have that

$$\begin{aligned}\left\|a_0\left(\Phi^{-1}(t)\right)\right\|_{\bar{A}_{q;K}} &\le \left(\int_0^{\Phi^{-1}(t)} K\left(s,a_0\left(\Phi^{-1}(t)\right)\right)^q \frac{ds}{s}\right)^{1/q} + \left(\int_{\Phi^{-1}(t)}^1 K\left(s,a_0\left(\Phi^{-1}(t)\right)\right)^q \frac{ds}{s}\right)^{1/q}\\ &\quad + \left(\int_1^\infty \left[\frac{K\left(s,a_0\left(\Phi^{-1}(t)\right)\right)}{s}\right]^q \frac{ds}{s}\right)^{1/q} = I_1 + I_2 + I_3.\end{aligned}$$

We estimate each of the integrals separately.

$$\begin{aligned} I_1 &\le \left(\int_0^{\Phi^{-1}(t)} K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^{\Phi^{-1}(t)} K\left(s,a_1\left(\Phi^{-1}(t)\right)\right)^q \frac{ds}{s}\right)^{1/q}\\ &= \left(\int_0^{\Phi^{-1}(t)} K\left(\min\left(s,\Phi^{-1}(t)\right),a\right)^q \frac{ds}{s}\right)^{1/q} + J_1,\end{aligned}$$

and

$$J_1 \le \left\|a_1\left(\Phi^{-1}(t)\right)\right\|_{A_1} \left(\int_0^{\Phi^{-1}(t)} s^q \frac{ds}{s}\right)^{1/q} \sim \Phi^{-1}(t)\left\|a_1\left(\Phi^{-1}(t)\right)\right\|_{A_1} \lesssim K\left(\Phi^{-1}(t),a\right).$$

Since $\int_{\Phi^{-1}(t)/2}^{\Phi^{-1}(t)} ds = \Phi^{-1}(t)/2$, it follows that

$$\begin{aligned} K\left(\Phi^{-1}(t),a\right) &= \Phi^{-1}(t)\frac{K\left(\Phi^{-1}(t),a\right)}{\Phi^{-1}(t)} \sim \int_{\Phi^{-1}(t)/2}^{\Phi^{-1}(t)} ds \frac{K\left(\Phi^{-1}(t),a\right)}{\Phi^{-1}(t)}\\ &\le \int_{\Phi^{-1}(t)/2}^{\Phi^{-1}(t)} \frac{K(s,a)}{s} ds \le \left(\int_{\Phi^{-1}(t)/2}^{\Phi^{-1}(t)} K(s,a)^q \frac{ds}{s}\right)^{1/q} \left(\log\frac{\Phi^{-1}(t)}{\Phi^{-1}(t)/2}\right)^{1/q'}\\ &\lesssim \left(\int_0^{\Phi^{-1}(t)} K\left(\min\left(s,\Phi^{-1}(t)\right),a\right)^q \frac{ds}{s}\right)^{1/q}\\ &\le \left(\int_0^1 K\left(\min\left(s,\Phi^{-1}(t)\right),a\right)^q \frac{ds}{s}\right)^{1/q}. \end{aligned} \tag{5.7}$$

This implies that

$$I_1 \lesssim \left(\int_0^1 K\left(\min\left(s,\Phi^{-1}(t)\right),a\right)^q \frac{ds}{s}\right)^{1/q}.$$

Furthermore, we have that

$$I_2 = \left(\int_{\Phi^{-1}(t)}^{1} K\left(s, a_0\left(\Phi^{-1}(t)\right)\right)^q \frac{ds}{s}\right)^{1/q} \lesssim K\left(\Phi^{-1}(t), a\right)\left(\int_{\Phi^{-1}(t)}^{1} \frac{ds}{s}\right)^{1/q}$$
$$\sim \left(\int_{\Phi^{-1}(t)}^{1} K\left(\Phi^{-1}(t), a\right)^q \frac{ds}{s}\right)^{1/q} \leq \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q},$$

and, by (5.7),

$$I_3 = \left(\int_1^\infty \left[\frac{K\left(s, a_0\left(\Phi^{-1}(t)\right)\right)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \lesssim K\left(\Phi^{-1}(t), a\right) \lesssim \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}.$$

This gives that $a_0\left(\Phi^{-1}(t)\right) \in \bar{A}_{q;K}$ and

$$\left\|a_0\left(\Phi^{-1}(t)\right)\right\|_{\bar{A}_{q;K}} \lesssim \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}.$$

On the other hand, since $t \sim \Phi^{-1}(t)(1 + \left|\log \Phi^{-1}(t)\right|^{1/q})$, we derive by (5.7) that

$$t\left\|a_1\left(\Phi^{-1}(t)\right)\right\|_{A_1} \lesssim \frac{t}{\Phi^{-1}(t)} K\left(\Phi^{-1}(t), a\right) \sim \frac{t - \Phi^{-1}(t)}{\Phi^{-1}(t)} K\left(\Phi^{-1}(t), a\right) + K\left(\Phi^{-1}(t), a\right)$$
$$\lesssim \left(-\log \Phi^{-1}(t)\right)^{1/q} K\left(\Phi^{-1}(t), a\right) + \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}$$
$$= \left(\int_{\Phi^{-1}(t)}^1 \frac{ds}{s}\right)^{1/q} K\left(\Phi^{-1}(t), a\right) + \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}$$
$$\lesssim \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}.$$

Therefore,

$$K\left(t, a; \bar{X}\right) \leq \left\|a_0\left(\Phi^{-1}(t)\right)\right\|_{\bar{A}_{q;K}} + t\left\|a_1\left(\Phi^{-1}(t)\right)\right\|_{A_1} \lesssim \left(\int_0^1 K\left(\min\left(s, \Phi^{-1}(t)\right), a\right)^q \frac{ds}{s}\right)^{1/q}.$$

This ends the proof. □

Next we show the corresponding formula for $t > 1$.

Lemma 5.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $1 \leq q \leq \infty$ and $1 < t < \infty$. Put $\bar{X} = (\bar{A}_{q;K}, A_1)$. Then, for any $a \in \bar{A}_{q;K} + A_1$, we have that*

$$K\left(t, a; \bar{X}\right) \sim \left(\int_0^1 K(s, a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s, a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Proof. Let $a \in \bar{A}_{q;K} + A_1$ and take any decomposition of a, $a = a_0 + a_1$, in $\bar{A}_{q,K} + A_1$. Then

$$\left(\int_0^1 K(s, a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s, a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}$$

$$\begin{aligned}
&\leq \left(\int_0^1 K(s,a_0)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s,a_0)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_0^1 K(s,a_1)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s,a_1)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\lesssim \|a_0\|_{\bar{A}_{q;K}} + \|a_1\|_{A_1} + t\,\|a_1\|_{A_1} \lesssim \|a_0\|_{\bar{A}_{q;K}} + t\,\|a_1\|_{A_1}.
\end{aligned}$$

Taking the infimum over all possible representations of a, we obtain that

$$\left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \lesssim K\left(t,a;\bar{X}\right).$$

Conversely, for each $t > 1$ choose a representation $a = a_0(t) + a_1(t)$ in $A_0 + A_1$ such that $\|a_0(t)\|_{A_0} + t\,\|a_1(t)\|_{A_1} \leq 2K(t,a)$. We claim that $a_0(e^{t^q}) \in \bar{A}_{q;K}$. Indeed,

$$\begin{aligned}
\left\|a_0(e^{t^q})\right\|_{\bar{A}_{q;K}} &\leq \left(\int_0^1 K\left(s,a_0(e^{t^q})\right)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K\left(s,a_0(e^{t^q})\right)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_{e^{t^q}}^{\infty} \left[\frac{K\left(s,a_0(e^{t^q})\right)}{s}\right]^q \frac{ds}{s}\right)^{1/q} = I_1 + I_2 + I_3.
\end{aligned}$$

We have that

$$I_1 \leq \left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^1 K\left(s,a_1(e^{t^q})\right)^q \frac{ds}{s}\right)^{1/q} \leq \left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q} + J_1,$$

and

$$J_1 \lesssim t\left\|a_1(e^{t^q})\right\|_{A_1} \lesssim t\frac{K\left(e^{t^q},a\right)}{e^{t^q}} \sim \left(\int_1^{e^{t^q}} \frac{ds}{s}\right)^{1/q} \frac{K\left(e^{t^q},a\right)}{e^{t^q}} \leq \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}.$$

It also follows that

$$\begin{aligned}
I_2 &\leq \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K\left(s,a_1(e^{t^q})\right)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&\leq \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q} + t\left\|a_1(e^{t^q})\right\|_{A_1} \lesssim \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)
\end{aligned}$$

and that

$$I_3 \lesssim \frac{\left\|a_0(e^{t^q})\right\|_{A_0}}{e^{t^q}} \lesssim t\frac{K\left(e^{t^q},a\right)}{e^{t^q}} \lesssim \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Therefore, $a_0(e^{t^q}) \in \bar{A}_{q;K}$ and

$$\left\|a_0(e^{t^q})\right\|_{\bar{A}_{q;K}} \lesssim \left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}.$$

On the other hand, we have that

$$t\left\|a_1(e^{t^q})\right\|_{A_1} \lesssim \frac{t}{e^{t^q}}K(e^{t^q},a) \lesssim \left(\int_1^{e^{t^q}}\left[\frac{K(s,a)}{s}\right]^q\frac{ds}{s}\right)^{1/q}.$$

So

$$K\left(t,a;\bar{X}\right) \lesssim \left(\int_0^1 K(s,a)^q\frac{ds}{s}\right)^{1/q} + \left(\int_1^{e^{t^q}}\left[\frac{K(s,a)}{s}\right]^q\frac{ds}{s}\right)^{1/q},$$

as desired. □

The following lemma can be proved as Lemma 5.3, but this time using the symmetry property for the K-method (3.1) and Lemmata 5.4 and 5.5.

Lemma 5.6. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple and let $1\le q\le\infty$.*

If $0<t<1$, then

$$K\left(t,a;A_0,\bar{A}_{q;K}\right) \sim t\left(\int_1^\infty\left[\frac{K(s,a)}{s}\right]^q\frac{ds}{s}\right)^{1/q} + t\left(\int_{e^{-t^{-q}}}^1 K(s,a)^q\frac{ds}{s}\right)^{1/q}.$$

If $1\le t<\infty$, then

$$K\left(t,a;A_0,\bar{A}_{q;K}\right) \sim t\left(\int_1^\infty\left[\frac{K\left(\max(1/\Phi^{-1}(t^{-1}),s),a\right)}{\max(1/\Phi^{-1}(t^{-1}),s)}\right]^q\frac{ds}{s}\right)^{1/q}.$$

5.2 Reiteration formulae

Next we establish reiteration results which follow from the Holmstedt type estimates of the previous section. The resulting spaces have the shape of an intersection $V\cap W$, where

$$\begin{cases} V=\{a\in A_0+A_1 : K(s,a)/v(s)\in L_q((0,1),ds/s)\},\\ W=\{a\in A_0+A_1 : K(s,a)/w(s)\in L_q((1,\infty),ds/s)\}.\end{cases} \tag{5.8}$$

Here v,w are functions in the form $s^ib(s)$ where $i=0$ or 1 and b is a certain slowly varying function.

Theorem 5.7. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple and let $1<q\le\infty$. Then we have with equivalent norms*

$$\left(A_0,\bar{A}_{q;J}\right)_{q;J} = \left\{ \begin{array}{l} a\in A_0+A_1 : \quad \|a\| = \left(\int_0^1\left[\frac{K(s,a)}{s(1-\log s)^2}\right]^q(1-\log s)\frac{ds}{s}\right)^{1/q} \\ \qquad + \left(\int_1^\infty\left[\frac{K(s,a)}{1+\log(1+\log s)}\right]^q(1+\log s)^{-1}\frac{ds}{s}\right)^{1/q}<\infty \end{array}\right\}$$

and

$$\left(\bar{A}_{q;J},A_1\right)_{q;J} = \left\{ \begin{array}{l} a\in A_0+A_1 : \|a\| = \left(\int_1^\infty\left[\frac{K(s,a)}{(1+\log s)^2}\right]^q(1+\log s)\frac{ds}{s}\right)^{1/q} \\ \qquad + \left(\int_0^1\left[\frac{K(s,a)}{s(1+\log(1-\log s))}\right]^q(1-\log s)^{-1}\frac{ds}{s}\right)^{1/q}<\infty \end{array}\right\}.$$

Proof. Let $\bar{Y} = \left(A_0, \bar{A}_{q;J}\right)$. According to Lemma 5.3, we get

$$\begin{aligned}
\|a\|_{\bar{Y}_{q;J}} &= \left(\int_0^1 \left[\frac{K\left(t,a;\bar{Y}\right)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K\left(t,a;\bar{Y}\right)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 (1-\log t)^{-q} \int_{e^{t^{q'}}}^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_0^1 (1-\log t)^{-q} \int_0^1 \left[\frac{K\left(\max(s,\Psi^{-1}(t)),a\right)}{\max(s,\Psi^{-1}(t))(1-\log s)}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[\frac{t}{1+\log t}\right]^q \int_{e^{t^{q'}}}^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

It is shown in [19, page 2339] that

$$I_3 \sim \left(\int_1^\infty \left[\frac{K(s,a)}{1+\log(1+\log s)}\right]^q (1+\log s)^{-1}\frac{ds}{s}\right)^{1/q}.$$

As for I_1, since $K(t,a)$ is increasing, we obtain

$$\begin{aligned}
I_1 &\sim \left(\int_1^\infty (1+\log t)^{-q} \int_{e^{t^{-q'}}}^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_1^\infty (1+\log t)^{-q} \int_{e^{t^{-q'}}}^{e^{t^{q'}}} \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_1^\infty (1+\log t)^{-q} \int_{e^{t^{q'}}}^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\lesssim \left(\int_1^\infty (1+\log t)^{-q} K(e^{t^{q'}},a)\left[(1+t^{-q'})^{1-q} - (1+t^{q'})^{1-q}\right]\frac{dt}{t}\right)^{1/q} + I_3.
\end{aligned}$$

Put $f(t) = (1+t^{-q'})^{1-q} - (1+t^{q'})^{1-q}$. Then we have that $f(1) = 0$, f is increasing on $[1,\infty)$ and $\lim_{t\to\infty} f(t) = 1$. Hence $f(t) \le 1$ for any $1 \le t < \infty$. Besides,

$$\int_{e^{t^{q'}}}^\infty (1+\log s)^{-q}\frac{ds}{s} \sim (1+t^{q'})^{1-q} \sim t^{-q}$$

for $t \ge 1$. Whence

$$\begin{aligned}
I_1 &\lesssim \left(\int_1^\infty \left[\frac{t}{1+\log t}\right]^q K(e^{t^{q'}},a) \int_{e^{t^{q'}}}^\infty (1+\log s)^{-q}\frac{ds}{s}\frac{dt}{t}\right)^{1/q} + I_3 \\
&\le \left(\int_1^\infty \left[\frac{t}{1+\log t}\right]^q \int_{e^{t^{q'}}}^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} + I_3 \lesssim I_3.
\end{aligned}$$

Consider now I_2. We have that

$$I_2 \sim \left(\int_0^1 (1-\log t)^{-q} \int_0^{\Psi^{-1}(t)} \left[\frac{K\left(\Psi^{-1}(t),a\right)}{\Psi^{-1}(t)(1-\log s)}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q}$$

$$+\left(\int_0^1 (1-\log t)^{-q} \int_{\Psi^{-1}(t)}^1 \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} = J_1 + J_2.$$

It is clear that

$$J_1 \sim \left(\int_0^1 \left[\frac{K\left(\Psi^{-1}(t),a\right)}{\Psi^{-1}(t)}\right]^q (1-\log t)^{-q}\left(1-\log\Psi^{-1}(t)\right)^{1-q}\frac{dt}{t}\right)^{1/q}.$$

Put $s=\Psi^{-1}(t)$. It follows that

$$\frac{dt}{t} = \left[1-\frac{1}{q'(1-\log s)}\right]\frac{ds}{s}.$$

Therefore,

$$\begin{aligned}J_1 &\sim \left(\int_0^1 \left[\frac{K(s,a)}{s\,(1-\log\Psi(s))}\right]^q (1-\log s)^{1-q}\left[1-\frac{1}{q'(1-\log s)}\right]\frac{ds}{s}\right)^{1/q}\\ &\leq \left(\int_0^1 \left[\frac{K(s,a)}{s\,(1-\log\Psi(s))}\right]^q \frac{1-\log s}{(1-\log s)^q}\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Besides, $-\log\Psi(s) = -\log s - \frac{1}{q'}\log(1-\log s)$. Hence, since

$$\lim_{s\to 0}\frac{1-\log s-\frac{1}{q'}\log(1-\log s)}{1-\log s} = 1,$$

we have that

$$1-\log\Psi(s)\sim 1-\log s. \tag{5.9}$$

This yields that

$$J_1 \lesssim \left(\int_0^1 \left[\frac{K(s,a)}{s(1-\log s)^2}\right]^q (1-\log s)\frac{ds}{s}\right)^{1/q}.$$

Concerning J_2, using (5.9), we derive

$$\begin{aligned}J_2 &= \left(\int_0^1 \left[\frac{K(s,a)}{s(1-\log s)}\right]^q \int_0^{\Psi(s)} (1-\log t)^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\ &\sim \left(\int_0^1 \left[\frac{K(s,a)}{s(1-\log s)}\right]^q (1-\log\Psi(s))^{-q+1}\frac{ds}{s}\right)^{1/q}\\ &\sim \left(\int_0^1 \left[\frac{K(s,a)}{s(1-\log s)}\right]^q (1-\log s)^{-q+1}\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

So, $I_2\sim J_1+J_2\sim J_2$. Collecting the estimates, the formula on $\left(A_0,\bar{A}_{q;J}\right)_{q;J}$ follows.

The case of $\left(\bar{A}_{q;J},A_1\right)_{q;J}$ can be derived from the previous formula by using the symmetry property $((A_0,A_1)_{q;J},A_1)_{q;J} = (A_1,\left(A_1,A_0\right)_{q;J})_{q;J}$. Namely,

$$\|a\|_{\left(\bar{A}_{q;J},A_1\right)_{q;J}} \sim \left(\int_0^1 \left[\frac{K\,(s,a;A_1,A_0)}{s(1-\log s)^2}\right]^q (1-\log s)\frac{ds}{s}\right)^{1/q}$$

$$+\left(\int_1^\infty\left[\frac{K(s,a;A_1,A_0)}{1+\log(1+\log s)}\right]^q(1+\log s)^{-1}\frac{ds}{s}\right)^{1/q}$$
$$=\left(\int_0^1\left[\frac{K(s^{-1},a)}{(1-\log s)^2}\right]^q(1-\log s)\frac{ds}{s}\right)^{1/q}$$
$$+\left(\int_1^\infty\left[\frac{sK(s^{-1},a)}{1+\log(1+\log s)}\right]^q(1+\log s)^{-1}\frac{ds}{s}\right)^{1/q}$$
$$=\left(\int_0^1\left[\frac{K(s,a)}{s(1+\log(1-\log s))}\right]^q(1-\log s)^{-1}\frac{ds}{s}\right)^{1/q}$$
$$+\left(\int_1^\infty\left[\frac{K(s,a)}{(1+\log s)^2}\right]^q(1+\log s)\frac{ds}{s}\right)^{1/q}.$$

This completes the proof. □

Remark 5.1. Theorem 5.7 is not true when $q=1$. Indeed, it follows from Lemma 3.6 that

$$(A_0,\bar{A}_{1;J})_{1;J}=A_0\cap A_1=(\bar{A}_{1;J},A_1)$$

and the norm of $A_0\cap A_1$ cannot be expressed by means of the K-functional (see Section 3.2).

With similar arguments but using now Lemmata 5.4, 5.5 and 5.6 we derive the following limiting reiteration formulae for K-spaces.

Theorem 5.8. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple and let $1\le q\le\infty$. Then we have with equivalent norms*

$$\left(\bar{A}_{q;K},A_1\right)_{q;K}=\left\{\begin{array}{l}a\in A_0+A_1:\quad \|a\|=\left(\int_0^1K(s,a)^q(1-\log s)\frac{ds}{s}\right)^{1/q}\\ \qquad\qquad+\left(\int_1^\infty\left[\frac{K(s,a)}{s(1+\log s)^{1/q}}\right]^q\frac{ds}{s}\right)^{1/q}<\infty\end{array}\right\}$$

and

$$\left(A_0,\bar{A}_{q;K}\right)_{q;K}=\left\{\begin{array}{l}a\in A_0+A_1:\quad \|a\|=\left(\int_0^1\left[\frac{K(s,a)}{(1-\log s)^{1/q}}\right]^q\frac{ds}{s}\right)^{1/q}\\ \qquad\qquad+\left(\int_1^\infty\left[\frac{K(s,a)}{s}\right]^q(1+\log s)\frac{ds}{s}\right)^{1/q}<\infty\end{array}\right\}.$$

Proof. Put $\bar{X}_{q;K}:=\left(\bar{A}_{q;K},A_1\right)_{q;K}$. Then, by Lemmata 5.4 and 5.5, we have that

$$\|a\|_{\bar{X}_{q;K}}=\left(\int_0^1K(t,a;\bar{X})^q\frac{dt}{t}\right)^{1/q}+\left(\int_1^\infty\left[\frac{K(t,a;\bar{X})}{t}\right]^q\frac{dt}{t}\right)^{1/q}$$
$$\sim\left(\int_0^1\int_0^1K\left(\min\left(\Phi^{-1}(t),s\right),a\right)^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q}$$
$$+\left(\int_1^\infty t^{-q}\left[\int_0^1K(s,a)^q\frac{ds}{s}+\int_1^{e^{t^q}}\left[\frac{K(s,a)}{s}\right]^q\frac{ds}{s}\right]\frac{dt}{t}\right)^{1/q}$$

$$\sim \left(\int_0^1 \int_0^{\Phi^{-1}(t)} K(s,a)^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \int_{\Phi^{-1}(t)}^1 K\left(\Phi^{-1}(t),a\right)^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q}$$
$$+ \left(\int_1^\infty t^{-q} \int_0^1 K(s,a)^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty t^{-q} \int_1^{e^{t^q}} \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q}$$
$$= I_1 + I_2 + I_3 + I_4.$$

It follows that

$$I_1 = \left(\int_0^1 \int_{\Phi(s)}^1 \frac{dt}{t} K(s,a)^q \frac{ds}{s}\right)^{1/q} = \left(\int_0^1 |\log \Phi(s)|\, K(s,a)^q \frac{ds}{s}\right)^{1/q}.$$

Furthermore,

$$I_2 = \left(\int_0^1 \left(-\log \Phi^{-1}(t)\right) K\left(\Phi^{-1}(t),a\right)^q \frac{dt}{t}\right)^{1/q}.$$

Put $u = \Phi^{-1}(t)$. Then

$$\frac{dt}{t} = \frac{du}{u}\left[1 - \frac{1}{q(1-\log u)}\right],$$

and therefore

$$I_2 = \left(\int_0^1 |\log u|\, K(u,a)^q \left[1 - \frac{1}{q(1-\log u)}\right] \frac{du}{u}\right)^{1/q}.$$

On the other hand,

$$I_3 = \left(\int_1^\infty t^{-q} \int_0^1 K(s,a)^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \sim \left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q}.$$

This implies that

$$I_1 + I_2 + I_3 \sim \left(\int_0^1 K(s,a)^q \left[1 + |\log \Phi(s)| + |\log s| - \frac{|\log s|}{q(1+|\log s|)}\right] \frac{ds}{s}\right)^{1/q}.$$

Next we find an equivalent expression for the term in brackets. Since

$$1 + |\log \Phi(s)| + |\log s| - \frac{|\log s|}{q(1+|\log s|)} \le 1 + |\log \Phi(s)| + |\log s|$$

and also

$$1 + |\log \Phi(s)| + |\log s| - \frac{|\log s|}{q(1+|\log s|)} \ge 1 - \frac{1}{q} + |\log \Phi(s)| + |\log s| \sim 1 + |\log \Phi(s)| + |\log s|,$$

it follows that

$$I_1 + I_2 + I_3 \sim \left(\int_0^1 K(s,a)^q \left[1 + |\log \Phi(s)| + |\log s|\right] \frac{ds}{s}\right)^{1/q}.$$

On the other hand, $\Phi(s) \ge s$ and $s, \Phi(s) \in (0,1)$. This gives that $|\log \Phi(s)| \le |\log s|$, and thus

$$I_1 + I_2 + I_3 \sim \left(\int_0^1 K(s,a)^q \left(1 + \max(|\log \Phi(s)|, |\log s|)\right) \frac{ds}{s}\right)^{1/q} = \left(\int_0^1 K(s,a)^q \left(1 + |\log s|\right) \frac{ds}{s}\right)^{1/q}.$$

Finally, a change in the order of integration yields that

$$\begin{aligned} I_4 &= \left(\int_1^e \left[\frac{K(s,a)}{s} \right]^q \int_1^\infty t^{-q} \frac{dt}{t} \frac{ds}{s} + \int_e^\infty \left[\frac{K(s,a)}{s} \right]^q \int_{(\log s)^{1/q}}^\infty t^{-q} \frac{dt}{t} \frac{ds}{s} \right)^{1/q} \\ &\sim \left(\int_1^e \left[\frac{K(s,a)}{s} \right]^q \frac{ds}{s} + \int_e^\infty \left[\frac{K(s,a)}{s} \right]^q (\log s)^{-1} \frac{ds}{s} \right)^{1/q} \\ &\sim \left(\int_1^\infty (1+|\log s|)^{-1} \left[\frac{K(s,a)}{s} \right]^q \frac{ds}{s} \right)^{1/q}, \end{aligned}$$

thereby giving the first formula.

Regarding the second formula, since

$$\left(A_0, (A_0, A_1)_{q;K}\right)_{q;K} = \left((A_1, A_0)_{q;K}, A_0\right)_{q;K}$$

(see (3.1)), we can apply the first formula and get

$$\begin{aligned} \|a\|_{\left(A_0, \bar{A}_{q;K}\right)_{q;K}} &\sim \left(\int_0^1 (1+|\log s|)\, K\left(s, a; A_1, A_0\right)^q \frac{ds}{s} \right)^{1/q} + \left(\int_1^\infty \left[\frac{K\left(s, a; A_1, A_0\right)}{s\,(1+|\log s|)^{1/q}} \right]^q \frac{ds}{s} \right)^{1/q} \\ &= \left(\int_0^1 (1+|\log s|) \left[sK\left(s^{-1}, a\right) \right]^q \frac{ds}{s} \right)^{1/q} + \left(\int_1^\infty (1+|\log s|)^{-1} K\left(s^{-1}, a\right)^q \frac{ds}{s} \right)^{1/q} \\ &= \left(\int_0^1 (1+|\log s|)^{-1} K(s,a)^q \frac{ds}{s} \right)^{1/q} + \left(\int_1^\infty (1+|\log s|) \left[\frac{K(s,a)}{s} \right]^q \frac{ds}{s} \right)^{1/q}, \end{aligned}$$

as desired. □

Remark 5.2. The formulae in Theorems 5.7 and 5.8 can be simplified if we are in the ordered case. For example, assume that $A_0 \hookrightarrow A_1$ and consider the space $\left(A_0, \bar{A}_{q;K}\right)_{q;K}$. Since

$$\begin{aligned} \left(\int_0^1 \left[\frac{K(s,a)}{(1-\log s)^{1/q}} \right]^q \frac{ds}{s} \right)^{1/q} &\leq \|a\|_{A_1} \left(\int_0^1 \left[\frac{s}{(1-\log s)^{1/q}} \right]^q \frac{ds}{s} \right)^{1/q} \\ &\lesssim \|a\|_{A_1} \lesssim \left(\int_1^\infty \left[\frac{K(s,a)}{s} \right]^q (1+\log s) \frac{ds}{s} \right)^{1/q}, \end{aligned}$$

we obtain

$$\|a\|_{\left(A_0, \bar{A}_{q;K}\right)_{q;K}} \sim \left(\int_1^\infty \left[\frac{K(s,a)}{s} \right]^q (1+\log s) \frac{ds}{s} \right)^{1/q}. \tag{5.10}$$

Similarly,

$$\|a\|_{\left(\bar{A}_{q;K}, A_1\right)_{q;K}} \sim \left(\int_1^\infty \left[\frac{K(s,a)}{s(1+\log s)^{1/q}} \right]^q \frac{ds}{s} \right)^{1/q}. \tag{5.11}$$

This recovers [19, Theorem 7.5].

For J-spaces with $A_0 \hookrightarrow A_1$, it turns out that

$$\|a\|_{\left(\bar{A}_{q;J}, A_1\right)_{q;J}} \sim \left(\int_1^\infty \left[\frac{K(s,a)}{(1+\log s)^2} \right]^q (1+\log s) \frac{ds}{s} \right)^{1/q} \tag{5.12}$$

and

$$\|a\|_{(A_0,\bar{A}_{q;J})_{q;J}} \sim \left(\int_1^\infty \left[\frac{K(s,a)}{(1+\log(1+\log s))}\right]^q (1+\log s)^{-1}\frac{ds}{s}\right)^{1/q}. \tag{5.13}$$

The last formula gives back [19, Theorem 4.6 (c)].

Using Holmstedt's formulae (5.1) to (5.3), we can describe the limiting spaces generated by couples of real interpolation spaces. Again the space comes in the shape (5.8) but now v, w are in the form $s^\theta h(s)$ where $0<\theta<1$ and h is a certain logarithmic function.

Theorem 5.9. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple and let $0<\theta_0\neq\theta_1<1$ and $1\le q\le\infty$. Put $\bar{X}=\big((A_0,A_1)_{\theta_0,q},(A_0,A_1)_{\theta_1,q}\big)$. Then we have with equivalence of norms*

$$\bar{X}_{q;J} = \left\{a\in A_0+A_1 : \|a\| = \left(\int_0^\infty \left[\frac{\max(s^{-\theta_0},s^{-\theta_1})K(s,a)}{(1+|\log s|)^{1/q'}}\right]^q \frac{ds}{s}\right)^{1/q} < \infty\right\}$$

and

$$\bar{X}_{q;K} = \left\{\ a\in A_0+A_1 : \|a\| = \left(\int_0^\infty \left[\min(s^{-\theta_0},s^{-\theta_1})K(s,a)(1+|\log s|)^{1/q}\right]^q \frac{ds}{s}\right)^{1/q} < \infty\ \right\}.$$

Proof. By the symmetry property for the J-method shown in Lemma 3.7, we may assume without loss of generality that $\theta_0<\theta_1$. If $q=1$, it follows from Lemma 3.6 that

$$\bar{X}_{1;J} = \bar{A}_{\theta_0,1}\cap\bar{A}_{\theta_1,1} = \left\{a\in A_0+A_1 : \|a\| = \int_0^1 s^{-\theta_1}K(s,a)\frac{ds}{s} + \int_1^\infty s^{-\theta_0}K(s,a)\frac{ds}{s} < \infty\right\}.$$

Assume now that $1<q\le\infty$. Using (5.1) and changing the order of integration, we obtain with $\lambda=\theta_1-\theta_0$

$$\begin{aligned}
\|a\|_{\bar{X}_{q;J}} &\sim \left(\int_0^1\int_0^{t^{1/\lambda}}\left[\frac{s^{-\theta_0}K(s,a)}{t(1-\log t)}\right]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_0^1\int_{t^{1/\lambda}}^\infty\left[\frac{s^{-\theta_1}K(s,a)}{1-\log t}\right]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q}\\
&\quad + \left(\int_1^\infty\int_0^{t^{1/\lambda}}\left[\frac{s^{-\theta_0}K(s,a)}{1+\log t}\right]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty\int_{t^{1/\lambda}}^\infty\left[\frac{ts^{-\theta_1}K(s,a)}{1+\log t}\right]^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q}\\
&\sim \left(\int_0^1\int_{s^\lambda}^1\left[\frac{s^{-\theta_0}K(s,a)}{t(1-\log t)}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_0^1\int_0^{s^\lambda}\left[\frac{s^{-\theta_1}K(s,a)}{1-\log t}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\
&\quad + \left(\int_1^\infty\int_0^1\left[\frac{s^{-\theta_1}K(s,a)}{1-\log t}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_0^1\int_1^\infty\left[\frac{s^{-\theta_0}K(s,a)}{1+\log t}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\
&\quad + \left(\int_1^\infty\int_{s^\lambda}^\infty\left[\frac{s^{-\theta_0}K(s,a)}{1+\log t}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty\int_1^{s^\lambda}\left[\frac{ts^{-\theta_1}K(s,a)}{1+\log t}\right]^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q} = \sum_{j=1}^6 I_j.
\end{aligned}$$

Now we estimate each of these six terms. By (5.4) we obtain

$$I_1 \lesssim \left(\int_0^1\left[\frac{K(s,a)}{s^{\theta_0}}s^{-\lambda}(1-\log s^\lambda)^{-1}\right]^q\frac{ds}{s}\right)^{1/q} \lesssim \left(\int_0^1\left[\frac{s^{-\theta_1}K(s,a)}{(1-\log s)^{1/q'}}\right]^q\frac{ds}{s}\right)^{1/q}.$$

Clearly

$$I_2 \sim \left(\int_0^1 \left[\frac{s^{-\theta_1}K(s,a)}{(1-\log s)^{1/q'}}\right]^q \frac{ds}{s}\right)^{1/q}, \quad I_3 \sim \left(\int_1^\infty \left[s^{-\theta_1}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q},$$

$$I_4 \sim \left(\int_0^1 \left[s^{-\theta_0}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} \quad \text{and} \quad I_5 \sim \left(\int_1^\infty \left[\frac{s^{-\theta_0}K(s,a)}{(1+\log s)^{1/q'}}\right]^q \frac{ds}{s}\right)^{1/q}.$$

In order to estimate I_6, note that, by (5.4),

$$\int_1^{s^\lambda} \left[\frac{t}{1+\log t}\right]^q \frac{dt}{t} = \int_{s^{-\lambda}}^1 [u(1-\log u)]^{-q} \frac{dt}{t} \lesssim \left[\frac{s^\lambda}{1+\log s}\right]^q.$$

Whence,

$$I_6 \lesssim \left(\int_1^\infty \left[\frac{s^{-\theta_0}K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q} \le I_5.$$

Having in mind that $\theta_0 < \theta_1$, we also get that $I_3 \lesssim I_5$ and $I_4 \lesssim I_2$. Consequently,

$$\|a\|_{\bar{X}_{q;J}} \sim I_2 + I_5 \sim \left(\int_0^\infty \left[\frac{\max(s^{-\theta_0}, s^{-\theta_1})K(s,a)}{(1+|\log s|)^{1/q'}}\right]^q \frac{ds}{s}\right)^{1/q}.$$

Now we prove the second formula. Once again, by the symmetry property for the K-method (3.1), we may assume without loss of generality that $\theta_0 < \theta_1$. If $q = \infty$, it follows from Lemma 3.1 that

$$\bar{X}_{\infty;K} = \bar{A}_{\theta_0,\infty} + \bar{A}_{\theta_1,\infty} = \left\{a \in A_0 + A_1 : \|a\| = \sup_{0<s<\infty} \min(s^{-\theta_0}, s^{-\theta_1})K(s,a) < \infty\right\}.$$

Assume now that $1 \le q < \infty$. Using (5.1) and changing the order of integration, we obtain with $\lambda = \theta_1 - \theta_0$

$$\begin{aligned}
\|a\|_{\bar{X}_{q;K}} &\sim \left(\int_0^1 \int_{s^\lambda}^1 \left[s^{-\theta_0}K(s,a)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_0^1 \int_0^{s^\lambda} \left[ts^{-\theta_1}K(s,a)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \int_0^1 \left[ts^{-\theta_1}K(s,a)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_0^1 \int_1^\infty \left[\frac{s^{-\theta_0}K(s,a)}{t}\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \int_{s^\lambda}^\infty \left[\frac{s^{-\theta_0}K(s,a)}{t}\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \int_1^{s^\lambda} \left[s^{-\theta_1}K(s,a)\right]^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[s^{-\theta_0}K(s,a)\right] |\log s| \frac{ds}{s}\right)^{1/q} + \left(\int_0^1 \left[s^{-\theta_0}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[s^{-\theta_1}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^1 \left[s^{-\theta_0}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[s^{-\theta_1}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[s^{-\theta_1}K(s,a)\right]^q (\log s) \frac{ds}{s}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[s^{-\theta_0}K(s,a)\right] (1+|\log s|) \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[s^{-\theta_1}K(s,a)\right]^q (1+\log s) \frac{ds}{s}\right)^{1/q},
\end{aligned}$$

as desired. □

With analogous arguments, but using now (5.2) and (5.3), we show the following characterisations:

Theorem 5.10. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple and let* $0 < \theta < 1$ *and* $1 < q \le \infty$. *Then we have with equivalence of norms*

$$\big(A_0, (A_0, A_1)_{\theta,q}\big)_{q;J} = \left\{ a \in A_0 + A_1 : \quad \|a\| = \left(\int_0^1 \left[\frac{t^{-\theta}K(t,a)}{(1-\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} < \infty \right\} \tag{5.14}$$

and

$$\big((A_0, A_1)_{\theta,q}, A_1\big)_{q;J} = \left\{ a \in A_0 + A_1 : \quad \|a\| = \left(\int_0^1 \left[\frac{K(t,a)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{t^{-\theta}K(t,a)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} < \infty \right\}. \tag{5.15}$$

Proof. First we prove (5.14). Let $a \in A_0 + A_1$ and put $\bar{X} = (A_0, \bar{A}_{\theta,q})$. Then, by (5.3), we have that

$$\|a\|_{\bar{X}_{q;J}} \sim \left(\int_0^1 \int_{t^{1/\theta}}^\infty \left[\frac{s^{-\theta}K(s,a)}{(1-\log t)}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty t^q \int_{t^{1/\theta}}^\infty \left[\frac{s^{-\theta}K(s,a)}{1+\log t}\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} = I_1 + I_2.$$

For I_1 we derive that

$$\begin{aligned} I_1 &\sim \left(\int_0^1 \left[s^{-\theta}K(s,a)\right]^q \int_0^{s^\theta} (1-\log t)^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[s^{-\theta}K(s,a)\right]^q \int_0^1 (1-\log t)^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\sim \left(\int_0^1 (1-\log s)^{1-q}\left[s^{-\theta}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[s^{-\theta}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q}. \end{aligned}$$

Regarding I_2, we get changing variables and by (5.4) that

$$\begin{aligned} I_2 &= \left(\int_1^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q \int_{s^{-\theta}}^1 [u(1-\log u)]^{-q}\frac{du}{u}\frac{ds}{s}\right)^{1/q} \lesssim \left(\int_1^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q s^{\theta q}(1+\log s)^{-q}\frac{ds}{s}\right)^{1/q} \\ &= \left(\int_1^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q}. \end{aligned}$$

Moreover, since $t^{-1}(1+\log t)$ is non-increasing on $(1,\infty)$,

$$\begin{aligned} I_2 &\ge \left(\int_{2^{1/\theta}}^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q \int_1^{s^\theta} t^q(1+\log t)^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\ge \left(\int_{2^{1/\theta}}^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q \int_{s^\theta/2}^{s^\theta} t^{q+1}t^{-1}(1+\log t)^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\gtrsim \left(\int_{2^{1/\theta}}^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q s^{-\theta}(1+\log s)^{-q} \int_{s^\theta/2}^{s^\theta} t^{q+1}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\sim \left(\int_{2^{1/\theta}}^\infty \left[\frac{K(s,a)}{s^\theta}\right]^q s^{-\theta}(1+\log s)^{-q}s^{\theta(q+1)}\frac{ds}{s}\right)^{1/q} = \left(\int_1^\infty \left[\frac{K(s,a)}{1+\log s}\right]^q \frac{ds}{s}\right)^{1/q}. \end{aligned}$$

Therefore,

$$\|a\|_{\bar{X}_{q;\widetilde{J}}} \sim \left(\int_0^1 (1-\log s)^{1-q}\left[s^{-\theta}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty K(s,a)^q \max(s^{-\theta},(1+\log s)^{-1})^q \frac{ds}{s}\right)^{1/q}.$$

Since $g(s)=s^{-\theta}(1+\log s)$, $s\geq 1$, is equivalent to a non-increasing function, it follows that for $s\geq 1$, $s^{-\theta}\lesssim (1+\log s)^{-1}$. Thus

$$\|a\|_{q;J} \sim \left(\int_0^1 \left[t^{-\theta}(1-\log t)^{-1/q'}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Next, we derive (5.15) from (5.14). Put $\bar{Y}=(\bar{A}_{\theta,q},A_1)$. Since

$$\big((A_0,A_1)_{\theta,q},A_1\big)_{q;J} = \big(A_1,(A_1,A_0)_{1-\theta,q}\big)_{q;J},$$

we have that

$$\begin{aligned}
\|a\|_{\bar{Y}_{q;J}} &\sim \left(\int_0^1 \left[t^{\theta-1}(1-\log t)^{-1/q'}K(t,a;A_1,A_0)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(t,a;A_1,A_0)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^1 \left[t^{\theta-1}(1-\log t)^{-1/q'}tK\left(t^{-1},a\right)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{tK\left(t^{-1},a\right)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^1 \left[\frac{K(t,a)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{-\theta}(1+\log t)^{-1/q'}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

This ends the proof. □

Remark 5.3. Formulae (5.14) and (5.15) do not hold when $q=1$. Indeed, let $\widetilde{A_j}$ denote the Gagliardo completion of A_j in A_0+A_1 (see [4, Section 5.1]). If equation (5.14) were also true for $q=1$ then we would have $(A_0,(A_0,A_1)_{\theta,1})_{1;J}=(\widetilde{A_0},(\widetilde{A_0},\widetilde{A_1})_{\theta,1})_{1;J}$ because $K(t,a;A_0,A_1)=K\big(t,a;\widetilde{A_0},\widetilde{A_1}\big)$. However, take $A_0=c_0$ and $A_1=\ell_\infty(2^{-m})$, where sequences are indexed by $\mathbb{N}$. Then, on the one hand, $(A_0,(A_0,A_1)_{\theta,1})_{1;J}=A_0=c_0$ because $A_0\hookrightarrow A_1$, and on the other hand, by [80, Theorem 1.18.2] and Lemma 3.6,

$$\begin{aligned}
(\widetilde{A_0},(\widetilde{A_0},\widetilde{A_1})_{\theta,1})_{1;J} &= (\ell_\infty,(\ell_\infty,\ell_\infty(2^{-m}))_{\theta,1})_{1;J} = (\ell_\infty,\ell_1(2^{-\theta m}))_{1;J} \\
&= \ell_\infty \cap \ell_1(2^{-\theta m}) = \ell_\infty \neq c_0.
\end{aligned}$$

The same couple shows that (5.15) is not true either when $q=1$.

We now prove the corresponding formulae for the K-method.

Theorem 5.11. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple and let $0<\theta<1$ and $1\leq q\leq\infty$. Then we have with equivalence of norms*

$$\big(A_0,(A_0,A_1)_{\theta,q}\big)_{q;K} = \left\{\begin{array}{ll} a\in A_0+A_1: & \|a\| = \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} \\ & \quad + \left(\int_1^\infty \left[t^{-\theta}(1+\log t)^{1/q}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} < \infty \end{array}\right\}$$

and

$$\big((A_0,A_1)_{\theta,q},A_1\big)_{q;K} = \left\{\begin{array}{ll} a\in A_0+A_1: & \|a\| = \left(\int_1^\infty \left[\frac{K(t,a)}{t}\right]^q \frac{dt}{t}\right)^{1/q} \\ & \quad + \left(\int_0^1 \left[t^{-\theta}(1-\log t)^{1/q}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} < \infty \end{array}\right\}.$$

Proof. This time we prove the second formula and then derive the first one. Let $a \in A_0 + A_1$ and put $\bar{Y} = (\bar{A}_{\theta,q}, A_1)$. By (5.2),

$$\begin{aligned}
\|a\|_{\bar{Y}_{q;K}} &\sim \left(\int_0^1 \int_0^{t^{1/(1-\theta)}} \left[s^{-\theta}K(s,a)\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty t^{-q} \int_0^{t^{1/(1-\theta)}} \left[s^{-\theta}K(s,a)\right]^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 \int_{s^{1-\theta}}^1 \frac{dt}{t} s^{-\theta q}K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_0^1 s^{-\theta q}K(s,a)^q \int_1^\infty t^{-q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\
&\quad + \left(\int_1^\infty s^{-\theta q}K(s,a)^q \int_{s^{1-\theta}}^\infty t^{-q}\frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 \left(1-\log s^{1-\theta}\right) s^{-\theta q}K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty s^{-(1-\theta)q}s^{-\theta q}K(s,a)^q \frac{ds}{s}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[s^{-\theta}(1-\log s)^{1/q}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(s,a)}{s}\right]^q \frac{ds}{s}\right)^{1/q}.
\end{aligned}$$

On the other hand, we have that

$$\left(A_0, (A_0,A_1)_{\theta,q}\right)_{q;K} = \left((A_1,A_0)_{1-\theta,q}, A_0\right)_{q;K}.$$

Put $\bar{X} = (A_0, \bar{A}_{\theta,q})$. Then

$$\begin{aligned}
\|a\|_{\bar{X}_{q;K}} &\sim \left(\int_0^1 \left[s^{\theta-1}(1-\log s)^{1/q}K(s,a;A_1,A_0)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[\frac{K(s,a;A_1,A_0)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&= \left(\int_0^1 \left[s^{\theta-1}(1-\log s)^{1/q}sK(s^{-1},a)\right]^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[\frac{sK(s^{-1},a)}{s}\right]^q \frac{ds}{s}\right)^{1/q} \\
&= \left(\int_0^1 K(s,a)^q \frac{ds}{s}\right)^{1/q} + \left(\int_1^\infty \left[s^{-\theta}(1+\log s)^{1/q}K(s,a)\right]^q \frac{ds}{s}\right)^{1/q},
\end{aligned}$$

as desired. □

Remark 5.4. With similar arguments to those used in Remark 5.2, we can show that, if $A_0 \hookrightarrow A_1$, $0 < \theta < 1$ and $0 < \theta_0 < \theta_1 < 1$, then

$$\|a\|_{(\bar{A}_{\theta_0,q},\bar{A}_{\theta_1,q})_{q;J}} \sim \left(\int_1^\infty \left[\frac{t^{-\theta_0}K(t,a)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q},$$

thereby recovering [19, Theorem 3.7], and that

$$\|a\|_{(\bar{A}_{\theta_0,q},\bar{A}_{\theta_1,q})_{q;K}} \sim \left(\int_1^\infty \left[t^{-\theta_1}K(t,a)(1+\log t)^{1/q}\right]^q \frac{dt}{t}\right)^{1/q},$$

obtaining [19, Remark 7.3]. Moreover, we can derive that

$$\|a\|_{(\bar{A}_{\theta,q},A_1)_{q;J}} \sim \left(\int_1^\infty \left[\frac{t^{-\theta}K(t,a)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} \tag{5.16}$$

and

$$\|a\|_{(A_0,\bar{A}_{\theta,q})_{q;J}} \sim \|a\|_{\bar{A}_{0,q;J}} \tag{5.17}$$

which gives us [19, Theorem 4.6 a) and b)], respectively. We can also simplify in this case the formulae in Theorem 5.11 to obtain

$$\|a\|_{(A_0,\bar{A}_{\theta,q})_{q;K}} \sim \left(\int_1^\infty \left[t^{-\theta}(1+\log t)^{1/q} K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} \tag{5.18}$$

and

$$\|a\|_{(\bar{A}_{\theta,q},A_1)_{q;J}} \sim \|a\|_{\bar{A}_{1,q;K}}\,. \tag{5.19}$$

5.3 Examples

First, we apply some of the reiteration formulae to the Banach couple of Lebesgue spaces (L_∞, L_1). Just as in Theorem 3.21(i), we will obtain intersections of Lorentz-Zygmund spaces.

Theorem 5.12. *Let (Ω,μ) be a σ-finite measure space, let $1 < p_0, p_1 < \infty, 1 \le q \le \infty$ and $1/q + 1/q' = 1$. Then we have with equivalence of norms*

$$(L_{p_0,q}, L_{p_1,q})_{q;J} = L_{p_0,q}(\log L)_{-1/q'} \cap L_{p_1,q}(\log L)_{-1/q'}.$$

Proof. Put $\theta_j = 1/p_j, j = 0,1$. Then

$$(L_\infty, L_1)_{\theta_j,q} = L_{p_j,q} \quad \text{and} \quad K(t,f) = t\int_0^{t^{-1}} f^*(s)\,ds = f^{**}(t^{-1}) \tag{5.20}$$

(see [5, Theorem 5.2.1]). It follows from Theorem 5.9 that

$$\begin{aligned}
\|f\|_{q;J} &\sim \left(\int_0^\infty \left[\frac{\max(t^{1/p_0}, t^{1/p_1})}{(1+|\log t|)^{1/q'}} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q}\\
&\sim \left(\int_0^\infty \left[t^{1/p_0}\frac{f^{**}(t)}{(1+|\log t|)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^\infty \left[t^{1/p_1}\frac{f^{**}(t)}{(1+|\log t|)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q}\\
&\sim \max\left[\left(\int_0^\infty \left[t^{1/p_0}\frac{f^{**}(t)}{(1+|\log t|)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q}, \left(\int_0^\infty \left[t^{1/p_1}\frac{f^{**}(t)}{(1+|\log t|)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q}\right]\\
&\sim \|f\|_{L_{p_0,q}(\log L)_{-1/q'} \cap L_{p_1,q}(\log L)_{-1/q'}},
\end{aligned}$$

as desired. □

Spaces $(L_{p_0,q}, L_{p_1,q})_{q;K}$ can be described as well with the help of Theorem 5.9. However, we will proceed using duality.

Theorem 5.13. *Let* (Ω,μ) *be a resonant,* σ*-finite measure space, and let* $1 < p_0, p_1 < \infty$ *and* $1 < q \leq \infty$. *Then we have with equivalence of norms*

$$(L_{p_0,q}, L_{p_1,q})_{q;K} = L_{p_0,q}(\log L)_{1/q} + L_{p_1,q}(\log L)_{1/q}.$$

Proof. By [44, Theorem 3.4.41 and Lemma 3.4.43],

$$\left(L_{p_j',q'}(\log L)_{-1/q}\right)^* = L_{p_j,q}(\log L)_{1/q}.$$

Using duality between limiting K- and J-spaces (that is, Theorem 3.18) and Theorem 5.12, we derive

$$\begin{aligned}(L_{p_0,q}, L_{p_1,q})_{q;K} &= \left[\left(L_{p_0',q'}, L_{p_1',q'}\right)_{q';J}\right]^* = \left[L_{p_0',q'}(\log L)_{-1/q} \cap L_{p_1',q'}(\log L)_{-1/q}\right]^* \\ &= L_{p_0,q}(\log L)_{1/q} + L_{p_1,q}(\log L)_{1/q}.\end{aligned}$$

This ends the proof. □

Next we consider couples where one of the spaces is L_∞ or L_1.

Theorem 5.14. *Let* (Ω,μ) *be a* σ*-finite measure space, let* $1 < p < \infty$, $1 < q \leq \infty$ *and* $1/q + 1/q' = 1$. *Then we have with equivalence of norms*

$$(L_\infty, L_{p,q})_{q;J} = L_{p,q}(\log L)_{-1/q'} \cap L_{\infty,q}(\log L)_{-1}$$

and

$$(L_{p,q}, L_1)_{q;J} = L_{p,q}(\log L)_{-1/q'} \cap L_{(1,q)}(\log L)_{-1}.$$

Proof. Recall that throughout the proof of Theorem 3.21(i) we showed that

$$\|f\|_{L_{\infty,q}(\log L)_{-1}} \sim \left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q}$$

(equation (3.27)). Hence,

$$\begin{aligned}\|f\|_{L_{p,q}(\log L)_{-1/q'}\cap L_{\infty,q}(\log L)_{-1}} \sim &\left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[\frac{t^{1/p}f^{**}(t)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} \\ &+ \left(\int_1^\infty \left[\frac{t^{1/p}f^{**}(t)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since $g(t) = t^{1/p}(1-\log t)^{1/q}$ is equivalent to an increasing function, we have

$$\left(\int_0^1 \left[\frac{t^{1/p}f^{**}(t)}{(1-\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q}.$$

So, the norm $\|f\|_{L_{p,q}(\log L)_{-1/q'}\cap L_{\infty,q}(\log L)_{-1}}$ is equivalent to

$$\left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{t^{1/p}f^{**}(t)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q}.$$

On the other hand, we have that $L_{p,q} = (L_\infty, L_1)_{1/p,q}$, so $(L_\infty, L_{p,q})_{q;J} = \big(L_\infty, (L_\infty, L_1)_{1/p,q}\big)_{q;J}$. Therefore, using (5.20) and (5.14) we obtain

$$\|f\|_{(L_\infty,L_{p,q})_{q;J}} \sim \left(\int_0^1 \left[t^{-1/p}(1-\log t)^{-1/q'} f^{**}(1/t)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{f^{**}(1/t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^1 \left[\frac{f^{**}(t)}{1-\log t}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{1/p}(1+\log t)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q}.$$

This ends the proof of the first equality.

The proof of the second formula follows similar lines. Indeed, using (5.20) and (5.15), we obtain that

$$\|f\|_{(L_{p,q},L_1)_{q;J}} \sim \left(\int_0^1 \left[\frac{f^{**}(1/t)}{t(1-\log t)}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{-1/p}(1+\log t)^{-1/q'} f^{**}(1/t)\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^1 \left[t^{1/p}(1-\log t)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q},$$

so clearly

$$\|f\|_{(L_{p,q},L_1)_{q;J}} \lesssim \left(\int_0^\infty \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^\infty \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= \|f\|_{L_{p,q}(\log L)_{-1/q'} \cap L_{(1,q)}(\log L)_{-1}}.$$

Since the function $h(t) = t^{1/p-1}(1+|\log t|)^{1/q}$ is equivalent to a decreasing function, we also have that

$$\left(\int_1^\infty \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_1^\infty \left[t^{1/p-1}(1+\log t)^{1/q}\frac{tf^{**}(t)}{1+\log t}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_1^\infty \left[tf^{**}(t)(1+|\log t|)^{-1}\right]^q \frac{dt}{t}\right)^{1/q}.$$

On the other hand, we derive that

$$\left(\int_0^1 \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^1 (1-\log t)^{-q}\left[\int_0^t f^*(s)\,ds\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\le \left(\int_0^1 f^*(s)\,ds\right)\left(\int_0^1 (1-\log t)^{-q}\frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_0^1 f^*(s)\,ds\right)\left(\int_1^\infty (1+\log t)^{-q}\frac{dt}{t}\right)^{1/q}$$
$$\le \left(\int_1^\infty \left[\left(\frac{1}{t}\int_0^t f^*(s)\,ds\right) t(1+|\log t|)^{-1}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Therefore,

$$
\begin{aligned}
&\left(\int_0^\infty \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^\infty \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad \lesssim \left(\int_0^1 \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\qquad + \left(\int_0^1 \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad \lesssim \left(\int_1^\infty \left[\frac{tf^{**}(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[t^{1/p}(1+|\log t|)^{-1/q'} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} \sim \|f\|_{(L_{p,q},L_1)_{q;J}}.
\end{aligned}
$$

This ends the proof. $\square$

Now we turn our attention to spaces of operators. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the Banach space of all bounded linear operators in $\mathcal{H}$. The *singular numbers of* $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$s_n(T) = \inf\left\{\|T-R\|_{\mathcal{L}(\mathcal{H})} : R \in \mathcal{L}(\mathcal{H}) \text{ with rank } R < n\right\},\ n \in \mathbb{N}.$$

Clearly, the sequence of singular numbers $(s_n(T))$ is non-increasing.

Given $1 \le p \le \infty$, let $\mathcal{L}_p(\mathcal{H})$ denote the *Schatten* p*-class*, that is, the collection of all those $T \in \mathcal{L}(\mathcal{H})$ which have a finite norm $\|T\|_{\mathcal{L}_p(\mathcal{H})} = (\sum_{n=1}^\infty s_n(T)^p)^{1/p}$. See [53]. Similarly, we can define the spaces $\mathcal{L}_{p,q}(\mathcal{H})$ for $1 \le p, q \le \infty$ as those consisting of the operators $T \in \mathcal{L}(\mathcal{H})$ that have a finite norm

$$\|T\|_{\mathcal{L}_{p,q}(\mathcal{H})} = \left(\sum_{n=1}^\infty \left[n^{1/p} s_n(T)\right]^q n^{-1}\right)^{1/q}.$$

We will also consider the spaces $\mathcal{L}_{p,q;\gamma}(\mathcal{H})$ for $1 \le p \le \infty$, $1 \le q \le \infty$ and $\gamma \in \mathbb{R}$ defined as the collection of operators $T \in \mathcal{L}(\mathcal{H})$ for which the following norm is finite

$$\|T\|_{\mathcal{L}_{p,q;\gamma}(\mathcal{H})} = \left(\sum_{n=1}^\infty \left[n^{1/p}(1+\log n)^\gamma s_n(T)\right]^q n^{-1}\right)^{1/q}.$$

If $\gamma = 0$, we write for simplicity $\mathcal{L}_{p,q}(\mathcal{H}) = \mathcal{L}_{p,q,0}(\mathcal{H})$. We refer to [11, 34, 31] for properties of these spaces. Other families that we will study are the spaces $\mathcal{L}_{\mathcal{M},q}(\mathcal{H})$ for $1 \le q < \infty$, defined as the set of all $T \in \mathcal{L}(\mathcal{H})$ that have a finite norm

$$\|T\|_{\mathcal{L}_{\mathcal{M},q}(\mathcal{H})} = \left(\sum_{n=1}^\infty \left[\frac{1}{n}\sum_{m=1}^{2^n} s_m(T)\right]^q\right)^{1/q},$$

see [53, 39], and also the so-called *Macaev ideal* $\mathcal{L}_{\mathcal{M},\infty}(\mathcal{H})$ which consists of all $T \in \mathcal{L}(\mathcal{H})$ such that

$$\|T\|_{\mathcal{L}_{\mathcal{M},\infty}(\mathcal{H})} = \sup_{n\in\mathbb{N}}\left\{(1+\log n)^{-1}\sum_{m=1}^{n} s_m(T)\right\} < \infty.$$

Theorem 5.15. *For* $1 < q \le \infty$ *and* $1 < p < \infty$, *we have with equivalence of norms*

$$\left(\mathcal{L}_1(\mathcal{H}), \mathcal{L}_{p,q}(\mathcal{H})\right)_{q;J} = \mathcal{L}_{\mathcal{M},q}(\mathcal{H})$$

and

$$\left(\mathcal{L}_{p,q}(\mathcal{H}), \mathcal{L}(\mathcal{H})\right)_{q;J} = \mathcal{L}_{p,q;-1/q'}(\mathcal{H}).$$

Proof. It is well-known that

$$\left(\mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H})\right)_{\theta,q} = \mathcal{L}_{p,q}(\mathcal{H}), \text{ where } 1/p = 1-\theta, \tag{5.21}$$

and clearly $\mathcal{L}_1(\mathcal{H}) \hookrightarrow \mathcal{L}(\mathcal{H})$. Therefore, by (5.17), we get that

$$\left(\mathcal{L}_1(\mathcal{H}), \mathcal{L}_{p,q}(\mathcal{H})\right)_{q;J} = \left(\mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H})\right)_{0,q;J},$$

and, according to [19, Corollary 4.4], $\left(\mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H})\right)_{0,q;J} = \mathcal{L}_{\mathcal{M},q}(\mathcal{H})$.

Regarding the second formula, since

$$K(t, T; \mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H})) = \sum_{n=1}^{[t]} s_n(T) \text{ for } t \ge 1,$$

we derive that

$$K(m, T; \mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H})) = \sum_{n=1}^{m} s_n(T), \quad m \in \mathbb{N}, \tag{5.22}$$

(see [80]). By (5.21) and (5.16), it follows that

$$\begin{aligned}
\|T\|_{(\mathcal{L}_{p,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J}} &\sim \left(\int_1^\infty \left[\frac{t^{-1/p'}K(t,T)}{(1+\log t)^{1/q'}}\right]^q \frac{dt}{t}\right)^{1/q} = \left(\sum_{m=1}^\infty \int_m^{m+1} \left[\frac{t^{-1/p'-1/q}K(t,T)}{(1+\log t)^{1/q'}}\right]^q dt\right)^{1/q} \\
&\sim \left(\sum_{m=1}^\infty \left[\frac{m^{-1/p'-1/q}K(m,T)}{(1+\log m)^{1/q'}}\right]^q \int_m^{m+1} dt\right)^{1/q} \\
&= \left(\sum_{m=1}^\infty m^{-q/p'-1}(1+\log m)^{-q/q'}\left[\sum_{n=1}^m s_n(T)\right]^q\right)^{1/q}.
\end{aligned}$$

On the one hand, we have by the monotonicity of $(s_n(T))$ that

$$\begin{aligned}
\|T\|_{(\mathcal{L}_{p,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J}} &\gtrsim \left(\sum_{m=1}^\infty m^{-q/p'-1+q}(1+\log m)^{-q/q'} s_m(T)^q\right)^{1/q} \\
&= \left(\sum_{m=1}^\infty \left[\frac{s_m(T)}{(1+\log m)^{1/q'}}\right]^q m^{q/p-1}\right)^{1/q}.
\end{aligned}$$

On the other hand, pick $\rho > -1/q'$ such that $\rho + 1/p < 1/q$. Then

$$\sum_{n=1}^m s_n(T) = \sum_{n=1}^m n^\rho \frac{s_n(T)}{n^\rho} \le \left(\sum_{n=1}^m \left[\frac{s_n(T)}{n^\rho}\right]^q\right)^{1/q} \left(\sum_{n=1}^m n^{\rho q'}\right)^{1/q'}.$$

We claim that $\sum_{n=1}^{m} n^{\rho q'} \lesssim m^{\rho q'+1}$. Indeed, if $\rho \geq 0$, then

$$\sum_{n=1}^{m} n^{\rho q'} = \sum_{n=1}^{m} n^{\rho q'} \int_{n}^{n+1} dt \leq \sum_{n=1}^{m} \int_{n}^{n+1} t^{\rho q'} dt = \int_{1}^{m+1} t^{\rho q'} dt \lesssim m^{\rho q'+1}$$

since $\rho q' > -1$. If $\rho < 0$ then

$$\sum_{n=1}^{m} n^{\rho q'} = 1 + \sum_{n=2}^{m} n^{\rho q'} \int_{n-1}^{n} dt \leq 1 + \sum_{n=2}^{m} \int_{n-1}^{n} t^{\rho q'} dt = 1 + \int_{1}^{m} t^{\rho q'} dt \sim m^{\rho q'+1}.$$

This gives that

$$\begin{aligned}\|T\|^{q}_{(\mathcal{L}_{p,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J}} &\lesssim \sum_{m=1}^{\infty} m^{-q/p'-1+\rho q+q/q'}(1+\log m)^{-q/q'} \sum_{n=1}^{m} \left[\frac{s_n(T)}{n^{\rho}}\right]^{q} \\ &= \sum_{n=1}^{\infty} s_n(T)^{q} n^{-\rho q} \sum_{m=n}^{\infty} m^{q(1/q'+\rho-1/p')-1}(1+\log m)^{-q/q'}.\end{aligned}$$

Since $q(1/q'+\rho-1/p')-1<0$, the second sum can be estimated by $n^{q(1/q'+\rho-1/p')}(1+\log n)^{-q/q'}$. We derive, thus, that

$$\|T\|^{q}_{(\mathcal{L}_{p,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J}} \lesssim \sum_{n=1}^{\infty} s_n(T)^{q} n^{-\rho q} n^{\rho q+q-q/p'-1}(1-\log n)^{-q/q'} = \sum_{n=1}^{\infty} \left[\frac{s_n(T)}{(1+\log n)^{1/q'}}\right]^{q} n^{q/p-1},$$

which gives the second formula. □

Next we find the corresponding K-spaces.

Theorem 5.16. *For* $1 \leq q < \infty$ *and* $1 < p < \infty$, *we have with equivalence of norms*

$$(\mathcal{L}_{p,q}(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{q;K} = \mathcal{L}_{\infty,q}(\mathcal{H})$$

and

$$(\mathcal{L}_{1}(\mathcal{H}), \mathcal{L}_{p,q}(\mathcal{H}))_{q;K} = \mathcal{L}_{\infty,q;-1/q}(\mathcal{H}).$$

Proof. By (5.21) and (5.19), we derive that $(\mathcal{L}_{p,q}(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{q;K} = (\mathcal{L}_{1}(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{1,q;K}$. According to [31, Corollary 4.3], $(\mathcal{L}_{1}(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{1,q;K} = \mathcal{L}_{\infty,q}(\mathcal{H})$.

Regarding the second formula, for $T \in \mathcal{L}(\mathcal{H})$ and $q < \infty$, we have by (5.18) and (5.22) that

$$\begin{aligned}\|T\|^{q}_{(\mathcal{L}_{1}(\mathcal{H}),\mathcal{L}_{p,q}(\mathcal{H}))_{q;K}} &\sim \int_{1}^{\infty} \left[s^{-1/p'}(1+\log s)^{1/q} K(s,T)\right]^{q} \frac{ds}{s} \\ &\sim \sum_{m=1}^{\infty} m^{-q/p'-1}(1+\log m) \left(\sum_{n=1}^{m} s_n(T)\right)^{q}.\end{aligned}$$

By the monotonicity of $(s_n(T))$, we get that

$$\|T\|^{q}_{(\mathcal{L}_{1}(\mathcal{H}),\mathcal{L}_{p,q}(\mathcal{H}))_{q;K}} \geq \sum_{m=1}^{\infty} m^{(1-1/p')q-1}(1+\log m) s_m(T)^{q} = \sum_{m=1}^{\infty} \left[s_m(T)(1+\log m)^{1/q}\right]^{q} m^{q/p-1}.$$

Furthermore, for $q > 1$ and any ρ such that $-1/q' < \rho < 1/q - 1/p$, we derive, just as before, that

$$\begin{aligned}\|T\|^q_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{p,q}(\mathcal{H}))_{q;K}} &\lesssim \sum_{m=1}^{\infty} m^{-q/p'-1}(1+\log m)m^{\rho q+q/q'}\sum_{n=1}^{m} n^{-\rho q}s_n(T)^q\\ &= \sum_{n=1}^{\infty} s_n(T)^q n^{-\rho q}\sum_{m=n}^{\infty} m^{q(1/q'-1/p'+\rho)-1}(1+\log m)\\ &\lesssim \sum_{n=1}^{\infty} s_n(T)^q n^{q/p-1}(1+\log n),\end{aligned}$$

and the same estimate can be obtained trivially for $q = 1$. This ends the proof. □

Theorem 5.17. *For $1 \le q < \infty$, we have with equivalence of norms*

$$(\mathcal{L}_1(\mathcal{H}), \mathcal{L}_{\infty,q}(\mathcal{H}))_{q;K} = \mathcal{L}_{\infty,q,1/q}(\mathcal{H})$$

and

$$(\mathcal{L}_{\infty,q}(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{q;K} = \mathcal{L}_{\infty,q,-1/q}(\mathcal{H}).$$

Proof. Using (5.10) and [31, Corollary 4.3], we obtain

$$\|T\|_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\infty,q}(\mathcal{H}))_{q;K}} \sim \left(\int_1^{\infty}(1+\log t)\left[\frac{K(t,T;\mathcal{L}_1(\mathcal{H}),\mathcal{L}(\mathcal{H}))}{t}\right]^q\frac{dt}{t}\right)^{1/q}.$$

By (5.22) and the monotonicity of $(s_n(T))$, we derive that

$$\begin{aligned}\|T\|_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\infty,q}(\mathcal{H}))_{q;K}} &\sim \left(\sum_{m=1}^{\infty}(1+\log m)m^{-q-1}\left[\sum_{n=1}^{m}s_n(T)\right]^q\right)^{1/q}\\ &\ge \left(\sum_{m=1}^{\infty}(1+\log m)m^{-1}s_m(T)^q\right)^{1/q} = \|T\|_{\mathcal{L}_{\infty,q,1/q}(\mathcal{H})}.\end{aligned}$$

To check the converse inequality, take $\rho > 0$ such that $\rho q - 1 < 0$. Applying Hölder's inequality, we get

$$\sum_{n=1}^{m}s_n(T) \lesssim m^{\rho+1/q'}\left(\sum_{n=1}^{m}\left[\frac{s_n(T)}{n^\rho}\right]^q\right)^{1/q}.$$

Consequently,

$$\begin{aligned}\|T\|_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\infty,q}(\mathcal{H}))_{q;K}} &\lesssim \left(\sum_{m=1}^{\infty}(1+\log m)m^{-q-1+\rho q+q/q'}\sum_{n=1}^{m}\left[\frac{s_n(T)}{n^\rho}\right]^q\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty}\left[\frac{s_n(T)}{n^\rho}\right]^q\sum_{m=n}^{\infty}m^{\rho q-2}(1+\log m)\right)^{1/q}\\ &\lesssim \left(\sum_{n=1}^{\infty}\left[\frac{s_n(T)}{n^\rho}\right]^q n^{\rho q-1}(1+\log n)\right)^{1/q} = \|T\|_{\mathcal{L}_{\infty,q,1/q}(\mathcal{H})}.\end{aligned}$$

This establishes the first formula.

The second one can be proved similarly. Indeed, by (5.11) we have that

$$\|T\|_{(\mathcal{L}_{\infty,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;K}} \sim \left(\int_1^\infty \left[\frac{K(t,T;\mathcal{L}_1(\mathcal{H}),\mathcal{L}(\mathcal{H}))}{t(1+\log t)^{1/q}}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$\sim \left(\sum_{m=1}^\infty \left[\sum_{n=1}^m s_n(T)\right]^q m^{-1-q}(1+\log m)^{-1}\right)^{1/q}.$$

Clearly,

$$\|T\|_{(\mathcal{L}_{\infty,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;K}} \gtrsim \left(\sum_{m=1}^\infty s_m(T)^q m^q m^{-1-q}(1+\log m)^{-1}\right)^{1/q}$$
$$= \left(\sum_{m=1}^\infty \left[s_m(T)(1+\log m)^{-1/q}\right]^q m^{-1}\right)^{1/q}.$$

On the other hand, take $\rho > 0$ such that $\rho q - 1 < 0$. Then, just as before,

$$\sum_{n=1}^m s_n(T) \lesssim m^{\rho+1/q'}\left(\sum_{n=1}^m \left[\frac{s_n(T)}{n^\rho}\right]^q\right)^{1/q}.$$

So:

$$\|T\|_{(\mathcal{L}_{\infty,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;K}} \lesssim \left(\sum_{m=1}^\infty m^{\rho q+q/q'}\sum_{n=1}^m \left[\frac{s_n(T)}{n^\rho}\right]^q m^{-1-q}(1+\log m)^{-1}\right)^{1/q}$$
$$= \left(\sum_{n=1}^\infty n^{-\rho q}s_n(T)^q \sum_{m=n}^\infty m^{\rho q-2}(1+\log m)^{-1}\right)^{1/q}$$
$$\lesssim \left(\sum_{n=1}^\infty \left[s_n(T)(1+\log n)^{-1/q}\right]^q n^{-1}\right)^{1/q},$$

and therefore

$$(\mathcal{L}_{\infty,q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;K} = \mathcal{L}_{\infty,q-1/q}(\mathcal{H}).$$ □

Next we consider the spaces $\mathcal{L}_{\mathcal{M},q}(\mathcal{H})$. We have this time $\mathcal{L}_1(\mathcal{H}) \hookrightarrow \mathcal{L}_{\mathcal{M},q}(\mathcal{H}) \hookrightarrow \mathcal{L}_r(\mathcal{H})$ for any $1 < q, r \le \infty$.

Theorem 5.18. *For $1 < q \le \infty$, we have with equivalence of norms*

$$(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\mathcal{M},q}(\mathcal{H}))_{q;J} = \left\{T \in \mathcal{L}(\mathcal{H}) : \|T\| = \left(\sum_{m=1}^\infty \left[\frac{1}{m}\sum_{n=1}^{2^m} s_n(T)\right]^q (1+\log m)^{-q}m^{q-1}\right)^{1/q} < \infty\right\}$$

and

$$(\mathcal{L}_{\mathcal{M},q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J} = \left\{T \in \mathcal{L}(\mathcal{H}) : \|T\| = \left(\sum_{m=1}^\infty \left[\frac{1}{m}\sum_{n=1}^{2^m} s_n(T)\right]^q m^{1-q}\right)^{1/q} < \infty\right\}.$$

Proof. Once again we use the fact that $\mathcal{L}_{\mathcal{M},q}(\mathcal{H}) = (\mathcal{L}_1(\mathcal{H}), \mathcal{L}(\mathcal{H}))_{q;J}$ ([19, Corollary 4.4]). By (5.13) and (5.22), we derive

$$\|T\|_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\mathcal{M},q}(\mathcal{H}))_{q;J}} \sim \left(\int_1^\infty \left[\frac{K(t,T;\mathcal{L}_1(\mathcal{H}),\mathcal{L}(\mathcal{H}))}{1+\log(1+\log t)}\right]^q (1+\log t)^{-1}\frac{dt}{t}\right)^{1/q}$$
$$= \left(\sum_{m=0}^\infty \int_{2^m}^{2^{m+1}} \left[\frac{K(t,T;\mathcal{L}_1(\mathcal{H}),\mathcal{L}(\mathcal{H}))}{1+\log(1+\log t)}\right]^q (1+\log t)^{-1}\frac{dt}{t}\right)^{1/q}.$$

In addition, if $t \in [2^m, 2^{m+1}]$,

$$\sum_{n=1}^{2^m} s_n(T) \le \sum_{n=1}^{[t]} s_n(T) \le \sum_{n=1}^{2^{m+1}} s_n(T)$$

and

$$\sum_{n=1}^{2^{m+1}} s_n(T) = \sum_{n=1}^{2^m} s_n(T) + \sum_{n=2^m+1}^{2^m+2^m} s_n(T) \le 2\sum_{n=1}^{2^m} s_n(T)$$

since $s_{2^m+k} \le s_k$, so

$$\sum_{n=1}^{[t]} s_n(T) \sim \sum_{n=1}^{2^{m+1}} s_n(T) \tag{5.23}$$

whenever $t \in [2^m, 2^{m+1}]$. Therefore, applying (5.22) we get

$$\|T\|_{(\mathcal{L}_1(\mathcal{H}),\mathcal{L}_{\mathcal{M},q}(\mathcal{H}))_{q;J}} \sim \left(\sum_{m=0}^\infty \left[\sum_{n=1}^{2^{m+1}} s_n(T)\right]^q (1+\log(m+1))^{-q}(1+m)^{-1}\right)^{1/q}$$
$$= \left(\sum_{m=1}^\infty \left[\frac{1}{m}\sum_{n=1}^{2^m} s_n(T)\right]^q (1+\log m)^{-q} m^{q-1}\right)^{1/q}.$$

The proof of the second formula is similar. Indeed, (5.12) and (5.23) give that

$$\|T\|_{(\mathcal{L}_{\mathcal{M},q}(\mathcal{H}),\mathcal{L}(\mathcal{H}))_{q;J}} \sim \left(\int_1^\infty \left[\sum_{n=1}^{[t]} s_n(T)\right]^q (1+\log t)^{1-2q}\frac{dt}{t}\right)^{1/q}$$
$$\sim \left(\sum_{m=0}^\infty (1+m)^{1-2q}\left[\sum_{n=1}^{2^{m+1}} s_n(T)\right]^q\right)^{1/q} = \left(\sum_{m=1}^\infty m^{1-2q}\left[\sum_{n=1}^{2^m} s_n(T)\right]^q\right)^{1/q}$$
$$= \left(\sum_{m=1}^\infty m^{1-q}\left[\frac{1}{m}\sum_{n=1}^{2^m} s_n(T)\right]^q\right)^{1/q}. \qquad \square$$

Chapter 6

Description of logarithmic interpolation spaces by means of the J-functional and applications

In this chapter we turn our attention to logarithmic interpolation spaces $(A_0, A_1)_{\theta,q}$, already defined in Chapter 2.

The theory of the case $0 < \theta < 1$ is covered by different papers (see, for instance, [55, 60, 46, 47]). We study here some open questions for the cases where $\theta = 0$ and $\theta = 1$. First of all we give the description of the spaces $(A_0, A_1)_{0,q,\mathbb{A}}$ and $(A_0, A_1)_{1,q,\mathbb{A}}$ by means of the J-functional. We will show that it depends on the relationship between $\mathbb{A}$ and q. Recall that this is not so in the case $0 < \theta < 1$.

Then we turn our attention to the behaviour of compact operators. Recently Edmunds and Opic [45] showed that if $T : L_{p_0} \longrightarrow L_{q_0}$ compactly and $T : L_{p_1} \longrightarrow L_{q_1}$ boundedly, then T is also compact when acting between Lorentz-Zygmund spaces which are very close to L_{p_0} and L_{q_0}. They supposed that the measure spaces involved were finite.

An abstract version of the results in [45] for Banach couples was obtained by Cobos, Fernández-Cabrera and Martínez in [23]. They required an embedding assumption on the last couple; this hypothesis corresponds to the finiteness of the measure spaces used in [45]. In this chapter we show that the results in [23] still hold without the embedding restriction and we obtain a version of the results in [45] for σ-finite measure spaces.

Furthermore, we use the J-representations to characterise weak compactness of interpolated operators when $\theta = 0$ or 1. In particular, we show that if $\alpha_0 + 1/q < 0 \leq \alpha_\infty + 1/q$, then $(A_0, A_1)_{1,q,\mathbb{A}}$ is reflexive if and only if the embedding $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ is weakly compact. However, if $\alpha_0 + 1/q < 0$ and $\alpha_\infty + 1/q < 0$ then the necessary and sufficient condition for the reflexivity of $(A_0, A_1)_{1,q,\mathbb{A}}$ is that the embedding $A_1 \hookrightarrow A_0 + A_1$ is weakly compact.

We also determine the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ and $(A_0, A_1)_{0,q,\mathbb{A}}$ in terms of the K-functional. In contrast to the classical theory, the duals of these spaces depend on the relationship between q and $\mathbb{A}$.

The plan of the chapter is as follows. In Section 6.1 we review the logarithmic K-interpolation spaces and we establish some basic properties. In Section 6.2 we study the equivalent description in terms of the J-functional when $\theta = 1, 0$. We also investigate the density of the intersection in the K-spaces. Compactness results are given in Section 6.3. Finally, Section 6.4 is devoted to weakly compact operators and duality. The main results of this chapter form the article [38].

6.1 Logarithmic interpolation methods

We start by recalling the definition of logarithmic interpolation methods. Put $\ell(t) = 1 + |\log t|$ and for $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ write

$$\ell^{\mathbb{A}}(t) = \ell^{(\alpha_0,\alpha_\infty)}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{if } 0 < t \le 1, \\ \ell^{\alpha_\infty}(t) & \text{if } 1 < t < \infty. \end{cases}$$

For $0 \le \theta \le 1$, $1 \le q \le \infty$ and $\mathbb{A} \in \mathbb{R}^2$, the *logarithmic interpolation space* $(A_0, A_1)_{\theta,q,\mathbb{A}}$ is formed by all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \left[t^{-\theta} \ell^{\mathbb{A}}(t) K(t,a) \right]^q \frac{dt}{t} \right)^{1/q}$$

(as usual, the integral should be replaced by the supremum when $q = \infty$). See [46, 47] for some of the properties of these spaces. When $\mathbb{A} = (0,0)$ and $0 < \theta < 1$, $(A_0, A_1)_{\theta,q,\mathbb{A}}$ coincides with the classical real interpolation space $(A_0, A_1)_{\theta,q}$ realised as a K-space (see [9, 5, 80, 4, 8]). Moreover, if $\mathbb{A} \neq (0,0)$ and $0 < \theta < 1$ then the resulting space is a special case of the real method with a function parameter (see [55, 60]). The properties of these two cases are well-known, for this reason we are only interested here in the values $\theta = 0$ and $\theta = 1$.

Remark 6.1. Since $K(t, a; A_0, A_1) = tK(t^{-1}, a; A_1, A_0)$, $a \in A_0 + A_1$, a change of variable yields that

$$(A_0, A_1)_{0,q,(\alpha_0,\alpha_\infty)} = (A_1, A_0)_{1,q,(\alpha_\infty,\alpha_0)} \tag{6.1}$$

with equality of norms. So it is enough to study the case $\theta = 1$.

Remark 6.2. It is shown in [47, Theorem 2.2] that $(A_0, A_1)_{1,q,\mathbb{A}} = \{0\}$ if $q < \infty$ and $\alpha_0 + 1/q \ge 0$ or $q = \infty$ and $\alpha_0 > 0$. Therefore, in what follows we assume

$$\begin{cases} \alpha_0 + 1/q < 0 & \text{if } q < \infty, \\ \alpha_0 \le 0 & \text{if } q = \infty. \end{cases} \tag{6.2}$$

In the assumption (6.2) it turns out that

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow A_0 + A_1.$$

Besides $(A_0, A_1)_{1,q,\mathbb{A}}$ is complete in this assumption provided that A_0 and A_1 are complete.

Next we show a special case where the norm of $(A_0, A_1)_{1,q,\mathbb{A}}$ is equivalent to the part of the integral over $(0,1)$.

Lemma 6.1. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple. Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \le q \le \infty$ *satisfying* (6.2). *Assume in addition that*

$$\begin{cases} \alpha_\infty + 1/q < 0 & \text{if } q < \infty, \\ \alpha_\infty \le 0 & \text{if } q = \infty. \end{cases} \tag{6.3}$$

Then

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} \sim \left(\int_0^1 \left[t^{-1} K(t,a) \ell^{\alpha_0}(t) \right]^q \frac{dt}{t} \right)^{1/q}.$$

Proof. Clearly

$$\left(\int_0^1 \left[t^{-1} K(t,a) \ell^{\alpha_0}(t) \right]^q \frac{dt}{t} \right)^{1/q} \le \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}}.$$

To check the converse inequality, assume first that $q < \infty$. Using that $t^{-1}K(t,a)$ is non-increasing and that $\alpha_\infty q + 1 < 0$ and $\alpha_0 q + 1 < 0$ we obtain

$$\begin{aligned} \left(\int_1^\infty \left[t^{-1} K(t,a) \ell^{\alpha_\infty}(t) \right]^q \frac{dt}{t} \right)^{1/q} &\le \frac{K(1,a)}{1} \left(\int_1^\infty \ell^{\alpha_\infty q}(t) \frac{dt}{t} \right)^{1/q} \lesssim \frac{K(1,a)}{1} \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t} \right)^{1/q} \\ &\le \left(\int_0^1 \left[t^{-1} K(t,a) \ell^{\alpha_0}(t) \right]^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

The proof when $q = \infty$ is analogous. □

Now we derive some consequences of this result.

Lemma 6.2. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple. Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \le q \le \infty$ *satisfying* (6.2) *and* (6.3). *Then we have with equivalent norms*

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,\mathbb{A}}.$$

Proof. Combining Lemma 6.1 with the fact that

$$K(\min(1,t), a; A_0, A_1) = K(t, a; A_0 + A_1, A_1) \tag{6.4}$$

(see [68, Theorem 2]), we get

$$\begin{aligned} \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} &\sim \left(\int_0^1 \left[t^{-1} K(\min(1,t), a; A_0, A_1) \ell^{\alpha_0}(t) \right]^q \frac{dt}{t} \right)^{1/q} \\ &= \left(\int_0^1 \left[t^{-1} K(t, a; A_0 + A_1, A_1) \ell^{\alpha_0}(t) \right]^q \frac{dt}{t} \right)^{1/q} \sim \|a\|_{(A_0+A_1,A_1)_{1,q;\mathbb{A}}}. \end{aligned}$$ □

Corollary 6.3. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple. Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \le q \le \infty$ *satisfying* (6.2) *and* (6.3). *Take any* $\alpha > -1/q$. *Then we have with equivalent norms*

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,(\alpha_0,\alpha)}.$$

Proof. Since $A_1 \hookrightarrow A_0 + A_1$, it follows that $K(t, a; A_0 + A_1, A_1) \sim \|a\|_{A_0+A_1}$ if $t \geq 1$ by reversing the order of the couple in (2.8). Hence, by Lemmata 6.1 and 6.2, we derive

$$\begin{aligned}
\|a\|_{(A_0+A_1,A_1)_{1,q,(\alpha_0,\alpha)}} &\sim \left(\int_0^1 \left[t^{-1}K(t,a;A_0+A_1,A_1)\ell^{\alpha_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_1^\infty \left[t^{-1}K(t,a;A_0+A_1,A_1)\ell^{\alpha}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} + \|a\|_{A_0+A_1} \left(\int_1^\infty t^{-q}\ell^{\alpha q}(t) \frac{dt}{t}\right) \\
&\sim \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} + \|a\|_{A_0+A_1} \sim \|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}}.
\end{aligned}$$

□

For later use, we establish now a result on the behaviour of the K-functional for elements in the space $(A_0, A_1)_{1,q,\mathbb{A}}$.

Lemma 6.4. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \leq q \leq \infty$ satisfying* (6.2). *If any of the following two conditions holds*

$$q < \infty \text{ and } \alpha_\infty + 1/q > 0 \quad \text{or} \quad q = \infty \text{ and } \alpha_\infty > 0, \tag{6.5}$$

or

$$q < \infty \text{ and } \alpha_\infty + 1/q = 0, \tag{6.6}$$

then for any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$ we have that

$$\min(1, t^{-1})K(t, a) \to 0 \text{ as } t \to 0 \text{ or as } t \to \infty.$$

Proof. Suppose $q < \infty$. The proof of the remaining case can be carried out in the same way.

Let $a \in (A_0, A_1)_{1,q,\mathbb{A}}$. Then $\int_0^1 [t^{-1}\ell^{\alpha_0}(t)K(t,a)]^q \frac{dt}{t} < \infty$. Since $\int_0^1 [t^{-1}\ell^{\alpha_0}(t)]^q \frac{dt}{t} = \infty$ and $K(t, a)$ is non-decreasing, it follows that $K(t, a) \to 0$ as $t \to 0$.

On the other hand $\int_1^\infty [t^{-1}\ell^{\alpha_\infty}(t)K(t,a)]^q \frac{dt}{t} < \infty$ and, by the assumption on α_∞, we also have that $\int_1^\infty \ell^{\alpha_\infty q}(t) \frac{dt}{t} = \infty$. Since $t^{-1}K(t, a)$ is non-increasing, we conclude that also $t^{-1}K(t, a) \to 0$ as $t \to \infty$. □

6.2 Representation in terms of the J-functional

Let $\ell\ell(t) = \ell(\ell(t)) = 1 + \log(1 + |\log t|)$ and if $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ put

$$\ell\ell^{\mathbb{B}}(t) = \begin{cases} \ell\ell^{\beta_0}(t) & \text{if } 0 < t \leq 1, \\ \ell\ell^{\beta_\infty}(t) & \text{if } 1 < t < \infty. \end{cases}$$

Write also $-\mathbb{B} = (-\beta_0, -\beta_\infty)$. As usual, if $1 \leq q \leq \infty$, we put $1/q + 1/q' = 1$.

Definition 6.1. Let $1 \le q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty), \mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ such that

$$\begin{cases} \text{if } 1 < q \le \infty & \text{then } 1 < \alpha_\infty + 1/q, \quad \text{or} \quad \alpha_\infty = 1 - 1/q \text{ and } \beta_\infty > 1 - 1/q, \\ \text{if } q = 1 & \text{then } 0 < \alpha_\infty, \quad \text{or} \quad \alpha_\infty = 0 \text{ and } \beta_\infty > 0. \end{cases} \tag{6.7}$$

Given any Banach couple $\bar{A} = (A_0, A_1)$, the space $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ is formed by all those vectors $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad (\text{convergence in } A_0 + A_1) \tag{6.8}$$

and

$$\left(\int_0^\infty \left[t^{-1} J(t, u(t))\, \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{6.9}$$

The norm in $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ is given by taking the infimum in (6.9) over all representations of the type (6.8), (6.9).

If $\mathbb{B} = (0,0)$, we write simply $(A_0, A_1)^J_{1,q,\mathbb{A}}$.

Remark 6.3. Conditions (6.7) and (6.9) yield that the integral $\int_0^\infty u(t) \frac{dt}{t}$ is absolutely convergent in $A_0 + A_1$. Indeed, since $\|u(t)\|_{A_0+A_1} \le \min(1, 1/t) J(t, u(t))$, using Hölder's inequality we obtain

$$\begin{aligned} \int_0^\infty \|u(t)\|_{A_0+A_1} \frac{dt}{t} &\le \left(\int_0^\infty \left[t^{-1} J(t, u(t))\, \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) \right]^q \frac{dt}{t} \right)^{1/q} \\ &\quad \times \left(\int_0^\infty \left[t \min(1, t^{-1}) \ell^{-\mathbb{A}}(t) \ell\ell^{-\mathbb{B}}(t) \right]^{q'} \frac{dt}{t} \right)^{1/q'}, \end{aligned}$$

and the last integral is finite by (6.7). This also shows that $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}} \hookrightarrow A_0 + A_1$.

Moreover, for any $a \in A_0 \cap A_1$ we have

$$\left(\int_0^\infty \left[\min(1, t) \ell^{-\mathbb{A}}(t) \ell\ell^{-\mathbb{B}}(t) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}} \le \|a\|_{A_0 \cap A_1}. \tag{6.10}$$

Indeed, let $\psi(t)$ be a non-negative function such that

$$\left(\int_0^\infty \left[\frac{\psi(t)}{t \ell^{-\mathbb{A}}(t) \ell\ell^{-\mathbb{B}}(t)} \right]^q \frac{dt}{t} \right)^{1/q} = 1.$$

Put

$$u(t) = \frac{\psi(t) \min(1, t^{-1})}{\int_0^\infty \psi(s) \min(1, s^{-1}) \frac{ds}{s}} a.$$

Clearly, $\int_0^\infty u(t) \frac{dt}{t} = a$. Furthermore, using that $J(t, a) \le \max(1, t) \|a\|_{A_0 \cap A_1}$, we obtain

$$\begin{aligned} \left(\int_0^\infty \psi(s) \min(1, s^{-1}) \frac{ds}{s} \right) \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}} &\le \left(\int_0^\infty \psi(s) \min(1, s^{-1}) \frac{ds}{s} \right) \\ &\quad \times \left(\int_0^\infty \left[t^{-1} J(t, u(t))\, \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) \right]^q \frac{dt}{t} \right)^{1/q} \end{aligned}$$

$$\leq \left(\int_0^\infty \left[\frac{\psi(t)}{t\ell^{-\mathbb{A}}(t)\ell\ell^{-\mathbb{B}}(t)} \right]^q \frac{dt}{t} \right)^{1/q} \|a\|_{A_0\cap A_1} = \|a\|_{A_0\cap A_1}.$$

Taking the supremum over all possible functions ψ we derive (6.10).

Note that (6.10) shows that if (6.7) is not satisfied the space $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ is meaningless because $\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}} = 0$ for any $a \in A_0 \cap A_1$. On the other hand, if (6.7) holds, then (6.10) yields that $A_0 \cap A_1 \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$.

Remark 6.4. Let $\bar{A} = (A_0, A_1)$ be any Banach couple and let A_j^o be the closure of $A_0 \cap A_1$ in A_j, $j = 0, 1$. Clearly $A_0 \cap A_1 = A_0^o \cap A_1^o$ and $J(t, a; A_0, A_1) = J(t, a; A_0^o, A_1^o)$ if $a \in A_0 \cap A_1$. Having Remark 6.3 in mind, it follows that

$$(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}} = (A_0^o, A_1^o)^J_{1,q,\mathbb{A},\mathbb{B}}. \tag{6.11}$$

More general J-spaces are investigated in [8], but equality (6.11) also holds for them because the assumptions on $J(t, u(t))$ still yield that $\int_0^\infty u(t)\frac{dt}{t}$ is absolutely convergent in $A_0 + A_1$ (see [8, page 362]).

Next we study whether the interpolation method $(A_0, A_1)_{1,q,\mathbb{A}}$ can be described using the J-functional. We start with a negative result.

Proposition 6.5. *Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \leq q \leq \infty$ *satisfying* (6.2) *and* (6.3). *Then the* $(1, q, \mathbb{A}; K)$-*method does not admit a description as a* J-*method.*

Proof. By Remark 6.4, a necessary condition for the $(1, q, \mathbb{A}; K)$-method to be described as a J-method is that for any Banach couple $\bar{A} = (A_0, A_1)$ we have that $(A_0, A_1)_{1,q,\mathbb{A}} = (A_0^o, A_1^o)_{1,q,\mathbb{A}}$. However, if we choose $A_0 = \ell_1$ and $A_1 = \ell_\infty$, then $A_0^o = \ell_1$, $A_1^o = c_0$ and $A_0 + A_1 = \ell_\infty$. According to Lemma 6.2, we get

$$(\ell_1, \ell_\infty)_{1,q,\mathbb{A}} = (\ell_\infty, \ell_\infty)_{1,q,\mathbb{A}} = \ell_\infty \neq c_0 = (c_0 + \ell_1, c_0)_{1,q,\mathbb{A}} = (\ell_1, c_0)_{1,q,\mathbb{A}} = (\ell_1^o, \ell_\infty^o)_{1,q,\mathbb{A}}. \qquad \square$$

In the following results, given any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we write $\mathbb{A} + 1 = (\alpha_0 + 1, \alpha_\infty + 1)$. We also put

$$\widetilde{K}(t, a) = K(t, a; A_1, A_0) \quad \text{and} \quad \widetilde{J}(t, a) = J(t, a; A_1, A_0).$$

Theorem 6.6. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple. Let* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \leq q \leq \infty$ *satisfying* (6.2) *and* (6.5). *Then we have with equivalence of norms*

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0, A_1)^J_{1,q,\mathbb{A}+1}.$$

Proof. Our assumptions on $\mathbb{A}$ and q are

$$\begin{cases} q < \infty, \alpha_0 + 1/q < 0 \text{ and } \alpha_\infty + 1/q > 0, \\ \qquad \text{or} \\ q = \infty, \alpha_0 \leq 0 \text{ and } \alpha_\infty > 0. \end{cases} \tag{6.12}$$

Hence $\mathbb{A}+1$ and q satisfy (6.7), and the J-space is meaningful. We deal first with the case where $q<\infty$.

For $\nu \in \mathbb{Z}$, we put

$$\eta_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

Given any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, we can decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$, $j = 0,1$ and

$$\eta_{\nu-1}^{-1} \|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} \le 2\widetilde{K}\left(\eta_{\nu-1}^{-1}, a\right).$$

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$. Then

$$\begin{aligned} \left\| a - \sum_{\nu=N}^{M} u_\nu \right\|_{A_0+A_1} &\le \|a_{0,N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} \le 2\eta_{N-2}\widetilde{K}\left(\eta_{N-2}^{-1}, a\right) + 2\widetilde{K}\left(\eta_{M-1}^{-1}, a\right) \\ &\sim K\left(\eta_{N-2}, a\right) + \frac{K\left(\eta_{M-1}, a\right)}{\eta_{M-1}} \xrightarrow[N\to-\infty]{M\to\infty} 0 \end{aligned}$$

by Lemma 6.4. Hence $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$.

Write $L_\nu = (\eta_{\nu-1}, \eta_\nu]$, $\nu \in \mathbb{Z}$, and consider the function

$$u(t) = \begin{cases} \dfrac{u_\nu}{2^{-\nu-1}\log 2} & \text{if } t \in L_\nu \text{ and } \nu < 0, \\ \dfrac{u_\nu}{\log 2} & \text{if } t \in L_\nu \text{ and } \nu = 0,1, \\ \dfrac{u_\nu}{2^{\nu-2}\log 2} & \text{if } t \in L_\nu \text{ and } \nu > 1. \end{cases} \tag{6.13}$$

We have that

$$\int_0^\infty u(t)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} u_\nu = a.$$

Put

$$\alpha = \begin{cases} \alpha_0 & \text{if } \nu \le 0 \\ \alpha_\infty & \text{if } \nu > 0 \end{cases} \qquad \text{and} \qquad \hat{\alpha} = \begin{cases} \alpha_0 & \text{if } \nu - 2 \le 0 \\ \alpha_\infty & \text{if } \nu - 2 > 0. \end{cases}$$

It is not hard to check that

$$\int_{L_\nu} (1+|\log t|)^{\alpha q}\frac{dt}{t} \sim 2^{(|\nu|-2)(\alpha q+1)} \sim \int_{L_{\nu-2}} (1+|\log t|)^{\hat{\alpha} q}\frac{dt}{t}. \tag{6.14}$$

Indeed, it is clear that $|\log\eta_\nu| = 2^{|\nu|-1}\log 2 \sim 2^{|\nu|} \sim |\log\eta_{\nu-1}|$. Thus if $t \in L_\nu$ and $\beta \in \mathbb{R}$ then $(1+|\log t|)^{\beta q} \sim (1+|\log\eta_\nu|)^{\beta q} \sim 2^{|\nu|\beta q}$. This gives (6.14).

Moreover, a change of variables gives

$$\int_{L_\nu}\left[\frac{(1+|\log t|)^{\alpha+1}}{t} J\left(t, u(t)\right)\right]^q \frac{dt}{t} = \int_{\eta_\nu^{-1}}^{\eta_{\nu-1}^{-1}} \left[(1+|\log s|)^{\alpha+1}\widetilde{J}\left(s, u(s^{-1})\right)\right]^q \frac{ds}{s}.$$

If $s \in \left[\eta_\nu^{-1}, \eta_{\nu-1}^{-1}\right)$, then $s^{-1} \in L_\nu$ and so

$$\begin{aligned}\widetilde{J}\left(s, u(s^{-1})\right) &\lesssim 2^{-|\nu|}\widetilde{J}\left(\eta_{\nu-1}^{-1}, u_\nu\right) \le 2^{-|\nu|}\left[\eta_{\nu-1}^{-1}\left(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0}\right) + \|a_{1,\nu}\|_{A_1} + \|a_{1,\nu-1}\|_{A_1}\right] \\ &\lesssim 2^{-|\nu|}\left(\widetilde{K}\left(\eta_{\nu-1}^{-1}, a\right) + \widetilde{K}\left(\eta_{\nu-2}^{-1}, a\right)\right) \lesssim 2^{-|\nu|}\widetilde{K}\left(\eta_{\nu-2}^{-1}, a\right). \end{aligned} \tag{6.15}$$

Consequently, using (6.14) and the fact that $t^{-1}K(t,a)$ is non-increasing, we obtain

$$\begin{aligned}\int_{L_\nu}\left[\frac{(1+|\log t|)^{\alpha+1}}{t} J(t,u(t))\right]^q \frac{dt}{t} &\lesssim 2^{-|\nu|q}\widetilde{K}\left(\eta_{\nu-2}^{-1}, a\right)^q \int_{L_\nu}(1+|\log t|)^{(\alpha+1)q}\frac{dt}{t} \\ &\sim 2^{-|\nu|q}\left[\frac{K(\eta_{\nu-2}, a)}{\eta_{\nu-2}}\right]^q (1+|\log \eta_\nu|)^q \int_{L_\nu}(1+|\log t|)^{\alpha q}\frac{dt}{t} \\ &\sim \left[\frac{K(\eta_{\nu-2}, a)}{\eta_{\nu-2}}\right]^q \int_{L_{\nu-2}}(1+|\log t|)^{\hat{\alpha} q}\frac{dt}{t} \\ &\le \int_{L_{\nu-2}}\left[\frac{K(t,a)}{t}(1+|\log t|)^{\hat{\alpha}}\right]^q \frac{dt}{t}. \end{aligned}$$

This shows that $(A_0, A_1)_{1,q,\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q,\mathbb{A}+1}$.

In order to establish the converse embedding, take any $a \in (A_0, A_1)^J_{1,q,\mathbb{A}+1}$ and choose a representation of a, $a = \int_0^\infty u(t)\frac{dt}{t}$, such that

$$\left(\int_0^\infty \left[t^{-1}\ell^{\mathbb{A}+1}(t)J(t,u(t))\right]^q \frac{dt}{t}\right)^{1/q} \le 2\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.$$

For any $t > 0$, we have by (3.16) that

$$\frac{1}{t}K(t,a) \le \frac{1}{t}\int_0^t J(s,u(s))\frac{ds}{s} + \int_t^\infty \frac{1}{s}J(s,u(s))\frac{ds}{s}.$$

Whence

$$\begin{aligned}\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}} &\le \left(\int_0^1 \left[\frac{(1-\log t)^{\alpha_0}}{t}\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_0^1 \left[(1-\log t)^{\alpha_0}\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_1^\infty \left[\frac{(1+\log t)^{\alpha_\infty}}{t}\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_1^\infty \left[(1+\log t)^{\alpha_\infty}\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Term I_1 can be estimated by the variant of Hardy's inequality given in [3, Theorem 6.4/ (6.7)]. In fact,

$$I_1 \lesssim \left(\int_0^1 \left[t^{-1}(1-\log t)^{\alpha_0}J(t,u(t))\right]^q \frac{dt}{t}\right)^{1/q} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.$$

As for I_2, we write

$$I_2 \leq \left(\int_0^1 \left[(1-\log t)^{\alpha_0}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[(1-\log t)^{\alpha_0}\int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}$$
$$= J_1 + J_2.$$

Take any $0 < \varepsilon < -(\alpha_0 + 1/q)$ and let $h(s) = s^{-1}J(s,u(s))(1-\log s)^{\alpha_0+1+\varepsilon}$. By Hölder's inequality, we have

$$\begin{aligned}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} &= \int_t^1 h(s)(1-\log s)^{-(\alpha_0+1+\varepsilon)}\frac{ds}{s} \leq \left(\int_t^1 h(s)^q\frac{ds}{s}\right)^{1/q} \\ &\quad\times\left(\int_t^1 (1-\log s)^{-(\alpha_0+1+\varepsilon)q'}\frac{ds}{s}\right)^{1/q'} \\ &\lesssim (1-\log t)^{1/q'-\alpha_0-1-\varepsilon}\left(\int_t^1 h(s)^q\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Using Fubini's theorem, we derive

$$\begin{aligned}J_1 &\lesssim \left(\int_0^1 (1-\log t)^{-1-\varepsilon q}\int_t^1 h(s)^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q} = \left(\int_0^1 h(s)^q\int_0^s (1-\log t)^{-1-\varepsilon q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\lesssim \left(\int_0^1 h(s)^q(1-\log s)^{-\varepsilon q}\frac{ds}{s}\right)^{1/q} \lesssim \left(\int_0^1\left[\frac{J(s,u(s))}{s}(1-\log s)^{\alpha_0+1}\right]^q\frac{ds}{s}\right)^{1/q} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.\end{aligned}$$

To proceed with J_2, note that (6.12) yields that $\int_0^1 (1-\log t)^{\alpha_0 q}\frac{dt}{t} < \infty$. Hence, using Hölder's inequality we obtain

$$J_2 \sim \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} \leq \left(\int_1^\infty (1+\log s)^{-(\alpha_\infty+1)q'}\frac{ds}{s}\right)^{1/q'}\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.$$

As for I_3, we have

$$\begin{aligned}I_3 &\leq \left(\int_1^\infty\left[\frac{(1+\log t)^{\alpha_\infty}}{t}\int_0^1 J(s,u(s))\frac{ds}{s}\right]^q\frac{dt}{t}\right)^{1/q} \\ &\quad+\left(\int_1^\infty\left[\frac{(1+\log t)^{\alpha_\infty}}{t}\int_1^t J(s,u(s))\frac{ds}{s}\right]^q\frac{dt}{t}\right)^{1/q} = J_3 + J_4.\end{aligned}$$

Now

$$\begin{aligned}J_3 &\sim \int_0^1 J(s,u(s))\frac{ds}{s} \leq \left(\int_0^1\left[\frac{(1-\log s)^{\alpha_0+1}}{s}J(s,u(s))\right]^q\frac{ds}{s}\right)^{1/q} \\ &\quad\times\left(\int_0^1\left[\frac{s}{(1-\log s)^{\alpha_0+1}}\right]^{q'}\frac{ds}{s}\right)^{1/q'} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.\end{aligned}$$

To estimate J_4, take any $0 < \varepsilon < 1$. Using Hölder's inequality, we derive for the interior integral that

$$\int_1^t J(s,u(s))\frac{ds}{s} \lesssim \left(\int_1^t\left[\frac{J(s,u(s))}{s^\varepsilon}\right]^q\frac{ds}{s}\right)^{1/q} t^\varepsilon.$$

Whence, by Fubini's theorem, we get

$$J_4 \lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s^\varepsilon}\right]^q \int_s^\infty t^{(\varepsilon-1)q}(1+\log t)^{\alpha_\infty q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q}$$
$$\lesssim \left(\int_1^\infty \left[\frac{(1+\log s)^{\alpha_\infty}}{s} J(s,u(s))\right]^q \frac{ds}{s}\right)^{1/q} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.$$

Finally, to estimate I_4, take any $0<\varepsilon<\alpha_\infty+1/q$, so $\varepsilon-\alpha_\infty-1<-1/q'$, and write

$$g(s)=s^{-1}J\left(s,u(s)\right)(1+\log s)^{1+\alpha_\infty-\varepsilon}.$$

Hölder's inequality implies that

$$\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} \le \left(\int_t^\infty g(s)^q\frac{ds}{s}\right)^{1/q}\left(\int_t^\infty (1+\log s)^{(-1-\alpha_\infty+\varepsilon)q'}\frac{ds}{s}\right)^{1/q'}$$
$$\lesssim (1+\log t)^{1/q'-1-\alpha_\infty+\varepsilon}\left(\int_t^\infty g(s)^q\frac{ds}{s}\right)^{1/q}.$$

Consequently, changing the order of integration, we obtain

$$I_4 \lesssim \left(\int_1^\infty (1+\log t)^{-1+\varepsilon q}\int_t^\infty g(s)^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q} = \left(\int_1^\infty g(s)^q\int_1^s (1+\log t)^{-1+\varepsilon q}\frac{dt}{t}\frac{ds}{s}\right)^{1/q}$$
$$\le \left(\int_1^\infty \left[\frac{J(s,u(s))}{s}(1+\log s)^{1+\alpha_\infty}\right]^q\frac{ds}{s}\right)^{1/q} \le \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}.$$

This completes the proof for $q<\infty$. The case $q=\infty$ can be treated analogously; the only difference is that, when estimating J_1, one should just take the value $\varepsilon=0$. In fact, we get the following with $h(s)=s^{-1}J(s,u(s))(1-\log s)^{\alpha_0+1}$.

$$\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \le \sup_{t\le s<1} h(s)\int_t^1 (1-\log s)^{-\alpha_0-1}\frac{ds}{s} \lesssim \sup_{t\le s<1} h(s)(1-\log t)^{-\alpha_0}$$
$$= \sup_{t\le s<1} s^{-1}J(s,u(s))(1-\log s).$$

This gives that

$$J_1 = \sup_{0<t<1}(1-\log t)^{\alpha_0}\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \lesssim \sup_{0<s<1} s^{-1}J(s,u(s))(1-\log s)\sup_{0<t\le s}(1-\log t)^{\alpha_0}$$
$$\lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}}. \qquad \square$$

For the remaining parameters the J-representation is different.

Theorem 6.7. *Let $\bar{A}=(A_0,A_1)$ be a Banach couple. Let $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ and $1\le q\le\infty$ satisfying* (6.2) *and* (6.6)*, and let $\mathbb{B}=(0,1)$. Then we have with equivalent norms*

$$(A_0,A_1)_{1,q,\mathbb{A}} = (A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}.$$

Proof. Note that the assumption on $\mathbb{A}$ and q reads

$$q < \infty, \ \alpha_0 + 1/q < 0 \text{ and } \alpha_\infty = -1/q. \tag{6.16}$$

In order to check that $(A_0, A_1)_{1,q;\mathbb{A}} \hookrightarrow (A_0, A_1)^J_{1,q;\mathbb{A}+1,\mathbb{B}}$, we put for $\nu \in \mathbb{Z}$

$$\tau_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{2^{\nu-1}}} & \text{if } \nu > 0. \end{cases}$$

As in the previous theorem, given any $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, we decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$ and

$$\tau_{\nu-1}^{-1} \|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} \le 2\widetilde{K}(\tau_{\nu-1}^{-1}, a).$$

Then $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu}$ belongs to $A_0 \cap A_1$ and $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$ because Lemma 6.4 still holds in the assumption (6.16). Besides

$$\widetilde{J}(\tau_{\nu-1}^{-1}, u_\nu) \lesssim \widetilde{K}(\tau_{\nu-2}^{-1}, a). \tag{6.17}$$

Let $M_\nu = (\tau_{\nu-1}, \tau_\nu]$ and $\lambda_\nu = \int_{M_\nu} \ell^{-1}(s)\ell\ell^{-1}(s)\frac{ds}{s} \sim 1$ if $\nu > 0$. This time we put

$$v(t) = \begin{cases} \dfrac{u_\nu}{2^{-\nu-1}\log 2} & \text{if } t \in M_\nu \text{ and } \nu < 0, \\ \dfrac{u_0}{\log 2} & \text{if } t \in M_0, \\ \dfrac{u_\nu}{\lambda_\nu \ell(t)\ell\ell(t)} & \text{if } t \in M_\nu \text{ and } \nu > 0. \end{cases}$$

It follows that $\int_0^\infty v(t)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} u_\nu = a$.

If $\nu > 2$, we have that

$$\int_{M_\nu} \ell^{-1}(t)\frac{dt}{t} \sim \int_{2^{2^{\nu-2}}}^{2^{2^{\nu-1}}} \frac{dt}{t} \sim 2^{\nu-2} \sim \int_{M_{\nu-2}} \ell^{-1}(t)\frac{dt}{t}.$$

Therefore, using that the functions $t^{-1}J(t,u)$ and $t^{-1}K(t,a)$ are non-increasing and (6.17), we derive that

$$\begin{aligned} \int_{M_\nu} \left[\frac{J(t, v(t))}{t} \ell^{1-1/q}(t)\ell\ell(t)\right]^q \frac{dt}{t} &\sim \int_{M_\nu} \left[\frac{J(t,u_\nu)}{t}\ell^{-1/q}(t)\right]^q \frac{dt}{t} \le \left[\frac{J(\tau_{\nu-1}, u_\nu)}{\tau_{\nu-1}}\right]^q \int_{M_\nu} \ell^{-1}(t)\frac{dt}{t} \\ &= \widetilde{J}(\tau_{\nu-1}^{-1}, u_\nu)^q \int_{M_\nu} \ell^{-1}(t)\frac{dt}{t} \lesssim \widetilde{K}(\tau_{\nu-2}^{-1}, a)^q \int_{M_{\nu-2}} \ell^{-1}(t)\frac{dt}{t} \\ &= \left[\frac{K(\tau_{\nu-2}, a)}{\tau_{\nu-2}}\right]^q \int_{M_{\nu-2}} \ell^{-1}(t)\frac{dt}{t} \le \int_{M_{\nu-2}} \left[\frac{K(t,a)}{t}\ell^{-1/q}(t)\right]^q \frac{dt}{t}. \end{aligned}$$

We also have that if $\nu = 1, 2$ then

$$\int_{M_\nu}\left[\frac{J(t,v(t))}{t}\ell^{1-1/q}(t)\ell\ell(t)\right]^q\frac{dt}{t}\lesssim\int_{M_{\nu-2}}\left[\frac{K(t,a)}{t}\ell^{\alpha_0}(t)\right]^q\frac{dt}{t},$$

and the same argument as in the proof of Theorem 6.6 shows that

$$\int_{M_\nu}\left[\frac{J(t,v(t))}{t}\ell^{\alpha_0+1}(t)\right]^q\frac{dt}{t}\lesssim\int_{M_{\nu-2}}\left[\frac{K(t,a)}{t}\ell^{\alpha_0}(t)\right]^q\frac{dt}{t}$$

whenever $\nu<0$. Consequently, $\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}\lesssim\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}}$.

Take now any $a\in(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}$ and let $a=\int_0^\infty u(t)\frac{dt}{t}$ be a J-representation with

$$\left(\int_0^\infty\left[t^{-1}J(t,u(t))\,\ell^{\mathbb{A}+1}(t)\ell\ell^{\mathbb{B}}(t)\right]^q\frac{dt}{t}\right)^{1/q}\le 2\,\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}.$$

Then we can estimate the K-norm just as in the proof of Theorem 6.6 obtaining that

$$\|a\|_{(A_0,A_1)_{1,q,\mathbb{A}}}\le I_1+J_1+J_2+J_3+J_4+I_4.$$

The terms I_1 and J_1 involve integrals with variables on $(0,1)$. Since $\alpha_0+1/q<0$, the same argument as in Theorem 6.6 yields that $I_1+J_1\lesssim\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}$. Also J_3 can be estimated as before. Regarding J_4, the same argument in the previous theorem yields that

$$\begin{aligned}J_4&\lesssim\left(\int_1^\infty\left[\frac{(1+\log s)^{\alpha_\infty}}{s}J(s,u(s))\right]^q\frac{ds}{s}\right)^{1/q}\le\left(\int_1^\infty\left[\frac{(1+\log s)^{\alpha_\infty+1}\ell\ell(s)}{s}J(s,u(s))\right]^q\frac{ds}{s}\right)^{1/q}\\&\lesssim\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}.\end{aligned}$$

As for J_2, using that $\alpha_0+1/q<0$ and $q<\infty$, we derive

$$\begin{aligned}J_2&=\left(\int_0^1\left[(1-\log t)^{\alpha_0}\int_1^\infty\frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q\frac{dt}{t}\right)^{1/q}\sim\int_1^\infty\frac{J(s,u(s))}{s}\frac{ds}{s}\\&\le\left(\int_1^\infty\left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell(s)\right]^q\frac{ds}{s}\right)^{1/q}\left(\int_1^\infty(1+\log s)^{-1}(1+\log(1+\log s))^{-q'}\frac{ds}{s}\right)^{1/q'}\\&\lesssim\|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}.\end{aligned}$$

Finally, we proceed with

$$I_4=\left(\int_1^\infty\left[\ell^{-1/q}(t)\int_t^\infty\frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q\frac{dt}{t}\right)^{1/q}.$$

Take any $0<\varepsilon<1/q=1-1/q'$. We have

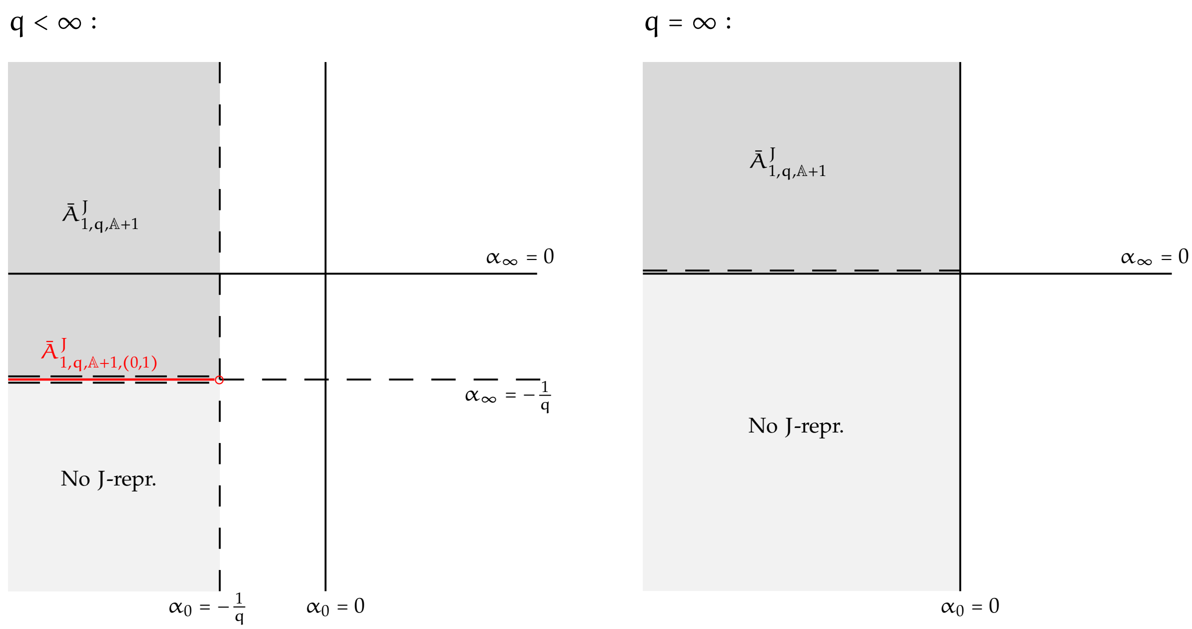
$$\begin{aligned}\int_t^\infty\frac{J(s,u(s))}{s}\frac{ds}{s}&\le\left(\int_t^\infty\left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{1-\varepsilon}(s)\right]^q\frac{ds}{s}\right)^{1/q}\left(\int_t^\infty\ell^{-1}(s)\ell\ell^{-q'+\varepsilon q'}(s)\frac{ds}{s}\right)^{1/q'}\\&\lesssim\left(\int_t^\infty\left[\frac{J(s,u(s))}{s}\ell^{1/q'}(s)\ell\ell^{1-\varepsilon}(s)\right]^q\frac{ds}{s}\right)^{1/q}\ell\ell^{\varepsilon-1/q}(t).\end{aligned}$$

Therefore, changing the order of integration, we derive

$$I_4 \lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell^{1-\varepsilon}(s) \right]^q \int_1^s \ell^{-1}(t) \ell\ell^{\varepsilon q - 1}(t) \frac{dt}{t} \frac{ds}{s} \right)^{1/q}$$

$$\lesssim \left(\int_1^\infty \left[\frac{J(s,u(s))}{s} \ell^{1/q'}(s) \ell\ell(s) \right]^q \frac{ds}{s} \right)^{1/q} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1,\mathbb{B}}}. \qquad \square$$

In order to clarify the situation, we include a diagram with the different areas of $\mathbb{R}^2$ in which $\mathbb{A}$ might be, and which J-representation corresponds to each of them.



Using these J-representations, we can now study the density of the intersection in the K-spaces.

Corollary 6.8. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Suppose that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q < \infty$ satisfy that $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$. Then $A_0 \cap A_1$ is dense in $(A_0, A_1)_{1,q,\mathbb{A}}$.*

Proof. Assume first that $0 < \alpha_\infty + 1/q$. Then (6.2) and (6.5) hold. By Theorem 6.6, given any vector $a \in (A_0, A_1)_{1,q,\mathbb{A}}$, there is a representation $a = \int_0^\infty u(t) \frac{dt}{t}$ such that

$$\left(\int_0^\infty \left[t^{-1} J(t,u(t))\, \ell^{\mathbb{A}+1}(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{6.18}$$

Take any $\varepsilon > 0$. Since $q < \infty$, we can find $M > 1$ such that

$$\left(\int_0^{1/M} \left[t^{-1} J(t,u(t))\, \ell^{\alpha_0+1}(t) \right]^q \frac{dt}{t} + \int_M^\infty \left[t^{-1} J(t,u(t))\, \ell^{\alpha_\infty+1}(t) \right]^q \frac{dt}{t} \right)^{1/q} < \varepsilon.$$

The integral $\int_{1/M}^M u(t) \frac{dt}{t}$ is absolutely convergent in $A_0 \cap A_1$ as follows by using that

$$\|u(t)\|_{A_0 \cap A_1} \le \max(1, t^{-1}) J(t,u(t)),$$

Hölder's inequality and (6.18). Let $w = \int_{1/M}^{M} u(t)\frac{dt}{t} \in A_0 \cap A_1$. We derive that

$$\|a - w\|_{(A_0,A_1)_{1,q,\mathbb{A}}} \sim \|a - w\|_{(A_0,A_1)^J_{1,q,\mathbb{A}+1}} \leq \left(\int_0^{1/M} \left[t^{-1} J\left(t, u(t)\right) \ell^{\alpha_0+1}(t)\right]^q \frac{dt}{t}\right.$$
$$\left. + \int_M^{\infty} \left[t^{-1} J\left(t, u(t)\right) \ell^{\alpha_\infty+1}(t)\right]^q \frac{dt}{t}\right)^{1/q} < \varepsilon.$$

Suppose now that $\alpha_\infty + 1/q = 0$. Then (6.2) and (6.6) are satisfied. The proof is similar but using this time Theorem 6.7. □

For the remaining case when $q < \infty$, as a direct consequence of Corollaries 6.3 and 6.8, we obtain the following.

Corollary 6.9. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Suppose that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \leq q < \infty$ satisfy that $\alpha_0 + 1/q < 0$ and $\alpha_\infty + 1/q < 0$. Then A_1 is dense in $(A_0, A_1)_{1,q,\mathbb{A}}$.*

Remark 6.5. In the assumptions of Corollary 6.9, the intersection $A_0 \cap A_1$ might not be dense in $(A_0, A_1)_{1,q,\mathbb{A}}$. Indeed, take $A_0 = \ell_1$, $A_1 = \ell_\infty$ and let $\alpha > -1/q$. By Corollary 6.3,

$$(\ell_1, \ell_\infty)_{1,q,\mathbb{A}} = (\ell_\infty, \ell_\infty)_{1,q,(\alpha_0,\alpha)} = \ell_\infty.$$

So $A_0 \cap A_1 = \ell_1$ is not dense in $(A_0, A_1)_{1,q,\mathbb{A}} = \ell_\infty$. However, if $A_0 \cap A_1$ is dense in A_1 then Corollaries 6.3 and 6.8 yield that $A_0 \cap A_1$ is also dense in $(A_0, A_1)_{1,q,\mathbb{A}}$.

Next, we consider the case $q = \infty$.

Proposition 6.10. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 < 0 < \alpha_\infty$. The closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,\mathbb{A}}$ is the space $(A_0, A_1)^o_{1,\infty,\mathbb{A}}$ formed by all those $a \in (A_0, A_1)_{1,\infty,\mathbb{A}}$ such that*

$$t^{-1}K(t, a)\ell^{\mathbb{A}}(t) \to 0 \textit{ as } t \to 0 \textit{ or } t \to \infty.$$

Proof. If a belongs to the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,\mathbb{A}}$, given any $\varepsilon > 0$, there is $w \in A_0 \cap A_1$ such that $\|a - w\|_{(A_0,A_1)_{1,\infty,\mathbb{A}}} \leq \varepsilon/2$. Since $\alpha_0 < 0$, we can find $M > 1$ such that

$$\ell^{\alpha_0}(t) \|w\|_{A_0 \cap A_1} \leq \varepsilon/2 \quad \text{if } 0 < t < 1/M$$

and

$$t^{-1}\ell^{\alpha_\infty}(t) \|w\|_{A_0 \cap A_1} \leq \varepsilon/2 \quad \text{if } t > M.$$

Besides,

$$K(t, a) \leq K(t, a - w) + K(t, w) \leq t\ell^{-\mathbb{A}}(t) \|a - w\|_{(A_0,A_1)_{1,\infty,\mathbb{A}}} + \min(1, t) \|w\|_{A_0 \cap A_1}.$$

Consequently, if $0 < t < 1/M$ or $t > M$, we derive that

$$t^{-1}K(t, a)\ell^{\mathbb{A}}(t) \leq \frac{\varepsilon}{2} + t^{-1}\min(1, t)\ell^{\mathbb{A}}(t) \|w\|_{A_0 \cap A_1} \leq \varepsilon.$$

So a belongs to $(A_0, A_1)^o_{1,\infty,\mathbb{A}}$.

Conversely, take any $a \in (A_0, A_1)^{\circ}_{1,\infty,\mathbb{A}}$ and any $\varepsilon > 0$. Since (6.2) and (6.5) hold, we are in the assumptions of Theorem 6.6. Let $u(t)$ be the function defined in (6.13) which is constant on each interval $L_\nu = (\eta_{\nu-1}, \eta_\nu]$, $\nu \in \mathbb{Z}$, and satisfies that $a = \int_0^\infty u(t)\frac{dt}{t}$. Using that $a \in (A_0, A_1)^{\circ}_{1,\infty,\mathbb{A}}$, we can find $N \in \mathbb{N}$ such that

$$\sup_{t \in L_{\nu-2}} t^{-1}\ell^{\mathbb{A}}(t)K(t,a) \leq \varepsilon \quad \text{for any } |\nu| > N.$$

Put $w = \int_{\eta_{-N-1}}^{\eta_N} u(t)\frac{dt}{t}$. Then $w \in A_0 \cap A_1$ and, by the construction of $u(t)$, we have that

$$\begin{aligned}\|a - w\|_{(A_0,A_1)_{1,\infty,\mathbb{A}}} \sim \|a - w\|_{(A_0,A_1)^J_{1,\infty,\mathbb{A}+1}} &= \sup_{|\nu|>N} \sup_{t \in L_\nu} t^{-1} J\left(t, u(t)\right) \ell^{\mathbb{A}+1}(t) \\ &\lesssim \sup_{|\nu|>N} \sup_{t \in L_{\nu-2}} t^{-1}K(t,a)\ell^{\mathbb{A}}(t) \leq \varepsilon.\end{aligned}$$

This shows that $(A_0, A_1)^{\circ}_{1,\infty,\mathbb{A}}$ is contained in the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,\mathbb{A}}$ and completes the proof. □

For the remaining case where $q = \infty$ and $\alpha_\infty \leq 0$, we obtain the following.

Corollary 6.11. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 < 0$ and $\alpha_\infty \leq 0$. Then the closure of A_1 in $(A_0, A_1)_{1,\infty,\mathbb{A}}$ is the space Z formed by all those $a \in (A_0, A_1)_{1,\infty,\mathbb{A}}$ such that*

$$t^{-1}K(t,a)\ell^{\alpha_0}(t) \to 0 \text{ as } t \to 0.$$

Moreover, if $A_0 \cap A_1$ is dense in A_1, then the closure $(A_0, A_1)^{\circ}_{1,\infty,\mathbb{A}}$ of $A_0 \cap A_1$ in $(A_0, A_1)_{1,\infty,\mathbb{A}}$ also coincides with Z.

Proof. Pick any $\alpha > 0$. Then we have that $(A_0, A_1)_{1,\infty,\mathbb{A}} = (A_0 + A_1, A_1)_{1,\infty,(\alpha_0,\alpha)}$ by Corollary 6.3. Take now $B_0 = A_0 + A_1$, $B_1 = A_1$ and $\mathbb{B} = (\alpha_0, \alpha)$. Applying Proposition 6.10 to the couple (B_0, B_1) and to $\mathbb{B}$, we obtain that the closure of $B_0 \cap B_1 = A_1$ in $(B_0, B_1)_{1,\infty,\mathbb{B}} = (A_0, A_1)_{1,\infty,\mathbb{A}}$ is the space Z formed by all those $a \in (A_0, A_1)_{1,\infty,\mathbb{A}}$ such that

$$t^{-1}K(t,a;A_0 + A_1, A_1)\ell^{\mathbb{B}}(t) \to 0 \text{ as } t \to 0 \text{ or } t \to \infty.$$

By (6.4), we get that this condition is equivalent to

$$\begin{aligned}&t^{-1}K(t,a;A_0,A_1)\ell^{\alpha_0}(t) \to 0 \text{ as } t \to 0 \quad \text{and} \\ &t^{-1}K(1,a;A_0,A_1)\ell^{\alpha}(t) \to 0 \text{ as } t \to \infty.\end{aligned} \tag{6.19}$$

Since (6.19) holds for any $a \in A_0 + A_1$, we derive the result. □

6.3 Compact operators

In this section, we turn our attention to the behaviour of compact operators. We establish first some notation. Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples. Recall that by $T \in \mathcal{L}(\bar{A}, \bar{B})$ we mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_i defines a bounded operator from A_i into B_i of norm M_i for $i = 0, 1$. If $A_0 = A_1 = A$ or $B_0 = B_1 = B$, then we write simply $T \in \mathcal{L}(A, \bar{B})$ and $T \in \mathcal{L}(\bar{A}, B)$, respectively.

As we mentioned in the Introduction, the origins of interpolation theory are the theorems of Riesz-Thorin and Marcinkiewicz. The Riesz-Thorin theorem is stated in Chapter 2 (Theorem 2.1). In 1960, Krasnosel'skiĭ [64] (see also [4] and [76]) showed that this result could be extended to involve compactness.

Theorem 6.12. [Krasnosel'skiĭ's theorem] *Let (Ω, μ) and (Θ, ν) be σ-finite measure spaces. Suppose $1 \le p_0, p_1, q_1 \le \infty$ and $1 \le q_0 < \infty$, and let T be a linear operator such that*

$$\begin{aligned} T &: L_{p_0}(\Omega, \mu) \longrightarrow L_{q_0}(\Theta, \nu) \quad \textit{is compact and} \\ T &: L_{p_1}(\Omega, \mu) \longrightarrow L_{q_1}(\Theta, \nu) \quad \textit{is bounded.} \end{aligned}$$

Take $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$T : L_p(\Omega, \mu) \longrightarrow L_q(\Theta, \nu) \ \textit{compactly.}$$

Very recently, Edmunds and Opic [45] proved the following version of Krasnosel'skiĭ's theorem, this time with Lorentz-Zygmund spaces which are very close to $L_{p_0}(\Omega, \mu)$ and $L_{q_0}(\Theta, \nu)$.

Theorem 6.13. [Edmunds and Opic [45]] *Let (Ω, μ) and (Θ, ν) be finite measure spaces and take $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ and $\alpha + 1/q > 0$. Put $\gamma_0 = \alpha + 1/\min(p_0, q)$ and $\gamma_1 = \alpha + 1/\max(q_0, q)$. If T is a linear operator such that*

$$\begin{aligned} T &: L_{p_0}(\Omega, \mu) \longrightarrow L_{q_0}(\Theta, \nu) \quad \textit{is compact and} \\ T &: L_{p_1}(\Omega, \mu) \longrightarrow L_{q_1}(\Theta, \nu) \quad \textit{is bounded,} \end{aligned}$$

then $T : L_{p_0,q}(\log L)_{\gamma_0}(\Omega, \mu) \longrightarrow L_{q_0,q}(\log L)_{\gamma_1}(\Theta, \nu)$ compactly.

Note that the measure spaces are finite. Moreover, the techniques that these authors used for the proof take advantage of dealing with Lebesgue spaces.

Later on, Cobos, Fernández-Cabrera and Martínez [23] showed an abstract version of Krasnosel'skiĭ's theorem for the logarithmic methods that we are discussing in this chapter. First of all, they showed that if $T \in \mathcal{L}(\bar{A}, \bar{B})$ then

$$\|T\|_{(A_0,A_1)_{1,q,\mathbb{A}},(B_0,B_1)_{1,q,\mathbb{A}}} \le c M_1 \ell\left(\frac{M_0}{M_1}\right)^{|\alpha_0|+|\alpha_\infty|} \tag{6.20}$$

(see also [45] for more sharp estimates).

As for compact operators, they showed in [23, Remark 2.4] that the compactness of $T : A_0 \longrightarrow B_0$ is not enough to imply that $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$ is compact. In addition, they proved the following.

Theorem 6.14. [Cobos, Fernández-Cabrera and Martínez [23]] *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples with $B_0 \hookrightarrow B_1$. Suppose that $T \in \mathcal{L}(\bar{A}, \bar{B})$ is a linear operator such that $T : A_1 \longrightarrow B_1$ is compact. Let $1 \le q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, where $\alpha_\infty \in \mathbb{R}$ and either $1 \le q < \infty$ and $\alpha_0 < -1/q$, or $q = \infty$ and $\alpha_0 \le 0$. Then $T : (A_0^o, A_1^o)_{1,q,\mathbb{A}} \longrightarrow (B_0^o, B_1^o)_{1,q,\mathbb{A}}$ is also compact.*

In this section, with the help of the J-representation of $(A_0, A_1)_{1,q,\mathbb{A}}$ and a different approach to the one used in [23], we show that one can get rid of the inclusion assumption in the couple $\bar{B}$ and also that one can replace the spaces A_j^o and B_j^o by the original spaces A_j and B_j. This will allow us to establish a version for σ-finite (not necessarily finite) measure spaces of Edmunds and Opic's Krasnosel'skiĭ-type compactness theorem [45].

For this aim, we shall work with the discrete representations of logarithmic K- and J-spaces and some properties of the associated vector-valued sequence spaces. This approach has its origin in the papers by Cobos and Peetre [32] and Cobos, Kühn and Schonbek [30].

It is easy to check that $\|\cdot\|_{(A_0,A_1)_{1,q,\mathbb{A}}}$ is equivalent to

$$\|a\|_{1,q,\mathbb{A}} = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) K(2^m, a) \right]^q \right)^{1/q}.$$

On the other hand, $(A_0, A_1)^J_{1,q,\mathbb{A},\mathbb{B}}$ coincides with the collection of all those $a \in A_0 + A_1$ for which there is a sequence $(u_m)_{m=-\infty}^{\infty} \subset A_0 \cap A_1$ such that

$$a = \sum_{m=-\infty}^{\infty} u_m \qquad \text{(convergence in } A_0 + A_1\text{)} \tag{6.21}$$

and

$$\left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) \ell\ell^{\mathbb{B}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} < \infty. \tag{6.22}$$

The infimum in (6.22) over all possible representations of the type (6.21), (6.22) gives the norm $\|\cdot\|^J_{1,q,\mathbb{A},\mathbb{B}}$ (denoted by $\|\cdot\|^J_{1,q,\mathbb{A}}$ if $\mathbb{B} = (0,0)$) which is equivalent to $\|\cdot\|_{(A_0,A_1)^J_{1,q,\mathbb{A},\mathbb{B}}}$. Note that (6.22) implies that $\sum_{m=-\infty}^{\infty} u_m$ is absolutely convergent in $A_0 + A_1$. One can show these facts with arguments that are very similar to those in the proofs of Lemmata 3.5 and 3.8 and Remark 6.3, respectively.

We start with an auxiliary result. Subsequently, if A is a Banach space, we write U_A for its closed unit ball.

Lemma 6.15. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let B be a Banach space. Take $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$ satisfying (6.2).*

(a) *If* $T \in \mathcal{L}(B, \bar{A})$ *with* $T : B \longrightarrow A_1$ *compactly, then* $T : B \longrightarrow (A_0, A_1)_{1,q,\mathbb{A}}$ *is compact.*

(b) *If* $T \in \mathcal{L}(\bar{A}, B)$ *with* $T : A_1 \longrightarrow B$ *compactly, then* $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow B$ *is compact.*

Proof. Note that there is a constant $c > 0$ such that for any $a \in A_0 \cap A_1$ we have

$$\|a\|_{1,q,\mathbb{A}} \le c\,\|a\|_{A_1}\,\ell\left(\frac{\|a\|_{A_0}}{\|a\|_{A_1}}\right)^{|\alpha_0|+|\alpha_\infty|}. \tag{6.23}$$

Indeed, let $\mathbb{K}$ be the scalar field and let $T \in \mathcal{L}((\mathbb{K}, \mathbb{K}), (A_0, A_1))$ be the operator defined by $T\lambda = \lambda a$. Clearly $\|T\|_{\mathbb{K},A_i} = \|a\|_{A_i}$ and $(\mathbb{K}, \mathbb{K})_{1,q,\mathbb{A}} = \mathbb{K}$ with equivalence of norms, so (6.20) yields (6.23).

Now take any bounded sequence $(b_n)_{n\in\mathbb{N}} \subset B$. Since $T : B \longrightarrow A_1$ is compact, there is a subsequence $(b_{n'})$ such that $(Tb_{n'})$ is a Cauchy sequence in A_1. Using (6.23), we derive

$$\begin{aligned}\|Tb_{n'} - Tb_{m'}\|_{1,q,\mathbb{A}} &\le c\,\|Tb_{n'} - Tb_{m'}\|_{A_1}\left(1 + \left|\log\frac{\|Tb_{n'} - Tb_{m'}\|_{A_0}}{\|Tb_{n'} - Tb_{m'}\|_{A_1}}\right|\right)^{|\alpha_0|+|\alpha_\infty|}\\ &\le c_1\,\|Tb_{n'} - Tb_{m'}\|_{A_1}\left(1 + \left|\log\|Tb_{n'} - Tb_{m'}\|_{A_1}\right|\right)^{|\alpha_0|+|\alpha_\infty|}.\end{aligned}$$

Hence $(Tb_{n'})$ is a Cauchy sequence in $(A_0, A_1)_{1,q,\mathbb{A}}$, which proves (a).

Next we prove (b). We claim that there is a constant $c > 0$ such that for any $a \in U_{\bar{A}_{1,q,\mathbb{A}}}$ we have

$$K(t, a) \le ct(1 + |\log t|)^{|\alpha_0|+|\alpha_\infty|}, \qquad t > 0. \tag{6.24}$$

Indeed, applying the Hahn-Banach theorem to $A_0 + A_1$ normed by $K(t, \cdot)$, we can find a functional f with norm 1 such that $f(a) = K(t, a)$. It follows that $\|f\|_{A_0,\mathbb{K}} \le 1$ and $\|f\|_{A_1,\mathbb{K}} \le t$. Then (6.24) follows from (6.20).

We proceed to show the compactness of the interpolated operator. Take any $\varepsilon > 0$ and choose t sufficiently small so that

$$2ct(1 + |\log t|)^{|\alpha_0|+|\alpha_\infty|}\,\|T\|_{A_0,B} \le \varepsilon/2. \tag{6.25}$$

Since $T : A_1 \longrightarrow B$ is compact, there is a finite set $\{b_1, \dots, b_k\} \subset B$ such that

$$T\left(2c(1 + |\log t|)^{|\alpha_0|+|\alpha_\infty|}U_{A_1}\right) \subset \bigcup_{j=1}^{k}\left\{b_j + \frac{\varepsilon}{2}U_B\right\}.$$

Given any $a \in U_{(A_0,A_1)_{1,q,\mathbb{A}}}$, we can decompose $a = a_0 + a_1$ with $a_i \in A_i$ and

$$\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} \le 2K(t, a).$$

By (6.24) and (6.25), we have that

$$\|a_0\|_{A_0} \le \frac{\varepsilon}{2\,\|T\|_{A_0,B}} \quad \text{and} \quad \|a_1\|_{A_1} \le 2c(1 + |\log t|)^{|\alpha_0|+|\alpha_\infty|}.$$

Therefore, for any $a \in U_{(A_0,A_1)_{1,q,\mathbb{A}}}$ there is b_j with $1 \le j \le k$ such that

$$\left\|Ta - b_j\right\|_B \le \left\|Ta_1 - b_j\right\|_B + \|Ta_0\|_B \le \frac{\varepsilon}{2} + \|T\|_{A_0,B}\,\frac{\varepsilon}{2\,\|T\|_{A_0,B}} = \varepsilon.$$

This shows that $T(U_{(A_0,A_1)_{1,q,\mathbb{A}}})$ is precompact and completes the proof. □

The next auxiliary result refers to interpolation of vector-valued sequence spaces.

Recall that, given a sequence $(\lambda_m)_{m\in\mathbb{Z}}$ of positive numbers and a sequence $(W_m)_{m\in\mathbb{Z}}$ of Banach spaces, we write $\ell_q(\lambda_m W_m)$ for the collection of all sequences $x = (x_m)$ such that $x_m \in W_m$ for any $m \in \mathbb{Z}$ and

$$\|x\|_{\ell_q(\lambda_m W_m)} = \left(\sum_{m=-\infty}^{\infty} \left[\lambda_m \|x_m\|_{W_m}\right]^q\right)^{1/q} < \infty.$$

We put

$$(1+|m|)^{\mathbb{A}} = \begin{cases}(1-m)^{\alpha_0} & \text{if } m \le 0,\\ (1+m)^{\alpha_\infty} & \text{if } m > 0,\end{cases}$$

and define $(1+\log(1+|m|))^{\mathbb{A}}$ similarly.

Lemma 6.16. *Let (W_m) be a sequence of Banach spaces, let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$.*

(a) If $\mathbb{A}$ and q satisfy (6.2), then $(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}} \hookrightarrow \ell_q\left(2^{-m}(1+|m|)^{\mathbb{A}}W_m\right)$.

(b) If $\mathbb{A}$ and q satisfy (6.2) and (6.5), then

$$\ell_q\left(2^{-m}(1+|m|)^{\mathbb{A}+1}W_m\right) \hookrightarrow (\ell_1(W_m), \ell_1(2^{-m}W_m))_{1,q,\mathbb{A}}.$$

(c) If $\mathbb{A}$ and q satisfy (6.16), then

$$\ell_q\left(2^{-m}(1+|m|)^{\mathbb{A}+1}(1+\log(1+|m|))^{\mathbb{B}}W_m\right) \hookrightarrow (\ell_1(W_m), \ell_1(2^{-m}W_m))_{1,q,\mathbb{A}},$$

where $\mathbb{B} = (0,1)$.

Proof. (a) Let $x = (x_m) \in (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}$. Given any decomposition $x = y + z$ with $y = (y_m) \in \ell_\infty(W_m)$ and $z = (z_m) \in \ell_\infty(2^{-m}W_m)$, we have

$$\|x_k\|_{W_k} \le \|y_k\|_{W_k} + \|z_k\|_{W_k} \le \|y\|_{\ell_\infty(W_m)} + 2^k \|z\|_{\ell_\infty(2^{-m}W_m)}, \quad k \in \mathbb{Z}.$$

So $\|x_k\|_{W_k} \le K(2^k, x)$ and

$$\|x\|_{\ell_q(2^{-m}(1+|m|)^{\mathbb{A}}W_m)} \le \left(\sum_{m=-\infty}^{\infty}\left[2^{-m}(1+|m|)^{\mathbb{A}}K(2^m,x)\right]^q\right)^{1/q} \sim \|x\|_{(\ell_\infty(W_m),\ell_\infty(2^{-m}W_m))_{1,q,\mathbb{A}}}.$$

(b) Let $x = (x_m) \in \ell_q\left(2^{-m}(1+|m|)^{\mathbb{A}+1}W_m\right)$. Let $u_k = (u_m^k)_{m\in\mathbb{Z}}$ where $u_m^k = 0$ for $m \ne k$ and $u_m^k = x_k$ if $m = k$. Since $x = \sum_{k=-\infty}^{\infty} u_k$ and $J(2^k, u_k; \ell_1(W_m), \ell_1(2^{-m}W_m)) = \|x_k\|_{W_k}$, using Theorem 6.6 we derive that

$$\begin{aligned}\|x\|_{(\ell_1(W_m),\ell_1(2^{-m}W_m))_{1,q,\mathbb{A}}} &\sim \|x\|_{(\ell_1(W_m),\ell_1(2^{-m}W_m))^J_{1,q,\mathbb{A}+1}}\\ &\lesssim \left(\sum_{k=-\infty}^{\infty}\left[2^{-k}(1+|k|)^{\mathbb{A}+1}J(2^k,u_k)\right]^q\right)^{1/q}\\ &= \|x\|_{\ell_q(2^{-m}(1+|m|)^{\mathbb{A}+1}W_m)}.\end{aligned}$$

(c) This case can be treated as (b) but using now Theorem 6.7. $\square$

Next we establish the compactness theorem for the $(1, q, \mathbb{A}; K)$-method.

Theorem 6.17. *Let* $\bar{A} = (A_0, A_1)$ *and* $\bar{B} = (B_0, B_1)$ *be Banach couples and let* $T \in \mathcal{L}(\bar{A}, \bar{B})$ *such that the restriction* $T : A_1 \longrightarrow B_1$ *is compact. For any* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *and* $1 \leq q \leq \infty$ *satisfying* (6.2), *we have that*

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$$

is compact.

Proof. Step 1. We first assume that $\mathbb{A}$ and q satisfy also (6.5) or (6.6). Let $\mathbb{B} = (0, 0)$ if (6.5) holds and $\mathbb{B} = (0, 1)$ if (6.6) is satisfied.

For $m \in \mathbb{Z}$, let $G_m = A_0 \cap A_1$ normed by $J(2^m, \cdot; A_0, A_1)$ and let $F_m = B_0 + B_1$ with the norm $K(2^m, \cdot; B_0, B_1)$. Consider the operators $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ and $jb = (\dots, b, b, b, \dots)$ and write

$$\lambda_m = 2^{-m}(1 + |m|)^{\mathbb{A}+1}(1 + \log(1 + |m|))^{\mathbb{B}} \text{ and } \mu_m = 2^{-m}(1 + |m|)^{\mathbb{A}}.$$

By Theorems 6.6 and 6.7,

$$\pi : \ell_q(\lambda_m G_m) \longrightarrow (A_0, A_1)_{1,q,\mathbb{A}}$$

is a metric surjection if we consider the discrete J-norm on $(A_0, A_1)_{1,q,\mathbb{A}}$. Moreover,

$$\pi : \ell_1(G_m) \longrightarrow A_0 \quad \text{and} \quad \pi : \ell_1(2^{-m} G_m) \longrightarrow A_1$$

are bounded with norm ≤ 1. On the other hand, if we consider the discrete K-norm on $(B_0, B_1)_{1,q,\mathbb{A}}$, then

$$j : (B_0, B_1)_{1,q,\mathbb{A}} \longrightarrow \ell_q(\mu_m F_m)$$

is a metric injection. Moreover, the restrictions

$$j : B_0 \longrightarrow \ell_\infty(F_m) \quad \text{and} \quad j : B_1 \longrightarrow \ell_\infty(2^{-m} F_m)$$

are bounded with norm ≤ 1.

Let $\hat{T} = jT\pi$. The properties of π and j yield that $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$ is compact if and only if $\hat{T} : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m)$ is compact. We shall check the compactness of $\hat{T}$ with the help of the following projections. For $n \in \mathbb{N}$, write

$$\begin{aligned} P_n(u_m) &= (\dots, 0, 0, u_{-n}, u_{-n+1}, \dots, u_{n-1}, u_n, 0, 0, \dots), \\ Q_n^+(u_m) &= (\dots, 0, 0, u_{n+1}, u_{n+2}, \dots), \\ Q_n^-(u_m) &= (\dots, u_{-n-2}, u_{-n-1}, 0, 0, \dots). \end{aligned}$$

The identity operator I on $\ell_1(G_m) + \ell_1(2^{-m} G_m)$ can be written as the sum $I = P_n + Q_n^+ + Q_n^-$ and the following properties hold:

$$P_n,\ Q_n^+ \text{ and } Q_n^- \text{ have norm 1 when acting on } \ell_1(G_m) \text{ and on } \ell_1(2^{-m}G_m). \tag{6.26}$$

$$\text{The restrictions } Q_n^+ : \ell_1(G_m) \longrightarrow \ell_1(2^{-m}G_m) \quad \text{and} \quad Q_n^- : \ell_1(2^{-m}G_m) \longrightarrow \ell_1(G_m)$$
$$\text{are bounded with norm } 2^{-(n+1)}. \tag{6.27}$$

$$P_n : \ell_1(G_m) + \ell_1(2^{-m}G_m) \longrightarrow \ell_1(G_m) \cap \ell_1(2^{-m}G_m) \text{ boundedly.} \tag{6.28}$$

On the couple $(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))$ we can define similar projections, denoted by R_n, S_n^+, S_n^-, which satisfy analogous properties.

We have

$$\begin{aligned} \hat{T} &= \hat{T}(P_n + Q_n^+ + Q_n^-) = \hat{T}P_n + \hat{T}Q_n^- + (R_n + S_n^+ + S_n^-)\hat{T}Q_n^+ \\ &= \hat{T}P_n + \hat{T}Q_n^- + R_n\hat{T}Q_n^+ + S_n^+\hat{T}Q_n^+ + S_n^-\hat{T}Q_n^+. \end{aligned}$$

Next we show that, acting from $\ell_q(\lambda_m G_m)$ into $\ell_q(\mu_m F_m)$, the operators $\hat{T}P_n$, $R_n\hat{T}Q_n^+$ and $S_n^-\hat{T}Q_n^+$ are compact and that the other two operators have norms converging to 0 as $n \to \infty$.

Using (6.28), we have the factorisation

$$\begin{tikzcd} & & \ell_1(G_m) \arrow[r, "\hat{T}"] & \ell_\infty(F_m) \\ \ell_q(\lambda_m G_m) \arrow[r, hook] & \ell_1(G_m) + \ell_1(2^{-m}G_m) \arrow[ur, "P_n"] \arrow[dr, "P_n"'] & & \\ & & \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m), \end{tikzcd}$$

which allows us to apply Lemma 6.15/(a) and Lemma 6.16/(a) to obtain that

$$\hat{T}P_n : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m) \text{ is compact.}$$

For $R_n\hat{T}Q_n^+$ we use the diagram

$$\begin{tikzcd} \ell_1(G_m) \arrow[r, "\hat{T}Q_n^+"] & \ell_\infty(F_m) \arrow[dr, "R_n"] & & \\ & & \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m) \arrow[r, hook] & \ell_q(\mu_m F_m), \\ \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}Q_n^+"] & \ell_\infty(2^{-m}F_m) \arrow[ur, "R_n"'] & & \end{tikzcd}$$

and again Lemmata 6.15 and 6.16 yield that $R_n\hat{T}Q_n^+ : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m)$ compactly.

As for $S_n^-\hat{T}Q_n^+$, we first use the factorisation

$$\begin{tikzcd} \ell_1(G_m) \arrow[dr, "Q_n^+"] & & \\ & \ell_1(2^{-m}G_m) \arrow[r, "\hat{T}"] & \ell_\infty(2^{-m}F_m), \\ \ell_1(2^{-m}G_m) \arrow[ur, "Q_n^+"'] & & \end{tikzcd}$$

and Lemmata 6.15 and 6.16 to get that $\hat{T}Q_n^+ : \ell_q(\lambda_m G_m) \longrightarrow \ell_\infty(2^{-m}F_m)$ is compact. Since we also have the diagram

$$\begin{tikzcd} & & \ell_\infty(F_m) \\ \ell_q(\lambda_m G_m) \arrow[r, "\hat{T}Q_n^+"] & \ell_\infty(2^{-m}F_m) \arrow[ur, "S_n^-"] \arrow[dr, "S_n^-"'] & \\ & & \ell_\infty(2^{-m}F_m), \end{tikzcd}$$

using again the lemmata we conclude that $S_n^-\hat{T}Q_n^+ : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m)$ compactly.

Now we estimate the norms of the operators $\hat{T}Q_n^-$, $S_n^+\hat{T}Q_n^+$ acting between $\ell_q(\lambda_m G_m)$ and $\ell_q(\mu_m F_m)$. By (6.20) and Lemma 6.16, in order to check that the norms go to 0 as $n \to \infty$, it suffices to show that the norms

$$\left\|\hat{T}Q_n^-\right\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} = \left\|\hat{T}Q_n^-\right\|_1 \quad \text{and} \quad \left\|S_n^+\hat{T}Q_n^+\right\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} = \left\|S_n^+\hat{T}Q_n^+\right\|_1$$

go to 0 as $n \to \infty$. As for $\hat{T}Q_n^-$, we proceed by contradiction. If $\left\|\hat{T}Q_n^-\right\|_1 \not\to 0$, since the sequence $\left(\left\|\hat{T}Q_n^-\right\|_1\right)$ is non-increasing, we would have $\lim_{n\to\infty} \left\|\hat{T}Q_n^-\right\|_1 = \tau > 0$. Take $(u_n) \subset U_{\ell_1(2^{-m}G_m)}$ such that

$$\lim_{n\to\infty} \left\|\hat{T}Q_n^- u_n\right\|_{\ell_\infty(2^{-m}F_m)} = \tau.$$

By (6.26), $(Q_n^- u_n)$ is bounded in $\ell_1(2^{-m}G_m)$. Since the restriction $\hat{T} : \ell_1(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}F_m)$ is compact, there is a vector $v \in \ell_\infty(2^{-m}F_m)$ and a subsequence $(Q_{n'}^- u_{n'})$ of $(Q_n^- u_n)$ such that $\lim_{n'\to\infty} \hat{T}Q_{n'}^- u_{n'} = v$ in $\ell_\infty(2^{-m}F_m)$. Therefore, $\|v\|_{\ell_\infty(2^{-m}F_m)} = \tau > 0$. However, (6.27) gives that $\lim_{n'\to\infty} Q_{n'}^- u_{n'} = 0$ in $\ell_1(G_m)$. So $\lim_{n'\to\infty} \hat{T}Q_{n'}^- u_{n'} = 0$ in $\ell_\infty(F_m)$. By compatibility, $v = 0$, which contradicts that $v \neq 0$.

Finally, for $S_n^+\hat{T}Q_n^+$, given any $\varepsilon > 0$, the compactness of $\hat{T} : \ell_1(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}F_m)$ yields that there are vectors $u_1, \dots, u_r$ having only a finite number of non-zero coordinates such that

$$\hat{T}\left(U_{\ell_1(2^{-m}G_m)}\right) \subset \bigcup_{j=1}^{r} \left\{\hat{T}u_j + \frac{\varepsilon}{2} U_{\ell_\infty(2^{-m}F_m)}\right\}.$$

Then $\hat{T}u_j \in \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m)$. By (6.27), there is $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\left\|S_n^+\hat{T}u_j\right\|_{\ell_\infty(2^{-m}F_m)} \leq \frac{\varepsilon}{2} \qquad \text{for } j = 1, \dots, r.$$

Now take any $u \in U_{\ell_1(2^{-m}G_m)}$ and any $n \geq N$. Since $v = Q_n^+ u$ belongs to $U_{\ell_1(2^{-m}G_m)}$, there is $1 \leq j \leq r$ such that $\left\|\hat{T}v - \hat{T}u_j\right\|_{\ell_\infty(2^{-m}F_m)} \leq \varepsilon/2$. Therefore,

$$\left\|S_n^+\hat{T}Q_n^+ u\right\|_{\ell_\infty(2^{-m}F_m)} \leq \left\|S_n^+\hat{T}v - S_n^+\hat{T}u_j\right\|_{\ell_\infty(2^{-m}F_m)} + \left\|S_n^+\hat{T}u_j\right\|_{\ell_\infty(2^{-m}F_m)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof when $\mathbb{A}$ and q satisfy (6.5) or (6.6).

Step 2. Suppose now that $\mathbb{A}$ and q satisfy (6.3). Let $\mathbb{D} = (\alpha_0, \alpha)$, where $\alpha > -1/q$. Then q and $\mathbb{D}$ satisfy (6.5). By Corollary 6.3, we have that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,\mathbb{D}} \quad \text{and} \quad (B_0, B_1)_{1,q,\mathbb{A}} = (B_0 + B_1, B_1)_{1,q,\mathbb{D}}.$$

Since $T \in \mathcal{L}((A_0 + A_1, A_1), (B_0 + B_1, B_1))$ and $T : A_1 \longrightarrow B_1$ compactly, it follows from Step 1 that

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$$

is also compact. □

The compactness theorem for the $(0, q, \mathbb{A}; K)$-method follows from Remark 6.1 and Theorem 6.17.

Corollary 6.18. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that the restriction $T : A_0 \longrightarrow B_0$ is compact. For any $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q \le \infty$ such that*

$$\alpha_\infty + 1/q < 0 \text{ if } q < \infty \quad \text{or} \quad \alpha_\infty \le 0 \text{ if } q = \infty,$$

we have that

$$T : (A_0, A_1)_{0,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{0,q,\mathbb{A}}$$

is also compact.

Next we apply the compactness theorem to extend to arbitrary σ-finite spaces a result proved by Edmunds and Opic [45] for finite measure spaces. The corollaries deal with *generalised Lorentz-Zygmund spaces*, which are defined in the Introduction, equation (1.6).

Using the well-known formula

$$K(t, f; L_1(\Omega), L_\infty(\Omega)) = t f^{**}(t),$$

it turns out that

$$L_{p,q}(\log L)_{\mathbb{A}}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1-1/p,q,\mathbb{A}}. \tag{6.29}$$

In what follows, if $\tau \in \mathbb{R}$, we put $\mathbb{A} + \tau = (\alpha_0 + \tau, \alpha_\infty + \tau)$.

Corollary 6.19. *Let (Ω, μ), (Θ, ν) be σ-finite measure spaces. Take $1 < p_0 < p_1 \le \infty$, $1 < q_0 < q_1 \le \infty$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$. Let T be a linear operator such that*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \text{ compactly and } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \text{ boundedly.}$$

Then

$$T : L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_0,q)}}(\Omega) \longrightarrow L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_0,q)}}(\Theta)$$

is also compact.

Proof. By Corollary 6.18,

$$T : (L_{p_0}(\Omega), L_{p_1}(\Omega))_{0,q,\mathbb{A}} \longrightarrow (L_{q_0}(\Theta), L_{q_1}(\Theta))_{0,q,\mathbb{A}} \text{ compactly.} \tag{6.30}$$

We shall work with these interpolation spaces using two theorems in [46]. First of all, according to [46, Theorem 5.9], for any Banach couple $\bar{A} = (A_0, A_1)$, if $0 < \theta_0 < \theta_1 < 1$, $0 < r_0, r_1, q \le \infty$, $\mathbb{A}_i \in \mathbb{R}^2$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ is such that $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$,

$$\left(\bar{A}_{\theta_0,r_0,\mathbb{A}_0}, \bar{A}_{\theta_1,r_1,\mathbb{A}_1}\right)_{0,q,\mathbb{A}} = \left(\bar{A}_{\theta_0,r_0,\mathbb{A}_0}, \bar{A}_1\right)_{0,q,\mathbb{A}} = \bar{A}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,r_0,\mathbb{A}_0}, \tag{6.31}$$

where $\bar{A}_1$ is an intermediate space of class 1. The spaces $\bar{A}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,r_0,\mathbb{A}_0}$ are defined in [46]. Suppose that $1 < r_0 < r_1 < \infty$ and that U is a σ-finite measure space. Then $L_{r_i}(U) = (L_1(U), L_\infty(U))_{\theta_i,r_i}$ with $\theta_i = 1 - 1/r_i$ and $\theta_0 < \theta_1$. On the other hand, if $r_1 = \infty$, then $L_\infty(U)$ is an intermediate space of class 1 with respect to the couple $(L_1(U), L_\infty(U))$. We derive, thus, that

$$(L_{r_0}(U), L_{r_1}(U))_{0,q,\mathbb{A}} = (L_1(U), L_\infty(U))^{\mathcal{L}}_{1-1/r_0;q,\mathbb{A},r_0,(0,0)} \quad \text{if } 1 < r_0 < r_1 \le \infty.$$

Next we apply [46, Theorem 4.7] to obtain that

$$(L_1, L_\infty)_{1-1/r_0,q,\mathbb{A}+\frac{1}{\min(q,r_0)}} \hookrightarrow (L_1, L_\infty)^{\mathcal{L}}_{1-1/r_0;q,\mathbb{A},r_0,(0,0)} \hookrightarrow (L_1, L_\infty)_{1-1/r_0,q,\mathbb{A}+\frac{1}{\max(q,r_0)}} \quad \text{if } 1 < r_0 < r_1 \le \infty.$$

Using now (6.29), we derive that

$$L_{r_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min(r_0,q)}}(U) \hookrightarrow (L_{r_0}(U), L_{r_1}(U))_{0,q,\mathbb{A}} \hookrightarrow L_{r_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(r_0,q)}}(U) \quad \text{if } 1 < r_0 < r_1 \le \infty. \tag{6.32}$$

Applying the first embedding in (6.32) to $r_i = p_i$ and $U = \Omega$, we obtain

$$L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(p_0,q)}}(\Omega) \hookrightarrow (L_{p_0}(\Omega), L_{p_1}(\Omega))_{0,q,\mathbb{A}}.$$

Similarly, the second embedding in (6.32) with $r_i = q_i$ and $U = \Theta$ yields that

$$(L_{q_0}(\Theta), L_{q_1}(\Theta))_{0,q,\mathbb{A}} \hookrightarrow L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_0,q)}}(\Theta).$$

These embeddings and (6.30) imply the wanted result. $\square$

We can obtain similar results for the p_i's and q_i's ordered in a different way. Indeed, note that if $1 < r_1 < r_0 < \infty$ we have that $L_{r_i}(U) = (L_\infty(U), L_1(U))_{\tilde{\theta}_i,r_i}$ with $\tilde{\theta}_i = 1/r_i$, so $\tilde{\theta}_0 < \tilde{\theta}_1$. Moreover, $L_1(U)$ is an intermediate space of class 1 with respect to the couple $(L_\infty(U), L_1(U))$. Whence we can still use (6.31), obtaining that

$$(L_{r_0}(U), L_{r_1}(U))_{0,q,\mathbb{A}} = (L_\infty(U), L_1(U))^{\mathcal{L}}_{1/r_0;q,\mathbb{A},r_0,(0,0)} \quad \text{if } 1 \le r_1 < r_0 < \infty.$$

Next we apply [46, Theorem 4.7] to obtain that

$$(L_\infty, L_1)_{1/r_0,q,\mathbb{A}+\frac{1}{\min(q,r_0)}} \hookrightarrow (L_\infty, L_1)^{\mathcal{L}}_{1/r_0;q,\mathbb{A},r_0,(0,0)} \hookrightarrow (L_\infty, L_1)_{1/r_0,q,\mathbb{A}+\frac{1}{\max(q,r_0)}} \quad \text{if } 1 \le r_1 < r_0 < \infty.$$

By (6.29) and a change of variable, we obtain the counterpart of (6.32) for $1 \le r_0 < r_1 < \infty$,

$$L_{r_0,q}(\log L)_{\tilde{\mathbb{A}}+\frac{1}{\min(r_0,q)}}(U) \hookrightarrow (L_{r_0}(U), L_{r_1}(U))_{0,q,\mathbb{A}} \hookrightarrow L_{r_0,q}(\log L)_{\tilde{\mathbb{A}}+\frac{1}{\max(r_0,q)}}(U) \quad \text{if } 1 \le r_1 < r_0 < \infty, \tag{6.33}$$

where $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$.

Thus, in order to obtain the corresponding results for the p_i's and q_i's ordered in a different way, all we have to do is choose the suitable embeddings in (6.32) or (6.33), depending on the order of the parameters, and apply (6.30).

In particular, if in Corollary 6.19 we change compactness to the second restriction, the result reads as follows.

Corollary 6.20. *Let* (Ω,μ), (Θ,ν) *be* σ*-finite measure spaces. Take* $1 \le p_0 < p_1 < \infty$, $1 \le q_0 < q_1 < \infty$, $1 \le q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *with* $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. *Let* T *be a linear operator such that*

$$T : L_{p_0}(\Omega) \longrightarrow L_{q_0}(\Theta) \text{ boundedly and } T : L_{p_1}(\Omega) \longrightarrow L_{q_1}(\Theta) \text{ compactly.}$$

Then

$$T : L_{p_1,q}(\log L)_{\mathbb{A}+\frac{1}{\min(p_1,q)}}(\Omega) \longrightarrow L_{q_1,q}(\log L)_{\mathbb{A}+\frac{1}{\max(q_1,q)}}(\Theta)$$

is also compact.

6.4 Weakly compact operators and duality

Let T be any bounded linear operator between vector-valued ℓ_q spaces with $1 < q < \infty$. As it was pointed out in [56], T is weakly compact provided that all its components (regarded T as a matrix) are weakly compact. This property is called the Σ_q*-condition* and makes a difference between compact and weakly compact operators. As we show next, this property is the key to establish the interpolation properties of weakly compact operators under logarithmic methods. Again the results depend on the relationship between q and $\mathbb{A}$.

Theorem 6.21. *Let* $1 < q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *satisfying* (6.2) *and also* (6.5) *or* (6.6). *Let* $\bar{A} = (A_0, A_1)$ *and* $\bar{B} = (B_0, B_1)$ *be Banach couples and assume that* $T \in \mathcal{L}(\bar{A}, \bar{B})$. *Then a necessary and sufficient condition for* $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$ *to be weakly compact is that the restriction* $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ *is weakly compact.*

Proof. The factorisation

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbb{A}} \xrightarrow{T} (B_0, B_1)_{1,q,\mathbb{A}} \hookrightarrow B_0 + B_1$$

shows that the condition is necessary.

In order to show that the condition is sufficient, we shall work with the representation of $(A_0, A_1)_{1,q,\mathbb{A}}$ as a J-space and with the discrete norms. We follow the same notation as in the proof of Theorem 6.17.

For $k, r \in \mathbb{Z}$, let $D_k : G_k \longrightarrow \ell_q(\lambda_m G_m)$ and $L_r : \ell_q(\mu_m F_m) \longrightarrow F_r$ be the operators defined by $D_k x = (\delta_m^k x)$ and $L_r(y_m) = y_r$. Here δ_m^k is the Kronecker delta. Consider the linear operator $\hat{T} = jT\pi : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m)$. Since $L_r \hat{T} D_k = T$ and $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is weakly compact, the Σ_q-condition yields that

$$\hat{T} : \ell_q(\lambda_m G_m) \longrightarrow \ell_q(\mu_m F_m)$$

is weakly compact. Now using that π is a metric surjection and j is a metric injection, we conclude that

$$T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}} \text{ is weakly compact.}$$

□

Theorem 6.22. *Let* $1 < q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *satisfying* (6.2) *and* (6.3). *Let* $\bar{A} = (A_0, A_1)$ *and* $\bar{B} = (B_0, B_1)$ *be Banach couples and* $T \in \mathcal{L}(\bar{A}, \bar{B})$. *Then a necessary and sufficient condition for the interpolated operator* $T : (A_0, A_1)_{1,q,\mathbb{A}} \longrightarrow (B_0, B_1)_{1,q,\mathbb{A}}$ *to be weakly compact is that* $T : A_1 \longrightarrow B_0 + B_1$ *is weakly compact.*

Proof. By Corollary 6.3, we have that

$$(A_0, A_1)_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)_{1,q,\mathbb{D}} \text{ and } (B_0, B_1)_{1,q,\mathbb{A}} = (B_0 + B_1, B_1)_{1,q,\mathbb{D}},$$

where $\mathbb{D} = (\alpha_0, \alpha)$ and $\alpha > -1/q$. Since q and $\mathbb{D}$ satisfy (6.2) and (6.5), the wanted result follows from Theorem 6.21. □

Remark 6.6. The techniques used in Theorems 6.21 and 6.22 still work for any injective and surjective operator ideal satisfying the Σ_q-condition. In particular, they apply also to Banach-Saks operators, Rosenthal operators and Asplund (or dual Radon-Nikodym) operators (see [59, 78, 42]). We refer to the paper of Fernández-Cabrera and Martínez [50] for interpolation properties of closed operator ideals under other limiting interpolation methods.

As a direct consequence of Theorems 6.21 and 6.22 we obtain the following characterisation for reflexive logarithmic spaces.

Corollary 6.23. *Assume that* $\bar{A} = (A_0, A_1)$ *is a Banach couple and let* $1 < q < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$.

(a) Suppose that $\alpha_0 + 1/q < 0 \le \alpha_\infty + 1/q$. *Then* $(A_0, A_1)_{1,q,\mathbb{A}}$ *is reflexive if and only if the embedding* $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ *is weakly compact.*

(b) If $\alpha_0 + 1/q < 0$ *and* $\alpha_\infty + 1/q < 0$, *then* $(A_0, A_1)_{1,q,\mathbb{A}}$ *is reflexive if and only if the embedding* $A_1 \hookrightarrow A_0 + A_1$ *is weakly compact.*

Remark 6.7. Using (6.1), one can easily write down the corresponding results to Theorems 6.21, 6.22 and Corollary 6.23 for the $(0, q, \mathbb{A}; K)$-method.

The rest of this section is devoted to the study of duality for logarithmic spaces. In what follows, we assume that $\bar{A} = (A_0, A_1)$ is a *regular* Banach couple, that is, $A_0 \cap A_1$ is dense in A_j for $j = 0, 1$. Recall that in this case the dual A_j^* of A_j can be identified with a subspace A_j' of $(A_0 \cap A_1)^*$, and (A_0', A_1') is a Banach couple. Moreover, by Corollary 6.8 and Remark 6.5, $A_0 \cap A_1$ is also dense in $(A_0, A_1)_{1,q,\mathbb{A}}$ provided that $q < \infty$.

As before, given $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we write $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ for the reverse pair.

For $0 < \theta < 1$, $1 \le q < \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, it follows from [41, Theorem 3.1] or [77, Theorem 2.4] that

$$(A_0, A_1)'_{\theta,q,\mathbb{A}} = (A_0', A_1')_{\theta,q',-\tilde{\mathbb{A}}}.$$

If $\theta = 1$ or 0, the dual space depends on the relationship between q and $\mathbb{A}$. Next we determine $(A_0, A_1)'_{1,q,\mathbb{A}}$ in terms of the K-functional.

Theorem 6.24. *Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Suppose that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q < \infty$ satisfy that $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$. Then*

$$(A_0, A_1)'_{1,q,\mathbb{A}} = (A'_0, A'_1)_{1,q',-\tilde{\mathbb{A}}-1}$$

with equivalent norms.

Proof. Given any scalar sequence $\xi = (\xi_m)_{m\in\mathbb{Z}}$, let

$$\Phi(\xi) = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) |\xi_m| \right]^q \right)^{1/q}.$$

If $a \in A_0 + A_1$, we have that

$$\Phi\left((K(2^m, a)) \right) = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbb{A}}(2^m) K(2^m, a) \right]^q \right)^{1/q} = \|a\|_{1,q,\mathbb{A}}.$$

Moreover,

$$\Phi'(\eta) = \sup \left\{ \frac{|\sum_{m=-\infty}^{\infty} \eta_{-m} \xi_m|}{\Phi(\xi)} \right\} = \left(\sum_{m=-\infty}^{\infty} \left[2^m \ell^{-\mathbb{A}}(2^m) |\eta_{-m}| \right]^{q'} \right)^{1/q'},$$

where the supremum is taken over all non-zero sequences having only a finite number of coordinates different from zero. Whence, we obtain that $(A_0, A_1)'_{1,q,\mathbb{A}} = (A'_0, A'_1)^J_{1,q',-\tilde{\mathbb{A}}}$ by [41, Theorem 3.1]. This equality can be also derived by using similar arguments to [19, Theorem 8.2]. Now, applying Theorem 6.6, we conclude that

$$(A_0, A_1)'_{1,q,\mathbb{A}} = (A'_0, A'_1)_{1,q',-\tilde{\mathbb{A}}-1}.$$ □

Remark 6.8. In [15] the dual of Besov spaces with logarithmic smoothness is determined by using Theorem 6.24.

To determine the dual of $(A_0, A_1)_{1,q,\mathbb{A}}$ with $\alpha_\infty = -1/q$, we need to introduce K-spaces with weights which include powers of iterated logarithms.

Let $1 < q \le \infty$ and $\alpha > -1/q$. We denote by $(A_0, A_1)_{1,q,(-1/q,\alpha),(-1,0)}$ the collection of all those $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{1,q,(-1/q,\alpha),(-1,0)} = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{(-1/q,\alpha)}(2^m)\, \ell\ell^{(-1,0)}(2^m) K(2^m, a) \right]^q \right)^{1/q}.$$

The J-description of these spaces is as follows.

Theorem 6.25. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple. Let $1 < q \le \infty$ and $\alpha > -1/q$. Then*

$$(A_0, A_1)_{1,q,(-1/q,\alpha),(-1,0)} = (A_0, A_1)^J_{1,q,(1/q',\alpha+1)}$$

with equivalent norms.

Proof. We follow the same lines as in the proofs of Theorems 6.6 and 6.7. Take any vector $a \in (A_0, A_1)^J_{1,q,(1/q',\alpha+1)}$ and let $a = \int_0^\infty u(t)\frac{dt}{t}$ be such that

$$\left(\int_0^1 \left[t^{-1}\ell^{1/q'}(t)J(t,u(t))\right]^q \frac{dt}{t} + \int_1^\infty \left[t^{-1}\ell^{\alpha+1}(t)J(t,u(t))\right]^q \frac{dt}{t}\right)^{1/q} \le 2\,\|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}.$$

Using that

$$K(t,a) \le \int_0^t J(s,u(s))\frac{ds}{s} + t\int_t^\infty s^{-1}J(s,u(s))\frac{ds}{s},$$

we obtain that

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{1,q,(-1/q,\alpha),(-1,0)}} \le & \left(\int_0^1 \left[\frac{\ell^{-1/q}(t)\ell\ell^{-1}(t)}{t}\int_0^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&+\left(\int_0^1 \left[\ell^{-1/q}(t)\ell\ell^{-1}(t)\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&+\left(\int_0^1 \left[\ell^{-1/q}(t)\ell\ell^{-1}(t)\int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&+\left(\int_1^\infty \left[\frac{\ell^{\alpha}(t)}{t}\int_0^1 J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&+\left(\int_1^\infty \left[\frac{\ell^{\alpha}(t)}{t}\int_1^t J(s,u(s))\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q}\\
&+\left(\int_1^\infty \left[\ell^{\alpha}(t)\int_t^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{1/q} = I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}$$

We are going to show that $I_j \lesssim \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}$ for $j = 1,\dots,6$ which will show the embedding

$$(A_0,A_1)^J_{1,q,(1/q',\alpha+1)} \hookrightarrow (A_0,A_1)_{1,q,(-1/q,\alpha),(-1,0)}.$$

We start by estimating the interior integral in I_1. Let $f(s) = s^{-1}J(s,u(s))\,\ell^{1/q'}(s)$. Using Hölder's inequality we obtain

$$\int_0^t J(s,u(s))\frac{ds}{s} \le \left(\int_0^t f(s)^q\frac{ds}{s}\right)^{1/q}\left(\int_0^t s^{q'}\ell^{-1}(s)\frac{ds}{s}\right)^{1/q'} \sim t\ell^{-1/q'}(t)\left(\int_0^t f(s)^q\frac{ds}{s}\right)^{1/q}.$$

Inserting this estimate in I_1, changing the order of integration and using that $q>1$, we derive that

$$\begin{aligned}
I_1 &\le \left(\int_0^1 \ell^{-q}(t)\ell\ell^{-q}(t)\int_0^t f(s)^q\frac{ds}{s}\frac{dt}{t}\right)^{1/q} = \left(\int_0^1\left[\frac{\ell^{1/q'}(s)J(s,u(s))}{s}\right]^q\int_s^1 \ell^{-q}(t)\ell\ell^{-q}(t)\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\
&\lesssim \left(\int_0^1\left[\frac{\ell^{1/q'}(s)J(s,u(s))}{s}\right]^q\frac{ds}{s}\right)^{1/q} \le \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}.
\end{aligned}$$

As for I_2, we take $1/q < \varepsilon < 1$ and $g(s) = s^{-1} J(s, u(s))\, \ell^{1/q'}(s)\ell\ell^{\varepsilon - 1/q}(s)$ and proceed similarly. We have

$$\int_t^1 \frac{J(s,u(s))}{s}\frac{ds}{s} \le \left(\int_t^1 g(s)^q \frac{ds}{s}\right)^{1/q}\left(\int_t^1 \ell^{-1}(s)\ell\ell^{-\varepsilon q' + q'/q}(s)\frac{ds}{s}\right)^{1/q'} \lesssim \ell\ell^{1-\varepsilon}(t)\left(\int_t^1 g(s)^q \frac{ds}{s}\right)^{1/q}.$$

So

$$\begin{aligned} I_2 &\lesssim \left(\int_0^1 \ell^{-1}(t)\ell\ell^{-\varepsilon q}(t) \int_t^1 g(s)^q \frac{ds}{s}\frac{dt}{t}\right)^{1/q} \\ &= \left(\int_0^1 \left[\frac{\ell^{1/q'}(s)}{s} J(s,u(s))\right]^q \ell\ell^{\varepsilon q - 1}(s) \int_0^s \ell^{-1}(t)\ell\ell^{-\varepsilon q}(t)\frac{dt}{t}\frac{ds}{s}\right)^{1/q} \\ &\lesssim \left(\int_0^1 \left[\frac{\ell^{1/q'}(s)}{s} J(s,u(s))\right]^q \frac{ds}{s}\right)^{1/q} \le \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}. \end{aligned}$$

For I_3, we obtain

$$\begin{aligned} I_3 &= \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s}\left(\int_0^1 \ell^{-1}(t)\ell\ell^{-q}(t)\frac{dt}{t}\right)^{1/q} \sim \int_1^\infty \frac{J(s,u(s))}{s}\frac{ds}{s} \\ &\lesssim \left(\int_1^\infty \left[\frac{\ell^{\alpha+1}(s)}{s} J(s,u(s))\right]^q \frac{ds}{s}\right)^{1/q}\left(\int_1^\infty \ell^{-(\alpha+1)q'}(s)\frac{ds}{s}\right)^{1/q'} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}, \end{aligned}$$

where in the last inequality we have used that $\alpha > -1/q$.

As for I_4, we have

$$\begin{aligned} I_4 &= \int_0^1 J(s,u(s))\frac{ds}{s}\left(\int_1^\infty t^{-q}\ell^{\alpha q}(t)\frac{dt}{t}\right)^{1/q} \sim \int_0^1 J(s,u(s))\frac{ds}{s} \\ &\le \left(\int_0^1 \left[\frac{\ell^{1/q'}(s)}{s} J(s,u(s))\right]^q \frac{ds}{s}\right)^{1/q}\left(\int_0^1 s^{q'}\ell^{-1}(s)\frac{ds}{s}\right)^{1/q'} \lesssim \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}. \end{aligned}$$

Term I_5 (respectively, I_6) coincides with J_4 (respectively, I_4) in the proof of Theorem 6.6 with $\alpha = \alpha_\infty$. Since $\alpha > -1/q$, the computations given there show that

$$I_5 + I_6 \lesssim \|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}}.$$

Conversely, take any a with

$$\|a\|_{(A_0,A_1)_{1,q,(-1/q,\alpha),(-1,0)}} = \left(\int_0^1 \left[\frac{\ell^{-1/q}(t)\ell\ell^{-1}(t)}{t}K(t,a)\right]^q \frac{dt}{t} + \int_1^\infty \left[\frac{\ell^{\alpha}(t)}{t}K(t,a)\right]^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Then

$$\min(1,t^{-1})K(t,a) \to 0 \text{ as } t \to 0 \text{ or as } t \to \infty \tag{6.34}$$

because
$$\int_0^1 \left[\frac{\ell^{-1/q}(t)\ell\ell^{-1}(t)}{t}\right]^q \frac{dt}{t} = \infty = \int_1^\infty \ell^{\alpha q}(t)\frac{dt}{t}.$$
(see the proof of Lemma 6.4).

For $\nu \in \mathbb{Z}$, we write
$$\mu_\nu = \begin{cases} 2^{-2^{2^{-\nu-1}}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0, \end{cases}$$
and we decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$ and
$$\mu_{\nu-1}^{-1}\|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} \le 2\widetilde{K}(\mu_{\nu-1}^{-1}, a),$$
where $\widetilde{K}$ is the K-functional for the couple (A_1, A_0) and $\widetilde{J}$ is defined similarly.

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$. Then
$$\widetilde{J}(\mu_{\nu-1}^{-1}, u_\nu) \lesssim \widetilde{K}(\mu_{\nu-2}^{-1}, a)$$
and, by (6.34), we have that $a = \sum_{\nu=-\infty}^{\infty} u_\nu$.

Put $D_\nu = (\mu_{\nu-1}, \mu_\nu]$. Then
$$\int_{D_\nu} \frac{dt}{t} = \begin{cases} \log 2 & \text{if } \nu = 1, \\ 2^{\nu-2}\log 2 & \text{if } \nu > 1. \end{cases}$$
For $\nu \le 0$ put
$$\delta_\nu = \int_{D_\nu} \ell^{-1}(t)\ell\ell^{-1}(t)\frac{dt}{t} \sim 1.$$
We define the function
$$w(t) = \begin{cases} \dfrac{u_\nu}{\delta_\nu \ell(t)\ell\ell(t)} & \text{if } t \in D_\nu \text{ and } \nu \le 0, \\ \dfrac{u_1}{\log 2} & \text{if } t \in D_1, \\ \dfrac{u_\nu}{2^{\nu-2}\log 2} & \text{if } t \in D_\nu \text{ and } \nu > 1. \end{cases}$$
It is clear that $a = \int_0^\infty w(t)\frac{dt}{t}$ in $A_0 + A_1$.

If $\nu > 2$, we can proceed as in the proof of Theorem 6.6 with $\alpha = \alpha_\infty$ to derive that

$$\int_{D_\nu}\left[\frac{\ell^{\alpha+1}(t)}{t}J(t,w(t))\right]^q\frac{dt}{t} \lesssim \int_{D_{\nu-2}}\left[\frac{\ell^{\alpha}(t)}{t}K(t,a)\right]^q\frac{dt}{t}.$$

A similar estimate is valid for $\nu = 2,1$ replacing in the last term $\ell^{\alpha}(t)$ by $\ell^{-1/q}(t)\ell\ell^{-1}(t)$. If $\nu \le 0$, we have

$$\begin{aligned}
\left(\int_{D_\nu}\left[\frac{\ell^{1/q'}(t)}{t}J(t,w(t))\right]^q\frac{dt}{t}\right)^{1/q} &\sim \left(\int_{D_\nu}\left[\frac{\ell^{-1/q}(t)\ell\ell^{-1}(t)}{t}J(t,u_\nu)\right]^q\frac{dt}{t}\right)^{1/q}\\
&\le \frac{J(\mu_{\nu-1},u_\nu)}{\mu_{\nu-1}}\left(\int_{D_\nu}\ell^{-1}(t)\ell\ell^{-q}(t)\frac{dt}{t}\right)^{1/q}\\
&\sim \frac{J(\mu_{\nu-1},u_\nu)}{\mu_{\nu-1}}2^{-\nu(1-q)/q} \sim \widetilde{J}(\mu_{\nu-1}^{-1},u_\nu)\left(\int_{D_{\nu-2}}\ell^{-1}(t)\ell\ell^{-q}(t)\frac{dt}{t}\right)^{1/q}\\
&\lesssim \widetilde{K}(\mu_{\nu-2}^{-1},a)\left(\int_{D_{\nu-2}}\ell^{-1}(t)\ell\ell^{-q}(t)\frac{dt}{t}\right)^{1/q}\\
&= \frac{K(\mu_{\nu-2},a)}{\mu_{\nu-2}}\left(\int_{D_{\nu-2}}\ell^{-1}(t)\ell\ell^{-q}(t)\frac{dt}{t}\right)^{1/q}\\
&\le \left(\int_{D_{\nu-2}}\left[\frac{\ell^{-1/q}(t)\ell\ell^{-1}(t)}{t}K(t,a)\right]^q\frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

Consequently, $\|a\|_{(A_0,A_1)^J_{1,q,(1/q',\alpha+1)}} \lesssim \|a\|_{(A_0,A_1)_{1,q,(-1/q,\alpha),(-1,0)}}$. This finishes the proof. □

Now we are ready to determine the dual of the K-space with $\alpha_\infty = -1/q$.

Theorem 6.26. *Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Suppose that $1 \le q < \infty$ and $\alpha_0 + 1/q < 0$. Then*

$$(A_0,A_1)'_{1,q,(\alpha_0,-1/q)} = (A_0',A_1')_{1,q',(-1/q',-\alpha_0-1),(-1,0)}$$

with equivalent norms.

Proof. By [41, Theorem 3.1], we get that

$$(A_0,A_1)'_{1,q,(\alpha_0,-1/q)} = (A_0',A_1')^J_{1,q',(1/q,-\alpha_0)}.$$

Now the result follows applying Theorem 6.25. □

The duality formula when $q = \infty$ reads as follows.

Theorem 6.27. *Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple and let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ with $\alpha_0 < 0 < \alpha_\infty$. Then*

$$\left((A_0,A_1)^{o}_{1,\infty,\mathbb{A}}\right)' = (A_0',A_1')_{1,1,-\tilde{\mathbb{A}}-1}$$

with equivalent norms.

Proof. Given any scalar sequence $\xi = (\xi_m)_{m\in\mathbb{Z}}$, let

$$\Psi(\xi) = \|(\xi_m)\|_{c_0(2^{-m}\ell^{\mathbb{A}}(2^m))}.$$

We derive by Proposition 6.10 that if $a \in A_0 + A_1$ then

$$\Psi((K(2^m, a))) = \|a\|_{(A_0,A_1)^o_{1,\infty,\mathbb{A}}}.$$

Moreover,

$$\Psi'(\eta) = \sup\left\{\frac{|\sum_{m=-\infty}^{\infty}\eta_{-m}\xi_m|}{\Psi(\xi)}\right\} = \sum_{m=-\infty}^{\infty}\left[2^m\ell^{-\mathbb{A}}(2^m)\,|\eta_{-m}|\right],$$

where the supremum is taken over all non-zero sequences having only a finite number of coordinates different from zero. Whence, we obtain by [41, Theorem 3.1] that

$$\left((A_0,A_1)^o_{1,\infty,\mathbb{A}}\right)' = (A_0', A_1')^J_{1,1,-\tilde{\mathbb{A}}}.$$

The result follows now from Theorem 6.6. □

Finally we consider the case when the K-space does not admit a description in terms of the J-functional.

Theorem 6.28. *Let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Suppose that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $1 \le q < \infty$ satisfy that $\alpha_0 + 1/q < 0$ and $\alpha_\infty + 1/q < 0$. Then we have with equivalence of norms*

$$(A_0, A_1)'_{1,q,\mathbb{A}} = A_1' \cap (A_0', A_1')_{1,q',(-1-1/q',-1-\alpha_0)}.$$

Proof. Take any $\alpha > -1/q$. Using Corollary 6.3 and Theorem 6.24, we get

$$(A_0, A_1)'_{1,q,\mathbb{A}} = (A_0 + A_1, A_1)'_{1,q,(\alpha_0,\alpha)} = (A_0' \cap A_1', A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}.$$

In particular, $(A_0, A_1)'_{1,q,\mathbb{A}} \subset A_1'$. Moreover, since $K(t, f; A_0' \cap A_1', A_1') \sim t\,\|f\|_{A_1'}$ for $t \le 1$, using that $-1/q < \alpha$, we obtain

$$\begin{aligned}
\left(\int_0^1\left[\frac{\ell^{-1-\alpha}(t)}{t}K(t,f;A_0'\cap A_1',A_1')\right]^{q'}\frac{dt}{t}\right)^{1/q'} &\sim \left(\int_0^1 \ell^{(-1-\alpha)q'}(t)\frac{dt}{t}\right)^{1/q'}\|f\|_{A_1'}\\
&\lesssim \left(\int_1^\infty\left[\frac{\ell^{-1-\alpha_0}(t)}{t}\right]^{q'}\frac{dt}{t}\right)^{1/q'}\|f\|_{A_1'}\\
&\lesssim \left(\int_1^\infty\left[\frac{\ell^{-1-\alpha_0}(t)}{t}K(t,f;A_0'\cap A_1',A_1')\right]^{q'}\frac{dt}{t}\right)^{1/q'}.
\end{aligned}$$

Whence

$$\|f\|_{(A_0'\cap A_1',A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}} \sim \left(\int_1^\infty\left[\frac{\ell^{-1-\alpha_0}(t)}{t}K(t,f;A_0'\cap A_1',A_1')\right]^{q'}\frac{dt}{t}\right)^{1/q'}. \tag{6.35}$$

On the other hand,

$$K(t,f;A_0'\cap A_1',A_1') \sim \|f\|_{A_1'} + K(t,f;A_0',A_1') \text{ for } t\geq 1. \tag{6.36}$$

Indeed, for $f\in A_1'$, given any decomposition $f=f_0+f_1$ with $f_0\in A_0'$ and $f_1\in A_1'$, we get

$$\begin{aligned}K(t,f;A_0'\cap A_1',A_1') &\leq \|f_0\|_{A_0'\cap A_1'} + t\|f_1\|_{A_1'} \leq \|f_0\|_{A_0'} + \|f_0\|_{A_1'} + t\|f_1\|_{A_1'}\\ &= \|f-f_1\|_{A_1'} + \|f_0\|_{A_0'} + t\|f_1\|_{A_1'} \lesssim \|f\|_{A_1'} + \Big(\|f_0\|_{A_0'} + t\|f_1\|_{A_1'}\Big).\end{aligned}$$

Conversely, if $f=g_0+g_1$ with $g_0\in A_0'\cap A_1'$ and $g_1\in A_1'$, we have that

$$\|f\|_{A_1'} + K(t,f;A_0',A_1') \leq \|g_0\|_{A_1'} + \|g_1\|_{A_1'} + \|g_0\|_{A_0'} + t\|g_1\|_{A_1'} \lesssim \|g_0\|_{A_0'\cap A_1'} + t\|g_1\|_{A_1'}.$$

Inserting (6.36) in (6.35), we obtain that

$$\begin{aligned}\|f\|_{(A_0'\cap A_1',A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}} \sim{}& \left(\int_1^\infty \left[\frac{\ell^{-1-\alpha_0}(t)}{t}\right]^{q'}\frac{dt}{t}\right)^{1/q'} \|f\|_{A_1'}\\ &+\left(\int_1^\infty \left[\frac{\ell^{-1-\alpha_0}(t)}{t}K(t,f;A_0',A_1')\right]^{q'}\frac{dt}{t}\right)^{1/q'}.\end{aligned}$$

Now, clearly

$$\|f\|_{(A_0'\cap A_1',A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}} \lesssim \|f\|_{A_1'} + \|f\|_{(A_0',A_1')_{1,q',(-1-1/q',-1-\alpha_0)}}.$$

On the other hand, since $(A_0'\cap A_1',A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}\subset A_1'$, we also have that

$$\left(\int_0^1 \left[\frac{\ell^{-1-1/q'}(t)}{t}K(t,f;A_0',A_1')\right]^{q'}\frac{dt}{t}\right)^{1/q'} \leq \|f\|_{A_1'}\left(\int_0^1 \ell^{-q'-1}(t)\frac{dt}{t}\right)^{1/q'}.$$

This implies that also $\|f\|_{A_1'}+\|f\|_{(A_0',A_1')_{1,q',(-1-1/q',-1-\alpha_0)}} \lesssim \|f\|_{(A_0'\cap A_1',A_1')_{1,q',(-1-\alpha,-1-\alpha_0)}}$ and completes the proof. □

With similar arguments but using now Theorem 6.27 we derive the following.

Theorem 6.29. *Let $\bar{A}=(A_0,A_1)$ be a regular Banach couple. Suppose that $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ satisfies that $\alpha_0<0$ and $\alpha_\infty\leq 0$. Then we have with equivalence of norms*

$$\big((A_0,A_1)^{o}_{1,\infty,\alpha}\big)' = A_1'\cap(A_0',A_1')_{1,1,(-1-1,-1-\alpha_0)}.$$

Remark 6.9. Using Remark 6.1, the duality results for $(A_0,A_1)_{0,q,\mathbb{A}}$ follow from those established for $(A_0,A_1)_{1,q,\tilde{\mathbb{A}}}$.

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