Thermalization in Kitaev's quantum double models via Tensor Network techniques

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Abstract

We show that the Davies generator associated to *any* 2D Kitaev's quantum double model has a non-vanishing spectral gap in the thermodynamic limit. This validates rigorously the extended belief that those models are useless as self-correcting quantum memories, even in the non-abelian case. The proof uses recent ideas and results regarding the characterization of the spectral gap for parent Hamiltonians associated to Projected Entangled Pair States in terms of a bulk-boundary correspondence.

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1 Introduction

In the seminal works [13, 2] it is shown that the 4D Toric Code is a self-correcting quantum memory, that is, it allows to keep quantum information protected against thermal errors (for all temperatures below a threshold) without the need for active error correction, for times that grow exponentially with the system size N. Since, after mapping the 4D Toric Code to a 2D or 3D geometry, interactions become highly non-local, it has been a major open question since then whether similar self-correction is possible in 2D or 3D, where the information is encoded in the degenerate ground space of a locally interacting Hamiltonian in a 2D or 3D geometry. We refer to the review [7] for a very detailed discussion of the many different contributions to the problem, that still remains open up to date.

Before focusing on the 2D case, which is the main goal of this work, let us briefly comment that in 3D, this question motivated the discovery of Haah's cubic code [5, 15], which was the opening door to a family of new ultra-exotic quantum phases of matter, currently known as *fractons* [30].

In 2D, it is a general belief that self-correction is not possible. There is indeed compelling evidence for that. For instance, Landon-Cardinal and Poulin [25], extending a result of Bravyi and Terhal [6], showed that commuting frustration free models in 2D display only a constant energy barrier. That is, it is possible to implement a sequence of poly(N) local operations that maps one ground state into an orthogonal one and, at the same time, the energy of all intermediate states is bounded by a constant independent of N. This seemed to rule out the existence of self-correction in 2D.

However, it was later shown in [8] that having a bounded energy barrier does not exclude self-correction, since it could happen that the paths implementing changes in the ground space are highly non-typical, and hence, the system could be *entropically* protected. Indeed, en example is shown in [8] where, in a very particular regime of temperatures though, entropic protection ocurs.

Therefore, in order to solve the problem in a definite manner, one needs to consider directly the mixing time of the thermal evolution operator which, in the weak coupling limit, is given by the Davies master equation [12]. Self-correction will not be possible if the noise operator relaxes fast to the Gibbs ensemble, where all information is lost. As detailed in [1] or [23] using standard arguments on Markovian semigroups, the key quantity that controls this relaxation time is the spectral gap of the Davies Lindbladian generator. Self-correction in 2D would be excluded if one is able to show that such gap is uniformly lower bounded independently of the system size. This is precisely the result proven for the Toric Code by Alicki et al, already in 2002, in the pioneer work [1]. The result was extended for the case of all abelian quantum double models by Komar et al in 2016 [23]. Indeed, up to now, these were the only cases for which the belief that self-correction does not exist in 2D have been rigorously proven. In particular, it remained an open question (as highlighted in the review [7]) whether the same result would hold for the case of *non-abelian* quantum double models.

In this work we address and solve this problem showing that non-abelian quantum double models behave exactly as their abelian counterparts. The main result of this work is the following

Theorem 1.1. *For any finite group G and for all inverse temperature β, the spectral gap of the Davies Lindbladian generator $\mathcal{L}$ associated to Kitaev's quantum double Hamiltonian H of group G has a uniform lower bound which is independent of the system size. Specifically, there exist positive constants C and $\lambda(G)$, independent of β and system size, such that*

$$\operatorname{gap}(\mathcal{L}) \geq \hat{g}_{\min}\, e^{-C\,\beta}\, \lambda(G), \tag{1}$$

where $\hat{g}_{\min}$ depends on the specific choice of thermal bath.

Note that while in principle $\hat{g}_{\min}$ could also scale with β, there are examples where it can be lower bounded by a strictly positive constant independent of the temperature.

The tools used to address the main Theorem are completely different from those used in the abelian case in [23]. There, following ideas of [31], the authors bound the spectral gap of $\mathcal{L}$ via a quantum version of the canonical-paths method in classical Markov chains.

We go back to the original idea of Alicki et al. for the Toric Code [1]: construct an artificial Hamiltonian from the Davies generator $\mathcal{L}$ so that the spectral gap of $\mathcal{L}$ coincides with the spectral gap above the ground state of that Hamiltonian, and then use techniques to bound spectral gaps of many body Hamiltonians.

This trick has found already other interesting implications in quantum information, especially in problems related to thermalization like the behavior of random quantum circuits [4] or the convergence of Gibbs sampling protocols [18].

In particular, we will follow closely the implementation of that idea used in [33], and reason as follows. We purify the Gibbs state ρ_β and consider the (pure) thermofield double state $|\rho_\beta^{1/2}\rangle$. The commutativity of the terms in the quantum double Hamiltonian H makes $|\rho_\beta^{1/2}\rangle$ a Projected Entangled Pair State (PEPS). One can show then that the gap of $\mathcal{L}$ can be lower bounded by the gap of the parent Hamiltonian of $|\rho_\beta^{1/2}\rangle$ in the PEPS formalism.

This opens the door to exploit the extensive knowledge gained in the area of Tensor Networks during the last decades. Tensor Networks, and in particular PEPS, have revealed themselves as an invaluable tool to understand, classify and simulate strongly correlated quantum systems (see e.g. the reviews [27], [32], [11]). The key reason is that they approximate well ground and thermal states of short-range Hamiltonians and, at the same time, display a local structure that allow to describe and manipulate them efficiently [11].

Such local structure manifests itself in a bulk-boundary correspondence that was first uncovered in [10], where one can associate to each patch of the 2D PEPS a 1D mixed state that lives on the boundary of the patch. It is conjectured in [10], and verified numerically for some examples, that the gap of the parent Hamiltonian in the bulk corresponds to a form of locality in the associated boundary state.

This bulk-boundary correspondence was made rigorous for the first time in [20] (see also the subsequent contribution [28]). In particular, it is shown in [20] that if the boundary state displays a locality property called *approximate factorization*, then the bulk parent Hamiltonian has a non-vanishing spectral gap in the thermodynamic limit.

Roughly speaking, approximate factorization can be defined as follows. Consider a 1D chain of N sites that we divide in 3 regions: left (L), middle (M) and right (R). A mixed state ρ_{LMR} is said to approximately factorize if if can be written as

$$\rho_{LMR} \approx (\Omega_{LM} \otimes \mathbb{1}_R)\,(\mathbb{1}_L \otimes \Delta_{MR})$$

where, for a particular notion of distance, the error in the approximation decays fast with the size of M.

It is one of the main contributions of [20, 28] to show that Gibbs states of Hamiltonians with sufficiently fast decaying interactions fulfill the approximate factorization property. Indeed, this idea has been used in [24] to give algorithms that provide efficiently Matrix Product Operator (MPO) descriptions of 1D Gibbs states.

We will precisely show that the boundary states associated to the thermofield double PEPS $|\rho_\beta^{1/2}\rangle$ approximately factorize. In order to finish the proof of our main theorem, we will also need to extend the validity of the results in [20] beyond the cases considered there (injective and MPO-injective PEPS), so that it applies to $|\rho_\beta^{1/2}\rangle$. Indeed, it has been a technical challenge in the paper to deal with a PEPS which is neither injective nor MPO-injective, the classes for which essentially all the analytical results for PEPS have been proven [11].

Let us finish this introduction by commenting that the results presented in this work can be seen as a clear illustration of the power of the bulk-boundary correspondence in PEPS, and in particular, the power of the ideas and techniques developed in [20].

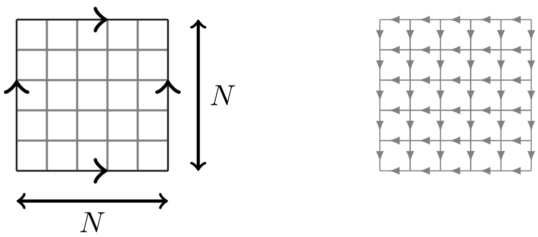
We are very confident that the result presented here can be extended, using similar techniques, to cover all possible 2D models that are renormalization fixed points, like string net models [26]. The reason is that all those models have shown to be very naturally described and analyzed in the language of PEPS [11]. We leave such extension for future work.

This paper is structured as follow. In Section 2, we recall the definition and elementary properties of the Quantum Double Models and their associated Davies generators. These will be shown to be equivalent to a frustration-free Hamiltonian having the thermofield double state $|\rho_\beta^{1/2}\rangle$ as unique ground state. In Section 3, we will construct the PEPS representation for the latter, and prove it satisfies an approximate factorization condition. We will then construct in Section 4 the parent Hamiltonian for this PEPS, show that it is gapped, and finally use this result to lower bound the gap of the Davies generator. The adaptation of the results relating the gap of the parent Hamiltonian and the approximate factorization of the PEPS from [20] are included in Section 5: these are stated more generally than the case we considered in the rest of the paper, as they might be of more general interest.

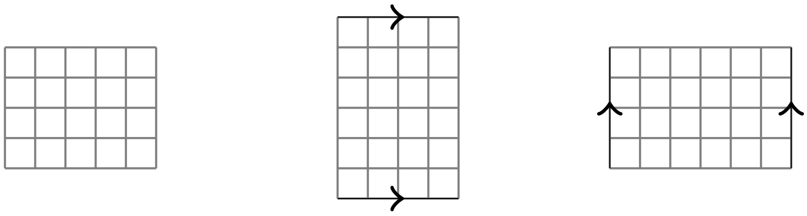
2 Davies generators for Quantum Double Models

2.1 Quantum Double Models

Let us stat by recalling the definition of the Quantum Double Models. They are defined on the lattice $\Lambda_N = \mathbb{Z}_N \times \mathbb{Z}_N$, a $N \times N$ square lattice with periodic boundary conditions, for some positive integer N. Let us denote by $\mathcal{V}$ the set of vertices, and by $\mathcal{E}$ the set of edges of Λ. Each edge is given an orientation: for simplicity, we will assume that all horizontal edges point to the left, while vertical edges point downwards.



By a rectangular (sub)region $\mathcal{R} \subset \Lambda$ we mean either the whole lattice Λ or any of the following three type of subregions



that is, a proper rectangle (open boundary conditions on its four sides), or a cylinder (open boundary conditions on two opposite sides). We will say that $\mathcal{R}$ has dimensions a, b if it has a plaquettes per row and b plaquettes per column.

Let us fix an arbitrary finite group G and denote let $\ell_2(G)$ be the complex finite dimensional Hilbert space with orthonormal basis given by $\{|g\rangle \mid g \in G\}$. At each edge $e \in \mathcal{E}$ we have a local Hilbert space and a space of observables

$$\mathcal{H}_e = \ell_2(G) \quad \text{and} \quad \mathcal{B}_e = \mathcal{B}(\mathcal{H}_e) = \mathcal{M}_{|G|}(\mathbb{C}).$$

For any subset $X \subset \mathcal{E}$, we will also denote

$$\mathcal{H}_X = \bigotimes_{e \in X} \mathcal{H}_e \quad , \quad \mathcal{B}_X = \bigotimes_{e \in X} \mathcal{B}_e.$$

In particular, we will use the alternative notation $\mathcal{H}_\Lambda = \mathcal{H}_\mathcal{E}$ and $\mathcal{B}_\Lambda = \mathcal{B}_\mathcal{E}$. As usual, we identify (isometrically) $\mathcal{B}_X$ with the subspace of $\mathcal{B}_\mathcal{E}$ consisting on elements with support in X through $Q \mapsto Q \otimes \mathbb{1}_{\mathcal{E} \setminus X}$.

Given $g \in G$ we define operators on $\ell_2(G)$ by

$$L^g := \sum_{h \in G} |gh\rangle\langle h| \,. \tag{2}$$

Then $g \mapsto L^g$ is a representation of the group G, known as the *left regular* representation.

For each finite group G, the Quantum Double Model on Λ is defined by a Hamiltonian H_Λ^{syst} of the form

$$H_\Lambda^{syst} = -\sum_{v \text{ vertex}} A(v) \; - \sum_{p \text{ plaquette}} B(p)\,;$$

where the terms $A(v)$ are *star operators*, supported on the four incident edges of v, which we will denote as ∂v, while $B(p)$ are *plaquette operators*, supported on the four edges forming the plaquette p. Both terms are projections and they commute, namely

$$[A(v), A(v')] \;=\; [B(p), B(p')] \;=\; [A(v), B(p)] \;=\; 0$$

for all vertices v, v' and plaquettes p, p'. We will now explicitly define these terms and check that they satisfy these properties.

Let v be a vertex and e an edge incident to v. For each $g \in G$ we define the operator $T^g(v,e)$ acting on $\mathcal{H}_e$ according to the orientation given to e as

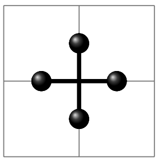
$$T^g(v,e) \quad = \quad \underset{\longmapsto}{\sum_{h\in G} |gh\rangle\!\langle h|} \qquad \text{or} \qquad \underset{\longleftarrow\!\!\!\!\!\!\!\!\!\!\vdash}{\sum_{h\in G} |hg^{-1}\rangle\!\langle h|}$$

In other words, the operator $T^g(v,e)$ acts on the basis vector of $\mathcal{H}_e$ by taking h into gh (resp. hg^{-1}) if the oriented edge e points away from (resp. to) v. It is easily checked that

$$T^g(v,e)\, T^h(v,e) = T^{gh}(v,e) \qquad \text{and} \qquad T^g(v,e)^\dagger = T^{g^{-1}}(v,e) \tag{3}$$

for every $g, h \in G$.

With this definition, the vertex operator $A(v)$ is given by

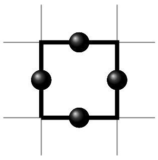


$$A(v) \;=\; \frac{1}{|G|} \sum_{g\in G} \bigotimes_{e\in\partial v} T^g(v,e)$$

Using (3) , it is easy to verify that $A(s)$ is a projection.

The plaquette operator $B(p)$ is defined as follows. Let us enumerate the four edges of p as e_1, e_2, e_3, e_4 following counterclockwise order starting from the upper horizontal edge. The plaquette operator on p acts on $\otimes_{j=1}^4 \mathcal{H}_{e_j}$ and is defined as the orthogonal projection $B(p)$ onto the subspace spanned by basis vectors of the form $|g_1 g_2 g_3 g_4\rangle$ with $\sigma_p(g_1)\sigma_p(g_2)\sigma_p(g_3)\sigma_p(g_4) = 1$. Here, $\sigma_p(g)$ is equal to g if the orientation of the corresponding edge agrees with the counter-clockwise labelling, otherwise it is equal to g^{-1}.

For the orientation we have previously fixed, we can give an explicit expression in terms of the regular character $g \mapsto \chi^{reg}(g) = \mathrm{Tr}(L^g) = |G|\,\delta_{g,1}$, namely



$$B(p) = \frac{1}{|G|} \sum_{g_1,g_2,g_3,g_4 \in G} \chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1}) \bigotimes_{j=1}^{4} |g_j\rangle\langle g_j|$$

2.2 Davies generators

We will now recall the construction of the generator of a semigroup of quantum channels which describes a weak-coupling limit of the joint evolution of the system with a local thermal bath, known as the Davies generator [12]. This construction applies to any commuting local Hamiltonian, but for the simplicity of the notation we will only consider the same setup of the previous section, i.e. the qudits live on the edges, and not the vertices, of a lattice $\Lambda = (\mathcal{V}, \mathcal{E})$.

For a bath at inverse temperature β the Davies generator in the Heisenberg picture takes the form

$$\mathcal{G}(Q) = i\Big[H_\Lambda^{syst}, Q\Big] + \sum_{e \in \mathcal{E}} \mathcal{L}_e(Q) \quad , \quad Q \in \mathcal{B}_\Lambda \tag{4}$$

where H_Λ^{syst} is the Hamiltonian of the system and the dissipative term $\mathcal{L}_e$ has the specific form

$$\mathcal{L}_e(Q) = \sum_{\alpha,\omega} \widehat{g}_{e,\alpha}(\omega)\, \mathcal{D}_{e,\alpha,\omega}(Q), \tag{5}$$
$$\mathcal{D}_{e,\alpha,\omega}(Q) = \frac{1}{2}\Big(S_{e,\alpha}^\dagger(\omega)[Q, S_{e,\alpha}(\omega)] + [S_{e,\alpha}^\dagger(\omega), Q]\, S_{e,\alpha}(\omega)\Big).$$

The variable ω runs over the finite set of Bohr frequencies of H_Λ^{syst} (the differences between energy levels), while the index α enumerates an orthonormal basis $S_{\alpha,e}$ of $\mathcal{B}_e$ for each edge e, which we can assume to be composed of self-adjoint operators. The non-negative scalar coefficients $\widehat{g}_{e,\alpha}(\omega) = \widehat{g}_\alpha(\omega)$ are the Fourier coefficients of the eigenvalues of the autocorrelation matrix of the environment, while the the jump operators $S_{e,\alpha}(\omega) = S_\alpha(\omega)$ are the Fourier components of S_α evolving under H_Λ^{syst}, namely

$$e^{itH_\Lambda^{syst}} S_\alpha e^{-itH_\Lambda^{syst}} = \sum_\omega S_\alpha(\omega)\, e^{-i\omega t}\,,\ t \in \mathbb{R}\,.$$

These satisfy the following properties for every α and ω:

(i) $S_\alpha^\dagger(\omega) = S_\alpha(-\omega)$,

(ii) $\widehat{g}_\alpha(-\omega) = e^{-\beta\omega}\, \widehat{g}_\alpha(\omega)$.

We can rewrite the sum in the definition of $\mathcal{L}_e$ only over $\omega \geq 0$:

$$\mathcal{L}_e = \sum_{\alpha} \left[\widehat{g}_{e,\alpha}(0)\, \mathcal{D}_{e,\alpha,0} + \sum_{\omega>0} \left(\widehat{g}_{e,\alpha}(\omega)\, \mathcal{D}_{e,\alpha,\omega} + \widehat{g}_{e,\alpha}(-\omega)\, \mathcal{D}_{e,\alpha,-\omega} \right) \right].$$

We now denote for each e, α and $\omega \geq 0$

$$\mathcal{L}_{e,\alpha,\omega} = \begin{cases} \widehat{g}_{e,\alpha}(0)\, \mathcal{D}_{e,\alpha,0} & \text{if } \omega = 0, \\ \widehat{g}_{e,\alpha}(\omega)\, \mathcal{D}_{e,\alpha,\omega} + \widehat{g}_{e,\alpha}(-\omega)\, \mathcal{D}_{e,\alpha,-\omega} & \text{otherwise.} \end{cases}$$

Moreover, if ρ_β denotes the Gibbs state associate to H_Λ^{syst} at inverse temperature β, then $\rho_\beta^s\, S_\alpha(\omega) = e^{s\beta\omega}\, S_\alpha(\omega)\, \rho_\beta^s$ for every $s \in \mathbb{R}$. This implies that $\mathcal{L}_{e,\alpha,\omega}$ satisfies the *detailed balance condition* with respect to ρ_β: defining a scalar product on $\mathcal{B}_\Lambda$ (known as the Liouville or GNS scalar product) by

$$\langle A, B \rangle_\beta := \operatorname{Tr}(\rho_\beta A^\dagger B),$$

we have that for every $A, B \in \mathcal{B}_\Lambda$

$$\langle A, \mathcal{L}_{e,\alpha,\omega}(B) \rangle_\beta = \langle \mathcal{L}_{e,\alpha,\omega}(A), B \rangle_\beta. \tag{6}$$

In fact a stronger condition holds: from $(i) - (ii)$ above it follows that

$$\begin{aligned} &-\langle A, \mathcal{L}_{e,\alpha,\omega}(A) \rangle_\beta \\ &\quad = \widehat{g}_{e,\alpha}(\omega)\, \|[A, S_{e,\alpha}(\omega)]\|_\beta^2 + \widehat{g}_{e,\alpha}(-\omega)\, \|[A, S_{e,\alpha}^\dagger(\omega)]\|_\beta^2 \geq 0, \end{aligned} \tag{7}$$

so that $-\mathcal{L}_{e,\alpha,\omega}$ is a positive self-adjoint operator w.r.t. the GNS scalar product. The Gibbs state ρ_β is an invariant state for $\mathcal{L}_{e,\alpha,\omega}$, in the sense that

$$\operatorname{Tr}(\rho_\beta e^{t\mathcal{L}_{e,\alpha,\omega}}(Q)) = \operatorname{Tr}(\rho_\beta Q) \quad \text{for every } Q, t \geq 0.$$

For any subset $X \subset \mathcal{E}$, the kernel of $\sum_{e \in X} \mathcal{L}_e$ is $\mathcal{B}_{\mathcal{E}\setminus X}$, that is, the subspace of operators whose support is contained in $\mathcal{E} \setminus X$. In particular, the kernel of $\mathcal{L} = \sum_{e\in\mathcal{E}} \mathcal{L}_e$ consists of multiples of the identity, and the generator is *primitive*: for any initial state ρ it holds that

$$\lim_{t\to\infty} \operatorname{Tr}(\rho e^{t\mathcal{L}}(Q)) = \operatorname{Tr}(\rho_\beta Q), \quad \text{for every } Q.$$

2.3 Explicit form of the Davies generator

We can now give an explicit description of the Davies generators for the Quantum Double Models. Since the local terms of H_Λ^{syst} are commuting, fixed $e \in \mathcal{E}$ for every $t \in \mathbb{R}$

$$\begin{aligned} &e^{itH_\Lambda^{syst}} S_{e,\alpha} e^{-itH_\Lambda^{syst}} \\ &\quad = e^{it(A(v_1)+A(v_2)+B(p_1)+B(p_2))}\, S_{e,\alpha}\, e^{-it(A(v_1)+A(v_2)+B(p_1)+B(p_2))} \end{aligned} \tag{8}$$

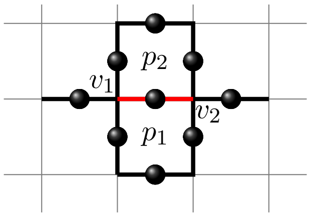


Figure 1: Region supporting $\mathcal{L}_e$ (the edge $e = (v_1, v_2)$ is marked in red).

where v_1, v_2 are the vertices of e and p_1, p_2 are the two plaquettes containing e. Hence the support $\mathcal{L}_e$ is strictly local and contained in (see Figure 1):

$$\operatorname{supp} \mathcal{L}_e \subset \partial v_1 \cup \partial v_2 \cup p_1 \cup p_2$$

The local terms $A(v)$ and $B(p)$ are both projections. It is easy to check that for a projection Π with orthogonal complement $\Pi^\perp := \mathbb{1} - \Pi$

$$e^{it\Pi} = \mathbb{1} + (e^{it} - 1)\Pi = \Pi^\perp + e^{it}\Pi$$

and so

$$e^{it\Pi} Q e^{-it\Pi} = \Pi Q \Pi + \Pi^\perp Q \Pi^\perp + e^{-it}\Pi^\perp Q \Pi + e^{it}\Pi Q \Pi^\perp .$$

Consequently, we can rewrite (8) as a sum

$$e^{itH_\Lambda^{syst}} S_{e,\alpha} e^{-itH_\Lambda^{syst}} = \sum_{\omega=-4}^{4} S_{e,\alpha}(\omega)\, e^{it\omega} .$$

where each $S_{e,\alpha}(\omega)$ has also support contained in $\partial v_1 \cup \partial v_2 \cup p_1 \cup p_2$ (see Figure 1).

2.4 Davies generator as a local Hamiltonian

We have seen in the previous sections that $-\mathcal{L}$ is local, self-adjoint, and positive with respect to the $\langle \cdot | \cdot \rangle_\beta$ scalar product, with a unique element in its kernel corresponding to the Gibbs state ρ_β. We will now describe how to convert it into a frustration free local Hamiltonian whose unique groundstate is the thermofield double of ρ_β (a local purification of the Gibbs state), defined as

$$|\rho_\beta^{1/2}\rangle = \frac{1}{Z_\beta^{1/2}} \sum_{\lambda \in \sigma(H_\Lambda^{syst})} e^{-\frac{\beta}{2}} |\lambda\rangle \otimes |\lambda\rangle = \left(\rho_\beta^{1/2} \otimes \mathbb{1}\right) |\Omega\rangle , \qquad (9)$$

where $|\Omega\rangle$ is a maximally entangled state on $\mathcal{H}_\Lambda^2 = \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$. Denoting

$$\iota(A) = A\rho_\beta^{1/2}, \quad A \in \mathcal{B}_\Lambda,$$

we can identify operators in $\mathcal{B}_\Lambda$ with vectors in $\mathcal{H}^2_\Lambda$ via

$$Q \longrightarrow |\iota(Q)\rangle = \left(Q\rho_\beta^{1/2} \otimes \mathbb{1}\right) |\Omega\rangle \,. \tag{10}$$

This identification preserves the $\langle\cdot|\cdot\rangle_\beta$ scalar product, making it an isometry between $(\mathcal{B}_\Lambda, \|\cdot\|_\beta)$ and $\mathcal{H}^2_\Lambda$:

$$\langle\iota(A)|\iota(B)\rangle = \langle\Omega|\left(\rho_\beta^{1/2} A^\dagger B \rho_\beta^{1/2}\right) \otimes \mathbb{1}|\Omega\rangle = \langle A, B\rangle_\beta.$$

This allows us to associate to $\mathcal{L}$ a Hamiltonian $\widetilde{H} = -\widetilde{L}$ on $\mathcal{H}^2_\Lambda$, where

$$\widetilde{L} = \sum_{e\in\mathcal{E}} \widetilde{L}_e \quad \text{and} \quad \widetilde{L}_e\,|\iota(Q)\rangle = |\iota(\mathcal{L}_e(Q)\rangle \,.$$

The properties of the Davies generator $\mathcal{L}$ are reflected into properties of $\widetilde{H}$ as follows:

1. With respect to the Hilbert-Schmidt scalar product on $\mathcal{H}^2_\Lambda$, $\widetilde{H}$ is self-adjoint and positive.

2. If $X \subset \mathcal{E}$ is a finite subset and $|\phi\rangle \in \mathcal{H}^2_\Lambda$ satisfies that $\sum_{e\in X} \widetilde{L}_e\,|\phi\rangle = 0$, then $\iota^{-1}(\phi) = Q \otimes \mathbb{1}_X \in \mathcal{B}_{\mathcal{E}\setminus X}$, and so

   $$|\phi\rangle = \frac{1}{Z_\beta^{1/2}}\left((Q \otimes e^{-\frac{\beta}{2}H_X})e^{-\frac{\beta}{2}(H_\mathcal{E}-H_X)}\right) \otimes \mathbb{1}\,|\Omega\rangle \,.$$

   In particular, the thermofield double $|\rho_\beta^{1/2}\rangle = |\iota(\mathbb{1})\rangle$ is the unique groundstate of $\widetilde{H}$.

In general $\widetilde{H}_e = -\widetilde{L}_e$ will not be a projection. Denoting $\Pi_e^\perp$ the projection on the subspace orthogonal to the groundstate of $\widetilde{H}_e$, we can always find a constant $C > 0$ such that

$$\widetilde{H}_e \geq C\,\Pi_e^\perp \quad , \quad \text{for every } e \in \mathcal{E},$$

from which it follows that the gap of $\widetilde{H}$ (which is the same as the gap of $\mathcal{L}$) is lower bounded by C times the gap of $\sum_e \Pi_e^\perp$. We will now compute an explicit expression for the constant C, which will turn out to depend on the inverse temperature β.

Proposition 2.1.

$$\widetilde{H}_e \geq \frac{C_2}{|\Omega_e|}\widehat{g}_{\min} e^{-C_1\beta}\,\Pi_e^\perp. \tag{11}$$

Proof. Let us fix the edge e (and drop the subscript to make notation lighter). We denote by

$$\delta_{\alpha,\omega}(A) = [A, S_\alpha(\omega)].$$

We can then rewrite (7) as

$$\langle\psi|\widetilde{H}_e\phi\rangle = -\left\langle \iota^{-1}\psi\middle|\mathcal{L}_e(\iota^{-1}\phi)\right\rangle_\beta = \sum_{\alpha,\,\omega}\widehat{g}_\alpha(\omega)\left\langle \delta_{\alpha,\omega}\iota^{-1}(\psi)\middle|\delta_{\alpha,\omega}\iota^{-1}(\psi)\right\rangle_\beta$$

$$= \sum_{\alpha,\omega}\widehat{g}_\alpha(\omega)\left\langle \iota\,\delta_{\alpha,\omega}\iota^{-1}(\psi)\middle|\iota\,\delta_{\alpha,\,\omega}\iota^{-1}(\phi)\right\rangle = \sum_{\alpha,\,\omega}\widehat{g}_\alpha(\omega)\;\langle\psi|k_{\alpha,\omega}\phi\rangle\,;$$

where $k_{\alpha,\omega}$ are positive operators defined by $k_{\alpha,\omega} = \left|\iota\,\delta_{\alpha,\omega}\,\iota^{-1}\right|^2$. We compute

$$\iota\,\delta_{\alpha,\omega}\,\iota^{-1} = e^{-\frac{\beta}{2}\omega}\,\mathbb{1}\otimes S_\alpha(\omega)^T - S_\alpha(\omega)\otimes\mathbb{1}. \tag{12}$$

Note that we can also write

$$\iota\,\delta_{\alpha,\omega}\,\iota^{-1} = \mathbb{1}\otimes(\rho_\beta^{-1/2}S_\alpha(\omega)\rho_\beta^{1/2})^T - S_\alpha(\omega)\otimes\mathbb{1}.$$

Now we observe that the term $\rho_\beta^{-1/2}S_\alpha(\omega)\rho_\beta^{1/2}$ only depends on the interaction terms near the edge e, in the sense that

$$\rho_\beta^{-1/2}S_\alpha(\omega)\rho_\beta^{1/2} = e^{-\frac{\beta}{2}H^{syst}_{R_e}}S_\alpha(\omega)e^{\frac{\beta}{2}H^{syst}_{R_e}}\,,$$

and R_e is the region supporting $\widetilde{L}_e$ that appears in Figure 2. We define a *localized* version of the norm $\|\cdot\|_\beta$, by

$$\|X\|^2_{R_e} = \operatorname{tr}\left[e^{-\beta H^{syst}_{R_e}}X^*X\right]\quad,\quad X\in\mathcal{B}_\Lambda.$$

Note that

$$\|Q\|^2_\beta = \frac{1}{Z_\beta}\left\|Qe^{-\beta(H^{syst}_\Lambda - H^{syst}_{R_e})}\right\|_{R_e}.$$

Moreover

$$e^{-\frac{\beta}{2}C_1}\|X\|_{R_e}\le\|X\|_{HS}\le\|X\|_{R_e},\quad C_1 = \left\|H^{syst}_{R_e}\right\|_\infty.$$

Let us now consider the ground space of $\widetilde{H}_e$, which we have seen is given by

$$\ker\widetilde{H}_e = \left\{(Q\otimes\mathbb{1}_e)(\rho_\beta^{1/2}\otimes\mathbb{1})\,|\Omega\rangle\mid Q\in\mathcal{B}_{\mathcal{E}\setminus\{e\}}\right\},$$

and denote by $\Pi_e^\perp$ the projection on the orthogonal complement. We observe that

$$\left\|\Pi_e^\perp\,|\phi\rangle\right\|^2 = \inf_{\psi\in\ker\widetilde{H}_e}\||\phi\rangle - |\psi\rangle\|^2 = \inf_{Q_0\in\mathcal{B}_{\mathcal{E}\setminus\{e\}}}\|Q - Q_0\otimes\mathbb{1}_e\|^2_\beta$$

$$= \inf_{Q_0\in\mathcal{B}_{\mathcal{E}\setminus\{e\}}}\frac{1}{Z_\beta}\left\|(Q-Q_0\otimes\mathbb{1}_e)e^{-\beta(H^{syst}_{\mathcal{E}} - H^{syst}_{R_e})}\right\|^2_{R_e}$$

$$= \inf_{Q_0\in\mathcal{B}_{\mathcal{E}\setminus\{e\}}}\left\|\frac{1}{Z_\beta}Qe^{-\beta(H^{syst}_{\mathcal{E}} - H^{syst}_{R_e})} - Q_0\otimes\mathbb{1}_e\right\|^2_{R_e}.$$

In other words, writing $|\phi\rangle = (Q\rho_\beta^{1/2} \otimes \mathbb{1})\,|\Omega\rangle = (Xe^{-\frac{\beta}{2}H_{R_e}^{syst}} \otimes \mathbb{1})\,|\Omega\rangle$, we have

$$\left\| \Pi_e^\perp \,|\phi\rangle \right\|^2 = \inf_{Q_0 \in \mathcal{B}_{\Lambda\setminus e}} \|X - Q_0 \otimes \mathbb{1}_e\|_{R_e}^2 .$$

Let us now compute $\langle \phi | \widetilde{H}_e \phi \rangle$. We start by observing that

$$\langle \phi | k_{\alpha,\omega} \phi \rangle = \left\| \iota\, \delta_{\alpha,\omega} \iota^{-1}(\phi) \right\|^2 = \|\delta_{\alpha,\omega}(Q)\|_\beta^2 = \|\delta_{\alpha,\omega}(X)\|_{R_e}^2 \,,$$

since the terms in $e^{-\beta(H_{\mathcal{E}}^{syst} - H_{R_e}^{syst})}$ commute with $S_{\alpha,\omega}$. We now sum over ω and α:

$$\begin{aligned} \langle \phi | \widetilde{H}_e \phi \rangle \geq \widehat{g}_{\min} \sum_\alpha \sum_\omega \|\delta_{\alpha,\omega}(X)\|_{R_e}^2 &\geq \frac{\widehat{g}_{\min}}{|\Omega_e|} \sum_\alpha \|\delta_\alpha(X)\|_{R_e}^2 \\ &\geq e^{-\beta C_1} \frac{\widehat{g}_{\min}}{|\Omega_e|} \sum_\alpha \|\delta_\alpha(X)\|_{HS}^2 \,, \end{aligned}$$

where $\widehat{g}_{\min} = \min_{\alpha,\omega} \widehat{g}_\alpha(\omega)$, and Ω_e is the set of Bohr frequencies over which we are summing (in this case, $\{-4, \dots, 4\}$).

Now let us consider the following quantity

$$|||X||| := \left(\sum_\alpha \|\delta_\alpha(X)\|_{HS}^2 \right)^{1/2} .$$

We can see that it defines a seminorm, and $|||X||| = 0$ if and only if $X \in \{S_\alpha\}_\alpha' = \mathcal{B}_{\mathcal{E}\setminus\{e\}}$. Therefore, there exists a constant $C_2 > 0$, independent of system size and β, such that for every X

$$|||X|||^2 \geq C_2 \inf_{Q_0 \in \mathcal{B}_{\mathcal{E}\setminus\{e\}}} \|X - Q_0 \otimes \mathbb{1}_e\|_{HS}^2 .$$

This implies that for $|\phi\rangle = (Xe^{-\frac{\beta}{2}H_{R_e}^{syst}} \otimes \mathbb{1})\,|\Omega\rangle$

$$|||X|||^2 \geq C_2 \inf_{Q_0 \in \mathcal{B}_{\mathcal{E}\setminus\{e\}}} \|X - Q_0 \otimes \mathbb{1}_e\|_{R_e}^2 = C_2 \left\| \Pi_e^\perp \,|\psi\rangle \right\|^2 .$$

Putting all the bounds together concludes the proof. □

Remark 2.2. *Note that the region supporting $\widetilde{L}_e$ will be larger in general, see Figure 2.*

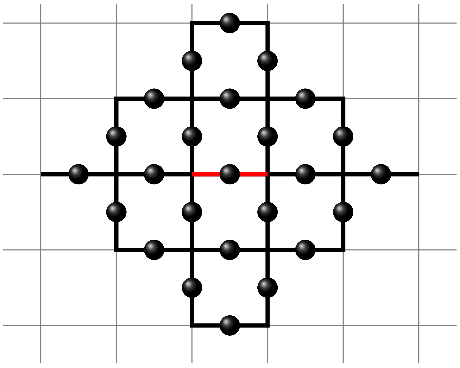


Figure 2: Region supporting $\widetilde{L}_e$ (the edge e is marked in red).

3 PEPS description of the thermofield double

Fixed $\beta > 0$, the associate Gibbs state at (inverse) temperature β is given by

$$\rho_\beta = e^{-\beta H_\Lambda^{syst}} / \operatorname{Tr}(e^{-\beta H_\Lambda^{syst}})$$

Since the star and plaquette operators commute, we can decompose

$$e^{-\frac{\beta}{2} H_\Lambda^{syst}} = \prod_{v \text{ vertex}} e^{\frac{\beta}{2} A(v)} \prod_{p \text{ plaquette}} e^{\frac{\beta}{2} B(p)} \,. \tag{13}$$

Using the fact that $A(v)$ and $B(p)$ are projections we can rewrite this expression as

$$e^{-\frac{\beta}{2} H_\Lambda^{syst}} = \prod_{v \text{ vertex}} \Big(\operatorname{Id} + (e^{\beta/2} - 1)\, A(v)\Big) \prod_{p \text{ plaquette}} \Big(\operatorname{Id} + (e^{\beta/2} - 1)\, B(p)\Big)$$

We will now construct a PEPO representation of the interactions $A(v)$ and $B(p)$, from which we will derive the PEPS representation of $|\rho_\beta^{1/2}\rangle$. We will use the notation

$$\gamma_\beta := \frac{e^\beta - 1}{|G|}$$

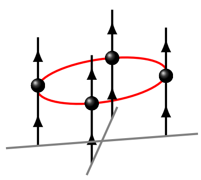
along the section.

3.1 PEPO elementary tensors

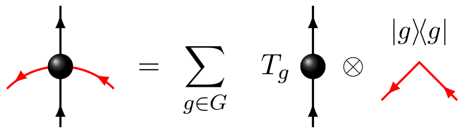
3.1.1 Star operator as a PEPO

The star operator $A(v)$ admits an easy representation as a PEPO, namely

$$A(v) = \frac{1}{|G|} \sum_{g \in G} \bigotimes_{e \in \partial v} T^g(e,v) = \frac{1}{|G|} \cdot$$



where we are denoting



Since $A(v)$ is a projection,

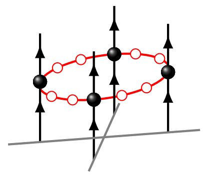
$$e^{\frac{\beta}{2}A(v)} \;=\; \mathrm{Id} + (e^{\beta/2}-1)\,A(v) \;=\; \mathrm{Id} + \gamma_{\beta/2}\,|G|\,A(v)\,,$$

or equivalently

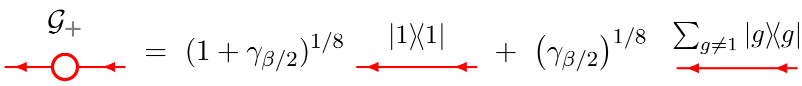
$$e^{\frac{\beta}{2}A(v)} \;=\; \left(1+\gamma_{\beta/2}\right) \bigotimes_{e\in\partial v} T^1(e,v) + \left(\gamma_{\beta/2}\right) \sum_{\substack{g\in G\\ g\neq 1}} \bigotimes_{e\in\partial v} T^g(e,v)\,.$$

Comparing with $A(v)$, we find a natural description as a PEPO

$$e^{\frac{\beta}{2}A(v)} \;=$$

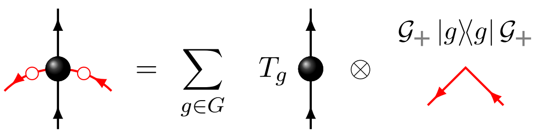


where we are adding to the above representation for $A(v)$ suitable *weights*



(14)

Therefore, we have a PEPO decomposition of $e^{\frac{\beta}{2}A(v)}$ into four identical tensors acting individually on each edge



where we can expand

$$\mathcal{G}_+\,|g\rangle\langle g|\,\mathcal{G}_+ = \left(\delta_{g,1}+\gamma_{\beta/2}\right)^{1/4}\,|g\rangle\langle g|\;.$$

3.1.2 Plaquette operator as a PEPO

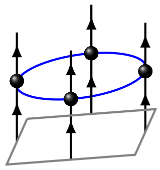
The plaquette operator

$$B(p) \;=\; \frac{1}{|G|} \sum_{g_1,g_2,g_3,g_4\in G} \chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1}) \bigotimes_{j=1}^{4} |g_j\rangle\langle g_j|$$

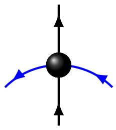
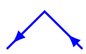
admits an easy PEPO representation. Using that

$$\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1}) = \mathrm{Tr}(L^{g_4} L^{g_3} L^{g_2^{-1}} L^{g_1^{-1}})\,.$$

we can decompose

$$B(p) \;=\; \frac{1}{|G|}\;\cdot$$ 

where

 $$= \sum_{g\in G} \;|g\rangle\langle g|$$  $$\otimes$$ 

As in the case of the star operator, $B(p)$ is a projection so

$$\begin{aligned} e^{\frac{\beta}{2}B(p)} &= \mathrm{Id} + (e^{\beta/2}-1)\,B(p) \\ &= \sum_{g_1,g_2,g_3,g_4\in G} \left(1 + \gamma_{\beta/2}\;\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1})\right)\;\bigotimes_{j=1}^{4}|g_j\rangle\langle g_j| \end{aligned}$$

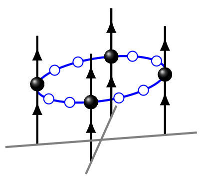
Recall that left regular representation is in general not irreducible. It has a unique irreducible sub-representation of dimension 1, which we denote by V_1, with associated projection P_1. If we denote $P_0 := P_1^{\perp} = \mathbb{1} - P_1$ then we have for every $g\in G$

$$L^g = P_1 L^g P_1 + P_0 L^g P_0 = P_1 + P_0 L^g P_0\,. \tag{15}$$

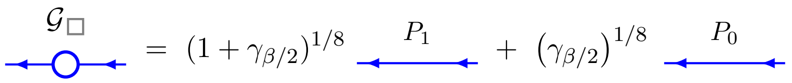
and so

$$\begin{aligned} 1 + (\gamma_{\beta/2})\,\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1}) &= 1 + (\gamma_{\beta/2})\;\mathrm{Tr}(L^{g_4} L^{g_3} L^{g_2^{-1}} L^{g_1^{-1}}) \\ &= \left(1+\gamma_{\beta/2}\right)\,\mathrm{Tr}(P_1\, L^{g_4} L^{g_3} L^{g_2^{-1}} L^{g_1^{-1}}) \\ &\quad + (\gamma_{\beta/2})\;\mathrm{Tr}(\,P_0\, L^{g_4} L^{g_3} L^{g_2^{-1}} L^{g_1^{-1}})\,. \end{aligned}$$

and so we have the following decomposition

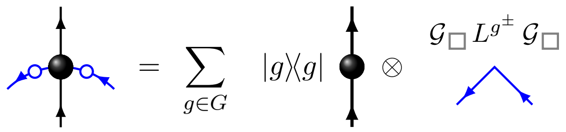
$$e^{\frac{\beta}{2}B(p)} \;=$$ 

where we are adding to the above representation for $B(p)$ suitable weights



(16)

Therefore, we have a PEPO decomposition of $e^{\frac{\beta}{2}B(p)}$ into four identical tensors acting individually on each edge can be given as

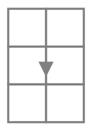


where we can expand

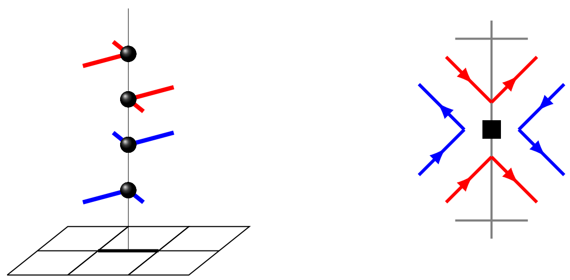
$$\mathcal{G}_\square \, L^{g^\pm} \, \mathcal{G}_\square = (1+\gamma_{\beta/2})^{1/4} \, P_1 \, + \, \left(\gamma_{\beta/2}\right)^{1/4} \, P_0 \, L^{g^\pm} \, P_0 \, .$$

3.1.3 PEPS tensor on an edge

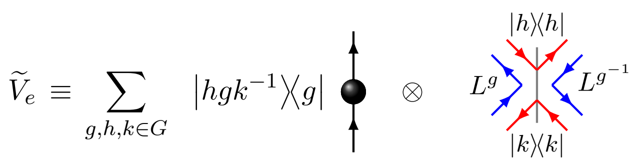
We have decomposed each star operator $e^{\frac{\beta}{2}A(s)}$, resp. plaquette operator $e^{\frac{\beta}{2}B(p)}$, into four tensors acting respectively on the incident, resp. surrounding, edges. Let us now fix an edge e with orientation:



Next, we contract the tensors acting on the edge (without weights)



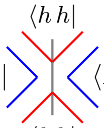
and obtain the *slim* tensor



(17)

Note that the two factors that appear in the previous expression correspond respectively to the physical part (left) and the virtual indices (right) of the tensor. In the usual notation for PEPS, we will also look at $\widetilde{V}_e$ as a map from the boundary Hilbert space $\mathcal{H}_{\partial e} = \ell_2(G)^{\otimes 8}$ into the physical Hilbert space $\mathcal{H}_e^2 = \ell_2(G)^{\otimes 2}$

$$\widetilde{V}_e : \mathcal{H}_{\partial e} \longrightarrow \mathcal{H}_e^2 \qquad , \qquad \widetilde{V}_e = \sum_{g,h,k\in G} \left|hgk^{-1}\right\rangle |g\rangle \quad \langle L^g| \; \overset{\langle h\,h|}{\underset{\langle k\,k|}{\times}} \; \langle L^{g^{-1}}| \tag{18}$$



Here we are using the identification $\mathcal{B}(\ell_2(G)) \equiv \ell_2(G)\otimes\ell_2(G)$ via the purification map $Q \mapsto |Q\rangle = Q\otimes \mathbb{1}\,|\Omega\rangle$, where $|\Omega\rangle = \sum_h |hh\rangle$, that we already used in the first section of the paper.

The *full* tensor V_e is then constructed from $\widetilde{V}_e$ by adding the corresponding weights $\mathcal{G}_+$ and $\mathcal{G}_\square$ on the boundary indices. Indeed, adding the weights to representation (17) leads to the corresponding representation for $\widetilde{V}_e$ simply replacing

$$|h\rangle\langle h| \;\mapsto\; \mathcal{G}_+\,|h\rangle\langle h|\,\mathcal{G}_+ \qquad , \qquad L^g \;\mapsto\; \mathcal{G}_\square L^g \mathcal{G}_\square\,,$$

whereas adding the weights to (18) leads to the same expression but replacing

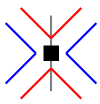
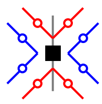
$$\langle h\,h| \;\mapsto\; \langle h\,h|\,(\mathcal{G}_+\otimes\mathcal{G}_+) \quad , \quad \langle L^g| \;\mapsto\; \langle L^g|\,(\mathcal{G}_\square\otimes\mathcal{G}_\square)\,.$$

Using this last representation, we can relate

$$V_e = \widetilde{V}_e\,\mathcal{G}_{\partial e}$$

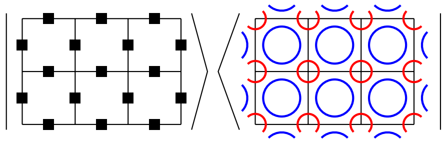
where $\mathcal{G}_{\partial e}$ is a suitable tensor product of (positive and invertible) operators of the form $\mathcal{G}_\square\otimes\mathcal{G}_\square$ and $\mathcal{G}_+\otimes\mathcal{G}_+$.

We will use a simpler picture for this *slim* tensor as well as for the *full* edge-tensor:

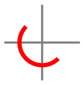
$\widetilde{V}_e$ =  $\qquad V_e$ = 

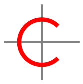
3.2 PEPS tensor on a rectangle

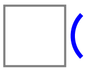
Let us consider a rectangular region $\mathcal{R}$ and denote by $\mathcal{E}_\mathcal{R}$ and $\mathcal{V}_\mathcal{R}$ the associate set of edges and vertices. The corresponding tensor $V_\mathcal{R}$ is constructed by placing at each edge e of the rectangle the tensor V_e obtained in the previous subsection, expanding the sum and contracting indices accordingly. Then it is easy to realize that the resulting operator $V_\mathcal{R} : \mathcal{H}_{\partial\mathcal{R}} \longrightarrow \mathcal{H}_\mathcal{R}^2$ can described as follows:

$V_\mathcal{R} = \sum$ 

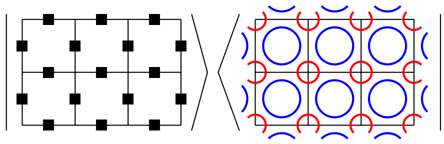
Here, the *bra* of each summand is actually a tensor product of *bra*'s of the form

 $= (\delta_{h,1} + \gamma_{\beta/2})^{1/2} \langle h\, h| = \langle h\, h|\, (\mathcal{G}_{+}^{2} \otimes \mathcal{G}_{+}^{2})$

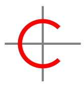
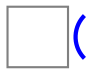
 $= (\delta_{h,1} + \gamma_{\beta/2})^{3/4} \langle h\, h| = \langle h\, h|\, (\mathcal{G}_{+}^{3} \otimes \mathcal{G}_{+}^{3})$

 $= \langle L^{g}|\, (\mathcal{G}_{\square} \otimes \mathcal{G}_{\square})$

The *slim* version $\widetilde{V}_{\mathcal{R}}$ obtained by removing the weights from the boundary will also have the form

$\widetilde{V}_{\mathcal{R}} = \sum$ 

but here the *bra* of each summand is a tensor product of *bra*'s of the form

 $= \langle h\, h|$,  $= \langle h\, h|$,  $= \langle L^{g}|$.

We then define $\mathcal{G}_{\mathcal{R}}$ as the suitable tensor product of (positive and invertible) elements

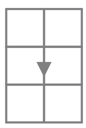
$$(\mathcal{G}_{+}^{2} \otimes \mathcal{G}_{+}^{2}) \quad , \quad (\mathcal{G}_{+}^{3} \otimes \mathcal{G}_{+}^{3}) \quad , \quad (\mathcal{G}_{\square} \otimes \mathcal{G}_{\square})$$

satisfying $V_{\partial\mathcal{R}} = \widetilde{V}_{\partial\mathcal{R}}\, \mathcal{G}_{\partial\mathcal{R}}$.

3.3 Boundary states

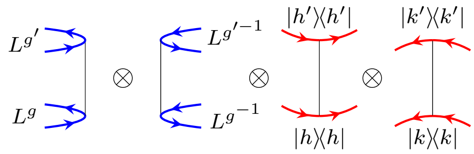
3.3.1 Edge

In the previous subsection we have described the *slim* and *full* tensors of the PEPS associated to an edge, namely $\widetilde{V}_{e}$ and V_{e}, both depending on the prefixed orientation. Next we are going to construct the corresponding boundary states $\widetilde{\rho}_{\partial e}$ and $\rho_{\partial e}$ by contracting the physical indices. Let us first consider an edge with fixed orientation:



The *slim* boundary state is constructed by considering $\widetilde{V}_e \otimes \widetilde{V}_e^\dagger$ and contracting the physical indices:

$$\widetilde{\rho}_{\partial e} \equiv \sum_{\substack{g,h,k\in G\\ g',h',k'\in G}} \left(\langle h'g'k'^{-1} | hgk^{-1}\rangle \, \langle g'|g\rangle \right)$$

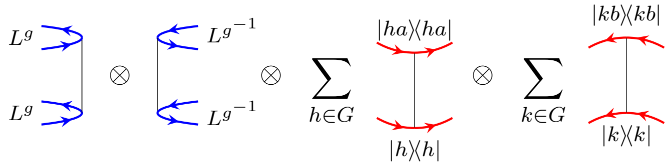


Note that the scalar factor given in terms of the trace will be zero or one, the latter if and only if

$$g = g' = (h^{-1}\, h')^{-1}\, g\, (k^{-1}\, k')\,,$$

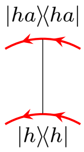
so if we denote $b := k^{-1}k'$ and $a := h^{-1}h'$, then we can rewrite

$$\widetilde{\rho}_{\partial e} = \sum_{\substack{g,a,b\in G:\\ a=gbg^{-1}}}$$



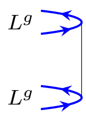
Let us introduce some notation for $a, g \in G$

$$\widetilde{\phi}_a \equiv \sum_{h\in G}$$



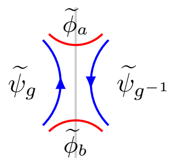
,

$$\widetilde{\psi}_g \equiv$$

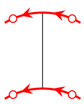


This allows us to rewrite the *slim* boundary state in the simpler form:

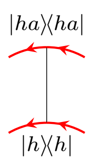
$$\widetilde{\rho}_{\partial e} \equiv \sum_{\substack{a,b,g\in G\\ a=gbg^{-1}}} \tag{19}$$



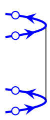
It will also be convenient to have the weighted version of these tensors. For that, we introduce the analog of $\widetilde{\phi}_a$ and $\widetilde{\psi}_g$ when contracting with the weights of the PEPO representation of $e^{\frac{\beta}{2}A(s)}$ and $e^{\frac{\beta}{2}B(p)}$.



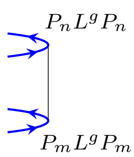
$$\phi_a \equiv \sum_{h\in G} \left(\delta_{h,1} + \gamma_{\beta/2}\right)^{1/4} \left(\delta_{ha,1} + \gamma_{\beta/2}\right)^{1/4}$$



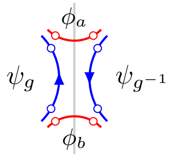
and

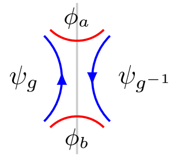


$$\psi_g \equiv \sum_{m,n\in\{0,1\}} \left(n + \gamma_{\beta/2}\right)^{1/4} \left(m + \gamma_{\beta/2}\right)^{1/4}$$



Thus, analogously to the *slim* case, we can represent the *full* boundary state

$$\rho_{\partial e} \equiv \sum_{\substack{a,b,g\in G\\ a=gbg^{-1}}}$$


$$= \sum_{\substack{a,b,g\in G\\ a=gbg^{-1}}}$$


(20)

where in the last expression we have simply omitted the weights to simplify the picture.

For the sake of applying the theory relating boundary states and the gap property of the parent Hamiltonian of a PEPS (see Section 5), we should look at the boundary states $\widetilde{\rho}_e$ and $\rho_{\partial e}$ as maps $\mathcal{H}_{\partial e} \longrightarrow \mathcal{H}_{\partial e}$. We could have taken the PEPS expressions for $\widetilde{V}_e$ and V_e described in the previous section as maps $\mathcal{H}_{\partial e} \longrightarrow \mathcal{H}_e^2$, see (18), and calculate $\widetilde{\rho}_{\partial e} = \widetilde{V}_e^\dagger \widetilde{V}_e$ and $\rho_{\partial e} = V_e^\dagger V_e$. This can be obtained also from (19) and (20) by reinterpreting for $a, g \in G$

$$\widetilde{\phi}_a = \sum_{h\in G} |ha\rangle|ha\rangle \langle h|\langle h| \qquad , \qquad \widetilde{\psi}_g = |L^g\rangle \langle L^g|$$

and so

$$\begin{aligned}
\phi_a &= \sum_{h\in G} \left(\delta_{h,1} + \gamma_{\beta/2}\right)^{1/4} \left(\delta_{ha,1} + \gamma_{\beta/2}\right)^{1/4} \; |ha\rangle|ha\rangle \langle h|\langle h| \\
\psi_g &= \sum_{n,m\in\{0,1\}} (n+\gamma_{\beta/2})^{1/4} (m+\gamma_{\beta/2})^{1/4} \, |P_n L^g P_n\rangle \langle P_m L^g P_m|
\end{aligned}$$

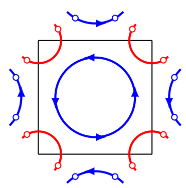
We will use this notation for the rest of the paper. Recall that

$$\phi_a = (\mathcal{G}_+ \otimes \mathcal{G}_+) \, \widetilde{\phi}_a \, (\mathcal{G}_+ \otimes \mathcal{G}_+) \quad , \quad \psi_g = (\mathcal{G}_\square \otimes \mathcal{G}_\square) \, \widetilde{\psi}_g \, (\mathcal{G}_\square \otimes \mathcal{G}_\square) \, ,$$

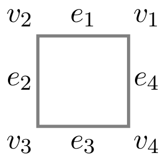
and $\rho_{\partial e} = \mathcal{G}_{\partial e} \, \widetilde{\rho}_e \, \mathcal{G}_{\partial e}$

3.3.2 Plaquette

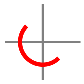
Let us next describe the boundary state of a plaquette, constructed by placing the boundary state of each edge, as it was described in the previous subsection, and contracting indices accordingly:

$$\rho_p =$$


For a more precise description, let us first label the edges and vertices of the plaquette e_1, e_2, e_3, e_4 and v_1, v_2, v_3, v_4 counterclockwise as in the next picture



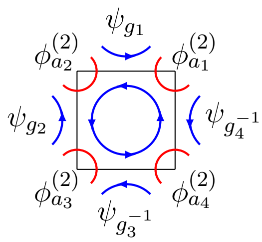
At each vertex v_j, when contracting the indices of ϕ_a and $\phi_{a'}$ coming from the two incident edges, note that this contraction will be zero if $a \neq a'$, while if $a = a'$ or equal to



$$\phi_a^{(2)} \;=\; \sum_{h \in G} (\delta_{h,1} + \gamma_{\beta/2})^{1/2} \, (\delta_{ha,1} + \gamma_{\beta/2})^{1/2} \;\; |ha\rangle |ha\rangle \;\langle h| \langle h|$$

Hence, we can expand

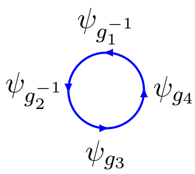
$$\rho_p \;=\; \sum_{\substack{g_1, g_2, g_3, g_4 \\ a_1, a_2, a_3, a_4}}$$



Remark that the elements g_j and a_j satisfy some *compatibility conditions* for the corresponding summand to be nonzero, namely

$$a_2 = g_1^{-1} a_1 g_1 \quad , \quad a_3 = g_2^{-1} a_2 g_2 \quad , \quad a_4 = g_3 a_3 g_3^{-1} \quad , \quad a_1 = g_4 a_4 g_4^{-1} \, .$$

The inner circunference is a constant factor that can be taken out as



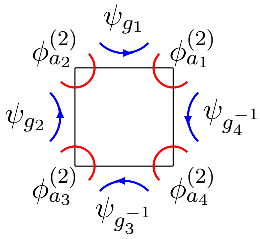
$$= \; \left(1 + \gamma_{\beta/2} \, \chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1})\right)^2 \tag{21}$$

Note that , since $\chi^{reg}(g)/|G|$ is equal to zero or one, we have for every $g \in G$

$$\begin{aligned} \left(1 + \frac{e^{\beta/2} - 1}{|G|} \chi^{reg}(g)\right)^2 \;&=\; 1 + 2(e^{\beta/2} - 1) \frac{\chi^{reg}(g)}{|G|} + (e^{\beta/2} - 1)^2 \, \frac{\chi^{reg}(g)}{|G|} \\ &=\; 1 + \left[\left((e^{\beta/2} - 1) + 1\right)^2 - 1\right] \frac{\chi^{reg}(g)}{|G|} \\ &=\; 1 + \frac{e^{\beta} - 1}{|G|} \chi^{reg}(g) \, . \end{aligned} \tag{22}$$

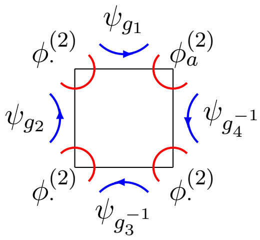
Therefore we can write

$$\rho_p \;=\; \sum_{\substack{g_1,g_2,g_3,g_4\\ a_1,a_2,a_3,a_4}} \left(1+\gamma_\beta\,\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1})\right)$$



Again remark that the sum expands over elements g_j and a_j satisfying the compatibility conditions. They yield in particular that knowing g_j for $j = 1, 2, 3, 4$ and one of the a_j's we can determine the rest. As a consequence we can rewrite

$$\rho_p \;=\; \sum_{a\in G}\;\sum_{g_1,g_2,g_3,g_4} \left(1+\gamma_\beta\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1})\right)$$

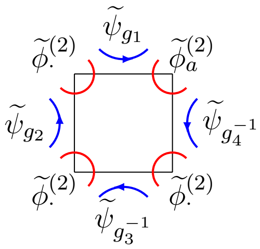


where the sum is extended over a, g_j satisfying

$$(g_1 g_2 g_3^{-1} g_4^{-1})\, a\, (g_1 g_2 g_3^{-1} g_4^{-1})^{-1} = a\,. \tag{23}$$

It will also be useful to consider the *slim* version $\widetilde{\rho}_p$ of ρ_p obtained by "removing" the weights from the boundary virtual indices:

$$\widetilde{\rho}_p \;=\; \sum_{a\in G}\;\sum_{g_1,g_2,g_3,g_4} \left(1+\gamma_\beta\,\chi^{reg}(g_1 g_2 g_3^{-1} g_4^{-1})\right)$$



where again the sum is extended over elements satisfying the compatibility condition (23).

3.3.3 Rectangular region

We aim at describing the boundary of the rectangular regions $\mathcal{R} \subset \Lambda$. We are going to make the details in the case of a proper rectangle, since the cylinder case follows analogously with few adaptations. First we need to introduce some further notation regarding the edges and vertices that form $\mathcal{R}$:

$$\mathcal{E}_{\mathcal{R}} := \text{edges contained in } \mathcal{R},$$

$$\mathcal{E}_{\partial\mathcal{R}} := \text{edges with only one adjacent plaquette inside } \mathcal{R},$$

$$\mathcal{E}_{\mathring{\mathcal{R}}} := \text{edges with both adjacent plaquettes inside } \mathcal{R},$$

$$\mathcal{V}_{\mathcal{R}} := \text{vertices contained in } \mathcal{R},$$

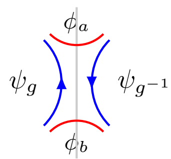
$$\mathcal{V}_{\partial\mathcal{R}} := \text{vertices with some but not all incident edges in } \mathcal{R},$$

$$\mathcal{V}_{\mathring{\mathcal{R}}} := \text{vertices with all four incident edges in } \mathcal{R},$$

$$n_{\mathcal{R}} := \text{ number of plaquettes contained in } \mathcal{R} \ .$$

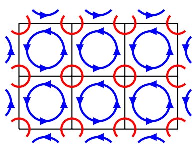
To formally construct the transfer operator or boundary state $\rho_{\mathcal{R}}$ on $\mathcal{R}$, we must place at each edge e contained in $\mathcal{R}$ the transfer operator ρ_e that was constructed in the previous subsection and contract the indices.

$$\rho_e \ = \ V_e^\dagger V_e \ = \sum_{\substack{a,b,g\in G\\ a=gbg^{-1}}}$$

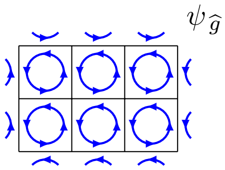


The philosophy is similar to the plaquette case, although more cumbersome to formalize. We are going to expand the expression for $\rho_{\partial\mathcal{R}}$

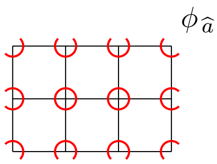
$$\rho_{\partial\mathcal{R}} \ =$$



$$= \sum_{\substack{\widehat{g}:\mathcal{E}_{\mathcal{R}}\to G\\ \widehat{a}:\mathcal{V}_{\mathcal{R}}\to G}}$$



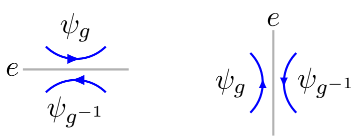
$$\otimes$$



as a sum over maps

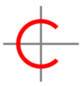
$$\widehat{g} : \mathcal{E}_{\mathcal{R}} \longrightarrow G \quad , \quad \widehat{a} : \mathcal{V}_{\mathcal{R}} \longrightarrow G$$

satisfying a certain compatibility condition. Let us explain the notation: at each edge $e \in \mathcal{E}_{\mathcal{R}}$, we set a value $g := \widehat{g}(e)$ according to whether we have



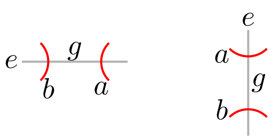
At each vertex $v \in \mathcal{V}_{\mathcal{R}}$, the contraction of indices coming from incident edges and corresponding to say ϕ_a and $\phi_{a'}$ will be zero unless $a = a'$. Thus, a nonzero contraction will be determined by a unique $\widehat{a}(v) = a = a'$:

$$\phi_a^{(2)} \ = \sum_{h\in G} (\delta_{h,1} + \gamma_{\beta/2})^{1/2} \, (\delta_{ha,1} + \gamma_{\beta/2})^{1/2} \ \ |ha\rangle|ha\rangle \ \langle h|\langle h|$$

 $\phi_a^{(3)} \;=\; \sum_{h\in G} (\delta_{h,1}+\gamma_{\beta/2})^{3/4}\,(\delta_{ha,1}+\gamma_{\beta/2})^{3/4}\;\;|ha\rangle|ha\rangle\;\langle h|\langle h|$

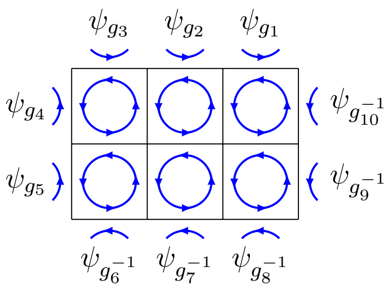

$$
\begin{aligned}
&= \sum_{h\in G} \left(\delta_{h,1}+\gamma_{\beta/2}\right)\,\left(\delta_{ha}+\gamma_{\beta/2}\right) \;=\; \delta_{a,1}+2\gamma_{\beta/2}+|G|\,(\gamma_{\beta/2})^2 \\
&= \delta_{a,1}+2\left(\tfrac{e^{\beta/2}-1}{|G|}\right)+|G|\,\left(\tfrac{e^{\beta/2}-1}{|G|}\right)^2 \;=\; \delta_{a,1}+\frac{e^{\beta}-1}{|G|} \;=\; \delta_{a,1}+\gamma_\beta\,.
\end{aligned} \tag{24}
$$

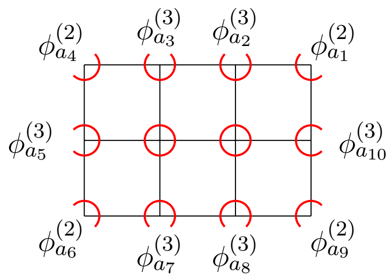
Moreover, the sum is extended over maps $\widehat{g}$ and $\widehat{a}$ satisfying the compatibility condition:



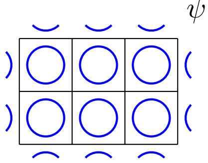
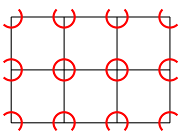
such that $\;a = gbg^{-1}$

The notation for the rectangles with $\psi_{\widehat{g}}$ and $\phi_{\widehat{a}}$ is the obvious



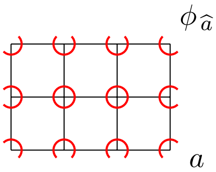


The compatibility condition yields an interesting consequence. For each pair $(\widehat{a},\widehat{g})$ the map $\widehat{a}$ can be reconstructed known only its value at one vertex using $\widehat{g}$. In particular, if we fix a vertex, namely the lower right corner, we can rewrite the above expression of $\rho_{\partial R}$ as

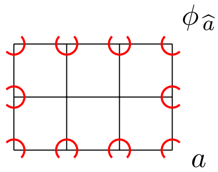
$\rho_{\partial\mathcal{R}} \;=\; \sum_{a\in G}\;\sum_{\widehat{g}:\mathcal{E}_{\mathcal{R}}\to G}$  $\otimes$ 

Here $\widehat{a}$ is the only map compatible with $\widehat{g}$ and the choice a in the prefixed vertex. In particular, all the elements $\widehat{a}(v)$, $v\in\mathcal{V}_{\mathcal{R}}$ belong to the same conjugation class.

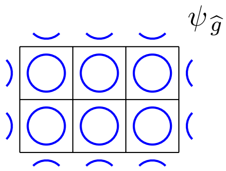
This means that we have two possibilities: If $a = 1$, resp. $a \neq 1$, then $\widehat{a}(v) = 1$, resp. $\widehat{a}(v) \neq 1$, for every vertex of $\mathcal{R}$, and thus by (24)



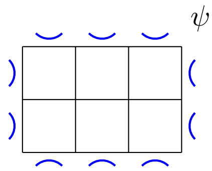
$$= (\delta_{a,1} + \gamma_\beta)^{\#\mathcal{V}_{\mathring{\mathcal{R}}}} \cdot$$



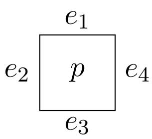
where we have extracted the constant factor resulting from full contraction in the inner vertices of the rectangle. Analogously, we can argue with the $\psi_{\widehat{g}}$ factor, where now we get a constant factor for each inner plaquette by (21) and (22):



$$= \prod_{\substack{p \text{ inner} \\ \text{plaquette}}} (1 + \gamma_\beta \, \chi^{reg}(\widehat{g}|_p)) \cdot$$



where we are denoting for each plaquette

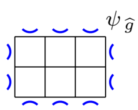


$$\chi^{reg}(\widehat{g}|_p) = \chi^{reg}(g_1 g_2 g_3^{-1} g_4), \quad g_i = \widehat{g}(e_i).$$

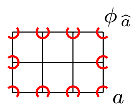
Note that the definition is independent of the enumeration of the edges as long as it is done counterclockwise (χ^{reg} is invariant under cyclic permutations) and respect the inverses on the lower and right edges.

We have then the following representation for the boundary state $\rho_{\mathcal{R}}$ of the rectangular region $\mathcal{R}$

$$\rho_{\partial\mathcal{R}} = \sum_{a \in G} \sum_{\widehat{g}:\mathcal{E}_{\mathcal{R}} \to G} (\delta_{a,1} + \gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \; c_\beta(\widehat{g})$$



$$\otimes$$

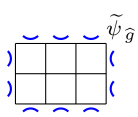


where we are denoting

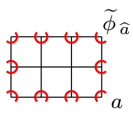
$$c_\beta(\widehat{g}) := \prod_{\substack{p \text{ inner} \\ \text{plaquette}}} (1 + \gamma_\beta \chi^{reg}(\widehat{g}|_p))$$

We are going to deal with the *slim* version of the boundary state resulting from removing the weights acting on the boundary indices:

$$\widetilde{\rho}_{\partial\mathcal{R}} = \sum_{a \in G} \sum_{\widehat{g}:\mathcal{E}_{\mathcal{R}} \to G} (\delta_{a,1} + \gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \, c_\beta(\widehat{g})$$



$$\otimes$$



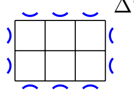
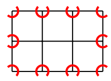
(25)

3.3.4 Leading Term and Approximate factorization

Finding a short explicit formula for the boundary states $\rho_{\partial\mathcal{R}}$ and $\widetilde{\rho}_{\partial\mathcal{R}}$ is not going to be feasible. But we will show that, after rearranging summands, there is a dominant term that we can explicitly describe in a short way. For that let us introduce

$$\widetilde{\Delta} := \frac{1}{|G|}\sum_{g\in G} \widetilde{\psi}_g \;=\; \frac{1}{|G|}\sum_{g\in G} |L^g\rangle\langle L^g|$$

and consider the (product) operator

$\widetilde{\mathcal{S}}_{\partial\mathcal{R}} :=$  $^{\widetilde{\Delta}^{\otimes \mathcal{E}_{\partial\mathcal{R}}}}$ $\otimes$  $^{\widetilde{\phi}_{\hat{1}}}$ (26)

We will later show that this is indeed an orthogonal projection on $\mathcal{H}_{\partial\mathcal{R}}$ made of local projections $\widetilde{\Delta}$ and $\widetilde{\phi}_{\hat{1}}$ on $\ell_2(G)\otimes\ell_2(G)$. We next state the first main (and most involved) result of the section.

Theorem 3.1 (Leading Term of the boundary)**.** *Let us define the scalars*

$$\kappa_{\mathcal{R}} := (1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|+n_{\mathcal{R}}}\,|G|^{|\mathcal{E}_{\mathcal{R}}|} \quad , \quad \epsilon_{\mathcal{R}} := 3\,|G|^2\left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}.$$

Then, we can decompose

$$\widetilde{\rho}_{\partial\mathcal{R}} = \kappa_{\mathcal{R}}\left(\widetilde{\mathcal{S}}_{\partial\mathcal{R}} + \widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}\right)$$

for some observable $\widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}$ with $\|\widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}\| \leq \epsilon_{\mathcal{R}}$.

Before proving the theorem, let us discuss two useful consequences. Recall that the *full* boundary state $\rho_{\partial\mathcal{R}}$ and its *slim* version $\widetilde{\rho}_{\partial\mathcal{R}}$ are related via a transformation $\mathcal{G}_{\partial\mathcal{R}}$ consisting of a tensor product of the (positive and invertible) weight-operators (see Section 3.2):

$$\rho_{\partial\mathcal{R}} = V_{\mathcal{R}}^\dagger V_{\mathcal{R}} = \mathcal{G}_{\partial\mathcal{R}}\,\widetilde{V}_{\mathcal{R}}^\dagger\widetilde{V}_{\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}} = \mathcal{G}_{\partial\mathcal{R}}\,\widetilde{\rho}_{\partial\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$$

Denote by $J_{\partial\mathcal{R}}$ and $\widetilde{J}_{\partial\mathcal{R}}$ the orthogonal projections onto $(\ker\rho_{\partial\mathcal{R}})^\perp$ and $(\ker\widetilde{\rho}_{\partial\mathcal{R}})^\perp$, respectively.

Theorem 3.2 (Approximate factorization of the boundary)**.** *Following the notation of the previous theorem, let us assume that $\epsilon_{\mathcal{R}} < 1$. Then, the following assertions hold:*

(i) $J_{\partial\mathcal{R}} = \widetilde{J}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}$. In other words, $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$ is the orthogonal projection onto the support of the slim *boundary state and the* full *boundary state.*

(ii) The operator

$$\sigma_{\partial\mathcal{R}} := \kappa_{\mathcal{R}}\,(\mathcal{G}_{\partial\mathcal{R}} J_{\partial\mathcal{R}}\, \mathcal{G}_{\partial\mathcal{R}}),$$

satisfies

$$\left\|\rho_{\partial\mathcal{R}}^{1/2}\sigma_{\partial\mathcal{R}}^{-1}\rho_{\partial\mathcal{R}}^{1/2} - J_{\partial R}\right\|_{\infty} < \epsilon_{\mathcal{R}}\ ,\ \left\|\rho_{\partial\mathcal{R}}^{-1/2}\sigma_{\partial\mathcal{R}}\rho_{\partial\mathcal{R}}^{-1/2} - J_{\partial R}\right\|_{\infty} \leq \frac{\epsilon_{\mathcal{R}}}{1-\epsilon_{\mathcal{R}}}\,.$$

where here the inverses are taken in the corresponding support.

In particular, $J_{\partial\mathcal{R}}$, $\widetilde{J}_{\partial\mathcal{R}}$ and $\sigma_{\partial\mathcal{R}}$ inherit the tensor product structure of $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$, $\mathcal{G}_{\partial\mathcal{R}}$.

In the rest of the subsection, we will develop the proofs of the above results. We have divided the whole argument into four parts. The first two parts are *Tool Box 1* and *Tool Box 2*, that contain some auxiliary results. Then, we will first prove Theorem 3.1, and finally Theorem 3.2.

Tool Box 1: Boundary projections

We start with a few useful observations on the operators $\widetilde{\phi}$ and $\widetilde{\psi}$. Using the explicit formula $\widetilde{\phi}_a = \sum_{h\in G} |ha\rangle|ha\rangle\,\langle h|\langle h|$, it is easy to check that

- ▷ $\widetilde{\phi}_a\widetilde{\phi}_{a'} = \widetilde{\phi}_{aa'}$ for each $a, a' \in G$,
- ▷ $\widetilde{\phi}_1$ is a projection.

For every $g, g' \in G$

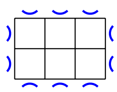
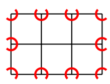
$$\langle L^g|L^{g'}\rangle = \mathrm{Tr}(L^{g^{-1}}L^{g'}) = \mathrm{Tr}(L^{g^{-1}g'}) = \delta_{g,g'}|G| \tag{27}$$

This means that vectors $\frac{1}{\sqrt{|G|}}\,|L^g\rangle$ are orthonormal vectors in $\ell_2[G]\otimes\ell_2[G]$, so

- ▷ $\frac{1}{|G|}\widetilde{\psi}_g = \frac{1}{\sqrt{G}}\,|L^g\rangle\,\langle L^g|$ is a one-dimensional orthogonal projection,
- ▷ $\frac{1}{|G|}\widetilde{\psi}_g$ and $\frac{1}{|G|}\widetilde{\psi}_{g'}$ are mutually orthogonal if g and g' are distinct.

These facts will be used throughout the forthcoming results.

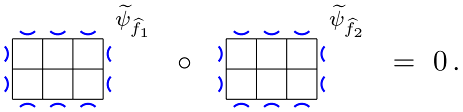
Proposition 3.3. *For each $\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G$ let*

$$\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}) := \frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$$  $\widetilde{\psi}_{\widehat{f}}$ $\otimes$  $\widetilde{\phi}_{\widehat{1}}$.

Then, $\mathcal{P}_{\partial\mathcal{R}}(\widehat{f})$ is an orthogonal projection on $\mathcal{H}_{\partial\mathcal{R}}$. Moreover if $\widehat{f}_1$ and $\widehat{f}_2$ are different, then $\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_1)$ and $\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_2)$ are mutually orthogonal, that is

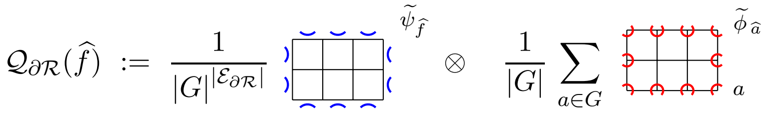
$$\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_1)\,\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_2) \ = \ \mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_2)\,\mathcal{P}_{\partial\mathcal{R}}(\widehat{f}_1) = 0$$

Proof. The first statement is clear since $\mathcal{P}_{\partial\mathcal{R}}(\widehat{f})$ is by definition a tensor product of projections $\widetilde{\phi}_1$ and $\frac{1}{|G|}\widetilde{\psi}_g$. Moreover, if $\widehat{f}_1$ and $\widehat{f}_2$ are different, then there is a boundary edge $e \in \mathcal{E}_{\partial\mathcal{R}}$ such that $\widehat{f}_1(e) \neq \widehat{f}_2(e)$. Thus,



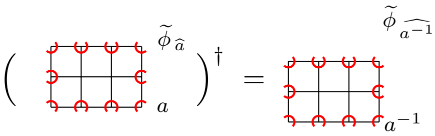
since $\widetilde{\psi}_{\widehat{f}_1(e)}$ and $\widetilde{\psi}_{\widehat{f}_2(e)}$ are mutually orthogonal by the above observations. □

Proposition 3.4. *For each $\widehat{f} : \mathcal{E}_{\partial\mathcal{R}} \longrightarrow G$ let us define*

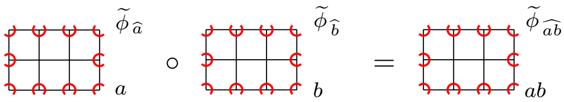


Recall that for each $a \in G$ the element $\widehat{a}$ that appears on the right hand side is the only choice compatible with $\widehat{f}$ and the fixed a. Then, $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f})$ is an orthogonal projection. Moreover if $\widehat{f}_1$ and $\widehat{f}_2$ are different, then $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f}_1)$ and $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f}_2)$ are mutually orthogonal.

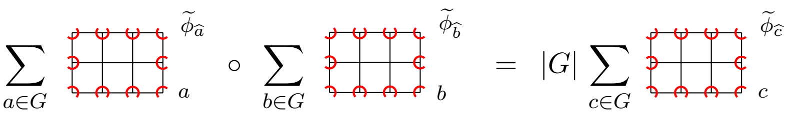
Proof. Let us first check that $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f})$ is an orthogonal projection. For that, it is enough to check that each of the two (tensor product) factors is a projection. The first factor is indeed a tensor product of projections $\frac{1}{|G|}\widetilde{\psi}_g$. For the second factor, we just need to check that it is self-adjoint and idempotent. Since for every a, the adjoint of $\widetilde{\phi}_a$ is $\widetilde{\phi}_{a^{-1}}$, it holds



and that the latter is again compatible with $\widehat{g}$, since at each edge the condition $a_2 = g a_1 g^{-1}$ is equivalent to $a_2^{-1} = g a_1^{-1} g^{-1}$. Thus, summing over all $a \in G$ we get self-adjointness, and so the first statement is proved. To see the second statement, note that



and that the resulting element $\widetilde{\phi}_{\widehat{ab}}$ is compatible with g, since $a_2 = g a_1 g^{-1}$ and $b_2 = g b_1 g^{-1}$ yield that $a_2 b_2 = g a_1 b_1 g^{-1}$. Hence, summing over $a, b \in G$ in the previous expression we get



This finishes the argument that $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f})$ is an orthogonal projection. Finally, if $\widehat{f}_1$ and $\widehat{f}_2$ are different, then we can check that $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f}_1)$ and $\mathcal{Q}_{\partial\mathcal{R}}(\widehat{f}_2)$ are mutually orthogonal arguing as in the proof of Proposition 3.3. $\square$

Lemma 3.5. *The operator*

$$\widetilde{\Delta} = \frac{1}{|G|}\sum_{g\in G}\widetilde{\psi}_g = \frac{1}{|G|}\sum_{g\in G}|L^g\rangle\langle L^g|$$

is a projection on $\ell_2(G)\otimes\ell_2(G)$ *satisfying*

$$\widetilde{\Delta}\widetilde{\psi}_g = \widetilde{\psi}_g\widetilde{\Delta} = \widetilde{\psi}_g \text{ for all } g\in G \qquad , \qquad (\mathcal{G}_\square\otimes\mathcal{G}_\square)\widetilde{\Delta} \;=\; \widetilde{\Delta}\,(\mathcal{G}_\square\otimes\mathcal{G}_\square)\,.$$

Proof. It is clear from the above observations on $\frac{1}{|G|}\widetilde{\psi}_g$ that $\widetilde{\Delta}$ is actually the projection onto the vector subspace generated by vectors of the form $|L^g\rangle$ and that $\widetilde{\Delta}\widetilde{\psi}_g = \widetilde{\psi}_g\widetilde{\Delta} = \widetilde{\psi}_g$ for all $g\in G$. To prove the last identity, let us apply (15) to decompose

$$|L^g\rangle\langle L^g| = |P_1\rangle\langle P_1| + |P_1\rangle\langle P_0L^gP_0| + |P_0L^gP_0\rangle\langle P_1| + |P_0L^gP_0\rangle\langle P_0L^gP_0|$$

Since $\frac{1}{|G|}\sum_{g\in G}L^g = P_1$, we get after summing over $g\in G$ in the previous expression

$$\widetilde{\Delta} = \frac{1}{|G|}\sum_{g\in G}|L^g\rangle\langle L^g| = |P_1\rangle\langle P_1| + \frac{1}{|G|}\sum_{g\in G}|P_0L^gP_0\rangle\langle P_0L^gP_0|\;.$$

Note that

$$(\mathcal{G}_\square\otimes\mathcal{G}_\square)\,|P_1\rangle = |\mathcal{G}_\square P_1\mathcal{G}_\square\rangle = (1+\gamma_{\beta/2})^{1/4}\,|P_1\rangle$$

$$(\mathcal{G}_\square\otimes\mathcal{G}_\square)\,|P_0L^gP_0\rangle = |\mathcal{G}_\square P_0L^gP_0\mathcal{G}_\square\rangle = (\gamma_{\beta/2})^{1/4}\;|P_0L^gP_0\rangle$$

As a consequence

$$\widetilde{\Delta}(\mathcal{G}_\square\otimes\mathcal{G}_\square) \;=\; (1+\gamma_{\beta/2})^{1/4}\,|P_1\rangle\langle P_1| + (\gamma_{\beta/2})^{1/4}\,\frac{1}{|G|}\sum_{g\in G}|P_0L^gP_0\rangle\langle P_0L^gP_0|$$

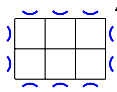
and so taking adjoints we immediately get that

$$\widetilde{\Delta}(\mathcal{G}_\square\otimes\mathcal{G}_\square) = (\mathcal{G}_\square\otimes\mathcal{G}_\square)\widetilde{\Delta}\,.$$

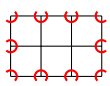
This concludes the proof. $\square$

Proposition 3.6. *The operator*

$\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$ =



$\widetilde{\Delta}^{\otimes\mathcal{E}_{\partial\mathcal{R}}}$ $\otimes$

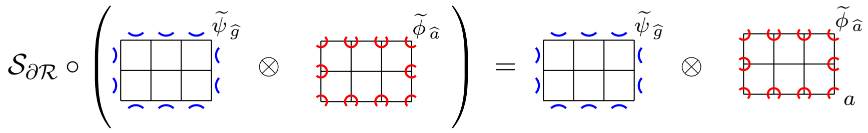


$\widetilde{\phi}_{\widehat{1}}$

is a projection on $\mathcal{H}_{\partial\mathcal{R}}$ *satisfying*

$$\widetilde{\rho}_{\partial\mathcal{R}} = \widetilde{\rho}_{\partial\mathcal{R}}\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}\,\widetilde{\rho}_{\partial\mathcal{R}} \qquad , \qquad \mathcal{G}_{\partial\mathcal{R}}\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$$

Proof. The first statement is clear as it is a tensor product of projections $\widetilde{\Delta}$ and $\widetilde{\phi}_{\widehat{1}}$. Let us check that $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}\,\widetilde{\rho}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}$. In view of the representation of $\widetilde{\rho}_{\partial\mathcal{R}}$ given in (25) it is enough to prove that



(28)

for all possible choices of $\widehat{g}$ and $\widehat{a}$. The composition on the left hand-side of (28) is again a tensor product of operators of the form $\widetilde{\phi}_1\,\widetilde{\phi}_a = \widetilde{\phi}_a$ and $\widetilde{\Delta}\,\widetilde{\psi}_g = \widetilde{\psi}_g$, so the equality (28) holds. Analogously, we can argue that $\widetilde{\rho}_{\partial\mathcal{R}}\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}$.

Finally, to see that $\mathcal{G}_{\partial\mathcal{R}}\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}} = \widetilde{\mathcal{S}}_{\partial\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$, note that both $\mathcal{G}_{\partial\mathcal{R}}$ and $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$ have a *compatible* tensor product structure, so that their product $\mathcal{G}_{\partial\mathcal{R}}\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$ is again a tensor product of elements of the form

$$(\mathcal{G}_{\square}\otimes\mathcal{G}_{\square})\widetilde{\Delta} \quad , \quad (\mathcal{G}_{+}^{3}\otimes\mathcal{G}_{+}^{3})\widetilde{\phi}_1 \quad , \quad (\mathcal{G}_{+}^{2}\otimes\mathcal{G}_{+}^{2})\widetilde{\phi}_1$$

and analogously for $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$,

$$\widetilde{\Delta}(\mathcal{G}_{\square}\otimes\mathcal{G}_{\square}) \quad , \quad \widetilde{\phi}_1(\mathcal{G}_{+}^{3}\otimes\mathcal{G}_{+}^{3}) \quad , \quad \widetilde{\phi}_1(\mathcal{G}_{+}^{2}\otimes\mathcal{G}_{+}^{2})\,.$$

By Lemma 3.5 we know that $(\mathcal{G}_{\square}\otimes\mathcal{G}_{\square})\widetilde{\Delta} = \widetilde{\Delta}(\mathcal{G}_{\square}\otimes\mathcal{G}_{\square})$. For the others, we only need to check that

$$(\mathcal{G}_{+}\otimes\mathcal{G}_{+})\,\widetilde{\phi}_1 = \sum_{h\in G}(\delta_{h,1}+\gamma_{\beta/2})^{1/4}\,|h\rangle|h\rangle\,\langle h|\langle h| \;=\; \widetilde{\phi}_1\,(\mathcal{G}_{+}\otimes\mathcal{G}_{+})\,.$$

This finishes the proof. $\square$

Tool Box 2: Plaquette constants

We need a couple of auxiliary results. First, let us introduce the notation

$$\widetilde{\chi}^{reg}(g) = \chi^{reg}(g) - 1\,, \quad g\in G\,.$$

If P_1 is the projection onto V_1, then

$$\widetilde{\chi}^{reg}(g) = \mathrm{Tr}(L^{g}(\mathbb{1}-P_1))\,.$$

Lemma 3.7. *Let us fix $u, v\in G$ and complex numbers $a_0, b_0, a_1, b_1 \in \mathbb{C}$. Then,*

$$\sum_{g_1,\dots,g_m\in G}(a_0+b_0\,\widetilde{\chi}^{reg}(ug_1\dots g_m))\,\big(a_1+b_1\widetilde{\chi}^{reg}(g_m^{-1}\dots g_2^{-1}g_1^{-1}v)\big)$$
$$= |G|^m\,\left(a_0a_1+b_0b_1\widetilde{\chi}^{reg}(uv)\right)\,.$$

Proof. Note that the left-hand side of the equality can be rewritten as

$$\sum_{g\in G} \sum_{\substack{g_1,\dots,g_m\in G\\ g_1\cdots g_m=g}} (a_0+b_0\widetilde{\chi}^{reg}(ug))\,\big(a_1+b_1\widetilde{\chi}^{reg}(g^{-1}v)\big)$$
$$= |G|^{m-1}\sum_{g\in G}(a_0+b_0\widetilde{\chi}^{reg}(ug))\,\big(a_1+b_1\widetilde{\chi}^{reg}(g^{-1}v)\big)$$

so we can restrict ourselves to the case $m=1$. First, let us expand

$$\sum_{g\in G}(a_0+b_0\widetilde{\chi}^{reg}(ug))\,\big(a_1+b_1\widetilde{\chi}^{reg}(g^{-1}v)\big)$$
$$= a_0a_1|G| \;+\; a_0b_1\sum_{g\in G}\widetilde{\chi}^{reg}(g^{-1}v) \;+\; b_0a_1\sum_{g\in G}\widetilde{\chi}^{reg}(ug)$$
$$+\; b_0b_1\sum_{g\in G}\widetilde{\chi}^{reg}(ug)\,\widetilde{\chi}^{reg}(g^{-1}v)$$

The second and third summands are equal to zero, since

$$\sum_{h\in G}\widetilde{\chi}^{reg}(h) = \sum_{h\in G}(|G|\,\delta_{h,1}-1) = |G|-|G| = 0\,.$$

Moreover,

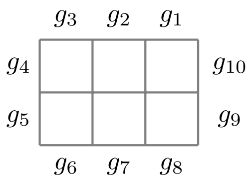
$$\begin{aligned}\sum_{g\in G}\widetilde{\chi}^{reg}(ug)\,\widetilde{\chi}^{reg}(g^{-1}v) &= \sum_{g\in G}(|G|\,\delta_{ug,1}-1)\,(|G|\delta_{g^{-1}v,1}-1)\\ &= \sum_{g\in G}|G|^2\,\delta_{ug,1}\delta_{g^{-1}v,1} - |G|\sum_{g\in G}\big(\delta_{ug,1}+\delta_{g^{-1}v,1}\big)+|G|\\ &= |G|^2\delta_{uv,1} \;-2|G|+|G|\\ &= |G|\,\big(|G|\delta_{uv,1}-1\big) = |G|\,\big(\chi^{reg}(uv)-1\big)\,,\end{aligned}$$

so the fourth summand in the aforementioned expansion is equal to $b_0b_1\widetilde{\chi}^{reg}(uv)$, leading to the desired statement. □

Next we need a result which help us to deal with the constants

$$c_\beta(\,\widehat{g}\,) := \prod_{\substack{p \text{ inner}\\ \text{plaquette}}}(1+\gamma_\beta\,\chi^{reg}(\widehat{g}|_p)\,)$$

that appear in $\widetilde{\rho}_{\partial\mathcal{R}}$, see (25). Let us first extend the notation $\chi^{reg}(\widehat{g}|_p)$ from plaquettes to the boundary of the rectangle $\mathcal{R}$: let us enumerate its boundary edges $\mathcal{E}_{\partial\mathcal{R}}$ counterclockwise by fixing any initial edge $e_1, e_2, \ldots, e_k$. Given a map $\widehat{f}_\partial : \mathcal{E}_{\partial\mathcal{R}} \longrightarrow G$ associating $e_j \longmapsto g_j$ let us define



$$\chi^{reg}(\widehat{f}_\partial) := \chi^{reg}(g_1^{\sigma_1} g_2^{\sigma_2} \ldots)$$

where $\sigma_j = 1$ if e_j is in the upper or left side of the rectangle, and $\sigma_j = -1$ otherwise. We will also use

$$\widetilde{\chi}^{reg}(\widehat{f}_\partial) := \chi^{reg}(\widehat{f}_\partial) - 1\,.$$

As in the case of plaquettes, note that the definition is independent of the enumeration of the edges as long as it is done counterclockwise (χ^{reg} is invariant under cyclic permutations) and respect the inverses on the lower and right edges.

Proposition 3.8. *Fixed $\widehat{f}_\partial : \mathcal{E}_{\partial\mathcal{R}} \longrightarrow G$ we have that*

$$\sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\longrightarrow G \\ \widehat{g}|_{\mathcal{E}_{\partial\mathcal{R}}}=\widehat{f}_\partial}} c_\beta(\widehat{g}) \;=\; |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}\,\Big((1+\gamma_\beta)^{n_\mathcal{R}} + \gamma_\beta^{n_\mathcal{R}}\;\widetilde{\chi}^{reg}(\widehat{f}_\partial)\Big)\,.$$

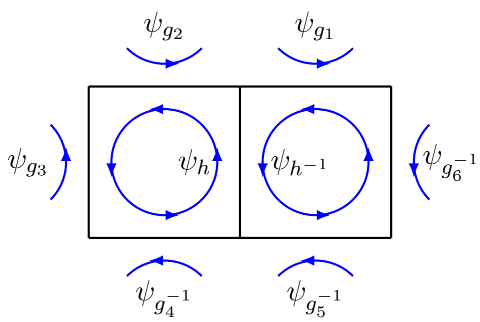
Proof. If the region $\mathcal{R}$ only consists of one plaquette, then the identity is trivial since there is only one summand $\widehat{g} = \widehat{f}_\partial$, so that

$$c_\beta(\widehat{g}) \;=\; 1 + \gamma_\beta \chi^{reg}(\widehat{g}|_p) = (1+\gamma_\beta) + \gamma_\beta\;\widetilde{\chi}^{reg}(\widehat{g}|_p)\,.$$

Let us denote to simplify notation

$$a_\beta := 1 + \gamma_\beta \quad , \quad b_\beta := \gamma_\beta\,.$$

For multiplaquette rectangular regions, the key is Lemma 3.7. Let us illustrate the case of a region $\mathcal{R}$ consisting of two (adjacent) plaquettes.



Then, as a consequence of the aforementioned lemma

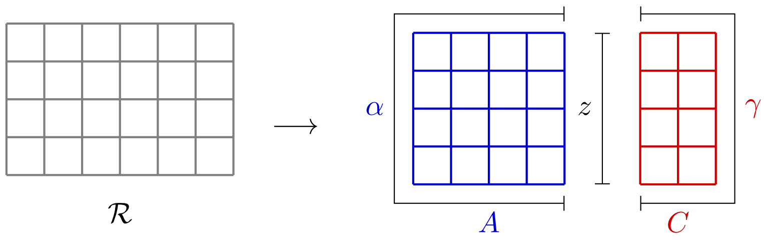
$$\sum_{h\in G} \left(1+\gamma_\beta \chi^{reg}(g_2 g_3 g_4^{-1} h^{-1})\right)\left(1+\gamma_\beta \chi^{reg}(h g_5^{-1} g_6^{-1} g_1)\right) =$$

$$= \sum_{h\in G} \left(a_\beta + b_\beta\, \widetilde{\chi}^{reg}(g_2 g_3 g_4^{-1} h^{-1})\right)\left(a_\beta + b_\beta\, \widetilde{\chi}^{reg}(h g_5^{-1} g_6^{-1} g_1)\right)$$

$$= |G| \left(a_\beta^2 + b_\beta^2\, \widetilde{\chi}^{reg}\,(g_2 g_3 g_4^{-1} g_5^{-1} g_6^{-1} g_1)\right)$$

The procedure is now clear and can be formalized by induction. Let $\mathcal{R}$ be a rectangular region. We can obviously split $\mathcal{R}$ into two adjacent rectangles sharing one side $\mathcal{R} = AC$ as below. The induction hypothesis yields that the identity is true for A and C. Let us set some notation for the boundary edges of A and C:

$$\alpha := \mathcal{E}_{\partial A} \setminus \mathcal{E}_{\partial C} \quad , \quad \gamma := \mathcal{E}_{\partial C} \setminus \mathcal{E}_{\partial A} \quad , \quad z := \mathcal{E}_{\partial A} \cap \mathcal{E}_{\partial C}\,.$$



Fixed a boundary selection $\widehat{f} : \mathcal{E}_{\partial \mathcal{R}} \to G$ (note that $\mathcal{E}_{\partial\mathcal{R}} = \alpha\gamma$) let us consider the sum over all choices $\widehat{g} : \mathcal{E}_{\mathcal{R}} \longrightarrow G$ such that $\widehat{g}$ coincides with $\widehat{f}$ on the boundary edges of $\mathcal{R}$

$$\sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\to G\\ \widehat{g}|_{\partial\mathcal{R}}=\widehat{f}}} c_\beta(\widehat{g}) = \sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\to G\\ \widehat{g}|_{\alpha\gamma}=\widehat{f}}} \prod_{\substack{p\subset\mathcal{R}\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p)) =$$

$$= \sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\to G\\ \widehat{g}|_{\alpha\gamma}=\widehat{f}}} \prod_{\substack{p\subset A\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p)) \prod_{\substack{p\subset C\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p))$$

Next we want to apply the induction hypothesis on each subregion A anc C. For that, we have to make the right expressions to appear. We can split the sum

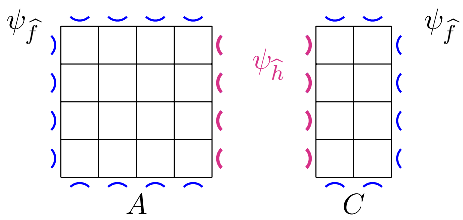
$$\sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\to G\\ \widehat{g}|_{\alpha\gamma}=\widehat{f}}} = \sum_{\widehat{h}:z\to G} \sum_{\substack{\widehat{g}:\mathcal{E}_\mathcal{R}\to G\\ \widehat{g}|_{\alpha\gamma}=\widehat{f}\\ \widehat{g}|_z=\widehat{h}}} = \sum_{\widehat{h}:z\to G} \sum_{\substack{\widehat{g}:\mathcal{E}_A\to G\\ \widehat{g}|_{\alpha}=\widehat{f}|_\alpha\\ \widehat{g}|_z=\widehat{h}}} \sum_{\substack{\widehat{g}:\mathcal{E}_C\to G\\ \widehat{g}|_{\gamma}=\widehat{f}|_\gamma\\ \widehat{g}|_z=\widehat{h}}} \tag{29}$$

Hence, we can rewrite (29) as

$$\sum_{\substack{\widehat{g}:\mathcal{E}_{\mathcal{R}}\to G\\ \widehat{g}|_{\alpha\gamma}=\widehat{f}}} \prod_{\substack{p\subset\mathcal{R}\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p)) =$$

$$= \sum_{\widehat{h}:z\to G} \Big(\sum_{\substack{\widehat{g}:\mathcal{E}_A\to G\\ \widehat{g}|_\alpha=\widehat{f}|_\alpha\\ \widehat{g}|_z=\widehat{h}}} \prod_{\substack{p\subset A\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p)) \Big)$$

$$\cdot \Big(\sum_{\substack{\widehat{g}:\mathcal{E}_C\to G\\ \widehat{g}|_\gamma=\widehat{f}|_\gamma\\ \widehat{g}|_z=\widehat{h}}} \prod_{\substack{p\subset C\\ \text{plaquette}}} (a_\beta + b_\beta \widetilde{\chi}^{reg}(\widehat{g}|_p)) \Big)$$



Applying the induction hypothesis we have that the above expression can be rewritten as

$$\sum_{\widehat{h}:z\to G} |G|^{|\mathcal{E}_{\mathring{A}}|} \left(a_\beta^{n_A} + b_\beta^{n_A}\, \widetilde{\chi}^{reg}(\widehat{g}|_\alpha \widehat{h}\,)\right) |G|^{|\mathcal{E}_{\mathring{C}}|} \left(a_\beta^{n_C} + b_\beta^{n_C}\, \widetilde{\chi}^{reg}(\,\widehat{h}\,\widehat{g}|_\gamma)\right)$$

$$= |G|^{|\mathcal{E}_{\mathring{A}}|+|\mathcal{E}_{\mathring{C}}|} \sum_{\widehat{h}:z\to G} \left(a_\beta^{n_A} + b_\beta^{n_A}\, \widetilde{\chi}^{reg}(\widehat{g}|_\alpha \widehat{h}\,)\right) \left(a_\beta^{n_C} + b_\beta^{n_C}\, \widetilde{\chi}^{reg}(\,\widehat{h}\,\widehat{g}|_\gamma)\right)$$

where n_A and n_C are the number of inner plaquettes of A and C respectively; $\widehat{g}|_\alpha \widehat{h}$ is the map $\mathcal{E}_{\partial A} \to G$ that coincides with $\widehat{g}$ on α and with $\widehat{h}$ on z; and $\widehat{h}\,\widehat{g}|_\gamma$ is the map $\mathcal{E}_{\partial C} \to G$ that coincides with $\widehat{g}$ on γ and with $\widehat{h}$ on z. Finally we use Lemma 3.7 to rewrite the above expression as

$$|G|^{|\mathcal{E}_{\mathring{A}}|+|\mathcal{E}_{\mathring{C}}|+|z|} \left(a_\beta^{n_A+n_C} + b_\beta^{n_A+n_C}\, \widetilde{\chi}^{reg}(\widehat{g}|_\alpha \widehat{g}|_\gamma)\right)$$

Finally observe that the set $\mathcal{E}_{\mathring{\mathcal{R}}}$ of inner edges of $\mathcal{R}$, is actually formed by the disjoint union of $\mathcal{E}_{\mathring{A}}$, $\mathcal{E}_{\mathring{C}}$ and z, whereas the number $n_{\mathcal{R}}$ of inner plaquettes of $\mathcal{R}$ is indeed the sum of n_A and n_B, so that

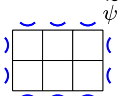
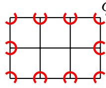
$$\sum_{\substack{\widehat{g}:\mathcal{E}_{\mathcal{R}}\to G \\ \widehat{g}|_{a\gamma}=\widehat{f}}} \prod_{\substack{p\subset\mathcal{R} \\ \text{plaquette}}} (a_\beta + b_\beta\ \widetilde{\chi}^{reg}(\widehat{g}|_p)) \ = \ |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|} \left(a_\beta^{n_\mathcal{R}} + b_\beta^{n_\mathcal{R}}\ \widetilde{\chi}^{reg}(\widehat{f}) \right) .$$

This finishes the proof of the Proposition.

□

Proof of the *Leading Term of the boundary* Theorem

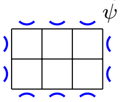
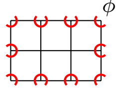
We are now ready to prove our main result about the boundary state of the thermofield double. The previous result allows us to rewrite (25) as

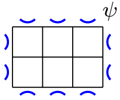
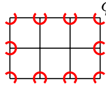
$$\widetilde{\rho}_{\partial\mathcal{R}} \ = \ \sum_{a\in G} (\delta_{a,1} + \gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$$  $\widetilde{\psi}_{\widehat{f}}$ $\otimes$  $\widetilde{\phi}_{\widehat{a}}$ a

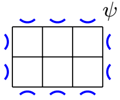
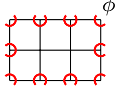
where

$$c_\beta(\widehat{f}) \ := \sum_{\substack{\widehat{g}:\mathcal{E}_{\mathcal{R}}\to G \\ \widehat{g}|_{\mathcal{E}_{\partial\mathcal{R}}}=\widehat{f}_\partial}} c_\beta(\,\widehat{g}\,) \ = \ |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|} \left[(1+\gamma_\beta)^{n_\mathcal{R}} + \gamma_\beta^{n_\mathcal{R}}\ \widetilde{\chi}^{reg}(\,\widehat{f}\,) \right] .$$

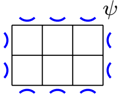
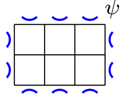
Let us now split the sum over $a \in G$ into a first summand with $a = 1$ (which forces $\widehat{a}(v) = 1$ for every $v \in \mathcal{V}_\mathcal{R}$, as we argued in previous sections) and $a \neq 1$:

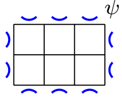
$$\widetilde{\rho}_{\partial\mathcal{R}} \ = \ (1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$$  $\widetilde{\psi}_{\widehat{f}}$ $\otimes$  $\widetilde{\phi}_{\widehat{1}}$ $+$

$$+\ (\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$$  $\widetilde{\psi}_{\widehat{f}}$ $\otimes \sum_{a\in G}$  $\widetilde{\phi}_{\widehat{a}}$ a

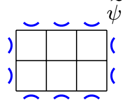
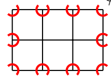
$$-\ (\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$$  $\widetilde{\psi}_{\widehat{f}}$ $\otimes$  $\widetilde{\phi}_{\widehat{1}}$

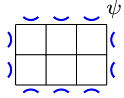
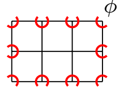
In the first summand, we can moreover decompose

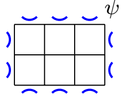
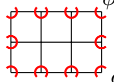
$$\sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$$  $\widetilde{\psi}_{\widehat{f}}$ $= \ |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}\ (1+\gamma_\beta)^{n_\mathcal{R}} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G}$  $\widetilde{\psi}_{\widehat{f}}$

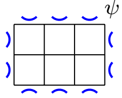
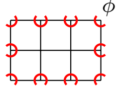
$+\ |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}\ \gamma_\beta^{n_\mathcal{R}} \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G} \widetilde{\chi}^{reg}(\widehat{f})$  $\widetilde{\psi}_{\widehat{f}}$

Thus, combining both expressions

$\widetilde{\rho}_{\partial\mathcal{R}} = (1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\,(1+\gamma_\beta)^{n_{\mathcal{R}}}\,|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}\sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G}$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

$+\ (1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\,(\gamma_\beta)^{n_{\mathcal{R}}}\,|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}\sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G}\widetilde{\chi}^{reg}(\widehat{f})$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

$+\,(\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$  $\widetilde{\psi}_{\widehat{f}} \otimes \sum_{a\in G}$  $\widetilde{\phi}_{\widehat{a}}$ a

$-\,(\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G} c_\beta(\,\widehat{f}\,)$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

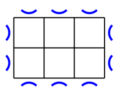
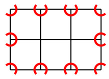
Dividing the above expression by

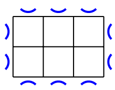
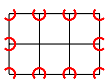
$$\kappa_{\mathcal{R}} = |G|^{|\mathcal{E}_{\mathcal{R}}|}\,(1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|+n_{\mathcal{R}}} = |G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|+|\mathcal{E}_{\partial\mathcal{R}}|}\,(1+\gamma_\beta)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|+n_{\mathcal{R}}}\,,$$

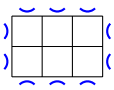
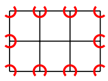
we obtain

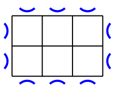
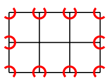
$$\frac{1}{\kappa_{\mathcal{R}}}\widetilde{\rho}_{\partial\mathcal{R}} = \mathcal{S}_1 + \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{n_\beta}\mathcal{S}_2 + \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\mathcal{S}_3 - \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|}\mathcal{S}_4 \quad (30)$$

where

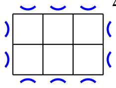
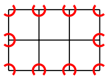
$\mathcal{S}_1 = \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G}\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

$\mathcal{S}_2 = \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G}\widetilde{\chi}^{reg}(\widehat{f})\,\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

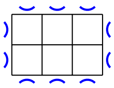
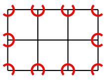
$\mathcal{S}_3 = \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G}\frac{c_\beta(\,\widehat{f}\,)}{(1+\gamma_\beta)^{n_\beta}\,|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}}\,\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$  $\widetilde{\psi}_{\widehat{f}} \otimes \sum_{a\in G}$  $\widetilde{\phi}_{\widehat{a}}$ a

$\mathcal{S}_4 = \sum_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\to G}\frac{c_\beta(\,\widehat{f}\,)}{(1+\gamma_\beta)^{n_\beta}\,|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}}\,\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$  $\widetilde{\psi}_{\widehat{f}} \otimes$  $\widetilde{\phi}_{\widehat{1}}$

Note that the first summand actually corresponds to

$$\mathcal{S}_1 \;=\;$$  $$^{\widetilde{\Delta}^{\otimes \mathcal{E}_{\partial\mathcal{R}}}} \otimes$$  $$^{\widetilde{\phi}_{\widehat{1}}} \;=\; \widetilde{\mathcal{S}}_{\partial\mathcal{R}}$$

To estimate the norm of $\mathcal{S}_2$ and $\mathcal{S}_4$ we are going to use that the operators

$$\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$$  $$^{\widetilde{\psi}_{\widehat{f}}} \otimes$$  $$^{\widetilde{\phi}_{\widehat{1}}} \quad , \quad \widehat{f} : \mathcal{E}_{\partial\mathcal{R}} \longrightarrow G\,.$$

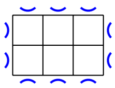
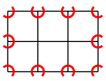
are (orthogonal) projections and mutually orthogonal, see Proposition 3.6. Then, we can estimate

$$\|\mathcal{S}_2\|_\infty \;\leq\; \sup_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G} |\widetilde{\chi}^{reg}(\widehat{f}\,)| \leq |G|\,,$$

and

$$\|\mathcal{S}_3\|_\infty \;\leq\; \sup_{\widehat{f}:\mathcal{E}_{\mathcal{R}}\longrightarrow G} \frac{c_\beta(\,\widehat{g}\,)}{(1+\gamma_\beta)^{n_\beta}\,|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}} \leq 1 + \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{n_\mathcal{R}} |G|\,.$$

To estimate the norm of $\mathcal{S}_3$ we are going to use that the operators

$$\frac{1}{|G|^{|\mathcal{E}_{\partial\mathcal{R}}|}}$$  $$^{\widetilde{\psi}_{\widehat{g}}} \otimes \sum_{a\in G}$$  $$^{\widetilde{\phi}_{\widehat{a}}}_{\;a} \quad , \quad \widehat{f} : \mathcal{E}_{\partial\mathcal{R}} \longrightarrow G\,.$$

are (orthogonal) projections and mutually orthogonal, see Proposition 3.4. Then,

$$\|\mathcal{S}_3\|_\infty = \sup_{\widehat{f}:\mathcal{E}_{\partial\mathcal{R}}\longrightarrow G} \frac{c_\beta(\widehat{f})}{(1+\gamma_\beta)^{n_\mathcal{R}}|G|^{|\mathcal{E}_{\mathring{\mathcal{R}}}|}} \;\leq\; 1 + \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)|G|\,.$$

Combining all these estimates, we can reformulate (30) as

$$\widetilde{\rho}_{\partial\mathcal{R}} \;=\; \kappa_\mathcal{R}\;\big(\,\widetilde{\mathcal{S}}_{\partial\mathcal{R}} \,+\, \widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}\,\big)$$

where the observable $\widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}$ satisfies

$$\begin{aligned}
\|\widetilde{\mathcal{S}}^{rest}_{\partial\mathcal{R}}\|_\infty \;&\leq\; \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{n_\mathcal{R}} |G| \,+\, \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} |G|\,\left[1 + \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{n_\mathcal{R}} |G|\right] \\
&\leq \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} |G|\,(2+|G|) \\
&\leq\; 3\,|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{V}_{\mathring{\mathcal{R}}}|} =: \epsilon_\mathcal{R}
\end{aligned}$$

having used in the second inequality that $|\mathcal{V}_{\mathring{\mathcal{R}}}| \leq n_\mathcal{R}$

Proof of the *Approximate Factorization of the boundary* Theorem

Let us assume that $\epsilon_{\mathcal{R}} < 1$. By Proposition 3.6, we have that $\widetilde{S}_{\partial\mathcal{R}}$ is a projection satisfying $\widetilde{\rho}_{\partial\mathcal{R}} = \widetilde{\rho}_{\partial\mathcal{R}}\widetilde{S}_{\partial\mathcal{R}} = \widetilde{S}_{\partial\mathcal{R}}\widetilde{\rho}_{\partial\mathcal{R}}$. Moreover, by Theorem 3.1

$$\left\| \kappa_{\mathcal{R}}^{-1}\,\widetilde{\rho}_{\partial\mathcal{R}} - \widetilde{S}_{\partial\mathcal{R}} \right\| \,=\, \|\widetilde{S}^{rest}_{\partial\mathcal{R}}\| \leq \epsilon_{\mathcal{R}} < 1\,. \tag{31}$$

These three properties yield that $\widetilde{S}_{\partial\mathcal{R}} = \widetilde{J}_{\partial\mathcal{R}}$, as a consequence of the next general observation:

Let T be a self-adjoint operator on $\mathcal{H} \equiv \mathbb{C}^d$, and Π an orthogonal projection onto a subspace $W \subset \mathcal{H}$ such that $T\,\Pi = \Pi\,T = T$ and $\|T - \Pi\| < 1$. Then, Π is the orthogonal projection onto $(\ker T)^{\perp}$. Indeed, since $T = \Pi T \Pi$ we have that $(\ker T)^{\perp}$ is contained in W. If they were different subspaces, then there would be a state $|u\rangle \in W$ with $T\,|u\rangle = 0$. But then, $1 = \langle u|u\rangle = \langle u|(T - \Pi)u\rangle \leq \|T - \Pi\| < 1$.

Next, observe that by Proposition 3.6

$$[\mathcal{G}_{\partial\mathcal{R}}, \widetilde{J}_{\partial\mathcal{R}}] \,=\, [\mathcal{G}_{\partial\mathcal{R}}, \widetilde{S}_{\partial\mathcal{R}}] \,=\, 0\,.$$

This yields that $\rho_{\partial\mathcal{R}}$ and $\widetilde{\rho}_{\partial\mathcal{R}}$ have the same support, that is $J_{\partial\mathcal{R}} = \widetilde{J}_{\partial\mathcal{R}}$, since they are related via $\rho_{\partial\mathcal{R}} = \mathcal{G}_{\partial\mathcal{R}}\,\widetilde{\rho}_{\partial\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$ where $\mathcal{G}_{\partial\mathcal{R}}$ is invertible. Moreover, the operators $\sigma_{\partial\mathcal{R}} = \kappa_{\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}\,\widetilde{J}_{\mathcal{R}}\,\mathcal{G}_{\partial\mathcal{R}}$ and $\widetilde{\sigma}_{\partial\mathcal{R}} = \kappa_{\mathcal{R}}\,\widetilde{J}_{\mathcal{R}}$ will also have the same support as $\rho_{\partial\mathcal{R}}$. We can thus argue as in the proof of Proposition 5.7 to get

$$\begin{aligned}
\|\rho_{\partial\mathcal{R}}^{1/2}\,\sigma_{\partial\mathcal{R}}^{-1}\,\rho_{\partial\mathcal{R}}^{1/2} - J_{\partial\mathcal{R}}\|_{\infty} &= \|\widetilde{\rho}_{\partial\mathcal{R}}^{\,1/2}\,(\kappa_{\mathcal{R}}\widetilde{J}_{\partial\mathcal{R}})^{-1}\,\widetilde{\rho}_{\partial\mathcal{R}}^{\,1/2} - \widetilde{J}_{\partial\mathcal{R}}\|_{\infty} \\
&= \|\kappa_{\mathcal{R}}^{-1}\,\widetilde{\rho}_{\partial\mathcal{R}}^{\,1/2}\,\widetilde{J}_{\partial\mathcal{R}}\widetilde{\rho}_{\partial\mathcal{R}}^{\,1/2} - \widetilde{J}_{\partial\mathcal{R}}\|_{\infty} \\
&= \|\kappa_{\mathcal{R}}^{-1}\,\widetilde{\rho}_{\partial\mathcal{R}} - \widetilde{J}_{\partial\mathcal{R}}\|_{\infty} \,\leq\, \epsilon_{\mathcal{R}}
\end{aligned}$$

where in the last line we have used again (31). Finally, since $\sigma_{\partial\mathcal{R}}$ and $\rho_{\partial\mathcal{R}}$ have the same support, we can use the identity

$$\rho_{\partial\mathcal{R}}^{-1/2}\,\sigma_{\partial\mathcal{R}}\,\rho_{\partial\mathcal{R}}^{-1/2} = \big(\rho_{\partial\mathcal{R}}^{1/2}\,\sigma_{\partial\mathcal{R}}^{-1}\,\rho_{\partial\mathcal{R}}^{1/2}\big)^{-1} = J_{\partial\mathcal{R}} + \sum_{m=1}^{\infty}\,\big(J_{\partial\mathcal{R}} - \rho_{\partial\mathcal{R}}^{1/2}\,\sigma_{\partial\mathcal{R}}^{-1}\,\rho_{\partial\mathcal{R}}^{1/2}\big)^{m}$$

so that

$$\|\rho_{\partial\mathcal{R}}^{-1/2}\,\sigma_{\partial\mathcal{R}}\,\rho_{\partial\mathcal{R}}^{-1/2} - J_{\partial\mathcal{R}}\| \,\leq\, \sum_{m=1}^{\infty}\,\epsilon_{\mathcal{R}}^{m} = \frac{\epsilon_{\mathcal{R}}}{1-\epsilon_{\mathcal{R}}}\,.$$

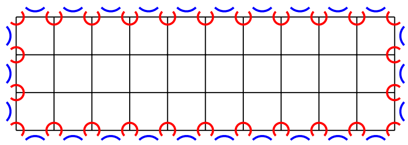
3.4 Approximate factorization of the ground state projections

In this section, we study for a given rectangular region $\mathcal{R}$ split into three suitable regions A, B, C where B shields A from C and such that AB, BC are again rectangular regions, how to estimate

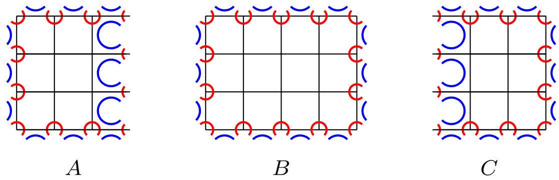
$$\|P_{ABC} - P_{AB}P_{BC}\|$$

in terms of the size of B. This problem is related to the approximate factorization property of the boundary states that we studied in the previous sections, (see Section 5.2.1 for details on this relation and the main results in the context of general PEPS). Our main tools will be Theorems 5.6 and 3.2.

Corollary 3.9. *Let us consider a rectangular region with open boundary conditions*



split into three subregions A, B, C *as in the next picture*



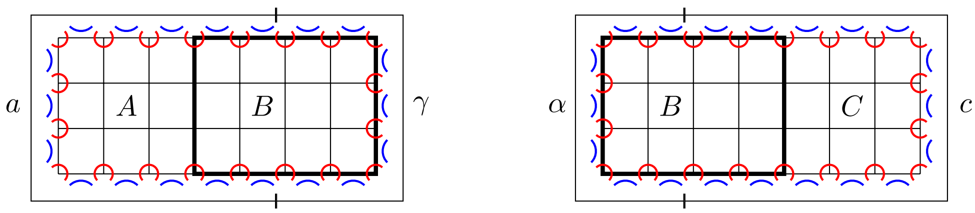
so that AB, BC, B *are again rectangular regions. If* B *has* M *plaquettes per row and* N *plaquettes per column with*

$$\epsilon_B := 3\,|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(M-1)(N-1)} < \frac{1}{2}\,, \tag{32}$$

then the orthogonal projections $P_{\mathcal{R}}$ *onto* $\mathrm{Im}(V_{\mathcal{R}})$ *satisfy*

$$\|P_{AB}P_{BC} - P_{ABC}\| \leq 6\epsilon_B(1+2\epsilon_B) \leq 12\epsilon_B\,.$$

Proof. In order to apply Theorem 5.6, we arrange the virtual indices of ABC, AB, BC and B into four sets a, c, α, γ as in the next picture:



so that

$$\partial ABC = ac \quad , \quad \partial AB = a\gamma \quad , \quad \partial BC = \alpha c \quad , \quad \partial B = \alpha\gamma\,.$$

The hypothesis $\epsilon_B < 1$ ensures that the hypothesis $\epsilon_{\mathcal{R}} < 1$ in Theorem 3.2 is satisfied for $\mathcal{R} \in \{B, AB, BC, ABC\}$, and so $J_{\partial\mathcal{R}}$ and $\sigma_{\partial\mathcal{R}}$ have a simple tensor product structure. This allows us to factorize

$$J_{\partial ABC} = J_a \otimes J_c \quad , \quad J_{\partial B} = J_\alpha \otimes J_\gamma\,,$$

$$\sigma_{\partial AB} = J_a \otimes J_\gamma \quad , \quad \sigma_{\partial BC} = J_\alpha \otimes J_c\,.$$

where J_a, J_c, J_α and J_γ are projections. The local structure of $\widetilde{\mathcal{S}}_{\partial\mathcal{R}}$ and $\mathcal{G}_{\partial\mathcal{R}}$ allows us to decompose both operators as a tensor product of operators acting on a, c, α, γ. Hence, we can define

$$\sigma_a := \frac{\kappa_{AB}}{\sqrt{\kappa_B}}\,\mathcal{G}_a\widetilde{\mathcal{S}}_a\mathcal{G}_a \quad , \quad \sigma_c := \frac{\kappa_{BC}}{\sqrt{\kappa_B}}\,\mathcal{G}_c\,\widetilde{\mathcal{S}}_c\,\mathcal{G}_c\,,$$

$$\sigma_\alpha := \sqrt{\kappa_B}\,\mathcal{G}_\alpha\,\widetilde{\mathcal{S}}_\alpha\,\mathcal{G}_\alpha \quad , \quad \sigma_\gamma := \sqrt{\kappa_B}\,\mathcal{G}_\gamma\,\widetilde{\mathcal{S}}_\gamma\,\mathcal{G}_\gamma\,.$$

Then, we can easily verify that $\kappa_{ABC}\kappa_B = \kappa_{AB}\kappa_{BC}$, and so

$$\sigma_{\partial ABC} = \sigma_a \otimes \sigma_c \quad , \quad \sigma_{\partial B} = \sigma_\alpha \otimes \sigma_\gamma\,,$$

$$\sigma_{\partial AB} = \sigma_a \otimes \sigma_\gamma \quad , \quad \sigma_{\partial BC} = \sigma_\alpha \otimes \sigma_c\,.$$

From Theorem 3.2, it follows that for each $\mathcal{R} \in \{ABC, AB, BC, B\}$

$$\|\rho_{\partial\mathcal{R}}^{1/2}\sigma_{\partial\mathcal{R}}^{-1}\rho_{\partial\mathcal{R}}^{1/2} - J_{\partial\mathcal{R}}\| < \epsilon_B\,.$$

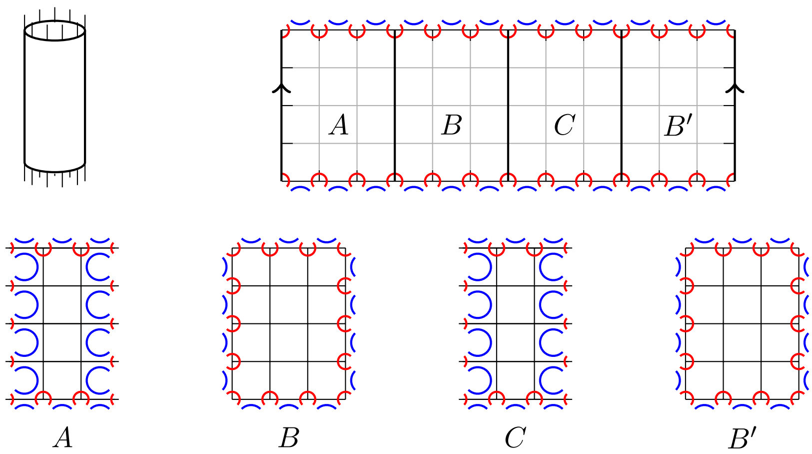
and

$$\|\rho_{\partial\mathcal{R}}^{-1/2}\,\sigma_{\partial\mathcal{R}}\,\rho_{\partial\mathcal{R}}^{-1/2} - J_{\partial\mathcal{R}}\| \le \frac{\varepsilon_B}{1-\varepsilon_B} \le 2\epsilon_B\,.$$

Thus, applying Theorem 5.6 we conclude the result. □

Analogously, we can prove results for the torus and cylinders.

Corollary 3.10. *Let us consider a cylinder with open boundary conditions and split it into four sections A, B, C, B' so that $B'AB$ and BCB' are two overlapping rectangles whose intersection is formed by two disjoint rectangles B and B', as in the next picture:*



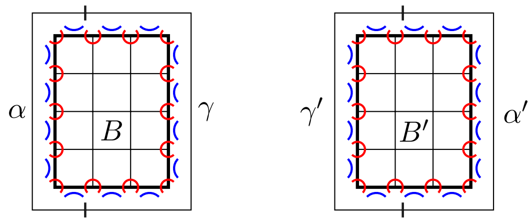
Let us assume that B and B' have at least M plaquettes per row and N plaquettes per column with

$$\epsilon_{BB'} := 3\,|G|^2\left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(M-1)(N-1)} \le \frac{1}{2}\,,$$

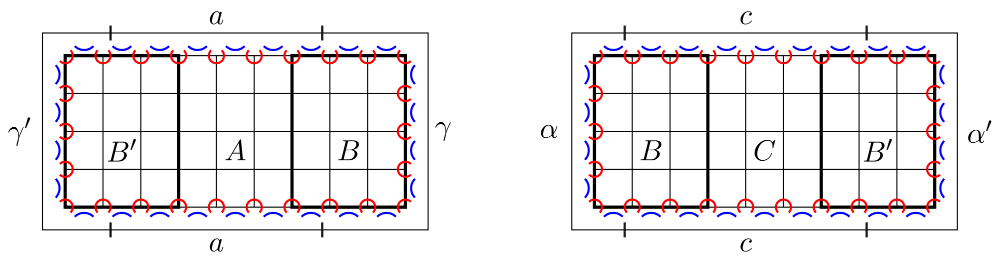
then the orthogonal projections $P_{\mathcal{R}}$ *onto* $\mathrm{Im}(V_{\mathcal{R}})$ *satisfy*

$$\|P_{B'AB}P_{BCB'} - P_{ABCB'}\| \leq 18\epsilon_{BB'}(1+6\epsilon_{BB'}) \leq 72\epsilon_{BB'}\,.$$

Proof. We are going to apply Theorem 5.6 for the three regions A, BB' and C. Firstly, we need to find a suitable arrangement of the virtual indices. First we split the boundary of BB' into two four parts $\gamma, \gamma', \alpha, \alpha'$ as in



and define regions a and c according to



so that

$$\partial B'AB = \gamma' a\gamma \quad , \quad \partial BCB' = \alpha c\alpha' \quad , \quad \partial ABCB' = ac$$
$$\partial B = \alpha\gamma \quad , \quad \partial B' = \gamma'\alpha'$$

The local structure of $\widetilde{S}_{\partial\mathcal{R}}$ and $\mathcal{G}_{\partial\mathcal{R}}$ allows us to write them as a tensor product of operators acting on the defined boundary segments, and define

$$\sigma_a := \frac{\kappa_{B'AB}}{\sqrt{\kappa_B\kappa_{B'}}}\mathcal{G}_a^\dagger\widetilde{\mathcal{S}}_a\mathcal{G}_a, \qquad \sigma_c := \frac{\kappa_{BCB'}}{\sqrt{\kappa_B\kappa_{B'}}}\mathcal{G}_c^\dagger\widetilde{\mathcal{S}}_c\mathcal{G}_c$$

$$\sigma_\alpha := \sqrt{\kappa_B\kappa_{B'}}\mathcal{G}_\alpha^\dagger\widetilde{\mathcal{S}}_\alpha\mathcal{G}_\alpha \qquad \sigma_{\alpha'} := \mathcal{G}_{\alpha'}^\dagger\widetilde{\mathcal{S}}_{\alpha'}\mathcal{G}_{\alpha'}$$

$$\sigma_\gamma := \sqrt{\kappa_B\kappa_{B'}}\mathcal{G}_\gamma^\dagger\widetilde{\mathcal{S}}_\gamma\mathcal{G}_\gamma \qquad \sigma_{\gamma'} := \mathcal{G}_{\gamma'}^\dagger\widetilde{\mathcal{S}}_{\gamma'}\mathcal{G}_{\gamma'}$$

Then, using that $\kappa_{ABC}\kappa_B\kappa_{B'} = \kappa_{B'AB}\kappa_{BCB'}$ we can easily check that

$$\sigma_{\partial B'AB} = \sigma_{\gamma'}\otimes\sigma_a\otimes\sigma_\gamma \quad , \quad \sigma_{\partial BCB'} = \sigma_{\alpha'}\otimes\sigma_c\otimes\sigma_{\alpha'} \quad , \quad \sigma_{\partial ABCB'} = \sigma_a\otimes\sigma_c$$

$$\sigma_{\partial B} = \sigma_\alpha\otimes\sigma_\gamma \quad , \quad \sigma_{\partial B'} = \sigma_{\gamma'}\otimes\sigma_{\alpha'}$$

By Theorem 3.2, we have for each $\mathcal{R}\in\{B, B', B'AB, BCB', ABCB'\}$

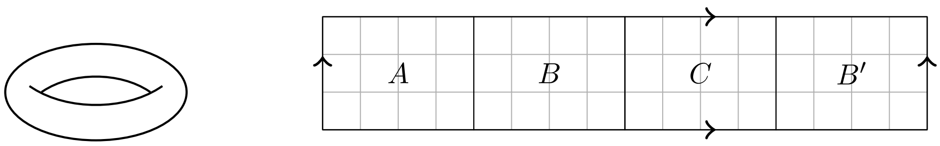
$$\|\rho_{\partial\mathcal{R}}^{1/2}\sigma_{\partial\mathcal{R}}^{-1}\rho_{\partial\mathcal{R}}^{1/2} - J_{\partial\mathcal{R}}\| \;,\; \|\rho_{\partial\mathcal{R}}^{-1/2}\sigma_{\partial\mathcal{R}}\rho_{\partial\mathcal{R}}^{-1/2} - J_{\partial\mathcal{R}}\| \;\leq\; 2\epsilon_{BB'}\,.$$

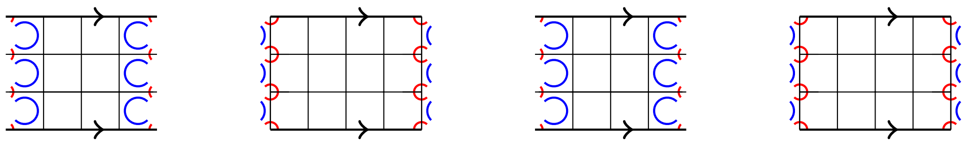
In particular, considering the joint region BB' we get

$$\begin{aligned}
\|\rho_{\partial BB'}^{-1/2}\sigma_{\partial BB'}\rho_{\partial BB'}^{-1/2} - J_{\partial BB'}\| &\leq \|(\rho_{\partial B}^{-1/2}\sigma_{\partial B}\rho_{\partial B}^{-1/2})(\rho_{\partial B'}^{-1/2}\sigma_{\partial B'}\rho_{\partial B'}^{-1/2}) - J_{\partial B}J_{\partial B'}\| \\
&\leq \|\rho_{\partial B}^{-1/2}\sigma_{\partial B}\rho_{\partial B}^{-1/2} - J_{\partial B}\| \cdot \|\rho_{\partial B'}^{-1/2}\sigma_{\partial B'}\rho_{\partial B'}^{-1/2}\| \\
&\quad + \|\rho_{\partial B'}^{-1/2}\sigma_{\partial B'}\rho_{\partial B'}^{-1/2} - J_{\partial B'}\| \cdot \|J_{\partial B}\| \\
&\leq 2\epsilon_{BB'}(1 + 2\epsilon_{BB'}) + 2\epsilon_{BB'} \\
&\leq 6\epsilon_{BB'} \,.
\end{aligned}$$

Applying Theorem 5.6, we conclude the result. □

Corollary 3.11. *We consider a decomposition of the torus into four regions A, B, C, B' as below. We have then two overlapping cylinders $B'AB$ and BCB', whose intersection is formed by two disjoint cylinders B and B'.*





A B C B'

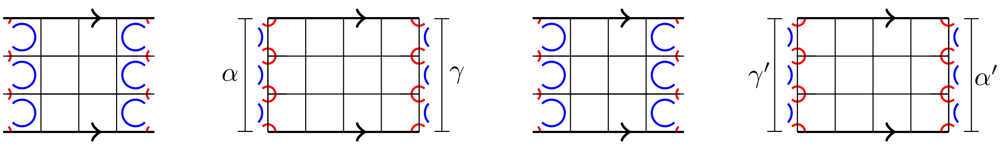
Let us assume that B and B' have at least M plaquettes per row and N plaquettes per column with

$$\epsilon_{BB'} := 3\,|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(M-1)(N-1)} \leq \frac{1}{2}\,,$$

then the orthogonal projections $P_{\mathcal{R}}$ onto $\operatorname{Im}(V_{\mathcal{R}})$ satisfy

$$\|P_{B'AB}P_{BCB'} - P_{ABCB'}\| \leq 18\epsilon_{BB'}(1 + 6\epsilon_{BB'}) \leq 72\epsilon_{BB'} \,.$$

Proof. We are going to apply Theorem 5.6 for the three regions A, BB' and C. Firstly, we need to find a suitable arrangement of the virtual indices. The boundary of B (resp. B') can be split into two regions: the part connecting with A, denoted by α (resp. α'); and the part connecting with C, denoted by γ (resp. γ'):



A B C B'

The local structure of $\widetilde{S}_{\partial\mathcal{R}}$ and $\mathcal{G}_{\partial\mathcal{R}}$ allows us to write them as a tensor product of operators acting on the defined boundary segments, and define

$$\sigma_\alpha := \sqrt{\kappa_B \kappa_{B'}}\, \mathcal{G}_\alpha^\dagger \widetilde{\mathcal{S}}_\alpha \mathcal{G}_\alpha \quad , \quad \sigma_\gamma := \sqrt{\kappa_B \kappa_{B'}}\, \mathcal{G}_\alpha^\dagger \widetilde{\mathcal{S}}_\alpha \mathcal{G}_\alpha$$

$$\sigma_{\alpha'} := \frac{\kappa_{BCB'}}{\sqrt{\kappa_B \kappa_{B'}}}\, \mathcal{G}_{\alpha'}^\dagger \widetilde{\mathcal{S}}_{\alpha'} \mathcal{G}_{\alpha'} \quad , \quad \sigma_{\gamma'} := \frac{\kappa_{B'AB}}{\sqrt{\kappa_B \kappa_{B'}}}\, \mathcal{G}_{\gamma'}^\dagger \widetilde{\mathcal{S}}_{\gamma'} \mathcal{G}_{\gamma'}$$

Using that $\kappa_{ABCB'} \kappa_B \kappa_{B'} = \kappa_{B'AB}\, \kappa_{BCB'}$, we can easily verify

$$\sigma_{\partial B} = \sigma_\alpha \otimes \sigma_\gamma \quad , \quad \sigma_{\partial B'} = \sigma_{\alpha'} \otimes \sigma_{\gamma'}$$

$$\sigma_{\partial B'AB} = \sigma_{\gamma'} \otimes \sigma_\gamma \quad , \quad \sigma_{\partial BCB'} = \sigma_\alpha \otimes \sigma_{\alpha'} \,.$$

By Corollary 3.10, we have for any cylinder $\mathcal{R} \in \{B'AB, BCB', B, B'\}$

$$\|\rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2} - J_{\partial\mathcal{R}}\| \,,\; \|\rho_{\partial\mathcal{R}}^{-1/2} \sigma_{\partial\mathcal{R}} \rho_{\partial\mathcal{R}}^{1/2} - J_{\partial\mathcal{R}}\| \;\leq\; 2\epsilon_{BB'} \,.$$

Reasoning as in the previous corollary, we have for the region joint BB'

$$\|\rho_{\partial BB'}^{1/2} \sigma_{\partial BB'}^{-1} \rho_{\partial BB'}^{1/2} - J_{\partial BB'}\| \;\leq\; 6\,\epsilon_{BB'} \,.$$

Applying Theorem 5.6, we conclude the result. □

4 Parent Hamiltonian of the thermofield double

The PEPS description of $|\rho_\beta^{1/2}\rangle$ is given in terms of a family of tensors whose contraction defines linear maps

$$V_{\mathcal{R}} : \mathcal{H}_{\partial\mathcal{R}} \longrightarrow \mathcal{H}_{\mathcal{R}}^2$$

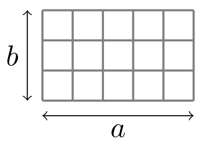
where $\mathcal{R}$ runs over all rectangular regions $\mathcal{R} \subset \Lambda_N$. We will denote by $P_{\mathcal{R}}$ be the orthogonal projection onto $\operatorname{Im}(V_{\mathcal{R}})$.

We aim at constructing a *parent Hamiltonian* of this PEPS, namely a local and frustration-free Hamiltonian whose local ground states spaces coincide with the range of $V_{\mathcal{R}}$ for all sufficiently large regions $\mathcal{R}$. In particular, this will imply that $|\rho_\beta^{1/2}\rangle$ is the unique ground state of the Hamiltonian on the torus.

For each $\beta > 0$, let us fix $n = n(\beta) \in \mathbb{N}$ large enough so that

$$216e|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{(n(\beta)-1)^2} < 1 \qquad \text{where} \qquad \gamma_\beta := \frac{e^\beta - 1}{|G|} . \tag{33}$$

We then consider the family $\mathcal{X} = \mathcal{X}_n$ formed by all rectangular regions $X \subset \Lambda_N$ with dimensions $a, b \in \mathbb{N}$ satisfying



$$n \leq a, b \leq 8n ,$$

and define the local Hamiltonian

$$H_{\mathcal{E}} = \sum_{X \in \mathcal{X}} P_X^\perp$$

where $P_X^\perp := \mathbb{1} - P_X$, that is, the orthogonal projection onto $\operatorname{Im}(V_X)^\perp$. Note that the range of interaction of the Hamiltonian $H_{\mathcal{E}}$ depends on the parameter $n(\beta)$.

Remark 4.1. *A sufficient condition ensuring that* (33) *holds is*

$$n(\beta) \geq 2 + (1 + \gamma_\beta) \log \left(216e|G|^2 \right) .$$

Indeed, in this case, using that $n \geq 2$ and that the map $x \mapsto (1 - 1/x)^x$ is bounded by $1/e$ for $x > 1$, we can estimate

$$\begin{aligned} \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{(n-1)^2} \leq \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{n} &= \left(1 - \frac{1}{1+\gamma_\beta} \right)^{(1+\gamma_\beta)\frac{n}{1+\gamma_\beta}} \\ &\leq e^{-\frac{n}{1+\gamma_\beta}} \leq \frac{1}{216e|G|^2} . \end{aligned}$$

4.1 Uniqueness of the ground state

The next result shows that $|\rho_\beta^{1/2}\rangle$ is the unique ground state of the aforementioned parent Hamiltonian.

Proposition 4.2. *Assume that $N > 2n(\beta)$. Then, for any rectangular region $\mathcal{R}$ of the lattice Λ_N with dimensions $a, b \geq 8n$ we have*

$$\ker\left(\sum_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} P_X^\perp\right) = \operatorname{Im}(V_\mathcal{R})\,. \tag{34}$$

Proof. There are three type of rectangular regions on the lattice we need to consider: proper rectangles, cylinders and the whole lattice (torus). One inclusion can be simultaneously proved for all of them:

$$\ker\left(\sum_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} P_X^\perp\right) = \bigcap_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} \ker P_X^\perp = \bigcap_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} \operatorname{Im}(V_X) \supset \operatorname{Im}(V_\mathcal{R})$$

since $\operatorname{Im}(V_\mathcal{R}) \subset \operatorname{Im}(V_X)$ whenever $X \subset \mathcal{R}$. To see the reversed inclusion, we are going to argue by induction on $a + b$ where a and b are the dimensions of $\mathcal{R}$.

Since we are considering rectangular regions with $a, b \geq 8n$, the first case is $a + b = 16n$. Then, necessarily $a = b = 8n$ and the equality (34) is clear since $\mathcal{R}$ actually belongs to $\mathcal{X}$. Let us then assume that $a + b > 16n$ and that the result holds for all rectangular regions $\mathcal{R}'$ with dimensions a', b' satisfying $a' + b' < a + b$. We claim that there exist rectangular subregions $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ such that:

(i) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$,

(ii) For each $j = 1, 2$, $\mathcal{R}_j$ has dimensions a_j, b_j satisfying $a_j + b_j < a + b$,

(iii) If $X \in \mathcal{X}$ is contained in $\mathcal{R}$, then $X \subset \mathcal{R}_1$ or $X \subset \mathcal{R}_2$,

(iv) $\|P_{\mathcal{R}_1} P_{\mathcal{R}_2} - P_\mathcal{R}\| < 1$.

Before proving the claim, let us check how it yields, together with the induction hypothesis, that $\mathcal{R}$ satisfies (34). Indeed, using (i) and (iii)

$$\begin{aligned}\ker\left(\sum_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} P_X^\perp\right) &= \bigcap_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}} \ker P_X^\perp \\ &= \bigcap_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}_1} \ker P_X^\perp \;\cap \bigcap_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}_2} \ker P_X^\perp \\ &= \ker\left(\sum_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}_1} P_X^\perp\right) \cap \ker\left(\sum_{X\in\mathcal{X}\,,\, X\subset\mathcal{R}_2} P_X^\perp\right)\,.\end{aligned}$$

By (ii), we can apply the induction hypothesis to $\mathcal{R}_1$ and $\mathcal{R}_2$ so that

$$\ker\left(\sum_{X\in\mathcal{X}\,,\,X\subset\mathcal{R}} P_X^{\perp}\right) = \operatorname{Im}(V_{\mathcal{R}_1}) \cap \operatorname{Im}(V_{\mathcal{R}_2})\,.$$

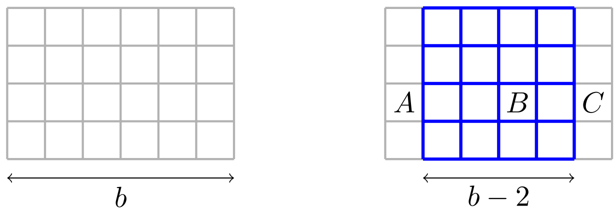
It remains to prove that

$$\operatorname{Im}(V_{\mathcal{R}_1}) \cap \operatorname{Im}(V_{\mathcal{R}_2}) \;=\; \operatorname{Im}(V_{\mathcal{R}})\,.$$

But since obviously $\operatorname{Im}(V_{\mathcal{R}}) \subset \operatorname{Im}(V_{\mathcal{R}_1}) \cap \operatorname{Im}(V_{\mathcal{R}_2})$, we simply apply Lemma 5.3 using condition (iv).

Thus, to finish the proof of the proposition, it remains to verify the claim. We now distinguish between three cases according to the type of rectangular region $\mathcal{R}$.

Case 1: If $\mathcal{R}$ is a proper rectangle, we can assume without loss of generality that $b > 8n$ and b corresponds to the number of plaquettes of each row. Then, we split $\mathcal{R}$ into A, B, C where B contains all plaquettes except for the first and last columns.



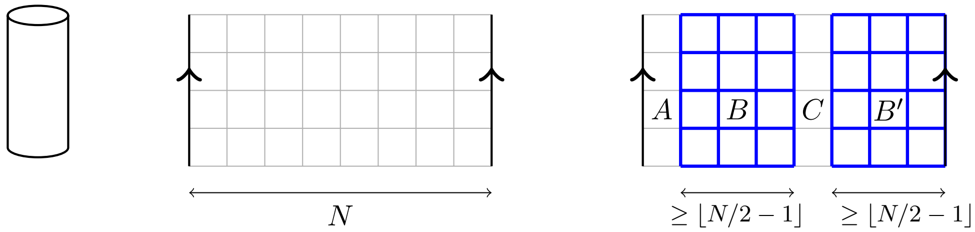
Then, taking $\mathcal{R}_1 = AB$ and $\mathcal{R}_2 = BC$ we have that (i) and (ii) clearly hold. To see that (iii) also holds, note that if X is a rectangular subregion of $\mathcal{R}$ not contained in $\mathcal{R}_1$ nor in $\mathcal{R}_2$, then it must contain a whole row of plaquettes of $\mathcal{R}$, and so its horizontal dimension will be b. But since $b > 8n$, we deduce that $X \notin \mathcal{X}$. Finally, property (iv) follows from Corollary 3.9 , stating that if $B = \mathcal{R}_1 \cap \mathcal{R}_2$ has dimensions a', b', then

$$\|P_{\mathcal{R}_1} P_{\mathcal{R}_2} - P_{\mathcal{R}_1\cup\mathcal{R}_2}\| \leq 36\,|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(a'-1)(b'-1)}\,.$$

In our case, note that $a' = a \geq n$ and $b' = b-2 \geq 8n-1$. Therefore,

$$\|P_{\mathcal{R}_1} P_{\mathcal{R}_2} - P_{\mathcal{R}_1\cup\mathcal{R}_2}\| \;\leq\; 36\,|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(n-1)^2} < 1.$$

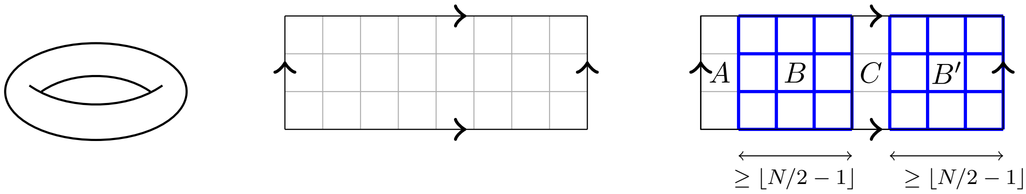
Case 2: Assume that $\mathcal{R}$ is a cylinder of dimensions $a = N$ and b. We can assume without loss of generality that its border lies on the horizontal sides, so that it contains N plaquettes per row. We then split $\mathcal{R}$ into four regions A, B, C, B' as in Corollary 3.10 where A and C correspond to columns of horizontal edges, and B, B' have horizontal dimension greater than $\lfloor N/2 - 1\rfloor$.



Taking $\mathcal{R}_1 = B'AB$ and $\mathcal{R}_2 = BCB'$ we immediately get that these are proper rectangles satisfying (i) and (ii). As in the previous case, we can argue that they also satisfy (iii) since $\lfloor N/2 - 1 \rfloor \geq 10n - 1 > 8n$ by the hypothesis. Property (iv) follows from Corollary 3.10 since it yields that

$$\|P_{B'AB}P_{BCB'} - P_{ABCB'}\| \leq 216|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{(8n-1)(n-1)} < 1\,.$$

Case 3: Assume $\mathcal{R}$ is the whole torus Λ_N. Then, we can split it into four regions as in Corollary 3.11 where B and B' have horizontal dimension greater than or equal to $\lfloor N/2 - 1 \rfloor$.



Taking $\mathcal{R}_1 = B'AB$ and $\mathcal{R}_2 = BCB'$ as rectangular subregions, we can argue analogously to the previous cases to deduce that they satisfy (i)-(iii). Property (iv) is consequence of Corollary 3.11, which yields

$$\begin{aligned} \|P_{\mathcal{R}_1}P_{\mathcal{R}_2} - P_{\Lambda_N}\| &\leq 216|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{(\lfloor N/2-1 \rfloor)^2} \\ &< 216|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta} \right)^{(n-1)^2} < 1\,. \end{aligned}$$

□

4.2 The parent Hamiltonian is gapped

In [19], a family $\mathcal{F}$ of rectangles is given (based on a previous construction form [9]), which serves to obtain lower bounds to the spectral gap local Hamiltonians over the plane, i.e. with open boundary conditions. But in the case of Quantum Double Models, what we are really interested in is to show a lower bound to the spectral gap over $\mathbb{Z}_N \times \mathbb{Z}_N$ uniform in N, i.e. when the consider periodic boundary conditions over a torus.

Due to the results on spectral gap thresholds [22, 14, 16, 17, 3], for the Hamiltonian on the plane to be gapped is sufficient that the Hamiltonian on the torus is gapped. Since the constants appearing in the spectral gap threshold are highly dependent on the specific shape and range of the interactions of the Hamiltonian, and in our case we will have to consider β-dependent interaction length. To obtain more explicit constants, here we will take a different route, and we will directly apply Theorem 5.2 to the case of the system defined on a torus.

To do so, we have to choose a particular family of sub-regions $\mathcal{F}$. The first two steps in the recursion will be used to pass from sub-regions of the torus to sub-regions of the plane (removing the periodic boundary condition in one of the dimensions each time): from then on, the family of sub-regions to be considered will be the same as in [19].

We refer the reader to Section 5.1 for a general presentation of the results and notation that we will use in this setting.

4.2.1 Periodic boundary conditions

Let us consider the set of edges $\mathcal{E}_{N+1}$ on the torus $\Lambda_{N+1} = \mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1}$ torus. We are going to define three families:

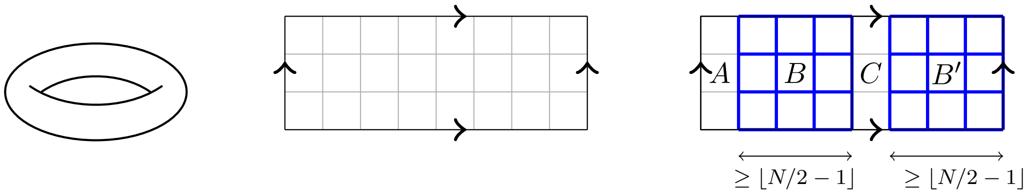
- ▷ $\mathcal{F}_N^{torus}$ is the family consisting of only one element, the whole lattice (torus).
- ▷ $\mathcal{F}_N^{cilin}$ is the family of all cylinders having $N+1$ plaquettes per column and N plaquettes per row.
- ▷ $\mathcal{F}_N^{rect}$ is the family of all proper rectangular regions $\mathcal{R} \subset \Lambda_{N+1}$ having N plaquettes per row and per column.

Recall that we have already defined a parent Hamiltonian in terms of local interactions $P_X^{\perp}$ where X runs over the family $\mathcal{X}$. The next result relates the gap associate to each family (see Definition 5.1).

Theorem 4.3. *Assume that $N > 2n(\beta)$. Then,*

$$\operatorname{gap}(\mathcal{F}_N^{torus}) \geq \frac{1}{4} \operatorname{gap}(\mathcal{F}_N^{cilin}) \geq \frac{1}{16} \operatorname{gap}(\mathcal{F}_N^{rect}) .$$

Proof. Let us first compare $\operatorname{gap}(\mathcal{F}_N^{torus})$ and $\operatorname{gap}(\mathcal{F}_N^{cilin})$. We have to apply Theorem 5.2, and for that we have to ensure that Λ_{N+1} admits a suitable decomposition in terms of cylinders of $\mathcal{F}_N^{cilin}$. We are going to use the same idea from the proof of Proposition 4.2: let us split Λ_{N+1} into four regions as in Corollary 3.11 where B and B' have horizontal dimension greater than or equal to $\lfloor N/2-1 \rfloor$.



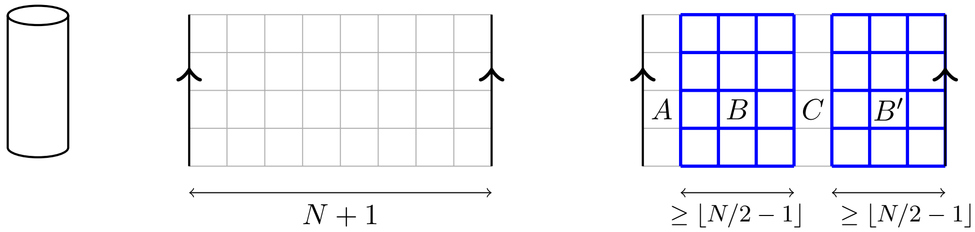
Then, cylinders $\mathcal{C}_1 = B'AB$ and $\mathcal{C}_2 = BCB'$ belong to $\mathcal{F}_N^{cilin}$ and satisfy, by Corollary 3.11,

$$\|P_{\mathcal{C}_1}P_{\mathcal{C}_2} - P_{\Lambda_{N+1}}\| \leq 216|G|^2\left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(\lfloor N/2-1\rfloor)^2} < \frac{1}{e}\,.$$

Thus, applying Theorem 5.2 with $s = 1$ and $\delta = 1/2$ we can bound

$$\mathrm{gap}(\mathcal{F}_N^{torus}) \geq \frac{1}{4}\,\mathrm{gap}(\mathcal{F}_N^{cilin})$$

Next, we compare $\mathcal{F}_N^{cilin}$ and $\mathcal{F}_N^{rect}$. Given $\mathcal{C} \in \mathcal{F}_N^{cilin}$ we can find a decomposition into four regions $ABCB'$ as in the next picture



so that $\mathcal{R}_1 := B'AB$ and $\mathcal{R}_2 := BCB'$ are proper rectangles having N plaquettes per row and per column, so that $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{F}_N^{rect}$. From Corollary 3.10, we deduce

$$\|P_{\mathcal{R}_1}P_{\mathcal{R}_2} - P_{\mathcal{C}}\| \leq 216|G|^2\left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{(8n-1)(n-1)} < 1\,.$$

Thus, applying Theorem 5.2 with $s = 1$ and $\delta = 1/2$ we can bound

$$\mathrm{gap}(\mathcal{F}_N^{cilin}) \geq \frac{1}{4}\,\mathrm{gap}(\mathcal{F}_N^{rect})$$

$\square$

4.2.2 Open boundary conditions

The next step is to show that the gap $\mathrm{gap}(\mathcal{F}_N^{rect})$ is lower bounded by a constant independent of N and β. For that, let us define the family $\mathcal{F} = \mathcal{F}_{n,N}^{rect}$ of all rectangular regions $\mathcal{R} \subset \Lambda_{N+1}$ with dimensions a, b satisfying $n \leq a, b \leq N$. Since $\mathcal{F}_N^{rect} \subset \mathcal{F}$, we have

$$\mathrm{gap}(\mathcal{F}_N^{rect}) \geq \mathrm{gap}(\mathcal{F})\,.$$

We are going to prove the next main result.

Theorem 4.4. *Assume that $N > n(\beta)$. Then, $\mathrm{gap}(\mathcal{F})$ (and so $\mathrm{gap}(\mathcal{F}_N^{rec})$) is lower bounded by a constant independent of β and N.*

For each $k \in \mathbb{N}$, let $\ell_k := (\sqrt{3/2})^k$ and $R(k) := [0, \ell_{k+1}] \times [0, \ell_{k+2}] \subset \Lambda_{N+1}$, and denote by $\mathcal{F}_k$ the collection of all rectangular regions $\mathcal{R}$ in $\mathcal{F}$ contained in $R(k)$ up to translations and permutation of coordinates. It is clear that this is an increasing sequence

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}\,.$$

such that $\mathcal{F}_k$ is equal to $\mathcal{F}$ if k is large enough.

First, we need an auxiliary result which appears in [19] and [9]. We have added in the statement additional information that implicitly appears in the original proof, and will be used in our setting.

Lemma 4.5. *Let $s_k \in \mathbb{N}$ with $s_k \leq \ell_k/32$ and $\mathcal{R} \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$ with dimensions a, b being $a \leq b$. Note that necessarily $\ell_k < b \leq \ell_{k+2}$ and $a \leq \ell_{k+1}$. Then, we can find s_k pairs $(A_i, B_i)_{i=1}^{s_k}$ of elements in $\mathcal{F}_{k-1}$ such that:*

(i) $\mathcal{R} = A_i \cup B_i$ for every $i = 1, \ldots, s_k$,

(ii) $(A_i \cap B_i) \cap (A_j \cap B_j) = \emptyset$ whenever $i \neq j$,

(iii) $A_i \cap B_i$ has dimensions a, b_i where $b_i \geq \lfloor \frac{\ell_k}{16 s_k} \rfloor$.

(iv) A_i has dimensions a, c_i with

$$\frac{3}{4}\ell_k \;\leq\; c_i \;\leq\; \frac{7}{8}\ell_k\,.$$

(v) B_i has dimensions a, d_i with

$$\frac{1}{8}\ell_k \leq d_i \leq \frac{3}{4}\ell_k\,.$$

Proof. Let us identify $\Lambda_{N+1} \equiv \mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1}$ and $\mathcal{R} = [0,a] \times [0,b]$. For each $i = 1, \ldots, s_k$, consider the subrectangles

$$\begin{aligned} A_i &:= \left([0,a] \times [0, \frac{\ell_{k+2}}{2} + 2i\frac{\ell_k}{16 s_k}] \right) \cap \Lambda_{N+1} \\ B_i &:= \left([0,a] \times [\frac{\ell_{k+2}}{2} + (2i-1)\frac{\ell_k}{16 s_k}, b] \right) \cap \Lambda_{N+1}\,. \end{aligned}$$

Clearly $\mathcal{R} = A_i \cup B_i$, and moreover the intersections

$$A_i \cap B_i = \left([0,a] \times [\,\frac{\ell_{k+2}}{2} + (2i-1)\frac{\ell_k}{16 s_k}, \frac{\ell_{k+2}}{2} + 2i\frac{\ell_k}{16 s_k}] \right) \cap \Lambda_{N+1}$$

satisfy

$$(A_i \cap B_i) \cap (A_j \cap B_j) = \emptyset \quad \text{whenever} \quad i \neq j\,,$$

so that $(i)-(iii)$ clearly hold. The dimensions of A_i are a and $c_i = \lfloor \frac{\ell_{k+2}}{2} + 2i\frac{\ell_k}{16 s_k} \rfloor$, the latter satisfying

$$\frac{3}{4}\ell_k \;=\; \frac{\ell_{k+2}}{2} \;\leq\; c_i \leq \frac{\ell_{k+2}}{2} + \frac{\ell_k}{8} \;=\; \frac{3}{4}\ell_k + \frac{\ell_k}{8} \;=\; \frac{7}{8}\ell_k$$

which gives (iv). The dimensions of B_i are a and $d_i = b - \lceil \frac{\ell_{k+2}}{2} + (2i-1)\frac{\ell_k}{16s_k} \rceil$, where d_i satisfies, using that $\ell_k < b \leq \ell_{k+2}$,

$$\frac{1}{8}\ell_k = \ell_k - \frac{7}{8}\ell_k \leq \ell_k - c_i \leq b - c_i < d_i \leq \frac{\ell_{k+2}}{2} = \frac{3}{4}\ell_k \,.$$

Thus, (v) also holds. □

Proposition 4.6. *Let $k_0 \in \mathbb{N}$ such that*

$$\ell_{k_0} \geq 8n(\beta) \geq \ell_{k_0-1} \,.$$

Then, the family $\mathcal{F}_{k_0}$ is contained in $\mathcal{X}$. Moreover, if $k > k_0$ and $\mathcal{R} \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$, then for any $s_k \in \mathbb{N}$ with $s_k \leq \ell_k/32$, there exist s_k pairs $(A_i, B_i)_{i=1}^{s_k}$ of elements in $\mathcal{F}_{k-1}$ such that:

(1) $\mathcal{R} = A_i \cup B_i$ for every $i = 1, \ldots, s_k$,

(2) $(A_i \cap B_i) \cap (A_j \cap B_j) = \emptyset$ if $i \neq j$,

(3) Rectangle $A_i \cap B_i$ contains $n(\beta) \cdot \lfloor \frac{\ell_k}{16s_k} - 1 \rfloor$ inner vertices.

Proof. To see that $\mathcal{F}_{k_0}$ is contained in the family $\mathcal{X}$, simply note that every rectangular region $\mathcal{R}$ in $\mathcal{F}_{k_0}$ has, by definition, dimensions a, b with

$$n(\beta) \leq a, b \leq \ell_{k_0+2} \;=\; (\sqrt{3/2})^3\, \ell_{k_0-1} \;\leq\; (\sqrt{3/2})^3\, n(\beta) \;<\; 8\, n(\beta) \,.$$

Let us check now the second statement. Fix $k > k_0$, $\mathcal{R} \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$ and s_k as above. Let $(A_i, B_i)_{i=1}^{k}$ be the pairs provided by Lemma 4.5. We are going to check that they satisfy the required properties of the theorem. First, we have to show that $A_i, B_i \in \mathcal{F}_{k-1}$ for every $i = 1, \ldots, s_k$. Denote by a, b the dimensions of $\mathcal{R}$. We can assume that $a \leq b$. Since $\mathcal{R}$ is in $\mathcal{F}$, we know that $n \leq a, b \leq N$. By Lemma 4.5.(iv)-(v), the dimensions of A_i and B_i are respectively (a, c_i) and (a, d_i) satisfying

$$n(\beta) \;\leq\; \frac{1}{8}\ell_{k_0} \;\leq\; \frac{1}{8}\ell_k \;\leq\; c_i, d_i \;\leq\; b \;\leq\; N \,.$$

Finally, the intersection (rectangular) region $A_i \cap B_i$ has dimensions (a, b_i) where $b_i = \frac{\ell_k}{16s_k}$.

□

We are now ready to prove the main result of this subsection.

Proof of Theorem 4.4. Let us fix, for every $k \in \mathbb{N}$ the value $s_k := \lfloor \frac{\ell_k}{32k^2} \rfloor$. By Proposition 4.6, we know that for every $k > k_0$ and every $\mathcal{R} \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}$ we can find pairs $(A_i, B_i)_{i=1}^{s_k}$ of elements in $\mathcal{F}_{k-1}$ such that $\mathcal{R} = A_i \cup B_i$ for every $i = 1, \ldots, s_k$, $(A_i \cap B_i) \cap (A_j \cap B_j) = \emptyset$ whenever $i \neq j$, and the intersection $A_i \cap B_i$ contains at least $n(\beta) \cdot (k^2 - 1)$ inner vertices. By Corollary 3.9, we have

$$\| P_{A_i} P_{B_i} - P_{\mathcal{R}} \| \leq 36 |G| \left(\frac{\gamma_\beta}{1 + \gamma_\beta} \right)^{n(\beta)(k^2-1)} \;\leq\; e^{-k^2+1} \,.$$

Hence, applying Theorem 5.2 with $\delta_k := e^{-k^2+1}$ we deduce

$$\mathrm{gap}(\mathcal{F}_k) \geq \exp\left[\frac{\delta_k}{1-\delta_k} - \frac{1}{s_k}\right] \mathrm{gap}(\mathcal{F}_{k-1}) \geq \ldots \geq \exp\left[\sum_{j=k_0}^{k} \frac{\delta_j}{1-\delta_j} - \frac{1}{s_j}\right] \mathrm{gap}(\mathcal{F}_{k_0})\,.$$

Since $\mathcal{F}_{k_0}$ is contained in $\mathcal{X}$ (see Proposition 4.6) we have $\mathrm{gap}(\mathcal{F}_{k_0}) = 1$. On the other hand, the two series $\sum_k \frac{1}{s_k}$ and $\sum_k \frac{\delta_k}{1-\delta_k}$ are absolutely convergent. Thus, $\mathrm{gap}(\mathcal{F}_k)$ is lower bounded by a constant independent of N, β uniformly on k. $\square$

4.3 Parent Hamiltonian vs Davies generator

We have two Hamiltonians,

$$\sum_{e\in\mathcal{E}} \widetilde{H}_e \quad \text{and} \quad H_{\mathcal{E}} = \sum_{X\in\mathcal{X}} P_X^{\perp}$$

both having the same (unique) ground state $|\rho_\beta^{1/2}\rangle$. We can relate them through

$$\sum_{e\in\mathcal{E}} \widetilde{H}_e \geq C \sum_{e\in\mathcal{E}} \Pi_e^{\perp} \geq \frac{C}{m(\mathcal{X})} \sum_{X\in\mathcal{X}} \left(\sum_{e\in\mathcal{E}} \Pi_e^{\perp} \right) = \frac{C}{m(\mathcal{X})} \sum_{X\in\mathcal{X}} \Pi_X^{\perp}\,.$$

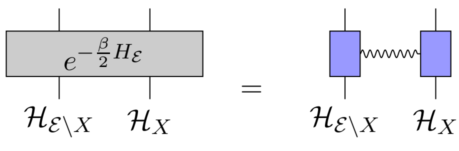
The number $m(\mathcal{X})$ is a uniform bound on the number of rectangles $X \in \mathcal{X}$ containing a prefixed edge e (uniform, in the sense that it is independent of the edge). To estimate $m(\mathcal{X})$, recall that each $X \in \mathcal{X}$ has dimension $a \times b$ with $n \leq a, b \leq 8n$, and so X contains at most $4(8n)^2$ edges. We can then roughly estimate

$$m(\mathcal{X}) \leq O(n(\beta)^4)\,.$$

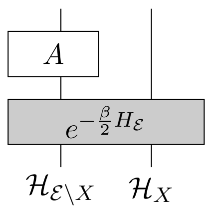
Next, we note that $P_X \geq \Pi_X$ for every rectangular region $X \subset \mathcal{E}$, or equivalently

$$\ker \Pi_X^{\perp} = \{\, (A \otimes e^{-\frac{\beta}{2}H_X}) e^{-\frac{\beta}{2}(H_{\mathcal{E}}-H_X)} \;:\; A \in \mathcal{B}_{\mathcal{E}\setminus X} \,\} \subseteq \mathrm{Im}(V_X)\,.$$

Indeed, using the PEPS decomposition

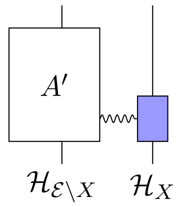


we can compare

$\ker \Pi_X^{\perp} = \Big\{$ 

$: A \in \mathcal{B}_{\mathcal{E}\setminus X} \Big\}$

and

$$\mathrm{Im}(V_X) = \Bigg\{$$ 

$$: A' \in \mathcal{B}_{\mathcal{E}\setminus X} \otimes \mathcal{H}_{\partial(\mathcal{E}\setminus X)} \Bigg\}$$

We summarize the results of this subsection in the following proposition.

Proposition 4.7. *The spectral gap $\mathcal{L}$ of the Davies generator for the Quantum Double Model with group G is bounded by*

$$\mathrm{gap}(\mathcal{L}) \geq C\, \hat{g}_{\min} \frac{e^{-c\beta}}{n(\beta)^2}\, \mathrm{gap}(H_{\mathcal{E}}), \tag{35}$$

where $H_{\mathcal{E}}$ is the parent Hamiltonian of the thermofield double state $|\rho_\beta^{1/2}\rangle$ with parameter $n(\beta)$, c and C are positive constants independent of β, G and the system size, and $\widehat{g}_{\min} = \min_{\alpha,\omega} \widehat{g}_\alpha(\omega)$.

The main Theorem 1.1 is then obtained combining Proposition 4.7 with Theorems 4.3 and 4.4.

5 Tools

In this section, we are presenting some of the concepts and auxiliary results that we have used in preceeding sections. Since we expect that they are useful in other contexts, we decided to present them in a more general setting.

Let us consider a general lattice Λ or even an arbitrary metric space, and associate to every site $x \in \Lambda$ a finite-dimensional Hilbert space $\mathcal{H}_x \equiv \mathbb{C}^d$. As usual, for a finite subset $X \subset \Lambda$ we define the corresponding $\mathcal{H}_X = \otimes_{x\in X}\mathcal{H}_x$ and space of observables $\mathcal{B}_X = \mathcal{B}(\mathcal{H}_X)$, identifying for $X \subset X' \subset \Lambda$ observables $\mathcal{B}_X \hookrightarrow \mathcal{B}_{X'}$ via $Q \mapsto Q \otimes \mathbb{1}_{X'\setminus X}$.

5.1 On the spectral gap of local Hamiltonians

Let us start by recalling the recursive strategy to obtain lower bounds to the spectral gap of frustration-free Hamiltonians described in [19], which we will also slightly improve over the original formulation.

A local Hamiltonian is defined by a family of local interactions, that is, a map Φ that associates to each finite subset $X \subset \Lambda$ an observable $\Phi_X = \Phi_X^\dagger \in \mathcal{B}_X$. For each finite $Y \subset \Lambda$, the corresponding Hamiltonian is given by

$$H_Y = \sum_{X\subset Y} \Phi_X \,.$$

Let us denote by P_Y the projection onto the groundspace of H_Y.

We will assume that the Hamiltonian is frustration-free, namely that for every finite $Y \subset \Lambda$

$$W_Y := \bigcap_{X\subset Y} \text{Groundspace}(\Phi_X) \neq 0\,.$$

This ensures that W_Y is the groundspace of H_Y, i.e. P_Y is the orthogonal projection onto W_Y. In addition, for every $X \subset Y \subset \Lambda$ we have $W_Y \subset W_X$ and therefore $P_Y P_X = P_Y$.

The frustration-free condition has further important implications on the relationships between the projections P_X, which we now recall (these statements are all consequences of Lemma 5.3). Let $X, Y \subset \Lambda$. Frustration-free condition yields $W_{X\cup Y} \subset W_X \cap W_Y$. As a consequence,

$$\|P_{X\cup Y} - P_X P_Y\| = \|(P_{X\cup Y} - P_X)(P_{X\cup Y} - P_Y)\| \in [0,1]\,.$$

Moreover, $W_{X\cup Y} = W_X \cap W_Y$ if and only if $\|P_{X\cup Y} - P_X P_Y\| \in [0,1)$. This happens whenever Φ has finite range $r > 0$ and the distance $d(X \setminus Y, Y \setminus X)$ is greater than r, since in this case every subset $Z \subset X \cup Y$ with $\Phi_Z \neq 0$ is either contained in X or Y.

Definition 5.1 (Spectral gap). *For each finite subset $Y \subset \Lambda$, let us denote by* $\text{gap}(H_Y)$*, or simply* $\text{gap}(Y)$*, the* spectral gap *of H_Y, namely the difference between* *the two lowest unequal eigenvalues of H_Y. Given a family $\mathcal{F}$ of finite subsets of Λ, we say that the system of Hamiltonians $(H_Y)_{Y\in\mathcal{F}}$ is* gapped *whenever*

$$\operatorname{gap}(\mathcal{F}) := \inf\{\operatorname{gap}(Y)\colon Y \in \mathcal{F}\} > 0\,.$$

Otherwise, it is said to be gapless *.*

The following result allows to relate the gap of two families. It adapts a result from [19, Section 4.2].

Theorem 5.2. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two families of finite subsets of Λ. Suppose that there are $s \in \mathbb{N}$ and $\delta \in [0,1)$ satisfying the following property: for each $Y \in \mathcal{F}'\setminus\mathcal{F}$ there exist $(A_i, B_i)_{i=1}^s$ pairs of elements in $\mathcal{F}$ such that:*

(i) $Y = A_i \cup B_i$ *for each* $i = 1, \dots, s$*,*

(ii) $(A_i \cap B_i) \cap (A_j \cap B_j) = \emptyset$ *whenever* $i \neq j$*,*

(iii) $\|P_{A_i} P_{B_i} - P_Y\| \leq \delta$ *for every* $i = 1, \dots, s$*.*

Then,

$$\operatorname{gap}(\mathcal{F}') \geq \frac{1-\delta}{1+\frac{1}{s}} \operatorname{gap}(\mathcal{F}) \geq \exp\left[\frac{\delta}{1-\delta} - \frac{1}{s}\right] \operatorname{gap}(\mathcal{F})\,.$$

The main tool is the following lemma, which is an improved version of [19, Lemma 14] in which the constant $(1-2c)$ has been improved to $(1-c)$. The argument here is different and inspired by [21, Lemma 14.4].

Lemma 5.3. *Let U, V, W be subspaces of a finite-dimensional Hilbert space $\mathcal{H}$ and assume that $W \subset U \cap V$. Then,*

$$\Pi_U^\perp + \Pi_V^\perp \geq (1-c)\,\Pi_W^\perp \qquad \textit{where} \qquad c := \|\Pi_U \Pi_V - \Pi_W\|\,.$$

Moreover, $c \in [0,1]$ always holds, and $c \in [0,1)$ if and only if $U \cap V = W$.

Proof. Let us start by observing that, $\Pi_W = \Pi_U \Pi_W = \Pi_V \Pi_W$. Thus, the constant c can be rewritten as

$$\begin{aligned}
c &= \|(\Pi_U - \Pi_W)\,(\Pi_V - \Pi_W)\| = \|\Pi_W^\perp\,\Pi_U\,\Pi_V\,\Pi_W^\perp\| \\
&= \sup\{\,|\,\langle a|\,\Pi_W^\perp\,\Pi_U\,\Pi_V\,\Pi_W^\perp\,|b\rangle\,|:\ \|a\| \leq 1,\ \|b\| \leq 1\rangle\} \\
&= \sup\{\,|\,\langle a|\,\Pi_U\,\Pi_V\,|b\rangle\,|:\ a, b \in W^\perp,\ \|a\| \leq 1,\ \|b\| \leq 1\rangle\} \\
&= \sup\{\,|\langle a|b\rangle|:\ a \in U \cap W^\perp,\ b \in V \cap W^\perp,\ \|a\| \leq 1,\ \|b\| \leq 1\rangle\}.
\end{aligned}$$

From here it immediately follows that $c \in [0,1]$. Moreover, a standard argument by compactness ($\mathcal{H}$ is finite-dimensional) shows that $c = 1$ if and only if there exists $a \in U \cap V \cap W^\perp$ with $\|a\| = 1$, or equivalently, if $W \subsetneq U \cap V$.

Next, let us reformulate the original inequality we aim to prove as

$$\begin{aligned}\Pi_U^\perp + \Pi_V^\perp \geq (1-c)\,\Pi_W^\perp \;\; &\Leftrightarrow \;\; \mathbb{1} - \Pi_U + \mathbb{1} - \Pi_V \geq (1-c)\Pi_W^\perp \\ &\Leftrightarrow \;\; 2\,\mathbb{1} - (1-c)\Pi_W^\perp \geq \Pi_U + \Pi_V \\ &\Leftrightarrow \;\; 2\Pi_W + 2\Pi_W^\perp - (1-c)\Pi_W^\perp \geq \Pi_U + \Pi_V \\ &\Leftrightarrow \;\; 2\,\Pi_W + (1+c)\,\Pi_W^\perp \;\geq\; \Pi_U + \Pi_V\end{aligned}$$

Moreover, since $\Pi_U = \Pi_W + \Pi_W^\perp\,\Pi_U\,\Pi_W^\perp$ and $\Pi_V = \Pi_W + \Pi_W^\perp\,\Pi_V\,\Pi_W^\perp$ we add one more line to the previous chain of equivalences:

$$\Pi_U^\perp + \Pi_V^\perp \geq (1-c)\,\Pi_W^\perp \;\Leftrightarrow\; (1+c)\Pi_W^\perp \geq \Pi_W^\perp\,(\Pi_U + \Pi_V)\Pi_W^\perp\,.$$

Let $|x\rangle$ be a norm-one eigenvector of $\Pi_W^\perp\,(\Pi_U + \Pi_V)\Pi_W^\perp$ with corresponding eigenvalue $\lambda > 0$. Note that $\Pi_W^\perp\,|x\rangle = |x\rangle$ necessarily, since eigenvectors with different eigenvalues are orthogonal, and W is contained in the kernel. We can write

$$\begin{aligned}\Pi_U\,|x\rangle = \lambda_U\;|x_U\rangle \quad &\text{for some} \quad |x_U\rangle \in U\cap W^\perp\,,\; \langle x_U|x_U|x_U|x_U\rangle = 1\,,\; \lambda_U \in \mathbb{R}\,,\\ \Pi_V\,|x\rangle = \lambda_V\;|x_V\rangle \quad &\text{for some} \quad |x_V\rangle \in V\cap W^\perp\,,\; \langle x_V|x_V|x_V|x_V\rangle = 1\,,\; \lambda_V \in \mathbb{R}\,.\end{aligned}$$

On the one hand, we have

$$\begin{aligned}\lambda = \langle x|\,\Pi_U + \Pi_V\,|x\rangle &= \langle x|\,\Pi_U\,|x\rangle + \langle x|\,\Pi_V\,|x\rangle \\ &= \langle x|\,\Pi_U^2\,|x\rangle + \langle x|\,\Pi_V^2\,|x\rangle \\ &= \;\lambda_U^2 + \lambda_V^2.\end{aligned}$$

and, on the other hand

$$\begin{aligned}\lambda^2 &= \langle x|\,(\Pi_U + \Pi_V)\,(\Pi_U + \Pi_V)\,|x\rangle \\ &= \lambda_U^2 + \lambda_V^2 + \langle x|\,\Pi_U\Pi_V\,|x\rangle + \langle x|\,\Pi_V\Pi_U\,|x\rangle \\ &= \lambda_U^2 + \lambda_V^2 + 2\,\lambda_U\,\lambda_V\;\mathrm{Re}\,\langle x_U|x_V|x_U|x_V\rangle\;.\end{aligned}$$

Combining both inequalities we get, denoting $c_x := |\,\mathrm{Re}\,\langle x_U|x_V|x_U|x_V\rangle\,|$,

$$\begin{aligned}(1+c_x)\lambda - \lambda^2 &= c_x\left(\lambda_U^2 + \lambda_V^2\right) - 2\lambda_U\lambda_V\,\mathrm{Re}\,\langle x_U|x_V|x_U|x_V\rangle \\ &\geq c_x(\lambda_U^2 + \lambda_V^2 - 2\lambda_U\lambda_V) = c_x(\lambda_U - \lambda_V)^2 \geq 0\,.\end{aligned}$$

Since $\lambda > 0$, we conclude that

$$\lambda \leq 1 + c_x \leq 1 + c,$$

where the last inequality follows from the observation at the beginning of the proof. □

Proof of Theorem 5.2. Let $Y \in \mathcal{F}' \setminus \mathcal{F}$ and let $(A_i, B_i)_{i=1}^s$ the family of pairs satisfying (i)-(iii) provided by the hypothesis. Applying Lemma 5.3, we can estimate

$$\begin{aligned}
\langle x| P_Y^\perp |x\rangle \leq \frac{1}{s}\sum_{i=1}^s \langle x| P_Y^\perp |x\rangle &\leq \frac{1}{s}\sum_{i=1}^s \frac{1}{1-\delta}\Big(\langle x| P_{A_i}^\perp |x\rangle + \langle x| P_{B_i}^\perp |x\rangle\Big) \\
&\leq \frac{1}{1-\delta}\,\frac{1}{s}\sum_{i=1}^s \frac{1}{\mathrm{gap}(\mathcal{F})}\left(\langle x| H_{A_i} |x\rangle + \langle x| H_{B_i} |x\rangle\right) \\
&= \frac{1}{1-\delta}\,\frac{1}{\mathrm{gap}(\mathcal{F})}\,\frac{1}{s}\sum_{i=1}^s \langle x| H_Y + H_{A_i\cap B_i} |x\rangle \\
&\leq \frac{1}{1-\delta}\,\frac{1}{\mathrm{gap}(\mathcal{F})}\,\ \langle x| H_Y + \frac{1}{s}\sum_{i=1}^s H_{A_i\cap B_i} |x\rangle \\
&\leq \frac{1}{1-\delta}\,\frac{1}{\mathrm{gap}(\mathcal{F})}\,\left(1+\frac{1}{s}\right)\,\langle x| H_Y |x\rangle\ .
\end{aligned}$$

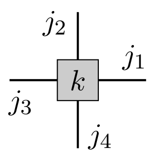
Therefore,

$$\mathrm{gap}(Y) \geq \frac{1-\delta}{1+\frac{1}{s}}\,\mathrm{gap}(\mathcal{F}) \quad \text{whenever} \quad Y \in \mathcal{F}' \setminus \mathcal{F}\,.$$

On the other hand, if $Y \in \mathcal{F}' \cap \mathcal{F}$, then $\mathrm{gap}(Y) \geq \mathrm{gap}(\mathcal{F})$ by definition. Hence, we conclude the result. $\square$

5.2 PEPS, boundary states and approximate factorization

Let us recall the notation and main concepts for PEPS. At each vertex $x \in \Lambda$ consider a tensor in the form of an operator



$$V_x : (\mathbb{C}^D)^{\otimes \partial x} \longrightarrow \mathbb{C}^d$$

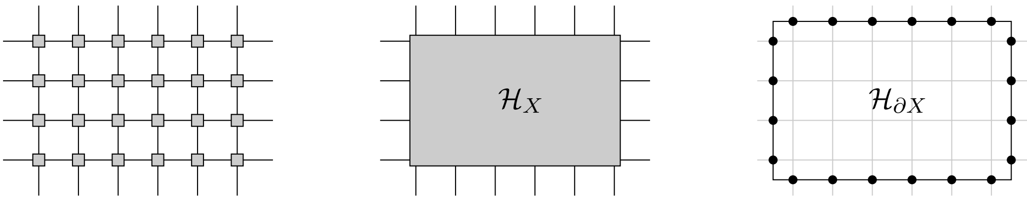
$$V_x = \sum_{k=1}^d \sum_{j_1,j_2,j_3,j_4=1}^D T^k_{j_1,j_2,j_3,j_4} |k\rangle \langle j_1 j_2 j_3 j_4|$$

Here, $\mathbb{C}^d$ is the *physical* space associated to v and each $\mathbb{C}^D$ is the *virtual* space corresponding to an edge $e \in \partial x$, i.e. incident to x.

Let us recall the notation for boundary states of a PEPS. For a finite region $X \subset \Lambda$, the contraction of the PEPS tensors gives a linear map from the virtual edges connecting X with its complement to the bulk physical Hilbert space, which we denote as

$$V_X : \mathcal{H}_{\partial X} \longrightarrow \mathcal{H}_X$$

The image of V_X are the physical states which can be represented by the PEPS with an appropriate choice of boundary condition, and so in our case these are exactly the ground states on X. In other words, P_X is (by assumption) the projection on the range of V_X.



The boundary state is then given by

$$\rho_{\partial X} = V_X^\dagger V_X \in \mathcal{B}(\mathcal{H}_{\partial X}). \tag{36}$$

If $\rho_{\partial X}$ has full rank, we say that the PEPS is *injective* on region X. We will also use the following notation

$$W_X = V_X \rho_{\partial X}^{-1/2}, \quad P_X = W_X W_X^\dagger = V_X \rho_{\partial X}^{-1} V_X^\dagger, \tag{37}$$

where W_X is an isometry.

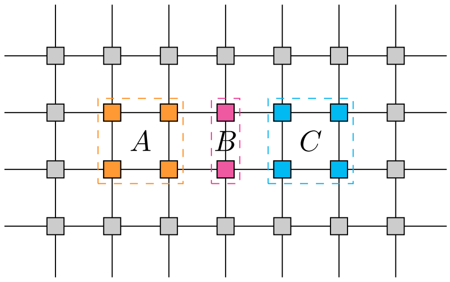
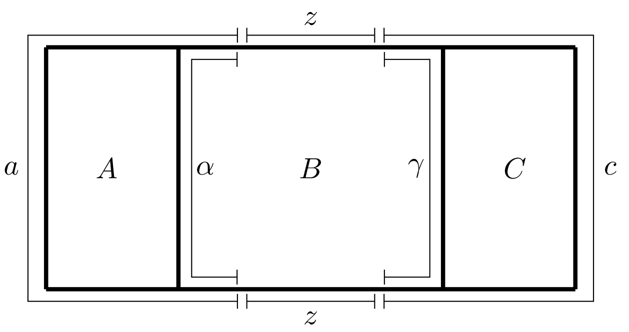


Figure 3: An example of three regions A, B, C.

Let us consider three (connected) regions $A, B, C \subset \Lambda$ and assume that B shields A from C, so that there is no edge joining vertices from A and C (see Figure 3). Let us consider the boundary states $\rho_{\partial ABC}, \rho_{\partial AB}, \rho_{\partial BC}$ and $\rho_{\partial B}$. In the case where they are all full rank, the approximate factorization condition is defined as follows.

Definition 5.4 (Approximate factorization for injective PEPS [20])**.** *Let $\varepsilon > 0$. We will say that the boundary states are ε-approximately factorizable, if we can divide the regions*



and find invertible matrices Δ_{az}, Δ_{zc}, $\Omega_{\alpha z}$, $\Omega_{z\gamma}$ with support in the regions indicated by the respective subindices such that the boundary observables

$$\sigma_{\partial AB} = \Omega_{z\gamma}\Delta_{az} \qquad \sigma_{\partial BC} = \Delta_{zc}\Omega_{\alpha z}$$
$$\sigma_{\partial ABC} = \Delta_{zc}\Delta_{az} \qquad \sigma_{\partial B} = \Omega_{z\gamma}\Omega_{\alpha z}$$

approximate the boundary states

$$\|\rho_{\partial\mathcal{R}}^{1/2}\sigma_{\partial\mathcal{R}}^{-1}\rho_{\partial\mathcal{R}}^{1/2} - \mathbb{1}\| \leq \varepsilon \quad \text{for each } \mathcal{R} \in \{ABC, AB, BC\},$$
$$\|\rho_{\partial B}^{-1/2}\sigma_{\partial B}\rho_{\partial B}^{-1/2} - \mathbb{1}\| \leq \varepsilon.$$

The approximate factorization of the boundary states implies a small norm of the overlaps of groundspace projections.

Theorem 5.5 ([20, Theorem 10]). *If the boundary states are ε-approximately factorizable, then*

$$\|P_{AB}P_{BC} - P_{ABC}\|_\infty < 8\varepsilon\,.$$

5.2.1 Approximate factorization for locally non injective PEPS

In [20], the approximate factorization condition was extended to non-injective PEPS satisfying what is known as the *pulling through condition*, which holds in the case of G-injective and MPO-injective PEPS. Unfortunately the PEPS representing the thermofield double state $|\rho_\beta^{1/2}\rangle$ will neither be injective nor satisfy such condition. At the same time, it will turn out to have some stronger property which will make up for the lack of it: it can be well approximated by a tensor product operator. We will now present the necessary modifications to the results of [20] required to treat this case.

There are three geometrical cases we need to consider in our decomposition of the torus $\mathbb{Z}_{N+1} \times \mathbb{Z}_{N+1}$ into sub-regions: two cylinders to cover the torus, two rectangles to cover a cylinder, and two rectangles to cover a rectangle. The following theorem is an adaptation of [20, Theorem 10] that covers each of these three cases.

Theorem 5.6. *Let A, B, C be three disjoint regions of Λ such that A and C do not share mutually contractible boundary indices. Let us moreover assume that the (boundary) virtual indices of ABC, AB, BC and B can be arranged into four sets a, c, α, γ so that*

$$\partial A \setminus \partial B \subset a \quad , \quad \partial C \setminus \partial B \subset c \quad , \quad \partial A \cap \partial B \subset \alpha \quad , \quad \partial C \cap \partial B \subset \gamma$$

$$\partial ABC = ac \quad , \quad \partial AB = a\gamma \quad , \quad \partial BC = \alpha c \quad , \quad \partial B = \alpha\gamma \,.$$

Let us also assume that the orthogonal projections $J_{\partial R}$ defined above admit a factorization in terms of projections $J_a, J_c, J_\alpha, J_\gamma$ (subindices indicate their corresponding support), namely

$$J_{\partial ABC} = J_a \otimes J_c \quad , \quad J_{\partial AB} = J_a \otimes J_\gamma \quad , \quad J_{\partial BC} = J_\alpha \otimes J_c \quad , \quad J_{\partial B} = J_\alpha \otimes J_\gamma \,,$$

and there also exist positive semi-definite operators $\sigma_a, \sigma_c, \sigma_\alpha, \sigma_\gamma$ with full-rank on $J_\alpha, J_{\alpha'}, J_\gamma, J_{\gamma'}$ such that

$$\sigma_{\partial ABC} := \sigma_a \otimes \sigma_c \quad , \quad \sigma_{\partial AB} := \sigma_a \otimes \sigma_\gamma \quad , \quad \sigma_{\partial BC} := \sigma_\alpha \otimes \sigma_c \quad , \quad \sigma_{\partial B} := \sigma_\alpha \otimes \sigma_\gamma$$

satisfy

$$\|J_{\partial\mathcal{R}} - \rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2}\| < \varepsilon \quad , \quad \mathcal{R} \in \{ABC, AB, BC\} \,,$$

$$\|J_{\partial B} - \rho_{\partial B}^{-1/2} \sigma_{\partial B} \rho_{\partial B}^{-1/2}\| < \varepsilon.$$

Then,

$$\|P_{AB} P_{BC} - P_{ABC}\| \, < \, 3\varepsilon(1+\varepsilon) \,.$$

In case the region $\mathcal{R}$ consists of the whole lattice (e.g. torus) then a and c would be empty. In this case σ_a and σ_c are simply scalars.

Proof. Let us define the approximate projections

$$Q_{\mathcal{R}} := V_{\mathcal{R}} \sigma_{\partial\mathcal{R}}^{-1} V_{\mathcal{R}}^\dagger \quad , \quad \mathcal{R} \in \{BC, AB, ABC\} \,.$$

which satisfy

$$\|P_{\mathcal{R}} - Q_{\mathcal{R}}\| = \|J_{\partial\mathcal{R}} - \rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2}\| < \varepsilon \,. \tag{38}$$

We are going to denote by $V_{C\to B}$ the tensor obtained from V_C by taking all input indices that connect with B into output indices, so that

$$V_{ABC} = V_{AB} V_{C\to B} \quad \text{and} \quad V_{BC} = V_B \, V_{C\to B}$$

Analogously, we define $V_{A\to B}$ satisfying

$$V_{ABC} = V_{BC} V_{A\to B} \quad \text{and} \quad V_{AB} = V_B V_{A\to B}$$

Then, we can rewrite

$$Q_{ABC} = V_{ABC}\sigma_{\partial ABC}^{-1}V_{ABC}^{\dagger} = V_{ABC}\sigma_a^{-1}\sigma_c^{-1}V_{ABC}^{\dagger}$$
$$= V_{AB}V_{C\to B}\sigma_a^{-1}\sigma_c^{-1}V_{A\to B}^{\dagger}V_{BC}^{\dagger}$$
$$= V_{AB}\sigma_a^{-1}V_{C\to B}\,V_{A\to B}^{\dagger}\,\sigma_c^{-1}V_{BC}^{\dagger}$$

At this point, we can use the local structure of the projections to write $V_{AB} = V_{AB}J_{\partial AB} = V_{AB}J_{\partial AB}J_{\gamma} = V_{AB}J_{\gamma} = V_{AB}\sigma_{\gamma}\sigma_{\gamma}^{-1}$. Analogously, $V_{BC} = V_{BC}\sigma_{\alpha}\sigma_{\alpha}^{-1}$. Inserting both identities above, we can rewrite

$$Q_{ABC} = V_{AB}\sigma_a^{-1}\sigma_{\gamma}^{-1}\,V_{A\to B}^{\dagger}\sigma_{\gamma}\sigma_{\alpha}\,V_{C\to B}\,\sigma_{\alpha}^{-1}\sigma_c^{-1}V_{BC}^{\dagger}$$
$$= V_{AB}\,\sigma_{\partial AB}^{-1}\,V_{A\to B}^{\dagger}\,\sigma_B\,V_{C\to B}\,\sigma_{\partial BC}^{-1}\,V_{BC}^{\dagger}$$

Similarly, we handle

$$Q_{AB}Q_{BC} = V_{AB}\sigma_{\partial AB}^{-1}V_{AB}^{\dagger}\,V_{BC}\sigma_{\partial BC}^{-1}V_{BC}^{\dagger}$$
$$= V_{AB}\sigma_{\partial AB}^{-1}V_{A\to B}^{\dagger}\,V_B^{\dagger}\,V_B\,V_{C\to B}\,\sigma_{\partial BC}^{-1}V_{BC}^{\dagger}$$
$$= V_{AB}\sigma_{\partial AB}^{-1}V_{A\to B}^{\dagger}\,\rho_{\partial B}\,V_{C\to B}\,\sigma_{\partial BC}^{-1}V_{BC}^{\dagger}$$

To compare the expressions for Q_{ABC} and $Q_{AB}\,Q_{BC}$ we introduce

$$\Delta_{AB} := V_{AB}\,\sigma_{\partial AB}^{-1}\,V_{A\to B}^{\dagger}\,\rho_{\partial B}^{1/2}\,,$$
$$\Delta_{BC} := V_{BC}\,\sigma_{\partial BC}^{-1}\,V_{C\to B}^{\dagger}\,\rho_{\partial B}^{1/2}\,.$$

It is easy to check that

$$Q_{ABC} - Q_{AB}Q_{BC} = \Delta_{AB}\big(\,J_{\partial B}\; - \;(\rho_{\partial B}^{-1/2}\sigma_{\partial B}\rho_{\partial B}^{-1/2})\,\big)\Delta_{BC}^{\dagger}\;. \tag{39}$$

Since $Q_{BC} = \Delta_{BC}\Delta_{BC}^{\dagger}$, we can apply (38) to estimate

$$\|\Delta_{BC}\|^2 \leq \|Q_{BC}\| \leq 1+\varepsilon\,.$$

Analogously $Q_{AB} = \Delta_{AB}\Delta_{AB}^{\dagger}$ and so $\|\Delta_{AB}\|^2 \leq 1+\varepsilon$. Combining these inequalities with (39), we get

$$\|Q_{ABC}\, - \,Q_{AB}Q_{BC}\| \;\leq\; (1\,+\,\varepsilon)\,\|J_{\partial B}\, - \,(\rho_{\partial B}^{-1/2}\sigma_{\partial B}\rho_{\partial B}^{-1/2})\,\| \;\;\leq\; \varepsilon(1\,+\,\varepsilon). \tag{40}$$

Finally, we combine the previous inequality with (38) to conclude

$$\begin{aligned}\|P_{ABC} - P_{AB}P_{BC}\| &\leq \|P_{ABC} - Q_{ABC}\| + \|Q_{ABC} - Q_{AB}Q_{BC}\| \\ &\quad + \|P_{AB}P_{BC} - Q_{AB}Q_{BC}\| \\ &\leq \varepsilon + \varepsilon(1+\varepsilon) + \|P_{AB} - Q_{AB}\|\,\|P_{BC}\| \\ &\quad + \|Q_{AB}\|\,\|P_{BC} - Q_{BC}\| \\ &\leq \varepsilon + \varepsilon(1+\varepsilon) + \varepsilon + (1+\varepsilon)\varepsilon \,\leq\, 3\varepsilon(1+\varepsilon),\end{aligned}$$

which gives the result. □

5.2.2 Gauge invariance of the approximate factorization condition

An interesting observation omitted in [20], is that the property of ε-approximately factorization is gauge invariant if the transformation does not change the support of the boundary state. Indeed, for every lattice site $x \in \Lambda$ and every edge e incident to x, let us fix an invertible matrix $\mathcal{G}(e,x) \in \mathbb{C}^D \otimes \mathbb{C}^D$. We assume that for every edge e with vertices x, y we have

$$\mathcal{G}(e,x) = \mathcal{G}(e,y)^{-1}\,. \tag{41}$$

We will simply write $\mathcal{G}(e)$ when the site is clear from the context. Let us assume that we have two PEPS related via this gauge, namely for every site $x \in \Lambda$ we have that the local tensors $\widetilde{V}_x$ and V_x are related via (see Figure 4)

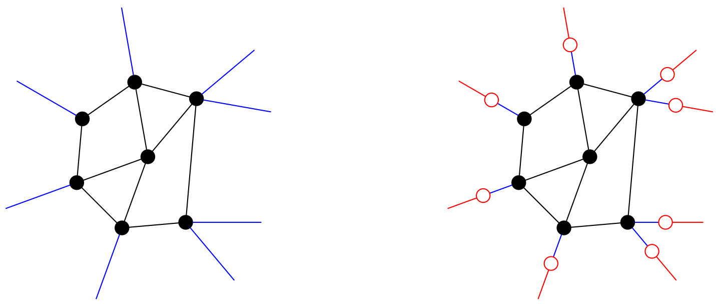
$$V_x = \widetilde{V}_x \circ \mathcal{G}_x \qquad \text{where} \qquad \mathcal{G}_x := \bigotimes_{e \sim x} \mathcal{G}(e,x)\,.$$

(a) $\tilde{V}_x$ (b) V_x

Figure 4: Tensors with (right) and without (left) gauge.

For a region $\mathcal{R} \subset \Lambda$, when contracting indices to construct $V_\mathcal{R}$ we have, as a consequence of (41), that contracting inner edges of $\mathcal{R}$ cancel the gauge matrices. Thus $\widetilde{V}_\mathcal{R}$ and $V_\mathcal{R}$ are related via (see Figure 5):

$$V_\mathcal{R} = \widetilde{V}_\mathcal{R} \circ \mathcal{G}_{\partial\mathcal{R}} \qquad \text{where} \qquad \mathcal{G}_{\partial\mathcal{R}} := \bigotimes_{\substack{x \in \mathcal{V}_{\partial\mathcal{R}} \\ e \in \mathcal{E}_{\partial\mathcal{R}} \cap \mathcal{E}_{\partial x}}} \mathcal{G}(e,x) \tag{42}$$



(a) $\tilde{V}_{\mathcal{R}}$ (b) $V_{\mathcal{R}}$

Figure 5: Tensor network with (right) and without (left) gauge (physical indices are not shown).

where we are denoting by $\mathcal{E}_{\partial\mathcal{R}}$ the set of edges incident to $\mathcal{R}$ but not inner, and $\mathcal{V}_{\partial\mathcal{R}}$ is the set of vertices having at least one incident edge in $\mathcal{E}_{\partial\mathcal{R}}$. The boundary state after the change of gauge is transformed as

$$\rho_{\partial\mathcal{R}} = \mathcal{G}_{\partial\mathcal{R}}^{\dagger} \widetilde{\rho}_{\partial\mathcal{R}} \mathcal{G}_{\partial\mathcal{R}} \,,$$

where $\widetilde{\rho}_{\partial\mathcal{R}} = \widetilde{V}_{\mathcal{R}}^{\dagger} \widetilde{V}_{\mathcal{R}}$.

Proposition 5.7. *Assume that* $[J_{\partial\mathcal{R}}, \mathcal{G}_{\partial\mathcal{R}}] = 0$. *Let* $\sigma_{\partial\mathcal{R}}$ *supported on* $J_{\partial\mathcal{R}}$, *and define*

$$\widetilde{\sigma}_{\partial\mathcal{R}} = \mathcal{G}_{\partial\mathcal{R}}^{\dagger}\, \sigma_{\partial\mathcal{R}}\, \mathcal{G}_{\partial\mathcal{R}}.$$

Then $\widetilde{\sigma}_{\partial\mathcal{R}}$ *is also supported on* $J_{\partial\mathcal{R}}$ *and it holds that*

$$\left\| \rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2} - J_{\partial R} \right\|_{\infty} = \left\| \widetilde{\rho}_{\partial\mathcal{R}}^{1/2} \widetilde{\sigma}_{\partial\mathcal{R}}^{-1} \widetilde{\rho}_{\partial\mathcal{R}}^{1/2} - J_{\partial R} \right\|_{\infty} .$$

Proof. Since $J_{\partial\mathcal{R}}$ and $\mathcal{G}_{\partial\mathcal{R}}$ commute, we have that $\rho_{\partial\mathcal{R}}$, $\widetilde{\rho}_{\partial\mathcal{R}}$, $\sigma_{\partial\mathcal{R}}$ and $\widetilde{\sigma}_{\partial\mathcal{R}}$ all have the same support, namely $J_{\partial\mathcal{R}}$, and so

$$\widetilde{\sigma}_{\partial\mathcal{R}}^{-1} = \mathcal{G}_{\partial\mathcal{R}}^{-1}\, \sigma_{\partial\mathcal{R}}^{-1}\, \mathcal{G}^{\dagger\,-1} \,.$$

Therefore, if $P_{\mathcal{R}}$ denotes the orthogonal projection onto $\mathrm{Im}(V_{\mathcal{R}}) = \mathrm{Im}(\widetilde{V}_{\mathcal{R}})$, then we can write

$$P_{\mathcal{R}} = V_{\mathcal{R}}\, \rho_{\partial\mathcal{R}}^{-1}\, V_{\mathcal{R}}^{\dagger} = \widetilde{V}_{\mathcal{R}}\, \widetilde{\rho}_{\partial\mathcal{R}}^{-1}\, \widetilde{V}_{\mathcal{R}}^{\dagger} .$$

From the definition of $\widetilde{\sigma}_{\partial\mathcal{R}}$, we similarly see that

$$V_{\mathcal{R}} \sigma_{\partial\mathcal{R}}^{-1}\, V_{\mathcal{R}}^{\dagger} = \widetilde{V}_{\mathcal{R}}\, \widetilde{\sigma}_{\partial\mathcal{R}}^{-1}\, \widetilde{V}_{\mathcal{R}}^{\dagger}.$$

The statement then follows from the fact that

$$\begin{aligned} W_{\mathcal{R}}(\rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2} - J_{\partial R}) W_{\mathcal{R}}^{\dagger} &= V_{\mathcal{R}} \sigma_{\partial\mathcal{R}}^{-1} V_{\mathcal{R}}^{\dagger} - V_{\mathcal{R}} \rho_{\partial\mathcal{R}}^{-1} V_{\mathcal{R}} \\ &= \widetilde{V}_{\mathcal{R}} \widetilde{\sigma}_{\partial\mathcal{R}}^{-1} \widetilde{V}_{\mathcal{R}}^{\dagger} - \widetilde{V}_{\mathcal{R}} \widetilde{\rho}_{\partial\mathcal{R}}^{-1} \widetilde{V}_{\mathcal{R}} \\ &= \widetilde{W}_{\mathcal{R}} (\widetilde{\rho}_{\partial\mathcal{R}}^{1/2} \widetilde{\sigma}_{\partial\mathcal{R}} \widetilde{\rho}_{\partial\mathcal{R}}^{1/2} - J_{\partial R}) \widetilde{W}_{\mathcal{R}} \end{aligned}$$

where $W_{\mathcal{R}} = V_{\mathcal{R}} \rho_{\partial\mathcal{R}}^{-1/2}$ and $\widetilde{W}_{\mathcal{R}} = \widetilde{V}_{\mathcal{R}} \widetilde{\rho}_{\partial\mathcal{R}}^{-1/2}$ are isometries. □

Corollary 5.8. *Let A, B, C be three regions of $\mathcal{V}$ as in the definition of approximate factorization, and assume that $[J_{\partial\mathcal{R}}, G_{\partial\mathcal{R}}] = 0$ for $\mathcal{R} \in \{ABC, AB, BC, B\}$. If the PEPS generated by $\widetilde{V}_x$ is ε-approximately factorizable, then so does the PEPS generated by V_x.*

Proof. Let us assume then that the PEPS with local tensors $\widetilde{V}_x$ is ε-approximately factorizable. Because of Proposition 5.7, it is sufficient to verify that $\sigma_{\partial\mathcal{R}}$ satisfies the necessary locality properties. If $\widetilde{J}_{\partial\mathcal{R}}$ and $\widetilde{\sigma}_{\partial\mathcal{R}}$ are product operators (as in Section 5.2.1), then so are $J_{\partial\mathcal{R}}$ and $\sigma_{\partial\mathcal{R}}$, and there is nothing to prove.

Let us now consider the case in which $\widetilde{\sigma}_{\partial\mathcal{R}}$ is not in a tensor product form (as in Definition 5.4). Let a, α, z, γ, c be the regions dividing the boundaries $\partial ABC = azc$, $\partial AB = az\gamma$, $\partial BC = \alpha zc$, $\partial B = \alpha z\gamma$ and let $\widetilde{\Delta}_{az}, \widetilde{\Delta}_{zc}, \widetilde{\Omega}_{\alpha z}, \widetilde{\Omega}_{z\gamma}$ be the corresponding matrices. Note that the gauge matrices $\mathcal{G}_{\partial\mathcal{R}}$ can be rearranged according to the boundary subregions, e.g.

$$\mathcal{G}_{\partial ABC} = \mathcal{G}_{azc} = \mathcal{G}_{az}\mathcal{G}_{c} = \mathcal{G}_{a}\mathcal{G}_{zc} \,.$$

If we define

$$\begin{aligned} \Delta_{az} &:= \mathcal{G}_a^\dagger \, \widetilde{\Delta}_{az} \, \mathcal{G}_{az}, \quad & \Delta_{zc} &:= \mathcal{G}_{zc}^\dagger \, \widetilde{\Delta}_{zc} \, \mathcal{G}_c \\ \Omega_{\alpha z} &:= \mathcal{G}_\alpha^\dagger \, \widetilde{\Omega}_{\alpha z} \mathcal{G}_{\alpha z}, \quad & \Omega_{z\gamma} &:= \mathcal{G}_{z\gamma}^\dagger \, \widetilde{\Omega}_{z\gamma} \, \mathcal{G}_\gamma \end{aligned} \tag{43}$$

then, we can directly check that $\sigma_{\partial\mathcal{R}}$ satisfies

$$\begin{aligned} \sigma_{\partial AB} &:= \Omega_{z\gamma}\Delta_{az}, \qquad & \sigma_{\partial BC} &:= \Delta_{zc}\Omega_{\alpha z}, \\ \sigma_{\partial ABC} &:= \Delta_{zc}\Delta_{az}, \qquad & \sigma_{\partial B} &:= \Omega_{z\gamma}\Omega_{\alpha z}. \end{aligned}$$

□

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