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Virtual Special Issue – In memory of Professor Sibe Mardesic

# Ultrametrics on Čech homology groups

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ABSTRACT

This paper is devoted to introducing additional structure on Čech homology groups. First, we redefine the Čech homology groups in terms of what we have called approximative homology by using approximative sequences of cycles, just as Borsuk introduced shape groups using approximative maps. From this point on, we are able to construct complete ultrametrics on Čech homology groups. The uniform type (and then the group topology) generated by the ultrametric leads to a shape invariant which we use to deduce topological consequences.



## 1. Introduction

During the 1990s we developed our program of equipping the sets of shape morphisms between spaces of useful structures in the sense that it could help us find new results in shape theory, as well as help to reinterpret known results in terms of such structures. See, in chronological order of conception, [18,19,10,20,21,9]. Although [9] was published recently, in fact it was presented as a talk, with the same title as the paper in the *II Congreso Iberoamericano de Topología y sus aplicaciones* hold at Morelia, Mexico, March 20-22, 1997. In [18,19] we developed the compact metrizable case using the Hilbert cube as ambient space

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[1] After the decision of the three authors to submit this paper to this Special issue honoring the memory of Sibe Mardesic, the first author Antonio Giraldo and his wife Sonia died in a car accident on August 24, 2016. The rest of the authors decided, painfully, to respect the initial decision.

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and Borsuk's shape theory. In particular, if the domain space is $(S^1, 1)$, the construction made in [18] allows us to give an ultrametric to the shape group $\check{\pi}_1(X, x_0)$ of a compact metrizable space $X$, as it was observed in [19] and extended later in [9] and in [25]. The main idea was to measure the closeness of two shape morphisms using a weighted version of the homotopy relation between maps into the Hilbert cube. The main tool was the use of *approximative maps* to represent shape morphisms. We recommend reading [3,4] for information about this, or any other concept in shape theory for compacta (compact metrizable spaces) used here such as *fundamental sequence*, *shape morphism*, *FANR-space*, *shape domination*, etc. In [10,9] we extended our results to the general setting of topological spaces using the description of shape theory given in [17] and some extensions of the concept of ultrametric introduced in [23,24]. In particular in [19] we described and used ultrametrics on shape groups which induce natural pseudo-ultrametrics on homotopy groups as deduced automatically from the first few lines of page 69 in [18].

Some years later a paper appeared [1], having [11] as an old antecedent, trying to give a topology on fundamental groups to convert them into topological groups with analogous objectives such as those described in the previous paragraph. Due to the amount of errors in [1] very recently the Journal of Topology and its Applications published a *Retraction Notice* to this article. Even so, the appearance of [1] attracted some authors to use topologies on the fundamental group and on the shape groups to get results in so called *wild topology*. Many of them found correct, significant and/or beautiful results. To find information on these results, one can look for citations to [1]; for example in Google Scholar. We also suggest, for example, [5–7], [22] and their references for the same purpose. We also have to say that, for example, in [8] were some of the topological constructions made in [10] or in [9] re-introduced again to get a topology on the fundamental group.

With the above antecedents we decided to look for appropriate ultrametrics for Čech homology groups following the line initiated in [18]. In this paper we construct ultrametrics for Čech homology groups of compacta. The natural extension for arbitrary topological spaces and the treatment of Čech cohomology groups using Pontryagin duality is developed in the Ph.D. thesis [26] of the third author co-supervised by the first two authors. There, the concept of generalized ultrametric in the sense of Priess-Crampe and Ribenboim [23,24] was also used.

In section 2 we define the necessary concept of *approximative cycles*, to represent Čech homology classes. In section 3, this paper's principal section, we define a weighted relation of *homologous approximative cycles*. This is the way to get a framework to apply a type of Cantor completion process to get complete ultrametrics on the Čech homology groups. As in [18], we use the Hilbert cube as ambient space. The uniform type generated by this ultrametric is a shape invariant. We also give a collection of properties of the topological group obtained from it. In addition, we give some topological conclusions about the compact metrizable space $X$. Moreover, we show in section 4 the correspondent uniform-topological version of the classical Hurewicz homomorphism.

It must be said as a warning that in order to unify results and proofs we are always considering the *reduced singular homology groups* in this paper. If we do not assume this, then some definitions, results and proofs should be changed accordingly for the case $n = 0$.

An analogous problem to introducing topologies on the Čech co-homology groups was also developed in [26]. There, it was pointed out that the unique *coherent* topology on the Čech co-homology groups of compacta, with integer coefficients, is the discrete one. This means that one can not hope for extra-help from the additional topological structure to solve problems depending on the algebraic structure of Čech co-homology groups, i.e. the algebraic structure is sufficient to get the same results. This is the reason why in [26] the Čech co-homology with coefficients in the circle $S^1$ was treated to get new shape invariants. We have to say that the topology found in [26] is intrinsically related to the metrical construction made here by means of Pontryagin duality. We suggest reading Keesling's very interesting papers [14–16] for previous use of Pontryagin duality in shape theory.

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## 2. Approximative homology

In order to fix the notation used throughout this paper, let $\Delta^n = [a_0, ..., a_n]$ be the geometric n-simplex in $\mathbb{R}^n$ and let $G$ be an Abelian group. We shall take $G = \mathbb{Z}$. If $X$ is a topological space, an *n-simplex in X* is a continuous function $\sigma : \Delta^n \to X$. An *n-chain in X* is a (finite) formal sum of the form

$$\sum_{i=1}^{k} n_i\sigma_i = n_1\sigma_1 + n_2\sigma_2 + ... + n_k\sigma_k$$

where each $\sigma_i$ is an n-simplex in $X$ and $n_i$ takes values in $G$ for all i. Given two n-chains $\sigma$ and $\tau$ their sum $\sigma + \tau$ is defined as the formal sum of the corresponding addends of each chain. We will denote by $\mathcal{C}_n(X, G)$ the group of all n-chains in $X$ with coefficients in $G$, or shortly $\mathcal{C}_n(X)$ if $G = \mathbb{Z}$.

Given an n-simplex $\sigma : \Delta^n \to X$, the *boundary of* $\sigma$ is an (n-1)-chain defined as follows:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[a_0,...,\hat{a}_i,...,a_n]}$$

where $a_0, ..., a_n$ are the vertices of $\Delta^n$ and, as usual, $[a_0, ..., \hat{a}_i, ..., a_n]$ is the proper face of $\Delta^n$ that does not contain the vertex $a_i$.

This boundary defined over n-simplices, is extended over all n-chains by linearity, i.e. if $\sigma = \sum_{i=1}^{k} n_i\sigma_i$ is a n-chain, its boundary is

$$\partial\sigma = \partial\big(\sum_{i=1}^{k} n_i\sigma_i\big) = \sum_{i=1}^{k} n_i\partial\sigma_i.$$

Hence, we have a boundary map

$$\partial : \mathcal{C}_n(X, G) \to \mathcal{C}_{n-1}(X, G).$$

If an n-chain $\sigma$ satisfies that $\partial\sigma = 0$, we say that $\sigma$ is an *n-cycle* and if there exists an (n+1)-chain $\gamma$ such that $\partial\gamma = \sigma$ we say that $\sigma$ is an *n-boundary*. We denote by $Z_n(X, G)$ and $B_n(X, G)$ the groups of cycles and boundaries respectively. It is obvious that $Z_n(X, G) = Ker\partial$ and $B_n(X, G) = Im\partial$ (in the corresponding dimensions). Hence, we have arrived to the classical definition of the singular homology group of $X$,

$$H_n(X, G) = \frac{Ker\partial}{Im\partial}.$$

It is also well-known that every continuous map $f : X \to Y$ induces a *chain map* $f_\# : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ such that $f_\#\partial = \partial f_\#$ and so also an homomorphism $f_* : H_n(X) \to H_n(Y)$ between singular homology groups is induced.

With every topological space $X$ one can associate an inverse system $\mathbf{C}(X) = \{X_\mathcal{U}, p_{\mathcal{U}\mathcal{U}'}, \Lambda\}$ called *the Čech system of* $X$. As we shall briefly recall here, this inverse system is in the category **HPol** of polyhedra (and continuous maps) up to homotopy.

The indexing set $\Lambda$ is the set of all normal coverings $\mathcal{U}$ of $X$ ordered by the relation of refinement of coverings ($\mathcal{U} \le \mathcal{U}'$ if and only if $\mathcal{U}'$ refines $\mathcal{U}$). Recall that a *normal covering* is an open covering $\mathcal{U}$ of $X$ which admits a partition of unity subordinated to $\mathcal{U}$.

Each term $X_\mathcal{U}$ is the nerve $|N(\mathcal{U})|$ associated to the covering $\mathcal{U}$. In $|N(\mathcal{U})|$ there exists a vertex corresponding to each set $U \in \mathcal{U}$ and $\{U_1, ..., U_n\}$ expand an $n$-simplex if and only if $U_1 \cap \cdots \cap U_n \neq \emptyset$.

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Finally, for $\mathcal{U} \leq \mathcal{U}'$, let $p_{\mathcal{U}\,\mathcal{U}'}$ be the simplicial projection $p_{\mathcal{U}\,\mathcal{U}'} : |N(\mathcal{U}')| \to |N(\mathcal{U})|$ sending a vertex $U' \in \mathcal{U}'$ to a vertex $U \in \mathcal{U}$ with $U' \subseteq U$. This map is not uniquely determined, but any other projection between the nerves is homotopic to this. Hence, the projection is unique up to homotopy.

Similarly, for $\mathcal{U} \in \Lambda$, $p_{\mathcal{U}} : X \to X_{\mathcal{U}}$ is a canonical projection uniquely determined up to homotopy. Furthermore, $p_{\mathcal{U}\,\mathcal{U}'} \circ p_{\mathcal{U}'} \simeq p_{\mathcal{U}}$.

If we apply the simplicial homology functor to the Čech inverse system, we get an inverse system of groups:

$$\mathbf{H_n}(\mathbf{X}) = \{H_n(X_\lambda), p_{\lambda\,\lambda'*}, \Lambda\}$$

where $p_{\lambda\,\lambda'*}$ is the corresponding induced map in homology. The inverse limit of this system

$$\check{H}_n(X) = \lim \mathbf{H}_n(X) = \varprojlim \{H_n(X_\lambda), p_{\lambda\,\lambda'*}, \Lambda\}$$

is called *Čech homology group* of $X$. This is also a functor which, for a continuous map $f : X \to Y$, induces an homomorphism $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.

Similarly, for cohomology we obtain a direct system of groups:

$$\mathbf{H}^n(X) = \{H^n(X_\lambda), p^*_{\lambda\,\lambda'}, \Lambda\}$$

where $p^*_{\lambda\,\lambda'}$ is the induced map in cohomology. The direct limit of this system

$$\check{H}^n(X) = \lim \mathbf{H}^n(X) = \varinjlim \{H^n(X_\lambda), p^*_{\lambda\,\lambda'}, \Lambda\}$$

is called *Čech cohomology group* of $X$.

In the particular case of compact metrizable spaces, it is possible to regain Čech homology and cohomology groups with a more geometrical inverse system of topological spaces. We shall call this special inverse system as *Borsuk's inverse system* and its construction is as follows: given a compact metrizable space $X$, it can be considered as a closed subset of the Hilbert cube $Q$, with a prefixed metric. Thus, for a sequence of real numbers $\{\varepsilon_k\}$ decreasing to zero, we can consider the balls of this metric $X_k = B(X, \varepsilon_k)$ as terms and the inclusion maps $i_{k\,k+1} : X_{k+1} \to X_k$ as bonding maps. Hence, $\{X_k, i_{k\,k+1}\}$ is an inverse sequence with inverse limit $\cap X_k = X$. Consequently,

$$\check{H}_n(X) = \varprojlim \{H_n(X_k), (p_{k\,k+1})_*\}$$

and

$$\check{H}^n(X) = \varinjlim \{H^n(X_k), p^*_{k\,k+1}\}.$$

It is necessary to define concepts of approximative sequences for cycles and a relation of homology between them. Actually, here we shall make use of the expansion associated to a compact metrizable space $X$ by its neighborhoods in the Hilbert cube $Q$. Let us introduce a suitable way to read the Čech homology, similar to Borsuk's construction of shape groups, using the neighborhoods of $X$.

**Definition 2.1.** Let $Z_n(Q)$ be the set of all n-dimensional cycles in $Q$. Given $A \subseteq Q$, we say that $\sigma, \sigma'$ in $Z_n(Q)$ are *homologous in A* if there exists $\gamma$ in $\mathcal{C}_{n+1}(A)$ such that $\partial\gamma = \sigma - \sigma'$. We shall denote it by $\sigma \sim \sigma'$ in $A$.

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**Remark 2.2.** The previous definition implies that each simplex of the $(n+1)$-chain $\gamma$ lies in $A$. In particular, the simplices of $\sigma$ and $\sigma'$ must lie in $A$. Otherwise, it is impossible that $\sigma$ and $\sigma'$ would be homologous in $A$. Also note that, for $n \geq 1$, every pair of cycles are homologous in $Q$, because $H_n(Q,G) = 0$ (since $Q$ is contractible).

As we have already mentioned, we follow the construction of Borsuk for the shape groups in order to obtain a more geometrical interpretation of the Čech homology groups.

**Definitions 2.3.** Let $X$ be a closed subspace space of $Q$.

a) A sequence $\sigma = \{\sigma_i\}_{i\in\mathbb{N}}$ in $\mathcal{C}_n(Q)$ is said to be an *infinite n-chain (in $Q$) for $X$* if there exists a sequence $\{\varepsilon_i\} \to 0$ such that $\sigma_i$ is an n-chain in $B(X,\varepsilon_i)$ for each $i \in \mathbb{N}$. (If it is clear, we shall omit the dimension $n$ of the chains in the sequel.)

b) The *boundary of an infinite chain* is defined as the classical boundary acting in each chain of the sequence, i.e.,

$$\partial\sigma = \partial\{\sigma_i\} = \{\partial\sigma_i\}$$

for an infinite chain $\sigma$. An infinite chain $\sigma$ is called *infinite cycle* if $\partial\sigma = 0$, or equivalently, if each chain of the sequence is a cycle.

c) An infinite cycle $\sigma = \{\sigma_i\}$ is said to be an *approximative cycle* if for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i+1}$ in $B(X,\varepsilon)$ for all $i \geq i_0$, or equivalently, if there exists an infinite chain $\gamma = \{\gamma_i\}$ such that $\partial\gamma_i = \sigma_i - \sigma_{i+1}$. We shall denote by $\mathcal{Z}_n^{\mathcal{A}}(X)$ the set of approximative cycles of dimension $n$.

d) Given two approximative cycles $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$, a sum $\sigma + \tau$ is defined as the (usual) sum of singular chains in each level, i.e.,

$$\sigma + \tau = \{\sigma_i\} + \{\tau_i\} = \{\sigma_i + \tau_i\}.$$

Immediately, $\mathcal{Z}_n^{\mathcal{A}}(X)$ with this operation is a group.

e) Two approximative cycles $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$ are said to be *homologous approximative cycles* if for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X,\varepsilon)$ for all $i \geq i_0$, or equivalently, if there exists an infinite chain $\gamma = \{\gamma_i\}$ such that $\partial\gamma = \sigma - \tau$. If an approximative cycle $\sigma$ is homologous to zero, $\sigma \sim 0$, i.e. if there exists an infinite chain $\gamma$ such that $\partial\gamma = \sigma$, we say that $\sigma$ is an *approximative boundary.* We shall denote by $\mathcal{B}_n^{\mathcal{A}}(X)$ the subgroup of $\mathcal{Z}_n^{\mathcal{A}}(X)$ of approximative boundaries of dimension $n$.

f) The *n-dimensional approximative homology group of $X$* is defined as the quotient group

$$H_n^{\mathcal{A}}(X) = \frac{\mathcal{Z}_n^{\mathcal{A}}(X)}{\mathcal{B}_n^{\mathcal{A}}(X)}.$$

It is clear how to define an operation between homology classes of approximative cycles, just as the sum of its representatives,

$$[\sigma] + [\tau] = [\{\sigma_i\}] + [\{\tau_i\}] = [\{\sigma_i\} + \{\tau_i\}] = [\{\sigma_i + \tau_i\}] = [\sigma + \tau].$$

The following fact is obvious.

**Proposition 2.4.** *$(H_n^{\mathcal{A}}(X), +)$ is an Abelian group.*

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**Remark 2.5.** Every singular cycle of $X$ gives an approximative cycle: if $s = \sum_{i=1}^{k} n_i s_i$ is a cycle where $s_i : \Delta^n \to X$, put $\sigma_i = s$ for every $i \in \mathbb{N}$ in order to obtain a constant sequence $\sigma = \{\sigma_i\}$ which obviously is an approximative cycle. In this case, we say that $\sigma$ *is generated by the singular cycle* $s$ *of* $X$.

Moreover, if $s \sim t$ as $n$-cycles in $X$, then the corresponding approximative cycles $\sigma$ and $\tau$ satisfies $\sigma \sim \tau$ as approximative cycles. In that way, for each $n \geq 0$, it is obtained a homomorphism of Abelian groups

$$\varphi^{\mathcal{A}} : H_n(X) \longrightarrow H_n^{\mathcal{A}}(X)$$

since $\varphi^{\mathcal{A}}([\sigma])$ is a well-defined approximative homology class.

The functorial properties of the approximative homology are resumed in the following results. For this, we follow the description of Borsuk for the (Vietoris) homology properties of shape, but in the framework of approximative cycles. For the definition of *fundamental sequence* between compact metrizable spaces, see [3,4].

**Lemma 2.6.** *If $\{f_k, X, Y\}$ is a fundamental sequence between compact metrizable spaces, then for every approximative cycle $\sigma = \{\sigma_i\}$ for $X$, there exists an increasing sequence $\{i_k\}$ of indices such that for every sequence of indices $\{j_k\}$ such that $j_k \geq i_k$ for $k \in \mathbb{N}$, the sequence $\{f_{k\#}(\sigma_{j_k})\}$ is an approximative cycle for $Y$, where $f_{k\#}$ is the induced map by each $f_k$ in the group of chains of $Q$.*

**Proof.** By definition of fundamental sequence, for every neighborhood $V$ of $Y$ there exists a neighborhood $U$ of $X$ such that $f_k|_U \simeq f_{k+1}|_U$ in $V$ for almost $k$. So for any sequence $\{\eta_k\}$ (without loss of generality it can be supposed that $\eta_k = \frac{diamQ}{k}$) of positive numbers tending to zero, there exists a sequence $\{\delta_k\}$, also of positive numbers converging to zero, such that

$$\text{if } \sigma \text{ is a cycle in } B(X, \delta_k) \text{ then } f_{k_\#}(\sigma) \sim f_{k+1_\#}(\sigma) \text{ as cycles in } B(Y, \eta_k) \text{ for } k \in \mathbb{N},$$

and

$$\text{if } \sigma \sim \sigma' \text{ as cycles in } B(X, \delta_k) \text{ then } f_{k_\#}(\sigma) \sim f_{k+1_\#}(\sigma') \text{ as cycles in } B(Y, \eta_k) \text{ for } k \in \mathbb{N}.$$

Let $\sigma = \{\sigma_i\}$ be an approximative cycle and take $\{\varepsilon_i\}$ such that $\sigma_i$ is a cycle in $B(X, \varepsilon_i)$. This sequence $\{\varepsilon_i\}$ converges to zero because of the definition of approximative cycle. Fix a sequence of indices $\{i_k\}$ such that $\varepsilon_j \leq \delta_k$ for every $j \geq i_k$. Then, if a sequence of indices $\{j_k\}$ satisfies $j_k \geq i_k$ for $k \in \mathbb{N}$, then $\sigma_{j_k}, \sigma_{j_{k+1}}$ are homologous cycles in $B(X, \delta_k)$, so

$$f_{k_\#}(\sigma_{j_{k+1}}) \sim f_{k+1_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k)$$

and

$$f_{k_\#}(\sigma_{j_k}) \sim f_{k_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k).$$

Therefore,

$$f_{k_\#}(\sigma_{j_k}) \sim f_{k+1_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k)$$

and it follows that $\{f_{k_\#}(\sigma_{j_k})\}$ is an approximative cycle for $Y$. □

**Remark 2.7.** From the previous proof, if $\sigma' = \{\sigma'_i\}$ is another approximative cycle homologous to $\sigma$, then the sequence of indices $\{i_k\}$ can be selected so that the inequality $j_k \geq i_k$ for $k \in \mathbb{N}$ implies that $\{f_{k_\#}(\sigma_{j_k})\}$ and $\{f_{k_\#}(\sigma'_{j_k})\}$ are homologous approximative cycles. The approximative homology class of the approximative cycle $\{f_{k_\#}(\sigma_{j_k})\}$ does not depend on the choice of the subsequence.

With the lemma above, we get the next.

**Proposition 2.8.** *Every continuous map $f : X \to Y$ between compact metrizable spaces induces, for each $n \geq 0$, a homomorphism $f_* : H_n^{\mathcal{A}}(X) \to H_n^{\mathcal{A}}(Y)$ between its approximative homology groups. If $g : Y \to Z$ is another continuous map to a compact metrizable space $Z$, then $(g \circ f)_* = g_* \circ f_*$. Moreover, the identity map $i : X \to X$ induces the identity homomorphism $i_* : H_n^{\mathcal{A}}(X) \to H_n^{\mathcal{A}}(X)$.*

**Proof.** Given an approximative class in $H_n^{\mathcal{A}}(X)$ with representative $\{\sigma_i\}$, let us assign the approximative class in $H_n^{\mathcal{A}}(Y)$ with representative $\{f_{k_\#}(\sigma_{j_k})\}$ given by 2.6. In that way, we obtain a map between the corresponding approximative homology groups, that clearly is an homomorphism of groups.[2]

The remaining properties are straightforward to check. □

The viewpoint of the homology from approximative cycles agrees with singular homology when $X$ is an ANR. We need to use the following assertion (see Lemma 3.8 of [4]).

**Lemma 2.9.** *If $r : U \to X$ is a retraction of a neighborhood $U$ of $X$ (in $Q$), then for every neighborhood $V$ of $X$ (in $Q$) there is a neighborhood $V'$ of $X$ (in $Q$) and a homotopy $h : V' \times [0, 1] \to V$ such that $h(x, 0) = x$ and $h(x, 1) = r(x)$ for every point $x \in V'$.*

**Lemma 2.10.** *Let $X$ be an ANR in $Q$. Every approximative cycle $\sigma$ is homologous to an approximative cycle $\tau$ generated by a singular chain of $X$.*

**Proof.** Let $\sigma = \{\sigma_i\} \subset Z_n(Q)$ be an approximative cycle. Since $X$ is an ANR, there exists a neighborhood $U$ of $X$ in $Q$ and a retraction $r : U \to X$. Let us take $\varepsilon_0 > 0$ such that $B(X, \varepsilon_0) \subseteq U$.

For this $\varepsilon_0$, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i+1}$ in $B(X, \varepsilon_0)$ for all $i \geq i_0$. Hence, $\sigma_{i_0} \sim \sigma_i$ in $B(X, \varepsilon_0)$ for all $i \geq i_0$.

The retraction $r$ induces $r_\# : \mathcal{C}_n(U) \to \mathcal{C}_n(X)$ by

$$r_\#(\sum_{i=1}^{k} n_i \gamma_i) = \sum_{i=1}^{k} n_i (r \circ \gamma_i).$$

Let us consider $r_\#(\sigma_{i_0})$, which is a singular cycle in $X$, and put $\tau_i = r_\#(\sigma_{i_0})$ for all $i \in \mathbb{N}$. Thus we have an approximative cycle $\tau = \{\tau_i\}$ generated by the singular chain $r_\#(\sigma_{i_0})$.

The induced infinite chain $r_\#(\sigma) = \{r_\#(\sigma_i)\}$ is again an approximative cycle such that $r_\#(\sigma) \sim \tau$. Indeed:

The relation

$$\sigma_{i_0} \sim \sigma_i \text{ in } B(X, \varepsilon_0) \text{ for all } i \geq i_0$$

implies

$$r_\#(\sigma_{i_0}) \sim r_\#(\sigma_i) \text{ in } X \text{ for all } i \geq i_0,$$

[2] Notice also that homotopic fundamental sequences induce the same homomorphism in homology. We do not need that result and, in fact, it shall be deduced from the isomorphism of the approximative homology to the Čech homology.

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but $r_\#(\sigma_{i_0}) = \tau_i$, so

$$\tau_i \sim r_\#(\sigma_i) \text{ in } X \text{ for all } i \geq i_0,$$

thus, $\tau \sim r_\#(\sigma)$.

On the other hand, using 2.9, for $\varepsilon > 0$ let $\delta > 0$ (we can assume $0 < \delta < \varepsilon < \varepsilon_0$) be such that there exists an homotopy

$$h : B(X, \delta) \times [0, 1] \to B(X, \varepsilon)$$

such that $h(x, 0) = x$ and $h(x, 1) = r(x)$ for every point $x \in B(X, \delta)$. For this $\delta$, there exists $i_1 \in \mathbb{N}$ such that $\sigma_i$ lies in $B(X, \delta)$ for every $i \geq i_1$ (and also in $B(X, \varepsilon)$). If we denote

$$h_0 = h|_{B(X,\delta)\times\{0\}} \quad \text{and} \quad h_1 = h|_{B(X,\delta)\times\{1\}},$$

then

$$\sigma_i = id_\#(\sigma_i) = h_{0\#}(\sigma_i) \sim h_{1\#}(\sigma_i) = r_\#(\sigma_i)$$

in $B(X, \varepsilon)$. Thus, $r_\#(\sigma) \sim \sigma$ and consequently, $\sigma \sim \tau$. □

**Proposition 2.11.** *Let $X$ be an ANR in $Q$. Then, $H_n^{\mathcal{A}}(X)$ is isomorphic to $H_n(X)$, for every $n \geq 0$.*

**Proof.** As it was pointed out in 2.5, every singular homology class generates an approximative one. Moreover, if two cycles are representatives of the same singular class, they are also in the same approximative class. Conversely, by the preceding lemma, every approximative class is generated by a well-defined singular class, so $\psi^{\mathcal{A}}$ is a bijective homomorphism. □

In fact, the groups defined in 2.3 are the same as classical Čech homology groups, as the next proposition shows.

**Proposition 2.12.** *Let $X$ be a closed subspace of $Q$. Then $H_n^{\mathcal{A}}(X)$ is isomorphic to $\check{H}_n(X)$, for each $n \geq 0$.*

**Proof.** Consider Borsuk's inverse system for a decreasing sequence of real numbers tending to zero, $\{\varepsilon_m\}$. That is, for each $m \in \mathbb{N}$ take $X_m = B(X, \varepsilon_m)$, $i_{mm+1}$ as the inclusion of $X_{m+1}$ in $X_m$, and compact ANR-spaces $P_m$ such that

$$B(X, \varepsilon_1) \supseteq P_1 \supseteq B(X, \varepsilon_2) \supseteq P_2 \supset \cdots \supset X.$$

See pages 104-105 in [2].

The inverse limit of this inverse sequence is $\bigcap_{m\in\mathbb{N}} X_m = X$, and therefore the same inverse limit is valid for the sequence of polyhedra and inclusions $p_{m\,m+1}$. Hence, $\check{H}_n(X)$ is the inverse limit of the inverse sequence of groups $\{H_n(P_m), (p_{m\,m+1})_*\}$.

By definition of inverse limit, the elements of $\check{H}_n(X)$ are sequences $\{\sigma_m\}$ such that each $\sigma_m$ lies in $P_m \subseteq B(X, \varepsilon_m)$ and $(p_{mm+1})_*(\sigma_{m+1}) = \sigma_m$. That is, $\sigma_{m+1} \sim \sigma_m$ in $B(X, \varepsilon_m)$ for every $m \in \mathbb{N}$. Therefore, it is clear that every Čech homology class is an approximative one.

On the other hand, let $\sigma = \{\sigma_i\}$ be an approximative cycle. We can choose a subsequence $\sigma' = \sigma_{i_m}$ with $\sigma_{i_m}$ lying in $B(X, \varepsilon_m)$. Obviously, $\sigma'$ is again an approximative cycle and $\sigma \sim \sigma'$. From that, $\sigma_{i_m} \sim \sigma_{i_{m+1}}$ in $B(X, \varepsilon_m)$. Consequently, $\sigma'$ is an element of $\check{H}_n(X)$.

Moreover, it is clear that the group operations of $\check{H}_n(X)$ and $H^{\mathcal{A}}(X)$ are equivalent. □

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**Corollary 2.13.** *Let $X$ be a compact metrizable space, and $\{X_n, p_{n\,n+1}\}$ an inverse sequence of ANRs such that $X = \varprojlim\{X_n, p_{n\,n+1}\}$. Then,*

$$H_k^{\mathcal{A}}(X) = \varprojlim\{H_k^{\mathcal{A}}(X_n), (p_{n\,n+1})_*\}$$

*for each $k \geq 0$.*

**Proof.** From 2.11 and 2.12:

$$H_k^{\mathcal{A}}(X) = \check{H}_k(X) = \varprojlim\{H_k(X_n), (p_{n\,n+1})_*\} = \varprojlim\{H_k^{\mathcal{A}}(X_n), (p_{n\,n+1})_*\}. \qquad \square$$

## 3. Cantor completion process for an ultrametric on $\check{H}_n(X)$

We shall consider the reduced homology groups and reduced cycles in the case $n = 0$ in order to avoid differences with $n \geq 1$ in the developing below. If the reader want to deal with the non-reduced homology group $\check{H}_0(X)$, it should be added

$$F(\sigma, \tau) = 1 + diamQ$$

if $\sigma$ and $\tau$ are not homologous in the Hilbert cube $Q$ in the definition of the map $F$ below.

Over $Z_n(Q)$, we define a map $F : Z_n(Q) \times Z_n(Q) \to \mathbb{R}$ as

$$F(\sigma, \tau) = \inf\{\varepsilon > 0 \,|\, \sigma \sim \tau \text{ in } B(X, \varepsilon)\}.$$

This map is well-defined since the singular reduced homology of the Hilbert cube is zero in any dimension. This implies that if $\varepsilon > diamQ$, then $B(X, \varepsilon) = Q$ and $\sigma \sim \tau$ in $Q$ for every $\sigma, \tau$ in $Z_n(Q)$. Thus,

$$0 \leq F(\sigma, \tau) \leq diamQ$$

for any pair of cycles $\sigma, \tau$ in $Q$.

It can be easily checked that the map $F$ enjoys the following properties for every $\sigma, \tau, \eta$ in $Z_n(Q)$:

i) $F(\sigma, \tau) \geq 0$;
ii) $F(\sigma, \tau) = F(\tau, \sigma)$;
iii) $F(\sigma, \tau) \leq \max\{F(\sigma, \eta), F(\eta, \tau)\}$ (in particular, this implies the triangle inequality).

**Remark 3.1.** In spite of the previous properties, $F$ is not a metric, nor even a pseudometric in $Z_n(Q)$. There are cycles such that $F(\sigma, \sigma) \neq 0$. If we restrict ourselves to $Z_n(X)$ (cycles formed by simplices lying in $X$) we obtain a pseudometric.

**Definitions 3.2.**

a) A sequence $\{\sigma_i\} \subset Z_n(Q)$ is said to be $F$-*Cauchy* if for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \sigma_{i'}) < \varepsilon$ for all $i, i' \geq i_0$.
b) Two $F$-Cauchy sequences $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$ are said to be $F$-*related*, denoted by $\{\sigma_i\}F\{\tau_i\}$, if the sequence $\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3...$ is again $F$-Cauchy.

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**Proposition 3.3.**

a) *The $F$-relation is an equivalence relation.*
b) *For every pair $\{\sigma_i\}, \{\tau_i\}$ of $F$-Cauchy sequences, there exists* $\lim_{i\to\infty} F(\sigma_i, \tau_i)$.
c) *Let $\{\sigma_i\}, \{\sigma'_i\}, \{\tau_i\}, \{\tau'_i\}$ be $F$-Cauchy sequences such that $\{\sigma_i\}F\{\sigma'_i\}$ and $\{\tau_i\}F\{\tau'_i\}$. Then* $\lim_{i\to\infty} F(\sigma_i, \tau_i) = \lim_{i\to\infty} F(\sigma'_i, \tau'_i)$.

**Proof.**

a) Reflexive and symmetric properties are obvious. For the transitive property, given $\{\sigma_i\}F\{\tau_i\}$ and $\{\tau_i\}F\{\omega_i\}$, let us take $\varepsilon > 0$. Then, there exist $i_1$ and $i_2$ realizing $\{\sigma_i\}F\{\tau_i\}$ and $\{\tau_i\}F\{\omega_i\}$, respectively. Also, there exist $i_3$, $i_4$ and $i_5$ for the fact that $\{\sigma_i\}, \{\tau_i\}$ and $\{\omega_i\}$ are $F$-Cauchy sequences. Let us take $i_0 = max\{i_1, i_2, i_3, i_4, i_5\}$. Hence, for $i, i' \geq i_0$, we have:

$$F(\sigma_i, \sigma_{i'}) < \varepsilon$$

$$F(\omega_i, \omega_{i'}) < \varepsilon$$

$$F(\sigma_i, \omega_{i'}) \leq max\{F(\sigma_i, \tau_i), F(\tau_i, \tau_{i'}), F(\tau_{i'}, \omega_{i'})\} < \varepsilon.$$

Thus, $\sigma_1, \omega_1, \sigma_2, \omega_2, \ldots$ is $F$-Cauchy, so $\{\sigma_i\}F\{\omega_i\}$.

b) We claim that $|F(\sigma_i, \tau_i) - F(\sigma_{i'}, \tau_{i'})| \leq F(\sigma_i, \sigma_{i'}) + F(\tau_i, \tau_{i'})$. Assuming this, the sequence $\{F(\sigma_i, \tau_i)\}$ is a Cauchy sequence in $\mathbb{R}$, so it is convergent.
In order to prove the claim, we distinguish two cases:
Case 1: $F(\sigma_i, \tau_i) \geq F(\sigma_{i'}, \tau_{i'})$.

$$\begin{aligned} |F(\sigma_i, \tau_i) - F(\sigma_{i'}, \tau_{i'})| =& F(\sigma_i, \tau_i) - F(\sigma_{i'}, \tau_{i'}) \\ \leq& F(\sigma_i, \sigma_{i'}) + F(\sigma_{i'}, \tau_{i'}) + F(\tau_{i'}, \tau_i) - F(\sigma_{i'}, \tau_{i'}) \\ =& F(\sigma_i, \sigma_{i'}) + F(\tau_i, \tau_{i'}). \end{aligned}$$

Case 2: $F(\sigma_i, \tau_i) \leq F(\sigma_{i'}, \tau_{i'})$.

$$\begin{aligned} |F(\sigma_i, \tau_i) - F(\sigma_{i'}, \tau_{i'})| =& F(\sigma_{i'}, \tau_{i'}) - F(\sigma_i, \tau_i) \\ \leq& F(\sigma_{i'}, \sigma_i) + F(\sigma_i, \tau_i) + F(\tau_i, \tau_{i'}) - F(\sigma_i, \tau_i) \\ =& F(\sigma_i, \sigma_{i'}) + F(\tau_i, \tau_{i'}). \end{aligned}$$

c) It is sufficient to apply b) to the sequences $\{a_i\}$ and $\{b_i\}$ formed respectively by $\sigma_1, \sigma'_1, \sigma_2, \sigma'_2 \ldots$ and $\tau_1, \tau'_1, \tau_2, \tau'_2 \ldots$, that are $F$-Cauchy by hypothesis. $\{F(\sigma_i, \tau_i)\}$ and $\{F(\sigma'_i, \tau'_i)\}$ are subsequences of $\{F(a_i, b_i)\}$, which has limit by b), so $\lim_{i\to\infty} F(\sigma_i, \tau_i) = \lim_{i\to\infty} F(\sigma'_i, \tau'_i)$. □

The next result is easily stated, and it connects with the definitions established in the preceding section.

**Proposition 3.4.**

a) *A sequence $\{\sigma_i\} \subset Z_n(Q)$ is $F$-Cauchy if and only if it is an approximative cycle.*
b) *Two $F$-Cauchy sequences $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$ are $F$-related if and only if $\{\sigma_i\} \sim \{\tau_i\}$ as approximative cycles.*

**Proof.**

a) Let $\{\sigma_i\} \subset Z_n(Q)$ be $F$-Cauchy and $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \sigma_{i'}) < \varepsilon'$ for every $i, i' \geq i_0$. This means that, for each $\delta > \varepsilon'$, $\sigma_i \sim \sigma_{i'}$ in $B(X, \delta)$ for all $i, i' \geq i_0$. In particular, $\sigma_i \sim \sigma_{i+1}$ in $B(X, \varepsilon)$ for every $i \geq i_0$, so $\{\sigma_i\}$ is an approximative cycle.
Conversely, let $\{\sigma_i\}$ be an approximative cycle and $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i'}$ in $B(X, \varepsilon')$ for all $i, i' \geq i_0$.
Let $i, i' \geq i_0$. Without loss of generality, suppose that $i' = i + j$. By the transitivity of the homology relation we obtain

$$\sigma_i \sim \sigma_{i+1} \sim \ldots \sim \sigma_{i+j-1} \sim \sigma_{i+j} = \sigma_{i'}$$

in $B(X, \varepsilon')$. Hence,

$$F(\sigma_i, \sigma_{i'}) = \inf\{\delta > 0 \,|\, \sigma_i \sim \sigma_{i'} \text{ in } B(X, \delta)\} \leq \varepsilon' < \varepsilon$$

so $\{\sigma_i\}$ is $F$-Cauchy.

b) Let $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$ be $F$-Cauchy sequences such that $\{\sigma_i\}F\{\tau_i\}$. Let $\varepsilon > 0$ and take $\varepsilon' < \varepsilon$. Then, there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \tau_i) < \varepsilon'$ for all $i \geq i_0$. That is, for each $\delta > \varepsilon'$, $\sigma_i \sim \tau_i$ in $B(X, \delta)$ for every $i \geq i_0$. In particular, $\sigma_i \sim \tau_i$ in $B(X, \varepsilon)$ for each $i \geq i_0$.
On the other hand, if $\{\sigma_i\}, \{\tau_i\}$ are homologous approximative cycles, they are $F$-Cauchy sequences by the previous point. We check now that the sequences are $F$-related. Let $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X, \varepsilon')$ for every $i \geq i_0$. Hence, $F(\sigma_i, \tau_i) \leq \varepsilon' < \varepsilon$ for all $i \geq i_0$. Therefore, the sequence $\sigma_1, \tau_1, \sigma_2, \tau_2 \ldots$ is $F$-Cauchy, so $\{\sigma_i\}F\{\tau_i\}$. □

The main result of this section is the following.

**Theorem 3.5.** *Let $\check{H}_n(X)$ be the n-dimensional Čech homology group of a compact metrizable space $X$. Given two homology classes in $\alpha, \beta \in \check{H}_n(X)$, define*

$$d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i),$$

*where $\{\sigma_i\}$ and $\{\tau_i\}$ are $F$-Cauchy sequences corresponding to $\alpha$ and $\beta$ respectively. Then, $(\check{H}_n(X), d_X)$ is a complete ultrametric space.*

**Proof.** As it has been proved in 2.12, for every Čech homology class $\alpha$, there exists one and only one class of approximative cycles with representative $\sigma$ associated to $\alpha$.

Let $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$ be approximative cycles representatives of the class of $\alpha$ and $\beta$ respectively. $d_X$ is well-defined since the limit

$$d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i)$$

always exists by 3.3.b) and others representatives $\sigma', \tau'$ in the classes of $\alpha$ and $\beta$ give us the same limit by 3.3.c). Moreover, $d_X$ satisfies the properties of an ultrametric as it shall be shown in the next part.

i) $d_X(\alpha, \beta) \geq 0$ since $F(\cdot, \cdot)$ is always non-negative. In addition, $d_X(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.
Suppose first that $d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i) = 0$. That is, for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X, \varepsilon)$ for all $i \geq i_0$. But that means that $\sigma$ and $\tau$ are homologous as approximative cycles, so that $\alpha = \beta$.

Conversely, if $\alpha = \beta$ then $\{\sigma_i\} \sim \{\tau_i\}$ hence, by 3.4, $\{\sigma_i\}F\{\tau_i\}$, and

$$d_X(\alpha, \beta) = \lim_{i\to\infty} F(\sigma_i, \tau_i) = 0.$$

ii) Since

$$d_X(\alpha, \beta) = \lim_{i\to\infty} F(\sigma_i, \tau_i) = \lim_{i\to\infty} F(\tau_i, \sigma_i) = d_X(\beta, \alpha),$$

then $F$ is symmetric.

iii) Let $\eta = \{\eta_i\}$ be an approximative cycle representing another Čech homology class $\gamma$. From the property iii) of the map $F$,

$$\begin{aligned} d_X(\alpha, \beta) &= \lim_{i\to\infty} F(\sigma_i, \tau_i) \leq \\ &\leq \lim_{i\to\infty} \max\{F(\sigma_i, \eta_i), F(\eta_i, \tau_i)\} = \\ &= \max\{\lim_{i\to\infty} F(\sigma_i, \eta_i), \lim_{i\to\infty} F(\eta_i, \tau_i)\} = \\ &= \max\{d_X(\alpha, \gamma), d_X(\gamma, \beta)\}. \end{aligned}$$

Finally, the completeness of $(\check{H}_n(X), d_X)$ is obtained in the same way as the completion of $\mathbb{R}$ from $\mathbb{Q}$ via Cauchy sequences. □

The equivalence between the approximative homology and Čech homology, allows to give an easy interpretation of the meaning of the ultrametric that we have defined. Recall that, for $n = 0$ we are using the reduced homology.

**Theorem 3.6.** *Let $X$ be a compact metrizable space and suppose it is embedded in the Hilbert cube $(Q, \rho)$ as a closed subset. If $\alpha, \beta \in \check{H}_n(X)$ are two homology classes, and $\varepsilon > 0$, then $d_X(\alpha, \beta) < \varepsilon$ if and only if $i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta)$ as homology classes in $H_n(B(X, \varepsilon))$, where $i_{\varepsilon *}$ is the induced homomorphism in homology by the inclusion map $i_\varepsilon : X \hookrightarrow B(X, \varepsilon)$.*

**Proof.** Let us prove only the first part. So let $\alpha, \beta \in \check{H}_n(X)$ be two homology classes. Using 2.12, $\alpha$ and $\beta$ are represented by approximative cycles $\{\alpha_k\}$ and $\{\beta_k\}$ respectively. If we consider a sequence $\{\varepsilon_k\} \to 0$ (e.g. $\varepsilon_k = \frac{\text{diam}Q}{k}$), without loss of generality, we can suppose that $\alpha_k, \beta_k$ are cycles lying in $B(X, \varepsilon_k)$ for each $k \in \mathbb{N}$ (if it would not be the case, pass to a subsequence).

First, if $d_X(\alpha, \beta) < \varepsilon$, then $\lim_{k\to\infty} F(\alpha_k, \beta_k) = l < \varepsilon$. Then, for any $l'$ such that $l < l' < \varepsilon$, there exists $k_0 \in \mathbb{N}$ satisfying $F(\alpha_k, \beta_k) < l'$ for $k \geq k_0$. Thus, $\alpha_k \sim \beta_k$ in $B(X, l') \subset B(X, \varepsilon)$.

On the other hand, we have $\alpha_k = i_{\varepsilon_k *}(\alpha)$ and $\beta_k = i_{\varepsilon_k *}(\beta)$. Hence,

$$i_{\varepsilon *}(\alpha) = i_{\varepsilon \varepsilon_k *} \circ i_{\varepsilon_k *}(\alpha) = i_{\varepsilon \varepsilon_n *} \circ i_{\varepsilon_k *}(\beta) = i_{\varepsilon *}(\beta)$$

in $H_n(B(X, \varepsilon))$.

For the converse, let us suppose $i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta)$. Using 3.4, let us take $k_0 \in \mathbb{N}$ such that

$$F(\alpha_k, \alpha_{k+1}) < \frac{\varepsilon}{2} \text{ and } F(\beta_k, \beta_{k+1}) < \frac{\varepsilon}{2} \text{ for each } k \geq k_0.$$

Hence, for every $k \geq k_0$ we have $\alpha_k \sim \alpha_{k+1}$ as cycles in $B(X, \varepsilon/2)$, so there exist $(n+1)$-chains $\gamma_k$ such that $\gamma_k = \alpha_{k+1} - \alpha_k$, and all simplices of $\gamma_k$ lie in $B(X, \varepsilon/2)$. Analogously, there exist $(n+1)$-chains $\gamma'_k$ lying in $B(X, \varepsilon/2)$ such that $\gamma'_k = \beta_{k+1} - \beta_k$.

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In addition,

$$\alpha_{k_0} = i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta) = \beta_{k_0}$$

as homology classes in $H_n(B(X,\varepsilon))$, using the hypothesis and because $\alpha_{k_0} = i_{\varepsilon_{k_0}*}(\alpha)$ is a cycle homologous to $i_\varepsilon(\alpha)$ in $B(X,\varepsilon)$ (resp. for $\beta_k$).

Let $\gamma$ be an $(n+1)$-chain lying in $B(X,\varepsilon)$ such that $\gamma = \alpha_{k_0} - \beta_{k_0}$. Recall that $\gamma$ is finite formal sum $\sum a_i\gamma_i$ where $a_i \in \mathbb{Z}$ and $\gamma_i : \Delta^n \to B(X,\varepsilon)$ is continuous. Hence, $\bigcup \gamma_i(\Delta^n)$ is compact, so there exists $0 < \delta < \varepsilon$ such that $\gamma$ lies in $B(X,\delta) \subset B(X,\varepsilon)$.

Putting it all together, we obtain $F(\alpha_k, \beta_k) \leq \max\{\delta, \varepsilon/2\}$ for $k \geq k_0$, so

$$d_X(\alpha,\beta) = \lim_{k\to\infty} F(\alpha_k,\beta_k) < \varepsilon,$$

as desired. □

**Corollary 3.7.** *Let $X$ be a compact metrizable space. Then:*

a) *Every ball (open or closed) is a clopen set in $(\check{H}_n(X), d_X)$.*
b) *$dim(\check{H}_n(X), d_X) = 0$ for every $n \in \mathbb{N}$.*

Both properties are consequence of well-known facts about ultrametrics.

**Example 3.8.** Let $\mathbb{P}^2$ be the real projective plane. It is well-known that $\check{H}_1(\mathbb{P}^2) = H_1(\mathbb{P}^2) = \mathbb{Z}_2$. Let $0, \alpha$ be the two elements of this group. By choosing a suitable embedding of $\mathbb{P}^2$ into $Q$, we can suppose that the metric in the group is given by

$$d_{\mathbb{P}^2}(0,0) = 0, \qquad d_{\mathbb{P}^2}(0,\alpha) = 1, \qquad d_{\mathbb{P}^2}(\alpha,\alpha) = 0.$$

In this case $B(0,1) = \{0\}$, so $\overline{B(0,1)} = \{0\}$, while $\overline{B}(0,1) = \{0,\alpha\} = \mathbb{Z}_2$.

The previous example is just a particular case of what happens with any polyhedron, since the additional metric structure that we have introduced in the groups gets reduced only to the algebraic information in the case of nice spaces such as ANRs. We show that in the next result.

**Proposition 3.9.** *If $X$ is a compact metrizable ANR (in $Q$), then $(\check{H}_n(X), d)$ is uniformly discrete.*

**Proof.** Since $X$ is an ANR, there exists a neighborhood $U$ of $X$ in $Q$ and a retraction $r : U \to X$. Take $\varepsilon_0 > 0$ such that $B(X,\varepsilon_0) \subseteq U$.

Let $\alpha, \beta \in \check{H}_n(X) = H_n(X)$ such that $d_X(\alpha,\beta) < \varepsilon_0$. By 2.10 and 2.11, there exist two singular cycles lying in $X$ such that the approximative cycles generated by them are representatives of the classes $\alpha$ and $\beta$ respectively. Then,

$$d_X(\alpha,\beta) = \lim_{i\to\infty} F(\sigma_i,\tau_i) = F(\sigma,\tau) = l < \varepsilon_0$$

which means in particular that $\sigma \sim \tau$ in $B(X,\varepsilon_0.)$ Hence, there exists a singular chain $\gamma$ lying in $B(X,\varepsilon_0)$ such that $\partial\gamma = \sigma - \tau$.

The retraction $s = r|_{B(X,\varepsilon_0)} : B(X,\varepsilon_0) \to X$ induces a map $s_\#$ over the chain groups such that $s_\#\partial = \partial s_\#$. Hence, $s_\#(\gamma)$ is chain lying in $X$ such that

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$$\partial s_{\#}(\gamma) = s_{\#}(\partial\gamma) = s_{\#}(\sigma - \tau) = s_{\#}(\sigma) - s_{\#}(\tau) = \sigma - \tau,$$

thus $\sigma \sim \tau$ in $X$. Consequently, $\alpha = \beta$. □

**Example 3.10.** The situation is completely different for a general compact metrizable space.

Let us consider the *Hawaiian Earring space* $X = HE = \bigcup_{n\in\mathbb{N}} C_n$, where $C_n$ is the circle with center $(0, 1/n)$ and radius $1/n$. Then, $x_0 = (0, 0)$ is the tangency point of all circles and we shall consider it as the base-point of $X$. We denote by $l_n$ the loop, based at $x_0$, which runs counterclockwise along $C_n$.

Alternatively, we can define $X$ as an inverse limit in the following way. Let $X_n$ be the union of the first $n$ circles, i.e.,

$$X_n = \bigcup_{m=1}^{n} C_m$$

and let $p_{n\,n+1} : X_{n+1} \to X_n$ be a map such that coincides with $1_{X_n}$ over $X_n$, and $p_{n\,n+1}(C_{n+1}) = \{x_0\}$. Then, we have an inverse sequence $\{X_n, p_{n\,n+1}\}$ whose inverse limit is $X_\infty = HE$. Also, the Hawaiian Earring space can be embedded in the Hilbert cube in a way that $B(HE, \frac{1}{n}) \simeq X_n$ and inclusions as bonding maps.

For some $n \in \mathbb{N}$, let $\langle l_n\rangle$ be the shape class of the loop $l_n$. It is clear that $l_n \simeq c_{x_0}$ in $B(HE, \frac{1}{m})$ if and only if $m > n$. Hence,

$$d_{HE}(\langle l_n\rangle, \langle c_{x_0}\rangle) = \frac{1}{n}.$$

In general, the induced sequence in homology is given by

$$\begin{array}{ccccccc} \mathbb{Z} & \longleftarrow & \mathbb{Z}\times\mathbb{Z} & \longleftarrow & \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} & \longleftarrow & \cdots \\ a & \longleftarrow\!\shortmid & (a,b) & \longleftarrow\!\shortmid & (a,b,c) & \longleftarrow\!\shortmid & \cdots \end{array}$$

which has $\prod \mathbb{Z} = \mathbb{Z}^{\mathbb{N}}$ as inverse limit. In the considered immersion, the ultrametric on $\check{H}_1(HE) = \mathbb{Z}^{\mathbb{N}}$ is given by

$$d_{HE}(\alpha, \beta) = \frac{1}{n_0}$$

where $n_0 = min\{n \in \mathbb{N} \mid \alpha_n = \beta_n \text{ in } H_1(X_n)\}$, where $\alpha_n, \beta_n$ are cycles representing homology classes in $H_1(X_n)$ and the topology generated by this ultrametric is the usual product topology on $\mathbb{Z}^{\mathbb{N}}$.

Analogous conclusion arises for $\check{H}_m(HE_m)$ where $HE_m$ is the m-dimensional Hawaiian Earring.

The ultrametric structure mixes properly with the algebraic one:

**Proposition 3.11.** *$(\check{H}_n(X), d_X)$ is a topological group.*

**Proof.** By 2.12, it is enough to check the continuity of the maps

$$\begin{array}{cccc} r: & H_n^{\mathcal{A}}(X) & \longrightarrow & H_n^{\mathcal{A}} \\ & [\{\sigma_i\}] & \longmapsto & -[\{\sigma_i\}] \end{array} \qquad \begin{array}{cccc} s: & H_n^{\mathcal{A}} \times H_n^{\mathcal{A}}(X) & \longrightarrow & H_n^{\mathcal{A}} \\ & ([\{\sigma_i\}], [\{\tau_i\}]) & \longmapsto & [\{\sigma_i\}] + [\{\tau_i\}] \end{array}$$

with the metric $d_X([\{\sigma_i\}], [\{\tau_i\}]) = \lim_{i\to\infty} F(\sigma_i, \tau_i)$.

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If $d_X([\{\sigma_i\}],[\{\tau_i\}]) = l$, then

$$d_X([\{\sigma_i\}],[\{\tau_i\}]) < l+\varepsilon \text{ for every } \varepsilon > 0.$$

Then there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i,\tau_i) < l+\varepsilon$ for every $i \geq i_0$, which means

$$\sigma_i \sim \tau_i \text{ in } B(X, l+\varepsilon) \text{ for every } i \geq i_0.$$

Hence, $-\sigma_i \sim -\tau_i$ in $B(X, l+\varepsilon)$, and therefore, $F(-\sigma_i, -\tau_i) < l+\varepsilon$. Thus,

$$d_X(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq l+\varepsilon,$$

which implies $d_X(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq l$.

This gives the inequality $d_X(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq d_X([\{\sigma_i\}],[\{\tau_i\}])$.

By symmetry, the same argument gives us the other inequality. So the equality

$$d_X([\{\sigma_i\}],[\{\tau_i\}]) = d_X(-[\{\sigma_i\}], -[\{\tau_i\}])$$

holds. Then, $r$ is an isometry. In particular, $r$ is continuous.

A similar argument is valid to see that

$$s(B([\{\sigma_i\}],\delta) \times B([\{\tau_i\}],\delta)) \subset B([\{\sigma_i\}] + [\{\tau_i\}], \varepsilon),$$

taking $\delta = \varepsilon$, for any $\varepsilon > 0$ given. Then $s$ and $r$ are continuous, and $H_n^{\mathcal{A}}(X)$ (or $\check{H}_n(X)$) is a topological group. $\square$

**Remark 3.12.** As in the proof above, it can be checked that the inequality

$$d_X([\sigma]+[\mu],[\tau]+[\eta]) \leq \max\{d_X([\sigma],[\tau]), d_X([\mu],[\eta])\}$$

is valid for every $[\sigma],[\tau],[\mu],[\eta] \in H_n^{\mathcal{A}}(X)$.

**Corollary 3.13.** *The translations in $H_n^{\mathcal{A}}(X)$ (hence in $\check{H}_n(X)$) are isometries.*

**Proof.** From the last remark,

$$d_X([\sigma]+[\mu],[\tau]+[\mu]) \leq \max\{d_X([\sigma],[\tau]), d_X([\mu],[\mu])\} = \max\{d_X([\sigma],[\tau]), 0\} = d_X([\sigma],[\tau])$$

Then,

$$d_X([\sigma],[\tau]) = d_X([\sigma]+[\mu]-[\mu],[\tau]+[\mu]-[\mu]) \leq d_X([\sigma]+[\mu],[\tau]+[\mu]) \leq d_X([\sigma],[\tau])$$

so $d_X([\sigma]+[\mu],[\tau]+[\mu]) = d_X([\sigma],[\tau])$ in $H_n^{\mathcal{A}}(X)$.

Hence,

$$d_X(\alpha+\gamma, \beta+\gamma) = d_X(\alpha,\beta) \text{ in } \check{H}_n(X). \qquad \square$$

The last result allows us to define a norm on Čech homology groups.

**Definition 3.14.** Given $\alpha \in \check{H}_n(X)$, we define *the norm of* $\alpha$ as

$$\|\alpha\| = d_X(\alpha, 0).$$

From 3.13, this norm recovers the definition of the metric $d$ by means of

$$d_X(\alpha, \beta) = \|\alpha - \beta\|.$$

**Corollary 3.15.** *For any $n \in \mathbb{Z}$, $\|n\sigma\| \leq \|\sigma\| \leq diamQ$.*

**Proof.** For $\alpha \in \check{H}_n(X)$,

$$\|\alpha + \alpha\| = d_X(\alpha + \alpha, 0) = d_X(\alpha, -\alpha) \leq \max\{d_X(\alpha, 0), d_X(-\alpha, 0)\} = d_X(\alpha, 0) = \|\alpha\|$$

and the statement holds by induction. □

Some consequences of $d_X$ being an ultrametric are discussed in the next result.

**Proposition 3.16.** *Let $X$ be a compact metrizable space. For any $\varepsilon > 0$, let us denote $H_\varepsilon = \{\alpha \in \check{H}_n(X) \,|\, \|\alpha\| < \varepsilon\}$. Then, $H_\varepsilon$ is a clopen normal subgroup and a neighborhood of the neutral element 0 of $\check{H}_n(X)$. In particular, $\{H_{\frac{1}{m}}\}_{m\in\mathbb{N}}$ is a numerable basis of neighborhoods of the identity in $\check{H}_n(X)$ formed by clopen normal subgroups.*

**Proof.** Let us fix $\varepsilon > 0$ and let $\alpha, \beta$ be in $H_\varepsilon$. Then,

$$\|\alpha + \beta\| \leq \max\{\|\alpha\|, \|\beta\|\} < \varepsilon,$$

so $\alpha + \beta$ is in $H_\varepsilon$. This subgroup is normal because $\check{H}_n(X)$ is Abelian. Moreover, $H_\varepsilon = B(0, \varepsilon)$ is clopen by 3.7. □

The metric $d_X$ has functorial properties with respect to homomorphisms induced by continuous maps between compact metrizable spaces or, more generally, by shape morphisms. These properties are immediate from the functoriality of approximative homology and its identification with Čech homology groups. By 2.12, we obtain also a homomorphism between $\check{H}_n(X)$ and $\check{H}_n(Y)$ which is in fact the classical homomorphism induced by a shape morphism, with its usual functorial properties.

**Proposition 3.17.** *Let $\{f_k, X, Y\}$ be a shape morphism between compact metrizable spaces in $Q$. Then $\{f_k\}$ induces a uniformly continuous homomorphism of topological groups $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.*

**Proof.** As it has been recalled before, every shape morphism induces a homomorphism between the corresponding Čech homology groups. It remains to verify the uniform continuity of this homomorphism.

Let $\varepsilon > 0$. Since $\{f_k\}$ is a fundamental sequence, for $0 < \varepsilon' < \varepsilon$ take $\delta > 0$ such that

$$f_k|_{B(X,\delta)} \simeq f_{k+1}|_{B(X,\delta)} \text{ in } B(Y, \varepsilon') \text{ for almost } k.$$

Let $\alpha, \beta$ be two Čech homology classes and $\sigma = \{\sigma_i\}, \tau = \{\tau_i\}$ be two approximative cycles for $X$ representing $\alpha$ and $\beta$ respectively such that $d_X(\alpha, \beta) = \lim_{i\to\infty} F(\sigma_i, \tau_i) < \delta$.

From the proof of 2.6, we can take a subsequence $\{i_k\}$ of indices such that the inequality $j_k \geq i_k$ implies that both $\{f_{k\#}(\sigma_{i_k})\}, \{f_{k\#}(\tau_{i_k})\}$ are approximative cycles for $Y$. The subsequences $\{\sigma_{i_k}\}$ and $\{\tau_{i_k}\}$ are again approximative cycles and obviously $\{\sigma_i\}F\{\sigma_{i_k}\}$ and $\{\tau_i\}F\{\tau_{i_k}\}$, which implies $\lim_{k\to\infty} F(\sigma_{i_k}, \tau_{i_k}) = \lim_{i\to\infty} F(\sigma_i, \tau_i) < \delta$.

Therefore, we have that

$$\sigma_{i_k} \sim \tau_{i_k} \text{ in } B(X, \delta)$$

for $k \in \mathbb{N}$ large enough, and consequently

$$f_{k\#}(\sigma_{i_k}) \sim f_{k\#}(\tau_{i_k}) \text{ in } B(Y, \varepsilon')$$

for $k$ sufficiently large.

As it has been remarked, the image of an approximative cycle is independent of the chosen subsequence, thus

$$d_Y(f_*(\alpha), f_*(\beta)) = \lim_{k\to\infty} F(f_{k\#}(\sigma_{i_k}), f_{k\#}(\tau_{i_k})) \leq \varepsilon' < \varepsilon,$$

which finishes the proof. □

**Corollary 3.18.** *Let $X, Y$ be two compact metrizable spaces and let $f : X \to Y$ be a map between them. Then $f$ induces a uniformly continuous homomorphism of topological groups $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.*

**Proof.** Since $Q$ is an absolute extensor for metrizable spaces and $Y$ is in $Q$, then there exists an extension map $\tilde{f} : Q \to Q$ of $f$. Setting $f_k = \tilde{f}$ for every $k \in \mathbb{N}$, we obtain a fundamental sequence generated by $f$. Now it is sufficient to apply 3.17 to that fundamental sequence. □

With these functorial properties, we can show now the invariance up to shape of the uniform type induced by the ultrametric and consequently the shape invariance of the induced topology.

**Theorem 3.19.** *Let $X, Y$ be compact metrizable spaces. If $X, Y$ are of the same shape, then $\check{H}_n(X)$ and $\check{H}_n(Y)$ are uniformly topologically isomorphic.*

**Proof.** Since $X, Y$ are of the same shape, there exist fundamental sequences $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ such that $\{g_k f_k\} \simeq \{id_X\}$ and $\{f_k g_k\} \simeq \{id_Y\}$. Hence,

$$g_* f_* = id_{\check{H}_n(X)} \text{ and } f_* g_* = id_{\check{H}_n(Y)}$$

which implies the statement of the theorem because of 3.17. □

**Corollary 3.20.** *If two compact metrizable spaces $X, Y$ have the same homotopy type, then $\check{H}_n(X)$ and $\check{H}_n(Y)$ are uniformly topologically isomorphic.*

**Proof.** Having the same homotopy type implies that, in particular, $X, Y$ have the same shape and the result comes from the previous theorem. □

In the following corollary we use the concept of shape domination as introduced by Borsuk in [3,4].

**Corollary 3.21.** *Let $X, Y$ be compact metrizable spaces such that $X$ is shape dominated by $Y$. Then $\check{H}_n(X)$ injects in $\check{H}_n(Y)$. Moreover, if $\check{H}_n(Y)$ is discrete, then so is $\check{H}_n(X)$. In particular, if $X$ is dominated in shape by an ANR, then $\check{H}_n(X)$ is uniformly discrete.*

**Proof.** By hypothesis, there exist fundamental sequences $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ such that $\{g_k f_k\} \simeq \{id_X\}$. So $g_* f_* = id_{\check{H}_n(X)}$ which in particular implies that $\check{H}_n(X)$ injects in $\check{H}_n(Y)$ via $f_*$. From that, it is obvious that if $\check{H}_n(Y)$ is discrete, then so is $\check{H}_n(X)$.

If in addition $Y$ is an ANR, $\check{H}_n(Y)$ is uniformly discrete by 3.9, so $\check{H}_n(X)$ must also be uniformly discrete. □

A stronger version of this result is the following.

**Proposition 3.22.** *Let $X$ and $Y$ be two compact metrizable spaces. If $Sh(X) \leq Sh(Y)$, then $\check{H}_n(X)$ is a factor subgroup of $\check{H}_n(Y)$ (for each $n \geq 0$). Moreover, there exists a continuous retraction from $\check{H}_n(Y)$ to $\check{H}_n(X)$.*

**Proof.** Let $F : X \to Y$ and $G : Y \to X$ two shape morphisms such that $G \circ F = 1_X$ in $Sh$.

The already known part can be sketched as follows: The induced maps on homology $F_* : \check{H}_n(X) \to \check{H}_n(Y)$ and $G_* : \check{H}_n(Y) \to \check{H}_n(X)$ satisfy $G_* \circ F_* = (G \circ F)_* = 1_{\check{H}_n(X)}$. Then, $F_*$ is injective and $G_*$ is onto. In addition, $\check{H}_n(Y) = \check{H}_n(X) \oplus KerG_*$ where, by abuse of notation, we have identified $\check{H}_n(X)$ with its image $F_*(\check{H}_n(X))$ in $\check{H}_n(Y)$.

To prove the topological part, observe that the induced map on homology

$$G_* : \check{H}_n(Y) \to \check{H}_n(X)$$

is continuous (by 3.17). Using the previous decomposition $\check{H}_n(Y) \cong \check{H}_n(X) \oplus KerG_*$, we have that every element $\gamma$ of $\check{H}_n(Y)$ can be decomposed (in a unique way) as $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1$ in $F_*(\check{H}_n(X))$ and $\gamma_2$ in $KerG_*$. Furthermore, $\gamma_1 = F_*(\gamma_1')$ for (a unique) $\gamma_1'$ of $\check{H}_n(X)$, and $\gamma_1$ is the same element, under identification, as $\gamma_1'$. Hence,

$$G_*(\gamma) = G_*(\gamma_1) = G_*(F_*(\gamma_1')) = \gamma_1' = \gamma_1.$$

In particular, if $\gamma_2 = 0$ (i.e. if $\gamma$ is in $\check{H}_n(X)$), then $G_*(\gamma) = \gamma$, hence $G_*$ is a continuous retraction from $\check{H}_n(Y)$ onto $\check{H}_n(X)$. □

**Corollary 3.23.** *If $X$ is a FANR, then $\check{H}_n(X)$ is uniformly discrete.*

**Proof.** Since $X$ is a FANR, there exists a neighborhood $U$ of $X$ which is an ANR and there exists a fundamental sequence $\{r_k, U, X\}$ which is a fundamental retraction. Moreover, there exists a compact polyhedron $P$, which is a neighborhood of $X$ in the Hilbert cube, such that $X \subset P \subset U$. This means that $\{r_k i_k\} \simeq \{id_X\}$, where $\{i_k, X, P\}$ is the fundamental sequence generated by the inclusion of $X$ in $P$. In particular, $X$ is shape dominated by $P$, so we can apply 3.21 joint with 3.9 in order to obtain the desired result. □

The above corollary shows us that FANR-spaces are adequate objects in shape theory to be studied by means of the algebra of Čech homology groups as ANRs are adequate in homotopy theory to be studied by means of algebraic properties of singular homology groups.

**Example 3.24.** Let us consider $X = \mathcal{W}$ the Warsaw circle. An inverse sequence $\{X_n, p_{n\,n+1}\}$ which represents the shape of this space is given by compact polyhedron $X_n$ contained, for each $n \in \mathbb{N}$, in the annulus $B(X, \frac{1}{n})$ and inclusions $p_{n\,n+1} = X_{n+1} \hookrightarrow X_n$. Thus, the induced inverse sequence in homology is given by $\{\mathbb{Z}, id_{\mathbb{Z}}\}$ which obviously has $\mathbb{Z}$ with discrete topology as inverse limit. Hence, $\check{H}_1(\mathcal{W}) = \mathbb{Z}$ and the ultrametric is just the discrete metric. This fact is a consequence of being $Sh(\mathcal{W}) = Sh(S^1)$.

Some properties that the metric space that we have constructed enjoys are resumed in the following.

**Theorem 3.25.** *$(\check{H}_n(X), d_X)$ is a second-countable space (hence, separable), homeomorphic to a closed subset of the irrationals.*

**Proof.** Let $\{X_m\}$ be a basis of neighborhoods of $X$ in $Q$ formed by compact ANRs such that $X_m \supset X_{m+1}$. By 2.11 and 2.12, for each $m \in \mathbb{N}$

$$H_n^{\mathcal{A}}(X_m) = H_n(X_m) = \check{H}_n(X_m)$$

and by 3.9 $H_n(X_m)$ is discrete.

Given any compact ANR, $A$, there exists a compact polyhedron $P$ which dominates $A$ in shape, so $H_n(A)$ injects in $H_n(P)$ by applying 3.21. In addition, $P$ has only a finite number of simplices in dimension $n$, so that $H_n(P)$ is finitely generated. Hence $H_n(P)$ has countable many elements, and then so has $H_n(A)$.

Last argument shows that $\mathrm{card}(H_n(X_m)) \leq \aleph_0$, so it is trivially separable. Countable products of separable metric spaces is again a separable metric space, thus

$$\prod_{m\in\mathbb{N}} H_n(X_m) \hookrightarrow \mathbb{Z}^{\mathbb{N}} \approx \mathbb{R} \setminus \mathbb{Q}$$

is separable metric and, hence, second countable (which is an hereditary property). Since

$$\check{H}_n(X) = \lim_{\leftarrow}\{H_n(X_m), i_{mm+1*}\} \leq \prod_{m\in\mathbb{N}} H_n(X_m)$$

is closed, the result follows. □

Using this topological information, we have the following characterization of discreteness on $\check{H}_n(X)$.

**Proposition 3.26.** *If $X$ is a compact metrizable space, then the topology generated by the ultrametric $d$ is discrete if and only if $\check{H}_n(X)$ is countable.*

**Proof.** Obviously, if $\check{H}_n(X)$ is discrete and separable (by 3.25), it must be countable.

On the other hand, if $\check{H}_n(X)$ is countable, we can put

$$\check{H}_n(X) = \bigcup_{\alpha\in\check{H}_n(X)} \{\alpha\}$$

where $\{\alpha\}$ is closed for each $\alpha \in \check{H}_n(X)$. By 3.25, $\check{H}_n(X)$ is complete (and $d$ is a metric), so we can apply the Baire's theorem. Thus, there exists an element $\alpha_0$ such that $int(\{\alpha_0\}) \neq \emptyset$ which implies $int(\{\alpha_0\}) = \{\alpha_0\}$ so $\{\alpha_0\}$ open. Since $\check{H}_n(X)$ is a topological group, this implies that every point is open and, therefore, $\check{H}_n(X)$ is discrete. □

We can also state the following.

**Proposition 3.27.** *Let $F : X \to Y$ be a shape morphism which induces an isomorphism $F_* : \check{H}_n(X) \to \check{H}_n(Y)$ for each $n \in \mathbb{N}$. Then $F_*$ is a uniform topological isomorphism.*

**Proof.** Without loss of generality, we can suppose that $F$ is given by a fundamental sequence, and apply 3.17 in order to obtain that $F_*$ is continuous. Now, the Banach open mapping theorem for separable and completely metrizable topological groups stays that $F_*$ is open, and this is equivalent to say that the inverse homomorphism of $F_*$ is continuous. Furthermore, since the ultrametric $d$ is both-sides invariant, any continuous homomorphism is uniformly continuous. Thus, $F_*$ is a uniform topological isomorphism. □

**Example 3.28.** Let $X = HE$ the Hawaiian Earring space and $Y = \prod_{k\in\mathbb{N}} S^1$. There exists a uniform topological isomorphism between $\check{H}_1(X)$ and $\check{H}_1(Y)$, despite $X$ and $Y$ are not of the same shape.

Let us consider $X, Y$ embedded in the Hilbert cube $Q$ in such a way that there exists a sequence $\{\varepsilon_n\}$ of positive real numbers converging to zero such that

$$B(HE, \varepsilon_n) \cong \bigvee_n S^1 \qquad \text{and} \qquad B(\prod_{k\in\mathbb{N}} S^1) \cong \prod_{k=1}^{n} S^1.$$

Then,

$$H_1(B(HE, \varepsilon_n)) \cong H_1(B(\prod_{k\in\mathbb{N}} S^1, \varepsilon_n)) \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$$

and let us denote $f_n$ the corresponding isomorphism of groups. This induces an isomorphism between the inverse limits

$$f = \lim_{\leftarrow} f_n : \check{H}_1(HE) \to \check{H}_1(\prod_{k\in\mathbb{N}} S^1).$$

It is enough to compare distances to the neutral element $0 \in \prod_{k\in\mathbb{N}} \mathbb{Z}$. Recall that a class $\alpha \in \check{H}_1(X)$ is represented by an approximative cycle $\{\sigma_n\}$ (by 2.12). By definition of the ultrametric $d$ for $HE$,

$$d(0, \alpha) < \varepsilon \Leftrightarrow \lim_{n\to\infty} F(0, \sigma_n) < \varepsilon$$

for an approximative cycle $\{\sigma_n\}$ associated to the class $\alpha$. Moreover, each cycle $\sigma_n$ lies in $B(HE, \varepsilon_n)$ and, by definition of approximative cycle,

$$\exists n_0 \in \mathbb{N} \text{ such that } \sigma_n \sim 0 \text{ in } B(HE, \varepsilon_n) \text{ for } n \geq n_0 \text{ (and } \varepsilon_n < \varepsilon).$$

This is equivalent to

$$f_n(\sigma_n) \sim 0 \text{ in } B(\prod_{k\in\mathbb{N}} S^1, \varepsilon_n) \text{ for } n \geq n_0 \text{ (and } \varepsilon_n < \varepsilon)$$

and, hence

$$\lim_{n\to\infty} F(0, \{f_n(\sigma_n)\}) < \varepsilon,$$

thus $d(0, f(\alpha)) < \varepsilon$ for the ultrametric $d$ for $\prod_{k\in\mathbb{N}} S^1$.

The following result is clear:

**Proposition 3.29.** *If $X$ is the inverse limit of an inverse sequence of compact polyhedron $\{X_k, p_{kk+1}\}$ all of them with finite $n$-homology groups, then $\check{H}_n(X)$ is compact.*

We have the following criterion for compactness.

**Theorem 3.30.** *Let $X$ be a compact subspace of the Hilbert cube with a fixed metric. Then, $(\check{H}_n(X), d_X)$ is compact if and only if for any compact polyhedron $P$ and any homotopy class of maps $f : X \longrightarrow P$ we have that $f_*(\check{H}_n(X))$ is a finite subgroup of $H_n(P)$.*

**Proof.** If $(\check{H}_n(X), d_X)$ is compact, then $f_*(\check{H}_n(X))$ is a compact subgroup of $H_n(P)$ and then finite because the topology in $H_n(P)$ is discrete. On the other hand, suppose $X$ is embedded in the Hilbert cube with a metric. Let $\varepsilon > 0$, consider a prism $T \subseteq B(X, \varepsilon)$ for such metric. Then there is a finite polyhedron $P$ with the same homotopy type as $T$. Consequently $i_*(\check{H}_n(X))$ is a finite subgroup of $H_n(T)$, where $i : X \longrightarrow T$ is the inclusion. Using Theorem 3.6 we get that $(\check{H}_n(X), d_X)$ is totally bounded and since $(\check{H}_n(X), d_X)$ is complete we obtain the result. See for example [12] for this description of compactness in metric spaces □

**Remark 3.31.** The homotopy groups are related with the shape groups, and singular (also simplicial) homology groups are related with Čech homology groups as well in a similar way.

If $X$ is compact metrizable (considered as a closed subset of the Hilbert cube $Q$) and $s$ is a singular cycle, we have described in 2.5 an homomorphism

$$\varphi^{\mathcal{A}} : H_n(X) \to H_n^{\mathcal{A}}(X)$$

from the singular homology to the approximative homology. Using 2.12, this homomorphism leads to

$$\varphi : H_n(X) \to \check{H}_n(X)$$

which is in fact the canonical homomorphism relating singular and Čech homology groups.

On the other hand, it was observed that the map $F$ used to define de ultrametric $d$ is not a metric. But if we restrict the domain of $F$ to cycles entirely lying on $X$ we get a pseudoultrametric (from the properties of $F$). This pseudoultrametric generates a topology on $H_n(X)$ and it coincides with the initial topology on $H_n(X)$ induced by $\psi$ and the topology on $\check{H}_n(X)$ induced by the ultrametric (this is the analogous result on the fundamental group with respect to the shape group). In the general case, the map $\varphi$ is easy to construct via the induced maps in homology of the projections of an expansion, and the same result is obtained.

## 4. Topological Hurewicz homomorphism

Homotopy and homology groups are related via the well-known *Hurewicz homomorphism* (see [13,27]), which for any pointed space $(X, x_0)$ and each $n \in \mathbb{N}$ is a natural group homomorphism

$$\varphi : \pi_n(X, x_0) \longrightarrow H_n(X)$$

such that $\varphi([\alpha]) = \alpha_*(Z_n)$ where $\alpha : (I^n, \partial I^n) \to (X, x_0)$ represents an element of $\pi_1(X, x_0)$ and $Z_n$ is the canonical generator of $H_n(I^n, \partial I^n)$.

It is natural to consider the analogous homomorphism but in the case of shape groups and Čech homology groups instead of homotopy groups and singular homology groups.

Given $(X, x_0)$ a pointed compact metrizable space, let $\{(X_m, x_m), p_{mm+1}\}$ be an inverse sequence of pointed spaces such that its inverse limit is $(X, x_0)$. In this case, taking the Hurewicz homomorphism (with $n \in \mathbb{N}$ fixed) in each level, we obtain a sequence of homomorphisms $\{\varphi_m\}$ such that all squares of the form

$$\begin{array}{ccc} \pi_n(X_{m+1}, x_{m+1}) & \xrightarrow{\varphi_{m+1}} & H_n(X_{m+1}) \\ \downarrow & \circlearrowright & \downarrow \\ \pi_n(X_m, x_m) & \xrightarrow{\varphi_m} & H_n(X_m) \end{array}$$

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are commutative (where vertical maps are the corresponding homomorphisms induced in homotopy and homology by $p_{mm+1}$). The corresponding limit homomorphism of $\{\varphi_m\}$

$$\check{\varphi} : \check{\pi}_n(X, x_0) \longrightarrow \check{H}_n(X)$$

is a homomorphism between the n-dimensional shape and Čech homology groups. It is also called the *Hurewicz homomorphism*.

It seems to be a natural question what happens when we endow these groups with the ultrametrics for $\check{\pi}_n(X, x_0)$ and $\check{H}_n(X)$, respectively. In order to answer that question, it is more convenient to consider again Borsuk's viewpoint of shape, considering elements of the shape group as approximative maps (see [3]) from $(I^n, \partial I^n)$ to $(X, x_0)$.

Recall that if $(X, x_0)$ is a pointed compact metrizable space in $Q$, a shape morphism $\alpha$ in $\check{\pi}_n(X, x_0)$ is represented by a sequence $\{\alpha_m\}$ of maps such that $\alpha_m : (I^n, \partial I^n) \to (Q, x_0)$ satisfies that for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\alpha_m \simeq \alpha_{m+1}$ in $B(X, \varepsilon)$ for every $m \geq m_0$. The definition of homotopy of approximative maps is defined in a similar way as homotopy of fundamental sequences. Given $\{\alpha_m\}$ and $\{\beta_m\}$ two approximative maps, they are homotopic provided for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\alpha_m \simeq \beta_m$ in $B(X, \varepsilon)$ for every $m \geq m_0$.

With this definition in mind, Hurewicz homomorphism can be rewritten as

$$\begin{array}{cccc} \check{\varphi} : & \check{\pi}_n(X, x_0) & \longrightarrow & \check{H}_n(X) \\ & \langle \alpha \rangle = \{[\alpha_m]\} & \longmapsto & \check{\varphi}(\langle \alpha \rangle) = \{\alpha_{m*}(Z_n)\} \end{array}$$

where $\{\alpha_m\}$ is an approximative map representing the shape class $\langle \alpha \rangle$ in $\check{\pi}_n(X, x_0)$, and $\alpha_{m*}$ is the induced map in singular homology between $H_n(I^n, \partial I^n)$ and $H_n(Q, x_0)$, being $Z_n$ the canonical generator of $H_n(I^n, \partial I^n)$.

**Remark 4.1.** The sequence of cycles $\{\alpha_{m*}(Z_n)\}$ is in fact an approximative cycle: given $\varepsilon > 0$, the approximative sequence $\{\alpha_m\}$ gives an index $m_0 \in \mathbb{N}$ such that

$$\alpha_m \simeq \alpha_{m+1} \quad \text{in } B(X, \varepsilon) \quad \text{for every } m \geq m_0$$

and, consequently,

$$\alpha_{m*}(Z_n) \sim \alpha_{m+1*}(Z_n) \quad \text{in } B(X, \varepsilon) \quad \text{for every } m \geq m_0,$$

since homotopic maps induce the same homomorphism in homology.

Hence, $\check{\varphi}(\langle \alpha \rangle)$ belongs to $H_n^{\mathcal{A}}(X)$, which is isomorphic to $\check{H}_n(X)$ by 2.12.

**Theorem 4.2.** *The Hurewicz homomorphism*

$$\check{\varphi} : \check{\pi}_n(X, x_0) \longrightarrow \check{H}_n(X)$$

*is a uniformly continuous homomorphism between topological groups. In fact, $\check{\varphi}$ is a non-expanding homomorphism.*

**Proof.** The fact that $\check{\varphi}$ is an homomorphism is in [17].

Let $\varepsilon > 0$ and let $\langle \alpha \rangle, \langle \beta \rangle$ be in $\check{\pi}_n(X, x_0)$ respectively represented by approximative maps $\{\alpha_k\}$ and $\{\beta_k\}$. Let us assume that $d(\alpha, \beta) < \varepsilon$, that is, $\lim_{k\to\infty} F(\alpha_k, \beta_k) = l < \varepsilon$ and take $l < \varepsilon' < \varepsilon$. There exists $k_0 \in \mathbb{N}$ such that

$$\alpha_k \simeq \beta_k \text{ in } B(X, \varepsilon') \text{ for } k \geq k_0$$

so $\alpha_{k*}$ and $\beta_{k*}$ induce the same homomorphism in homology for $k \geq m_0$.

Therefore,

$$\alpha_{k*}(Z_n) \sim \beta_{k*}(Z_n) \quad \text{in } B(X, \varepsilon') \quad \text{for every } k \geq k_0,$$

so $\check{\varphi}(\alpha) = \{\alpha_{k*}(Z_n)\}$ and $\check{\varphi}(\beta) = \{\beta_{k*}(Z_n)\}$ are approximative cycles such that

$$d(\check{\varphi}(\alpha), \check{\varphi}(\beta)) = \lim_{k \to \infty} F(\alpha_{k*}(Z_n), \beta_{k*}(Z_n)) \leq \varepsilon' < \varepsilon,$$

as desired. □

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