On first passage times in discrete skeletons and uniformized versions of a continuous-time Markov chain

A. Gómez-Corral[1], M.J. Lopez-Herrero[2], and M.T. Rodríguez-Bernal[3]

Abstract In this paper, the aim is to study similarities and differences between a continuous-time Markov chain and its uniformized Markov chains and discrete skeletons in terms of first passage times when the taboo subset of states is assumed to be accessible from a class of communicating states. Under the assumption of a finite communicating class, we characterize the first-passage times in terms of either continuous or discrete phase-type random variables. For illustrative purposes, we show how first passage times in uniformized Markov chains and discrete skeletons can be used to approximate the random duration of an outbreak in the SIS epidemic model.

1 Introduction

The use of uniformized Markov chains and discrete skeletons has been shown to be a keystone in the derivation of theoretical results for the underlying continuous-time Markov chain (CTMC) by applying well-known theorems for discrete-time Markov chains, as well as the performance analysis of systems modelled by a CTMC; see e.g. Anderson [1, Chapter 5] and van Dijk et al. [19].

Specifically, the uniformization method was first described by Jensen [12] in 1953 for time-homogeneous CTMCs with uniformly bounded transition rates. Uniformization allows one to interpret a CTMC in terms of a discrete-time Markov chain by replacing the constant unit of time by random jump times, which are selected from a suitably defined Poisson process. By a simple conditioning argument on the number of Poisson events up to time t, it is

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[1] Department of Statistics and Operations Research, Complutense University of Madrid, Spain; `agcorral@ucm.es` · [2] Department of Statistics and Data Science, Complutense University of Madrid, Spain; `lherrero@ucm.es` · [3] Department of Statistics and Operations Research, Complutense University of Madrid, Spain; `mayter@mat.ucm.es`

easy to compute the matrix exponential for transient probabilities in the CTMC by using the Fox-Glynn method [5] via iterative computation of the n-step transition probability matrix in the uniformized Markov chain, with global error control at any time t. For a related work, see Section 2.8 in the monograph of Latouche and Ramaswami [14], where uniformization is used to evaluate numerically the density and distribution functions of a continuous phase-type random variable without recourse to the Kolmogorov forward equations.

An interesting analytical method to determine expected sojourn time averages and the expected number of events is provided by Gross and Miller [9, Section 6] in the special case of CTMCs with finitely many states. For CTMCs with a countable state space, Melamed and Yadin [15] present upper and lower bounds on cumulative-time distributions by using a computational methodology that utilizes and generalizes the uniformization technique of Jensen [12]. In the queueing-theoretic context, a cumulative time amounts to the time spent in a specified set of states up until hitting another set of states, whence the Melamed-Yadin method [15, Section 2] is seen to be a powerful tool for computing sojourn times and waiting times in queueing systems with exponential servers and Poisson arrivals, such as tandem Jackson networks. Uniformization is also seen to be an appealing technique applied to time-inhomogeneous systems [17] and unbounded transition rates [18]. The paper by van Dijk et al. [19] is an excellent mathematical and intuitive review of the uniformization technique and some of its exact and approximate extensions, including steady state detection, adaptative uniformation and unbounded Markov decision processes.

An elementary but striking property shown by Jensen [12] states that an irreducible CTMC and any of its uniformized Markov chains behave asymptotically the same in the limit of large time index. As argued by Kingman [13, Section 3], this property also holds for h-skeletons (or discrete skeletons at scale h), which are obtained by recording the state of the CTMC at a sequence of inspection times $t_n = nh$, for a fixed length $h > 0$. We refer the reader to Chapter 5 of Anderson [1] for further details about communicating classes and classification of states —which are equivalent for the CTMC and any of its h-skeletons— and ergodic theorems.

In this paper, we complement the classical work of Jensen [12] and Kingman [13] (see also Anderson [1, Chapter 5], van Dijk et al. [19], and references therein) by focusing on first passage times for a time-homogenous CTMC and their discrete counterparts in the resulting uniformized Markov chains and h-skeletons. More concretely, we first establish that, in the original setting of Jensen [12], the matrix of expected sojourn times in a proper non-closed communicating subset $\mathcal{D}$ of states for the CTMC is the same as the scaled matrix of expected sojourn times for any of its uniformized Markov chains. In a more general framework, we then study the dynamics of the CTMC before leaving states in $\mathcal{D}$ and, demonstrate that the first passage times to states in the outside of $\mathcal{D}$ for the h-skeleton are stochastically greater than the analogous first-passage times for the CTMC, for any time step $h > 0$, in the usual stochastic order.

2 The continuous-time process $\mathcal{X}$ under a taboo

We consider a conservative time-homogeneous CTMC $\mathcal{X} = \{X(t) : t \geq 0\}$ with values on a countable state space $\mathcal{S}$, generator matrix $Q = (q_{i,j} : i, j \in \mathcal{S})$ and standard transition function $P_{i,j}(t) = P(X(t) = j|X(0) = i)$, for $i, j \in \mathcal{S}$ and $t \geq 0$. Let $\mathcal{D}$ be a proper communicating subset of $\mathcal{S}$ satisfying that at least one state in its complement $\mathcal{D}^c = \mathcal{S} \setminus \mathcal{D}$ is accessible from $\mathcal{D}$; i.e., $\mathcal{D}$ is a non-closed communicating class or non-essential class.

For our purposes, we assume that $X(0) \in \mathcal{D}$ and define

$$T = \inf \{t > 0 : X(t) \notin \mathcal{D}\}$$

as the first passage time to $\mathcal{D}^c$, or the sojourn time in $\mathcal{D}$. In Section 2.1, the interest is in a scaled version of the matrix $S_{\mathcal{D}}$ of expected sojourn times in $\mathcal{D}$ before the first visit of process $\mathcal{X}$ to any state in $\mathcal{D}^c$. This matrix has the form

$$S_{\mathcal{D}} = \int_0^\infty P_{\mathcal{D},\mathcal{D}}(t; \mathcal{D}^c) dt,$$

where $P_{\mathcal{D},\mathcal{D}}(t; \mathcal{D}^c)$ is the taboo transition function with elements $P_{i,j}(X(t) = j, T \geq t|X(0) = i)$, for states $i, j \in \mathcal{D}$ and $t > 0$.

It is well known that $P_{\mathcal{D},\mathcal{D}}(t; \mathcal{D}^c)$ may be written in terms of the sub-matrix $Q_{\mathcal{D}} = (q_{i,j} : i, j \in \mathcal{D})$ as

$$P_{\mathcal{D},\mathcal{D}}(t; \mathcal{D}^c) = exp\{Q_{\mathcal{D}} t\},$$

from which it follows (see e.g. [4, Chapter 10]) that the matrix $S_{\mathcal{D}}$ of expected sojourn times is the minimal nonnegative solution of the following systems of equations:

$$S_{\mathcal{D}}(-Q_{\mathcal{D}}) \; = \; I, \qquad -Q_{\mathcal{D}} S_{\mathcal{D}} \; = \; I, \tag{1}$$

where I denotes the identity matrix.

Remark 1. For any finite subset $\mathcal{D}$, the first visit to the taboo subset $\mathcal{D}^c$ occurs almost surely in a finite expected time from any initial state $i \in \mathcal{D}$, and $S_{\mathcal{D}} = -Q_{\mathcal{D}}^{-1}$. Indeed, T behaves as a continuous phase-type random variable of order d with representation $(\alpha, Q_{\mathcal{D}})$, where d denotes the cardinality of $\mathcal{D}$ and α is a row vector with entries $P(X(0) = i)$, for $i \in \mathcal{D}$; see e.g. Latouche and Ramaswami [14, Theorem 2.4.3]. As a result, $E[T^k|X(0) = i] = \alpha(-Q_{\mathcal{D}}^{-1})^k \mathbf{1}$, for $k \in \mathbb{N}$, where $\mathbf{1}$ is a column vector of 1's.

2.1 Expected sojourn times for uniformized Markov chains

Under the assumption that $|q_{i,i}| \leq h^{-1} < \infty$, for states $i \in \mathcal{S}$, it can be readily seen that the random variable $X(t)$ is identically distributed to $Y_{N(t)}$ at any time t, where $\mathcal{N} = \{N(t) : t \geq 0\}$ is the counting process of a Poisson process with rate h^{-1}, $\mathcal{Y} = \{Y_n : n \in \mathbb{N}_0\}$ is an aperiodic discrete-time Markov chain —termed *uniformized* Markov chain— with one-step transition probability matrix $P = I + hQ$, and $\mathcal{N}$ and $\mathcal{Y}$ are assumed to be independent; see e.g. Çinlar [4, Chapter 8].

Remark 2. For the process $\mathcal{X}$, one has that

$$P(X(t+h) = j|X(t) = i) = \begin{cases} q_{i,j}h + o(h), & \text{if } j \neq i, \\ 1 - \sum_{j\in\mathcal{S}\setminus\{i\}} q_{i,j}h + o(h), & \text{if } j = i, \end{cases}$$

for states $i, j \in \mathcal{S}$ and $t \geq 0$, with $h^{-1}o(h) \to 0$ as $h \to 0$. Thus, the one-step transition probabilities $P_{i,j}$ of the uniformized Markov chain $\mathcal{Y}$ can be seen as approximations of $P(X(t+h) = j|X(t) = i)$ at time steps $t = nh$, for $n \in \mathbb{N}_0$, provided that h is sufficiently small. Note that the condition $|q_{i,i}| \leq h^{-1} < \infty$, for $i \in \mathcal{S}$, is equivalent to the inequality $0 < h \leq \inf\{|q_{i,i}^{-1}| : i \in \mathcal{S}\}$.

For the uniformized Markov chain $\mathcal{Y}$, Lemma 5.1.2 in Latouche and Ramaswami [14] tells us how to compute the matrix $S'_{\mathcal{D}}$ of expected sojourn times in the subset $\mathcal{D}$ of states, before the first passage to $\mathcal{D}^c$. Here, we recall that the entries of $S'_{\mathcal{D}}$ are given by

$$\sum_{n=0}^{\infty} P(Y_n = j, T' \geq n|Y_0 = i), \quad i, j \in \mathcal{D},$$

where $T' = \inf\{n \in \mathbb{N} : Y_n \notin \mathcal{D}\}$, so that

$$S'_{\mathcal{D}} = \sum_{n=0}^{\infty} P^n_{\mathcal{D},\mathcal{D}}(\mathcal{D}^c),$$

where $P^n_{\mathcal{D},\mathcal{D}}(\mathcal{D}^c)$ is the n-step transition probability matrix of $\mathcal{Y}$ under the taboo of $\mathcal{D}^c$. This means that $S'_{\mathcal{D}}$ is the minimal nonnegative solution of the systems

$$S'_{\mathcal{D}}(I - P_{\mathcal{D}}) \;=\; I, \qquad (I - P_{\mathcal{D}})S'_{\mathcal{D}} \;=\; I. \tag{2}$$

Observe that the matrices $S_{\mathcal{D}}$ and $S'_{\mathcal{D}}$ are uniquely characterized from (1) and (2), respectively. Since $I - P_D = -hQ_{\mathcal{D}}$, it is then seen that

$$S_{\mathcal{D}} = hS'_{\mathcal{D}}. \tag{3}$$

Moreover, since the transition rates from states in $\mathcal{D}^c$ are not used in the underlying arguments, the equality (3) holds under the less restrictive assumption of uniformly bounded transition rates on $\mathcal{D}$; i.e., $|q_{i,i}| \leq h^{-1} < \infty$, for states $i \in \mathcal{D}$.

It is worth bearing in mind that, since T is a continuous random variable and hT' is a discrete one, T and hT' are not identically distributed, but their expectations are identical by (3), irrespectively of the value h satisfying $|q_{i,i}| \leq h^{-1} < \infty$, for $i \in \mathcal{D}$. This first-order property does not necessarily extend to moments of higher order, as was noticed by Gómez-Corral et al. [8, Section 2.2] for the random duration T of an outbreak and its discrete counterpart hT' in the SIS epidemic model.

Remark 3. For a finite subset $\mathcal{D}$, Equation (3) becomes $S'_{\mathcal{D}} = (-hQ_{\mathcal{D}})^{-1}$. It is also verified that the first passage time T' to $\mathcal{D}^c$ is a discrete phase-type random variable of order d with representation $(\alpha, I+hQ_{\mathcal{D}})$; see e.g. Latouche and Ramaswami [14, Section 2.5].

2.2 A simple stochastic ordering property

Given a fixed value $h > 0$, the h-skeleton of a conservative time-homogeneous CTMC $\mathcal{X}$ is defined by Kingman [13] as the discrete-time Markov chain $\mathcal{Z} = \{Z_n : n \in \mathbb{N}_0\}$ with $Z_n = X(nh)$, which takes values in $\mathcal{S}$ with one-step transition probabilities $P_{i,j}(h)$, for $i, j \in \mathcal{S}$.

It is worth noting that the division of the state space $\mathcal{S}$ into communicating classes for the h-skeleton is exactly the same as results from the continuous-time process $\mathcal{X}$. This implies that, under our assumptions on $\mathcal{D}$, the submatrix $P_{\mathcal{D}}(h) = (P_{i,j}(h) : i, j \in \mathcal{D})$ consists of strictly positive entries and, consequently, $\mathcal{Z}$ is aperiodic. In addition, the submatrix $P_{\mathcal{D},\mathcal{D}^c}(h) = (P_{i,j}(h) : i \in \mathcal{D}, j \in \mathcal{D}^c)$ of one-step transition probabilities is a non-null matrix.

It results from the definition of $P_{\mathcal{D},\mathcal{D}}(h; \mathcal{D}^c)$ that

$$P_{\mathcal{D},\mathcal{D}}(h; \mathcal{D}^c) \leq P_{\mathcal{D}}(h), \tag{4}$$

since the matrix $P_{\mathcal{D},\mathcal{D}}(h; \mathcal{D}^c)$ is related to the dynamics of process $\mathcal{X}$ up to time h under the taboo of $\mathcal{D}^c$ and the proper subset $\mathcal{D}$ is assumed to be non-closed. Similarly, since $h[h^{-1}t] \leq t$, it is seen that

$$P_{\mathcal{D},\mathcal{D}}(t; \mathcal{D}^c) \leq P_{\mathcal{D},\mathcal{D}}\left(h\left[h^{-1}t\right]; \mathcal{D}^c\right), \tag{5}$$

for any time $t > 0$, where $[\cdot]$ denotes integer part.

By defining $T'' = \inf\{n \in \mathbb{N} : Z_n \notin \mathcal{D}\}$ as the sojourn time of the h-skeleton on $\mathcal{D}$ up until hitting the subset $\mathcal{D}^c$, the inequalities (4) and (5) yield

$$P_{\mathcal{D},\mathcal{D}}(t;\mathcal{D}^c) \leq \left(P_{\mathcal{D}}(h)\right)^{[h^{-1}t]}, \tag{6}$$

where the entries of matrix $\left(P_{\mathcal{D}}(h)\right)^{[h^{-1}t]}$ are equivalent to the taboo probabilities $P(Z_n = j, T'' \geq n | Z_0 = i)$ with $n = [h^{-1}t]$, for states $i, j \in \mathcal{D}$.

Then, by postmultiplying both sides of (6) by vector $\mathbf{1}$, it is found that

$$P\left(T > t | X(0) = i\right) \leq P\left(hT'' > t | Z_0 = i\right), \quad t > 0,$$

for any initial state $i \in \mathcal{D}$, so that

$$T \leq_{st} hT'', \tag{7}$$

in the usual stochastic order. For any nondecreasing function f, this implies that $E[f(T)|X(0) = i] \leq E[f(hT'')|Z_0 = i]$, for $i \in \mathcal{D}$; more particularly, $E[T|X(0) = i] \leq hE[T''|Z_0 = i]$.

Remark 4. In a similar manner to the first passage time T' in Remark 3, the first passage time T'' to $\mathcal{D}^c$ for the h-skeleton is a discrete phase-type random variable if $\mathcal{D}$ is assumed to be finite, and its representation of order d is given by $(\alpha, P_{\mathcal{D}}(h))$.

3 Discussion and concluding remarks

For practical use, the scaled versions hT' and hT'' —the former for any process $\mathcal{X}$ with a bounded sub-matrix $Q_{\mathcal{D}}$ and the latter regardless of this sub-matrix— could be used to approximate the probability law of the first passage time T to the complement of a proper non-closed communicating subset $\mathcal{D}$ by selecting the time step $h > 0$ in an appropriate manner.

In Figures 1-2 and Table 1, we briefly illustrate this issue for the random duration T of an outbreak in the SIS model with contact and recovery rates $\beta > 0$ and $\gamma > 0$, respectively. We assume that a single initially infective individual is hosted within a closed and homogeneously well-mixed population with constant size N. The process $\mathcal{X}$, describing the number of infective individuals at any time t, is a finite birth-death process and T is a continuous phase-type random variable of order $d = N$ with representation $(\alpha, Q_{\mathcal{D}})$, where $\mathcal{D} = \{1, ..., N\}$, $\alpha = (1, 0, ..., 0)$ and

$$Q_{\mathcal{D}} = \begin{pmatrix} -(\lambda_1+\mu_1) & \lambda_1 & & & \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{N-1} & -(\lambda_{N-1}+\mu_{N-1}) & \lambda_{N-1} \\ & & & \mu_N & -\mu_N \end{pmatrix},$$

with $\lambda_i = \beta N^{-1} i(N-i)$, for $i \in \{1, ..., N-1\}$, and $\mu_i = \gamma i$, for $i \in \{1, ..., N\}$.

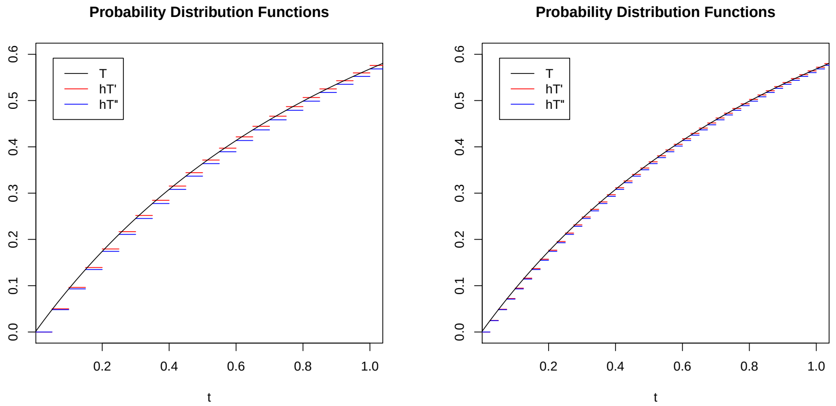

Fig. 1 The probability distribution function $F_T(t)$ versus its discrete counterparts $F_{hT'}(t)$ and $F_{hT''}(t)$ for $h = h_0$ (left) and $2^{-1}h_0$ (right) with $h_0 = \min\{(\lambda_1 + \mu_1)^{-1}, ..., (\lambda_{N-1} + \mu_{N-1})^{-1}, \mu_N^{-1}\}$, in the SIS model with $\mathcal{R}_0 = 0.5$, $N = 20$ and $X(0) = 1$.

In Figures 1-2, the probability distribution functions $F_T(t)$, $F_{hT'}(t)$ and $F_{hT''}(t)$ of T, hT' and hT'', respectively, are plotted as a function of t for values $h \in \{h_0, 2^{-1}h_0\}$, where $h_0 = \min\{(\lambda_1+\mu_1)^{-1}, ..., (\lambda_{N-1}+\mu_{N-1})^{-1}, \mu_N^{-1}\}$. The population consists of $N = 20$ individuals, $\gamma = 1.0$ and $\beta \in \{0.5, 2.0\}$, whence the basic reproduction number $\mathcal{R}_0 = \gamma^{-1}\beta \in \{0.5, 2.0\}$. The specific interval on the ox axis in both figures (i.e., $t \in [0, 1]$) is selected to make the probability distribution functions graphically distinguishable. In this sense, it should be pointed out that, from numerical experiments additional to those reported here, the probability distribution functions $F_{hT'}(t)$ and $F_{hT''}(t)$ are seen to approximate $F_T(t)$ in a very accurate manner for time instants $t \leq K_{0.99}$, where $K_{0.99}$ denotes the 99% percentile of $F_T(t)$. Without going into details, we may remark that the estimation error of $F_T(t)$ by $F_{hT'}(t)$ (respectively, $F_{hT''}(t)$) can be routinely measured in terms of the supremum of the differences $|F_T(t) - F_{hT'}(t)|$ (respectively, $F_T(t) - F_{hT''}(t)$) over subintervals $\mathcal{C}_k = \{t \in [0, K_{0.99}] : [h^{-1}t] = k\}$, for integers $k \in \mathbb{N}_0$.

It is observed that, as intuition tells us, the smaller the value of h, the better approximation of $F_T(t)$ is obtained by $F_{hT'}(t)$ and $F_{hT''}(t)$, regardless of the expected duration of the outbreak shown in Table 1. More particularly, it is observed that $F_{hT'}(t)$ results in a better approximation of $F_T(t)$ than $F_{hT''}(t)$, as long as the specific value of h yields a well defined uniformized Markov chain $\mathcal{Y}$. Furthermore, Figures 1-2 show how the scaled length hT' of the outbreak in the corresponding uniformized Markov chain $\mathcal{Y}$ is neither stochastically greater than, nor lesser than, nor equal to the random duration T of the outbreak in the SIS model.

To conclude, we remak that the theoretical and methodological aspects in Sections 2.1 and 2.2 extend results by Gómez-Corral et al. [8, Sections

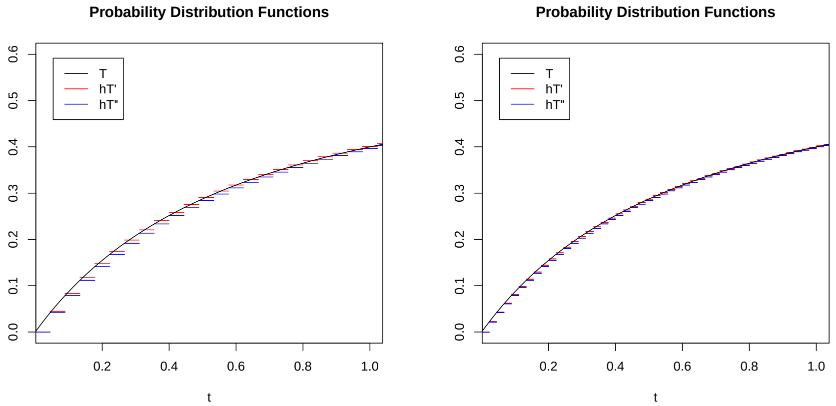

Fig. 2 The probability distribution function $F_T(t)$ versus its discrete counterparts $F_{hT'}(t)$ and $F_{hT''}(t)$ for $h = h_0$ (left) and $2^{-1}h_0$ (right) with $h_0 = \min\{(\lambda_1 + \mu_1)^{-1}, ..., (\lambda_{N-1} + \mu_{N-1})^{-1}, \mu_N^{-1}\}$, in the SIS model with $\mathcal{R}_0 = 2.0$, $N = 20$ and $X(0) = 1$.

Table 1 Expected values $E[T]$ versus $hE[T']$ and $hE[T'']$, for $h \in \{h_0, 2^{-1}h_0\}$ with $h_0 = \min\{(\lambda_1 + \mu_1)^{-1}, ..., (\lambda_{N-1} + \mu_{N-1})^{-1}, \mu_N^{-1}\}$, in the SIS model with $\mathcal{R}_0 \in \{0.5, 2.0\}$, $N = 20$ and $X(0) = 1$.

$\mathcal{R}_0$	$E[T] = hE[T']$	h	$hE[T'']$
0.5	1.34194	h_0	1.36715
		$2^{-1}h_0$	1.35449
2.0	31.21071	h_0	31.23309
		$2^{-1}h_0$	31.22187

2 and 3], linked to a specific finite birth-death process, to the more general setting of CTMCs with a state space $\mathcal{S}$ containing a countable communicating subset $\mathcal{D}$ of states. Therefore, our results on the first passage time T and its discrete analogues, hT' and hT'', in Sections 2.1-2.2 can be readily applied to absorption times for non-finite birth-death processes (Artalejo et al. [2]) and competition processes (Iglehart [11]; Reuter [16]), including the two-species competition process and the host-parasitoid process, where first passage times amount to extinction times; see also Billard [3], Gómez-Corral and López García [6, 7], and Hitchcock [10], among others. More work is needed to investigate how the scaled first passage times hT' and hT'' could be used to improve the phase-type approximation of T in [6, Section 3], which is based on truncation of the state space and extreme values.

In a general framework, there is clearly future work to be done on the comparison between our results on the discrete versions T in Sections 2.1-2.2 and the upper and lower bounds of Melamed and Yadin [15] on cumulative-time distributions. As a last remark, we note that an interesting open problem is related to the derivation of exact or approximate results on T, hT' and hT'' for time-inhomogeneous CTMCs, as well as their application to the analysis of seasonal fluctuations in epidemic models.

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References

1. Anderson WJ (1991) Continuous-Time Markov Chains. An Applications-Oriented Approach. Springer-Verlag, New York
2. Artalejo JR, Gómez-Corral A, López-García M, Molina-Paris C (2017) Stochastic descriptors to study the fate and potential of naive T cell clonotypes in the periphery. J Math Biol 74:673–708
3. Billard L (1974) Competition between two species. Stoch Process Their Appl 2:391-398
4. Çinlar E (1975) Introduction to Stochastic Processes. Prentice-Hall, Englewood Cliffs, NJ
5. Fox BL, Glynn PW (1988) Computing Poisson probabilities. Commun ACM 31:440–445
6. Gómez-Corral A, López García M (2012) Extinction times and size of the surviving species in a two-species competition process. J Math Biol 64:255–289
7. Gómez-Corral A, López García M (2013) Maximum population sizes in host-parasitoid models. Int J Biomath 6:1350002
8. Gómez-Corral A, López-García M, Rodríguez-Bernal MT (2020) On time-discretized versions of the stochastic SIS epidemic model: A comparative analysis. J Math Biol, under review
9. Gross D, Miller DR (1984) The randomization technique as a modelling tool and solution procedure for transient Markov processes. Oper Res 32:343–361
10. Hitchcock SE (1986) Extinction probabilities in predator-prey models. J Appl Probab 23:1–13
11. Iglehart DL (1964) Multivariate competition processes. Ann Math Stat 35:350–361
12. Jensen A (1953) Markoff chains as an aid in the study of Markoff processes. Scand Actuar J 1953(sup 1):87–91
13. Kingman JFC (1963) Ergodic properties of continuous-time Markov processes and their discrete skeletons. Proc Lond Math Soc 13:593–604
14. Latouche G, Ramaswami V (1999) Introduction to Matrix Analytic Methods in Stochastic Modelling. ASA-SIAM Series on Statistics and Applied Probability, Philadelphia
15. Melamed B, Yadin M (1984) Randomization procedures in the computation of cumulative-time distributions over discrete state Markov processes. Oper Res 32:926–944
16. Reuter GEH (1961) Competition processes. In: Neyman J (ed) Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol II: Contributions to Probability Theory. University of California Press, Berkeley, pp 421–430

17. Van Dijk NM (1992) Uniformization for nonhomogeneous Markov chains. Oper Res Lett 12:283–291
18. Van Dijk NM (1992) Approximate uniformization for continuous-time Markov chains with an application to performability analysis. Stoch Process Their Appl 40:339–357.
19. Van Dijk NM, van Brummelen SPJ, Boucherie RJ (2018) Uniformization: Basics, extensions and applications. Perform Eval 118:8–32