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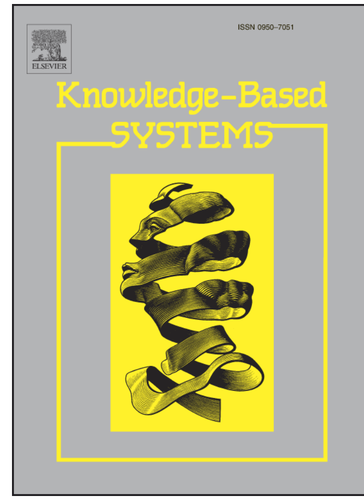

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Fuzzy Image Segmentation based upon Hierarchical Clustering

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Abstract

In this paper we introduce the concept of Fuzzy Image Segmentation, providing an algorithm to build fuzzy boundaries based on the existing relations between the fuzzy boundary set problem and the (crisp) hierarchical image segmentation problem. In particular, since a crisp image segmentation can be characterized in terms of the set of edges that separates the adjacent regions of the segmentation, from these edges we introduce the concept of fuzzy image segmentation. Hence, each fuzzy image segmentation is characterized by means of a fuzzy set over the set of edges, which can be then understood as the fuzzy boundary of the image. Some computational experiences are included in order to show the obtained fuzzy boundaries of some digital images.

Keywords:
Fuzzy set, Image segmentation, Hierarchical segmentation, Graph-based segmentation.

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1. Introduction

When some object of interest must be detected in a given image, an standard technique is segmentation, which produces a partition of the pixels in regions [1–4]. Regions are delimited by their boundaries. But such boundaries are not always *crisply* defined (i.e., borders are not always necessarily sharp and clear). Fuzzy approaches have provided popular methods for crisp image segmentation due to the fact that they are able to efficiently model certain kind of noisy images, or simply because of the fuzzy nature of objects. Nevertheless, few efforts has been dedicated to extend the concept of crisp image segmentation into a fuzzy framework, not to be confused with the frequent (crisp) image segmentation obtained by means of any fuzzy-based technique (see e.g. [5, 6] for instance of fuzzy-based Image Segmentation technique that produce a crisp output).

A crisp image segmentation (see for example the classical work [7]), can be defined as a partition of the set of pixel into a family of connected subsets or regions. The concept of partition is easily translated into a fuzzy framework. But connectivity is not so easily translated into a fuzzy framework, and it is sometimes neither clear what the output of a fuzzy image segmentation should be. Some authors have extended some classical concepts in image analysis into a fuzzy framework (see [8] for example), proposing some possible definitions for concepts as fuzzy area, fuzzy perimeter or fuzzy connectivity, although still drawn for crisp objects and crisp regions. Following these ideas, in [9–11]) a fuzzy connectedness definition was provided within a image segmentation framework, in such a way that connectivity is modelled as a fuzzy relation that expresses the degree up to which two elements are connected in a given subset. It is therefore possible to build a membership function that represents the degree of connectivity of a specific region, as well as to measure some fuzzy concepts as area, perimeter, or connectivity for a crisp subset of pixels in an image, an object or a scene (see [11] for more details). But it is not so clear whether a set of fuzzy classes $R_1, \ldots, R_c$ is suitable to be a fuzzy image segmentation, or what should be a suitable fuzzy output of an image segmentation.

Generally speaking, a fuzzy image segmentation should be a set of fuzzy regions in the image, $R_1, \ldots, R_k$, each one of these regions with a membership function $\mu_{R_1}, \ldots, \mu_{R_k}$ that represents the degree up to which each pixel of the image belongs to the region. In order to differentiate this definition from a standard fuzzy classification, we should impose some covering, connectivity and redundancy properties to these set of fuzzy regions $R_1, \ldots, R_k$, so a suitable notion of fuzzy image segmentation output can be reached (see also [12, 13]).

In this paper we propose an innovative definition for fuzzy image segmentation, which has not been clearly defined in the past, or does not have a generally accepted definition. Our approach is based on the fact that a crisp image segmentation can be characterized in terms of the set of edges that separates adjacent regions of such segmentation. We take into account that there exists a bijection between the classical definition of crisp image segmentation (i.e., a partition of the set of nodes into connected regions) and the set of edges that connects nodes of different regions (i.e. the boundary edges set). In this way we introduce an alternative way to define the concept of fuzzy image segmentation. Similarly to the crisp case, in this paper we propose a definition of fuzzy image segmentation by means of a fuzzy set over the set of edges, that could be understood as the fuzzy boundary of the image.

It is important to remark that the concept of *fuzzy boundary* in image analysis has been introduced in the framework of fuzzy edge detection problems (see for example [14–17]), where each pixel has a degree of membership to the border or frontier class). Again, we have to emphasize the differences between considering the boundary of an image as a set of nodes and considering the boundary of an image as a set of edges. Crisp edge detection and image segmentation problems are related problems, but they are indeed different. In general, the output of an edge detection problem does not produce a suitable image segmentation and the opposite is also true. Similarly, in a fuzzy framework a fuzzy edge detection output (i.e. a fuzzy boundary of nodes) does not produce a suitable fuzzy image segmentation output (i.e. a fuzzy boundary of edges).

Once an alternative concept of fuzzy image segmentation is provided (by means of the fuzzy boundary set), in this paper we show the existing key relations between fuzzy image segmentation problems and the hierarchical segmentation of an image. Based on these relations, we construct the fuzzy boundary based on an adaptation of a hierarchical segmentation algorithm designed by the authors ([18–22]). This algorithm can be classified as an unsupervised, graph-based image segmentation technique (see [1, 2, 23–25] for more details). The hierarchical approach defined here shares key elements of the segmentation algorithm proposed in [21] and extended in [20].

This paper is organized as follows: In Section 2 it is introduced the graph-based approach of the (crisp) image segmentation problem, formalizing the concept of node-based image segmentation and the concept of edge-based image segmentation image. In Section 3 we propose a definition for fuzzy image segmentation. In Section 4 the fuzzy image segmentation problem is related to the hierarchical segmentation problem. In order to show that the proposed fuzzy image segmentation concept is operative, Section 5 is devoted to depict a hierarchical segmentation algorithm which allows the visualization of the fuzzy segmentation set of several images. A final Section 6 is included for conclusions and future research.

2. The crisp image segmentation problem. A graph-based approach

Before introducing the fuzzy image segmentation concept, two definitions of crisp image segmentation in a graph context are introduced: the most natural one, based on the elements to be separated (the nodes of the graph), and the equivalent one based on the edges which link those elements.

2.1. The image network

A digital image I can be considered as a graph whose nodes are the pixels. Formally, given an $(r \times s)$-digital image, let $V = \{p_{ij} = (i,j) \mid 1 \leq i \leq r\ ; 1 \leq j \leq s\}$ be the finite set of pixels of the image. Let $E = \big\{e = \{p_{ij}; p_{i'j'}\} \mid p_{ij}, p_{i'j'} \in V\big\}$ be the set of non-ordered pairs of neighbour pixels: if two pixels p and p' are neighbours, then there exists an edge $e = \{p; p'\} \in E$; otherwise, $\{p; p'\} \notin E$. Hence, we define a graph $G = (V, E)$ that shows the neighbourhood between the pixels of the image. The graph G can be assumed to be connected; otherwise, its connected components must be analyzed separately.

One useful topology for images could be that one where a pixel is linked with four neighbors (see Fig. 1(a)); other topologies are depicted in Fig. 1 (b), (c) and (d). To start with, we have chosen the standard 4-connectivity.

Given $e = \{p; p'\} \in E$, an edge joining two pixels p and p', we shall need to evaluate the degree $d_e \geq 0$ of dissimilarity between both pixels, in such a way that the greater d_e is, the more dissimilar p and p' are. Any particular dissimilarity measure d should be defined taking into account the specific problem and the characteristics of the elements under consideration. Once a measure d verifying desirable properties is chosen, we denote by $D = \{d_e \mid e \in E\}$ the set of all dissimilarities between adjacent nodes.

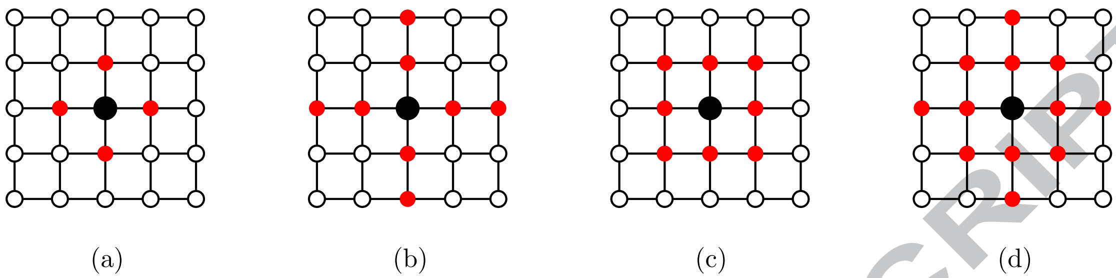


Figure 1: Some topologies applied to image networks. (a): four, (b-c): eight, and (d): twelve neighbors.

The available information of a digital image I can therefore be summarized by the image network

$$N(I) = \{ \, G \, ; \, D \, \}. \tag{1}$$

Example 2.1. *In Figure 2 it is depicted a* 4×5*-image network with three different types of pixels: Red, Blue and White; the distances between these types are* 0 *if they are equals and* 0.3*,* 0.6 *and* 0.9 *respectively between white and blue, blue and red and red and white.*

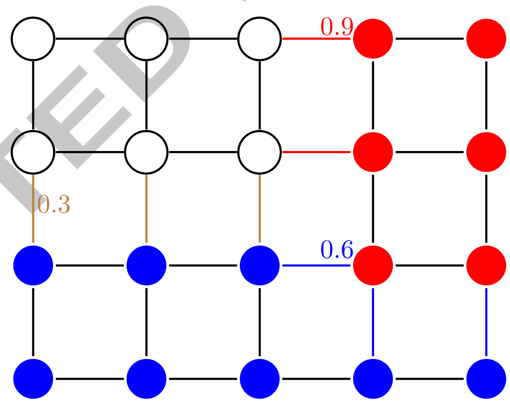


Figure 2: Image network of Example 2.1

2.2. The node-based segmentation concept

A crisp image segmentation (see, e.g., [7]) can be defined as a partition of the set of pixels (nodes in our graph-based approach) into a family of connected subsets or regions. In this subsection, once the image has been modelled as a network, we shall provide a formal graph-based definition of image segmentation, which is viewed as a partition of the set of pixels with some properties (but see again [12, 13]).

Definition 2.1. *Given a network image* $N(I) = \{\ G\ = (V, E);\ D\ \}$*, we will say that the family* $S = \{R_1, \ldots, R_t\}$*, with* $R_j \subset V$ *for all* $j \in \{1, 2, \ldots, t\}$*, is a segmentation of the image* $N(I)$ *if and only if the following holds:*

- *(1) Non Overlapping regions (i.e., for all* $i \neq j$*,* $R_i \cap R_j = \emptyset$*).*
- *(2) Covering:* $\bigcup_{j=1}^{t} R_j = V$ *(all pixels are covered by regions).*
- *(3) Connectivity of all regions: for all* $j \in \{1, \ldots, t\}$*, the subgraph* $(R_j, E_{|R_j})$ *is a connected graph.*

According to the above definition, pixels belonging to the same region are graph-connected. Obviously, two different and not adjacent regions can share the same characteristics (in terms of the dissimilarity distance d), and these regions will be associated to the same class if a consistent classification procedure is applied on the segmented image.

Remark 1. Let us note that some authors define image segmentation in a similar way but without considering the connectivity condition (3) over the set of feasible image segmentation solutions. However, solutions satisfying (1)-(2) can be easily transformed into a solution also satisfying condition (3). From now on, and following [7] among others, we will associate image segmentation problems to connected images, meanwhile non-connected images will be associated to image classification problems.

2.3. The edge-based segmentation concept

Definition 2.1 of a segmented image is based on the pixel regions $R_i \subset V$. But if the image is modelled by a network, there is an alternative and equivalent definition based on the boundary edges set $B \subset E$ between the different regions of the segmentation of the image network $N(I)$, and the segmentation can be univocally characterized (see, e.g., [26]) through the minimal set of edges which separate the regions of the segmentation:

Definition 2.2. *Given a network image* $N(I) = \{\ G\ = (V, E);\ D\ \}$*, a subset* $B \subset E$ *characterizes an image segmentation if and only if the number of connected components of the partial graph generated by the edges* $E - B$*, denoted as* $G(E - B) = (V, E - B)$*, decreases when any edge of* B *is deleted.*

In Figure 3 we show the two concepts of segmentation (focussing on nodes and focussing on edges) applied to the image network of Example 2.1: (a) three regions (white, blue and red); and (b) the edges set which separates them (thick lines).

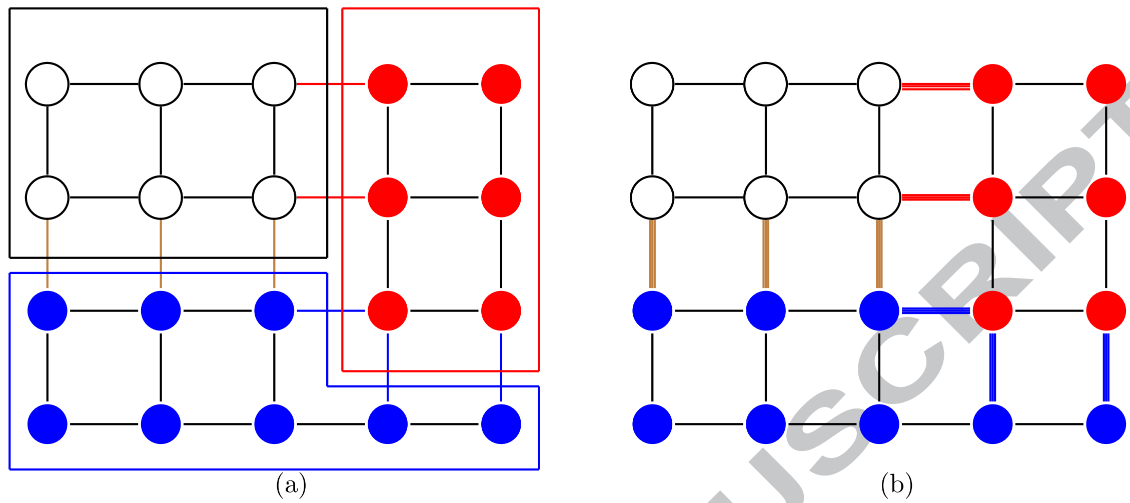


Figure 3: Segmentation of image network of Example 2.1 by nodes (a) and by edges (b)

Remark 2. Given a boundary edges set B verifying Definition 2.2, the family $S = \{R_1, \ldots, R_t\}$ of connected components of the partial graph $G(E - B)$ is a segmentation.

It is important to emphasize that there exists a bijection between the set of image segmentations $S = \{R_1, \ldots, R_t\}$ and the set of links $B \subset E$ that produce image segmentations. Hence, it is equivalent to talk about finding a partition of the set of pixels V in the sense of Definition 2.1, and about finding a subset of edges in the sense of Definition 2.2. In the following theorem we formally establish this equivalence.

Theorem 2.1. *Given a network image $N(I) = \{G(V, E); D\}$, let $\mathcal{S}^n(N(I))$ be the class of all node segmentations (in the sense of Definition 2.1), and let $\mathcal{S}^e(N(I))$ be the set of all edge segmentations (in the sense of Definition 2.2). Then, there exists a bijection $\phi : \mathcal{S}^n(N(I)) \longrightarrow \mathcal{S}^e(N(I))$.*

Proof.

- In one hand, given a segmentation defined by its boundary set $B \in \mathcal{S}^e(N(I))$, the segmentation $\phi^{-1}(B)$ is defined as the family of connected components of the partial graph $G(E - B)$.

  In general, given any arbitrary subset $B \subset E$, the connected components of the partial graph $G(E - B)$ are a segmentation in the sense of Definition 2.1. From this segmentation, we can obtain a minimal set of boundary edges that satisfies Definition 2.2 by deleting from B those edges that link nodes in the same connected component.

- In the other hand, given $\mathcal{S} = \{R_1, \ldots, R_t\} \in \mathcal{S}^n(N(I))$, the set $\phi(\mathcal{S}) \in \mathcal{S}^e(N(I))$ is constructed in the following way:

$$\phi(\mathcal{S}) = E - \bigcup_{h=1}^{t} \{e = \{p; q\} \in E \mid p, q \in R_h\}$$

By definition, the connected components of $G(E-\phi(\mathcal{S})) = G(\bigcup_{h=1}^{t}\{e = \{p;q\} \in E \mid p,q \in R_h\})$ are the subsets $R_h \in \mathcal{S}$, being t be the number of these connected components.

Let $e = \{p,q\} \in \phi(\mathcal{S})$ be an edge to be deleted from $\phi(\mathcal{S})$; then there exist two different sets $R_h, R_{h'} \in \mathcal{S}$ in such a way that $p \in R_h$ and $q \in R_{h'}$, and the partial graph $G(E-\phi(\mathcal{S})-\{e\}) = G(\bigcup_{h=1}^{t}\{e = \{p;q\} \in E \mid p,q \in R_h\} \cup \{e\})$ has exactly $t-1$ connected components (sets R_h and $R_{h'}$ are now in the same connected component). Therefore, $\phi(\mathcal{S}) \in \mathcal{S}^e(N(I))$.

So, if a subset of edges $B \subset E$ characterizes an image segmentation of $N(I)$, then any of its edges links two different regions of the segmentation. Moreover, if an edge is deleted from B, the two adjacent regions of its endpoints are joined in one region. The following lemma gives necessary and sufficient conditions for this property. Notice that in this lemma we introduce the structure of forest in an image network, a partial graph without cycles, which will be useful to understand the construction algorithm of a fuzzy image segmentation.

Lemma 2.1. *Given a network image $N(I) = \{\ G = (V,E);\ D\ \}$, a subset of edges $B \subset E$ is the boundary of a segmentation if and only if there exists a spanning forest $G(F) = (V,F)$ of G and there exists a subset of edges $F' \subset F$ verifying:*

1. $F' \subset B$
2. $(F - F') \not\subset B$
3. *$\forall e = \{p,q\} \in E-F$, then $e \in B$ if and only if p and q belong to different connected components of the partial graph $G(F-F')$.*

Proof:

$\Rightarrow$) Given the boundary $B \subset E$ of a segmentation $S = \{R_1, R_2, \ldots, R_t\}$, where any of the sets R_h is a connected component of the partial graph $G(E-B)$, let $\{F_1, F_2, \ldots, F_t\}$ be arbitrary spanning trees of these connected components. Let $F = \bigcup_{h=1}^{t} F_h$ be the initialization of the set $F \subset E$.

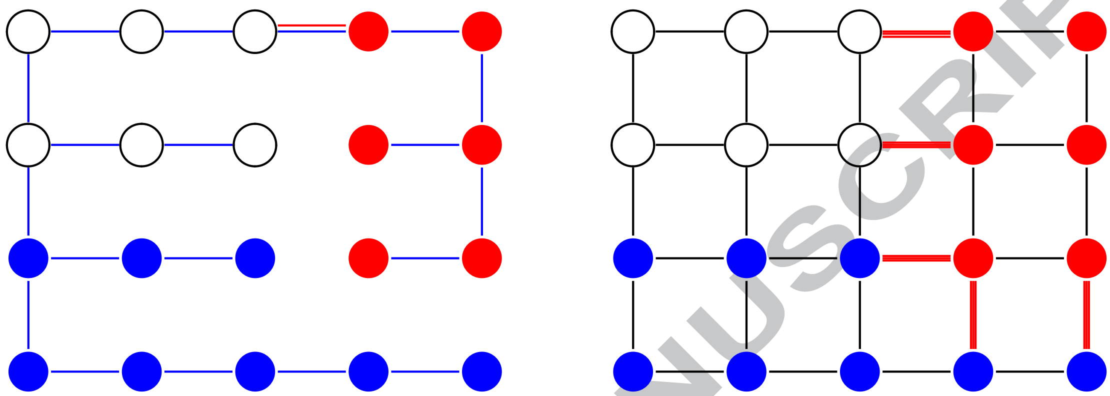
Let $F' \subset B$ be the maximal (in the inclusion sense) set of those edges in B such that $F \cup F'$ does not include any cycle. The set $F = F \cup F'$ is updated, and properties 1 and 2 hold. To see property 3, let $e = \{p,q\} \in E - F$. Two exhaustive and exclusive cases are possible:

- $p,q \in R_h$ for some $h \in \{1,2,\ldots,t\}$. In this case, $e \notin B$.
- $p \in R_h$ and $q \in R_{h'}$ for some $h \neq h'$ in $\{1,2,\ldots,t\}$. In this case, $e \in B$ (otherwise, if $e \notin B$, R_h and $R_{h'}$ would belong to the same connected component of $G(E-B)$).

$\Leftarrow$) Let $F \subset E$ be a subset of edges such that the partial graph $G(F) = (V,F)$ defines a spanning forest of G. Let $F' \subset F$ be an arbitrary subset, could be the empty set; the set B is initialized as $B = F'$. By definition of spanning forest, the partial graph $G(F-F')$ defines a partition $S = \{R_1, R_2, \ldots, R_t\} \subset V$ in such a way that the subgraphs induced by any of these t subsets is a connected graph.

The set B is then completed with all edges $e = \{p,q\} \in E - F$ linking different subsets of the partition, i.e. $p \in R_h$ and $q \in R_{h'}$ with $h \neq h'$. The set B in this way constructed is $B = B(S)$, i.e. the minimal set of boundary edges for the segmentation S previously introduced.

In Figure 4, for the image network of Example 2.1 we show in (a) the relation between a spanning forest $G = (V, F)$ (colored in blue) and a selected subset of edges $F' \subset F$ (colored in red); also, in (b) it is shown the final set B (colored in thick red) obtained through this selection.



(a) (b)

Figure 4: In Example 2.1: A spanning forest F and a subset of it $F' \subset F$ (a) and the subset B deduced by these subsets (b)

3. The fuzzy image segmentation problem. A graph-based approach

At this point it is important to emphasize that humans make fuzzy segmentation in a natural way, as can be observed when we see some human segmentation test images. The traces we make depend on how clear we see each border, showing our doubts about the significativeness of the line we are drawing.

Figure 5: Human segmentation http://www.eecs.berkeley.edu/Research/ Projects/ CS/ vision/.

In this way, in Figure 5 we can distinguish some lines that are thicker than others. The black colour that is used to separate the objects in the segmentation present gradations. Human segmentation does not correspond to a crisp image segmentation, where its visualization should show segmentation with a unique intensity for all segmentation lines.

Taking into account such a consideration, it seems reasonable to define in a formal way the concept of fuzzy segmentation through the fuzzyfication of the edge-based segmentation concept introduced in Definition 2.2:

Definition 3.1. *[27] Given a network image* $N(I) = \left\{ G = (V, E);\ D \right\}$*, we will say that the fuzzy set* $\widetilde{B} = \{(e, \mu_B(e)),\ e \in E\}$ *produces a fuzzy image segmentation if and only for all* $\alpha \in [0, 1]$ *the crisp set* $B(\alpha) = \{e \in E \ :\ \mu_B(e) \geq \alpha\}$ *produces an image segmentation in the sense of Definition 2.2.*

Let us note that in the previous definition, the membership function of the fuzzy set $\widetilde{B}$ for a given edge represents the degree of separation between these two adjacent pixels in the segmentation process.

Remark 3. This definition extends Definition 2.2:

$$\widetilde{B} = B \Longleftrightarrow \mu_B(e) = 1 \forall e \in B$$

For instance, Example 2.1 can be linked to a fuzzy segmentation $\widetilde{B}$ of the image network, as depicted in Figure 6: e.g. assume that the thin lines e (colored in black) have $\mu_B(e) = 0$, that the thicker blue lines have $\mu_B(e) = 0.3$ and the thickest red lines have $\mu_B(e) = 0.6$.

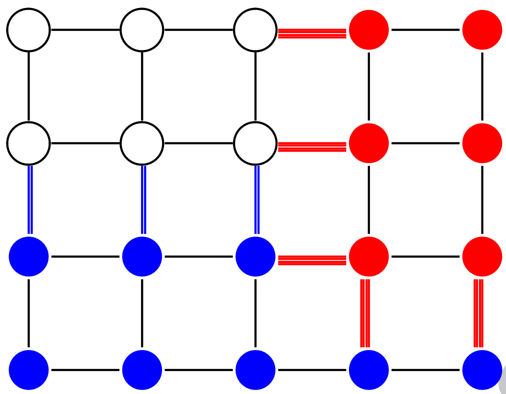


Figure 6: Fuzzy Segmentation $\widetilde{B}$ of Example 2.1

Two crisp segmentations of this fuzzy segmentation $\widetilde{B}$ are $B(0.3)$ and $B(0.6)$, as depicted in Figure 7. Indeed, there are four possible different segmentations, associated to the intervals:

$$\alpha \in \{[0,0]; (0,0.3]; (0.3,0.6]; (0.6,1.0]\}$$

which are associated respectively to the trivial segmentation $B^0 = E$; $B^1 = B(0.3)$; $B^2 = B(0.6)$ and $B^3 = \emptyset$.

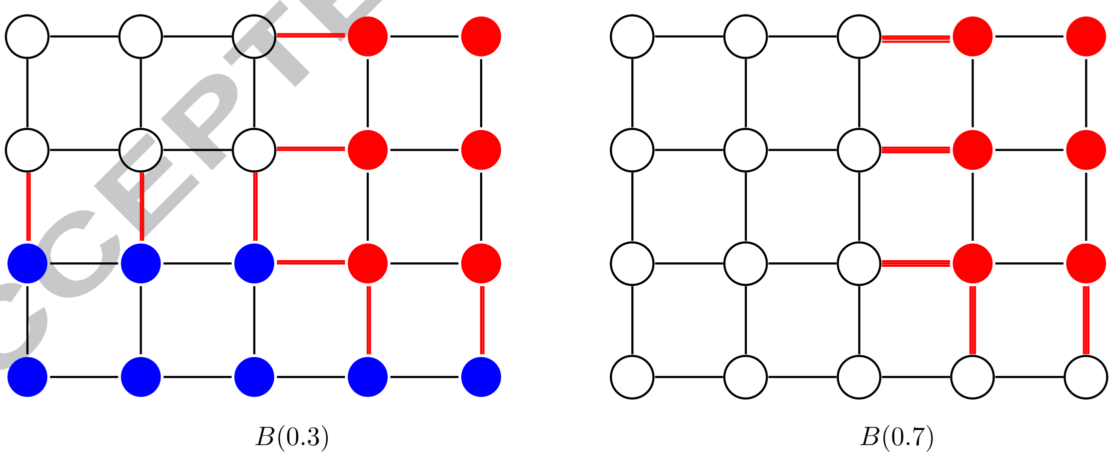


Figure 7: Crisp segmentations $B(0.3)$ and $B(0.7)$ of fuzzy image segmentation $\widetilde{B}$ of Example 2.1

Remark 4. In this section we have defined the concept of fuzzy image segmentation based on the *boundary* fuzzy class over the edges set. In this sense, the membership function of this class represents, for a particular edge, the degree of separation between two adjacent pixels in the segmentation process. It is very important to distinguish between this concept (fuzzy boundary) and the classical concept of dissimilarity between two adjacent pixels. We would like to emphasize that these two concepts (although related) are not equivalent. Dissimilarity measures between two adjacent pixels use to take only into account the spectral information of each pixel (or node), without considering the segmentation process. For example, in edge detection or fuzzy edge detection (see [28] for more details) the measure associated with each edge represents the dissimilarity (in terms of spectral information) between these two pixels, that does not necessarily meets the idea of boundary between regions.

In general, the direct construction of a fuzzy image segmentation based on a dissimilarity measure is not possible, as we can see in the following example.

Example 3.1. *Let us denote by $\widetilde{D}$ a fuzzy class over the edges set E based on the initial dissimilarities depicted in Figure 2. The membership function $\mu_{\widetilde{D}} : E \longrightarrow [0,1]$ of this fuzzy class coincides with the degree of dissimilarity between two adjacent pixels. So $\mu_{\widetilde{D}}(e) = d(e)$, that takes values* 0, 0.3*,* 0, 6 *and* 0.9*. It is easy to see that this fuzzy class does not produce a fuzzy image segmentation.*

- *For $\alpha > 0.9$, $\widetilde{D}(\alpha) = \{e \in E \ / \ \mu_{\widetilde{D}}(e) \geq \alpha\} = \{\emptyset\}$. The image segmentation produced by this set of links is the whole picture $S = \{P_{i,j} \ i = 1, \dots, 4 \ j = 1, \dots, 5\}$.*
- *For $0.6 < \alpha \leq 0.9$, $\widetilde{D}(\alpha) = \{(P_{1,3}, P_{1,4})(P_{2,3}, P_{2,4})\}$ (i.e. only the red links), which does not produce an image segmentation in the sense of Definition 2.2.*
- *For $0.3 < \alpha \leq 0.6$, $\widetilde{D}(\alpha) = \{(P_{1,3}, P_{1,4}), (P_{2,3}, P_{2,4}), (P_{3,3}, P_{3,4}), (P_{3,4}, P_{4,4}), (P_{3,5}, P_{4,5})\}$ (i.e. the red and blue links of Figure 2). The image segmentation produced by this set of links is $S = \{R_1, R_2\}$ where R_1 is the set of white and blue nodes and R_2 is the set of red nodes of Figure 2.*
- *For $0 < \alpha \leq 0.3$, $\widetilde{D}(\alpha) = E$. The image segmentation produced by this set of links is $S = \{\{P_{ij}\} \forall i, j\}$.*

Since the crisp set $\widetilde{D}(\alpha)$ is not an edges set that induced an image segmentation for $\alpha \in (0.6, 0.9]$, we can conclude that $\widetilde{D}$ is not a fuzzy image segmentation.

As a consequence, the problem of finding a fuzzy image segmentation is not a trivial task. Usually, a distance or dissimilarity measure between adjacent pixels is not enough to build a fuzzy segmentation. However, as we have already pointed out, there exists a strong relation between the fuzzy image segmentation problem and the hierarchical image segmentation problem, and it will be in general possible to build a hierarchical image segmentation solution from a fuzzy image segmentation, and viceversa. In the next section we introduce the concept of hierarchical image segmentation in order to show it.

4. The fuzzy image segmentation problem versus the hierarchical segmentation problem

Once we have provided two formal definitions of image segmentation in a graph-based framework, let us now recall in the definition of a hierarchical image segmentation, but first let us define the concept of *finer partition.*

Definition 4.1. *Given a network image $N(I) = \{G = (V, E); D\}$, let $\mathcal{S}, \mathcal{S}' \in \mathcal{S}^n(N(I))$ be two segmentations, we will say that $\mathcal{S}$ is finer than $\mathcal{S}'$ (and we will denote it by $\mathcal{S}\widetilde{\subseteq}\mathcal{S}'$) if for all $R \in \mathcal{S}$, there exist $R' \in \mathcal{S}'$, such that $R \subseteq R'$.*

The finest segmentation will be $\mathcal{S}^0 = \{\{v_1\}; \{v_2\}; \cdots\}$, i.e. each node of the network image is an element of the partition; on the other hand, the coarsest segmentation will be $\mathcal{S}^K = \{V\}$, with one cluster containing all nodes.

Between both two extreme segmentations, it is possible to construct a inclusive family of segmentations, ordered from the finest one to the coarsest.

Definition 4.2. *Given a network image $N(I) = \{G = (V, E); D\}$, we will say that the family $\boldsymbol{\mathcal{S}} = (\mathcal{S}^0, \mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^K)$ of segmentations of $N(I)$ is a hierarchical image segmentation of $N(I)$ when the following holds:*

- *$S^t \in \mathcal{S}^n(N(I))$ for all $t \in \{0, 1, \ldots, K\}$, i.e. each S^t is an image segmentation of $N(I)$.*
- *There are two trivial partitions: the first ($\mathcal{S}^0 = \{\{v\}, v \in V\}$), which has all pixels as singleton clusters, and the last one, ($\mathcal{S}^K = \{V\}$) with one cluster of all pixels.*
- *$|\mathcal{S}^t| > |\mathcal{S}^{t+1}|$ for all $t = 0, 1, \ldots, K-1$ (i.e., in each iteration the number of regions in the image decreases).*
- *$\mathcal{S}^t \widetilde{\subseteq} \mathcal{S}^{t+1}$ for all $t = 0, 1, \ldots, K-1$.*

Notice that it is possible to stat an equivalent definition of hierarchical segmentation based on the edge segmentation definition.

It is not difficult to prove that given two segmentations $B, B' \in \mathcal{S}^e(N(I))$, then B is finer than B' if $B \supset B'$.

Definition 4.3. *Given a network image $N(I) = \{G = (V, E); D\}$, we will say that the sequence $\boldsymbol{\mathcal{B}} = \{B^0, \ldots, B^K\}$, with $B^t \subset E$ for all $t \in \{0, 1, \ldots, K\}$ is a hierarchical segmentation of $N(I)$ if and only if the following holds:*

- *$B^t \in \mathcal{S}^e(N(I))$ for all $t \in \{0, 1, \ldots, K\}$, i.e. each B^t is an image segmentation of $N(I)$.*
- *The sequence $\{\phi^{-1}(B^0), \ldots, \phi^{-1}(B^K)\}$ is a hierarchical segmentation of $N(I)$.*

As a consequence, $B^0 = E \supset B^1 \supset \ldots \supset B^K = \emptyset$.

In the following example we illustrate these two definitions.

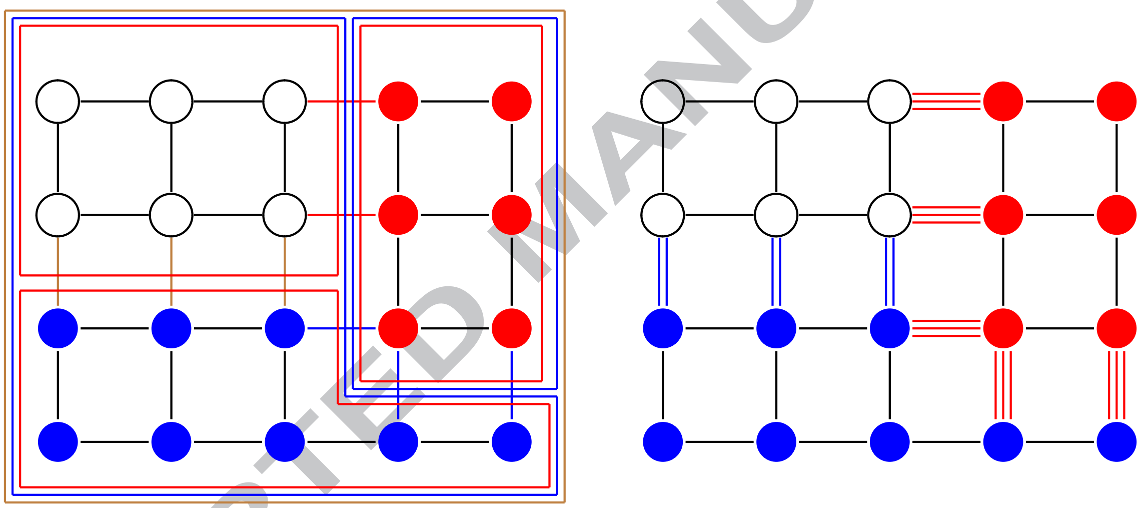
Example 4.1. *Figure 8 depicts two hierarchical segmentations of the image network of Example 2.1:*

In (a), a hierarchical segmentation specified by nodes:

- *S^0, twenty clusters (the nodes).*
- *S^1, the three clusters delimited by red lines.*
- *S^2, the two clusters delimited by blue lines.*
- *S^3, just one cluster with all the nodes, delimited by the brown line.*

In (b), the same hierarchical segmentation, now specified in terms of edges:

- *B^0, given by the red, blue and black edges, leading to twenty clusters (the nodes).*
- *B^1, given by the red and blue edges, leading to three clusters.*
- *B^2, given by the red edges, defining two clusters.*
- *$B^3 = \emptyset$, defining a unique cluster with all nodes.*



(a) (b)

Figure 8: Hierarchical segmentations of image network of Example 2.1 by nodes (a) and by edges (b)

The following result is then extremely relevant.

Theorem 4.1. *Given a network image $N(I) = \{G = (V,E); D\}$, if $\widetilde{B} = \{(e, \mu_B(e)),\ e \in E\}$ is a fuzzy image segmentation, then for any sequence $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_{K-1} = 1 < \alpha_K$, the set $\{B(\alpha_0) = E, B(\alpha_1), \ldots, B(\alpha_{K-1}), B(\alpha_K) = \emptyset\}$ produces a hierarchical image segmentation.*

Proof. Given a sequence $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_{K-1} = 1 < \alpha_K$, we can denote by $\widetilde{S} = \{S^0 = \phi^{-1}(B(\alpha_0)), \ldots, S^K = \phi^{-1}(B(\alpha_K))\}$ the sequence of segmentations induced by the set $\{B(\alpha_0) = E, \ldots, B(\alpha_K) = \emptyset\}$. In order to prove that $\widetilde{S}$ is a hierarchical segmentation of $N(I)$, we have to check the following properties:

1. For all $t = 0, \dots, K$ each segmentation $S^t \in \widetilde{S}$ is an image segmentation of $N(I)$.
2. The trivial segmentations S^0 and S^K induced respectively by $\alpha_0 = 0$ and α_K are the two trivial segmentations (the segmentation which has all pixels as singleton clusters and the segmentation with one cluster containing all pixels).
3. $|S^t| > |S^{t+1}|$ for all $t = 0, 1, \dots, K-1$ (i.e., in each iteration the number of regions in the image decreases).
4. $S^t \widetilde{\subseteq} S^{t+1}$ for all $t = 0, 1, \dots, K-1$.

From the definition of fuzzy image segmentation it is clear that property 1 holds, since it is required that for all $\alpha \in (0,1)$, the set of links $B(\alpha)$ produces an image segmentation. Property 2 also holds, since the links set $B(0) = E$ produces the trivial partition in which all pixels belong to singleton cluster (i.e., $R_0 = \{\{v\}, v \in V\}$) and the set $\emptyset$ produces a partition with only one cluster in which all pixels belong. And properties 3 and 4 also hold, since for any iteration t the segmentations $B(\alpha_t)$ and $B(\alpha_{t+1})$ induced by the partitions S^t and S^{t+1} satisfies that $B(\alpha_t) \supset B(\alpha_{t+1})$.

Theorem 4.2. *Given a network image $N(I) = \{G = (V, E); D\}$, let $\boldsymbol{\mathcal{B}} = \{B^0 = E, B^1, \dots, B^K = \emptyset\}$ be a hierarchical image segmentation, and for all $t \in \{0, 1, \dots, K\}$ let $\mu^t : E \longrightarrow \{0, 1\}$ be the membership function associated to the boundary set $B^t \subset E$. Then, the fuzzy set $\widetilde{B}$ defined as:*

$$\mu_B(e) = \sum_{t=0}^{K} w_t \mu^t(e) \qquad \forall e \in E$$

induces a fuzzy image segmentation of $N(I)$ for any sequence $w = (w_0, w_1, \dots, w_K)$ such that

$$w_t \geq 0 \quad \forall t \in \{0, 1, \dots, K\} \qquad \sum_{t=0}^{K} w_t = 1$$

Proof. In order to prove that the fuzzy set $\widetilde{B}$ produces a fuzzy image segmentation, we only have to check that any α-cut over the fuzzy set $\widetilde{B}$ produces a crisp image segmentation.

Notice that the simplest segmentations, with $\alpha = 0$ and $\alpha = 1$, are direct.

Let $\alpha \in (0,1)$ be a value in the interval $(0,1)$, and let $B(\alpha) \subset E$ be the subset of edges with $\mu_B(e) \geq \alpha$.

Let $h_0 \in \{0, 1, \dots, K\}$ be the first hierarchical level such that $\sum_{t=0}^{h_0} w_t \geq \alpha$, and let us then see that $B(\alpha) = B^{h_0}$.

- $B(\alpha) \subset B^{h_0}$

  Taking into account that $\boldsymbol{\mathcal{B}}$ is a hierarchical image segmentation of $N(I)$, the following holds: $B^0 \supset B^1 \dots \supset B^{h_0} \supset \dots \supset B^K$.

  Let $e \in B(\alpha)$ be an arbitrary edge, then $\mu_B(e) = \sum_{t=0}^{K} w_t \mu^t(e) \geq \alpha > 0$. Since $\mu_B(e) > 0$ there exists at least one hierarchical level $i \in \{0, 1, \dots, K\}$ such that $e \in B^i$; let i_e be the greatest of such hierarchical levels with this property, *i.e.*, $e \in B^{i_e}$ and $e \notin B^{i_e+1}$.

By definition of i_e and taking into account the hierarchical property of $\boldsymbol{\mathcal{B}}$,

$$\alpha \leq \mu_B(e) = \sum_{t=0}^{K} w_t \mu^t(e) = \sum_{t=0}^{i_e} w_t \mu^t(e) + \sum_{t=i_e+1}^{K} w_t \mu^t(e) = \sum_{t=0}^{i_e} w_t 1 + \sum_{t=i_e+1}^{K} w_t 0$$

Therefore, $\sum_{t=0}^{i_e} w_t \geq \alpha$, and taking into account that h_0 is the first level that satisfies $\sum_{t=0}^{h_0} w_t \geq \alpha$, we have that $i_e \geq h_0$. Moreover, since $\boldsymbol{\mathcal{B}}$ is hierarchical, $B^{h_0} \supset B^{i_e}$.
Now, $e \in B^{i_e} \subset B^{h_0}$ and it is proven that $B(\alpha) \subset B^{h_0}$.

- $B^{h_0} \subset B(\alpha)$
  Let now $e \in B^{h_0}$ be a generic edge; since $\boldsymbol{\mathcal{B}}$ is hierarchical, $e \in B^t$ for any $t \leq h_0$; in this way:

$$\mu_B(e) = \sum_{t=0}^{K} w_t \mu^t(e) = \sum_{t=0}^{h_0} w_t \mu^t(e) + \sum_{t=h_0+1}^{K} w_t \mu^t(e) \geq \sum_{t=0}^{h_0} w_t 1 + \sum_{t=h_0+1}^{K} w_t \mu^t(e) \geq \alpha$$

  So $e \in B(\alpha)$ and it is proven that $B^{h_0} \subset B(\alpha)$.

Hence, for any α value there exist $t \in \{0, \ldots, K\}$ such that $B(\alpha) = B^t$. And since B^t produces a crisp image segmentation by definition, the result follows.

Remark 5. Let us observe that the previous theorem gives a way to build a family (that depends on the weights) of fuzzy image segmentations from a hierarchical one. How to determine the desirable weights is a question that will be addressed in the following sections.

In Example 2.1, from the hierarchical segmentation $\boldsymbol{\mathcal{B}} = \{B^0, B^1, B^2, B^3\}$*, see Figure 8 (b), the fuzzy image segmentation depicted in Figure 6 can be recovered with the weights* $(w_0 = 0.4; w_1 = 0.3; w_2 = 0.3, w_3 = 0)$.

5. Computational Experiences

The key consequence of the previous section is that the construction of a fuzzy image segmentation can be based on the construction of a hierarchical segmentation of the image network, $N(I) = (G = (V, E), D)$.

This section is organized in three subsections: a hierarchical segmentation algorithm will be outlined in the first subsection; the visualization process of a fuzzy segmented image network is explained in the second subsection; and some fuzzy segmented images are shown in the third subsection.

5.1. A hierarchical segmentation algorithm

The hierarchical segmentation algorithm we introduce next is divisive, i.e. it starts from the unique cluster V and ends with the trivial segmentation having as many clusters as pixels or nodes in the image network. This process is driven by successive partitions of the pixels: separately for any region of a given previous segmentation, the pixels will be segmented again. The hierarchical process finishes when there are as many regions as pixels.

Let K be the number of hierarchical levels; the hierarchical segmentation will be denoted as $\widetilde{S} = \{\mathcal{S}^0, \mathcal{S}^1, \ldots, \mathcal{S}^K\}$ where $\mathcal{S}^0 = \{R_1^0 = V\}$ is the initial trivial segmentation and $\mathcal{S}^K = \big\{R_1^K = \{p_1\}; R_2^K = \{p_2\}; \ldots; R_n^K = \{p_n\}\big\}$ is the terminal segmentation. The intermediate segmentations depend on the threshold values $\{\alpha_0, \alpha_1, \ldots, \alpha_K\}$, which are related to the family $\{d_e\ e \in E\}$:

$$\alpha_K > \overline{d} = \alpha_{K-1} > \cdots > \alpha_0 = \underline{d} \qquad (\overline{d} \equiv \max_{e\in E} d_e\ ; \underline{d} \equiv \min_{e\in E} d_e) \tag{2}$$

The keystone of the algorithm (see [18] for more details), is the following: given a segmentation $\mathcal{S}^{t+1} = \big\{R_1^{t+1}, R_2^{t+1}, \ldots, R_{s_{t+1}}^{t+1}\big\}$ for a given $t \in \{K-1, K-2, \ldots, 2, 1\}$, given an edge $e = \{p, q\} \in R_i^{t+1}$ for some $i \in \{1, 2, \ldots, s_{t+1}\}$, then: *the pixels p and q will belong to different regions of $\mathcal{S}^t$ if and only if $d_e \geq \alpha_t$*

To avoid inconsistencies, induced by different chains joining two pixels in the image network, cycles must be avoided; in this way, only the edges of the spanning forest defined by the spanning trees of the regions R_i^t will be used at each level t. This spanning forest is constructed following a Kruskal scheme where the edges are ordered: in a decreasing way for those edges with $d_e \geq \alpha_t$ and, afterwards, in an increasing way for the edges with $d_e < \alpha_t$.

The following pseudocode summarizes these ideas:

```
1.- Input:
      N(I) = (G = (V,E), {d_e, e ∈ E})      % the image network
      K      % the number of hierarchical levels
      {α_0, α_1, ..., α_K}      % threshold values for the hierarchical levels verifying:
      α_K > α_{K-1} = d̄ > α_{K-2} ··· > α_1 > α_0 = d_
2.- Initialization:
      t = K   E^t = E   G^t = (G,E)      % The initial graph
      s_K = 1   S^K = {R_1^K}      % (R_1^K = V, the initial partition)
3.- Process:
do t=K-1,0,-1
      E_f ≡ {e = {p,q} ∈ E^{t+1} | p ∈ R_i^{t+1}; q ∈ R_j^{t+1}; i ≠ j; i,j ∈ {1,2,...,s_{t+1}}}
      E^t = E^{t+1} − E_f
      E_d = {e_1, e_2, ..., e_{n_1}} ⊂ E^t      % list of divisive edges verifying:
            d_{e_1} ≥ d_{e_2} ≥ ··· ≥ d_{e_{n_1}} ≥ α_t
      E_l = {e_{n_1+1}, e_{n_1+2}, ..., e_{n_1+n_2}} ⊂ E^t      % list of linking edges verifying:
            d_{e_i} < α_t   ∀i ∈ {n_1+1, n_1+2, ..., n_1+n_2}
            d_{e_{n_1+1}} ≤ d_{e_{n_1+2}} ≤ ··· ≤ d_{e_{n_1+n_2}}
      L = E_d ∪ E_l = {e_1, e_2, ..., e_{n_1}, e_{n_1+1}, ..., e_{n_1+n_2}}      % List of edges
      F^t = (V, W^t) ⊂ G^t = (V, E^t)      % F^t is the spanning forest of G^t, construction:
         W^t = ∅
         do j=1,n_1 + n_2
            if (V, W^t ∪ e_j) is acyclic
               W^t = W^t ∪ e_j
            endif
         enddo
      S^t = {R_1^t, R_2^t, ..., R_{s_t}^t} is the partition of G^t, construction:
         W_l^t = W^t ∩ E_l   , the divisive edges are deleted from W^t
         R_i^t, for i = 1, ..., s_t are the connected components of H^t = (V, W_l^t)
enddo
```

In we apply this pseudocode to the example 2.1 with the parameters

$$K = 4 \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.0, 0.3, 0.6, 0.9, 1.0)$$

the hierarchical segmentation obtained is given by the four different segmentations depicted in Figure 8 (the segmentations obtained for values α_2 and α_3 are the same).

The computational complexity of this algorithm is $\mathcal{O}(Kf)$ where K is the number of hierarchical levels, being K bounded by $|V| = n$; and f is the maximum computational complexity of the following procedures at each iteration level:

1. Arrange the edges,
2. Compute the spanning forest,
3. Compute the new segmentation.

The three steps above are bounded by $\mathcal{O}(n^2)$, then, the complete algorithm has polynomial complexity.

5.2. Visualizing an image segmentation as an edge detection

How to visualize an image segmentation output is not a trivial task. A crisp image segmentation is usually visualized as a picture in which the pixels that belong to the same region are drawn by a colour that is obtained as an aggregation of the pixels that belong to this region. At this point, two crisp image segmentation output, obtained in [29], are shown in Figure 9. In columns three and four are depicted two crisp image segmentations, one (column 3) finer than the other one (column 4). Nevertheless, this approach presents some inconveniences: not all images are three dimensional and thus you cannot visualize them; if the number of homogenous regions are high the segmented image are quite identical than the original; and two adjacent but different incorrectly segmented regions could appear as only one in the visualization.

Alternatively, image segmentation can also be visualized by means of the contour of the regions that has been obtained. In this approach, the pixels are classified into black and white pixels, where the pixels represent the boundary pixels (those pixels that are between homogeneous regions) and the black pixels represent the core of the homogeneous regions. This way of visualization is more common in edge detection or contour problems (that are different problems since they are only interested in showing spectral differences between adjacent pixels). As can be seen in the picture A-B(2) of figure 9, a graduation of these white and black classes is possible as is done in the visualization of fuzzy edge detection solutions.

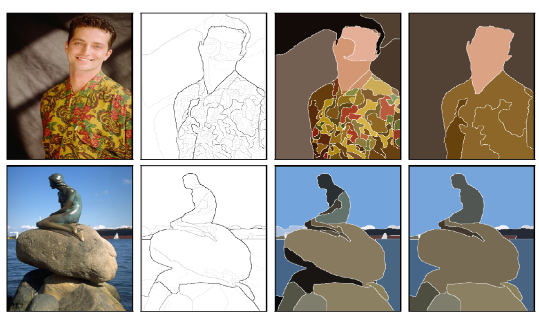


Figure 9: A-B(1) original image. A-B(2) A fuzzy edge detection visualization. A-B(3-4) Two crisp segmentation visualization. A-B(3) is finner than A-B(4)

Taking into account that our final aim is to visualize fuzzy image segmentations solutions, the first approach looks insufficient since it does not allow a gradation in the visualization process. So, we better propose to use the second approach in order to visualize our crisp image segmentation, our hierarchical image segmentation and, at the end, our fuzzy image segmentation.

The fist step in this section is how to classify the pixels, within a given crisp image segmentation S, into black and white. Formally, if we denote by $S(p)$ the region to which the pixel p belongs, we propose to classify the set of pixels $V = black \cup white$ in the following form:

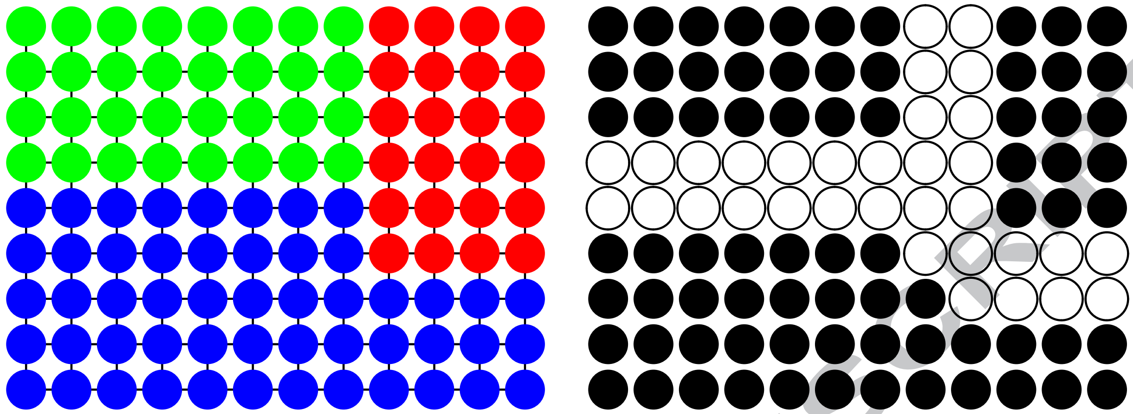
Definition 5.1. *Given a network image* $N(I) = \{\ G = (V, E);\ D\ \}$*, and given a segmentation* $S = \{R_1, \ldots, R_k\}$*, we will define the white and black class in the following way:*

- $black = \{p \in V \textit{ such that } S(p) = S(p')\ \forall e = (p, p') \in E\}$
- $white = \{p \in V \textit{ such that there exist } e = (p, p') \in E \textit{with } S(p) \neq S(p')\}$.

where $S(p)$ *is the region to which the pixel* p *belongs.*

Let us point out again that one pixel is coloured white if it has at least one neighbour in other region. The pixels that are coloured black are those that are bounded by pixels of the same region. It is important to emphasize that it is possible to go from an image segmentation solution into an edge detection solution but in general they are different problems. The edge detection problems (see [30] for example) are defined as a partition of the set of nodes into two classes: the region and the boundary (usually painted as black and white colours respectively). As it is pointed in [30], an edge detection solution should satisfy some desirable properties, and in general an edge detection solution no necessarily disconnects the image into the homogeneous regions, so the image segmentation solution obtained from an edge detection solution is usually not suitable. Since the opposite is true, it is not trivial how to go from a suitable image segmentation output into a suitable edge detection solution (although this implication is easier than the other).

Example 5.1. *In order to visualize a crisp image segmentation let us consider the following picture.*



(a) (b)

Figure 10: A segmented image network (a) and its visualization (b)

Once the problem of visualizing a crisp image segmentation is solved, we have to deal with the problem of how to visualize a fuzzy image segmentation. Now the problem here is how to build the fuzzy class *White* over the nodes set. In this work we will consider that the *Black* fuzzy class is built as a negation of the *White* class.

Given a fuzzy image segmentation $\widetilde{B}$, with membership function $\mu_B : E \longrightarrow [0,1]$, a natural way (but not the only one) to build the membership functions μ_{white} and μ_{black} is the following:

Definition 5.2. *Given a network image* $N(I) = \{\ G\ =\ (V,E);\ D\ \}$*, and given a fuzzy image segmentation* $\widetilde{B}$ *with membership function* $\mu_B : E \longrightarrow [0,1]$*, we define the white and black fuzzy classes in the following way:*

- $\mu_{white}(p) = MAX\{\mu_B(e), for\ all\ e = (p,q) \in E\}\ \forall p \in V.$
- $\mu_{black}(p) = N(\mu_{white}(p)) = 1 - \mu_{white}(p)\ \forall p \in V.$

Remark 6. Let us note that in the previous definition, the degree of membership of a given pixel p to the fuzzy class *boundary-white* is the result of the aggregation of the values$\{\mu_{\widetilde{B}}(p,q_i), i = 1,\dots,4\}$, where $q_1,\dots,q_4$ are the neighbours of the pixel p. In general, the aggregation operator $\phi : [0,1]^4 \longrightarrow [0,1]$ used in this process should satisfy the following properties:

1. ϕ is symmetric.
2. $\phi(x_1,\dots,x_4) = 0$ if and only if $x_i = 0$, for all $i = 1,\dots,4$ (i.e., a pixel that is always in the same region as its neighbours should have degree zero to the boundary class).
3. $\phi(x_1,\dots,x_4) = 1$ if and only if there exist $i \in \{1,\dots,4\}$ with $x_i = 1$ (i.e., a pixel that is always in the boundary since it always has an adjacent pixel in a different region should have degree one to the boundary class).

In order to simplify this visualization process, we have used the *MAX* aggregation operator, but other aggregation functions can be obviously considered.

We would like to emphasize again that a black/white segmentation-visualization is the way in which our concept of fuzzy image segmentation should be shown, since this kind of visualization permits in a natural way a representation of the different intensities that represent the different degrees up to which the nodes belong to the frontier class. As it happens with any clustering or classification problem, a fuzzy output presents many advantages over crisp output. The information provided by a fuzzy output is clearly more rich as it enables to show the different graduations of the objects to the different classes in the clustering/classification problem. Obviously, if a crisp output is needed it is always possible to obtain from a fuzzy output through deffuzification. Let us recall again Figure 9, in which two crisp image segmentations were shown (one of them is finer that the other). One of the problems in clustering (and image segmentation is not different at this regard) is to determine how many segmented regions in image segmentation are. Figure 9 shows two crisp image segmentations that depend on the detail-level of the segmentation output being demanded by the users. However, a fuzzy image segmentation output amalgamates all this information in just one picture, since its allows to see the regions that should appear in the first steps of the clustering problem colored with a higher white intensity, and the regions that appear in finer (or more detailed) segmentation outputs with a lower white intensity. In addition with all of these visualization advantages, it is possible to build a crisp (or hierarchical sequence of crisp) segmentation outputs from a fuzzy image segmentation by just giving an alpha-cut (or a sequence of alpha values). Higher values of alpha give a crisp segmentation with lower details, and lower values of alpha will provide a segmentation output with higher details. Anyway, we stress again the importance and the advantages of a fuzzy (image segmentation) output over any crisp one.

5.3. Some fuzzy segmented images

In this section, we show some computational experiences by applying the algorithm outlined in Subsection 5.1, that gives a hierarchical segmentation, to some RGB-color and W/B images. These images have been obtained from the Berkeley database http://www.eecs.berkeley.edu/Research/ Projects/ CS/ vision/bsds/ We will show the fuzzy segmented image obtained from the hierarchical image segmentation procedure described in previous sections.

Let $P_1^{\text{rgb}} = (r_1, g_1, b_1)$ and $P_2^{\text{rgb}} = (r_2, g_2, b_2)$ be two colours on the RGB-space, its representation on the CIELAB space is given by $P_1^{\text{lab}} = (L_1, a_1, b_1)$ and $P_2^{\text{lab}} = (L_2, a_2, b_2)$ respectively. Now, one way to measure the dissimilarity between P_1^{lab} and P_2^{lab} is

$$d_{1,2}^{\text{lab}} = \sqrt{(\Delta L)^2 + (\Delta a)^2 + (\Delta b)^2} \tag{3}$$

where $\Delta L = L_2 - L_1$, $\Delta a = a_2 - a_1$ and $\Delta b = b_2 - b_1$.

The distance (3) is known as CIE76, and it is one of the definitions of dissimilarity between colors widely used, given by the CIE (International Commission on Illumination) in 1976. Distance CIE76 can be considered a standard distance, once it has been widely used in various applications. There are certainly alternative versions and corrections to (3), as are CIE94 and CIE2000 for example, but indeed CIE76 looks good enough for our current purposes.

a b

Figure 11: (a) Original Church image. (b) Visualization of a fuzzy segmented image

a b

Figure 12: (a) Original Japon image. (b) Visualization of a fuzzy segmented image

a b

Figure 13: (a) Original Pilots image. (b) Visualization of a fuzzy segmented image

Figure 14: (a) Original Reno image. (b) Visualization of a fuzzy segmented image

6. Final Comments

First at all, we would like to remind the amount of papers that use fuzzy techniques to obtain crisp segmentations, suggesting that the terms fuzzy and image segmentation are strongly connected. Such a connection is also present within classification, where there also exist a huge amount of papers that (at the end) produce a crisp classification based on fuzzy techniques. In such a classification framework we should distinguish between the term *fuzzy* classification (that produces a set of fuzzy classes, each one with a membership function) and the term *crisp fuzzy based* classification (that produces a crisp classification using fuzzy algorithms). But this distinction is sometimes difficult to realize in image segmentation, once the concept of *fuzzy image segmentation* has not been clearly defined. Such a formal clarification is a major consequence of our approach in this paper, since the term fuzzy image segmentation has been quite often associated to crisp fuzzy based segmentation.

In this paper, we have proposed a formal definition of the concept *fuzzy image segmentation*, by means of a fuzzy class over the links set. We think that this approach is more natural and consistent to human segmentation. When we try to segment an image into homogeneous regions, we might be able to identify some regions first than others, since their contour or boundary appear more clearly to us.

In order to build our proposal, we have first proven that there exists a bijection between the set of all crisp image segmentations and the set of what we have called *boundary links*. A crisp image segmentation is univocally determined by such *boundary links*. For a given segmentation, these links are those that connects pixels of different regions, and thus represent the boundary of the segmentation process.

Once this association between boundary links and the set of homogeneous regions is established within a crisp framework, fuzzy image segmentation is defined as a fuzzy class over the set of links. For a given edge, the membership function of this fuzzy class represents the degree of separation between two adjacent pixels in the segmentation process. Nevertheless, the task of building a fuzzy image segmentation is not trivial, since a dissimilarity measure between adjacent pixels might not be an adequate solution. In order to build a fuzzy image segmentation in a natural way, in this paper we study and analyze the strong relations between the concept of fuzzy image segmentation and that of hierarchical image segmentation. The connection we have proven allows the construction of a fuzzy image segmentation from a hierarchical image segmentation, and viceversa.

In addition with the definition of a new problem and its relations with some existing ones, in this paper we have proposed a method to build a fuzzy image segmentation from an hierarchical image segmentation. The hierarchical image segmentation problem has not been studied very much, but we have introduced some modifications to the $D\&L$ algorithm (previously developed by the authors in another paper), in such a way that this adaptation allows to build the hierarchical image segmentation and the posterior fuzzy image segmentation. Some computational experiments reflects the advantages of using fuzzy image segmentation instead of crisp image segmentation.

One of the main problems when working on theories involving segmentation thresholds is to choose a set of appropriate thresholds for the segmentation process. If we want to provide the threshold set explicitly, we need to manipulate the pixel values in any colour space, and the distances between them in a measure space. However, in our proposed algorithm it is more important the number of cutting edges (edges with large values) included in a given step of segmentation, than the threshold value α_i. Because of this, the only thing that matters to the hierarchical segmentation proposed is the number of steps (the K value). Our proposed algorithm can be executed using coarse nets, which reduces the computational time substantially. The theory of aggregation functions can be used for this purpose, i.e., to relate sets of pixels through some aggregation function, so that results can be obtained more quickly and preserving a good classification. Performances of different aggregator functions in networks can be a future research work.

How to define a fuzzy image segmentation or which characteristics should be imposed to a fuzzy classification of the nodes, in order to guarantee a suitable fuzzy image segmentation, is a question that should be also explored.

Finally, we would like to remind that the main aim in this paper is to clarify what should be understood by Fuzzy Image Segmentation, as well as its relationship with other well-known Image Segmentation approaches. Our point is the need of a visualization technique for a proper fuzzy segmentation. In this sense, the proposed algorithm has been included just to show our approach to be operative. Moreover, it is not obvious how the crisp output of any (either fuzzy-based or not) Image Segmentation algorithm and the fuzzy output of any Fuzzy Image Segmentation method should be compared (see [31] for a first attempt in that direction). Indeed, future work will be devoted to study this question.

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