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NUMERICAL EVALUATION OF RENEWAL EQUATIONS: APPLICATIONS TO RISK THEORY AND FINANCIAL MODELS

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NUMERICAL EVALUATION OF RENEWAL EQUATIONS: APPLICATIONS TO RISK THEORY AND FINANCIAL MODELS

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Numerical evaluation of Renewal equations: applications to Risk Theory and financial models.

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ABSTRACT The so-called Renewal Theory is a frequently used methodology in applied mathematics. Renewal Theory is mainly focussed on solving a Volterra integral equation of the second kind known as Renewal Integral Equation:

$$\Phi(u) = h(u) + \int_0^u \Phi(u-x) d_x F(x)$$

An interesting problem arises when choosing the appropriate numerical tool in order to approximate the solution of the former integral. The decision will be based on the degree of knowledge of function F(x) and some properties of $\Phi(u)$.
Three methods based in classical methodologies (simulation, product integration and inverting Laplace transform) will be presented and applied to the calculation of ultimate ruin probabilities in the classical case of Risk Theory.
The first one is an original simulation scheme, based on the importance sampling technique, that leads to tight interval estimations of the solution of the Renewal equation.
In the second one, the use of the so-called Product Integration technique will be considered and compared with other techniques based on the Newton-Cotes methodology.
The last method considered is the Gaver-Stehfest algorithm of inverting Laplace transform. This last one, under certain conditions, could be considered as a very fast and accurate method.

1. MONTE CARLO SIMULATION APPROACH

Feller ([8]. *pg. 183)* proved that the solution of a renewal equation could be expressed using the following convolution:

$$\Phi(u) = R * h(u)$$

where $R(x)$ is the renewal function:

$$R(x) = \sum_{t=0}^{\infty} F^{*n}(x)$$

and $F^{*t}(x)$ the t-fold convolution of function F(x) (F(0)=0).

In many situations the function F(x) could be approximated for certain values of its argument and even random numbers could be obtained but the first derivative can be hard to find.

In order to solve the renewal equation we will focus our attention on the evaluation of the renewal function.

Let us express the t-fold convolution of the function F(x) using a multiple integral:

$$\int_{\mathcal{R}(\mathbf{X})} \varphi(\mathbf{S}) = F^{*t}(x) =$$

$$= \int_0^x \int_{S_1}^x \cdots \int_{S_{t-2}}^x F(x - S_{t-1})\, d_{S_{t-1}} F(S_{t-1} - S_{t-2}) \cdots$$
$$\cdots d_{S_2} F(S_2 - S_1)\, d_{S_1} F(S_1)$$

Citing **Fishman** ([9]. *Chapter 2.),* "... the convergence of deterministic methods seems clearly better (specially if the dimension of the integral, t, is not large) when they can be applied. Nevertheless, the applicability matters make Monte Carlo techniques competitive because the verification of the conditions we cited for deterministic methods becomes very difficult as t increases or very restrictive for function $\varphi(\mathrm{X})$. Monte Carlo methods can be applied considerably more broadly to functions that merely satisfy $\int_{\Re} \varphi^2(\mathbf{X}) < \infty$. Also, the Monte Carlo error depends on φ only through this integral, and in no way on the continuity and variational properties of φ. Another interesting fact is that Monte Carlo methods allow one to estimate error from generated data, whereas one needs to rely in considerably more global measures of error when employing deterministic techniques. Finally, the Monte Carlo convergence is always $O\left(n^{-\frac{1}{2}}\right)$ regardless the dimension of the integral t, this is an interesting aspect when t is large because in the deterministic methods this convergence worsen as t increases."

It is clear that when we work with approximations of function F(x), the restrictive conditions for deterministic methods are really difficult to prove, that is an important statement to introduce Monte Carlo simulation methods.

In **Usábel** (*[22] [23])*, a method based in Monte Carlo simulation is designed to solve a general type of multiple integrals frequently used in Risk Theory that models the probability that the paths of an discrete stochastic process $\{S_t\}$ will be bounded by vector $\mathbf{X} = (x_1, x_2, \ldots, x_t)$ and $\mathbf{0}$:

$$\int_{\mathcal{R}(\mathbf{X})} \varphi(\mathbf{S}) =$$

$$= \int_0^{x_1} \int_{S_1}^{x_2} \cdots \int_{S_{t-2}}^{x_{t-1}} G_t(x_t - S_{t-1})\, g_{t-1}(S_{t-1} - S_{t-2}) \cdots$$
$$\cdots g_2(S_2 - S_1)\, g_1(S_1)\, dS_1 \cdots dS_{t-1}$$

and :

$$S_t = \sum_{i=1}^{t} z_i \qquad z_i \geq 0$$

where $z_i \quad i = 1, ..., t$ are independent random variables with p.d.fs. $g_i(x)$, c.d.fs. $G_i(x)$ and $E_{g_i}\{z_i\} = \mu_i < \infty$.

The t-fold convolution of a function, $F^{*t}(x)$, is a particular case of these family of integrals.Let us define now an estimator using an importance sampling scheme for $F^{*t}(x)$:

$$\aleph^{*t}(x,n) = \frac{\sum_{i=1}^{n} \mathcal{F}_i^{*t}(x)}{n} =$$

$$= \frac{\sum_{i=1}^{n} F\left(x - S_{t-1}^i\right) F\left(x - S_{t-2}^i\right) \ldots F\left(x - S_1^i\right) F(x)}{n}$$

where S_j^i are random numbers generated from the p.c.fs. for $i = 1, \ldots, n$ and $j = 1, \ldots, t-1$:

$$S_1^i \longrightarrow \mathfrak{d}_1^i\left(S_1^i\right) = \frac{F\left(S_1^i\right)}{F(x)} \qquad S_1^i \in [0, x]$$

$$S_j^i \longrightarrow \mathfrak{d}_j^i(S_j) = \frac{F\left(S_j^i - S_{j-1}^i\right)}{F\left(x - S_{j-1}^i\right)} \qquad S_j^i \in \left[S_{j-1}^i, x\right] \qquad j > 1$$

and $\{\mathcal{F}_i^{*t}(x)\}_{i=1}^n$ is a sample of independent estimators

$$\mathcal{F}_i^{*t}(x) = F\left(x - S_{t-1}^i\right) F\left(x - S_{t-2}^i\right) \ldots F\left(x - S_1^i\right) F(x)$$

As it is proved in **Usábel** ([23]) , $\aleph^{*t}(x,n)$ is an unbiased and consistent estimator with variance bounds (in the non trivial case):

$$Var\left\{\aleph^{*t}(x,n)\right\} = \frac{Var\left\{\mathcal{F}^{*t}(x)\right\}}{n} < \frac{F^{*t}(x) - \left[F^{*t}(x)\right]^2}{n}$$

$$Var\left\{\aleph^{*t}(x,n)\right\} \leq \frac{\left(F(x)\right)^t F^{*t}(x) - \left[F^{*t}(x)\right]^2}{n}$$

In the numerical illustrations presented in the former reference, the percentage of the true variance with respect to the first upper bound (direct simulation variance) ranged from 2% to 47%.

Under fairly general conditions that include $\int_{\Re} \varphi^4(\mathbf{X}) < \infty$ (Fishman) :

$$\lim_{n \to \infty} \quad \aleph^{*t}(x,n) \longrightarrow N\left[F^{*t}(x), \sqrt{\frac{Var\left\{\mathcal{F}^{*t}(x)\right\}}{n}}\right]$$

and the interval estimation with a confidence level 1-α :

$$\left[\aleph^{*t}(x,n) \mp \phi(1-\alpha)\sqrt{\frac{Var\left\{\mathcal{F}^{*t}(x)\right\}}{n}}\right]$$

we can use an estimator of the variance of the $\mathcal{F}^{*t}(\mathbf{X})$:

$$Var\left\{\mathcal{F}^{*t}(x)\right\} \simeq k = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(\mathcal{F}_i^{*t}(x)\right)^2 - n\sum_{i=1}^{n}\mathcal{F}_i^{*t}(x)\right)$$

as recommended in **Fishman**([9] pg 68), k is a strongly consistent estimator of $Var\left\{\mathcal{F}^{*t}(x)\right\}$,then an asymptotically valid confidence interval can be:

$$\mathcal{F}^{*t}(x) \in \left[\aleph^{*t}(x,n) \mp \phi(1-\alpha)\sqrt{\frac{k}{n}}\right]$$

One of the main features of this method is the fact that if we increase the dimension t, we do not have to start again the simulation process as should be done in direct simulation. When we get the estimator and store these pairs of values:

$$\left(\mathcal{F}_i^{*t}(x), S_{t-1}^i\right) \qquad i = 1, \ldots, n$$

then:

$$\aleph^{*t+1}(x,n) = \frac{\sum_{i=1}^{n}\mathcal{F}_i^{*t+1}(x)}{n} =$$

$$\frac{\sum_{i=1}^{n} F\left(x - S_t^i\right) F\left(x - S_{t-1}^i\right) F\left(x - S_{t-2}^i\right) \ldots F\left(x - S_1^i\right) F(x)}{n}$$

$$= \frac{\sum_{i=1}^{n} F\left(x - S_t^i\right) \mathcal{F}_i^{*t}(x)}{n}$$

using :

$$S_t^i \longrightarrow \mathfrak{d}_t^i(S_t) = \frac{F\left(S_t^i - S_{t-1}^i\right)}{F\left(x - S_{i-1}^i\right)} \quad S_t^i \in \left[S_{t-1}^i, x\right] \quad i = 1, \ldots, n$$

This last result means that increasing one unit the dimension of the multiple integral only imply generating n random numbers more and the total amount of random numbers required is $n(t-1)$, where t is the dimension considered. The save of number of steps – random numbers in our case – become even more obvious when we need to evaluate the integral $F^{*t}(x)$ for $t = 1, 2, \ldots, l$, one by one up to a certain integer l, as the convolutions in the solution of a renewal equation, in these cases the total amount of steps still remains $n(l-1)$:

$$R(x) \simeq \sum_{t=0}^{l} F^{*n}(x)$$

As an application to Risk Theory the probability of ultimate survival with initial reserves U could be written as a compound process(see for example **Grandell** ([11]):

$$\Phi(U) = \sum_{t=0}^{\infty} \Phi(0)\left(1 - \Phi(0)\right)^t F^{*t}(U)$$

solution of the renewal equation:

$$\Phi(u) = \Phi(0) + \int_0^u \Phi(u-x) d_x \left(1 - \Phi(0)\right) F(x) \qquad u \geq 0$$

The infinite sums were calculated up to a certain number of terms (l) for which the rest of the terms of the sums were smaller than 10^{-10} and $F^{*t}(U)$ is the t-fold convolution of the distribution of the record heights of the risk process random walk. This last distribution could be approximated using a sample of record heights and then use the former method as stated in **Usábel** *([21])*. In order to compare results with other methods we will test examples in the Classical case of Risk Theory (although an explicit formula for F(x) can be found in this very case):

$$F(x) = \int_0^x \frac{1 - \mathfrak{C}(t)}{p_1} dt$$

and $\mathfrak{C}(x)$ is the c.d.f. of the claim size , $E_c\{x\} = p_1$ and θ the security loading.

The t-fold convolution $\mathrm{F}^{*t}(U)$ could be expressed :

$$F^{*t}(U) = H^{*t}(u)$$

$$\simeq \frac{\sum_{i=1}^{bn} F\left(U - S_{t-1}^{i}\right) F\left(U - S_{t-2}^{i}\right) \ldots F\left(U - S_{1}^{i}\right) F(U)}{n}$$

This approach was tested for Pareto claim size and different values of θ and the initial reserves(u),

$$\left(\mathfrak{C}(x) = 1 - \left(\frac{\mu}{\mu + x}\right)^{\mu+1} \mu = 1\right)$$

Table I.

Confidence intervals for survival probability. $\alpha = 0.01 \qquad n = 5,000$

$\Phi(u) = + \int_0^u \Phi(u-x) d_x \left(1 - \Phi\left(0\right)\right) F(x) \qquad u \geq 0 \qquad \Phi\left(0\right) = \frac{\theta}{1+\theta}$

$F\left(x\right) = \int_0^x \left(\frac{\mu}{\mu+t}\right)^{\mu+1} dt \qquad \mu = 1$ Pareto Claim size

Security Loading	initial reserves(U)	lower limit	upper limit	terms of the sum(l)
	20	0.4994	0.5058	38
	100	0.8305	0.8369	89
$\theta = 0.10$	500	0.9742	0.9756	189
	1000	0.9878	0.9889	226
	20	0.7532	0.7585	35
	100	0.9469	0.9492	70
$\theta = 0.25$	500	0.9913	0.9915	100
	1000	0.9956	0.9959	103
	20	0.8792	0.8823	30
	100	0.9766	0.9775	50
$\theta = 0.50$	500	0.9957	0.9959	57
	1000	0.9979	0.9979	57
	20	0.9230	0.9249	27
	100	0.9853	0.9857	39
$\theta = 0.75$	500	0.9972	0.9973	41
	1000	0.9986	0.9986	42
	20	0.9448	0.9460	24
$\theta = 1.00$	100	0.9890	0.9894	32
	500	0.9979	0.9979	34
	1000	0.9989	0.9989	34

2. Product Integration approach

Let us consider the case now when the function f(x) is known explicitly, then the renewal equation could be written:

$$\Phi(u) = h(u) + \int_0^u \Phi(u-x)f(x)dx \qquad u \geq 0$$

that it is a Volterra integral equation of the second kind. In many applications of renewal Theory the value of the function at u=0 is the only initial condition; this last fact means that the quadrature rules of the methods used should be of lower order. In this context, the rate of convergence of the methods could be significantly slow when f(x) is a heavy-tailed or bad-behaved function, specially when u is large. Under these constraints, a reasonable approach to test in order to improve the rate of convergence, could be a numerical tecnique that includes more information about the function f(x) than merely its values in certain points of the domain.

The so-called product integration uses a quadrature rule whose weights are define as integrals of the function f(x).

Following **Delves and Mohamed** ([7]) we discompose the interval [0,u] into n subintervals $\{h_i\}$ where:

$$h_i = s_{i+1} - s_i \qquad i = 0, 1, ..., n-1$$

and $0 = s_0 < s_1 < ... < s_n = u$.

Product integration proceeds by approximating the renewal equation for s=$s_i, i = 1, 2, ..., n$ using a quadrature rule of the form

$$\int_0^{s_i} \Phi(s_i - t)f(t)dt \approx \sum_{j=0}^{i} w_{ij}\Phi(t_j)$$

where $t_i = s_i$ for $i = 0, 1, 2, ..., n$. The weights are determined by insuring that the rule of the former expression is exact when $\Phi(t)$ is a polynomial in t of degree $\leq d$. Product integration is only applicable if the following (d+1) moments μ_{ij} exist and can be calculated for each i, where

$$\mu_{ij} = \int_0^{s_i} t^j f(s_i - t)dt, \qquad j = 0, 1, ..., d.$$

We will also assume that $\Phi(t)$ is linear (d=1) in t, i.e.,

$$\Phi(t) \approx \frac{(t_{j+1} - t)}{h_j}\Phi(t_j) + \frac{(t - t_j)}{h_j}\Phi(t_{j+1})$$

it follows that:

$$\int_0^{s_i} \Phi(s_i - t) f(t) dt = \int_0^{s_i} \Phi(t) f(s_i - t) dt \approx$$

$$\approx \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} f(s_i - t) \left[\frac{(t_{j+1} - t)}{h_j} \Phi(t_j) + \frac{(t - t_j)}{h_j} \Phi(t_{j+1}) \right] dt =$$

$$= \sum_{j=0}^{i} w_{ij} \Phi(t_j)$$

where:

$$w_{i0} = \int_{t_0}^{t_1} \frac{(t_1 - t)}{h_0} f(s_i - t) dt \qquad for \text{ j=0}$$

$$\begin{aligned} w_{ij} &= \int_{t_j}^{t_{j+1}} \frac{(t_{j+1} - t)}{h_j} f(s_i - t) dt + \int_{t_{j-1}}^{t_j} \frac{(t - t_{j-1})}{h_{j-1}} f(s_i - t) dt \\ j &= 1, 2, ..., i - 1 \end{aligned}$$

$$w_{ii} = \int_{t_{i-1}}^{t_i} \frac{(t - t_{i-1})}{h_{i-1}} f(s_i - t) dt \qquad for \text{ j=i}$$

it is obvious that the weights include much more information about the function f(x) than a pure Newton-Cotes based method for integral equations.

Thus, the approximate solution of the renewal equation is determined recursively using

$$\Phi^a(s_i) = h(s_i) + \sum_{j=0}^{i} w_{ij} \Phi^a(t_j)$$

for $i = 1, 2, ..., n$, with

$$\Phi^a(s_0) = h(0)$$

the resulting estimate of $\Phi(u)$ is $\Phi^a(s_n)$.

As stated in **Ramsay**(*[17]*) the convergence could be accelerated using Richardson extrapolation tecnique.

This method was tested in solving the ultimate non-ruin probability function in the classical case of risk theory, **Ramsay and Usábel**([21]). An example is included considering Pareto claim size, obtaining a rate of convergence significantly better than other Newton-Cotes based methods used in actuarial literature.

Table II.

$\Phi(u) = \Phi(0) + \int_0^u \Phi(u-x)(1-\Phi(0)) f(x)dx \qquad u \geq 0$

$\Phi(0) = \frac{\theta}{1+\theta}$ minimum value of h= $\frac{u}{320}$,

Pareto Claim size $F(x) = \int_0^x \left(\frac{\mu}{\mu+t}\right)^{\mu+1} dt \qquad \mu = 1$

Significant digits 6-8

Security Loading	initial reserves(u)	$\Phi(u)$
	20	0.501858
	100	0.835140
$\theta = 0.10$	500	0.974877
	1000	0.988659
	20	0.754740
	100	0.947773
$\theta = 0.25$	500	0.991293
	1000	0.995806
	20	0.880726
	100	0.977161
$\theta = 0.50$	500	0.995835
	1000	0.997954
	20	0.924092
$\theta = 0.75$	100	0.985483
	500	0.997263
	1000	0.998647
	20	0.944951
$\theta = 1.00$	100	0.989370
	500	0.997962
	1000	0.998989

3. Numerical inversion of Laplace transform approach

The former approach when f(x) is known, although it can be considered outstanding, it is not the most accurate and fast for functions that fullfil some requirements (not very restrictive in applied mathematics models). The last method of solving renewal equations is devoted to numerical inversion of Laplace transforms.

The Laplace transform of a renewal equation is expressed,

$$\Phi^*(s) = \int_0^{+\infty} e^{-sx}\Phi(x)\,dx = H^*(s) + \Phi^*(s)\,F^*(s) \qquad s > 0$$

finally:

$$\Phi^*(s) = \frac{H^*(s)}{1 - F^*(s)} \quad s > 0$$

where:

$$H^*(s) = \int_0^{+\infty} e^{-sx} h(x)\,dx$$

$$F^*(s) = \int_0^{+\infty} e^{-sx} f(x)\,dx \qquad s > 0$$

when the former two integrals exist, the solution of a renewal equation can be obtained by inversion of $\Phi^*(s)$.

Exact inversion of Laplace transforms using complex analysis can be considered a difficult task in many cases, that is the reason we consider more reliable, for applied mathematics purposes, the use of a numerical tecnique of inversion .

It is clear that many numerical techniques can be used for our goal, see for example **Davies and Martin** *([5])*, and that accuracy and applicability rely on the properties of $\Phi^*(s)$ and $\Phi(s)$. Nevertheless, when the solution of the Laplace transform $\Phi^*(s)$ is a continuous function, with neither salient points nor rapid oscillations, the Gaver-Stehfest algorithm of inverting Laplace transform could be considered as "very reliable" in terms of number of calculations vs. significant digits compared with other numerical techniques of inverting Laplace transforms. **Stehfest** *([20])*.

Citing **Davies and Martin**, an approximate inversion algorithm for the Laplace transform can be obtained by computing a sample,

$$I_n(t) = \int_0^{\infty} \delta_n(t,x)\Phi(x)\,dx$$

where $I_n(t)$ tends to $\Phi(x)$ with increasing n, using the function

$$\delta_n(t,x) = (2n)!\,(n!(n-1)!)\,a(1-e^{-ax})^n e^{-nax}$$

Gaver([10]) proved that $I_n(t) - \Phi(x)$ may be expanded in an asymptotic expansion in inverse powers of n; consequently the result may be improved using extrapolation. The most useful extrapolation formula has been derived by Stehfest to give the algorithm:

$$\Phi(u) \simeq z \sum_{n=1}^{N} k_n^N \Phi^*(nz)$$

where:

$$z = \frac{\ln(2)}{u}$$

$$k_n^N = (-1)^{n+\frac{N}{2}} \sum_{i=\left[\frac{(n+1)}{2}\right]}^{Min\left(n,\frac{N}{2}\right)} \frac{i^{\frac{N}{2}}(2i)!}{\left(\frac{N}{2}-i\right)!i!\,(i-1)!\,(n-i)!\,(2i-n)!}$$

and N is even.

Theoretically the method becomes more accurate with increasing N, but the result may not be very accurate when $\Phi(u)$ is not a "well-behaved" function with discontinuities or rapid oscillations. Another advantage of the method is that $\Phi^*(s)$ is only evaluated for real values of its argument.

The functions included in the models of Risk Theory usually fulfill the "well-behaved" condition and with an example we will prove that the use of the Gaver-Stehfest algorithm of inverting Laplace transforms is a very powerful tool to consider when solving renewal equations.

Let us consider the same illustration presented in the former two methodologies.

4. Conclusions

The former three methods presented are designed to deal with three different degrees of knowledge about function F(x). As it was expected, the more information about F(x) the more accurate the approximation of the solution of the renewal equation is, for a fixed computational time.

Properties of the estimator $\aleph^{*t}(x,n)$, presented in the first section, make Monte Carlo simulation a suitable method when the values of function F(x) can only be approximated and samples are obtained from these "truncated" distribution functions:

$$\mathfrak{d}_j(S_j) = \frac{F(S_j - S_{j-1})}{F(x - S_{j-1})} \qquad S_j \in [S_{j-1}, x]$$

In order to make the approach efficient we should choose the optimal method for sample generation. For a detailed review on sample generation see **Fishman** ([9]) or **Bratley, Fox and Schrage** *([2])*.

When the first order derivative of function f(x) exist and the following "moments" can be obtained or approximated up to s_i=u,

$$\mu_i = \int_0^{s_i} t\ f(s_i - t)dt$$

we show in section 2 that the use of product integration can certainly lead us to very accurate results with much less computations than Newton-Cotes based methods, as it was suppose to be, due to the optimal use of the information about f(x).

Finally, when these two integrals exist and can be calculated or approximated:

$$H^*(s) = \int_0^{+\infty} e^{-sx} h(x)\, dx$$

$$F^*(s) = \int_0^{+\infty} e^{-sx} f(x)\, dx \qquad s > 0$$

we can get the Laplace transform of the solution of a renewal equation:

$$\Phi^*(s) = \frac{H^*(s)}{1 - F^*(s)}$$

and use the Gaver-Stehfest algorithm of inverting Laplace transforms. Under constrains of continuity and smooth behavior for function $\Phi(u)$, not very restrictive for Risk Theory models, this approach can be considered highly reliable in terms of the ratio accuracy-number of computations. We should bear in mind that we use more information about f(x) than in the second approach because the limits of the integral of the Laplace transform are 0 to infinity.

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