

WEAK GEODESICS ON PROX-REGULAR SUBSETS OF RIEMANNIAN MANIFOLDS

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ABSTRACT. We give a definition of weak geodesics on prox-regular subsets of Riemannian manifolds as continuous curves with some weak regularities. Then obtaining a suitable Lipschitz constant of the projection map, we characterize weak geodesics on a prox-regular set with assigned end points as viscosity critical points of the energy functional.

1. INTRODUCTION

A classical topic in differential geometry and global nonlinear analysis is the study of geodesics on a Riemannian manifold M without boundary. Considering the paths with assigned extreme points, the problem of existence and multiplicity of geodesics on manifolds without boundary was studied in the classical works [28, 30]. Indeed, it was proved that they are critical points of the energy functional on the smooth manifold X of the admissible paths and by means of Morse and Lusternik-Schnirelman theory, the multiplicity results are obtained.

In the case in which the manifold M has boundary, even if M is smooth, different kind of irregularities may be developed. For example, the natural domain of the energy functional, i.e., the Sobolev space of $W^{1,2}$ -paths on M has no more the structure of a Hilbert manifold and there is no uniqueness for the Cauchy problem. Moreover, in this case the geodesics are not in general C^2 , however they are differentiable curves with locally Lipschitz derivative. To obtain the results regarding this issue, the theory of critical points and gradient flows for some lower semicontinuous functions is employed; see [2, 22, 29, 33].

The above manifolds are basically smooth and it is natural to study manifolds with a certain degree of irregularity. Various extensions have been considered in this regards, for instance one can refer to conical manifolds. A conical manifold M is a complete m -dimensional C^0 submanifold of \mathbb{R}^n which is everywhere smooth, except for a finite set of points, see [16]. Another development in which the manifold M is the closure of a bounded open subset of \mathbb{R}^n with Lipschitz boundary has been started in [14]. Then for this case, a new definition of geodesic on a general subset M of \mathbb{R}^n which

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is related to the nonsmooth critical point theory developed in [13], is used. The intrinsic case in which M is the closure of a bounded open subset of a differentiable manifold N has been studied more recently in [20].

Another natural development was provided in [6, 7, 8], where geodesics on certain nonsmooth sets of \mathbb{R}^n , called p -convex (or φ -convex) sets, are considered. In spite of the lack of regularity in the set M , using a new definition of geodesics in the framework of Sobolev spaces the author characterized these geodesics as critical points of the energy functional on a suitable path space. The class of p -convex sets includes submanifolds (possibly with boundary) of class $C_{loc}^{1,1}$, images under $C_{loc}^{1,1}$ -diffeomorphism of convex sets, but also subsets which are not topological manifolds, although they are absolute neighborhood retracts. In particular, it contains subsets with corners of convex type and concave parts of class C^2 . In [14], the authors proved that their notion of geodesic agrees with that of [8], when M is a C^2 -submanifold of \mathbb{R}^n , possibly with boundary.

The notions of φ -convexity (as titled p -convexity) and prox-regularity of sets were introduced in [10] and [24], respectively. In [5] the concept of φ -convex sets was extended to Hadamard manifolds and it was shown that if S is a φ -convex subset of an infinite-dimensional Hadamard manifold M , then there exists a neighborhood U of S in M such that the metric projection $P_S : U \rightarrow S$ is single-valued and locally Lipschitz. On the other hand, in [18] the notion of prox-regular sets was introduced on Riemannian manifolds as a subclass of regular sets. Prox-regular sets have significant applications in the theory of Moreau sweeping process, crowd motion and second order analysis; see, for instance, [23, 31].

In [26] we proved that the two classes of φ -convex sets and prox-regular sets coincide in the setting of finite-dimensional Riemannian manifolds. Moreover, in [25] we verified that for a prox-regular subset S of a Riemannian manifold M , P_S is a locally Lipschitz retraction from a neighborhood of S .

In [25], the subject of minimizing curves on a prox-regular set $S \subset M$ with C^2 boundary was considered. In this paper, we employed an adapted variational technique and by applying the first variation formula, we obtained a necessary condition for an admissible curve to be minimizing on S . Indeed, this curve is a piecewise C^2 curve and it has the property that

$$D_t \dot{\gamma}(t) \in N_S^P(\gamma(t)),$$

for every $t \in [a, b]$ except for finitely many points, where $N_S^P(x)$ denotes the proximal normal cone at $x \in S$. When the prox-regular set S does not possess a C^2 boundary, the problem becomes more complicated and the first variation formula can no longer be applied. Therefore motivated by [8], we want to study weak geodesics on a prox-regular set S in the intrinsic case where S is a subset of a Riemannian manifold M . Since a prox-regular set S is not necessarily smooth, a geodesic on S needs to be defined with some weaker regularities as a curve in the Sobolev space $W^{2,2}(I, M)$. The aim of

this paper is to demonstrate that the so-called weak geodesics on S are critical points of the energy functional. In Section 2 we review some preliminary concepts and results from nonsmooth analysis and Riemannian manifolds. In Section 3 we give a definition of weak geodesics on S with assigned end points and also we define a constraint minimization problem using the energy functional. Moreover, we obtain a suitable Lipschitz constant of the projection map and we prove some auxiliary theorems. Section 4 is devoted to the study of critical points of the energy functional and the main results of the paper are proved

2. PRELIMINARIES AND NOTATIONS

Let us recall some notions of Riemannian manifolds and nonsmooth analysis; see, e.g., [3, 4, 9, 15, 27]. Throughout this paper, I denotes the closed interval $[0, 1]$ and (M, g) is an n -dimensional Riemannian manifold endowed with a Riemannian metric $g_x = \langle \cdot, \cdot \rangle_x$ on each tangent space $T_x M$ and ∇ is the Riemannian connection of g . For every $x, y \in M$, the Riemannian distance from x to y is denoted by $d(x, y)$. Moreover, $B(x, r)$ and $\overline{B}(x, r)$ signify the open and closed metric ball centered at x with radius r , respectively. For a smooth curve $\gamma : I \rightarrow M$ and $t_0, t \in I$, the notation $L_{t_0, t}^\gamma$ is used for the parallel transport along γ from $\gamma(t_0)$ to $\gamma(t)$. When γ is the unique minimizing geodesic joining y to x , we use the notation $L_{y, x}$.

For $x \in M$, let $r(x)$ be the convexity radius at x , then the function $x \mapsto r(x)$ from M to $\mathbb{R}^+ \cup \{+\infty\}$ is continuous; see [27]. The map $\exp_x : U_x \rightarrow M$ will stand for the exponential map at x , where U_x is an open subset of the tangent space $T_x M$ containing $0_x \in T_x M$. Note that if x and y belong to a convex set, then both $\exp_x^{-1} y$ and $\exp_y^{-1} x$ are defined and

$$\|\exp_x^{-1} y\| = d(x, y) = \|\exp_y^{-1} x\|, \quad L_{y, x}(\exp_y^{-1} x) = -\exp_x^{-1} y.$$

Moreover, for a fixed point $z \in M$, the function $\psi : M \rightarrow \mathbb{R}$ defined by $\psi(x) = d^2(x, z)$ is C^∞ on any convex neighborhood of z and for every x in a convex neighborhood of z , $\nabla \psi(x) = -2 \exp_x^{-1} z$.

For $x \in M$, let $B(x, R)$ be a convex ball with compact closure and δ, Δ be such that $\delta \leq \sec \leq \Delta$ for all sectional curvatures of M on $B(x, R)$. Then using Rauch's theorem, it can be derived that for any $0 < r \leq R$, \exp_x^{-1} is Lipschitz on $B(x, r)$ with the Lipschitz constant $k = k(r)$ defined by

$$k = \begin{cases} \frac{r\sqrt{\Delta}}{\sin(r\sqrt{\Delta})} & \Delta > 0 \\ 1 & \Delta \leq 0 \end{cases}.$$

Let S be a nonempty closed subset of M . Recall that the distance function to S is $d_S(z) = \inf_{x \in S} d(x, z)$ and the metric projection to S , denoted by P_S , is defined by

$$P_S(z) = \{x \in S : d_S(z) = d(x, z)\} \quad \forall z \in M.$$

The proximal normal cone to S at $x \in S$, is denoted by $N_S^P(x)$ and $\xi \in N_S^P(x)$ if and only if there exists $\sigma > 0$ such that

$$\langle \xi, \exp_x^{-1} y \rangle \leq \sigma d^2(x, y),$$

for every $y \in U \cap S$, where U is a convex neighborhood of x . Moreover, the Bouligand tangent cone to S at x is defined as

$$T_S^B(x) := \left\{ \lim_{i \rightarrow \infty} \frac{\exp_x^{-1} z_i}{t_i} : z_i \in U \cap S, z_i \rightarrow x \text{ and } t_i \downarrow 0 \right\},$$

where U is a convex neighborhood of x in M .

Let $f : M \rightarrow (-\infty, +\infty]$ be a lower semicontinuous function and $x \in \text{dom}(f) := \{y \in M : f(y) < \infty\}$. The viscosity (or Fréchet) subdifferential of f at x , denoted by $D^-f(x)$, is the set

$$D^-f(x) := \{dg(x) : g \in C^1(M, \mathbb{R}), f - g \text{ attains a local minimum at } x\}.$$

Using [4, Theorem 4.3], $\xi \in D^-f(x) \subset T_x M$ if and only if

$$\liminf_{v \rightarrow 0} \frac{(f \circ \exp_x)(v) - f(x) - \langle \xi, v \rangle_x}{\|v\|} \geq 0.$$

It is worth mentioning that if f has a local minimum at x , then $0 \in D^-f(x)$.

Let us now take a brief look at the subject of prox-regular sets and present some of their properties. A closed subset S of M is prox-regular at $\bar{x} \in S$ if there exist $\varepsilon > 0$ and $\sigma > 0$ such that $B(\bar{x}, \varepsilon)$ is convex and for every $x \in S \cap B(\bar{x}, \varepsilon)$ and $v \in N_S^P(x)$ with $\|v\| < \varepsilon$,

$$\langle v, \exp_x^{-1} y \rangle \leq \sigma d^2(x, y) \quad \forall y \in S \cap B(\bar{x}, \varepsilon).$$

Moreover, S is called prox-regular if it is prox-regular at each point of S ; for more details, see [18].

In [26, Theorem 3.4], it was shown that for every prox-regular subset S of M there exists a continuous function $\varphi : S \rightarrow [0, \infty)$ such that S is φ -convex. Recall that a nonempty closed subset $S \subset M$ is called φ -convex if for every $x \in S$ and $v \in N_S^P(x)$

$$\langle v, \exp_x^{-1} y \rangle \leq \varphi(x) \|v\| d^2(x, y),$$

for every $y \in U \cap S$, where U is a convex neighborhood of x . Since we need to utilize the function φ , we prefer to work with φ -convex sets.

In [25], we proved that for a closed φ -convex set S , the metric projection P_S is locally Lipschitz on an open set containing S . Moreover, P_S is directionally differentiable at each point $x \in S$ and for every $x \in S$ and $v \in T_x M$ we have

$$(2.1) \quad \lim_{t \rightarrow 0^+} \frac{\exp_x^{-1}(P_S(\exp_x(tv)))}{t} = P_{T_S^B(x)}(v),$$

where $P_{T_S^B(x)}$ denotes the metric projection to the Bouligand tangent cone $T_S^B(x)$.

3. LIPSCHITZ CONSTANT OF PROJECTION MAP

Our first task in this section is to define weak geodesics on a prox-regular set and the energy functional on a suitable constraint and to study the lower semicontinuity of the energy functional.

In the case in which the prox-regular set S has C^2 boundary, we were able to obtain a necessary condition for a curve γ to be a minimizing curve between its endpoints in S , see [25, Theorem 6]. In this situation, an admissible curve is a piecewise C^2 curve $\gamma : I \rightarrow M$ with nonzero derivatives that is entirely in S . When S is an arbitrary prox-regular set without any smoothness assumption on its boundary, the set of admissible curves needs to be considered more broadly. Since a prox-regular set is locally Lipschitz path connected, an admissible curve can be chosen among curves belonging to the Sobolev space $W^{1,2}(I, M)$ or so-called H^1 -curves. An H^1 -curve $\gamma : I \rightarrow M$ can be considered as an absolutely continuous curve for which $\dot{\gamma}(t)$ exists for almost all $t \in I$ and $\int_0^1 \|\dot{\gamma}(t)\|^2 dt < \infty$, see [19].

In order to remind the Sobolev space of manifold valued curves, we present the following proposition which is a slight modification of [32, Lemma B.5]. By $C^0(I, M)$ we denote the space of continuous curves, endowed with the metric $d_\infty(\gamma, \eta) = \sup_{t \in I} d(\gamma(t), \eta(t))$ and $C^\infty(I, M)$ denotes the set of smooth curves. For more details about Sobolev spaces, see for instance [1].

Let $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ be an atlas on M and $\Phi : M \rightarrow \mathbb{R}^{2n+1}$ be an embedding which exists by the Whitney theorem.

Proposition 3.1. *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > 1$. Then for $\gamma \in C^0(I, M)$ the following statements are equivalent:*

- (i) $\phi_\alpha \circ \gamma \in W^{k,p}(\gamma^{-1}(U_\alpha), \mathbb{R}^n)$ for all α ;
- (ii) $\Phi \circ \gamma \in W^{k,p}(I, \mathbb{R}^{2n+1})$;
- (iii) $\gamma = \exp_c(V)$ for some $c \in C^\infty(I, M)$ and $V \in W^{k,p}(I, \gamma^{-1}TM)$,

where $\gamma^{-1}TM$ denotes the pullback bundle of TM by γ .

According to the previous proposition, the Sobolev space $W^{k,p}(I, M)$ of curves on M is defined as the set of continuous curves that satisfy these equivalent statements. In the case $k = 0$ and $p = 2$, we consider the space $L^2(I, M)$ as

$$L^2(I, M) := \{\gamma : I \rightarrow M : \Phi \circ \gamma \in L^2(I, \mathbb{R}^{2n+1})\},$$

with the topology given by the following convergence criteria:

$$\gamma_i \rightarrow \gamma \text{ in } L^2(I, M) \iff \Phi \circ \gamma_i \rightarrow \Phi \circ \gamma \text{ in } L^2(I, \mathbb{R}^{2n+1}).$$

The Sobolev space $W^{1,2}(I, M)$ usually denotes by $H^1(I, M)$ and as known it is a Hilbert manifold. Moreover, the Sobolev space $H^1(I, \gamma^{-1}TM)$ is a Hilbert space containing all continuous vector fields $V : I \rightarrow M$ along γ with the properties that

$$\int_0^1 \|V(t)\|^2 dt < \infty \quad \text{and} \quad \int_0^1 \|D_t V(t)\|^2 dt < \infty.$$

Note that for $V \in H^1(I, \gamma^{-1}TM)$, the weak covariant derivative of V , denoted by $D_t V$, is defined as

$$D_t V := k_{TM} \circ DV,$$

where $DV : I \rightarrow \mathbb{R} \otimes TTM$ is the weak derivative of V and $k_{TM} : TTM \rightarrow TM$ is the connection map on the tangent bundle TM . Since $\|D_t V(t)\|_{\gamma(t)} \leq \|DV(t)\|_{\gamma(t)}$, we have $D_t V \in L^2(I, \gamma^{-1}TM)$; see [11, 12].

Now let S be a closed and connected prox-regular subset of M and U be an open neighborhood of S on which P_S is single-valued and locally Lipschitz. In order to obtain a necessary condition for a curve γ to be a minimizing curve between its endpoints, we consider the set of admissible curves as follows:

$$\mathcal{A} = \mathcal{A}_{x,y} := \{\gamma \in H^1(I, M) : \gamma(t) \in S, \forall t \in I, \gamma(0) = x, \gamma(1) = y\}.$$

Since S is locally Lipschitz path connected, \mathcal{A} is nonempty. Note that without considering the constraint “ $\gamma(t) \in S, \forall t \in I$ ”, the set $\mathcal{A}_{x,y}$ is a submanifold of $H^1(I, M)$ and with the distance deduced from the Riemannian metric on $H^1(I, M)$, $\mathcal{A}_{x,y}$ is a complete metric space, see [19].

Motivated by [25, Theorem 6] and [8], we introduce a weak geodesic on S as follows:

Definition 3.2. *A continuous curve $\gamma : I \rightarrow M$, is called a weak geodesic on S joining $x, y \in S$ if $\gamma(0) = x, \gamma(1) = y$ and*

- (a) $\gamma(t) \in S \quad \forall t \in I$;
- (b) $\gamma \in W^{2,2}(I, M)$;
- (c) $D_t \dot{\gamma}(t) \in N_S^P(\gamma(t)) \quad a. e. t \in I$.

It is worth mentioning that when S is considered to be all of M or S is a C^2 -submanifold of M possibly with boundary, this definition corresponds to the usual definition of geodesics.

We intend to characterize weak geodesics on S as nonsmooth critical points of the energy functional defined on the space $L^2(I, M)$. We define the energy functional as follows,

$$f : L^2(I, M) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$f(\gamma) = \begin{cases} \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|^2 dt & \gamma \in \mathcal{A} \\ +\infty & \gamma \in L^2(I, M) \setminus \mathcal{A}, \end{cases}$$

and we consider the optimization problem $\min_{\gamma \in L^2(I, M)} f$. In the following proposition, we present some topological properties of \mathcal{A} and the energy functional f .

Proposition 3.3. *The admissible set \mathcal{A} is closed in $L^2(I, M)$ and the energy functional f is lower semicontinuous.*

Proof. In order to use the continuity of P_S , we consider a subset \mathcal{D} of $H^1(I, M)$ as

$$\mathcal{D} = \{\gamma \in H^1(I, M) : \gamma(t) \in U, \forall t \in I\}.$$

Using (iii) of Proposition 3.1, it can be proved that \mathcal{D} is open in $H^1(I, M)$. On the other hand, since $\|\Phi \circ \gamma\|_{L^2} \leq \|\Phi \circ \gamma\|_{H^1}$ for every $\gamma \in H^1(I, M)$, it follows that \mathcal{D} is open in $L^2(I, M)$.

We now define the functional map $P_S : \mathcal{D} \rightarrow \mathcal{D}$ by $P_S(\gamma)(t) := P_S(\gamma(t))$ for all $t \in I$. Since the metric projection P_S is locally Lipschitz on U , the map $P_S : \mathcal{D} \rightarrow \mathcal{D}$ is well defined. We claim that $P_S : \mathcal{D} \rightarrow \mathcal{D}$ is continuous. Indeed, let γ_n be a sequence in \mathcal{D} such that $d_\infty(\gamma_n, \gamma) \rightarrow 0$. Since P_S is locally Lipschitz on U and $\text{Im}(\gamma)$ is compact, there exists $l > 0$ such that

$$d(P_S(\gamma_n(t)), P_S(\gamma(t))) \leq ld(\gamma_n(t), \gamma(t)) \leq ld_\infty(\gamma_n, \gamma),$$

for all $t \in I$. This implies that $d_\infty(P_S(\gamma_n), P_S(\gamma)) \rightarrow 0$ and then $P_S(\gamma_n) \rightarrow P_S(\gamma)$ in $L^2(I, M)$.

We now define the map $g : \mathcal{D} \rightarrow \mathbb{R} \times M \times M$ as

$$g(\gamma) := (d_\infty(\gamma, P_S(\gamma)), \gamma(0), \gamma(1)),$$

hence g is continuous and then $\mathcal{A} = g^{-1}\{(0, x, y)\}$ is closed. Therefore the characteristic function $1_{\mathcal{A}}$ of \mathcal{A} is lower semicontinuous. Then as a product of a nonnegative continuous function and a lower semicontinuous function, f is lower semicontinuous, because $f = E \times 1_{\mathcal{A}}$ where $E : H^1(I, M) \rightarrow \mathbb{R}$ is a smooth functional (see [19]) defined by

$$E(\gamma) := \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|^2 dt.$$

□

We now improve the Lipschitz constant of P_S obtained in [25] and then we prove some auxiliary theorems that are used in the proof of main results of the paper in Section 4.

Let S be a nonempty, closed and φ -convex subset of M and using [25, Theorem 4], let U be an open set containing S such that P_S is single-valued and locally Lipschitz on U . For simplicity, we use the notation $\phi_x := \exp_x^{-1}$.

Lemma 3.4. *For every $x_0 \in M$ there exist $R > 0$ and $C > 0$ such that the Lipschitz constant A of*

$$L_{y,x} \circ \phi_y - \phi_x : B(x_0, R) \rightarrow T_x M,$$

satisfies $A \leq Cd(x, y)$ for every $x, y \in B(x_0, R)$.

Proof. Let $x_0 \in M$ and \mathcal{U} be a convex neighborhood of x_0 . It is enough to prove that there exist $R > 0$ and $C > 0$ such that the norm of

$$L_{y,x} \circ D\phi_y(z) - D\phi_x(z) : T_z M \rightarrow T_x M,$$

is bounded by $Cd(x, y)$ for every $x, y, z \in B(x_0, R)$. The function

$$J : \mathcal{U} \times \mathcal{U} \times T\mathcal{U} \rightarrow T\mathcal{U}$$

defined by

$$J(x, y, (z, h_z)) = L_{y,x} \circ D\phi_y(z)(h_z) - D\phi_x(z)(h_z)$$

is C^∞ . Indeed, the map \exp^{-1} is differentiable as a function defined on $\mathcal{U} \times \mathcal{U}$, and the parallel transport is the solution of an ordinary differential equation which depends C^∞ -wise on the initial data x, y , and consequently itself depends C^∞ -wise on x, y .

Hence there exist $R > 0$ and $C_0 > 0$ such that $\|DJ(x, y, (z, h_z))\| \leq C_0$, provided that $x, y, z \in B(x_0, R)$ and $\|h_z\| \leq R$, this implies that $\|DJ(x, y, (z, h_z))\| \leq \frac{C_0}{R} = C$ provided that $x, y, z \in B(x_0, R)$ and $\|h_z\| \leq 1$, and consequently

$$d^*(J(x_1, y_1, (z_1, h_1)), J(x_2, y_2, (z_2, h_2))) \leq C(d(x_1, x_2) + d(y_1, y_2) + d^*((z_1, h_1), (z_2, h_2))),$$

provided that $x_i, y_i, z_i \in B(x_0, R)$ and $\|h_i\| \leq 1$, $i = 1, 2$, where d^* is the distance induced in TM . Particularizing $x_1 = x_2 = y_2 = x$, $y_1 = y$, $z_1 = z_2 = z$ and $h_1 = h_2 = h$, we deduce

$$\|L_{y,x} \circ D\phi_y(z)(h) - D\phi_x(z)(h)\| = \|J(x, y, (z, h))\| \leq Cd(x, y),$$

for every $x, y, z \in B(x_0, R)$ and $\|h\| \leq 1$. Thus

$$\|L_{y,x} \circ D\phi_y(z) - D\phi_x(z)\| \leq Cd(x, y).$$

□

We denote $\tilde{x} = P_S(x)$ and $\tilde{y} = P_S(y)$ for $x, y \in U$. We also shortened $L = L_{\tilde{y}, \tilde{x}}$.

Theorem 3.5. *If $S \subset M$ is φ -convex and $x_0 \in S$, then there exist $R > 0$ and $C > 0$ such that for every $0 < r < R$ and $x, y \in B(x_0, r)$*

$$(3.1) \quad d(\tilde{x}, \tilde{y}) \leq \frac{k(\tilde{x})}{1 - \beta - \alpha} d(x, y),$$

where $k(\tilde{x})$ is the Lipschitz constant of $\phi_{\tilde{x}}$ on a convex ball $B(\tilde{x}, \sigma)$ containing x, y , $\beta = \beta(x, y) := \varphi(\tilde{x})d(x, \tilde{x}) + \varphi(\tilde{y})d(y, \tilde{y})$, and $\alpha = \alpha(y) := Cd(y, \tilde{y})$.

Proof. Let $x_0 \in S$ and $R_0 > 0$, $C > 0$ be the constants obtained from Lemma 3.4 and let R_0 be small enough such that $B(x_0, R_0)$ is convex with compact closure and $B(x_0, R_0) \subset U$. We put $\bar{\varphi} := \max_{x \in S \cap \overline{B(x_0, R_0)}} \varphi(x)$ and $\bar{r} := \min_{x \in S \cap \overline{B(x_0, R_0)}} r(x)$ and then we define

$$R := \min \left\{ \frac{R_0}{2}, \frac{\bar{r}}{3}, \frac{1}{2\bar{\varphi} + C} \right\}.$$

We now assume that $0 < r < R$ and $x, y \in B(x_0, r)$. Hence $x, y, \tilde{x}, \tilde{y} \in B(x_0, R_0)$ and we have that

$$\langle \phi_{\tilde{x}}(x), \phi_{\tilde{x}}(\tilde{y}) \rangle_{\tilde{x}} \leq \varphi(\tilde{x}) \|\phi_{\tilde{x}}(x)\| d^2(\tilde{x}, \tilde{y}),$$

and

$$\langle \phi_{\tilde{y}}(y), \phi_{\tilde{y}}(\tilde{x}) \rangle_{\tilde{y}} \leq \varphi(\tilde{y}) \|\phi_{\tilde{y}}(y)\| d^2(\tilde{x}, \tilde{y}).$$

As inner product is invariant throughout parallel transport, we have also

$$\langle \phi_{\tilde{y}}(y), \phi_{\tilde{y}}(\tilde{x}) \rangle_{\tilde{y}} = \langle L(\phi_{\tilde{y}}(y)), L(\phi_{\tilde{y}}(\tilde{x})) \rangle_{\tilde{x}}.$$

As $L(\phi_{\tilde{y}}(\tilde{x})) = -\phi_{\tilde{x}}(\tilde{y})$, we deduce

$$\langle L(\phi_{\tilde{y}}(y)), -\phi_{\tilde{x}}(\tilde{y}) \rangle_{\tilde{x}} \leq \varphi(\tilde{y}) \|\phi_{\tilde{y}}(y)\| d^2(\tilde{x}, \tilde{y}),$$

and consequently

$$\begin{aligned} \langle \phi_{\tilde{x}}(x) - L(\phi_{\tilde{y}}(y)), \phi_{\tilde{x}}(\tilde{y}) \rangle_{\tilde{x}} &\leq (\varphi(\tilde{x}) \|\phi_{\tilde{x}}(x)\| + \varphi(\tilde{y}) \|\phi_{\tilde{y}}(y)\|) d^2(\tilde{x}, \tilde{y}) \\ &= \beta d^2(\tilde{x}, \tilde{y}), \end{aligned}$$

where $\beta = \beta(x, y) = \varphi(\tilde{x}) \|\phi_{\tilde{x}}(x)\| + \varphi(\tilde{y}) \|\phi_{\tilde{y}}(y)\|$. Hence

$$\beta d^2(\tilde{x}, \tilde{y}) \geq \langle \phi_{\tilde{x}}(x) - L(\phi_{\tilde{y}}(y)) - \phi_{\tilde{x}}(\tilde{y}), \phi_{\tilde{x}}(\tilde{y}) \rangle_{\tilde{x}} + d^2(\tilde{x}, \tilde{y}),$$

since $\|\phi_{\tilde{x}}(\tilde{y})\| = d(\tilde{x}, \tilde{y})$. This implies

$$\begin{aligned} (1 - \beta) d^2(\tilde{x}, \tilde{y}) &\leq \langle -\phi_{\tilde{x}}(x) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}), \phi_{\tilde{x}}(\tilde{y}) \rangle_{\tilde{x}} \\ &\leq \| -\phi_{\tilde{x}}(x) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}) \| d(\tilde{x}, \tilde{y}), \end{aligned}$$

therefore

$$\begin{aligned} (1 - \beta) d(\tilde{x}, \tilde{y}) &\leq \| -\phi_{\tilde{x}}(x) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}) \| \\ &\leq \| -\phi_{\tilde{x}}(x) + \phi_{\tilde{x}}(y) \| + \| -\phi_{\tilde{x}}(y) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}) \| \\ &\leq k(\tilde{x}) d(x, y) + \| -\phi_{\tilde{x}}(y) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}) \|, \end{aligned}$$

where $k(\tilde{x})$ is the Lipschitz constant of $\phi_{\tilde{x}}$ on $B(\tilde{x}, 3r) \subset B(\tilde{x}, \bar{r})$. Following with the second term, we have

$$\begin{aligned} \| -\phi_{\tilde{x}}(y) + L(\phi_{\tilde{y}}(y)) + \phi_{\tilde{x}}(\tilde{y}) \| &= \| (L \circ \phi_{\tilde{y}} - \phi_{\tilde{x}})(y) - (L \circ \phi_{\tilde{y}} - \phi_{\tilde{x}})(\tilde{y}) \| \\ &\leq Ad(y, \tilde{y}), \end{aligned}$$

and consequently

$$(1 - \beta) d(\tilde{x}, \tilde{y}) \leq k(\tilde{x}) d(x, y) + Ad(y, \tilde{y}),$$

where A is the Lipschitz constant of

$$L \circ \phi_{\tilde{y}} - \phi_{\tilde{x}} : B(x_0, R) \rightarrow T_{\tilde{x}}M.$$

Using Lemma 3.4, $A \leq Cd(\tilde{x}, \tilde{y})$ and we deduce

$$(1 - \beta) d(\tilde{x}, \tilde{y}) \leq k(\tilde{x}) d(x, y) + \alpha d(\tilde{x}, \tilde{y}),$$

where $\alpha = Cd(y, \tilde{y})$. Our choice of R and r implies that $1 - \beta - \alpha > 0$ and then we get the result. Indeed, we have

$$\begin{aligned} \beta + \alpha &= \varphi(\tilde{x}) d(x, \tilde{x}) + \varphi(\tilde{y}) d(y, \tilde{y}) + Cd(y, \tilde{y}) \\ &\leq \bar{\varphi} d(x, x_0) + \bar{\varphi} d(y, x_0) + Cd(y, x_0) \\ &\leq (2\bar{\varphi} + C) r < 1, \end{aligned}$$

since $x_0 \in S$.

□

Corollary 3.6. *Let $S \subset M$ be φ -convex and $x_0 \in S$. Then for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$P_S : B(x_0, R) \rightarrow S$$

has Lipschitz constant less or equal than $1 + \varepsilon$.

Let $\gamma \in H^1(I, M)$ be such that $\gamma(t) \in S$ for all $t \in I$ and V be a vector field along γ such that $V \in H^1(I, \gamma^{-1}TM)$. We define

$$P_{\gamma(t)}V(t) := P_{T_S^B(\gamma(t))}V(t), \quad \forall t \in I,$$

where $T_S^B(\gamma(t))$ is the Bouligand tangent cone to S at the point $\gamma(t)$. Using [25, Theorem 4.2], we have $P_\gamma V \in H^1(I, \gamma^{-1}TM)$ and

$$\|P_{\gamma(t)}V(t)\| = \lim_{s \rightarrow 0^+} \frac{d\left(\gamma(t), P_S\left(\exp_{\gamma(t)} sV(t)\right)\right)}{s}.$$

We now consider a variation Γ of γ defined as follows

$$\Gamma_s(t) := \exp_{\gamma(t)}(sV(t)), \quad \forall t \in I, s \geq 0,$$

and for sufficiently small s , we define $\tilde{\Gamma}_s(t) := P_S(\Gamma_s(t))$ for all $t \in I$. Hence $\Gamma_s, \tilde{\Gamma}_s \in H^1(I, M)$.

To proceed, we need to estimate the following statement from below

$$\liminf_{s \rightarrow 0^+} \frac{f(\Gamma_s) - f(\tilde{\Gamma}_s)}{s}.$$

Let $\text{Im}(\gamma)$ denote the image of γ and $\rho := \max_{t \in I} \|V(t)\|$.

Theorem 3.7. *Let $\gamma \in H^1(I, M)$ be such that $\gamma(t) \in S$ for all $t \in I$ and V be a vector field along γ such that $V \in H^1(I, \gamma^{-1}TM)$. Then there exists a piecewise constant function $\tau : I \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} \liminf_{s \rightarrow 0^+} \frac{\frac{1}{2} \int_0^1 \|\dot{\Gamma}_s(t)\|^2 dt - \frac{1}{2} \int_0^1 \|\dot{\tilde{\Gamma}}_s(t)\|^2 dt}{s} \geq \\ - \int_0^1 (\varphi(\gamma) + \tau) \|V - P_\gamma V\| \|\dot{\gamma}\|^2 dt. \end{aligned}$$

Proof. We first show that there exists a piecewise constant function τ on I such that for all sufficiently small s and for almost all $t \in I$,

$$(3.2) \quad \|\dot{\tilde{\Gamma}}_s(t)\| \leq \frac{K(s)}{1 - \left(2\varphi(\tilde{\Gamma}_s(t)) + \tau(t)\right) d(\tilde{\Gamma}_s(t), \Gamma_s(t))} \|\dot{\Gamma}_s(t)\|,$$

where K is a function with the property that $K \rightarrow 1$ as $s \rightarrow 0^+$.

Indeed, using Theorem 3.5 and the compactness of $\text{Im}(\gamma)$, there exist finitely many points $x_1, \dots, x_m \in \text{Im}(\gamma)$ and some constants $R_i > 0, C_i > 0$

such that the inequality (3.1) holds on $B_i := B(x_i, R_i)$ for $i = 1, \dots, m$ and the balls B_i , $i = 1, \dots, m$ cover $\text{Im}(\gamma)$.

Let t_i , $i = 0, \dots, m$ be such that $t_0 = 0$, $t_m = 1$, and $\gamma([t_{i-1}, t_i]) \subset B_i$ for $i = 1, \dots, m$. Since $\Gamma_s \rightarrow \gamma$ uniformly on I , we can find $s_1 > 0$ small enough such that $s_1\rho < \min\left\{\frac{\bar{r}}{2}, \frac{R_i}{3}, i = 1, \dots, m\right\}$ and

$$\Gamma_s(t) \in B_i, \quad \forall t \in [t_{i-1}, t_i], \quad \forall s < s_1,$$

where $\bar{r} := \min\left\{r(x) : x \in S \cap \left(\bigcup \overline{B(x_i, 2R_i)}\right)\right\}$.

Let $s < s_1$ and $t \in (t_{j-1}, t_j)$ for some $1 \leq j \leq m$ be such that $\dot{\Gamma}_s(t)$ and $\ddot{\Gamma}_s(t)$ exist. Then for $h \in \mathbb{R}$ with sufficiently small $|h|$ we have $t+h \in (t_{j-1}, t_j)$ and $\Gamma_s(t+h) \in B(\tilde{\Gamma}_s(t), s\rho)$, and hence using Theorem 3.5 we obtain that

$$(3.3) \quad d\left(\tilde{\Gamma}_s(t+h), \tilde{\Gamma}_s(t)\right) \leq \Theta d\left(\Gamma_s(t+h), \Gamma_s(t)\right),$$

where

$$\Theta := \frac{k\left(\tilde{\Gamma}_s(t)\right)}{1 - \beta\left(\Gamma_s(t), \Gamma_s(t+h)\right) - C_j d\left(\tilde{\Gamma}_s(t+h), \Gamma_s(t+h)\right)},$$

and k is the Lipschitz constant of $\phi_{\tilde{\Gamma}_s(t)}$ on $B(\tilde{\Gamma}_s(t), s\rho)$. We now define the piecewise constant function τ on I as $\tau(t) = C_i$ for all $t \in [t_{i-1}, t_i)$, $i = 1, \dots, m$ and $\tau(t_m) = C_m$. Moreover, let the sectional curvature of any plane of M on $\bigcup B(x_i, R_i)$ be bounded by $\Delta > 0$, i.e. $|\text{sec}| \leq \Delta$, then putting

$$K(s) := \frac{2s\rho\sqrt{\Delta}}{\sin\left(2s\rho\sqrt{\Delta}\right)},$$

we have $k\left(\tilde{\Gamma}_s(t)\right) \leq K(s)$. Hence taking the limit of (3.3) as $h \rightarrow 0$, we get the inequality (3.2).

For simplicity, we denote $x := \Gamma_s(t)$ and $\tilde{x} := \tilde{\Gamma}_s(t)$. Using (3.2), we have

$$\begin{aligned} & \frac{1}{2s} \int_0^1 \left(\|\dot{\Gamma}_s(t)\|^2 - \|\dot{\tilde{\Gamma}}_s(t)\|^2 \right) dt \geq \\ & \frac{1}{2s} \int_0^1 \left(\|\dot{\Gamma}_s(t)\|^2 - \left(\frac{K(s)}{1 - (2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x)} \right)^2 \|\dot{\tilde{\Gamma}}_s(t)\|^2 \right) dt \\ (3.4) \quad & = \frac{1}{2s} \int_0^1 \frac{1 - K^2(s)}{[1 - (2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x)]^2} \|\dot{\tilde{\Gamma}}_s(t)\|^2 dt + \end{aligned}$$

$$(3.5) \quad \frac{1}{2s} \int_0^1 \frac{(2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x) [(2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x) - 2]}{[1 - (2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x)]^2} \|\dot{\tilde{\Gamma}}_s(t)\|^2 dt.$$

Note that $(2\varphi(\tilde{x}) + c) d(\tilde{x}, x) \rightarrow 0$ uniformly on I and $\|\dot{\Gamma}_s\|^2 \rightarrow \|\dot{\gamma}\|^2$ in L^1 as $s \rightarrow 0^+$. Moreover, we have $x, \tilde{x} \in B(\gamma(t), 2s\rho)$ and then

$$\frac{d(\tilde{\Gamma}_s(t), \Gamma_s(t))}{s} \geq \frac{1}{K(s)} \frac{\|sV(t) - \exp_{\gamma(t)}^{-1} P_S(\exp_{\gamma(t)} sV(t))\|}{s},$$

and

$$\frac{1}{K(s)} \frac{\|sV(t) - \exp_{\gamma(t)}^{-1} P_S(\exp_{\gamma(t)} sV(t))\|}{s} \rightarrow \|V(t) - P_{\gamma(t)} V(t)\|,$$

in L^1 as $s \rightarrow 0^+$. Therefore

$$\lim_{s \rightarrow 0^+} (3.5) \geq - \int_0^1 (2\varphi(\gamma) + \tau) \|V - P_{\gamma} V\| \|\dot{\gamma}\|^2 dt.$$

On the other hand, in (3.4), we have

$$\frac{1}{[1 - (2\varphi(\tilde{x}) + \tau) d(\tilde{x}, x)]^2} \|\dot{\Gamma}_s\|^2 \rightarrow \|\dot{\gamma}\|^2,$$

in L^1 and

$$\frac{1 - K^2(s)}{s} \rightarrow 0$$

as $s \rightarrow 0^+$ and consequently $\lim_{s \rightarrow 0^+} (3.4) = 0$. \square

In the sequel, we need to compute the derivative $\dot{\Gamma}_s(t) = \frac{\partial}{\partial t} \Gamma(s, t)$, where it exists. Let $t \in I$ be such that both $\dot{\gamma}(t)$ and $D_t V(t)$ exist. Note that for fixed t , $\Gamma(\cdot, t)$ is a geodesic and Γ is a variation of $\Gamma(\cdot, t)$ among geodesics. Let (x^i) be the normal coordinates on M centered at $\gamma(t)$ and (x^i, v^i) be the corresponding coordinates on TM . Then using the smooth approximations of γ and V , we have

$$\dot{\Gamma}_s(t) = \left(\dot{\gamma}^i(t) + s(D_t V)^i(t) \right) \partial_i|_{\Gamma(s,t)},$$

for sufficiently small s , where $\partial_i := \partial/\partial x^i$ and $\dot{\gamma}^i(t)$, $(D_t V)^i(t)$ are the components of $\dot{\gamma}(t)$ and $D_t V(t)$ in these coordinates, respectively. Hence according to [19], we obtain that

$$\|\dot{\Gamma}_s(t)\|^2 = \|\dot{\gamma}(t) + sD_t V(t)\|^2 + O(s^2),$$

where $O(s^2)/s \rightarrow 0$ as $s \rightarrow 0$.

Theorem 3.8. *Let $\gamma \in \mathcal{A}$ and $\xi \in D^-f(\gamma) \subset L^2(I, \gamma^{-1}TM)$. Then there exists a piecewise constant function $\tau = \tau(\gamma)$ on I such that for every vector field V along γ with the properties that $V \in H^1(I, \gamma^{-1}TM)$ and $V(0) = V(1) = 0$,*

$$\int_0^1 \langle \dot{\gamma}, D_t V \rangle dt \geq \int_0^1 \langle \xi, P_{\gamma} V \rangle dt - \int_0^1 (2\varphi(\gamma) + \tau) \|V - P_{\gamma} V\| \|\dot{\gamma}\|^2 dt.$$

Proof. Note that

$$\begin{aligned} \langle \dot{\gamma}(t), D_t V(t) \rangle &= \frac{1}{2s} \|\dot{\gamma}(t) + s D_t V(t)\|^2 - \frac{1}{2s} \|\dot{\gamma}(t)\|^2 - \frac{s}{2} \|D_t V(t)\|^2 \\ &= \frac{1}{s} \left(\frac{1}{2} \|\dot{\Gamma}_s(t)\|^2 - \frac{1}{2} \|\dot{\gamma}(t)\|^2 \right) + \left(\frac{O(s^2)}{s} - \frac{s}{2} \|D_t V(t)\|^2 \right), \end{aligned}$$

and hence

$$\langle \dot{\gamma}(t), D_t V(t) \rangle = \lim_{s \rightarrow 0^+} \frac{1}{s} \left(\frac{1}{2} \|\dot{\Gamma}_s(t)\|^2 - \frac{1}{2} \|\dot{\gamma}(t)\|^2 \right).$$

Therefore we have

$$\begin{aligned} \int_0^1 \langle \dot{\gamma}, D_t V \rangle dt - \int_0^1 \langle \xi, P_\gamma V \rangle dt &= \\ \lim_{s \rightarrow 0^+} \frac{1}{s} \int_0^1 \left(\frac{1}{2} \|\dot{\Gamma}_s(t)\|^2 - \frac{1}{2} \|\dot{\gamma}(t)\|^2 - \left\langle \xi, \exp_{\gamma(t)}^{-1} P_S \left(\exp_{\gamma(t)} s V(t) \right) \right\rangle \right) dt & \\ \geq \liminf_{s \rightarrow 0^+} \frac{1}{s} \int_0^1 \left(\frac{1}{2} \|\dot{\tilde{\Gamma}}_s(t)\|^2 - \frac{1}{2} \|\dot{\gamma}(t)\|^2 - \left\langle \xi, \exp_{\gamma(t)}^{-1} \tilde{\Gamma}_s(t) \right\rangle \right) dt & \\ + \liminf_{s \rightarrow 0^+} \frac{1}{2s} \int_0^1 \left(\|\dot{\Gamma}_s(t)\|^2 - \|\dot{\tilde{\Gamma}}_s(t)\|^2 \right) dt. & \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \liminf_{s \rightarrow 0^+} \frac{1}{s} \int_0^1 \left(\frac{1}{2} \|\dot{\tilde{\Gamma}}_s(t)\|^2 - \frac{1}{2} \|\dot{\gamma}(t)\|^2 - \left\langle \xi, \exp_{\gamma(t)}^{-1} \tilde{\Gamma}_s(t) \right\rangle \right) dt &\geq \\ \liminf_{s \rightarrow 0^+} \frac{f(\tilde{\Gamma}_s) - f(\gamma) - \left\langle \xi, \exp_{\gamma}^{-1} \tilde{\Gamma}_s \right\rangle_{L^2}}{s} &= \\ \liminf_{s \rightarrow 0^+} \frac{f(\tilde{\Gamma}_s) - f(\gamma) - \left\langle \xi, \exp_{\gamma}^{-1} \tilde{\Gamma}_s \right\rangle_{L^2}}{\|\exp_{\gamma}^{-1} \tilde{\Gamma}\|_{L^2}} \times \frac{\|\exp_{\gamma}^{-1} \tilde{\Gamma}\|_{L^2}}{s} &\geq 0, \end{aligned}$$

because $\xi \in D^-f(\gamma)$ and the function $\frac{\|\exp_{\gamma}^{-1} \tilde{\Gamma}\|_{L^2}}{s}$ is bounded by $2\|V\|_{L^2}$. Then Theorem 3.7 implies that

$$\int_0^1 \langle \dot{\gamma}, D_t V \rangle dt - \int_0^1 \langle \xi, P_\gamma V \rangle dt \geq - \int_0^1 (2\varphi(\gamma) + \tau) \|V - P_\gamma V\| \|\dot{\gamma}\|^2 dt.$$

□

Let $H_0^1(I, \gamma^{-1}TM)$ denote the space of all proper vector fields $V \in H^1(I, \gamma^{-1}TM)$ with $V(0) = V(1) = 0$. For $\gamma \in H^1(I, M)$, we can find an open neighborhood W of $\text{Im}(\gamma)$ with compact closure. Let $\delta \leq \text{sec} \leq \Delta$ on U , $\Delta \geq 0$ and $\bar{r} = \min_{t \in I} r(\gamma(t))$. In the following lemma, we want to compute the covariant derivative of the H^1 -vector field $\exp_{\gamma}^{-1} \eta$ along γ with respect to $\dot{\gamma}, \dot{\eta}$ for $\eta \in H^1(I, M)$ sufficiently near γ .

Lemma 3.9. *Suppose that $\eta \in H^1(I, M)$. If $d_\infty(\gamma, \eta) < \bar{r}$, then for almost all $t \in I$,*

$$(3.6) \quad \langle \dot{\gamma}, D_t(\exp_\gamma^{-1} \eta) \rangle \leq \left\langle \dot{\gamma}, L_{\eta, \gamma} \dot{\eta} - \frac{1}{2} c(\gamma, \eta) \dot{\gamma} \right\rangle \\ + \frac{1}{2} |R|_\infty d^2(\gamma, \eta) \|\dot{\gamma}\| \|\dot{\eta}\|,$$

where $c(\gamma, \eta)(t) := 2\sqrt{\Delta} d(\gamma(t), \eta(t)) \cot(\sqrt{\Delta} d(\gamma(t), \eta(t)))$ for all $t \in I$ and R denotes the curvature tensor on M .

Proof. We define $V(t) := \phi_{\gamma(t)} \eta(t)$ for all $t \in I$. Then $V \in H^1(I, \gamma^{-1}TM)$ and for almost all $t \in I$,

$$(3.7) \quad D_t V(t) = D\phi_{\gamma(t)}(\eta(t))(\dot{\eta}(t)) + \nabla V_1(\gamma(t))(\dot{\gamma}(t)),$$

where V_1 is a vector field on $B(\gamma(t), \bar{r})$ defined by $V_1(x) := \exp_x^{-1}(\eta(t))$. Thus $\nabla V_1(\gamma(t)) = \text{Hess}\left(-\frac{1}{2}d_{\eta(t)}^2\right)(\gamma(t))$ and so using [25, Lemma 3], we deduce that

$$(3.8) \quad \text{Hess}\left(-\frac{1}{2}d_{\eta(t)}^2\right)(\gamma(t))(\dot{\gamma}(t))^2 \leq -\frac{1}{2}c(\gamma, \eta)(t)\|\dot{\gamma}(t)\|^2,$$

where

$$c(\gamma, \eta)(t) = 2\sqrt{\Delta} d(\gamma(t), \eta(t)) \cot(\sqrt{\Delta} d(\gamma(t), \eta(t))).$$

On the other hand, using [17, p. 110], we have

$$\langle \dot{\gamma}(t), D\phi_{\gamma(t)}(\eta(t))(\dot{\eta}(t)) - L_{\eta(t), \gamma(t)} \dot{\eta}(t) \rangle \\ \leq \|D\phi_{\gamma(t)}(\eta(t)) - L_{\eta(t), \gamma(t)}\| \|\dot{\gamma}(t)\| \|\dot{\eta}(t)\| \\ \leq \frac{1}{2} |R|_\infty d^2(\gamma(t), \eta(t)) \|\dot{\gamma}(t)\| \|\dot{\eta}(t)\|.$$

Hence we derive that

$$(3.9) \quad \langle \dot{\gamma}(t), D\phi_{\gamma(t)}(\eta(t))(\dot{\eta}(t)) \rangle \leq \langle \dot{\gamma}(t), L_{\eta(t), \gamma(t)} \dot{\eta}(t) \rangle + \\ \frac{1}{2} |R|_\infty d^2(\gamma(t), \eta(t)) \|\dot{\gamma}(t)\| \|\dot{\eta}(t)\|.$$

Therefore (3.6) is obtained from (3.7), (3.8) and (3.9). \square

Lemma 3.10. *If $\eta \in H^1(I, M)$ and $d_\infty(\gamma, \eta) < \bar{r}$, then for almost all $t \in I$,*

$$\langle \exp_\gamma^{-1} \eta, D_t(\exp_\gamma^{-1} \eta) \rangle = \langle \exp_\gamma^{-1} \eta, L_{\eta, \gamma} \dot{\eta} - \dot{\gamma} \rangle.$$

Proof. Let $t \in I$ be such that $\dot{\gamma}(t), \dot{\eta}(t)$ exist, then we have

$$2 \langle (\exp_\gamma^{-1} \eta)(t), D_t(\exp_\gamma^{-1} \eta)(t) \rangle = \frac{d}{dt} \|(\exp_\gamma^{-1} \eta)(t)\|^2 \\ = \frac{d}{dt} d^2(\gamma(t), \eta(t)) \\ = \left\langle -2 \exp_{\gamma(t)}^{-1} \eta(t), \dot{\gamma}(t) \right\rangle + \left\langle -2 \exp_{\eta(t)}^{-1} \gamma(t), \dot{\eta}(t) \right\rangle$$

$$= 2 \left\langle \exp_{\gamma(t)}^{-1} \eta(t), L_{\eta(t), \gamma(t)} \dot{\eta}(t) - \dot{\gamma}(t) \right\rangle.$$

□

Theorem 3.11. *Let $\gamma \in \mathcal{A} \cap W^{2,2}(I, M)$ and $\xi \in L^2(I, \gamma^{-1}TM)$ be such that*

$$\xi + D_t \dot{\gamma} \in N_S^P(\gamma), \text{ a. e.}$$

Then for all $\eta \in \mathcal{A}$ with the property that $d_\infty(\gamma, \eta) < \bar{r}$,

$$\begin{aligned} \frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|^2 dt &\geq \frac{1}{2} \int_0^1 (c(\gamma, \eta) - 1) \|\dot{\gamma}(t)\|^2 dt + \int_0^1 \langle \xi, \exp_{\gamma}^{-1} \eta \rangle dt \\ &\quad - d_\infty(\gamma, \eta) \|\exp_{\gamma}^{-1} \eta\|_{L^2} (\bar{\varphi} \|\xi + D_t \dot{\gamma}\|_{L^2} + \frac{1}{2} |R|_\infty \|\dot{\gamma}\|_{L^\infty} \|\dot{\eta}\|_{L^2}), \end{aligned}$$

where $\bar{\varphi} := \max_{t \in I} \varphi(\gamma(t))$.

Proof. Let $\eta \in \mathcal{A}$ and $d_\infty(\gamma, \eta) < \bar{r}$, then we have

$$\|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 = \|\dot{\eta}\|^2 + \|\dot{\gamma}\|^2 - 2 \langle \dot{\gamma}, L_{\eta, \gamma} \dot{\eta} \rangle,$$

and so

$$\frac{1}{2} \|\dot{\eta}\|^2 - \frac{c(\gamma, \eta) - 1}{2} \|\dot{\gamma}\|^2 = \frac{1}{2} \|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 + \left\langle \dot{\gamma}, L_{\eta, \gamma} \dot{\eta} - \frac{c(\gamma, \eta)}{2} \dot{\gamma} \right\rangle.$$

Moreover, using Lemma 3.9 we have

$$\begin{aligned} \frac{d}{dt} \langle \dot{\gamma}, \exp_{\gamma}^{-1} \eta \rangle &\leq \langle D_t \dot{\gamma}, \exp_{\gamma}^{-1} \eta \rangle + \left\langle \dot{\gamma}, L_{\eta, \gamma} \dot{\eta} - \frac{1}{2} c(\gamma, \eta) \dot{\gamma} \right\rangle \\ &\quad + \frac{1}{2} |R|_\infty d^2(\gamma, \eta) \|\dot{\gamma}\| \|\dot{\eta}\|. \end{aligned}$$

We now obtain that

$$\begin{aligned} \frac{1}{2} \|\dot{\eta}\|^2 - \frac{c(\gamma, \eta) - 1}{2} \|\dot{\gamma}\|^2 &\geq \frac{1}{2} \|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 + \frac{d}{dt} \langle \dot{\gamma}, \exp_{\gamma}^{-1} \eta \rangle \\ &\quad - \langle D_t \dot{\gamma}, \exp_{\gamma}^{-1} \eta \rangle - \frac{1}{2} |R|_\infty d^2(\gamma, \eta) \|\dot{\gamma}\| \|\dot{\eta}\|. \end{aligned}$$

Hence by integrating from both side and noting that $\xi + D_t \dot{\gamma} \in N_S^P(\gamma)$, a. e., we have

$$\begin{aligned} &\frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|^2 dt - \frac{1}{2} \int_0^1 (c(\gamma, \eta) - 1) \|\dot{\gamma}(t)\|^2 dt - \int_0^1 \langle \xi, \exp_{\gamma}^{-1} \eta \rangle dt \geq \\ &\frac{1}{2} \int_0^1 \|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 dt - \int_0^1 \langle \xi + D_t \dot{\gamma}, \exp_{\gamma}^{-1} \eta \rangle dt - \frac{1}{2} |R|_\infty \int_0^1 d^2(\gamma, \eta) \|\dot{\gamma}\| \|\dot{\eta}\| dt \\ &\geq \frac{1}{2} \int_0^1 \|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 dt - \int_0^1 \varphi(\gamma) \|\xi + D_t \dot{\gamma}\| \|\exp_{\gamma}^{-1} \eta\|^2 dt \\ &\quad - \frac{1}{2} |R|_\infty \int_0^1 \|\dot{\gamma}\| \|\dot{\eta}\| \|\exp_{\gamma}^{-1} \eta\|^2 dt. \end{aligned}$$

On the other hand, using Hölder inequality we have

$$\begin{aligned} \int_0^1 \|\xi + D_t \dot{\gamma}\| \|\exp_{\gamma}^{-1} \eta\|^2 dt &\leq d_{\infty}(\gamma, \eta) \int_0^1 \|\xi + D_t \dot{\gamma}\| \|\exp_{\gamma}^{-1} \eta\| dt \\ &\leq d_{\infty}(\gamma, \eta) \|\xi + D_t \dot{\gamma}\|_{L^2} \|\exp_{\gamma}^{-1} \eta\|_{L^2}, \end{aligned}$$

and the following inequality,

$$(3.10) \quad \|\dot{\gamma}\|_{L^{\infty}} \leq \|\dot{\gamma}\|_{L^2} + \|D_t \dot{\gamma}\|_{L^1},$$

implies that $\dot{\gamma} \in L^{\infty}(I, \gamma^{-1}TM)$. Therefore

$$\int_0^1 \|\dot{\gamma}\| \|\dot{\eta}\| \|\exp_{\gamma}^{-1} \eta\|^2 dt \leq \|\dot{\gamma}\|_{L^{\infty}} \|\dot{\eta}\|_{L^2} \|\exp_{\gamma}^{-1} \eta\|_{L^2} d_{\infty}(\eta, \gamma),$$

by Hölder inequality again, hence

$$\begin{aligned} \frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|^2 dt - \frac{1}{2} \int_0^1 (c(\gamma, \eta) - 1) \|\dot{\gamma}(t)\|^2 dt &\geq \\ \int_0^1 \langle \xi, \exp_{\gamma}^{-1} \eta \rangle dt + \frac{1}{2} \int_0^1 \|L_{\eta, \gamma} \dot{\eta} - \dot{\gamma}\|^2 dt & \\ -d_{\infty}(\gamma, \eta) \|\exp_{\gamma}^{-1} \eta\|_{L^2} (\bar{\varphi} \|\xi + D_t \dot{\gamma}\|_{L^2} + \frac{1}{2} |R|_{\infty} \|\dot{\gamma}\|_{L^{\infty}} \|\dot{\eta}\|_{L^2}). & \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|^2 dt - \frac{1}{2} \int_0^1 (c(\gamma, \eta) - 1) \|\dot{\gamma}(t)\|^2 dt - \int_0^1 \langle \xi, \exp_{\gamma}^{-1} \eta \rangle dt &\geq \\ -d_{\infty}(\gamma, \eta) \|\exp_{\gamma}^{-1} \eta\|_{L^2} (\bar{\varphi} \|\xi + D_t \dot{\gamma}\|_{L^2} + \frac{1}{2} |R|_{\infty} \|\dot{\gamma}\|_{L^{\infty}} \|\dot{\eta}\|_{L^2}), & \end{aligned}$$

and we get the result. \square

4. WEAK GEODESICS AS CRITICAL POINTS OF THE ENERGY FUNCTIONAL

In this section, we characterize weak geodesics on S as viscosity critical points of the energy functional.

Theorem 4.1. *Suppose that $\gamma \in \mathcal{A}$. Then $D^-f(\gamma) \neq \emptyset$ if and only if $\gamma \in W^{2,2}(I, M)$. Moreover, if $\xi \in L^2(I, \gamma^{-1}TM)$, then $\xi \in D^-f(\gamma)$ if and only if*

$$\xi(t) + D_t \dot{\gamma}(t) \in N_S^P(\gamma(t)), \text{ a. e. } t \in I.$$

Proof. Let $\xi \in D^-f(\gamma)$, then using Theorem 3.8, there exists a piecewise constant function τ on I such that

$$(4.1) \quad \int_0^1 \langle \dot{\gamma}, D_t V \rangle dt \geq \int_0^1 \langle \xi, P_{\gamma} V \rangle dt - \int_0^1 (2\varphi(\gamma) + \tau) \|V - P_{\gamma} V\| \|\dot{\gamma}\|^2 dt,$$

for every $V \in H_0^1(I, \gamma^{-1}TM)$.

Since $T_S^B(\gamma(t))$ is a closed convex cone for all t , we have $\|V(t) - P_{\gamma(t)}V(t)\| \leq \|V(t)\|$ and $\|V(t)\| \geq \|P_{\gamma(t)}V(t)\|$ for all $t \in I$. Then similar to the proof of [8, Lemma 3.5], we obtain that

$$(4.2) \quad \int_0^1 \langle \dot{\gamma}, D_t V \rangle dt \geq -\|\xi\|_{L^2} \|V\|_{L^2} - \int_0^1 (2\varphi(\gamma) + \tau) \|V\| \|\dot{\gamma}\|^2 dt,$$

by Hölder inequality. Therefore we have

$$(4.3) \quad \left| \int_0^1 \langle \dot{\gamma}, D_t V \rangle dt \right| \leq (\|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2) \|V\|_{L^\infty},$$

for every $V \in H_0^1(I, \gamma^{-1}TM)$, where $C = \max_I \tau$. Hence taking a suitable sequence of vector fields $V_n \in H_0^1(I, \gamma^{-1}TM)$ in (4.3) and passing to the limit as $n \rightarrow \infty$, we conclude that

$$(4.4) \quad \|\dot{\gamma}\|_{L^\infty} \leq \|\dot{\gamma}\|_{L^1} + \|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2,$$

and hence $\dot{\gamma} \in L^\infty(I, \gamma^{-1}TM)$. Indeed, for given $t_0 \in I$, let $w \in T_{\gamma(t_0)}M$ be such that $\|w\| = 1$ and $\langle \dot{\gamma}(t_0), w \rangle = \|\dot{\gamma}(t_0)\|$. We now consider a sequence of functions $u_n \in C_0^\infty(\mathbb{R})$ with the properties that

$$0 \leq u_n \leq 1, \quad -tu_n'(t) \leq 1 \quad \forall t \in \mathbb{R}, \quad \text{Supp } u_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right] \quad \forall n \in \mathbb{N}.$$

These functions can be obtained as follows. Let ψ be the smooth function defined by

$$\psi(t) := \begin{cases} \lambda \exp\left(\frac{-1}{1-t^2}\right) & |t| < 1 \\ 0 & |t| \geq 1, \end{cases}$$

where λ is the constant such that $\int \psi(t) dt = 1$ and we put $u_n(t) := \frac{\varepsilon}{\lambda} \psi(nt)$ for all $t \in \mathbb{R}$.

We now define a sequence of vector fields V_n along γ as

$$V_n(t) := u_n(t - t_0) (t - t_0) L_{t_0, t}(w) \quad \forall t \in I,$$

where $L_{t_0, t}$ denotes the parallel transport along γ from $\gamma(t_0)$ to $\gamma(t)$. Then for n sufficiently large, $V_n \in H_0^1(I, \gamma^{-1}TM)$ and $\|V_n\|_{L^\infty} \leq 1$. Putting V_n in (4.3), we have

$$\begin{aligned} & \left\langle \int_0^1 u_n(t - t_0) L_{t, t_0} \dot{\gamma}(t) dt, w \right\rangle + \int_0^1 u_n'(t - t_0) (t - t_0) \langle L_{t, t_0} \dot{\gamma}(t), w \rangle dt \\ & + \int_0^1 u_n(t - t_0) (t - t_0) \langle \dot{\gamma}(t), D_t(L_{t_0, t}(w)) \rangle dt \leq \mathcal{C} \quad \forall n, \end{aligned}$$

where $\mathcal{C} := (\|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2)$. Hence passing to the limit as $n \rightarrow \infty$, we conclude that $\|\dot{\gamma}(t_0)\| \leq \mathcal{C} + \|\dot{\gamma}\|_{L^1}$.

On the other hand, using (4.4) and by Hölder inequality, we have

$$\begin{aligned} \int_0^1 (2\bar{\varphi}(\gamma) + \tau) \|V\| \|\dot{\gamma}\|^2 dt &\leq (2\bar{\varphi} + C) \|V\|_{L^2} \left(\int_0^1 \|\dot{\gamma}\|^4 dt \right)^{1/2} \\ &\leq (2\bar{\varphi} + C) \|V\|_{L^2} \|\dot{\gamma}\|_{L^2} \|\dot{\gamma}\|_{L^\infty} \\ &\leq (2\bar{\varphi} + C) \|V\|_{L^2} \|\dot{\gamma}\|_{L^2} \\ &\quad \times (\|\dot{\gamma}\|_{L^1} + \|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2). \end{aligned}$$

Hence applying this to (4.2), we get

$$(4.5) \quad \left| \int_0^1 \langle D_t \dot{\gamma}, V \rangle dt \right| \leq (1 + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}) (\|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2) \|V\|_{L^2},$$

for every $V \in H_0^1(I, \gamma^{-1}TM)$. Hence taking a suitable sequence of vector fields $V_n \in H_0^1(I, \gamma^{-1}TM)$ in (4.5) and passing to the limit as $n \rightarrow \infty$, we obtain that

$$\|D_t \dot{\gamma}\|_{L^2} \leq (1 + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}) (\|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2).$$

Indeed, for given $t_0 \in I$, let $w \in T_{\gamma(t_0)}M$ be the vector such that $\|w\| = 1$ and $\langle D_t \dot{\gamma}(t_0), w \rangle = \|D_t \dot{\gamma}(t_0)\|$. We now consider a sequence of functions $u_n \in C_0^\infty(\mathbb{R})$ with the properties that

$$0 \leq u_n \leq 1, \quad \text{Supp } u_n \subseteq \left[-\frac{1}{n}, \frac{1}{n} \right], \quad \forall n \in \mathbb{N},$$

and we define a sequence of vector fields V_n along γ as

$$V_n(t) := u_n(t - t_0) L_{t_0, t}(w) \quad \forall t \in I.$$

Then for n sufficiently large, $V_n \in H_0^1(I, \gamma^{-1}TM)$ and $\|V_n\|_{L^2} \leq 1$. Putting V_n in (4.5), we have

$$\left\langle \int_0^1 u_n(t - t_0) L_{t_0, t} D_t \dot{\gamma}(t) dt, w \right\rangle \leq \mathcal{D} \quad \forall n,$$

where $\mathcal{D} := (1 + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}) (\|\xi\|_{L^2} + (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^2}^2)$. Hence passing to the limit as $n \rightarrow \infty$, we conclude that $\|D_t \dot{\gamma}(t_0)\| \leq \mathcal{D}$. Thus $D_t \dot{\gamma} \in L^2(I, \gamma^{-1}TM)$ and consequently $\gamma \in W^{2,2}(I, M)$.

We now show that

$$D_t \dot{\gamma}(t) + \xi(t) \in N_S^P(\gamma(t)), \quad a. e. t \in I.$$

Using [25, Lemma 1], we have $N_S^P(x) = (T_S^B(x))^\circ$ and it suffices to prove that

$$\langle D_t \dot{\gamma}(t) + \xi(t), \omega \rangle \leq 0 \quad \forall \omega \in T_S^B(\gamma(t)), \quad a. e. t \in I.$$

Let $t_0 \in I$ and $\omega \in T_S^B(\gamma(t_0))$. Using a sequence of functions $u_n \in C_0^\infty(\mathbb{R})$ with the properties that

$$u_n \geq 0, \quad \text{Supp } u_n \subseteq \left[-\frac{1}{n}, \frac{1}{n} \right], \quad \int u_n = 1, \quad \forall n \in \mathbb{N}$$

we construct a sequence of proper vector fields $V_n \in H_0^1(I, \gamma^{-1}TM)$ along γ defined by

$$V_n(t) := u_n(t - t_0) L_{t_0, t}(\omega) \quad \forall t \in I,$$

where $L_{t_0, t}$ denotes the parallel transport along γ from $\gamma(t_0)$ to $\gamma(t)$. Since $\gamma \in W^{2,2}(I, M)$, for very $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_0^1 \langle \dot{\gamma}, D_t V_n \rangle dt &= - \int_0^1 \langle D_t \dot{\gamma}, V_n \rangle dt \\ &= - \int_0^1 \langle u_n(t - t_0) L_{t, t_0}(D_t \dot{\gamma}), \omega \rangle dt. \end{aligned}$$

Then from (4.1) we obtain that

$$\begin{aligned} - \left\langle \int_0^1 u_n(t - t_0) L_{t, t_0}(D_t \dot{\gamma}) dt, \omega \right\rangle &\geq \int_0^1 u_n(t - t_0) \langle \xi, P_\gamma(L_{t_0, t} \omega) \rangle dt \\ &\quad - (2\bar{\varphi} + C) \|\dot{\gamma}\|_\infty^2 \int_0^1 u_n(t - t_0) \|L_{t_0, t} \omega - P_\gamma(L_{t_0, t} \omega)\| dt. \end{aligned}$$

Therefore when $n \rightarrow \infty$, we derive that

$$- \langle D_t \dot{\gamma}(t_0), \omega \rangle \geq \langle \xi(t_0), P_{\gamma(t_0)} \omega \rangle - (2\bar{\varphi} + C) \|\dot{\gamma}\|_\infty^2 \|\omega - P_{\gamma(t_0)} \omega\|,$$

and since $\omega \in T_S^B(\gamma(t_0))$, $P_{\gamma(t_0)} \omega = \omega$ and then we get the result.

For the converse, we assume that $\gamma \in W^{2,2}(I, M)$ and $\xi \in L^2(I, \gamma^{-1}TM)$ are such that

$$\xi + D_t \dot{\gamma} \in N_S^P(\gamma), \quad a. e.$$

Then using Theorem 3.11, for every $\eta \in \mathcal{A}$ with $d_\infty(\gamma, \eta) < \bar{r}$ we have

$$\begin{aligned} \frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|^2 dt &\geq \frac{1}{2} \int_0^1 (c(\gamma, \eta) - 1) \|\dot{\gamma}(t)\|^2 dt + \int_0^1 \langle \xi, \exp_\gamma^{-1} \eta \rangle dt \\ &\quad - d_\infty(\gamma, \eta) \|\exp_\gamma^{-1} \eta\|_{L^2} (\bar{\varphi} \|\xi + D_t \dot{\gamma}\|_{L^2} + \frac{1}{2} |R|_\infty \|\dot{\gamma}\|_{L^\infty} \|\dot{\eta}\|_{L^2}), \end{aligned}$$

where $c(\gamma, \eta) = 2\sqrt{\Delta} d(\gamma, \eta) \cot(\sqrt{\Delta} d(\gamma, \eta))$.

Since $c(\gamma, \eta) - 2 = O(d(\gamma, \eta)^2)$, we have

$$\left| \int_0^1 (c(\gamma, \eta) - 2) \|\dot{\gamma}(t)\|^2 dt \right| \leq K \int_0^1 d(\gamma, \eta)^2 dt = K \|\exp_\gamma^{-1} \eta\|_{L^2}^2$$

for a suitable constant K that depends on the Taylor expansion of the tangent function at 0 and $\|\dot{\gamma}\|_{L^\infty}^2$, and consequently,

$$\liminf_{d_\infty(\eta, \gamma) \rightarrow 0} \frac{f(\eta) - f(\gamma) - \langle \xi, \exp_\gamma^{-1} \eta \rangle_{L^2}}{\|\exp_\gamma^{-1} \eta\|_{L^2}} \geq 0.$$

It follows that $\xi \in D^- f(\gamma)$.

In particular, since

$$-D_t \dot{\gamma}(t) + D_t \dot{\gamma}(t) = 0 \in N_S^P(\gamma(t)) \quad a. e. \quad t \in I,$$

we deduce that $-D_t \dot{\gamma} \in D^- f(\gamma)$ and $D^- f(\gamma) \neq \emptyset$. \square

Corollary 4.2. *If $\gamma \in \mathcal{A} \cap W^{2,2}(I, M)$, then $-P_\gamma(D_t\dot{\gamma}) \in D^-f(\gamma)$ and*

$$\| -P_\gamma(D_t\dot{\gamma}) \|_{L^2} \leq \| \xi \|_{L^2} \quad \forall \xi \in D^-f(\gamma).$$

Proof. Since $D_t\dot{\gamma} \in L^2(I, \gamma^{-1}TM)$, we also have $-P_\gamma(D_t\dot{\gamma}) \in L^2(I, \gamma^{-1}TM)$. Moreover,

$$D_t\dot{\gamma} - P_\gamma(D_t\dot{\gamma}) \in N_{T_S^B(\gamma(t))}^P(0) = N_S^P(\gamma(t)) \quad \text{a. e. } t \in I,$$

because $T_S^B(\gamma(t))$ is a closed convex subset of $T_{\gamma(t)}M$. Hence Theorem 4.1 implies that $-P_\gamma(D_t\dot{\gamma}) \in D^-f(\gamma)$.

We now assume that $\xi \in D^-f(\gamma)$, thus $\xi + D_t\dot{\gamma} \in N_S^P(\gamma(t))$ for almost all $t \in I$. It follows that

$$\langle \xi + D_t\dot{\gamma}, P_\gamma(D_t\dot{\gamma}) \rangle \leq 0, \quad \text{a. e.}$$

and hence

$$\begin{aligned} \langle \xi, -P_\gamma(D_t\dot{\gamma}) \rangle_{L^2} &\geq \langle D_t\dot{\gamma}, P_\gamma(D_t\dot{\gamma}) \rangle_{L^2} \\ &= \langle P_\gamma(D_t\dot{\gamma}), P_\gamma(D_t\dot{\gamma}) \rangle_{L^2}, \end{aligned}$$

that completes the proof. \square

Theorem 4.3. *Let $\gamma \in \mathcal{A}$, then $0 \in D^-f(\gamma)$ if and only if γ is a weak geodesic on S .*

Proof. Using Theorem 4.1, we have $0 \in D^-f(\gamma)$ if and only if $\gamma \in W^{2,2}(I, M)$ and $D_t\dot{\gamma}(t) \in N_S^P(\gamma(t))$ for almost all $t \in I$. \square

Proposition 4.4. *If $\gamma \in \mathcal{A}$ is a weak geodesic on S , then $\gamma \in W^{2,\infty}(I, M)$ and γ has constant speed.*

Proof. If $\gamma \in \mathcal{A}$ is a weak geodesic on S , then $0 \in D^-f(\gamma)$ and hence using Theorem 3.8, there exists a piecewise constant function τ on I such that for every vector field $V \in H_0^1(I, \gamma^{-1}TM)$,

$$\int_0^1 \langle \dot{\gamma}, D_tV \rangle dt \geq - \int_0^1 (2\varphi(\gamma) + \tau) \|V - P_\gamma V\| \|\dot{\gamma}\|^2 dt.$$

Then similar to the proof of Theorem 4.1 we deduce that $\dot{\gamma} \in L^\infty(I, \gamma^{-1}TM)$ and hence

$$\left| \int_0^1 \langle \dot{\gamma}, D_tV \rangle dt \right| \leq (2\bar{\varphi} + C) \|\dot{\gamma}\|_{L^\infty}^2 \|V\|_{L^1},$$

for all $V \in H_0^1(I, \gamma^{-1}TM)$. This implies that $D_t\dot{\gamma} \in L^\infty(I, \gamma^{-1}TM)$. Hence $\gamma \in W^{2,\infty}(I, M)$ and the function $\|\dot{\gamma}\|^2$ is Lipschitz on I . Indeed, we have

$$\left| \frac{d}{dt} \|\dot{\gamma}(t)\|^2 \right| = |2\langle \dot{\gamma}(t), D_t\dot{\gamma}(t) \rangle| \leq 2\|\dot{\gamma}\|_{L^\infty} \|D_t\dot{\gamma}\|_{L^\infty}.$$

Therefore similar to the proof of [8, Theorem 3.8], we show that $\frac{d}{dt} \|\dot{\gamma}\|^2 = 0$, a.e. on I . To this end, since $D_t\dot{\gamma}(t) \in N_S^P(\gamma(t))$, a.e. on I , It suffices to prove that $\langle \eta, \dot{\gamma}(t) \rangle = 0$ for all $\eta \in N_S^P(\gamma(t))$.

Let $\eta \in N_S^P(\gamma(t))$ for some $t \in (0, 1)$. Then for all $s > t$ and close enough to t we have

$$\left\langle \eta, \frac{\exp_{\gamma(t)}^{-1} \gamma(s)}{s-t} \right\rangle \leq \varphi(\gamma(t)) \|\eta\| \frac{d^2(\gamma(t), \gamma(s))}{s-t}.$$

Taking the limit as $s \rightarrow t^+$, we conclude that $\langle \eta, \dot{\gamma}(t) \rangle \leq 0$. Similarly, we have $\langle \eta, \dot{\gamma}(t) \rangle \geq 0$ and then we get the result. \square

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