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# Expressing complementarity and the $x$ – $p$ commutation relation through further quantum inequalities

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## Abstract

Complementarity and the commutation relation of position ( $x$ ) and momentum ( $p$ ) imply much more than the fundamental  $x$ – $p$  uncertainty inequality. Here, we display some further consequences of the former that could have certain pedagogical interest and, so, contribute to the teaching of quantum mechanics. Inspired by an elementary derivation of the  $x$ – $p$  uncertainty inequality, based upon a positive quadratic polynomial, we explore one possible extension, via quartic polynomials and simple algebra and integrations. Our analysis, aimed at providing some further pedagogic expression of genuine quantum behaviours, yields other quantum inequalities for expectation values, expressed through suitable discriminants associated with quartic algebraic equations, which differ from (and are not a strict consequence of) the  $x$ – $p$  uncertainty inequality. Those quantum inequalities are confirmed, and genuine non-classical behaviours are exhibited, for simple cases: a harmonic oscillator, a hydrogenic atom and free Gaussian wave packets. The physical interest of the expectation values involved in the quantum inequalities and of the latter is discussed, in the framework of quantum optics and squeezing phenomena.

## 1. Introduction

Bohr's complementarity principle [1], a fundamental principle of quantum mechanics, expresses the wave–particle duality in nature, namely, that the wave aspect and the corpuscular aspect are complementary aspects at the microscopic level, which become manifest only in mutually exclusive experiments. For a detailed account in the historical context, see [2]. Bohm [3] has expressed compactly that principle as follows. ‘At the quantum level, the most general physical properties of any system must be expressed in terms of complementary pairs of variables, each of which can be better defined only at the expense of a corresponding loss in the degree of definition of the other’. More precisely, once the superposition

principle, the association of physical variables (observables) with self-adjoint operators and the probabilistic interpretation have been formulated, commutation relations of complementary pairs of variables (like that for position and momentum) express complementarity. See also [4, 5]. In particular, the position–momentum uncertainty inequalities, which follow from the position–momentum commutation relation (and give a precise meaning to Heisenberg’s uncertainty relation), can be regarded as a consequence of complementarity, as stressed in [2]. See also [6]. Research programmes, based upon quantum optics, have been devoted in the last few decades to test complementarity, and to study the performance of repeated precise measurements (named quantum non-demolition measurements) of a quantum-mechanical observable, which will not be contaminated by uncertainties in its complementary variable, due to the uncertainty inequality: see [6, 7].

One could ask whether the commutation relations of complementary pairs of variables also imply further quantum conditions on expectation values of the latter, which will not be a strict consequence of their uncertainty inequalities (although be related to them). It is open, *a priori*, whether those additional restrictions would be much more severe than those following from the uncertainty inequalities. In spite of that, such further quantum conditions may well have, at least, pedagogical interest (say, at about the graduate level) and provide additional insight into the peculiarities of the genuine quantum regime. This paper will be devoted to their search for position and momentum, to display them in simple cases, at an elementary level, and to outline their possible physical interest regarding the squeezing phenomenon. In particular, the possibility of quantum noise reduction has triggered the interest in squeezing. We recall that squeezed quantum states at second order are those in which the uncertainty in one variable (position or momentum) is adequately reduced, at the expense that the uncertainty in the complementary variable increases, consistently with the uncertainty inequality. Second-order squeezing has been demonstrated experimentally for photons: by 1985 for the first time, and repeatedly, and with increasing improvements later. See, for instance, [8]. Squeezing behaviours of quantum states at even  $N$ th order ( $N = 4, 6, \dots$ ) in some variable have been characterized theoretically [9, 10]. For  $N$ th-order squeezing with  $N > 2$ , to understand the sort of restrictions imposed by complementarity and by the  $x$ – $p$  commutation relation on the variable which is complementary to the one being squeezed appears to be interesting physically and to require, eventually, further analysis. To see whether this question could receive some clarification from the further quantum conditions to be treated here seems also physically interesting. For an additional appreciation of the physical relevance of squeezing, we shall just mention that, very recently, the latter has also been demonstrated experimentally for surface-plasmon polaritons (namely, combined electron oscillations and electromagnetic waves propagating along the interface between a conductor and a dielectric medium) in a gold waveguide [11].

Section 2 revisits the position–momentum uncertainty inequality, a textbook derivation of it based upon second-degree polynomials, and an interesting extension of it [12, 13], which exhibits an additional quantum contribution. Section 3 presents one simple extension, using a fourth-degree polynomial, and the resulting quantum inequalities for certain expectation values. Section 4 analyses the latter in some simple cases and examples, with a pedagogical purpose. For brevity, we shall omit the direct computations involved in those examples, which may constitute exercises for students (who have previously followed an introductory course on quantum physics or quantum mechanics and have then become acquainted with harmonic oscillators and their coherent states, free Gaussian wave packets and hydrogenic atoms). Section 5 discusses briefly the physical interest of the expectation values appearing in the new quantum inequalities (and the significance of the latter) in connection with quantum optics and squeezing.

Section 6 contains conclusions and discussions. An appendix recalls, in outline, the elementary solution of quartic algebraic equations, which leads to the quantum inequalities in section 3.

## 2. The position–momentum uncertainty inequality and its variants

To fix the ideas, let a quantum system be represented, at a given time  $t$ , by a single state (the normalized ket  $|\psi\rangle$ ,  $\langle\psi|\psi\rangle = +1$ ). Then, if  $O$  is a generic observable, its real expectation value  $\langle O \rangle$  equals  $\langle\psi|O|\psi\rangle$ . Let  $[C, D] = CD - DC$  and  $[C, D]_+ = CD + DC$  denote, respectively, the commutator and the anticommutator of the observables  $C$  and  $D$ . We shall focus on one single microscopic entity or constituent of those forming our quantum system and, specifically, on two observables of that entity: one (cartesian) coordinate  $x$  and its associated (complementary) momentum  $p$  ( $= -i\hbar\partial/\partial x$ , in coordinate representation).  $\hbar$  is Planck's constant. One has the  $x$ - $p$  commutation relation ( $x$ - $p$  c.r.):  $[x, p] = i\hbar$ .

The  $x$ - $p$  uncertainty inequality ( $x$ - $p$  u.i.) was derived first, using Schwartz's inequality, by Weyl [14] (on a suggestion by W Pauli). That derivation appears in various books (see, for instance, [15]). Among various extensions of the  $x$ - $p$  u.i., we shall comment on one by Schrödinger [12], in the form presented in a lucid elaboration by de la Torre [13]. The latter reads

$$\Delta_x^2 \Delta_p^2 \geq \left(\frac{\hbar}{2}\right)^2 + \delta_+, \quad (1)$$

$$\Delta_x = [(\langle x - \langle x \rangle \rangle^2)]^{1/2}, \quad \Delta_p = [(\langle p - \langle p \rangle \rangle^2)]^{1/2}, \quad (2)$$

$$\delta_+ = \left[ \frac{1}{2} \langle [x, p]_+ \rangle - \langle x \rangle \langle p \rangle \right]^2. \quad (3)$$

If  $\delta_+$  vanishes, equation (1) becomes the standard  $x$ - $p$  u.i.:  $\Delta_x \Delta_p \geq \hbar/2$  (see comments in [13]). Our specific interest in (1) is due to the appearance of  $\langle [x, p]_+ \rangle$  in (3). The student of quantum mechanics is used to seeing commutators ubiquitously: to meet other objects (namely, anticommutators), as another signal of the departure from classical behaviour, could contribute to open one's mind. That is one reason for focusing here on the anticommutator  $[x, p]_+$ , as a correction in (1).  $[x, p]_+$  is also a forerunner of another anticommutator, to appear in section 3.

Let us recall briefly the interesting role of  $\langle [x, p]_+ \rangle$  in other contexts.  $[x, p]_+$  appears in the analysis of the  $\hbar \rightarrow 0$  limit (section 6.3 in [4]) and in the quantum-mechanical virial theorem [16]. One also meets  $\langle [x, p]_+ \rangle$  and  $\delta_+$  in quantum optics: we shall discuss those features in section 5.

Another well-known textbook derivation of the  $x$ - $p$  u.i. employs a non-negative real second-degree polynomial  $f_2(\lambda)$ : see, for instance, [4, 16, 17]. We shall work with  $x' = x - x_0$  and  $p' = p - p_0$ ,  $x_0$  and  $p_0$  being real constants, to be chosen adequately later. Primes will denote dependences on  $x_0$  and  $p_0$ .  $\lambda$  stands for a real (dimensionful) variable.  $f_2(\lambda)$  reads

$$f_2(\lambda) = \langle (x' + i\lambda p')(x' - i\lambda p') \rangle = \langle p'^2 \rangle \lambda^2 + \hbar \lambda + \langle x'^2 \rangle \geq 0. \quad (4)$$

$[x', p'] = i\hbar$  has also been used. The second-degree equation  $f_2(\lambda) = 0$  cannot have two real different solutions, so that its discriminant fulfils  $(\hbar/2)^2 - \langle p'^2 \rangle \langle x'^2 \rangle \leq 0$ . The last inequality becomes, first,  $\Delta_x \Delta_p \geq \hbar/2$  (provided that  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ ) and, second,  $\langle p'^2 \rangle \langle x'^2 \rangle \geq (\hbar/2)^2$  (if  $x_0 = 0$  and  $p_0 = 0$ ) [4]. Both of them constitute fundamental quantum restrictions.  $f_2(\lambda)$  (which has not given rise to anticommutators as corrections to the  $x$ - $p$

u.i., unlike equation (1)) will give inspiration in the search for further quantum restrictions in section 3.

### 3. Fourth-degree polynomial $f_4(\lambda)$ : further quantum inequalities

In order to generalize  $f_2$ , we shall define the new function:

$$f_4(\lambda) = \langle (x' + i\lambda p')(x' + i\lambda p')(x' - i\lambda p')(x' - i\lambda p') \rangle. \quad (5)$$

It follows that (1)  $f_4(\lambda)$  is real, (2)  $f_4(\lambda) \geq 0$  for any real  $\lambda$  in  $-\infty < \lambda < +\infty$  and (3)  $f_4(\lambda)$  is a polynomial in  $\lambda$  of degree 4. Equation (5), and a little bit of elementary algebra using  $[x', p'] = i\hbar$ , yields

$$f_4(\lambda) = a'_0 \lambda^4 + a'_1 \lambda^3 + a'_2 \lambda^2 + a'_3 \lambda + a'_4, \quad (6)$$

$$a'_0 = \langle p'^4 \rangle, \quad a'_1 = 4\hbar \langle p'^2 \rangle, \quad a'_3 = 4\hbar \langle x'^2 \rangle, \quad a'_4 = \langle x'^4 \rangle, \quad (7)$$

$$a'_2 = 3\hbar^2 + \langle [x'^2, p'^2]_+ \rangle. \quad (8)$$

All  $a'_j$ ,  $j = 0, 1, \dots, 4$  are real and  $a'_j$ ,  $j = 0, 1, 3, 4$ , are  $\geq 0$ . We shall suppose that  $a'_j > 0$ ,  $j = 0, 1, 3, 4$  (which hold typically for normalized states  $|\psi\rangle$ ). Equation (6) and the crucial property (2) will lead to the quantum inequalities on the  $a'_j$  announced in section 1. To exhibit those inequalities for the fourth-degree polynomial  $f_4(\lambda)$  is a task more difficult than that for  $f_2(\lambda)$ . Such an algebraic task is outlined in the appendix. We shall summarize here the practical recipes.

Let us consider the roots of the quartic algebraic equation

$$f_4(\lambda) = 0. \quad (9)$$

In principle, the four solutions of the quartic equation in (9), with real coefficients and with the crucial condition  $f_4(\lambda) \geq 0$ , could be (a) either all complex, grouped as two pairs of complex conjugate solutions (both pairs may coincide), (b) or a double real solution and one pair of complex conjugate solutions, (c) or two double real solutions (both pairs being different from each other), (d) or a fourthfold real solution.

Let us introduce

$$p'_q = -\frac{3a_1'^2}{8a_0'} + a_2', \quad q'_q = \frac{a_1'^3}{8a_0'^2} - \frac{a_1'a_2'}{2a_0'} + a_3', \quad (10)$$

$$r'_q = -\frac{3a_1'^4}{4(4a_0')^3} + a_2' \left( \frac{a_1'}{4a_0'} \right)^2 - \frac{a_1'a_3'}{4a_0'} + a_4'. \quad (11)$$

First, let us assume that  $q'_q = 0$  (the special case). We anticipate that  $q'_q = 0$  will occur near the classical limit. Then,  $f_4(\lambda) \geq 0$  is true in any of the following alternative cases.

- (i)  $p_q'^2 - 4r_q'a_0' > 0$ ,  $p_q' > 0$  and  $r_q' \geq 0$ . The case  $r_q' = 0$  corresponds to (b), while  $r_q' > 0$  corresponds to (a).
- (ii)  $p_q'^2 - 4r_q'a_0' = 0$ . The case  $p_q' = 0$  corresponds to (d). The case  $p_q' > 0$  corresponds to (a). The case  $p_q' < 0$  corresponds to (c).
- (iii)  $p_q'^2 - 4r_q'a_0' < 0$ . This corresponds to (a).

Second, let us suppose that  $q'_q \neq 0$  (the general case). Let us introduce  $D'_r$  as

$$D'_r \equiv \left[ \frac{a_0' p_r}{3} \right]^3 + \left[ \frac{a_0'^2 q_r}{2} \right]^2, \quad \frac{a_0' p_r}{3} = \frac{1}{2^2} \left[ \frac{1}{3} \frac{a_1'}{2} \frac{a_3'}{2} - \frac{1}{3} a_0' a_4' - \frac{1}{2^2} \left( \frac{a_2'}{3} \right)^2 \right], \quad (12)$$

$$\frac{a_0'^2 q_r}{2} = \frac{1}{2^4} \left[ \frac{1}{2} \frac{a_1'}{2} \frac{a_3'}{2} \frac{a_2'}{3} - \frac{1}{2^2} \left( \frac{a_2'}{3} \right)^3 + a_0' a_4' \frac{a_2'}{3} - \frac{1}{2} \left( a_0' \left( \frac{a_3'}{2} \right)^2 + a_4' \left( \frac{a_1'}{2} \right)^2 \right) \right]. \quad (13)$$

The motivation for  $D_r'$  is explained in the appendix.  $D_r'$  will play a role for  $f_4$  similar to the discriminant for  $f_2$ . Note that  $D_r'$  (which involves  $\hbar^2$ ) is a symmetric function of the pair  $(a_0', a_4')$  and of the pair  $(a_1', a_3')$ . One finds that  $f_4(\lambda) \geq 0$  is true if either

$$D_r' < 0, \quad (14)$$

which implies case (a) above (the four roots of (9) constituting two pairs of complex conjugate solutions), or

$$D_r' = 0, \text{ with } q_r \neq 0. \quad (15)$$

$D_r' = 0$  (with  $q_r \neq 0$ ) implies case (b) above (the four roots of (9) grouping into a double real root and two complex conjugate ones).  $f_4(\lambda) < 0$  if  $D_r' > 0$  which, then, is discarded.

A new feature is that  $f_4(\lambda)$  does give rise to the anticommutator  $[x'^2, p'^2]_+$  in  $a_2'$  (equation (8)) which, in turn, contributes to the quantum inequalities. The anticommutator  $[x'^2, p'^2]_+$  is a Hermitian operator and, then,  $\langle [x'^2, p'^2]_+ \rangle$  is real. It may be interesting to realize that  $[x'^2, p'^2]_+$  can be recast in terms of the simpler anticommutator  $[x', p']_+$  considered in section 2. In fact, a straightforward and elementary algebraic calculation gives successively

$$[x'^2, p'^2]_+ = x' p'^2 x' + p' x'^2 p' - 2\hbar^2 = \frac{([x', p']_+)^2}{2} - \frac{3\hbar^2}{2}. \quad (16)$$

By using  $\langle \psi | ([x', p']_+)^2 | \psi \rangle \geq 0$  and (16), one finds  $\langle [x'^2, p'^2]_+ \rangle \geq -(3/2)\hbar^2$ . Then,  $a_2' \geq (3/2)\hbar^2$ . Without attempting to overemphasize it,  $\langle [x'^2, p'^2]_+ \rangle$  could be tentatively interpreted as the correlation between the squared deviations of  $x$  and  $p$  with respect to  $x_0$  and  $p_0$ , at equal times.

Having in mind  $N$ th-order squeezing ( $N = 4, 6, \dots$ ) [9, 10], one may wonder whether at least one inequality involving only  $\langle x'^4 \rangle$  and  $\langle p'^4 \rangle$  (but no other expectation value) could exist, which could play a role similar to the uncertainty inequality for  $\langle x'^2 \rangle$  and  $\langle p'^2 \rangle$ . To the best of the author's knowledge, any non-trivial inequality containing  $\langle x'^4 \rangle$  and  $\langle p'^4 \rangle$  should also include other expectation values, as (14) and (15) illustrate. In this connection, we shall present briefly an alternative inequality (different from (14) and (15)) containing  $\langle x'^4 \rangle$ ,  $\langle p'^4 \rangle$ , but now including the anticommutator  $[x', p']_+$ . In fact, let us consider the real second-degree polynomial  $g_2(\lambda) = \langle (x'^2 + i\lambda p'^2)(x'^2 - i\lambda p'^2) \rangle$ . One has  $g_2(\lambda) \geq 0$  for any real  $\lambda$ . Like for  $f_2(\lambda) = 0$ , the discriminant of  $g_2(\lambda) = 0$  yields

$$\langle p'^4 \rangle \langle x'^4 \rangle \geq \hbar^2 \langle [x', p']_+ \rangle^2. \quad (17)$$

Inequality (17) will turn out to be less interesting than (14) and (15), as we shall see at the end of subsection 4.2.

Thus far, we have supposed that the quantum system was represented by a pure state  $|\psi\rangle$ . The above analysis in this section can be directly generalized and also holds if the system is represented by a statistical mixture of states, namely, by a density operator  $\rho$  (self-adjoint, positive and with unit trace).

In section 4, we shall confirm and explore the above inequalities for several simple cases. In genuine quantum regimes and for suitable  $x_0$  and  $p_0$ , we shall see that  $D_r' (< 0)$  will get close (and, in some case, equal) to 0, and that  $\langle [x'^2, p'^2]_+ \rangle$  can become negative (anticorrelation of  $x'^2$  and  $p'^2$ ).

#### 4. Inequalities implied by $f_4(\lambda)$ : simple cases and examples

##### 4.1. Near the classical limit ( $\hbar$ close to 0)

Let  $x_0 = 0$  and  $p_0 = 0$ , and let  $\hbar$  be very small. One has  $a'_1 \rightarrow 0$ ,  $a'_3 \rightarrow 0$  and  $a'_2 \rightarrow \langle [x^2, p^2]_+ \rangle$ . Then,  $f_4(\lambda) \rightarrow a'_0 \lambda^4 + a'_2 \lambda^2 + a'_4 \equiv f_{4,c}$ , which is biquadratic and  $\geq 0$ . Hence, as a first approximation,  $q'_q \simeq 0$  and the solutions of  $f_4(\lambda) = 0$  approach those for  $f_{4,c}(\lambda) = 0$  which, in turn, are

$$\lambda^2 = \frac{-a'_2 \pm [a'^2_2 - 4a'_0 a'_4]^{1/2}}{2a'_0}. \quad (18)$$

If  $a'^2_2 - 4a'_0 a'_4 < 0$ , the four  $\lambda$  provided by equation (18) are complex. This corresponds, by recalling the classification in section 3, to  $p'^2_q - 4r'_q a'_0 < 0$  (that is, to (iii) and, hence, to case (a)). Similarly,  $a'^2_2 - 4a'_0 a'_4 = 0$  corresponds to either (d) (if  $a'_2 = 0$ ) or to (a) (if  $a'_2 > 0$ ). As  $\hbar$  is close to 0, the conditions for  $f_4(\lambda) \geq 0$  involving  $D'_r$ , given in section 3, are consistent with  $a'^2_2 - 4a'_0 a'_4 \leq 0$ . Also, for very small  $\hbar$ ,  $x$  and  $p$  become commuting variables, and both  $a'_2 \rightarrow 2\langle x^2 p^2 \rangle$  and  $a'_2 \geq 0$  hold. On the other hand, the expectation that  $\langle p^4 \rangle \rightarrow \langle p^2 \rangle^2$  and  $\langle x^4 \rangle \rightarrow \langle x^2 \rangle^2$  would hold if  $\hbar$  is very small fails in general (compare with section 6.3 in [4]): see subsection 4.2.

##### 4.2. The harmonic oscillator (1)

We consider a one-dimensional quantum harmonic oscillator with unit mass, frequency  $\omega (> 0)$ , coordinate  $x$ , momentum  $p$  and Hamiltonian  $H = \hbar\omega(a^+ a + 1/2)$ . We employ the destruction ( $a$ ) and creation ( $a^+$ ) operators:

$$a = \left[ \frac{1}{2\hbar} \right]^{1/2} \left[ \omega^{1/2} x + i \frac{p}{\omega^{1/2}} \right], \quad a^+ = \left[ \frac{1}{2\hbar} \right]^{1/2} \left[ \omega^{1/2} x - i \frac{p}{\omega^{1/2}} \right]. \quad (19)$$

Let  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$  and let  $|\psi\rangle$  be a normalized stationary state  $|n\rangle$  (an eigenstate of  $H$ , with energy  $\hbar\omega(n + 1/2)$ ,  $n = 0, 1, 2, 3, \dots$ ). Using  $a|0\rangle = 0$  and  $[a, a^+] = 1$ , all the following computations are straightforward. One finds  $\langle x \rangle = 0$ ,  $\langle p \rangle = 0$  and  $\langle [x, p]_+ \rangle = 0$ . Let  $\lambda = \lambda_1/\omega$ . Then, one gets

$$f_4(\lambda) = \left[ \frac{\hbar}{2\omega} \right]^2 f_{4,1}(\lambda_1), \quad (20)$$

$$f_{4,1}(\lambda_1) = (6n^2 + 6n + 3)\lambda_1^4 + 4(4n + 2)\lambda_1^3 + (4n^2 + 4n + 10)\lambda_1^2 + 4(4n + 2)\lambda_1 + 6n^2 + 6n + 3. \quad (21)$$

The comparison of (6) and (21) yields all  $a'_j$ ,  $j = 0, 1, \dots, 4$ . (17) becomes a triviality. Numerical computation confirms that  $f_{4,1}(\lambda_1) \geq 0$  for any  $n \geq 0$  and any real  $\lambda_1$ . For very large  $n$  (close to the classical limit), by keeping only the dominant terms (in  $n^2$ ), one has  $f_{4,1}(\lambda_1) \simeq 6n^2\lambda_1^4 + 4n^2\lambda_1^2 + 6n^2$ . Then, all solutions of  $f_{4,1}(\lambda_1) = 0$ , which is biquadratic, are complex and different from one another (which corresponds to the special case  $q'_q = 0$  and (iii) in section 3). Some quantum remnants still remain near the classical limit, since  $\langle p^4 \rangle \neq \langle p^2 \rangle^2$  and  $\langle x^4 \rangle \neq \langle x^2 \rangle^2$ . Next, with large  $n$ , we keep the two small terms in  $\lambda_1^3$  and  $\lambda_1$  in  $f_{4,1}(\lambda_1)$ . Then,  $q'_q$  is small but it does not vanish (the general case). In this case, one finds  $D'_r < 0$ . As  $n$  decreases,  $D'_r$  increases, with  $D'_r < 0$  for  $n \geq 2$ , and it takes on its maximum value ( $D'_r = 0$ ) for both  $n = 0$  and  $n = 1$ . For  $n = 0$  and  $n = 1$ ,  $f_{4,1}(\lambda_1) = 0$

has as a double real solution ( $\lambda_1 = -1$ ) and two complex conjugate solutions. Thus, for both  $n = 0$  and  $n = 1$ ,  $q'_q \neq 0$  and  $D'_r = 0$  (with  $q_r \neq 0$ ) hold (case (b) in section 3). This is a counterpart for  $f_4(\lambda)$  of the property (implied by  $f_2(\lambda)$ ) that  $\Delta_x^2 \Delta_p^2 - (\hbar^2/4)$  decreases as  $n$  does and vanishes for  $n = 0$  for the harmonic oscillator. Another manifestation of genuine quantum behaviour:  $\langle [x^2, p^2]_+ \rangle = (\hbar^2/2)(2n^2 + 2n - 1)$  equals  $-(\hbar^2/2)$  for  $n = 0$  (but it is  $> 0$  for  $n \geq 1$ ). Since  $\langle x \rangle = 0$ ,  $\langle p \rangle = 0$  and  $\langle [x, p]_+ \rangle = 0$ , inequality (17) is trivially satisfied here, while inequalities (14) and (15) are not trivial.

#### 4.3. The harmonic oscillator(2): minimal uncertainty states, coherent states

Let  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ . We shall look for normalized states  $|\psi\rangle$  fulfilling  $(x' - i\lambda p')^2 |\psi\rangle = 0$  for a suitable  $\lambda$ , so that  $f_4(\lambda) = 0$ . Let  $|\psi\rangle$  be a normalized minimal uncertainty state ( $(x' - i\lambda p') |\psi\rangle = 0$ ) [4, 6, 15–17]. Then, one finds that  $f_4(\lambda) = 0$ , with  $\lambda = -\hbar/[2\langle (p - \langle p \rangle)^2 \rangle]$  (a double real root).

There is also one normalized state  $|\psi\rangle (\neq 0)$ , such that both (1)  $(x' - i\lambda p') |\psi_1\rangle = 0$  and (2)  $|\psi_1\rangle = (x' - i\lambda p') |\psi\rangle \neq 0$  hold. One finds that  $|\psi\rangle = a^+ |0\rangle$ ,  $|\psi_1\rangle = |0\rangle$ ,  $a|0\rangle = 0$ ,  $\langle x \rangle = 0$  and  $\langle p \rangle = 0$ , with  $f_4(\lambda) = 0$  for  $\lambda = -\omega^{-1}$  (a double real root). In terms of wavefunctions  $|\psi_1\rangle \rightarrow [\pi\hbar/\omega]^{-1/4} \exp(-2\hbar^{-1}\omega x^2)$  and  $|\psi\rangle \rightarrow 2^{1/2}[\pi(\hbar/\omega)^3]^{-1/4} x \exp(-2\hbar^{-1}\omega x^2)$ .

If  $|\psi\rangle = |\alpha\rangle$  is a normalized coherent state ( $a|\alpha\rangle = \alpha|\alpha\rangle$ ), with complex eigenvalue  $\alpha = \text{Re}\alpha + i\text{Im}\alpha$ ,  $f_4(\lambda) \geq 0$  holds, of course. We shall state a few of its properties.

First, let  $x_0 = 0$  and  $p_0 = 0$ . Then,  $f_4(\lambda) = [\hbar/(2\omega)]^2 f_{4,2}(\lambda_1)$ , with

$$f_{4,2}(\lambda_1) = [16(\text{Im}\alpha)^4 + 24(\text{Im}\alpha)^2 + 3]\lambda_1^4 + [32(\text{Im}\alpha)^2 + 8]\lambda_1^3 + (2/\hbar)^2 a_2' \lambda_1^2 + [32(\text{Re}\alpha)^2 + 8]\lambda_1 + 16(\text{Re}\alpha)^4 + 24(\text{Re}\alpha)^2 + 3, \quad (22)$$

with  $(2/\hbar)^2 a_2' = 8|\alpha|^2 + 16(\text{Re}\alpha)^2(\text{Im}\alpha)^2 + 10$ .  $f_{4,2}(\lambda_1) = 0$  has a double real root and two complex conjugate ones for small  $|\alpha|$ , and four complex roots for large  $|\alpha|$ .  $\langle [x^2, p^2]_+ \rangle$  is  $< 0$  for  $8|\alpha|^2 + 16(\text{Re}\alpha)^2(\text{Im}\alpha)^2 < 2$  (that is, if  $|\alpha| \leq$  about 0.5).

Second, let  $x_0 = \langle x \rangle = (2\hbar/\omega)^{1/2} \text{Re}\alpha$  and  $p_0 = \langle p \rangle = (2\hbar\omega)^{1/2} \text{Im}\alpha$ . Then,  $f_4(\lambda) = 0$  holds, with  $\lambda = -\omega^{-1}$  (a double real root, so that  $D'_r = 0$ ), and one has  $\langle [x^2, p^2]_+ \rangle = -(\hbar^2/2)$  for any  $\alpha$ .

#### 4.4. Free Gaussian wave packets

Let  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ . We consider a free non-relativistic particle with mass  $m$ , described by the normalized time ( $t$ ) dependent Gaussian wave packet

$$|\psi\rangle \rightarrow \psi(x, t)_0 = \left[ \frac{2^{1/2}a}{\pi^{1/2}(a^2 + (2i\hbar t/m))} \right]^{1/2} \exp\left[ -\frac{x^2}{a^2 + (2i\hbar t/m)} \right], \quad (23)$$

with width  $a(> 0)$  and  $\langle \psi | \psi \rangle = \int_{-\infty}^{+\infty} \psi(x, t)^* \psi(x, t)$ . All integrations are Gaussian and can be done straightforwardly. For (23) at any  $t \geq 0$ , one has  $\langle x \rangle = 0$ ,  $\langle p \rangle = 0$  and  $\langle [x, p]_+ \rangle = 2\hbar^2 t/(ma^2)$ , and the coefficients of  $f_4(\lambda)$  become

$$a'_{0,0} = 3 \left[ \frac{\hbar}{a} \right]^4, \quad a'_{0,1} = 4\hbar \left[ \frac{\hbar}{a} \right]^2, \quad a'_{0,2} = \hbar^2 \left[ \frac{5}{2} + \frac{6\hbar^2 t^2}{m^2 a^4} \right], \quad (24)$$

$$a'_{0,3} = \hbar \left[ a^2 + \frac{4\hbar^2 t^2}{m^2 a^2} \right], \quad a'_{0,4} = \frac{3}{16a^4} \left[ a^4 + \frac{4\hbar^2 t^2}{m^2} \right]^2. \quad (25)$$

$a'_{0,3}$  displays the standard spreading of a free Gaussian wave packet [18]. Such a spreading is negligible for  $t(> 0)$  appreciably smaller than  $\tau = (ma/\hbar)$  ( $a'_{0,3} \simeq \hbar a^2$ ). Recall that spreading

of incoming free Gaussian wave packets in (at least, non-relativistic) scattering experiments is negligible for time durations smaller than  $\tau$  [18]. On the other hand,  $\langle [x'^2, p'^2]_+ \rangle = a'_{0,2} - 3\hbar^2$  becomes negative for  $0 < t < 12^{-1/2}\tau$ . Then,  $\langle [x'^2, p'^2]_+ \rangle < 0$  would mean a genuine quantum behaviour (anticorrelation of  $x^2$  and  $p^2$ ) in typical time durations smaller than  $\tau$ .

Next, we consider  $|\psi\rangle \rightarrow \psi(x, t)_1 = [2ax/(a^2 + (2i\hbar t/m))]\psi(x, t)_0$  for  $t \geq 0$ , which implies  $\langle x \rangle = 0$  and  $\langle p \rangle = 0$ . Four coefficients of  $f_4(\lambda)$  for  $\psi(x, t)_1$  are  $a'_{1,0} = 5a'_{0,0}$ ,  $a'_{1,1} = 3a'_{0,1}$ ,  $a'_{1,3} = 3a'_{0,3}$ ,  $a'_{1,4} = 5a'_{0,4}$ . Finally, we find for  $\psi(x, t)_1$   $\langle [x'^2, p'^2]_+ \rangle = \hbar^2[(3/2) + (30\hbar^2 t^2/m^2 a^4)] > 0$ .

$\psi(x, t = 0)_0$  and  $\psi(x, t = 0)_1$  have the same properties as the  $n = 0$  and  $n = 1$  states of a quantum harmonic oscillator (see subsections 4.2 and 4.3).

#### 4.5. Hydrogenic atom

We now consider a hydrogenic atom in its ground state, with  $|\psi\rangle \rightarrow \psi(r) = (\pi a^3)^{-1/2} \exp[-(r/a)]$  and  $a = a_0/Z$  [4, 16, 17].  $r$ ,  $a_0$  and  $Z$  are the three-dimensional radial coordinate, the Bohr radius and the atomic number, respectively.  $x$  is now the  $z$  Cartesian coordinate (used to define polar  $\theta$  and azimuthal angles  $\phi$ ), with  $\langle \psi|\psi \rangle = \int_0^{+\infty} dr r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \psi(r)^* \psi(r)$ . We choose  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ . All integrations turn out to be like the standard ones for a hydrogenic atom [4, 16, 17]. One finds  $\langle x \rangle = 0$ ,  $\langle p \rangle = 0$  and  $\langle [x, p]_+ \rangle = 0$ . The coefficients of  $f_4(\lambda)$  are

$$\begin{aligned} a'_{0,0} &= \left[\frac{\hbar}{a}\right]^4, & a'_{0,1} &= \frac{4\hbar}{3} \left[\frac{\hbar}{a}\right]^2, \\ a'_{0,2} &= \hbar^2 \frac{11}{5}, & a'_{0,3} &= 4\hbar a^2, & a'_{0,4} &= \frac{9a^4}{2}. \end{aligned} \quad (26)$$

Note that  $\langle [x'^2, p'^2]_+ \rangle = -(4/5)\hbar^2$ . One finds  $D'_r \simeq -0.06\hbar^{12} < 0$ .

### 5. Quantum optics: physics with the $a'_j$ and with $\langle [x', p']_+ \rangle$

In this section, we shall concentrate on certain single mode of light of interest (in short, smli), with some determined frequency  $\omega$ , wavevector and polarization, and on the associated photon states. We shall work with the operators  $x$  and  $p$  given by (19), in terms of the creation and destruction operators for the smli.  $x$  and  $p$  coincide (up to overall constant factors) with the so-called quadratures of the radiation field. The analysis for a harmonic oscillator in section 4 can be trivially extended for those photons. For the latter,  $\langle [x^2, p^2]_+ \rangle$  (with  $x_0 = 0$  and  $p_0 = 0$ ) could be tentatively interpreted as the correlation between the squared quadratures of the radiation field in a given state of the latter, at equal times, and so on for  $\langle [x'^2, p'^2]_+ \rangle$  (with  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ ). The one-photon state (the counterpart of the  $n = 1$  state of the harmonic oscillator) fulfils  $D'_r = 0$ , while  $\langle [x^2, p^2]_+ \rangle = +3\hbar^2/2$ . More generally, the smli can also be described by some density operator  $\rho$ .

We shall treat, in outline, the relationships of the  $a'_j$  and of  $\langle [x', p']_+ \rangle$  (with  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ ) to some physical processes in quantum optics. We anticipate that, thus far and to the best of the author's knowledge, those physical processes have already given rise to real experiments for  $a'_1$  and  $a'_3$ , while they are related to idealized experiments for  $a'_0$  and  $a'_4$  [8–10]. As we shall see,  $\langle [x', p']_+ \rangle$  does appear in idealized experiments, which are closely related to and have inspired the real experiments measuring  $a'_1$  or  $a'_3$ . Thus far,  $a'_2$  seems not so directly related to typical idealized experiments, but it is not excluded that it could be.

Let  $x_0 = \langle x \rangle$  and  $p_0 = \langle p \rangle$ . The smli describes, by definition, squeezed light in  $x$  (respectively, in  $p$ ) at even order  $N$ ,  $N = 2, 4, \dots$ , if  $\langle (x - \langle x \rangle)^N \rangle < (N - 1)!(\hbar/(2\omega))^{N/2}$

(respectively,  $\langle (p - \langle p \rangle)^N \rangle < (N-1)!!(\hbar\omega/(2))^{N/2}$ , with  $(N-1)!! = (N-1)(N-3)\dots 3 \cdot 1$  [9, 10]. Then, the possibility that the smli be squeezed in  $p$  at second or fourth orders is directly related to the values of  $a'_j$  for  $j = 0$  or 1, respectively. Similarly, the possibility that the smli be squeezed in  $x$  at second or fourth orders is directly related to the values of  $a'_j$  for  $j = 3$  or 4, respectively. Let  $y$  be one quadrature (either  $x$  or  $p$ ), and let  $y_c$  denote the quadrature complementary to  $y$  (either  $p$  or  $x$ , respectively). For some chosen  $N$  ( $N = 2, 4$ ), by using suitably prepared quantum states of the smli, one would perform an experiment and measure whether  $y$  is squeezed, namely, whether  $\langle (y - \langle y \rangle)^N \rangle$  is smaller than the bound given above. If it does and  $N = 2$  (as it occurs in experiments already performed), then the uncertainty inequality implies that  $\langle (y_c - \langle y_c \rangle)^2 \rangle$  is larger than the corresponding bound. Let us now turn to  $N = 4$ . As commented in section 3, no non-trivial inequality involving only  $\langle (x - \langle x \rangle)^4 \rangle$  and  $\langle (p - \langle p \rangle)^4 \rangle$  appears to exist, to the best of the author's knowledge. Then, if the result of an experiment (thus far, only idealized and not yet performed, seemingly) states that  $\langle (y - \langle y \rangle)^4 \rangle$  lies below the corresponding bound given above ( $y$  being squeezed), it is not warranted that  $\langle (y_c - \langle y_c \rangle)^4 \rangle$  be necessarily above the corresponding bound. On the other hand, inequalities involving  $\langle (x - \langle x \rangle)^4 \rangle$ ,  $\langle (p - \langle p \rangle)^4 \rangle$  and other expectation values do exist: (14) or (15) does the job (as also does (17), but in a less interesting way). Then, if, say, the experiment showed that  $\langle (x - \langle x \rangle)^4 \rangle$  is squeezed in a given quantum state, then, the set formed by  $\langle (p - \langle p \rangle)^4 \rangle$  and the other expectation values (including those of anticommutators) is restricted to fulfil either (14) or (15) (and so on with (17)).

Following [19], we shall consider an (idealized) interference experiment in which the light from a laser with the same frequency  $\omega$  as the smli is superimposed on the latter (say, by a beam splitter). Such an interference experiment is also known as a homodyne one. The interfering laser light is in a coherent state with complex amplitude proportional to  $E = |E| \exp i\theta$ . We consider the photon statistics of the combined field (formed by the superposition of the smli and the laser light), for a short time interval  $t$  (compared to relevant coherence times). Let  $\langle n \rangle$  and  $\langle (n - \langle n \rangle)^2 \rangle$  be, respectively, the average number of photons of the combined field and its variance, as counted by a photodetector, by repeating the measurements many times. For large  $|E|$  (intense coherent laser light beam, the latter being close to the classical limit), one finds

$$\begin{aligned} \langle (n - \langle n \rangle)^2 \rangle - \langle n \rangle &\simeq (\eta t)^2 |E|^2 \left[ \cos^2 \theta \left( \langle (x - \langle x \rangle)^2 \rangle - \frac{\hbar}{2\omega} \right) \right. \\ &\quad \left. + \sin^2 \theta \left( \langle (p - \langle p \rangle)^2 \rangle - \frac{\hbar\omega}{2} \right) + \frac{\sin 2\theta}{2\omega} \langle [x - \langle x \rangle, p - \langle p \rangle]_+ \rangle \right], \end{aligned} \quad (27)$$

which agrees with equation (12) in [19] (the components of the real electromagnetic field considered in [19] being interpreted as  $x$  and  $p$  for the smli). The constant  $\eta$  characterizes the collection and quantum efficiencies of the detector. Compare also with equation (21.6.5) in [8]. We emphasize that  $\langle (x - \langle x \rangle)^2 \rangle$ ,  $\langle (p - \langle p \rangle)^2 \rangle$  and  $\langle [x - \langle x \rangle, p - \langle p \rangle]_+ \rangle$  are evaluated with  $|\psi\rangle$  or  $\rho$  describing the actual smli. Suitably chosen values of the phase  $\theta$  enhance different contributions in (27). Thus, for  $\theta = 0, \pi$  and if  $|\psi\rangle$  or  $\rho$  describes squeezed light at second order, then one has  $\langle (n - \langle n \rangle)^2 \rangle - \langle n \rangle < 0$  (named, in turn, sub-Poissonian photon statistics) [19]. Equation (27) also enhances the contribution of  $\langle [x', p']_+ \rangle$  to the photon statistics in the actual (idealized) homodyne experiment, if  $\theta = \pi/4$  (although  $\langle (x - \langle x \rangle)^2 \rangle > -\hbar/(2\omega)$ ) and  $\langle (p - \langle p \rangle)^2 \rangle > -\hbar/(2\omega)$  still contribute).

As commented on in [8], the presence of higher order squeezing ( $N = 4, \dots$ ) could also be detected in a homodyne experiment of the kind just discussed (which led to (27)), but it would require measurements of higher order moments: such measurements would be likely more difficult than the corresponding second-order ones. Then,  $a'_0$  and  $a'_4$  could also

become manifest in real experiments, as there are no reasons of principle preventing that [8]. It has been shown theoretically that the processes of degenerate parametric down conversion, harmonic generation and resonance fluorescence all exhibit higher order squeezing [9]. A detailed theoretical study of higher order squeezing in single-mode multiphoton absorption processes appears in [20].

As announced in section 2, we treat another manifestation of the role of  $\langle [x, p]_+ \rangle$  in quantum optics. We shall recall an interesting theoretical proposal of a different kind of squeezing phenomenon, namely the one regarding the square of the field amplitude [21, 22], as the uncertainty in  $[x, p]_+$  contributes to it. We note that  $(2\hbar)^{-1}[x, p]_+$  equals precisely the imaginary part of the square of the field amplitude for the smli, denoted as  $Y_2$  in [21]. Let  $x_0 = 0$  and  $p_0 = 0$ . Following [21], let us consider a second harmonic generation process, in which a smli (with the destruction and creation operators ( $a$  and  $a^+$ ) given in (19)) couples, through a nonlinear medium (a crystal), with another photon mode at frequency  $2\omega$  (the second harmonic) with coupling constant  $\kappa$ . Let  $X_B$  be the coordinate operator for the second harmonic (say, the counterpart for the second harmonic of  $x$  in (19) for the smli). Incident radiation corresponding to the smli will be partially converted, after passage through the (nonlinear) crystal, to the second harmonic. It is supposed that, at the initial time ( $t = 0$ ), both light modes are uncorrelated. Let the second harmonic be in a coherent state. For a short interaction time  $t$ , the squared uncertainty in  $X_B$  equals, approximately,

$$[1/(4\omega)] + (\kappa t)^2 [(\langle Y_2^2 \rangle - \langle Y_2 \rangle^2) - \langle (a^+ a + 1/2) \rangle], \quad (28)$$

[21]. In turn, at a later stage, the second harmonic thus produced could be subject to a homodyne experiment of the kind considered above (say, by interference with another light mode, different from the initial smli). Measurements in the latter homodyne experiment would give information of the squared uncertainty in  $X_B$  for the second harmonic and, by using (28), one would get information on  $(\langle Y_2^2 \rangle - \langle Y_2 \rangle^2) - \langle (a^+ a + 1/2) \rangle$  (say, on the squared uncertainty in  $[x, p]_+$  minus the mean photon number in the smli).

## 6. Conclusion and discussion

Complementarity and its representation through commutation relations (c.r.) for pairs of complementary variables imply not only the fundamental uncertainty inequalities (u.i.), but they also yield further inequalities having, at least, possible pedagogical and/or methodological interest. Here, we have worked out an extension of a standard derivation of the  $x$ - $p$  u.i. [4, 16, 17], through non-negative real fourth-degree polynomials  $f_4(\lambda)$  and also based on the  $x$ - $p$  c.r. and on the Hilbert space structure genuine of quantum mechanics.  $f_4(\lambda)$  yields (through suitable discriminants for quartic algebraic equations) further quantum inequalities involving expectation values of up to quartic powers of  $x$  and  $p$ , and the anticommutator  $[x^2, p^2]_+$ . The inequalities (which hold independently on the interaction) restrict several expectation values, part of which do not appear in the standard  $x$ - $p$  u.i. (although others do). The inequalities implied by  $f_4(\lambda)$  get close to equalities in genuine quantum situations, become strict equalities at least for the lowest states of the harmonic oscillator (thereby enjoying some sort of optimality or saturation) and, hence, could be helpful as a contribution to the teaching of quantum mechanics (at about the graduate level, say, addressed to students who have already become acquainted with it, through an introductory course). We argue that, for a quantum particle bound by an attractive potential, the new inequalities become closer to equalities as the bound states approach the ground state, thereby extending a behaviour already met for the  $x$ - $p$  u.i. As an intriguing surprise (from a pedagogical standpoint), the inequalities display a non-classical object: the expectation value of the anticommutator  $[x^2, p^2]_+$ . The latter,

contrary to what happens near the classical limit, can be  $< 0$  in genuine quantum regimes (in the examples treated in subsections 4.2–4.5). The anticommutator  $[x^2, p^2]_+$  is given easily in terms of another anticommutator which, in turn, appeared previously [12, 13], related to the  $x$ - $p$  u.i.

The physical interest of the expectation values involved in the quantum inequalities and of the latter has been discussed, in the frameworks of quantum optics and of squeezing of quantum states at fourth order. Thus, if either  $\langle (x - \langle x \rangle)^4 \rangle$  or  $\langle (p - \langle p \rangle)^4 \rangle$  becomes small (squeezed) in a given quantum state, then, the other and other expectation values (in particular, that of an anticommutator) are restricted to fulfil either (14) or (15).

The analysis of the consequences of the  $x$ - $p$  c.r. carried out in this paper can be extended beyond fourth-degree polynomials only to a limited extent. Thus, one can construct real polynomials  $f_{2n}(\lambda)$  of degree  $2n$ ,  $n = 3, 4, 5, \dots$  (containing  $(x' + i\lambda p')^n (x' - i\lambda p')^n$ ), also with  $f_{2n}(\lambda) \geq 0$  for any real  $\lambda$ . However, algebraic equations of degree higher than four cannot be solved explicitly by any closed formulae (involving a finite number of radicals) [23, 24]. One expects that the quantum inequalities, implied by  $f_{2n}(\lambda) \geq 0$ ,  $n = 3, 4, 5, \dots$ , will not be expressed in any closed form, through a finite number of operations. This, of course, does not exclude the possibility that other functions of  $x$  and  $p$  (together with the  $x$ - $p$  c.r.) could be used in a similar way to get other expressions of complementarity and other quantum restrictions. The author encourages the reader to search for other ones and so contribute to displaying pedagogically other expressions of complementarity.

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## Appendix

We shall study the conditions ensuring  $f_4(\lambda) \geq 0$ . By using  $\lambda = \lambda_q - a'_1/(4a'_0)$ , equations (6) and (9) become

$$f_4(\lambda) \equiv a'_0 f_{4,q}(\lambda_q), \quad (\text{A.1})$$

$$f_{4,q}(\lambda_q) \equiv \lambda_q^4 + p'_q \lambda_q^2 + q'_q \lambda_q + r'_q = 0. \quad (\text{A.2})$$

$p'_q$ ,  $q'_q$  and  $r'_q$  are given in equations (10) and (11). The quartic equation  $f_{4,q}(\lambda) = 0$  in (A.2) can be solved by reducing it to either a biquadratic equation in a special case or, in general, to a cubic one. Succinct treatments of the solutions of both quartic and cubic equations are given in section 3.8 in [25]. Based upon it, and through a direct but somewhat cumbersome analysis, one finds the general conditions on the coefficients  $a'_j$ ,  $j = 0, 1, \dots, 4$ , ensuring that  $f_4(\lambda) \geq 0$  holds. Here, we shall limit ourselves to summarize those steps and conditions.

First, let us assume that  $q'_q = 0$  (the special case). Then, the quartic equation (A.2) can be reduced to a biquadratic one, namely

$$\lambda_q^4 + p'_q \lambda_q^2 + r'_q = 0. \quad (\text{A.3})$$

By solving the quadratic equation (A.3) for  $\lambda_q^2$ , one finds directly the conditions (i)–(iii), given in section 3, ensuring  $f_4(\lambda) \geq 0$ .

Second, let us suppose that  $q'_q \neq 0$  (the general case). Then, the quartic equation (A.2) can be reduced to an associated resolvent cubic algebraic equation [25], which poses less difficulties. The associated resolvent cubic equation has the general structure  $\lambda_0^3 + q\lambda_0 + r = 0$ , with a new unknown  $\lambda_0$  and new coefficients  $q$  and  $r$  (the expressions of which in terms of  $p'_q$ ,  $q'_q$  and  $r'_q$  being unnecessary for our outline). Subsequently, the resolvent cubic algebraic equation is, in turn, explicitly solved [25] and its solution gives back that of  $f_{4,q}(\lambda_q)$ . Let us introduce the so-called discriminant  $q^3 + r^2$  of the associated resolvent cubic equation. The values of  $q^3 + r^2$  determine the solutions of the associated resolvent cubic equation and, through the latter, those of (A.2). An elementary but cumbersome analysis shows that  $f_{4,q}(\lambda_q) \geq 0$  holds if either  $q^3 + r^2 < 0$  (the so-called irreducible case) or  $q^3 + r^2 = 0$  (with  $r \neq 0$ ). On the other hand,  $q^3 + r^2 > 0$  can be shown to imply  $f_{4,q}(\lambda_q) < 0$ .

The quantities  $D'_r$ ,  $a'_0 p_r/3$  and  $a'^2_0 q_r/2$  introduced in equations (12) and (13) correspond to  $q^3 + r^2$ ,  $q$  and  $r$  in section 3.8 in [25], except for trivial scalings. The subscript  $r$  denotes the resolvent cubic equation. The cases  $D'_r < 0$ ,  $D'_r = 0$  and  $D'_r > 0$  correspond, respectively, to  $q^3 + r^2 < 0$ ,  $q^3 + r^2 = 0$  and  $q^3 + r^2 > 0$ .

The student does not need to go through the elementary but time-consuming analysis yielding the above conditions which ensure  $f_4(\lambda) \geq 0$ . In order to convince himself/herself of their correctness, he/she could compare them with the numerical analysis and solutions of his/her own examples for quartic algebraic equations.

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