

# Robust adaptive Lasso in high-dimensional logistic regression with an application to genomic classification of cancer patients.

Basu, A.<sup>1</sup>, Ghosh, A.<sup>1</sup>; Jaenada, M.<sup>2</sup> and Pardo, L.<sup>2</sup>

<sup>1</sup>Indian Statistical Institute, India

<sup>2</sup>Complutense University of Madrid, Spain

September 8, 2021

## Abstract

Penalized logistic regression is extremely useful for binary classification with a large number of covariates (significantly higher than the sample size), having several real life applications, including genomic disease classification. However, the existing methods based on the likelihood based loss function are sensitive to data contamination and other noise and, hence, robust methods are needed for stable and more accurate inference. In this paper, we propose a family of robust estimators for sparse logistic models utilizing the popular density power divergence based loss function and the general adaptively weighted LASSO penalties. We study the local robustness of the proposed estimators through its influence function and also derive its oracle properties and asymptotic distribution. With extensive empirical illustrations, we clearly demonstrate the significantly improved performance of our proposed estimators over the existing ones with particular gain in robustness. Our proposal is finally applied to analyse four different real datasets for cancer classification, obtaining robust and accurate models, that simultaneously performs gene selection and patient classification.

**Software availability:** A ready implementation of our proposal is available as an R package awD-PDlasso in Github.

**MSC 2020:** 62F35, 62J12

**Keywords:** Density power divergence, High-dimensional data, Logistic regression, Oracle properties, Variable Selection

## 1 Introduction

Real life scientific experiments often include dichotomous response, which requires the use of binary classification procedures. Logistic regression can be used as a discriminative classification technique, having a direct probabilistic interpretation. Let  $Y_1, \dots, Y_n$  be independent binary response variables following a Bernoulli model, i.e.,

$$\Pr(Y_i = 1) = \pi_i \text{ and } \Pr(Y_i = 0) = 1 - \pi_i, \quad i = 1, \dots, n.$$

Often,  $\pi_i \in [0, 1]$  depends on the observation through a series of explanatory variables  $x_{i0}, \dots, x_{ik}$  associated with each  $Y_i$  ( $x_{i0} = 1, x_{ij} \in \mathbb{R}, i = 1, \dots, n, j = 1, \dots, k$ ). We consider a  $(k + 1)$ -dimensional vector of parameters  $\beta = (\beta_0, \dots, \beta_k)^T$ , with  $\beta_i \in (-\infty, \infty)$ , such that the binomial parameter,  $\pi_i$ , is linked to the linear predictor via the logit function, i.e.,  $\text{logit}(\pi_i) = \sum_{j=0}^k \beta_j x_{ij}$ , where  $\text{logit}(p) =$

$\log(p/(1-p))$ . This relation links the expectation of  $Y_i$  (or equivalently the probability of belonging to the group labelled as 1) given the explanatory variables  $\mathbf{x}_i$  as

$$E[Y_i|\mathbf{x}_i] = \pi_i = \pi(\mathbf{x}_i^T \boldsymbol{\beta}) = \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}}}{1 + e^{\mathbf{x}_i^T \boldsymbol{\beta}}}, \quad i = 1, \dots, n, \quad (1)$$

where  $\mathbf{x}_i^T = (x_{i0}, \dots, x_{ik})$ . The likelihood function for the logistic regression model is then given by

$$\mathcal{L}(\boldsymbol{\beta}) = \prod_{i=1}^n \pi(\mathbf{x}_i^T \boldsymbol{\beta})^{y_i} (1 - \pi(\mathbf{x}_i^T \boldsymbol{\beta}))^{1-y_i}. \quad (2)$$

where  $y_1, \dots, y_n$  denote the observed values of the random variables  $Y_1, \dots, Y_n$ . Then, the MLE of  $\boldsymbol{\beta}$ ,  $\hat{\boldsymbol{\beta}}$ , is obtained minimizing the negative log-likelihood function over  $\boldsymbol{\beta} \in \Theta$  with

$$\Theta = \left\{ (\beta_0, \dots, \beta_k)^T : \beta_j \in (-\infty, \infty), \quad j = 0, \dots, k \right\} = \mathbb{R}^{k+1}.$$

In practice, if the number of observations,  $n$ , is not large enough compared the parameter dimension  $k$  (high dimensional set-up), the classical logistic regression may cause over-fitting. A useful technique to overcome this weakness is the regularization by adding a non-negative penalty  $p_{\lambda_n}$  to the negative log-likelihood function (2) where  $\lambda_n > 0$  is the regularization parameter controlling the strength of shrinkage in the regression coefficients: the greater the value of  $\lambda_n$  is, the greater weight will be given to the penalty term and consequently the amount of shrinkage. Without loss of generality, let us assume that the covariates are standardized. As a result, the intercept,  $\beta_0$ , is not penalized and the estimation of the vector  $\boldsymbol{\beta}$  under a high-dimensional logistic regression model is performed through the minimization of the penalized likelihood-based objective function given by

$$Q_{n,\lambda}(\boldsymbol{\beta}) = -\log \mathcal{L}(\boldsymbol{\beta}) + \sum_{j=1}^k p_{\lambda_n}(|\beta_j|). \quad (3)$$

The regularized estimate of  $\boldsymbol{\beta}$  is obtained by minimizing (3), i.e.,

$$\hat{\boldsymbol{\beta}}_P = \arg \min_{\boldsymbol{\beta} \in \Theta} Q_{n,\lambda}(\boldsymbol{\beta}).$$

The value  $\lambda_n$  depends on data and can be computed using a cross-validation method (James et al. (2013)), or a generalized information criterion (Fan and Tang (2013)) which consistently identifies the true model with asymptotic probability 1 in the ultra-high dimensional setting. From now on, we denote  $\lambda_n = \lambda$  for the sake of simplicity. Note that  $\lambda = 0$  gives the MLE, whereas if  $\lambda \rightarrow \infty$  all  $\beta_j$  tend to 0.

Among several penalties discussed in the literature, the general  $\ell_q$ -penalization is defined as  $\sum_{j=1}^k p_{\lambda_n}(|\beta_j|) = \lambda \|\boldsymbol{\beta}\|_q^q = \lambda \sum_{j=1}^k |\beta_j|^q$ . The value  $q = 2$  yields  $l_2$ -regularized logistic regression, i.e., the Ridge procedure, which is particularly appropriate when there is multicollinearity between the explanatory variables (see Duffy and Santner (1989), Schaefer, Roi and Wolfe (1984) and Le Cessie and Van Houwelingen (1992)), but not useful in high-dimensional set-ups. The value  $q = 1$  results in the  $\ell_1$ -penalization (LASSO) proposed by Tibshirani (1996) in the context of linear regression. In this case, the penalty function continuously shrinks the coefficients toward zero, yielding a sparse subset of variables with nonzero regression coefficients; the larger the value of  $\lambda$ , greater the number of zero coefficients. Shevade and Keerthi (2003) proposed sparse logistic regression based on the LASSO penalty and Cawley and Talbot (2006) investigated the same with Bayesian penalty. Some interesting results as well as applications of LASSO procedure in logistic regression can be seen in Fokianos (2008), Park and Hastie (2008), Koh, Kim and Boyd (2007), Yang and Qian (2016), Plan

and Vershynin (2013), Zhu and Hastie (2004) and Sun and Wang (2012). However, the LASSO penalty has been criticized for its biasedness, as it tends to select many noisy features (false positives) with high probability (Huang, Ma and Zhang (2008)). To overcome the bias drawback, Zou (2006) proposed the adaptive Lasso penalty in the context of linear regression models. Precisely, the adaptive LASSO criterion for logistic regression considers the objective function,

$$Q_{n,\lambda}(\boldsymbol{\beta}) = -\log \mathcal{L}(\boldsymbol{\beta}) + \lambda \sum_{j=1}^k \frac{|\beta_j|}{|\tilde{\beta}_j|}, \quad (4)$$

where  $\tilde{\beta}_j$  is a consistent (initial) estimator of  $\beta_j$  for all  $j$ . Note that, for any fixed  $\lambda$ , the penalty for zero initial estimation of  $\beta_j$  goes to infinity, while the weights for nonzero initials converge to a finite constant. Consequently, by allowing a relatively higher penalty for zero coefficients and lower for nonzero coefficients, the adaptive LASSO estimator reduces the estimation bias and improves variable selection accuracy. Some interesting applications of adaptive LASSO can be seen in Algamal and Lee (2015), Algamal and Lee (2019) and Guo et al. (2015).

However, all the above methods are based on the negative log-likelihood function associated with the logistic regression model. The lack of robustness of the likelihood-based loss is well known and thus, leverage points or influence points can individually or jointly have significant influence on the regularization procedures leading to erroneous conclusions. To achieve robustness against data contamination, we propose an alternative robust loss function based on the density power divergence (DPD) measure (Ghosh and Basu (2013)) along with the general adaptive loss. This idea of using DPD based loss function in high-dimensional context has been used previously in general linear regression models by Ghosh and Majumdar (2020) and Ghosh, Jaenada and Pardo (2020) with non-concave and adaptive penalties, respectively. We follow a similar idea in order to develop our robust DPD-based loss for the logistic regression model. In this respect, one may find the definitions of our adaptively weighted DPD-LASSO estimators (see Section 2) to be rather straightforward; but the derivations of their theoretical properties for the ultra-high dimensional logistic regression as well as its efficient computation need non-trivial extensions of the existing literature as we will see throughout the rest of the paper. These theoretical computations and numerical evaluation of the performance of the proposed estimator (over the existing procedures) are our major contributions in this paper. In particular, we will show that our proposed estimators are indeed locally robust, through their influence function analysis (Section 3), and enjoy oracle properties, i.e., consistently perform model selection and parameter estimation (Section 4). Additionally, we extend the IRLS algorithm, commonly used for obtaining the MLE in logistic regression, to the DPD-loss function in order to develop an efficient computational algorithm for our proposal (Section 5). Further, we empirically compare our proposed estimators to some state-of-the-art robust and non-robust methods under the high-dimensional logistic regression (Section 6). Results show significant improvement in robustness of our proposal over all existing robust procedures under different contaminated scenarios, while keeping it competitive to the classical loglikelihood based methods in the absence of contamination.

A major focus in cancer research is identifying genetic markers. The cancer disease is caused by abnormalities in the genetic material resulting from acquired mutations and epigenetic changes that influence the gene expression of a patient. Microarray technology measures the expression level of genes of an individual and that genetic expression can be used in the diagnosis, through the classification of different types of cancerous genes leading to a cancer disease. Clinical diagnosis of cancer based on gene expression data pursue two goals, correctly diagnosing a patient and identifying the genes involved in this diagnosis. Moreover, microarray data are generally high dimensional data, having a large number of genes in comparison to the number of samples. Therefore, these two purposes can be achieved using regularized logistic regression models as a classifier, which also performs gene selection. In addition, it is well known that microarray datasets with many genes often contain outliers and several studies

have pointed out that errors in labelling and gene expression measure are not uncommon, motivating the use of robust procedures.

We will apply our proposed estimation methods on four publicly available datasets which have been widely used to study different techniques in cancer classification, regarding three types of tumours; colon cancer, leukaemia and breast cancer. These four datasets have been flagged as being suspected of containing wrongly labelled samples or outlying observations, making robust methods particularly suitable for their analysis.

## 2 Adaptive LASSO based on the density power divergence loss

Firstly, we recall that the likelihood-based loss (3) under a logistic regression model could also be justified from an information theoretic perspective by considering the famous Kullback-Leibler (KL) divergence measure between the probability vectors  $\hat{\mathbf{p}} = \frac{1}{n} (y_1, 1 - y_1, y_2, 1 - y_2, \dots, y_n, 1 - y_n)^T$  and

$$\mathbf{p}(\boldsymbol{\beta}) = \frac{1}{n} (\pi(\mathbf{x}_1^T \boldsymbol{\beta}), (1 - \pi(\mathbf{x}_1^T \boldsymbol{\beta})), \dots, \pi(\mathbf{x}_n^T \boldsymbol{\beta}), (1 - \pi(\mathbf{x}_n^T \boldsymbol{\beta})))^T.$$

Clearly, the vector  $\hat{\mathbf{p}}$  could be interpreted as the empirical estimator of  $\mathbf{p}(\boldsymbol{\beta})$ . Mathematically, the KL divergence between  $\hat{\mathbf{p}}$  and  $\mathbf{p}(\boldsymbol{\beta})$  is given by

$$d_{KL}(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta})) = \sum_{i=1}^n \left( \frac{y_i}{n} \log \frac{y_i}{\pi(\mathbf{x}_i^T \boldsymbol{\beta})} + \frac{1 - y_i}{n} \log \frac{1 - y_i}{1 - \pi(\mathbf{x}_i^T \boldsymbol{\beta})} \right). \quad (5)$$

With some algebra, it is not difficult to establish that

$$d_{KL}(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta})) = c - \frac{1}{n} \log \mathcal{L}(\boldsymbol{\beta}), \quad (6)$$

and therefore, the MLE of  $\boldsymbol{\beta}$  can be defined by

$$\hat{\boldsymbol{\beta}} = \arg \min_{\boldsymbol{\beta} \in \Theta} d_{KL}(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta})). \quad (7)$$

Based on (7), we might replace  $d_{KL}$  by any divergence measure between the probability vectors  $\hat{\mathbf{p}}$  and  $\mathbf{p}(\boldsymbol{\beta})$  in order to define a minimum distance estimator (MDE) for  $\boldsymbol{\beta}$ . In this paper we shall use the density power divergence (DPD) measure defined by Basu et al. (1998), since it produces MDEs with excellent robustness properties (see, e.g., Basu et al. (2011, 2013, 2015, 2016), Ghosh et al. (2015, 2016)). The DPD between the probability vectors  $\hat{\mathbf{p}}$  and  $\mathbf{p}(\boldsymbol{\beta})$  is given by

$$\begin{aligned} d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta})) &= \frac{1}{n^{1+\alpha}} \sum_{i=1}^n \left\{ \left( \pi^{1+\alpha}(\mathbf{x}_i^T \boldsymbol{\beta}) + (1 - \pi(\mathbf{x}_i^T \boldsymbol{\beta}))^{1+\alpha} \right) \right. \\ &\quad \left. - \left( 1 + \frac{1}{\alpha} \right) \left( y_i \pi^\alpha(\mathbf{x}_i^T \boldsymbol{\beta}) + (1 - y_i) (1 - \pi(\mathbf{x}_i^T \boldsymbol{\beta}))^\alpha \right) + \frac{1}{\alpha} (y_i^{\alpha+1} + (1 - y_i)^{\alpha+1}) \right\} \\ &= \frac{1}{n^{1+\alpha}} \sum_{i=1}^n \rho_\alpha(\mathbf{x}_i^T \boldsymbol{\beta}, y_i). \end{aligned} \quad (8)$$

for  $\alpha > 0$  (see Basu et al. (2017)) where

$$\rho_\alpha(\mathbf{x}^T \boldsymbol{\beta}, y) = \frac{1}{n^\alpha} \left[ \left( \pi^{1+\alpha} + (1 - \pi)^{1+\alpha} \right) - \left( 1 + \frac{1}{\alpha} \right) (y \pi^\alpha + (1 - y) (1 - \pi)^\alpha) + \frac{1}{\alpha} (y^{\alpha+1} + (1 - y)^{\alpha+1}) \right].$$

Note that the third term does not depend on  $\beta$  and therefore it can be excluded in the minimization. The value  $\alpha = 0$  is derived from the general formula by taking its continuous limit as  $\alpha$  tends to 0, which indeed yields

$$d_0(\hat{\mathbf{p}}, \mathbf{p}(\beta)) = \lim_{\alpha \rightarrow 0} d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\beta)) = d_{KL}(\hat{\mathbf{p}}, \mathbf{p}(\beta)).$$

Hence, the minimum density power divergence estimator (MDPDE),  $\hat{\beta}_\alpha$ , for the parameter  $\beta$ , in the logistic regression model is defined as the minimizer of  $d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\beta))$  in (8), which coincides with the MLE at  $\alpha = 0$  and provides a robust generalization of it at  $\alpha > 0$ .

Taking derivatives respect to  $\beta$  we obtain the estimating equations of the MDPDE at  $\beta > 0$  as given by

$$\sum_{i=1}^n \Psi_\alpha(\mathbf{x}_i, y_i, \beta) \mathbf{x}_i = \mathbf{0}_{k+1}, \quad (9)$$

with

$$\Psi_\alpha(\mathbf{x}_i, y_i, \beta) = (e^{\alpha \mathbf{x}_i^T \beta} + e^{\mathbf{x}_i^T \beta}) \frac{e^{\mathbf{x}_i^T \beta} - y_i(1 + e^{\mathbf{x}_i^T \beta})}{(1 + e^{\mathbf{x}_i^T \beta})^{\alpha+2}}, \quad (10)$$

which reconfirms that the MDPDE in logistic regression models is an M-estimator but with a model-dependent  $\Psi$ -function (10). For the value  $\alpha = 0$  in (9) we get the estimating equations for the MLE which are

$$\sum_{i=1}^n (\pi(\mathbf{x}_i^T \beta) - y_i) \mathbf{x}_i = \mathbf{0}_{k+1}.$$

In this paper, utilizing the idea of adaptive LASSO penalty discussed in Section 1, we propose to minimize the regularized objective function with a general adaptively weighted LASSO penalty as given by

$$Q_{n,\alpha,\lambda}(\beta) = d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\beta)) + \lambda \sum_{j=1}^k w(|\tilde{\beta}_j|) |\beta_j|, \quad (11)$$

where  $w$  is a suitable weight function,  $\lambda$  is the usual regularization parameter and  $\tilde{\beta} = (\tilde{\beta}_j)_{j=1,\dots,k}$  is any consistent estimator of  $\beta$ , referred to as the initial estimator. We will denote the estimator obtained minimizing by  $Q_{n,\alpha,\lambda}(\beta)$  in (11) by  $\hat{\beta}_{\lambda,\alpha}$  and call it an adaptive weighted DPD-LASSO estimator (AW-DPD-LASSO) of parameter  $\beta$  under the high-dimensional logistic regression model. When the weight function is  $w = 1$ , we obtain the DPD-based regularized with standard LASSO estimator (which we refer to as the DPD-LASSO). When the conventional adaptive LASSO penalty, obtained by using the hard-thresholding weight function  $w(s) = \frac{1}{s} I(s \neq 0)$  is used the corresponding estimator will be referred to as the adaptive DPD-LASSO (Ad-DPD-LASSO).

**Remark 1** *Ghosh and Basu (2016) derived the expression of the DPD loss for generalized regression models, where  $Y_i$  follows the general exponential family of distributions. For independent but not identically distributed observations, they minimized the average divergence between the data points and the models. Using Bernoulli distributions, they obtained the DPD loss*

$$H_n(\beta) = \frac{1}{n} \sum_{i=1}^n \left\{ (1 - \pi(\mathbf{x}_i^T \beta))^{\alpha+1} + \pi^{\alpha+1}(\mathbf{x}_i^T \beta) - \frac{\alpha+1}{\alpha} \pi^{\alpha y_i}(\mathbf{x}_i^T \beta) (1 - \pi(\mathbf{x}_i^T \beta))^{\alpha(1-y_i)} \right\} \quad (12)$$

Note that our DPD-based loss function given in (8) differs from the above expression in (12) only on the scale  $\frac{1}{n^\alpha}$  since  $Y_i$  can take only values 0 or 1. Therefore, both expressions achieve the minimum at the same point without regularization.

**Remark 2** *Adaptively weighted penalties can also provide computationally flexible approximation to any general non-concave penalty functions,  $p_{\lambda_n}(\cdot)$ , using appropriate first order Taylor expansion series around an initial estimator, when the weight function is chosen as  $w(s) = p'_{\lambda_n}(s)$ ; see Fan and Li (2001) for details. In particular, the weight function approximating the popular SCAD penalty is given by*

$$w(|\beta_j|) = I(|\beta_j| \leq \lambda_n) + \frac{(a\lambda_n - |\beta_j|)_+}{(a-1)\lambda_n} I(|\beta_j| > \lambda_n), \quad (13)$$

with  $a > 2$  being a tuning constant, suggested as  $a = 3.7$  on an empirical basis.

### 3 Robustness: Influence Function Analysis

We study the local robustness of the AW-DPD-LASSO through its influence function (IF), which quantifies the impact that entails a infinitesimal contamination in the sample on parameter estimation. An estimator is said locally robust if its associated IF is bounded. We consider  $(Y_1, \mathbf{X}_1), \dots, (Y_n, \mathbf{X}_n)$ , a random sample from the random vector  $(Y, \mathbf{X})$  with true distribution function  $G(y, \mathbf{x})$ . We assume that  $\mathbf{X}_1, \dots, \mathbf{X}_n$  is a random sample from a random variable  $\mathbf{X}$  with marginal distribution function  $H(\mathbf{x})$ . The statistical functional  $\mathbf{T}_{\lambda, \alpha}^\beta(G)$  corresponding to the AW-DPD-LASSO estimator  $\widehat{\beta}_{\lambda, \alpha}$  is defined as the minimizer of

$$Q_{\alpha, \lambda}(\beta) = \int \frac{1}{n^\alpha} \rho_\alpha(\mathbf{x}^T \beta, y) dG(y, \mathbf{x}) + \lambda \sum_{j=1}^k w(|U_j(G)|) |\beta_j|, \quad (14)$$

with respect to  $\beta$ , where  $\mathbf{U}(G) = (U_1(G), \dots, U_k(G))$  is the statistical functional corresponding to the initial estimator  $\widetilde{\beta}$ . Note that the objective function  $Q_{\alpha, \lambda}(\beta)$  in (14) coincides with the empirical objective function (11) when  $G$  is substituted by the empirical distribution function  $G_n$ , as  $\mathbf{U}(G_n) = \widetilde{\beta}$ , and hence,  $\mathbf{T}_{\lambda, \alpha}^\beta(G_n) = \widehat{\beta}_{\lambda, \alpha}$ . As mentioned in Section 2, the MDPDE belongs to the family of M-estimators, so we can apply the extended IF theory developed in Avella-Medina (2017) for our proposed MDPDE, with their  $L(\theta, \mathbf{Z})$  being our  $L_\alpha((y, \mathbf{x}); \beta)$ . They justified that the IF of a M-estimator of the form (14) can be attained as

$$\text{IF} \left( (y_t, \mathbf{x}_t), \mathbf{T}_{\lambda, \alpha}^\beta(G), G \right) = \lim_{m \rightarrow \infty} \text{IF}_{p_m} \left( (y_t, \mathbf{x}_t), \mathbf{T}_{\lambda, \alpha, m}^\beta(G), G \right), \quad (15)$$

where  $\text{IF}_{p_m}$  are the influence functions of estimators penalized with continuous and infinitely differentiable functions in both arguments  $p_m(s, t(G))$ , for all  $m = 1, 2, \dots$ , and  $p_m(s, t(G))$  converges in the Sobolev space to our adaptive weighted penalty  $w(|t(G)|) |s|$  as  $m \rightarrow \infty$ . Avella-Medina (2017) proved that the expression of  $\text{IF} \left( (y_t, \mathbf{x}_t), \mathbf{T}_{\lambda, \alpha}^\beta(G), G \right)$  does not depend on the election of the sequence  $p_m(s, t(G))$ , and further, if the parameter space is compact and satisfies certain regularity conditions involving the continuity in  $\beta$  of some relevant functions and the invertibility of the Hessian of the corresponding objective function at  $\beta$  for all  $m$ , the IF exists for all  $(y_t, \mathbf{x}_t)$ .

Let us consider  $s$  to be the model size of true data-generating model and let us denote  $\beta_1$ ,  $\mathbf{x}_1$  and  $\mathbf{x}_{t,1}$  the  $s$ -vectors containing the first  $s$  elements of  $\beta$ ,  $\mathbf{x}$  and  $\mathbf{x}_t$ , and the corresponding functional  $\mathbf{T}_{\lambda, \alpha}^\beta(G) = \left( \mathbf{T}_{\lambda, \alpha}^{\beta_1}(G), \mathbf{T}_{\lambda, \alpha}^{\beta_{k-s}}(G) \right)$ . Then, the IF of our AW-DPD-LASSO estimator,  $\mathbf{T}_{\lambda, \alpha}^\beta$ , is presented in the following theorem.

**Theorem 3** *Consider the previous set-up and the true parameter value  $\beta^g = \mathbf{T}_{\lambda, \alpha}^\beta(G)$ , where  $\beta^g$  is sparse with only  $s < k$  nonzero components and  $k \gg n$ . Without loss of generality, assume  $\beta^g = \left( \beta_1^{gT}, \beta_{k-s}^{gT} \right)^T$ , where  $\beta_1^g$  contains all  $s$ -nonzero elements of  $\beta^g$ . Define  $\mathbf{P}^*(\beta, \mathbf{U}(G))$  a  $k$ -vector having*

$j$ -th element as  $w(|U_j(G)|) \text{sign}(\beta_j)$  and  $\mathbf{P}^{*(2)}(\boldsymbol{\beta}, \mathbf{U}(G))$  as the  $s \times s$  diagonal matrix with  $j$ -th entry being  $\frac{\partial w}{\partial s}(U_j(G)) \text{sign}(U_j(G)\beta_j)$  for  $j = 1, 2, \dots, s$  and  $\mathbf{S}_\alpha(G, \boldsymbol{\beta}^g)$  the  $s \times s$  submatrix containing the first  $s$  columns and rows of  $J_\alpha(G, \boldsymbol{\beta}^g)$ , with

$$J_\alpha(G, \boldsymbol{\beta}) = \frac{\alpha + 1}{n^\alpha} \mathbb{E}_{\mathbf{X}} \left[ \left(1 + e^{\mathbf{x}^T \boldsymbol{\beta}}\right)^{-(\alpha+3)} \left[ e^{(\alpha+1)\mathbf{x}^T \boldsymbol{\beta}} - e^{2\mathbf{x}^T \boldsymbol{\beta}} \right] \mathbf{x} \mathbf{x}^T \right] + \mathbf{P}_{m,\lambda}^{*(1)}(\boldsymbol{\beta}, \mathbf{U}(G)). \quad (16)$$

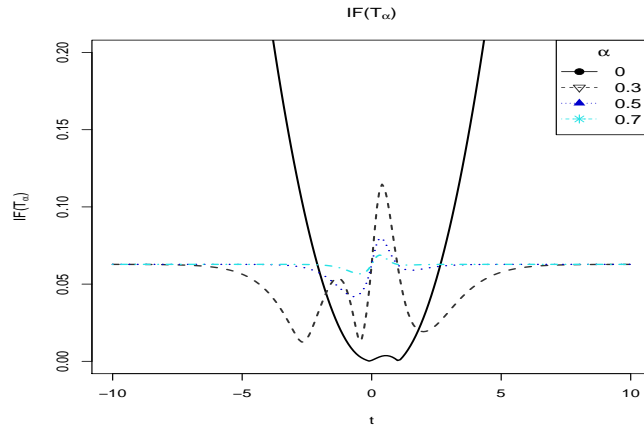
Then, whenever the associated quantities exist finitely, the  $IF((y_t, \mathbf{x}_t), \mathbf{T}_{\lambda,\alpha}^{\beta_{k-s}}, G)$  is identically zero and the  $IF$  of  $\mathbf{T}_{\lambda,\alpha}^{\beta_1}$  is given by

$$IF((y_t, \mathbf{x}_t), \mathbf{T}_{\lambda,\alpha}^{\beta_1}, G) = -\mathbf{S}_\alpha(G, \boldsymbol{\beta}^g)^{-1} \left[ \frac{1 + \alpha}{n^\alpha} \boldsymbol{\Psi}_\alpha((y_t, \mathbf{x}_{t,1}); \boldsymbol{\beta}_1) \mathbf{x}_{t,1} + \lambda \mathbf{P}_\lambda^*(\boldsymbol{\beta}_1, \mathbf{U}(G)) \right. \\ \left. + \lambda \mathbf{P}^{*(2)}(\boldsymbol{\beta}_1, \mathbf{U}(G)) IF((y_t, \mathbf{x}_{t,1}), \mathbf{U}_1, G) \right] \quad (17)$$

where  $\mathbf{U}_1$  denotes the first  $s$  elements of  $\mathbf{U}(G)$ .

If we consider the fixed design matrix set-up, the vector  $\mathbf{x}$  is non-stochastic and the expectation in (16) should be ignored. Using Theorem 3 we can compute the expression of the  $IF$  of  $\mathbf{T}_{\lambda,\alpha}^{\beta_1}$  at the true model distribution  $G$ . Note that  $\boldsymbol{\beta}_1^g = \boldsymbol{\beta}_1$  at the true model  $G$ , and  $\mathbf{U}(G) = \boldsymbol{\beta}$  due to its consistency. It is clear that the boundness of  $IF((y_t, \mathbf{x}_{t,1}), \mathbf{T}_{\lambda,\alpha}^{\beta_1}, G)$  depends only on  $\boldsymbol{\Psi}_\alpha((y_t, \mathbf{x}_{t,1}); \boldsymbol{\beta}_1) \mathbf{x}_{t,1}$  and  $IF((y_t, \mathbf{x}_{t,1}), \mathbf{U}_1, G)$ . The second function is bounded only if the initial estimator  $\tilde{\boldsymbol{\beta}}$  is robust; thus a robust initial estimator is needed to achieve robustness for our proposed estimator AW-DPD-LASSO. Finally, note that  $\boldsymbol{\Psi}_\alpha((y_t, \mathbf{x}_{t,1}); \boldsymbol{\beta}_1) \mathbf{x}_{t,1}$  as defined in (10) is bounded either or both of the contamination point  $(y_t, \mathbf{x}_t)$ . Figure 1 shows the  $\ell_2$ -norm of the influence function of Ad-DPD-LASSO with identical contamination in all directions,  $IF(t(1, \mathbf{1}), \mathbf{T}_{\lambda,\alpha}^{\beta_1}, G)$ ,  $t \in \mathbb{R}$ , for different values of the parameter  $\alpha$  and using the DPD-LASSO as initial estimator. A particular choice of true regression coefficients  $\boldsymbol{\beta} = (3, 2)$ , parameter  $\lambda = 0.1$  and sample size  $n = 100$  was made.  $IF((y_t, \mathbf{x}_{t,1}), \widehat{\boldsymbol{\beta}}_{\lambda,\alpha,1}^{\text{LASSO}}, G)$  can be easily obtained from expression (17) by substituting  $w(s) \equiv 1$ . As shown, the  $IF$  of the Ad-DPD-LASSO is bounded for positives values of  $\alpha$ , whereas remains unbounded for  $\alpha = 0$ .

Figure 1:  $\ell_2$ -norm of the Ad-DPD-LASSO Influence Function, with fixed parameters  $\boldsymbol{\beta} = (3, 2)$ ,  $\lambda = 0.1$  and  $n = 100$ . The  $IF$  for the value  $\alpha = 0$  is rescaled ( $\times 10^{-2}$ ).



## 4 Oracle properties

We now study the asymptotic properties of the proposed AW-DPD-LASSO estimators under the ultra-high dimensional set-up with non polynomial order, i.e., when  $\log(k) = \mathcal{O}(n^\zeta)$ , for some  $\zeta \in (0, 1)$ , by establishing its oracle properties under certain necessary conditions. For simplicity we assume that the design matrix  $\mathbb{X}$  is fixed with each column being standardized to have  $\ell_1$ -norm  $\sqrt{n}$ .

We first introduce some useful notation. Given any  $\mathcal{S} \subset \{1, 2, \dots, p\}$  and any  $p$ -vector  $\mathbf{v} = (v_1, \dots, v_p)^T$ , we will denote  $\mathcal{S}^c = \{1, 2, \dots, p\} \setminus \mathcal{S}$ ,  $\mathbf{v}_{\mathcal{S}} = (v_j : j \in \mathcal{S})$  whereas  $\text{Supp}(\mathbf{v}) = \{j : v_j \neq 0\}$ . Recall that the true model is assumed to be sparse, and so if we denote  $\mathcal{S}_0$  to be the subset of non zero elements of the true regression vector,  $\boldsymbol{\beta}_0$ , then the cardinality of  $\mathcal{S}_0$  is  $s \ll n$ . Without loss of generality, we can assume that the first  $s$  elements of  $\boldsymbol{\beta}_0$  are non-zero. In particular, let us allow that  $s = s_n = o(n)$  slowly diverges with the sample size  $n$ . Further, let  $\mathbb{X}_{\mathcal{S}}$  consist of the  $j$ -th column of  $\mathbb{X}$  for all  $j \in \mathcal{S}$  for any  $\mathcal{S}$ , and put  $\mathbb{X}_1 = \mathbb{X}_{\mathcal{S}_0}$  and  $\mathbb{X}_2 = \mathbb{X}_{\mathcal{S}_0^c}$  (so that  $\mathbb{X} = [\mathbb{X}_1, \mathbb{X}_2]$ ); the corresponding partition of the  $i$ -th row of  $\mathbb{X}$  would be denoted by  $\mathbf{x}_i = (\mathbf{x}_{1i}^T, \mathbf{x}_{2i}^T)^T$ . The gradient and Hessian matrix of the DPD loss is given by  $\frac{\partial d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta}} = \frac{1}{n} \mathbb{X}^T \mathbf{H}_\alpha^{(1)}(\boldsymbol{\beta})$ , and  $\frac{\partial^2 d_\alpha(\hat{\mathbf{p}}, \mathbf{p}(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} = \frac{1}{n} \mathbb{X}^T \mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta}) \mathbb{X}$ , with  $\mathbf{H}_\alpha^{(1)}(\boldsymbol{\beta})$  being a  $(k+1)$ -dimensional vector with  $i$ -component  $\mathbf{H}_\alpha^{(1)}(\boldsymbol{\beta})_i = \frac{(\alpha+1)}{n^\alpha} \boldsymbol{\Psi}_\alpha(\mathbf{x}_i, y_i, \boldsymbol{\beta})$ , where  $\boldsymbol{\Psi}_\alpha(\mathbf{x}_i, y_i, \boldsymbol{\beta})$  is as defined in (10) and  $\mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta})$  is a diagonal matrix with  $i$ -diagonal entries

$$\begin{aligned} \mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta})_{ii} = & \frac{(\alpha+1)}{n^\alpha} \sum_{i=1}^n \left[ \pi_i^{\alpha+1} (1 - \pi_i) \left( (1 + \alpha) - (2 + \alpha)\pi_i \right) + (1 - \pi_i)^\alpha \pi^2 (2 - (2 + \alpha)\pi_i) \right. \\ & \left. - y \left[ \pi_i^\alpha (1 - \pi_i) (\alpha - (1 + \alpha)\pi_i) + \pi_i (1 - \pi_i)^\alpha (1 - (\alpha + 1)\pi_i) \right] \right]. \end{aligned} \quad (18)$$

Since AW-DPD-LASSO estimators coincide with the negative loglikelihood-based weighted adaptive LASSO at  $\alpha = 0$  we only derive the results for  $\alpha > 0$ . We first establish the oracle property for the AW-DPD-LASSO with the assistance of the oracle information on the location of signal covariates,  $\mathcal{S}_0$ , and then extend the result to the cases of unknown  $\mathcal{S}_0$ , and the  $s$  first elements of the estimated regression vector are consistent estimators of  $\boldsymbol{\beta}_{\mathcal{S}_0}$ . We start with the following assumptions, where all expectations are taken with respect to the conditional distribution  $Y_i | \mathbf{X}_i = \mathbf{x}_i$  (which is also the case for the rest of the paper).

- (A1) The function  $\rho_\alpha(\cdot, y_i)$  is Lipschitz with Lipschitz constant  $L$  for all  $i = 1, 2, \dots, n$ .
- (A2) The eigenvalues of the matrix  $n^{-1}(\mathbf{X}_1^T \mathbf{X}_1)$  are bounded below and above by positive constants  $c_0$  and  $c_0^{-1}$ , respectively. Also  $\kappa_n := \max_{i,j} |x_{ij}| = o(n^{1/2} s^{-1})$ .
- (A3) The diagonal elements of  $E[\mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta}_0)]$  are all finite and bounded from below by a constant  $c_1 > 0$ .
- (A4) Expectation of all third order partial derivatives of  $\rho_\alpha(\mathbf{x}_i^T \boldsymbol{\beta}, y_i)$ ,  $i = 1, \dots, n$ , with respect to the components of  $\boldsymbol{\beta}_{\mathcal{S}_0}$  are uniformly bounded in a neighborhood of  $\boldsymbol{\beta}_{01}$ .

The first two Conditions are common in the literature of robust high-dimensional inference; see, e.g., Ghosh, Jaenada and Pardo (2020) and Fan, Fan and Barut (2014). In particular, Lipschitz property of a density or a loss function is commonly assumed to achieve robust inference. Next two assumptions (A3)-(A4) on the DPD-loss function are taken from the literature of classical minimum divergence inference, which hold for common (bounded) design matrices. Let us first restrict to the cases of the fixed weights  $\mathbf{w} = (w_1, \dots, w_p)$  with  $\mathbf{w}_0 = \mathbf{w}_{\mathcal{S}_0}$ ,  $\mathbf{w}_1 = \mathbf{w}_{\mathcal{S}_0^c}$ , and define  $\delta_n = \sqrt{s(\log n)/n} + \lambda_n \|\mathbf{w}_0\|_2$ . Let us denote  $\hat{\boldsymbol{\beta}}^o$  the oracle estimator that minimizes the objective function (11) over the restricted (to the true model) parameter space,  $\{\boldsymbol{\beta} = (\boldsymbol{\beta}_1^T, \boldsymbol{\beta}_2^T)^T \in \mathbb{R}^p : \boldsymbol{\beta}_2 = \mathbf{0}_{p-s}\} \equiv \mathbb{R}^s \times \{0\}^{p-s}$ .

Then, we obtain the  $\ell_2$ -consistency of the oracle estimator  $\widehat{\boldsymbol{\beta}}^o$  as presented in the following theorem; the proof is given in the Supplementary Material.

**Theorem 4** *Consider the cases of the fixed weights  $\mathbf{w} = (w_1, \dots, w_k)$  which satisfy the condition  $\lambda_n \|\mathbf{w}_0\|_2 \sqrt{s} \kappa_n \rightarrow 0$  with  $\mathbf{w}_0 = w_{S_0}$ , and let Assumptions (A1)-(A4) hold. Then, given any constant  $C_1 > 0$  and  $\delta_n = \sqrt{s(\log n)/n} + \lambda_n \|\mathbf{w}_0\|_2$ , there exists some  $c > 0$  such that the oracle estimator  $\boldsymbol{\beta}^o = (\boldsymbol{\beta}_1^o, \mathbf{0})$  satisfies*

$$P\left(\left\|\widehat{\boldsymbol{\beta}}_1^o - \boldsymbol{\beta}_{10}\right\|_2 \leq C_1 \delta_n\right) \geq 1 - n^{-cs}. \quad (19)$$

Further, if  $\delta_n^{-1} \min_{1 \leq j \leq s} |\beta_{j0}| \rightarrow \infty$ , then the sign of each component of  $\widehat{\boldsymbol{\beta}}_1^o$  matches with that of the true  $\boldsymbol{\beta}_{10}$ .

Note that Theorem 4 indicates the oracle rate to be within a factor of  $\log n$  and bias due to penalization. This result can be further extended to show that the AW-DPD-LASSO estimator enjoys the oracle property even when the oracle information is not known, for a convenient choice of the weight vector. Since the regularized estimator depends on the full design matrix, we need to make additional assumptions controlling the correlation between the significant and insignificant explanatory variables allowed to have oracle model selection consistency. Following Ghosh, Jaenada and Pardo (2020) and Fan, Fan and Barut (2014), we assume

(A5)  $\left\|n^{-1}(\mathbf{X}_2^T E[\mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta}_0)] \mathbf{X}_1)\right\|_{2,\infty} < \frac{\lambda_n \min(|\mathbf{w}_1|)}{2C_1 \delta_n}$ , for some constant  $C_1 > 0$ , where we denote  $\|\mathbf{A}\|_{2,\infty} = \sup_{\mathbf{x} \in \mathbb{R}^q \setminus \{0\}} \frac{\|\mathbf{A}\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}$  for any  $p \times q$  matrix  $\mathbf{A}$  and  $\min(|\mathbf{w}_1|) = \min_{j>s} |w_j|$ .

The following theorem then shows that the (unrestricted) oracle estimator is indeed an asymptotic global minimizer of our objective function over the whole parameter space with probability tending to one.

**Theorem 5** *Consider the cases of the fixed weights. Suppose that Assumptions (A1)-(A5) hold with  $\lambda_n > 2\sqrt{(c+1)\log p/n}$  and  $\min(|\mathbf{w}_1|) > c_3$  for some constant  $c, c_3 > 0$ , and*

$$\lambda_n \|\mathbf{w}_0\|_2 \kappa_n \max\{\sqrt{s}, \|\mathbf{w}_0\|_2\} \rightarrow 0, \quad \delta_n s^{3/2} \kappa_n^2 (\log_2 n)^2 = o(n\lambda_n^2).$$

Then, with probability at least  $1 - O(n^{-cs})$ , there exists a global minimizer  $\widehat{\boldsymbol{\beta}} = \left((\widehat{\boldsymbol{\beta}}_1^o)^T, \widehat{\boldsymbol{\beta}}_2^T\right)^T$  of the AW-DPD-LASSO objective function  $Q_{n,\gamma,\lambda}(\boldsymbol{\beta})$  in (11) such that

$$\left\|\widehat{\boldsymbol{\beta}}_1^o - \boldsymbol{\beta}_{10}\right\|_2 \leq C_1 \delta_n, \quad \text{and} \quad \widehat{\boldsymbol{\beta}}_2 = \mathbf{0}_{p-s}.$$

We next establish the asymptotic distribution of the AW-DPD-LASSO. For this purpose, we need to make additional assumptions to put further restrictions to ensure that the bias term due to penalization does not diverge. Let us define  $\mathbf{V}_n = \left[\mathbf{X}_1^T E[\mathbf{H}_\alpha^{(2)}(\boldsymbol{\beta}_0)] \mathbf{X}_1\right]^{-1/2}$ ,  $\mathbf{Z}_n = \mathbf{X}_1 \mathbf{V}_n$  and  $\boldsymbol{\Omega}_n = \text{Var}\left[\mathbf{H}_\alpha^{(1)}(\boldsymbol{\beta}_0)\right]$ , and consider the following assumption.

(A6)  $\mathbf{Z}_n^T \boldsymbol{\Omega}_n \mathbf{Z}_n$  is positive definite,  $\lambda_n \|\mathbf{w}_0\|_2 = O(\sqrt{s/n})$ , and  $\sqrt{n/s} \min_{1 \leq j \leq s} |\beta_{j0}| \rightarrow \infty$ .

Under these conditions we can now state the following asymptotic distribution of the AW-DPD-LASSO with fixed weights.

**Theorem 6** *Suppose that, under the cases of fixed weights, assumptions of Theorem 5 hold along with Assumption (A6). Then, with probability tending to one, there exists a global minimizer  $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_1^o, \hat{\boldsymbol{\beta}}_2)^T$  of the AW-DPD-LASSO objective function  $Q_{n,\alpha,\lambda}(\boldsymbol{\beta})$  in (11), such that  $\hat{\boldsymbol{\beta}}_2 = \mathbf{0}_{p-s}$  and*

$$\mathbf{u}^T [\mathbf{Z}_n^T \boldsymbol{\Omega}_n \mathbf{Z}_n]^{-1/2} \mathbf{V}_n^{-1} \left[ \left( \hat{\boldsymbol{\beta}}_1^o - \boldsymbol{\beta}_{10} \right) + n \lambda_n \mathbf{V}_n^2 \widetilde{\mathbf{w}}_0 \right] \xrightarrow{\mathcal{L}} N(0, 1), \quad (20)$$

for any arbitrary  $s$ -dimensional vector  $\mathbf{u}$  satisfying  $\mathbf{u}^T \mathbf{u} = 1$ , where  $\widetilde{\mathbf{w}}_0$  is an  $s$ -dimensional vector with  $j$ -th element  $w_j \text{sign}(\beta_{j0})$ .

Note that, the consistency and asymptotic distribution of the standard DPD-LASSO can now be derived from the above theorems as a special case, by defining the weight vector  $\mathbf{w} = (1, \dots, 1)$ . It is important to note that these properties crucially depend on the structure of the oracle estimator through different assumptions on the weight vector applied to important and unimportant covariates. This dependency encourages the use of adaptive weights from the data.

So, we now derive the oracle and asymptotic properties of the general AW-DPD-LASSO with adaptive (stochastic) weights based on a weight function  $w(\cdot)$ . We consider an initial estimator  $\tilde{\boldsymbol{\beta}}$  of the regression vector and denote  $\hat{w}_j = w(|\beta_j|)$ . Let the true regression parameter  $\boldsymbol{\beta}_0$  produces the weight vector  $\mathbf{w}^* = (w_1^*, \dots, w_p^*)$  and we define  $\delta_n^* = \left[ \sqrt{s(\log n)/n} + \lambda_n \left( \|\mathbf{w}_0^*\|_2 + C_2 c_5 \sqrt{s(\log p)/n} \right) \right]$ , where the constants  $C_2, c_5$  are as defined in the following assumptions.

- (A7) The initial estimator  $\tilde{\boldsymbol{\beta}}$  satisfies  $\|\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|_2 \leq C_2 \sqrt{s(\log p)/n}$  for some constant  $C_2 > 0$ , with probability tending to one.
- (A8) The weight function  $w(\cdot)$  is non-increasing over  $(0, \infty)$  and is Lipschitz continuous with Lipschitz constant  $c_5 > 0$ . Further,  $w(C_2 \sqrt{s(\log p)/n}) > \frac{1}{2} w(0+)$  for large enough  $n$ , where  $C_2$  is as in Assumption (A7).
- (A9) With  $C_2$  being as in Assumption (A7),  $\min_{1 \leq j \leq s} |\beta_{0j}| > 2C_2 \sqrt{s(\log p)/n}$ . Further, the derivative of the wight satisfies  $w'(|b|) = o(s^{-1} \lambda_n^{-1} (n \log p)^{-1/2})$  for any  $|b| > \frac{1}{2} \min_{1 \leq j \leq s} |\beta_{0j}|$ .

These Assumptions (A7)-(A9) are again extremely common in the literature of adaptive penalty and hold in several practical scenarios. Particularly, Assumption (A7) puts a weaker restriction (only consistency) on the initial estimator in order to achieve the variable selection consistency of the second stage AW-DPD-LASSO estimators. So, it justifies the usage of most simple LASSO estimators, including the DPD-LASSO, as the initial estimator in our proposed AW-DPD-LASSO for high-dimensional logistic regression models. Assumptions (A8)-(A9) can also be verified for standard weight functions including the one obtained from SCAD penalty in (13). The main results are presented in the following two theorems.

**Theorem 7** *Consider the cases of stochastic weights and suppose that the assumptions of Theorem 5 hold with  $\mathbf{w} = \mathbf{w}^*$  and  $\delta_n = \delta_n^*$ . Additionally, suppose that Assumptions (A7)-(A8) hold with  $\lambda_n s \kappa_n \sqrt{(\log p)/n} \rightarrow 0$ . Then, with probability tending to one, there exists a global minimizer  $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_1^T, \hat{\boldsymbol{\beta}}_2^T)^T$  of the AW-DPD-LASSO objective function  $Q_{n,\alpha,\lambda}(\boldsymbol{\beta})$  in (11), with adaptive weights, such that*

$$\left\| \hat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10} \right\|_2 \leq C_1 \delta_n, \quad \text{and} \quad \hat{\boldsymbol{\beta}}_2 = \mathbf{0}_{p-s}.$$

**Theorem 8** *Consider the cases of stochastic weights and suppose that the assumptions of Theorem 6 hold with  $\mathbf{w} = \mathbf{w}^*$  and  $\delta_n = \delta_n^*$ . Additionally, suppose that Assumptions (A7)-(A9) hold. Then, with probability tending to one, there exists a global minimizer  $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_1^T, \hat{\boldsymbol{\beta}}_2^T)^T$  of the AW-DPD-LASSO objective function  $Q_{n,\alpha,\lambda}(\boldsymbol{\beta})$  in (11), with adaptive weights, having the same asymptotic properties as those described in Theorem 6.*

## 5 Computational algorithm for the proposed estimator

We present a computational algorithm using an iteratively re-weighted least squares (IRLS) approach appropriately adjusted for our DPD loss. Roughly IRLS, a popular approach for obtaining the (non-penalized) MLE under the logistic model, uses a Newton-Raphson optimization technique where the step direction is computed by solving a weighted ordinary least squares problem by using some efficient existing algorithm. This technique has been easily extended for minimizing the penalized negative log likelihood, simply solving a penalized least squares problem at each step (Lee et al. (2006)).

We first fit IRLS for the minimization of the non-penalized DPD loss function. More specifically, at a current solution  $\boldsymbol{\beta}^{(m)}$  at  $m$ -th iteration, the step direction is computed as

$$\boldsymbol{\gamma}^{(m)} = \boldsymbol{\beta}^{(m)} - \frac{\partial^2 d_\alpha(\hat{\boldsymbol{p}}, \boldsymbol{p}(\boldsymbol{\beta}))^{-1}}{\partial \boldsymbol{\beta} \boldsymbol{\beta}^T} (\boldsymbol{\beta}^{(m)}) \frac{\partial d_\alpha(\hat{\boldsymbol{p}}, \boldsymbol{p}(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta}} (\boldsymbol{\beta}^{(m)}). \quad (21)$$

But

$$\frac{\partial d_\alpha(\hat{\boldsymbol{p}}, \boldsymbol{p}(\boldsymbol{\beta}))}{\partial \boldsymbol{\beta}} (\boldsymbol{\beta}^{(m)}) = \frac{1}{n} \mathbb{X}^T \boldsymbol{H}_n^{(1)}(\boldsymbol{\beta}^{(m)}) = \frac{1}{n} \mathbb{X}^T \boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)}) \left( \boldsymbol{z}_m + \mathbb{X} \boldsymbol{\beta}^{(m)} \right) \quad (22)$$

with  $\boldsymbol{z}_m = [\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})]^{-1} \boldsymbol{H}_n^{(1)}(\boldsymbol{\beta}^{(m)}) - \mathbb{X} \boldsymbol{\beta}^{(m)}$ , where matrices  $\boldsymbol{H}_n^{(1)}$  and  $\boldsymbol{H}_n^{(2)}$  are defined in Section 4. Thus, Equation (21) can be rewritten as

$$\boldsymbol{\gamma}^{(m)} = - \left( \mathbb{X}^T \boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)}) \mathbb{X} \right)^{-1} \mathbb{X}^T \boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)}) \boldsymbol{z}_m.$$

From this representation, we see that the step direction can be computed by solving the least squares problem as

$$\boldsymbol{\gamma}^{(m)} = - \min_{\boldsymbol{\gamma}} \left[ (\boldsymbol{y}_m - \mathbb{X}_m \boldsymbol{\gamma})^T \cdot (\boldsymbol{y}_m - \mathbb{X}_m \boldsymbol{\gamma}) \right], \quad (23)$$

with  $\mathbb{X}_m = [\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})]^{1/2} \mathbb{X}$  and  $\boldsymbol{y}_m = [\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})]^{1/2} \boldsymbol{z}$ , i.e., solving a surrogate linear regression problem with design matrix  $\mathbb{X}_m$  and response vector  $\boldsymbol{y}_m$ . Note that the linear least squares problem does not include any intercept, but the vector  $\boldsymbol{\beta}$  obtained from (21) includes the logistic coefficient  $\beta_0$  implicitly. For penalized DPD logistic regression, it suffices to include the desirable penalization term in (23), yielding to

$$\min_{\boldsymbol{\gamma}} \left[ (\boldsymbol{y}_m - \mathbb{X}_m \boldsymbol{\gamma})^T \cdot (\boldsymbol{y}_m - \mathbb{X}_m \boldsymbol{\gamma}) + \sum_{j=1}^k w(|\tilde{\beta}_j|) |\gamma_j| \right]. \quad (24)$$

The penalty function for  $\gamma_0$  is defined as the constant zero so the intercept term is not penalized and the least squares estimation can be compute using efficient algorithms for adaptive LASSO. Ghosh, Jaenada and Pardo (2020) addressed the robust estimation using adaptive LASSO procedure based in the DPD loss in ultra-high dimensional for the linear regression model, which could be also used for solving (24). The previous discussion can be summarized in the following algorithm.

**Algorithm 1** [IRLS for the computation of AW-DPD-LASSO Estimator under the penalized logistic regression]

1. Set  $m = 0$ . Choose values of the hyper-parameters  $\lambda$  and  $\alpha$ , and set a robust initial value  $\boldsymbol{\beta}^{(0)}$  using any suitable robust algorithm.
2. Set  $m = m + 1$  and do:
  - Compute  $\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})$  and  $\boldsymbol{z}_m$  using (18) and (22) respectively,
  - Define the reweighted design matrix  $\mathbb{X}_m = [\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})]^{1/2} \mathbb{X}$  and response vector  $\boldsymbol{y}_m = [\boldsymbol{H}_n^{(2)}(\boldsymbol{\beta}^{(m)})]^{1/2} \boldsymbol{z}_m$ ,

- Solve the regularized linear regression problem (24) using any convenient algorithm to obtain the step direction  $\gamma^{(m)}$ ,
- Update  $\beta^{(m+1)} = t\beta^{(m)} + (1-t)\gamma^{(m)}$  by a line search over the step size  $t$  to minimize the objective function (11).

3. If (the stopping criteria is satisfied): Stop.

Else: Return to step 2.

**Remark 9** *The weights are practically calculated as  $w_j = \max(\delta, w_j)$ , with  $\delta$  sufficiently large in the implementation of Algorithm 1 so as to avoid zero division.*

Note that at each iteration of the algorithm the initial estimator of  $\beta$  is updated. Thus, our algorithm uses stochastic weights. The previous estimation depends sharply of the value of  $\lambda$ . We apply the high-dimensional adaptation of the Generalized Information Criterion (GIC), following the discussions in Fan and Tang (2013), which is given by

$$\text{HGIC}(\lambda) = \frac{-2 \log \mathcal{L}(\hat{\beta}_\lambda)}{n} + \frac{\log \log(n) \log(k)}{n} \|\hat{\beta}\|_0 \quad (25)$$

where  $\mathcal{L}(\hat{\beta}_\lambda)$  is as defined in (2). We then select the optimal  $\lambda$  by minimizing HGIC in (25) over a predefined grid.

Our algorithm is implemented in R<sup>1</sup>, and it uses the R package *glmnet* to solve the auxiliar linear regression problem (24).

## 6 Simulation study

In this section we evaluate the performance of our proposed methods through an extensive simulation study. The response variable is generated from the logistic regression model (1) with  $k = 500$ , the design matrix is drawn from a multivariate normal distribution with zero vector mean and variance-covariance matrix  $\Sigma$  having Toeplitz structure, where the  $(i, j)$ -th element is  $0.5^{|i-j|}$ . The true regression vector  $\beta_0$  is defined to be sparse,  $(\beta_{01}, \beta_{02}, \beta_{03}, \beta_{04}, \beta_{05}) = (3, 1.5, 0, 0, 2)$  for the first five components and zero in the rest. In order to evaluate the robustness of our proposed methods, we additionally modify the previous set-up introducing contamination in the response variable and leverage points. For this purpose, the contaminated response data are generated using a contaminated logistic model,  $Y_i \sim \text{Be}(\pi(\mathbf{x}_i^T \beta))$  with probability  $1 - \varepsilon$ , and  $Y_i \sim \text{Be}(1 - \pi(\mathbf{x}_i^T \beta))$  with probability  $\varepsilon$ , where  $\varepsilon$  represent the contamination level. In case of leverage points, we introduce contamination in observations with response  $Y = 1$  by adding to one variable with true non-zero coefficient (randomly chosen for each sample) a random variable generated from a normal distribution,  $\mathcal{N}(-5, 0.01)$  or to five variables with true zero coefficient (randomly chosen for each sample) a random variable generated from  $\mathcal{N}(5, 0.01)$ . Contaminated observations of both types are selected randomly up to a total of  $\varepsilon = 5, 10\%$  of the samples

The performance of our method is studied and compared via different error measures. Particularly, model size (MS), true positive proportion (TP) and true negative proportion (TN) are used to evaluate the variable selection accuracy, whereas the mean square error for the true non-zero coefficients (MSES) and the mean absolute error (MAE) of the estimated parameters are used to evaluate the accuracy on the estimation. Formally, these measures are defined as

$$\text{MS}(\hat{\beta}) = |\text{supp}(\hat{\beta})|, \quad \text{TP}(\hat{\beta}) = \frac{|\text{supp}(\hat{\beta}) \cap \text{supp}(\beta_0)|}{|\text{supp}(\beta_0)|}, \quad \text{TN}(\hat{\beta}) = \frac{|\text{supp}^c(\hat{\beta}) \cap \text{supp}^c(\beta_0)|}{|\text{supp}^c(\beta_0)|}.$$

---

<sup>1</sup>package *awDPDlasso*

$$\text{MSES}(\hat{\beta}) = \frac{1}{s} \|\hat{\beta}_S - \beta_{0S}\|^2, \quad \text{MAE}(\hat{\beta}) = \frac{1}{p} \|\hat{\beta} - \beta_0\|_1.$$

For comparison, we also fit the logistic model with the classical least squares LASSO (LS-LASSO) and its adaptive version (Ad-LS-LASSO), as well as the existing robust methods, namely the elastic net-regularized least trimmed squares (LTS) estimator (Park and Konishi (2016)) and the robust M-estimation based on Huber loss (RLASSO) proposed recently by Avella and Ronchetti (2018) along with its adaptive version (Ad-RLASSO). Further, we consider three different weight functions in our proposal, corresponding to the DPD-LASSO, Ad-DPD-LASSO and a general formulation of AW-DPD-LASSO with weight function as in (13) in Remark 2 of Section 2.

To compute LS-LASSO and Ad-LS-LASSO we use the R package *glmnet*. The LTS method is carried out using the R package *enetLTS*, whereas RLASSO and Ad-RLASSO are computed as per the implementation proposed by the authors but using HGIC criterion to select the optimal  $\lambda$ . Our algorithm is also implemented in R, and uses the R package *glmnet* to solve the auxiliary linear regression problem (24).

Tables 1-3 show the performance evaluation measures produced by the different methods considered in our simulation exercises. All estimators perform well and competitively in the absence of contamination, and it is straightforward to see that adaptive methods enhance the variable selection property. Further, our proposed algorithm decreases the estimation error and rejects a major number of insignificant variables, specially with weighted penalties. In a contaminated scenario, the results show a clear significant improvement on the estimation accuracy and robustness. Non-robust adaptive method Ad-LS-LASSO dismiss significant variables, while Ad-DPD-LASSO and AW-DPD-LASSO maintain the rate of TP. Moreover, the improvement in the model selection and estimation accuracy is preserved in all settings. The LTS method selects the largest number of variables, but has the best TP rate in the presence of contaminated data. On the other hand, RLASSO and Ad-RLASSO estimators lack robustness under leverage points.

Besides, there exists a considerable difference between Ad-DPD-LASSO and AW-DPD-LASSO estimators. The AW-DPD-LASSO generally selects a greater number of variables, containing mostly the truly important variables, but registers lower error in estimation.

## 7 Real Data Applications: Genomic Classification of Cancer Patients

### 7.1 Breast Cancer

We apply our methods to two different datasets regarding breast cancer. The first dataset is from van't Veer et al. (2002), which contains  $p = 24481$  gene expression levels of  $n = 78$  breast cancer patients, 34 of whom had developed distant metastasis within 5 years and 44 continued to be disease-free after that period. All cancer samples have the same histology, and from each patient,  $5\mu\text{g}$  total RNA was isolated from snap-frozen tumour material and used to derive complementary RNA. A reference cRNA pool was made by pooling equal amounts of cRNA from each of the sporadic carcinomas. Two hybridizations were carried out for each tumour using a fluorescent dye reversal technique on microarrays containing approximately 25,000 human genes synthesized. Fluorescence intensities of scanned images were quantified, normalized, and corrected to yield the transcript abundance of a gene as an intensity ratio with respect to that of the signal of the reference pool. Some 5,000 genes were significantly regulated across the group of samples. Moreover, 19 additional breast cancer patients' gene expressions are used as test data. The aim is to discriminate between relapse and non-relapse observations.

We first delete genes with null variation through samples, and apply to the resultant microarray data a feature selection by ranking the genes by their correlation to the response variable, and selecting

Table 1: Error measures obtained by different methods under pure data.

	MS	TP	TN	MSES $\times 10^2$	MAE
LS-LASSO	15.41	1.00	0.98	16.07	2.63
Ad-LS-LASSO	5.56	1.00	0.99	6.25	1.54
LTS	29.34	1.00	0.95	11.68	2.78
RLASSO	4.32	1.00	1.00	10.37	1.94
Ad-RLASSO	4.18	1.00	1.00	3.83	1.17
DPD-LASSO $\alpha = 0.1$	14.69	1.00	0.98	15.70	2.59
DPD-LASSO $\alpha = 0.3$	15.28	1.00	0.98	16.03	2.63
DPD-LASSO $\alpha = 0.5$	15.40	1.00	0.98	16.07	2.63
DPD-LASSO $\alpha = 0.7$	15.41	1.00	0.98	16.07	2.63
DPD-LASSO $\alpha = 1$	15.41	1.00	0.98	16.28	2.63
Ad-DPD-LASSO $\alpha = 0.1$	4.30	1.00	1.00	3.42	1.15
Ad-DPD-LASSO $\alpha = 0.3$	4.51	1.00	1.00	3.59	1.21
Ad-DPD-LASSO $\alpha = 0.5$	4.57	1.00	1.00	4.00	1.27
Ad-DPD-LASSO $\alpha = 0.7$	4.65	1.00	1.00	4.43	1.35
Ad-DPD-LASSO $\alpha = 1$	4.66	1.00	1.00	4.90	1.41
AW-DPD-LASSO $\alpha = 0.1$	4.17	1.00	1.00	1.20	0.69
AW-DPD-LASSO $\alpha = 0.3$	4.25	1.00	1.00	1.07	0.68
AW-DPD-LASSO $\alpha = 0.5$	4.25	1.00	1.00	1.13	0.69
AW-DPD-LASSO $\alpha = 0.7$	4.23	1.00	1.00	1.26	0.71
AW-DPD-LASSO $\alpha = 1$	4.21	1.00	1.00	1.34	0.72

the genes which have correlations higher than 0.3 with the class distinction. This feature selection reduces the number of explanatory genes to  $p = 457$ . We also scale the data so all gene expressions have zero mean and unit variance. According to Zervakis et al. (2006), observations numbered 37, 38, 54, 60 and 76 can be characterized as outliers and hence, we should study the influence of these five outlying observations on the performance of the model on the rest of the sample.

Table 4a shows the mean model size (MS) and the mean accuracy of the different methods with the unclean data (training set) as well as the cleaned data after removal of outliers and also the test data for  $R = 100$  fitted models. The accuracy of the methods with cleaned and test data is measured using the fitted model with the original dataset. DPD-based methods are initialized with the LASSO estimator as in the simulation study. As illustrated, the number of selected genes is lower with the adaptive methods, around 11-14 genes in contrast to the standard LASSO RLASSO and LTS methods, which select over 20 and 37 genes, respectively. Conversely, the highest accuracy with cleaned data is achieved with Ad-RLASSO jointly with the proposed robust methods, Ad-DPD-LASSO and AW-DPD-LASSO and in test data with RLASSO, LTS and the proposed Ad-DPD-LASSO methods, indicating a slight over-fitting of the AW-DPD-LASSO. Figure 2 (top left) shows a boxplot of the mean absolute error (MAE) between the predicted probabilities and real classes for all methods based on  $R = 100$  iterations. The superiority of the Ad-DPD-LASSO and AW-DPD-LASSO is obvious, which indicates better predictive performance of these methods for the present dataset

Additionally, we consider the breast cancer dataset investigated in West et al. (2001). These data contain gene expression profiles of 49 patients with breast cancer, 25 estrogen receptor (ER+) tumours (cells of this type of breast cancer have receptors that allow them to use the hormone estrogen to grow) and 24 non estrogen receptors breast cancers (ER-). All assays used the human HuGeneFL GENECHIP microarray. Arrays were hybridized and DNA chips were scanned with the GENECHIP

scanner and processed by GENECHIP Expression Analysis algorithm. All patients had the same histology (invasive ductal carcinoma) and each tumour was between 1.5 and 5 cm in maximal dimension. The aim is to distinguish between ER+ and ER- tumours.

These data have been widely studied in the cancer classification field, specially for mislabelling detection algorithms. West et al. (2001) pointed out there is biological evidence that observations 14, 16, 31, 33, 40, 43, 45 and 46 are mislabelled, and observation 11 is clearly characterized as unusual. The original data consisted of  $p = 7129$  gene expression levels, which were reduced to the  $p = 799$  most correlated genes with the class distinction as in the previous dataset. We scale the data so that all gene expression levels have zero mean and unit variance. We could assess the robustness of the estimators evaluating the estimated models fitted with the original contaminated dataset in the cleaned data (without outliers) and comparing the performance with the different estimating methods. Table 4b contains the mean model size and mean accuracy obtained with the different estimating methods for  $R = 100$  iterations. The high accuracy of the DPD based methods together with M-estimators RLASSO and Ad-RLASSO are clearly noticeable; in particular the AW-DPD-LASSO succeeds in selecting half of the genes. Additionally, the certainty of the predictions can be evaluated using the MAE between predicted probabilities and the real class labels. A boxplot of the MAE produced in  $R = 100$  iterations with the different methods is plotted in Figure 2 (top right), where the low error produced by Ad-DPD-LASSO and AW-DPD-LASSO methods stand out.

## 7.2 Colon Tumour

Next we study the Colon tumour dataset published in Alon et al. (1999). It contains gene expression levels of 40 tumours and 22 normal colon tissues for 6500 human genes obtained with an Affymetrix oligonucleotide array. A subset of  $p = 2000$  genes with the highest minimal intensity across the samples was used. The aim is to discriminate between tumour and normal observations. Additionally, there is biological evidence that the samples T2, T30, T33, T36, T37, N8, N12, N34, N36 may be mislabelled and moreover, almost of all these observations have been flagged as possible outliers by different outlier detection algorithms. Again, we are interested in studying the influence of the outlying observations on the fitted models performance. We apply a base-10 logarithm transform and scale the data before applying the logistic model. Mislabelled observations represent approximately 15% of the sample size, a proportion critically significant considering the high dimensionality of the data. Table 5a contains the mean predicted model size and mean accuracy with the unclean and cleaned (after the removal of mislabelled observations) data for  $R = 100$  fitted models. The number of selected genes with adaptive methods is again the lowest, and the large number of genes selected by the LTS method stands out compared to the remaining methods. However, the hardship of mislabelled observation turns out in a significantly higher accuracy with cleaned data of the robust methods, especially Ad-RLASSO and the proposed Ad-DPD-LASSO and AW-DPD-LASSO: most of the incorrectly classified observations in the training set were actually mislabelled observations. Finally, Figure 2 (bottom left) shows the boxplot of the MAE between predicted probabilities and real classes belonging with all methods, where AW-DPD-LASSO with low values of  $\alpha = 0.1, 0.3$  stands out for its low errors.

## 7.3 Leukemia

Our final example is based on the leukemia dataset published in Golub et al. (1999). The dataset contains the expression levels of 6817 human genes obtained from  $n = 38$  bone marrow mononuclear cells hybridized to high-density oligonucleotide microarrays (HU6800 chip) produced by Affymetrix. The chip actually contains  $p = 7129$  different probe sets from 27 cases of lymphoid precursors (acute lymphoblastic leukemia, ALL) and 11 cases of myeloid precursors (acute myeloid leukemia, AML). Samples were subjected to a priori quality control standards regarding the amount of labelled RNA and the quality of the scanned microarray image. The aim is the classification of acute leukemias into

those arising from ALL or from AML. Golub et al. (1999) also included an independent collection of 34 leukemia samples, consisting on 24 bone marrow and 10 peripheral blood samples from different reference laboratories that used different sample preparation protocols. These independent data can be used to test the quality of the predictors. The literature does not indicate any biological evidence that the dataset contains mislabelling, though Filzmoser, Maronna and Werner (2008) labelled 296 possibly multivariate outlying genes from the original dataset with 7129 predictive genes, using both training and test set and there is consensus in the literature that the 66th sample observation is an outlier. We first apply a preprocessing step to the data as suggested in Golub et al. (1999), thresholding a floor of 100 and a ceiling of 16000, applying base-2 logarithm and standardization of the data so that the expression measures for each array have mean 0 and variance 1 across genes. Additionally we filter the genes excluding genes with  $\max/\min \leq 5$  and  $(\max - \min) \leq 500$ , where  $\max$  and  $\min$  refer respectively to the maximum and minimum expression levels of a particular gene across mRNA samples. Table 5b contains the mean model size and mean accuracy with the training and test data produced by the different methods in  $R = 100$  iterations. The high correct classification rate for all methods with leukemia data indicates a lower contamination, where RLASSO, Ad-RLASSO and DPD-based methods are the most accurate in training and test. Moreover, the proposed methods Ad-DPD-LASSO and AW-DPD-LASSO remain competitive with respect to least squares based methods, and classify observations with lower MAE (Figure 2, bottom right), that is, with more certainty.

## 8 Concluding remarks

In this paper we have presented a robust adaptive method for sparse logistic regression. Based on the DPD loss function and general adaptive weighted penalties, we have proposed a family of estimators depending on a parameter  $\alpha \geq 0$  controlling the trade-off between efficiency and robustness. Moreover, these estimators are indeed robust for  $\alpha > 0$ , as is shown by the IF analysis, and enjoy oracle properties (consistently perform model selection and parameter estimation) for all  $\alpha$  under some necessary assumptions. Besides, we have developed a computational algorithm based on the well known IRLS to compute our proposed estimators and we have compared the performance of our proposed estimators to some existing robust and nonrobust methods under pure data and contaminated data through a simulation study. Our results show a clear improvement in robustness for our proposals while remaining competitive in efficiency. Finally, we employed our proposed method on four real datasets involving different types of cancer classification, obtaining promising results in each of the four cases. The logistic regression model for high dimensional data works conveniently to study gene expression profiling, as it performs gene identification and patient classification. Further, measurement errors on gene expression could influence the gene selection, and thus, robust methods are particularly advantageous.

The competent performance of the proposed methods encourages the extension to more general regression models, as the Poisson regression. Concurrently, robust methodology for high dimensional data based on the DPD could be developed to define robust statistical tests for high dimensional data, an inadequately explored field. Besides, a data driven study may be carried out so as to select optimal values of the tuning parameter  $\alpha$ .

## Acknowledgements

This research is supported by the Spanish Grants PGC2018-095 194-B-100 and FPU 19/01824. Research of AG is also partially supported by an INSPIRE Faculty Research Grant from Department of Science and Technology, Government of India, India.

## References

- ALGAMAL, Z. Y. and LEE, M. H. (2019). *A two-stage sparse logistic regression for optimal gene selection in high-dimensional microarray data classification*. *Advances in Data Analysis and Classification*, **13**, 753–771.
- ALGAMAL, Z. Y. and LEE, M. H. (2015). *Penalized logistic regression with the adaptive LASSO for gene selection in high-dimensional cancer classification*. *Expert Systems with Applications*, **42**, 9326–9332.
- ALON, U., BARKAI, N., NOTTERMAN, D. A., GISH, K., YBARRA, S., MACK, D., and LEVINE, A. J. (1999). *Broad patterns of gene expression revealed by clustering analysis of tumor and normal colon tissues probed by oligonucleotide arrays*. *Proceedings of the National Academy of Sciences*, **96**(12), 6745–6750.
- AVELLA-MEDINA, M. (2017). *Influence functions for penalized M-estimators*. *Bernoulli*, **7**, 23(4B), 3178–3196.
- Avella-Medina, M. and Ronchetti, E. (2018). *Robust and consistent variable selection in high-dimensional generalized linear models*. *Biometrika*, **105**(1), 31–44.
- BASU, A., HARRIS, R., HJORT, N., and JONES, M. C., (1998). *Robust and efficient estimation by minimising a density power divergence*. *Biometrika*, **85**, 549–559, 1998.
- BASU, A. , SHIOYA, H. and PARK, C. (2011). *The minimum distance approach*. *Monographs on Statistics and Applied Probability*. CRC Press, Boca Raton.
- BASU, A., MANDAL, A., MARTÍN, N. and PARDO, L. (2013). *Testing statistical hypotheses based on the density power divergence*. *Annals of the Institute of Statistical Mathematics*, **65**, 319–348.
- BASU, A., MANDAL, A., MARTÍN, N. and PARDO, L. (2015). *Robust tests for the equality of two normal means based on the density power divergence*. *Metrika*, **78**, 611–634.
- BASU, A., MANDAL, A., MARTÍN, N. and PARDO, L. (2016). *Generalized Wald-type tests based on minimum density power divergence estimators*. *Statistics*, **50**, 1–26.
- BASU, A., GHOSH, A., MANDAL, A., MARTÍN, N. and PARDO, L. (2017). *A Wald-type test statistic for testing linear hypothesis in logistic regression models based on minimum density power divergence estimator*. *Electron. J. Stat.*, **11**, 2, 2741–2772.
- CASTILLA, E., GHOSH, A., JAENADA, M., and PARDO, L (2020). *On regularization methods based on Rényi’s pseudodistances for sparse high-dimensional linear regression models*. arXiv preprint arXiv:2007.15929.
- CAWLEY, G. C. and TALBOT, N. L. C. 2006(). *Gene Selection in Cancer Classification Using Sparse Logistic Regression with Bayesian Regularization*. *Bioinformatics*, **22**(19), 2348–2355.
- DUFFY, D. E. and SANTNER, T. J. (1989). *On a Small Sample Properties of Norm-Restricted Maximum Likelihood Estimators for Logistic Regression Models*. *Communication Statistics (Theory Methods)* ,**18**, 959–980.
- FAN, J., FAN, Y., and BARUT, E. (2014). Adaptive robust variable selection. *Annals of Statistics*, **42**(1), 324 – 351.

- FAN, J. and LI, R. (2001). Variable selection via nonconcave penalized likelihood and its oracle properties. *Journal of the American Statistical Association*, **96**, 1348–1360.
- FAN, Y. and TANG, C. Y. (2013). *Tuning parameter selection in high dimensional penalized likelihood*. *Journal of the Royal Statistical Society, Series B Statistical Methodology*, 531–552.
- FILZMOSER, P., MARONNA, R., and WERNER, M. (2008). *Outlier identification in high dimensions*. *Computational statistics & data analysis*, **52**(3), 1694–1711.
- FOKIANOS, K. (2008). *Comparing two samples by penalized logistic regression*. *Electronic Journal of Statistics*, **2**, 564–580,
- GUO, P. , ZENG, F., HU, X. , ZHANG, D., ZHU, S., DENG, Y. and HAO, Y. (2015). *Improved Variable Selection Algorithm Using a LASSO-Type Penalty, with an Application to Assessing Hepatitis B Infection Relevant Factors in Community Residents*. *PloS, ONE*, **10** (7).
- GHOSH, A. and BASU, A. (2013). *Robust Estimation for Independent but Non-Homogeneous Observations using Density Power Divergence with application to Linear Regression*. *Electronic Journal of Statistics*, **7**, 2420–2456.
- GHOSH, A., BASU, A. and PARDO, L. (2015). *On the robustness of adivergence based test of simple statistical hypotheses*. *Journal of Statistical Planning and Inference*, **161**, 91–108.
- GHOSH, A. and BASU, A. (2016). *Robust estimation in generalized linear models: the density power divergence approach*. *Test*, **25**(2), 269–290.
- GHOSH, A., MANDAL, A., MARTÍN, N. and PARDO, L. (2016). *Influence Analysis of Robust Wald-type Tests*. *Journal of Multivariate Analysis*, **147**, 102–126.
- GHOSH, A. and MAJUMDAR, S. (2020). *Ultrahigh-dimensional Robust and Efficient Sparse Regression using Non-Concave Penalized Density Power Divergence*. *IEEE Transactions on Information Theory*, doi: 10.1109/TIT.2020.3013015.
- GHOSH, A., JAENADA, M. and PARDO, L. (2020). *Robust adaptive variable selection in ultra-high dimensional regression models based on the density power divergence loss*.
- GOLUB, T. R., SLONIM, D. K., TAMAYO, P., HUARD, C., GAASENBEEK, M., MESIROV, J. P., COLLIER, H., LOH, M. L., DOWNING, J. R., CALIGIURI, M. A., BLOOMFIELD, C. D. and LANDER, E. S. (1999). *Molecular classification of cancer: class discovery and class prediction by gene expression monitoring*. *Science*, **286**(5439), 531–537.
- HUANG, J. MA, S. and ZHANG, C. (2008). *The iterative LASSO for high-dimensional regression*. Technical Report 392. University of Iowa.
- HOERL, A. E. and KENNARD, R. W. (1970). *Ridge regression: Biased estimation for nonorthogonal problems*. *Tecnometrics*, **12**(1), 55–67.
- HOBZA, T., PARDO, L. and VAJDA, I. (2008). *Robust median estimator in logistic regression*. *Journal of Statistical Planning and Inference*, **138** (12), 3822–3840.
- JAMES, G., WITTEN, D., HASTIE, T. and TIBSHIRANI, R. (2013). *An introduction to statistical learning*, New York, Springer.
- KOH, K., KIM, S. and BOYD, S. (2007). *An interior-point method for large-scale  $l_1$ -regularized logistic regression*. *Journal of Machine Learning Research*, **8** (4), 1519–1555.

- LE CESSIE, S. and VAN HOUWELINGEN, J. C. (1992). *Ridge Estimators in Logistic Regression*. Journal of Royal Statistical Society. Series C, **41**(1), 191–201.
- LEE, S. I., LEE, H., ABBEEL, P. and NG, A. Y. (2006). *Efficient  $\ell_1$  regularized logistic regression*. Association for the advancement of artificial intelligence. **6**, 401-408.
- LEE, A. H. and SILVAPULLE, M. J. (1988). *Ridge Estimation in Logistics Regression*. Journal of Communication Statistics Simulation and Computation, **17** (4), 1231–1257.
- PARK, M. Y. and HASTIE, T. (2008). *Penalized logistic regression for detecting gene interactions*. Biostatistics, **9**, 30-50.
- PARK, M. Y. and HASTIE, T. (2007).  *$l_1$ -regularization path algorithm for generalized linear models*. Journal of the Royal Statistical Society. Series B, **69**, 659–677.
- Park, H., and Konishi, S. (2016). *Robust logistic regression modelling via the elastic net-type regularization and tuning parameter selection*. Journal of Statistical Computation and Simulation, **86**(7), 1450-1461.
- PLAN, Y. and VERSHYNIN, R. (2013). *Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach*. IEEE Transactions on Information Theory, **59**(1), 482–494.
- SCHAEFER, R. L., ROI, L. D., and WOLFE, R. A. (1984). *A ridge logistic estimator*. Communications in Statistics (Theory and Methods), **13**(1), 99–113.
- SHEVADE, S. K. and KEERTHI, S. S. (2003). *A simple and efficient algorithm for gene selection using sparse logistic regression*. Bioinformatics, **19** (17), 2246–2253.
- SUN, H. and WANG, S. (2012). *Penalized logistic regression for high-dimensional DNA methylation data with case-control studies*. Bioinformatics, **28**, 1368–1375.
- TIBSHIRANI, R. (1996). *Regression shrinkage and selection via the lasso*. Journal of the Royal Statistical Society. Series B (Methodological), **58**, 267–288.
- VAN’T VEER, L. J., DAI, H., VAN DE VIJVER, M. J., HE, Y. D., HART, A. A., MAO, M., PETERSE, H., VAN DER KOOY, K., MARTON, M. J., WITTEVEEN, A. T., SCHREIBER, G.J., KERKHOVEN, R. M., ROBERTS, C. LINSLEY, P. S., BERNARDS, R. and FRIEND, S. H. (2002). *Gene expression profiling predicts clinical outcome of breast cancer*. nature, **415**(6871), 530-536.
- YANG, Z., LIANG, Y., ZHANG, H., CHAI, H., ZHANG, B., and PENG, C. (2018). *Robust Sparse Logistic Regression With the  $L_q$  ( $0 < q < 1$ ) Regularization for Feature Selection Using Gene Expression Data*. IEEE Access, **6**, 68586-68595.
- YANG, L. and QIAN, Y. (2016). *A sparse logistic regression framework by different of convex functions programming*. Applied Intelligence, **45**, 241-254.
- WEST, M., BLANCHETTE, C., DRESSMAN, H., HUANG, E., ISHIDA, S., SPANG, R., ZUZAN, H., OLSON, J.A., MARKS, J.R. and NEVINS, J. R. (2001). *Predicting the clinical status of human breast cancer by using gene expression profiles*. Proceedings of the National Academy of Sciences, **98**(20), 11462-11467.
- ZERVAKIS, M., BLAZADONAKIS, M. E., TSILIKI, G., DANILATOU, V., TSIKNAKIS, M., and KAFETZOPOULOS, D. (2009). *Outcome prediction based on microarray analysis: a critical perspective on methods*. BMC bioinformatics, **10**(1), 1-22.

ZHU, J. and HASTIE, T. (2004). *Classification of Expressions Arrays by Penalized logistic regression*. Biostatistics, **5**(3), 427-443.

ZOU, H. and HASTIE, T. (2006). *The adaptive lasso and its oracle properties*. Journal of American Statistical Association **101**, 1418–1429.

Table 2: Error measures obtained by different methods under 5% contaminated data.

	Size	TP	TN	MSES( $\hat{\beta}$ ) $\times 10^2$	MAE( $\hat{\beta}$ )
5% of outliers at the response variable					
LS-LASSO	9.77	0.98	0.99	20.18	2.81
Ad-LS-LASSO	5.26	0.96	1.00	15.55	2.47
LTS	29.76	0.99	0.95	13.98	3.10
RLASSO	3.80	0.85	1.00	15.59	2.38
AdRLASSO	3.75	0.85	1.00	10.32	1.89
DPD-LASSO $\alpha = 0.1$	9.52	0.98	0.99	19.84	2.79
DPD-LASSO $\alpha = 0.3$	9.64	0.98	0.99	20.03	2.80
DPD-LASSO $\alpha = 0.5$	9.69	0.98	0.99	20.10	2.80
DPD-LASSO $\alpha = 0.7$	9.70	0.98	0.99	20.13	2.81
DPD-LASSO $\alpha = 1$	9.71	0.98	0.99	20.15	2.81
Ad-DPD-LASSO $\alpha = 0.1$	4.60	0.97	1.00	10.93	2.12
Ad-DPD-LASSO $\alpha = 0.3$	4.68	0.97	1.00	10.80	2.11
Ad-DPD-LASSO $\alpha = 0.5$	4.48	0.97	1.00	11.00	2.10
Ad-DPD-LASSO $\alpha = 0.7$	4.78	0.97	1.00	10.55	2.12
Ad-DPD-LASSO $\alpha = 1$	4.64	0.97	1.00	10.89	2.11
AW-DPD-LASSO $\alpha = 0.1$	5.01	0.97	1.00	7.19	1.98
AW-DPD-LASSO $\alpha = 0.3$	5.05	0.97	1.00	5.64	1.82
AW-DPD-LASSO $\alpha = 0.5$	5.03	0.95	1.00	5.67	1.79
AW-DPD-LASSO $\alpha = 0.7$	4.84	0.94	1.00	6.37	1.77
AW-DPD-LASSO $\alpha = 1$	4.90	0.94	1.00	6.55	1.78
5% of leverage points					
LS-LASSO	8.59	0.91	0.99	20.90	2.84
Ad-LS-LASSO	4.90	0.85	1.00	16.93	2.56
LTS	28.00	0.93	0.95	14.71	2.97
RLASSO	3.25	0.71	1.00	16.67	2.43
Ad-RLASSO	3.23	0.71	1.00	13.56	2.16
DPD-LASSO $\alpha = 0.1$	8.22	0.92	0.99	20.61	2.81
DPD-LASSO $\alpha = 0.3$	8.43	0.92	0.99	20.74	2.82
DPD-LASSO $\alpha = 0.5$	8.48	0.91	0.99	20.81	2.83
DPD-LASSO $\alpha = 0.7$	8.51	0.91	0.99	20.81	2.83
DPD-LASSO $\alpha = 1$	8.50	0.91	0.99	20.81	2.83
Ad-DPD-LASSO $\alpha = 0.1$	4.52	0.90	1.00	12.68	2.27
Ad-DPD-LASSO $\alpha = 0.3$	4.43	0.89	1.00	13.02	2.28
Ad-DPD-LASSO $\alpha = 0.5$	4.24	0.88	1.00	13.41	2.28
Ad-DPD-LASSO $\alpha = 0.7$	4.25	0.87	1.00	13.66	2.31
Ad-DPD-LASSO $\alpha = 1$	4.16	0.87	1.00	14.05	2.33
AW-DPD-LASSO $\alpha = 0.1$	5.46	0.93	0.99	8.33	2.23
AW-DPD-LASSO $\alpha = 0.3$	5.86	0.89	0.99	8.42	2.49
AW-DPD-LASSO $\alpha = 0.5$	5.07	0.84	0.99	9.85	2.33
AW-DPD-LASSO $\alpha = 0.7$	4.83	0.83	1.00	10.57	2.30
AW-DPD-LASSO $\alpha = 1$	4.82	0.82	1.00	11.11	2.29

Table 3: Error measures obtained by different methods under 10% contaminated data.

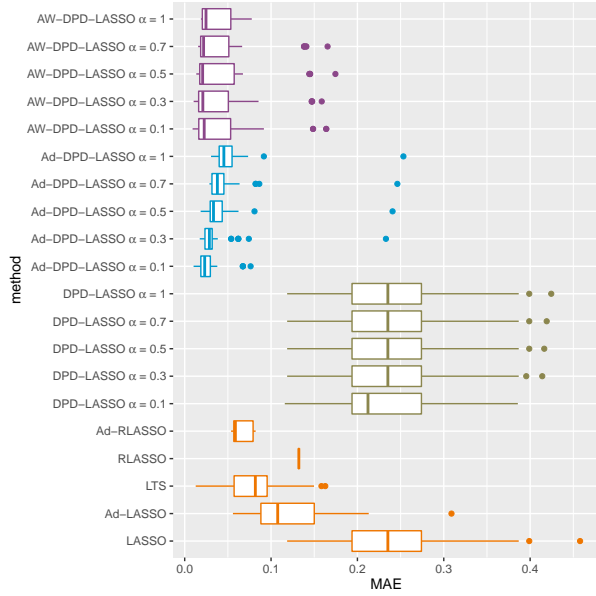
	Size	TP	TN	MSES( $\hat{\beta}$ ) $\times 10^2$	MAE( $\hat{\beta}$ )
10% of outliers at the response variable					
	Size	TP	TN	MSES	MAE
LS-LASSO	7.93	0.87	0.99	22.06	2.91
Ad-LS-LASSO	5.08	0.82	0.99	18.96	2.75
LTS	27.45	0.93	0.95	16.89	3.25
RLASSO	3.06	0.63	1.00	19.43	2.66
Ad-RLASSO	3.05	0.63	1.00	16.37	2.43
DPD-LASSO $\alpha = 0.1$	7.79	0.87	0.99	21.80	2.90
DPD-LASSO $\alpha = 0.3$	7.83	0.87	0.99	21.89	2.90
DPD-LASSO $\alpha = 0.5$	7.86	0.87	0.99	21.96	2.90
DPD-LASSO $\alpha = 0.7$	7.86	0.87	0.99	21.98	2.91
DPD-LASSO $\alpha = 1$	7.86	0.87	0.99	22.00	2.91
Ad-DPD-LASSO $\alpha = 0.1$	4.77	0.86	1.00	15.20	2.57
Ad-DPD-LASSO $\alpha = 0.3$	4.45	0.85	1.00	15.77	2.54
Ad-DPD-LASSO $\alpha = 0.5$	4.42	0.84	1.00	15.72	2.53
Ad-DPD-LASSO $\alpha = 0.7$	4.27	0.84	1.00	16.03	2.53
Ad-DPD-LASSO $\alpha = 1$	4.23	0.84	1.00	15.89	2.51
AW-DPD-LASSO $\alpha = 0.1$	6.16	0.91	0.99	10.12	2.74
AW-DPD-LASSO $\alpha = 0.3$	6.06	0.90	0.99	9.92	2.82
AW-DPD-LASSO $\alpha = 0.5$	5.07	0.85	0.99	11.43	2.49
AW-DPD-LASSO $\alpha = 0.7$	4.77	0.83	1.00	12.40	2.45
AW-DPD-LASSO $\alpha = 1$	4.45	0.82	1.00	13.20	2.39
10% of leverage points					
LS-LASSO	7.53	0.72	0.99	21.62	2.89
Ad-LS-LASSO	4.90	0.69	0.99	18.60	2.72
LTS	26.58	0.77	0.95	16.17	3.09
RLASSO	3.09	0.63	1.00	16.93	2.45
Ad-RLASSO	3.09	0.63	1.00	16.40	2.44
DPD-LASSO $\alpha = 0.1$	7.26	0.72	0.99	21.29	2.86
DPD-LASSO $\alpha = 0.3$	7.41	0.72	0.99	21.44	2.87
DPD-LASSO $\alpha = 0.5$	7.46	0.72	0.99	21.51	2.88
DPD-LASSO $\alpha = 0.7$	7.45	0.72	0.99	21.54	2.88
DPD-LASSO $\alpha = 1$	7.47	0.72	0.99	21.56	2.88
Ad-DPD-LASSO $\alpha = 0.1$	3.84	0.72	1.00	15.80	2.49
Ad-DPD-LASSO $\alpha = 0.3$	3.89	0.71	1.00	16.14	2.53
Ad-DPD-LASSO $\alpha = 0.5$	3.73	0.70	1.00	16.52	2.53
Ad-DPD-LASSO $\alpha = 0.7$	3.77	0.70	1.00	16.57	2.54
Ad-DPD-LASSO $\alpha = 1$	3.74	0.70	1.00	16.70	2.54
AW-DPD-LASSO $\alpha = 0.1$	5.40	0.76	0.99	11.59	2.70
AW-DPD-LASSO $\alpha = 0.3$	5.51	0.76	0.99	11.55	2.81
AW-DPD-LASSO $\alpha = 0.5$	4.74	0.73	0.99	12.73	2.68
AW-DPD-LASSO $\alpha = 0.7$	4.37	0.72	1.00	13.51	2.59
AW-DPD-LASSO $\alpha = 1$	3.90	0.71	1.00	14.13	2.44

Table 4: Model size and accuracy of the different methods with unclean data (training set), cleaned data (after outlier removal) and test data for (a) van't Veer et al. (2002) and (b) West et al. (2001) breast cancer datasets.

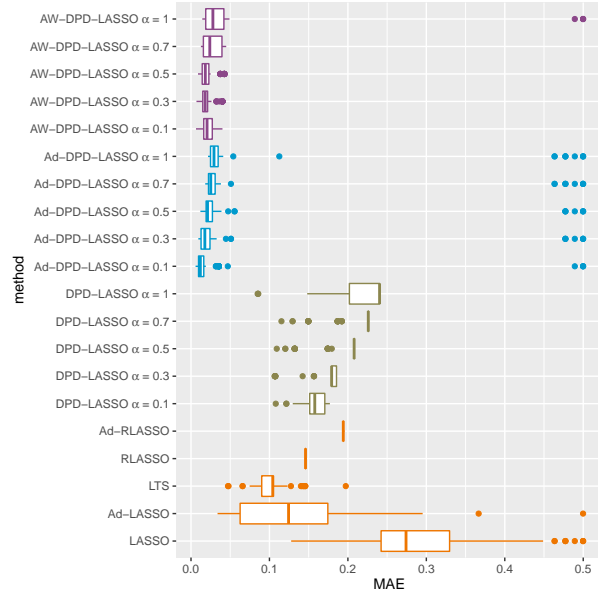
	MS	Accuracy (%)			MS	Accuracy (%)	
		unclean	cleaned	test		unclean	cleaned
LASSO	26.09	96.83	96.79	63.53	14.74	97.73	97.62
Ad-LASSO	13.35	96.69	97.15	65.21	6.74	98.04	97.72
LTS	37.38	93.09	94.19	74.74	24.60	90.86	91.25
RLASSO	20.00	94.87	94.52	84.21	19.00	100.00	100.00
AD-RLASSO	12.73	100.00	100.00	73.68	9.00	100.00	100.00
DPD-LASSO $\alpha = 0.1$	25.51	98.01	98.05	68.16	12.18	100.00	100.00
DPD-LASSO $\alpha = 0.3$	26.10	97.56	97.60	65.47	12.39	100.00	100.00
DPD-LASSO $\alpha = 0.5$	26.19	97.04	97.03	63.74	12.59	100.00	100.00
DPD-LASSO $\alpha = 0.7$	26.18	97.00	96.99	63.63	12.70	100.00	100.00
DPD-LASSO $\alpha = 1$	26.17	96.99	96.97	63.74	13.11	100.00	100.00
Ad-DPD-LASSO $\alpha = 0.1$	14.06	99.90	99.95	72.95	7.95	98.31	98.33
Ad-DPD-LASSO $\alpha = 0.3$	12.07	99.73	99.77	73.47	7.20	98.08	98.05
Ad-DPD-LASSO $\alpha = 0.5$	11.90	99.73	99.77	74.74	6.59	98.10	98.08
Ad-DPD-LASSO $\alpha = 0.7$	11.76	99.74	99.78	72.89	6.28	98.02	97.97
Ad-DPD-LASSO $\alpha = 1$	11.16	99.73	99.77	74.63	6.15	97.98	97.92
AW-DPD-LASSO $\alpha = 0.1$	11.57	98.46	98.64	64.58	7.26	100.00	100.00
AW-DPD-LASSO $\alpha = 0.3$	11.46	98.60	98.82	64.89	7.25	100.00	100.00
AW-DPD-LASSO $\alpha = 0.5$	11.30	98.64	98.97	66.05	6.99	100.00	100.00
AW-DPD-LASSO $\alpha = 0.7$	11.27	98.83	99.07	66.32	6.97	99.96	99.95
AW-DPD-LASSO $\alpha = 1$	11.39	99.33	99.52	66.68	6.72	98.29	98.30

(a) Error measures for van't Veer et al. (2002) breast cancer dataset.

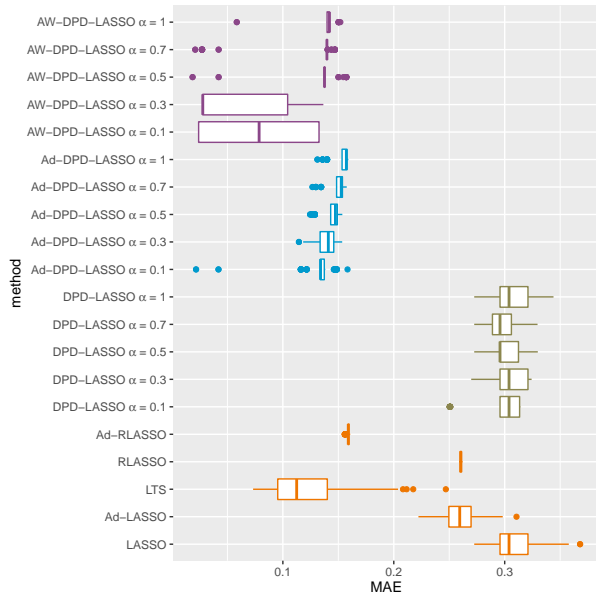
(b) Error measures for West et al. (2001) breast cancer dataset .



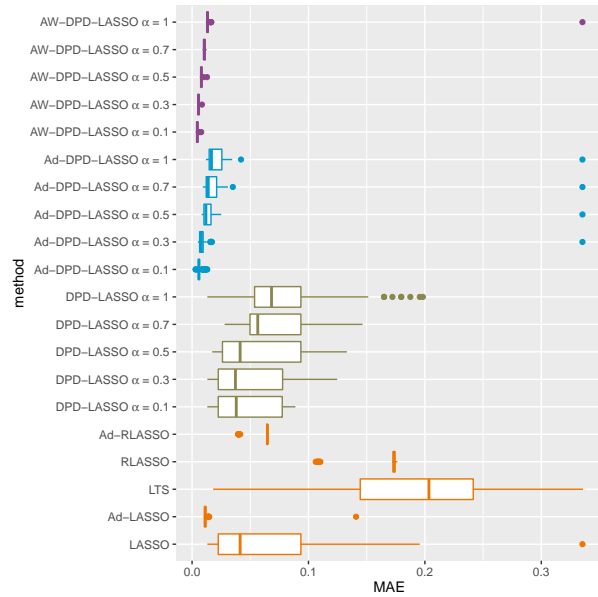
(a) Breast Cancer van't Veer et al. (2002)



(b) Breast Cancer West et al. (2001)



(c) Colon tumour Alon et al. (1999)



(d) Leukemia Golub et al. (1999)

Figure 2: Boxplot of the mean square error (MAE) produced with the different methods in  $R = 100$  fits of the real datasets.

Table 5: Model size and accuracy of the different methods with unclean data (training set), cleaned data (after outlier removal) and test data for (a) for Alon et al. (1999) colon tumour and (b) for Golub et al. (1999) Leukemia datasets.

	MS	Accuracy (%)		MS	Accuracy (%)	
		unclean	cleaned		train	test
LASSO	7.52	87.58	92.63	14.34	99.58	91.41
Ad-LASSO	4.00	84.53	89.19	5.01	99.97	94.12
LTS	19.24	91.95	96.85	9.94	96.39	91.41
RLASSO	13.00	79.03	79.63	4.08	100.00	91.41
AD-RLASSO	7.00	90.32	94.44	4.00	100.00	94.12
DPD-LASSO $\alpha = 0.1$	8.09	89.60	94.61	13.16	100.00	94.12
DPD-LASSO $\alpha = 0.3$	7.87	89.27	94.44	13.68	100.00	91.41
DPD-LASSO $\alpha = 0.5$	7.82	88.95	94.13	14.36	100.00	94.12
DPD-LASSO $\alpha = 0.7$	7.91	88.56	93.94	15.48	100.00	91.41
DPD-LASSO $\alpha = 1$	7.82	88.26	93.67	12.57	99.87	94.12
Ad-DPD-LASSO $\alpha = 0.1$	5.25	90.24	96.17	7.62	100.00	91.41
Ad-DPD-LASSO $\alpha = 0.3$	4.98	90.11	96.52	6.99	99.71	94.12
Ad-DPD-LASSO $\alpha = 0.5$	4.95	90.47	96.52	6.59	99.71	91.41
Ad-DPD-LASSO $\alpha = 0.7$	4.95	90.47	96.52	6.81	99.71	94.12
Ad-DPD-LASSO $\alpha = 1$	4.92	90.52	96.52	7.04	99.71	91.41
AW-DPD-LASSO $\alpha = 0.1$	5.95	95.15	98.15	8.64	100.00	94.12
AW-DPD-LASSO $\alpha = 0.3$	6.44	97.52	99.00	8.58	100.00	91.41
AW-DPD-LASSO $\alpha = 0.5$	4.98	90.52	96.26	8.53	100.00	94.12
AW-DPD-LASSO $\alpha = 0.7$	5.20	91.74	96.85	8.44	100.00	91.41
AW-DPD-LASSO $\alpha = 1$	4.92	91.90	96.37	8.54	99.71	94.12

(a) Error measures for Alon et al. (1999) colon tumour dataset. (b) Error measures for Golub et al. (1999) Leukemia dataset.