Arcs and jets on toric singularities and quasi-ordinary singularities

Pedro Daniel González Pérez
(joint work with Helena Cobo Pablos)

We denote by $(S, 0)$ a reduced irreducible equidimensional germ of algebroid singularity of dimension d defined over $\mathbf{C}$, an algebraically closed field of zero characteristic. A *formal arc* on the germ $(S, 0)$ is a morphism of germs $h : (D, 0) \to (S, 0)$ where $D :=$ Spec $\mathbf{C}[[t]]$. If we fix an embedding $(S, 0) \subset (\mathbf{C}^n, 0)$ and local coordinates $(x_1, \dots, x_d)$ at 0 then the arc h is given by n power series in $\mathbf{C}[[t]]$: $x_i(t) = a_1^{(i)} t + a_2^{(i)} t^2 + \cdots + a_r^{(i)} t^r + \cdots$, $i = 1, \dots, n$, such that $F(x_1(t), \dots, x_n(t)) = 0$, for any F in the ideal of $(S, 0)$. The set of arcs H on $(S, 0)$ can be seen as an affine subscheme of Spec $\mathbf{C}[a_1^{(i)}, a_2^{(i)}, \dots]_{i=1}^n$. An *s-jet* on the germ $(S, 0)$ is a morphism of germs $h : (D_s, 0) \to (S, 0)$ where $D_s =$ Spec $\mathbf{C}[[t]]/(t)^{s+1}$. The set H^s of s-jets on $(S, 0)$ is an affine subscheme of $\mathbf{A}^{sn}_{\mathbf{C}} :=$ Spec $\mathbf{C}[a_1^{(i)}, \dots, a_s^{(i)}]_{i=1}^n$.

Any arc $h \in H$ has a *s-jet* $j^s(h) \in H^s$. A theorem of Greenberg implies that the set $j^s(H) \subset H^s$ is a *constructible set* of H^s for every $s \geq 0$. It has an image $[j^s(H)]$ in the *Grothendieck ring* $K_0(\mathrm{Var}_{\mathbf{C}})$ of **C**-*varieties*. This ring is generated by the symbols $[X]$ for X an algebraic variety, subject to relations: $[X] = [X']$ if X is isomorphic to X', $[X] = [X - X'] + [X']$ if $[X']$ is closed in X and $[X][X'] = [X \times X']$. If X is an algebraic variety the map $X' \mapsto [X']$, for X' closed in X extends to constructible subsets W of X, $W \mapsto [W]$ in a unique way if $[W \cup W'] = [W] + [W'] - [W \cap W']$, see [2].

The *geometric Poincaré series* $P_{\mathrm{geom}}(T) := \sum_{s \geq 0} [j^s(H)] T^s \in K_0(\mathrm{Var}_{\mathbf{C}})[[T]]$. is an invariant of the germ $(S, 0)$. We denote by $\mathbf{L}$ the class $\mathbf{L} = [\mathbf{A}^1_{\mathbf{C}}]$ of the affine line and by $\mathcal{M}_{\mathbf{C}}$ the ring $K_0(\mathrm{Var}_{\mathbf{C}})[\mathbf{L}^{-1}]$. A theorem of Denef and Loeser states that the series $P_{\mathrm{geom}}(T)$, when viewed in $\mathcal{M}_{\mathbf{C}}[[T]]$, is a rational function, i.e., it belongs to $\mathcal{M}_{\mathbf{C}}[T]$ (see [2]). The proof of this deep result concerns quantifier elimination for semi-algebraic sets of power series in zero characteristic, the theory of motivic integration introduced by Kontsevich and the existence of resolution of singularities of varieties over a field of zero characteristic. The invariants of $(S, 0)$ encoded by this series are not well understood, see Nicaise work for some particular cases [6]. Lejeune and Reguera gave an explicit description of this series in the case of an affine normal toric surface, see [4]. We describe the classes $[j^s(H)]$ associated to a *quasi-ordinary hypersurface singularity* or to a germ of affine toric variety, needless to say non necesarily normal, in terms of the convexity properties of certain monomial ideals which we associate to the singularity, if the singularity is locally unibranched along the singular locus. Rond [9] studies inductively the the series $P_{geom}(T)$ in the quasi-ordinary case.

1. Toric case

Let M be a rank d lattice, N its dual lattice and $\Lambda \subset M$ a submonoid generated by $e_1, \dots, e_n$, such that the cone $\sigma^\vee := \mathbf{R}_{\geq 0} e_1 + \cdots + \mathbf{R}_{\geq 0} e_n \subset M_{\mathbf{R}} := M \otimes \mathbf{R}$ is strictly convex of dimension d. Denote by $\sigma \subset N_{\mathbf{R}}$ the dual cone of $\sigma^\vee \subset M_{\mathbf{R}}$. If

$e \in \Lambda$ we denote by $\chi^e \in \mathbf{C}[\Lambda]$ the corresponding monomial. We denote the affine toric variety $\operatorname{Spec} \mathbf{C}[\Lambda]$ by Z^Λ. The toric morphism $Z^{\sigma^\vee \cap M} \to Z^\Lambda$ corresponding to the inclusion $\Lambda \subset \sigma^\vee \cap M$ is the normalization map. The embedding $Z^\Lambda \subset \mathbf{C}^n$, defined by $x_i = \chi^{e_i}$ for $i = 1, \dots, n$, is equivariant.

We study the class $[j^s(H)]$ if $(S, 0) = (Z^\Lambda, 0)$. An arc $h \in H$ has its generic point in the torus if and only if $\chi^m \circ h \neq 0$, $\forall m \in M$. We expand $\chi^m \circ h = t^{\nu_h(m)} u_h(m)$, where $u_h(m)$ is a unit in $\mathbf{C}[[t]]$. It follows that $\nu_h \in \overset{\circ}{\sigma} \cap N$. Denote by H^* the set of arcs $h \in H$ with generic point in the torus.

Lemma 1. *If S is locally unibranched along its singular locus and if then we have $j^s(H) = j^s(H^*)$, for $s \geq 0$ (*see [7] in the normal toric case*).*

We have a partition $H^* = \sqcup_{\nu \in \overset{\circ}{\sigma} \cap N} H^*_\nu$ where $H^*_\nu := \{h \in H^* / \nu_h = \nu\}$. It follows that $j^s(H) = j^s(H^*) = \cup_{\nu \in \overset{\circ}{\sigma} \cap N} j^s(H^*_\nu)$, but this union is non-finite and non-disjoint since different arcs $h \in H^*_\nu$ and $h' \in H^*_{\nu'}$ may have the same s-jet. We follow the strategy of Lejeune and Reguera [4]: we show that $j^s(H^*_\nu)$ is locally closed in H^s and we compute the class $[j^s(H^*_\nu)]$; then we exhibit a finite subset $\Xi(s)$ of $\overset{\circ}{\sigma} \cap N$ determining a partition $j^s(H) = \sqcup_{\nu \in \Xi(s)} j^s(H^*_\nu)$. This description is given in terms of a sequence of monomial ideals of the local ring $\mathbf{C}[[\sigma^\vee \cap M]]$. In the case of Lejeune and Reguera these ideals are the *maximal ideal* and the *logarithmic jacobian ideal*, which determines the *Nash modification* of S, see [4].

Let us define the following subsets of Γ, for $k = 1, \dots, d$:

(1) $\qquad \mathcal{J}_k := \{e_{i_1} + \cdots + e_{i_k} / e_{i_1}, \dots, e_{i_k} \text{ linearly independent } \}_{1 \leq i_1 < \cdots < i_k \leq d}.$

If $\mathcal{J} \subset \sigma^\vee \cap M$ we denote also by $\mathcal{J}$ the corresponding monomial ideal of $\mathbf{C}[[\sigma^\vee \cap M]]$. The *Newton polyhedron* $\mathcal{N}(\mathcal{J})$ of the monomial ideal $\mathcal{J}$ is the convex hull of $\mathcal{J} + \sigma^\vee$. We denote by $\operatorname{ord}_{\mathcal{J}}$ the support function of the polyhedron $\mathcal{N}(\mathcal{J})$, defined by: $\operatorname{ord}_{\mathcal{J}} : \sigma \to \mathbf{R}$, $\nu \mapsto \inf_{\omega \in \mathcal{N}(\mathcal{J})} \langle \nu, \omega \rangle$. We use the following notations: $\phi_1 := \operatorname{ord}_{\mathcal{J}_1}$, $\phi_2 := \operatorname{ord}_{\mathcal{J}_2} - \operatorname{ord}_{\mathcal{J}_1}$, $\dots$, $\phi_d := \operatorname{ord}_{\mathcal{J}_d} - \operatorname{ord}_{\mathcal{J}_{d-1}}$. We have that $\phi_1 \leq \phi_2 \cdots \leq \phi_d$ on σ (point-wise). For any $s \geq 0$, we have a partition of $\overset{\circ}{\sigma} \cap N$:

$$
\begin{array}{lclllrcll}
\rho_0(s) & := & \{ & \mu \in \overset{\circ}{\sigma} \cap N & / & & s & < \phi_1(\mu) & \} \\
\rho_1(s) & := & \{ & \mu \in \overset{\circ}{\sigma} \cap N & / & \phi_1(\mu) \leq & s & < \phi_2(\mu) & \} \\
\dots & \dots & & & \dots & \dots & & & \\
\rho_{d-1}(s) & := & \{ & \mu \in \overset{\circ}{\sigma} \cap N & / & \phi_{d-1}(\mu) \leq & s & < \phi_d(\mu) & \} \\
\rho_d(s) & := & \{ & \mu \in \overset{\circ}{\sigma} \cap N & / & \phi_d(\mu) \leq & s & & \}.
\end{array}
$$

Theorem 1. *If $s \geq 0$ and $\nu \in \overset{\circ}{\sigma} \cap N$, let k be the unique integer such that $\nu \in \rho_k(s)$ then we have that if $k = 0$ the jet space $j^s(H^*_\nu)$ is equal to $\{0\}$ otherwise it is isomorphic to $(\mathbf{C}^*)^k \times \mathbf{A}_{\mathbf{C}}^{sk - \operatorname{ord}_{\mathcal{J}_k}(\nu)}$.*

Remark 2. *The ideal $\mathcal{J}_d$ gives the Nash modification of a normal $(S, 0)$,* see [4].

We define an equivalence relation $\sim$ in the set $\rho_k(\sigma)$ for any $s > 0$ and $1 \leq k \leq d$:

$$
\nu \sim \nu' \in \rho_k(s) \Leftrightarrow \left\{ \begin{array}{c} \nu \text{ and } \nu' \text{ define the same face of } \mathcal{N}(\mathcal{J}_j) \text{ and} \\ \operatorname{ord}_{\mathcal{J}_j}(\nu) = \operatorname{ord}_{\mathcal{J}_j}(\nu'). \end{array} \right\} \text{ for } 1 \leq j \leq k.
$$

We denote by $\bar{\nu}$ the class of ν. The quotient set $\rho_k(s)/\sim$ is finite.

Corollary 2. *If S is locally unibranched along its singular locus the class of $j^s(H)$ in the Grothendieck ring is of the form:*

$$[j^s(H)] = 1 + \sum_{k=1}^{d} \sum_{\bar{\nu} \in \rho_k(s)/\sim} (\mathbf{L}-1)^k \mathbf{L}^{ks - \mathrm{ord}_{\mathcal{J}_k}(\nu)}.$$

2. Quasi-ordinary hypersurface case

An equidimensional germ $(S, 0)$ is *quasi-ordinary* (QO) if there exists a finite projection $\pi : (S, 0) \to (\mathbf{C}^d, 0)$ which is a local isomorphism outside a normal crossing divisor. The class of QO-singularities contains curve singularities and simplicial toric singularities; it is important in Jung's approach to resolution of singularities, see [5]. The normalization of a QO-singularity is a toric singularity, see [8]. A *QO-hypersurface* $(S, 0)$ has an equation $f = 0$, where $f \in \mathbf{C}[[x_1, \ldots, x_d]][y]$ is the minimal polynomial over $\mathbf{C}[[x_1, \ldots, x_d]]$ of a especial type of fractional power series $\zeta \in \mathbf{C}[[x_1^{1/m}, \ldots, x_d^{1/m}]]$ possessing a finite set $\lambda_1, \ldots, \lambda_g \in \mathbf{Q}^d$ of *characteristic exponents*, see [1] and [5]. This series generalizes Newton-Puiseux expansions. The normalization is the toric singularity $Z^{\sigma^\vee \cap M}$ where $\sigma^\vee := \mathbf{R}^d_{\geq 0}$ and $M := \sum_{i=1}^{d+g} \mathbf{Z} e_i$, for $x_i = \chi^{e_i}$, $i = 1, \ldots, d$; and $e_{d+j} := \lambda_j$, $j = 1, \ldots, g$, see [3]. The strategy is similar to the toric case, though proofs are more involved. We denote by H^* the set of arcs $h \in H$ such that $\pi \circ h$ has its generic point in the torus, we set H^*_ν analogously for $\nu \in \overset{\circ}{\sigma} \cap N$. We define the sets for $k = 1, \ldots, d$:

(2) $$\mathcal{J}_k := \{\textstyle\sum_{r=1}^{k} e_{i_r} / e_{i_1}, \ldots, e_{i_k} \text{ linearly independent } \}^{i_{k-1} < i_k \leq d+g}_{1 \leq i_1 < \cdots < i_{k-1} \leq d,}.$$

Theorem 3. *With the notations and hypothesis analogous to those of section 1, the statement of Theorem 1 and Corollary 2 holds with respect to the ideals (2).*

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