

# James' Theorem Fails for Starlike Bodies

D. Azagra

*Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas,  
Universidad Complutense, Madrid 28040, Spain; and ,  
Université Pierre et Marie Curie—Paris 6, 4, place Jussieu,  
75005 Paris, France*

and

R. Deville

*Université Bordeaux I, 351, cours de la Libération, 33405 Talence cedex, France*  
E-mail: daniel@sunam1.mat.ucm.es, deville@math.u-bordeaux.fr

Starlike bodies are interesting in nonlinear functional analysis because they are strongly related to bump functions and to  $n$ -homogeneous polynomials on Banach spaces, and their geometrical properties are thus worth studying. In this paper we deal with the question whether James' theorem on the characterization of reflexivity holds for (smooth) starlike bodies, and we establish that a feeble form of this result is trivially true for starlike bodies in nonreflexive Banach spaces, but a reasonable strong version of James' theorem for starlike bodies is never true, even in the smooth case. We also study the related question as to how large the set of gradients of a bump function can be, and among other results we obtain the following new characterization of smoothness in Banach spaces: a Banach space  $X$  has a  $C^1$  Lipschitz bump function if and only if there exists another  $C^1$  smooth Lipschitz bump function whose set of gradients contains the unit ball of the dual space  $X^*$ . This result might also be relevant to the problem of finding an Asplund space with no smooth bump functions.

## 1. INTRODUCTION AND MAIN RESULTS

A closed subset  $A$  of a Banach space  $X$  is said to be a starlike body provided  $A$  has a non-empty interior and there exists a point  $x_0 \in \text{int } A$  such that each ray emanating from  $x_0$  meets the boundary of  $A$  at most once. In this case we will say that  $A$  is *starlike with respect to*  $x_0$ . When dealing with starlike bodies, we can always assume that they are starlike

with respect to the origin (up to suitable translations), and we will do so unless otherwise stated. For a starlike body  $A$ , we define the characteristic cone of  $A$  as

$$ccA = \{x \in X \mid rx \in A \text{ for all } r > 0\}$$

and the Minkowski functional of  $A$  as

$$q_A(x) = \inf \left\{ \lambda > 0 \mid \frac{1}{\lambda} x \in A \right\}$$

for all  $x \in X$ . It is easily seen that for every starlike body  $A$  its Minkowski functional  $q_A$  is a continuous function which satisfies  $q_A(rx) = rq_A(x)$  for every  $r \geq 0$  and  $q_A^{-1}(0) = ccA$ . Moreover,  $A = \{x \in X \mid q_A(x) \leq 1\}$ , and  $\partial A = \{x \in X \mid q_A(x) = 1\}$ , where  $\partial A$  stands for the boundary of  $A$ . Conversely, if  $\psi: X \rightarrow [0, \infty)$  is continuous and satisfies  $\psi(\lambda x) = \lambda\psi(x)$  for all  $\lambda \geq 0$ , then  $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$  is a starlike body. Convex bodies (that is, closed convex sets with nonempty interior) are an important kind of starlike bodies. For a convex body  $U$ ,  $ccU$  is always a convex set, but in general the characteristic cone of a starlike body will not be a convex set. We will say that  $A$  is a  $C^p$  smooth starlike body provided its Minkowski functional  $q_A$  is  $C^p$  smooth on the set  $X \setminus ccA = X \setminus q_A^{-1}(0)$ .

The reader might wonder why we should care about smooth starlike bodies at all. Such objects appear in nonlinear functional analysis as natural substitutes of convex bodies. Indeed, on the one hand, smooth starlike bodies are strongly related to polynomials in Banach spaces since for every  $n$ -homogeneous polynomial  $P: X \rightarrow \mathbb{R}$  the sets  $A_c = \{x \in X \mid P(x) \leq c\}$ ,  $c \in \mathbb{R}$ , are either starlike bodies or complements of starlike bodies; therefore the level sets of every  $n$ -homogeneous polynomials are boundaries of starlike bodies, and if one is interested in the geometrical behaviour of  $n$ -homogeneous polynomials then one should also pay some attention to the geometrical properties of starlike bodies. On the other hand, smooth bounded starlike bodies also arise in a natural way from smooth bump functions; indeed, for every Banach space  $(X, \|\cdot\|)$  with a  $C^p$  smooth bump function there exist a functional  $\psi$  and constants  $a, b > 0$  such that  $\psi$  is  $C^p$  smooth away from the origin,  $\psi(\lambda x) = |\lambda| \psi(x)$  for every  $x \in X$  and  $\lambda \in \mathbb{R}$ , and  $a \|x\| \leq \psi(x) \leq b \|x\|$  for every  $x \in X$  (see [9], Proposition II.5.1). The function  $\psi$  has a useful conical shape and can sometimes take the role of a smooth norm in spaces which in general are not know to possess such norms. The level sets of this function are precisely the boundaries of the  $C^p$  smooth bounded starlike bodies  $A_c = \{x \in X \mid \psi(x) \leq c\}$ ,  $c > 0$ . Conversely, if a Banach space  $X$  has a  $C^p$  smooth bounded starlike body then it has a  $C^p$  smooth bump function as well.

It is therefore reasonable to ask to what extent the geometrical properties of convex bodies are shared with the more general class of starlike bodies. Surprisingly enough, to the best of the authors' knowledge, very little work concerning smooth starlike bodies and their geometrical properties has been attempted.

One of the deepest classical results of functional analysis is James' theorem [11] on the characterization of reflexivity. Let us recall what James' theorem reads. A Banach space  $X$  is reflexive if and only if, for a given bounded convex body  $B$  in  $X$ , every continuous linear functional  $T \in X^*$  attains its supremum on  $B$ . In this paper we investigate to what extent this fundamental result can be generalized for starlike bodies.

There are two problems to be considered, one for each direction in the equivalence given by James' theorem. The difficult and more interesting part of James' theorem tells us that for every bounded convex body  $B$  in a nonreflexive Banach space  $X$  there exists  $T \in X^*$  so that  $T$  does not attain its supremum on  $B$ . Since  $B$  is convex this amounts to saying that  $T$  does not attain any local extrema on  $B$ . Moreover, if  $B$  is smooth then this means that there is some one-codimensional subspace  $H$  of  $X$  so that the hyperplanes  $y + H$  are not tangent to  $B$  at any point  $y \in \partial B$ . At this point we face two possible generalizations of this result for starlike bodies, one for each of those formulations (which, as we just said, are equivalent in the case of convex bodies, but not for starlike bodies). The first one yields a statement which is true but not very interesting; we call it a "weak form of James' theorem" for starlike bodies:

**PROPOSITION 1.1.** *Let  $A$  be a bounded starlike body in a nonreflexive Banach space  $X$ . Then there exists a continuous linear functional  $T \in X^*$  such that  $T$  does not attain its supremum on  $A$ .*

However, when one considers the second formulation of the difficult part of James' theorem, things turn out very different in the case of starlike bodies. In this new setting it is natural to ask whether a "strong form of James' theorem" is true for starlike bodies (at least when they are smooth). By a strong James' theorem we mean the following: if  $A$  is a bounded starlike body in a nonreflexive Banach space  $X$ , does there exist  $T \in X^*$  so that  $T$  does not attain any local extrema on  $A$ ? For a smooth starlike body  $A$  the question should even be made stronger: is there some one-codimensional subspace  $H$  in  $X$  such that the hyperplanes  $z + H$  are not tangent to  $A$  at any point  $z \in \partial A$ ? Equivalently, if  $q_A$  is the Minkowski functional of  $A$ , is there some  $T \in X^*$  such that  $T$  does not belong to the cone generated by the set  $\{q'_A(x): x \neq 0\}$  in  $X^*$ ? (Of course, if  $A$  is a convex body then the answer is "yes", it satisfies this strong form of James' theorem.)

We will prove that both questions have negative answers, that is, a strong James' theorem fails for bounded starlike bodies, even when they are smooth, in nonreflexive Banach spaces.

**THEOREM 1.2.** *Let  $X$  be an infinite-dimensional Banach space. Then there exists a bounded starlike body  $A \subset X$  such that every  $T \in X^*$  attains infinitely many local maxima and minima on  $A$ .*

*Moreover, if  $X$  is a separable Asplund space then there exists a bounded  $C^1$  smooth starlike body  $A \subset X$  with the property that for every  $T \in X^* \setminus \{0\}$  there exists  $y \in \partial A$  such that the hyperplane  $y + \text{Ker } T$  is tangent to  $\partial A$  at  $y$ . In other words, if  $q_A$  is the Minkowski functional of  $A$  then  $\{\lambda q'_A(x) : \lambda \geq 0, x \neq 0\} = X^*$ .*

The starlike body provided by the first part of this result is not smooth. Our construction of a general counterexample in the smooth case (second part of this statement) is strongly related to the following natural question concerning the size of the sets of gradients of bump functions, which we also deal with in this paper. If  $X$  is a Banach space and  $b: X \rightarrow [0, \infty)$  is a smooth bump function (that is, a smooth function with bounded support, not identically zero), how large can the set of gradients  $b'(X)$  be in  $X^*$ ?

In general, as a consequence of Ekeland's variational principle, one has that the cone generated by the set of gradients of  $b$ ,  $\mathcal{C}(b) = \{\lambda b'(x) : \lambda \geq 0, x \in X\}$ , is norm-dense in the dual space  $X^*$  (see [9], p. 58, Proposition 5.2). It seems natural to think that there should be no upper bound on the size of  $\mathcal{C}(b)$ , and it might well happen that  $b'(X) = X^*$ . We will show that this is indeed true. In fact we have the following.

**THEOREM 1.3.** *Let  $X$  be an infinite-dimensional Banach space. The following statements are equivalent.*

- (1)  $X$  has a  $C^1$  smooth Lipschitz bump function;
- (2)  $X$  has a  $C^1$  smooth Lipschitz bump function  $f$  such that  $f'(X)$  contains the unit ball of  $X^*$ ; in particular  $\mathcal{C}(f) = X^*$ .

*Moreover, if  $X$  satisfies (1) then there exists a  $C^1$  smooth (non-Lipschitz) bump function  $b$  on  $X$  with the property that  $b'(X) = X^*$ .*

A straightforward corollary to this result is that a separable infinite-dimensional space  $X$  is Asplund if and only if  $X$  has a  $C^1$  smooth bump function  $b$  so that  $b'(X) = X^*$ .

It is worth mentioning that the result provided by Theorem 1.3 is the keystone for our proof of the "smooth" part of Theorem 1.2. On the other hand, Theorem 1.3 might also be relevant to the problem of finding an Asplund space with no smooth bumps.

Now let us consider the other direction of the equivalence given by James' theorem, the "easy" part of this result. Namely, if  $X$  is reflexive, every bounded convex body  $B \subset X$  satisfies that, for every  $T \in X^*$ ,  $T$  attains its supremum on  $B$ . Equivalently, every one-codimensional subspace  $H$  of  $X$  has the property that  $z + H$  supports and touches  $B$  at some point  $z \in \partial B$ . When  $B$  is smooth this means that  $z + H$  is tangent to  $B$  at some point  $z \in \partial B$ . Does this part of James' theorem remain true when one replaces the term "convex body" with "starlike body"?

The next result tells us precisely that, whatever the formulation we choose for this part of James' theorem, the answer to the above question is negative.

**THEOREM 1.4.** *In the Hilbert space  $\ell_2$  there exist a  $C^\infty$  smooth bounded starlike body  $A$  and a one-codimensional subspace  $H$  with the property that for no  $y \in \partial A$  is the hyperplane  $y + H$  tangent to  $A$  at  $y$ . In other words,  $\{\lambda q'_A(x) : \lambda \geq 0, x \neq 0\} \neq X^*$ .*

It comes as no surprise that this result is a consequence of the failure of Rolle's theorem and the existence of deleting diffeomorphisms in infinite-dimensional Banach spaces (see [1-5, 14]). Indeed, James' theorem trivially implies that the classical Rolle's theorem is true for the class of convex functions in a Banach space  $X$  if and only if  $X$  is reflexive. Namely, for every Banach space  $X$  and every bounded convex body  $B \subset X$ , the following statements are equivalent:

(1)  $X$  is reflexive

(2) For every continuous convex function  $f: B \rightarrow \mathbb{R}$  such that  $f = 0$  on  $\partial B$ , there exists  $x_0 \in \text{int } B$  so that  $0 \in \partial f(x_0)$ ,

where  $\partial f(x)$  stands for the classical subdifferential of  $f$  at  $x$ ,  $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq x^*(y - x) \text{ for all } y\}$ . Hence, it is only natural that the failure of the "easy" part of James' theorem for starlike bodies is closely related to the failure of Rolle's theorem for bump functions in infinite dimensions.

As a corollary to 1.4 we have a result that sheds some light on the natural question as to how *small* the cone generated by the set of gradients of a bump can be. Namely,

**COROLLARY 1.5.** *In the Hilbert space  $\ell_2$  there exist a  $C^\infty$  smooth bump function  $f: X \rightarrow \mathbb{R}$  (with starlike support) and a linear functional  $T \in X^*$  such that the vectors  $\{T, b'(x)\}$  are linearly independent for all  $x \in X$  with  $b'(x) \neq 0$ .*

In order to highlight the link between bump functions and starlike bodies let us make one final remark. Theorems 1.2 and 1.4, taken together, tell us that James' theorem fails for (smooth) starlike bodies, while their "bump"

counterparts, Theorem 1.3 and Corollary 1.5, could be summed up by saying that James' theorem fails for smooth bump functions. This similarity between starlike bodies and bump functions is also stressed by the interdependence of the proofs: 1.2 will be deduced from 1.3, while 1.5 is a corollary to 1.4.

## 2. THE PROOFS

We will start with the easy proof of 1.1, then we will proceed with the proof of 1.3, which is the keystone in the proof of 1.2. Finally we will prove Theorem 1.4 and Corollary 1.5.

*Proof of Proposition 1.1.* Let  $C$  be the closed convex hull of  $A$ , which is a (bounded) convex body of  $X$ . Since  $X$  is not reflexive, James' theorem gives us a continuous linear functional  $T \in X^*$  such that  $T$  does not attain its sup on  $C$ . Let us see that

$$\sup_{x \in A} T(x) = \sup_{x \in C} T(x) := \alpha > 0$$

and  $T$  does not attain  $\sup_{x \in A} T(x)$  either. It is obvious that  $\sup_{x \in A} T(x) \leq \alpha$ . We only have to see that  $\alpha \leq \sup_{x \in A} T(x)$ .

Let  $\varepsilon > 0$ , and choose  $y \in C$  so that  $T(y) \geq \alpha - \varepsilon/2$ . Since  $C = \overline{\text{conv}}(A)$  we can pick  $x \in \text{conv}(A)$  such that  $x$  is so close to  $y$  as to satisfy  $|T(y) - T(x)| \leq \varepsilon/2$ . Then we have  $T(x) + \varepsilon/2 \geq T(y) \geq \alpha - \varepsilon/2$  and hence  $T(x) \geq \alpha - \varepsilon$ . Now, since  $x \in \text{conv}(A)$ , there exist points  $a_i \in A$  and numbers  $t_i \in (0, 1]$ ,  $i = 1, \dots, n$ , such that  $\sum_{i=1}^n t_i = 1$  and  $x = \sum_{i=1}^n t_i a_i$ . There must be some  $i$  such that  $T(a_i) \geq \alpha - \varepsilon$ ; otherwise we would have

$$T(x) = \sum_{i=1}^n t_i T(a_i) < \sum_{i=1}^n t_i (\alpha - \varepsilon) = (\alpha - \varepsilon),$$

a contradiction. This shows that for each  $\varepsilon > 0$  there exists an  $a \in A$  such that  $T(a) \geq \alpha - \varepsilon$ . Therefore  $\alpha \leq \sup_{x \in A} T(x)$  and consequently  $\sup_{x \in A} T(x) = \sup_{x \in C} T(x)$ . Finally, since  $T$  does not attain  $\sup_{x \in C} T(x)$  and  $A$  is contained in  $C$ , it is obvious that  $T$  cannot attain  $\sup_{x \in A} T(x)$ .

*Proof of Theorem 1.3.* The proof of 1.3 finds inspiration in Bates's construction of smooth surjections between Banach spaces (see [6]). For the reader's convenience we first give the proof in the separable case, which is easier to grasp, and then we say a few words about the way one can adapt this proof to the general (nonseparable) case.

So let us assume that  $X$  is a separable Banach space with a  $C^1$  smooth Lipschitz bump function. In particular  $X$  is an Asplund space and the dual

$X^*$  is also separable. In these conditions it is known (see [9]) that  $X$  has an equivalent  $C^1$  smooth norm  $\|\cdot\|$ . Let us denote  $B = \{x \in X : \|x\| \leq 1\}$ , and fix a number  $\varepsilon$  such that  $0 < \varepsilon < 1/32$ . For this  $\varepsilon$  take a sequence  $\{z_n\}$  of points in  $B$  such that  $B(z_n, 4\varepsilon) \subset B$  for every  $n$  and  $\|z_n - z_m\| > 8\varepsilon$  whenever  $n \neq m$ . Let  $T_n: X \rightarrow X$  be the affine contraction defined by  $T_n(x) = z_n + \varepsilon x$  for each  $n$ . Then the balls  $B_n := T_n(B) = B(z_n, \varepsilon)$  are pairwise disjoint and lie in the interior of the unit ball  $B$ . Let  $\mathcal{B}_0 = \{B\}$ , and for  $k \geq 1$  let us define subcollections of balls within  $B$  by

$$\mathcal{B}_k = \{T_n(B') \mid n \in \mathbb{N}, B' \in \mathcal{B}_{k-1}\}.$$

By a chain of balls we will mean a sequence  $(U_j)$  such that  $U_j \in \mathcal{B}_j$  and  $U_{j+1} \subset U_j$  for each  $j \in \mathbb{N}$ . It should be noted that there is a bijection between the chains of balls  $(U_j)$  and the sequences of natural numbers  $s = (n_1, n_2, \dots, n_j, \dots) \in \mathbb{N}^{\mathbb{N}}$  by means of the relation  $U_j^s = T_{n_1} T_{n_2} \cdots T_{n_j}(B)$ . We will use the notation

$$U_j^s := B_{(n_1, n_2, \dots, n_j)} = z_{n_1} + \varepsilon z_{n_2} + \cdots + \varepsilon^{j-1} z_{n_j} + \varepsilon^j B.$$

It is quite clear that the intersection of any chain of these balls consists exactly of one point. Indeed,

$$\bigcap_{j=1}^{\infty} B_{(n_1, n_2, \dots, n_j)} = \sum_{j=1}^{\infty} \varepsilon^{j-1} z_{n_j};$$

this series being absolutely convergent because  $\|z_n\| \leq 1$  for all  $n$  and  $\varepsilon < 1$ . Note also that for every  $(n_1, n_2, \dots, n_j) \in \mathbb{N}^j$  the balls  $\{B_{(n_1, n_2, \dots, n_j, m)}\}_{m \in \mathbb{N}}$  are contained in the ball  $B_{(n_1, n_2, \dots, n_j)}$  and are the image of the balls  $\{B_m\}_{m \in \mathbb{N}}$  under the affine contraction  $T_{n_1} T_{n_2} \cdots T_{n_j}$ .

Now, by composing the  $C^1$  smooth norm  $\|\cdot\|$  with a suitable real function we can obtain a  $C^1$  smooth Lipschitz function  $h: X \rightarrow [0, 1]$  such that  $h(x) = 1$  whenever  $\|x\| \leq 2$ ,  $h(x) = 0$  if  $\|x\| \geq 3$ , and  $\|h'\|_{\infty} = \sup_{x \in X} \|h'(x)\| \leq 2$ .

Next we are going to define a series of  $C^1$  mappings as follows. Take a sequence  $(x_n^*)_{n \in \mathbb{N}}$  which is dense in the unit ball of  $X^*$ , and for  $k \geq 1$  put

$$f_k(x) = \sum_{(n_1, n_2, \dots, n_k) \in \mathbb{N}^k} \varepsilon^{2(k-1)} h \left( \frac{x - \sum_{j=1}^k \varepsilon^{j-1} z_{n_j}}{\varepsilon^k} \right) [x_{n_k}^*(x) + 1]$$

for all  $x \in X$ . It is clear that  $f_k$  is  $C^1$  smooth and Lipschitz and its support is contained in  $\bigcup_{(n_1, n_2, \dots, n_k) \in \mathbb{N}^k} B(\sum_{j=1}^k \varepsilon^{j-1} z_{n_j}, 3\varepsilon^k) \subset B$  (note that this is a disjoint union of balls). Moreover, by the construction, it is clear that  $f_k'(B_{(n_1, n_2, \dots, n_k)}) = \varepsilon^{2(k-1)} x_{n_k}^*$  for all  $(n_1, n_2, \dots, n_k) \in \mathbb{N}^k$ .

We claim that the series  $\sum_{k=1}^{\infty} f_k$  converges to a  $C^1$  Lipschitz mapping  $f: X \rightarrow [0, 4]$  which has the property that  $f'(X)$  contains the unit ball of the

dual  $X^*$ . For a given  $k \in \mathbb{N}$  we have that for every  $x \in X$  either  $x$  belongs to one and only one of the disjoint balls  $z_{n_1} + \varepsilon z_{n_2} + \dots + \varepsilon^{j-1} z_{n_j} + 3\varepsilon^j B$  or  $x$  is not in any of these balls; in any case we will have that there exist some  $n_1, \dots, n_k$  such that

$$f_k(y) = \varepsilon^{2(k-1)} h \left( \frac{y - \sum_{j=1}^k \varepsilon^{j-1} z_{n_j}}{\varepsilon^k} \right) [x_{n_k}^*(y) + 1]$$

for all  $y$  in a neighbourhood of  $x$ ; then, taking into account that  $h$  is 2-Lipschitz and  $\|x_n^*\| \leq 1$  for all  $n$ , we can estimate

$$\begin{aligned} \|f'_k(x)\| &= \left\| \varepsilon^{k-2} h' \left( \frac{x - \sum_{j=1}^k \varepsilon^{j-1} z_{n_j}}{\varepsilon^k} \right) [x_{n_k}^*(x) + 1] \right. \\ &\quad \left. + \varepsilon^{2(k-1)} h \left( \frac{x - \sum_{j=1}^k \varepsilon^{j-1} z_{n_j}}{\varepsilon^k} \right) x_{n_k}^* \right\| \\ &\leq 4\varepsilon^{k-2} + \varepsilon^{2(k-1)} \leq 5\varepsilon^{k-2}. \end{aligned}$$

Then we have  $\|f'_k(x)\| \leq 5\varepsilon^{k-2}$  for all  $x \in X$  and all  $n \in \mathbb{N}$ , which implies that  $\sum_{k=1}^\infty \|f'_k(x)\| \leq 5 \sum_{k=1}^\infty \varepsilon^{k-2}$  for all  $x$ . In the same way one can easily check that  $\sum_{k=1}^\infty \|f_k(x)\| \leq \sum_{k=1}^\infty 2\varepsilon^{2k-2} \leq 4$  for all  $x$ . This means that the series  $\sum_{k=1}^\infty f_k(x)$  converges uniformly on  $X$  to a  $C^1$  smooth Lipschitz function  $f: X \rightarrow [0, 4]$  whose derivative is  $f'(x) = \sum_{k=1}^\infty f'_k(x)$ . Moreover, according to the above estimation for the series of derivatives, we have  $\|f'\|_\infty = \sup_{x \in X} \|f'(x)\| \leq 5\varepsilon^{-2}$ , that is,  $f$  is  $5\varepsilon^{-2}$ -Lipschitz.

Let us now see that  $f'(X)$  contains the unit ball of  $X^*$ . By the construction of the  $f_k$  and  $f$  it is clear that  $f'_k(\partial B_{(n_1, \dots, n_k)}) = \varepsilon^{2(k-1)} x_{n_k}^*$  and

$$f'(\partial B_{(n_1, \dots, n_k)}) = x_{n_1}^* + \varepsilon^2 x_{n_2}^* + \dots + \varepsilon^{2(k-1)} x_{n_k}^*$$

for all chains of balls  $(B_{(n_1, \dots, n_k)})_{k \in \mathbb{N}}$ . Let  $x^* \in X^*$ ,  $\|x^*\| \leq 1$ . Since  $(x_n^*)$  is dense in the unit ball of  $X^*$  we can choose  $n_1$  so that  $\|x^* - x_{n_1}^*\| \leq \varepsilon^2$ . Then we can choose  $n_2$  so that  $\|(x^* - x_{n_1}^*) - \varepsilon^2 x_{n_2}^*\| \leq \varepsilon^4$ . We apply this argument inductively to choose a chain of balls  $(B_{(n_1, \dots, n_k)})_k$  such that

$$\begin{aligned} \|x^* - f'(\partial B_{(n_1, \dots, n_k, n_{k+1})})\| \\ = \|x^* - x_{n_1}^* - \varepsilon^2 x_{n_2}^* - \dots - \varepsilon^{2(k-1)} x_{n_k}^* - \varepsilon^{2k} x_{n_{k+1}}^*\| \leq \varepsilon^{2(k+1)} \end{aligned}$$

for all  $k \in \mathbb{N}$ . Then, if we put  $x := \bigcap_{j=1}^\infty B_{(n_1, n_2, \dots, n_j)} = \sum_{j=1}^\infty \varepsilon^{j-1} z_{n_j}$ , by continuity of  $f'$  we will have that  $f'(x) = x^*$ .

So far we have proved that there exists a  $C^1$  smooth  $5\varepsilon^{-2}$ -Lipschitz bump function  $f: X \rightarrow [0, 4]$  such that  $f'(X)$  contains the unit ball of  $X^*$  and the support of  $f$  is contained in  $B$ . This shows that (1) implies (2) in

Theorem 1.3 (and it is obvious that (2) implies (1) too). Finally, in order to obtain a  $C^1$  smooth bump function  $b: X \rightarrow \mathbb{R}$  with the property that  $b'(X) = X^*$  we can define

$$b(x) = \sum_{n=1}^{\infty} n f\left(\frac{x - z_n}{\varepsilon}\right)$$

for all  $x \in X$ . Since the maps  $f_n(x) = f\left(\frac{x - z_n}{\varepsilon}\right)$  have pairwise disjoint supports and their derivatives have the property that  $f'_n(X) = \varepsilon^{-1}f'(X)$  contains the ball of center 0 and radius  $1/\varepsilon$  of the dual  $X^*$ , it is quite clear that  $b'(X) = X^*$ . This concludes the proof of 1.3 in the separable case.

In order to extend this kind of results to nonseparable Banach spaces with smooth Lipschitz bumps we will need the following fact about Asplund spaces. Recall that the density character  $\text{dens}(X)$  of a Banach space  $X$  is the smallest cardinality that a dense subset of  $X$  can have.

**LEMMA 2.1.** *Let  $X$  be an infinite-dimensional Asplund space. Then  $\text{dens}(X) = \text{dens}(X^*)$ , and for every  $\varepsilon \in (0, 1/2)$  there exists a  $\varepsilon$ -separated collection of points  $(z_\alpha)_{\alpha \in \Gamma}$  in the unit ball of  $X$  with  $\text{card}(\Gamma) = \text{dens}(X^*)$ .*

*Proof.* In [13] it is shown that for an Asplund space  $X$  we have  $\text{dens}(X) = \text{dens}(X^*)$ . Let us briefly see that if  $\text{dens}(X) = \kappa$  then there exists an  $\varepsilon$ -separated collection of points  $(z_\alpha)_{\alpha \in \Gamma}$  in  $B_X$  with  $\text{card}(\Gamma) = \kappa$ . By Zorn's lemma we can take a maximal set  $(z_\alpha)_{\alpha \in \Gamma}$  of  $\varepsilon$ -separated points in  $B_X$ . Since  $X$  is infinite-dimensional it is clear that  $\text{card}(\Gamma) \geq \aleph_0$ . Let  $A$  be the set of finite linear combinations with rational coefficients of elements of  $(z_\alpha)_{\alpha \in \Gamma}$ . Then  $\text{card}(A) = \text{card}(\Gamma \times \mathbb{N}) = \text{card}(\Gamma) \geq \aleph_0$ . It is easy to see that  $\bar{A} = X$  (indeed,  $\bar{A}$  is a closed subspace of  $X$ , and if  $\bar{A} \neq X$  then we can take a point  $z \in S_X$  such that  $\text{dist}(z, \bar{A}) > 1/2$ , but this contradicts the maximality of  $(z_\alpha)_{\alpha \in \Gamma}$ ). Therefore, by definition of  $\text{dens}(X)$  we have  $\kappa \leq \text{card}(A) = \text{card}(\Gamma)$ . On the other hand, since the points of  $(z_\alpha)_{\alpha \in \Gamma}$  are isolated, it is clear that  $\text{card}(\Gamma) \leq \kappa$ . ■

Now we are ready to make a sketch of the proof of Theorem 1.3 in the nonseparable case. Since  $X$  has a  $C^1$  Lipschitz bump function, by composing it with a suitable real function, we can obtain a  $C^1$  Lipschitz function  $h: X \rightarrow [0, 1]$  such that  $h(x) = 1$  whenever  $\|x\| \leq 2$ ,  $h(x) = 0$  if  $\|x\| \geq M$ , and  $\|h'\|_\infty = \sup_{x \in X} \|h'(x)\| \leq M$ , for some  $M \geq 3$ .

Now, for a fixed  $\varepsilon > 0$  such that  $\varepsilon < 1/32M$ , according to Lemma 2.1, we can take a  $2M\varepsilon$ -separated collection of points  $(z_\alpha)_{\alpha \in \Gamma}$  in  $\frac{1}{2}B$  (where  $B$  is the unit ball of  $X$ ) with  $\text{card}(\Gamma) = \text{dens}(X^*)$ . Then the balls  $B(z_\alpha, M\varepsilon)$ ,  $\alpha \in \Gamma$ , are all disjoint and contained in  $B$ . As in the separable case we can define chains of balls

$$U_j^s := B_{(\alpha_1, \alpha_2, \dots, \alpha_j)} = z_{\alpha_1} + \varepsilon z_{\alpha_2} + \dots + \varepsilon^{j-1} z_{\alpha_j} + \varepsilon^j B$$

for  $s = (\alpha_1, \alpha_2, \dots, \alpha_j, \dots) \in \Gamma^{\mathbb{N}}$ , and there is a bijection between the chains of balls  $(U_j)$  and the set of sequences  $\Gamma^{\mathbb{N}}$ ; moreover, the intersection of any chain of these balls consists exactly of one point, namely,  $\bigcap_{j=1}^{\infty} B_{(\alpha_1, \alpha_2, \dots, \alpha_j)} = \sum_{j=1}^{\infty} \varepsilon^{j-1} z_{\alpha_j}$ .

Next, since  $\text{dens}(X^*) = \text{card}(\Gamma)$ , we can take a subset  $(x_{\alpha}^*)_{\alpha \in \Gamma}$  which is dense in the unit ball of  $X^*$ , and for every  $k \geq 1$  we can define

$$f_k(x) = \sum_{(\alpha_1, \alpha_2, \dots, \alpha_k) \in \Gamma^k} \varepsilon^{2(k-1)} h \left( \frac{x - \sum_{j=1}^k \varepsilon^{j-1} z_{\alpha_j}}{\varepsilon^k} \right) [x_{\alpha_k}^*(x) + 1]$$

for all  $x \in X$ . As in the separable case one can check that the series  $\sum_{k=1}^{\infty} f_k$  converges to a  $C^1$  Lipschitz function  $f$  which has the property that  $f'(X)$  contains the unit ball of  $X^*$ . This shows the equivalence between (1) and (2) of 1.3 in the general case.

Finally, in order to obtain a  $C^1$  smooth bump  $b$  on  $X$  such that  $b'(X) = X^*$ , it is enough to take a sequence  $(\alpha_1, \alpha_2, \dots, \alpha_k, \dots) \in \Gamma^{\mathbb{N}}$  with  $\alpha_i \neq \alpha_j$  if  $i \neq j$ , and put

$$b(x) = \sum_{n=1}^{\infty} n f \left( \frac{x - z_{\alpha_n}}{\varepsilon} \right)$$

for all  $x \in X$ .

*Proof of Theorem 1.2 (Nonsmooth case).* Let us fix some  $\varepsilon \in (0, 1/8)$  and pick a continuous real function  $g = g_{\varepsilon}: [-\varepsilon, \varepsilon] \rightarrow [0, 2]$  such that  $g$  is twice differentiable away from the origin and  $g$  satisfies the following conditions:

- (1)  $g(t) = g(-t)$  for all  $t$ ;
- (2)  $g'(t) < 0$  for all  $t \in (0, \varepsilon)$ ;
- (3)  $g''(t) > 0$  for all  $t \in (0, \varepsilon)$ ;
- (4)  $\lim_{t \rightarrow 0^+} g'(t) = -\infty$ ; and
- (5)  $g(0) = 2, g(\varepsilon) = g(-\varepsilon) = 0$ .

In particular  $g$  is not differentiable at 0 and the graph of  $g$  has the nice property that every line but  $t = 0$  passing through the point  $(0, 2) \in \mathbb{R}^2$  is above the graph.

Now, for every closed hyperplane  $H = \{x \in X \mid f(x) = 0\}$ , where  $f \in X^*, \|f\| = 1$ , and for every vector  $z$  such that  $\|z\| = 1$  and  $f(z) = 1$ , we are going to define what we will call a "spike directed by  $z$  and  $H$ " and we will denote  $S_{H,z}$ . Let us write  $X = H \oplus [z]$ , and define

$$S_{H,z} = \{h + tz \in X : 0 \leq t \leq g(\|h\|), \|h\| \leq \varepsilon\} \cup \{h + tz \in X : t \leq 0, \|h + tz\| \leq \varepsilon\}$$

as the spike directed by  $z$  and  $H$ . This spike is easily checked to be a bounded starlike body which has the property that for every hyperplane  $M$  passing through the point  $2z$  and not containing the line  $\{\lambda z: \lambda \in \mathbb{R}\}$  there exists a neighbourhood  $V$  of  $2z$  such that  $M$  touches  $S_{H,z} \cap V$  only at the “cusp” of  $S_{H,z}$ , that is the point  $2z$ . This means that for every  $T \in X^*$  such that  $z \notin \text{Ker } T$ ,  $T$  attains either a local maximum or a local minimum on the body  $S_{H,z}$  at the point  $2z$ .

Next we consider the “ball with a spike directed by  $z$  and  $H$ ”,  $U_{H,z} = B_X \cup S_{H,z}$ , where  $B_X$  is the unit ball of  $X$ . Again, it is easy to check that  $U_{H,z}$  is a bounded starlike body (because the union of two bodies which are starlike with respect to the origin is a starlike body with respect to the origin), and this “spiky ball” has the property that for every  $T \in X^*$  such that  $z \notin \text{Ker } T$ ,  $T$  attains either a local maximum or a local minimum on  $U_{H,z}$  at the point  $2z$ .

Now, let  $\{z_\alpha\}_{\alpha \in I}$  be a  $4\varepsilon$ -net in the unit sphere  $S_X$ , that is, a maximal subset of  $4\varepsilon$ -separated points of  $S_X$  (two points  $x, y$  are said to be  $\delta$ -separated provided  $\|x - y\| \geq \delta$ ). For each  $z_\alpha$  take  $f_\alpha \in X^*$  with  $\|f_\alpha\| = 1$  such that  $f_\alpha(z_\alpha) = 1$  (this  $f_\alpha$  always exists thanks to the Hahn–Banach theorem), and consider the ball with spike  $U_{H_\alpha, z_\alpha}$  as defined above, where  $H_\alpha = \text{Ker } f_\alpha$ ; we will denote  $U_\alpha = U_{H_\alpha, z_\alpha}$  for short. For each  $\alpha \in I$  let  $\psi_\alpha$  be the Minkowski functional of the starlike body  $U_\alpha$ . Next consider the union of all these balls with spikes, what we could call a “porcupine body”,

$$A = \bigcup_{\alpha \in I} U_\alpha.$$

We claim that  $A$  is a bounded starlike body. Indeed, let us define  $\psi: X \rightarrow [0, \infty)$  by

$$\psi(x) = \inf_{\alpha \in I} \psi_\alpha(x).$$

Since all the functions  $\psi_\alpha$  are positive homogeneous,  $\psi$  is also positive homogeneous. Let us see that  $\psi$  is continuous on  $X$ . Since  $\psi$  is positive homogeneous it is enough to check that  $\psi$  is continuous on the unit sphere  $S_X$ . Take a point  $z \in S_X$  and consider  $\eta = \inf\{\|z - z_\alpha\|: \alpha \in I\}$ . If  $\eta \leq \varepsilon$  then, since the points  $z_\alpha$  are  $4\varepsilon$ -separated there is a unique  $\alpha = \alpha_0 \in I$  such that  $\|z - z_{\alpha_0}\| \leq \varepsilon$ ; for all the other  $z_\alpha$  we have  $\|z - z_\alpha\| \geq 3\varepsilon$ . Then for every  $x \in S_X$  with  $\|x - z\| < \varepsilon$  we have  $\|x - z_{\alpha_0}\| < 2\varepsilon$ , and  $\|x - z_\alpha\| \geq 2\varepsilon$  for  $\alpha \neq \alpha_0$ ; by the construction of the  $U_\alpha$  and  $\psi_\alpha$  this implies that  $\psi_{\alpha_0}(x) \leq \psi_\alpha(x)$  for all  $\alpha \neq \alpha_0$  and for all  $x \in S_X$  such that  $\|x - z\| < \varepsilon$ . Then it is clear that  $\psi(x) = \psi_{\alpha_0}(x)$  for all  $x \in S_X$  with  $\|x - z\| < \varepsilon$  and  $\psi$  is continuous at  $z$  on  $S_X$  because so is  $\psi_{\alpha_0}$ . On the other hand, if  $\eta > \varepsilon$  then we can take  $r = (\eta - \varepsilon)/2 > 0$  and we have that  $\|x - z_\alpha\| > \varepsilon$  for all  $\alpha$  whenever  $\|x - z\| < r$ , so that,

by the construction of  $U_\alpha$  and  $\psi_\alpha$ ,  $\psi(x) = \psi_\alpha(x) = \|x\|$  for all  $\alpha$  and for all  $x \in S_X$  with  $\|x - z\| < r$ , and therefore  $\psi$  is continuous at  $z$  on  $S_X$ . In either case we see that  $\psi$  is continuous at  $z$ .

Since  $\psi$  is continuous and positive homogeneous, the set  $\{x \in X \mid \psi(x) \leq 1\}$  is a starlike body. Let us see that  $A = \{x \in X \mid \psi(x) \leq 1\}$ . By using the fact that the points  $z_\alpha$  are  $4\varepsilon$ -separated and the construction of  $A$  one can easily check that  $A$  is a closed set. If  $\psi(x) < 1$  then, by the definition of  $\psi$ , there exists some  $\alpha$  such that  $\psi_\alpha(x) < 1$ , so that  $x \in U_\alpha$  and therefore  $x \in A$ ; since  $A$  is closed and  $\psi$  is continuous and positive homogeneous this implies that  $\{x \in X \mid \psi(x) \leq 1\} \subseteq A$ . On the other hand, if  $x \in A$  then there exists some  $\alpha$  such that  $x \in U_\alpha$ , which means  $\psi_\alpha(x) \leq 1$  and therefore  $\psi(x) \leq 1$ .

So we have that  $A = \bigcup_{\alpha \in I} U_\alpha$  is a starlike body, and  $A$  is bounded (it is contained in the ball of radius 2 with center at the origin). It only remains to check that every  $T \in X^*$  attains a local maximum or minimum on  $A$ . Take  $T \in X^*$ ,  $T \neq 0$ , and consider the hyperplane  $\text{Ker } T$ . Since  $\{z_\alpha\}_{\alpha \in I}$  is a  $4\varepsilon$ -net in the sphere  $S_X$  and  $\varepsilon < 1/8$ , there must be some  $z_\alpha$  such that  $z_\alpha \notin \text{Ker } T$ . Then, as said above,  $T$  attains a local maximum or minimum at the point  $2z_\alpha$  on the body  $U_\alpha$ . It is easy to see that the bodies  $A$  and  $U_\alpha$  are locally the same around the point  $2z_\alpha$  (that is, there exists some  $r > 0$  such that  $A \cap B(2z_\alpha, r) = U_\alpha \cap B(2z_\alpha, r)$ ). Then it is clear that  $T$  also attains a local maximum or minimum at the point  $2z_\alpha$  on the body  $A$ .

Finally, when  $X$  is an infinite-dimensional space, for any hyperplane  $M = \text{Ker } T$  of  $X$  there must be infinitely many points  $z_\alpha$  of our  $4\varepsilon$ -net in  $S_X$  such that  $z_\alpha \notin M$ , and hence  $T$  attains infinitely many local maxima and minima on  $A$ .

*Proof of Theorem 1.2 (Smooth case).* First of all let us note that from the proof of Theorem 1.3 we know that for every separable Asplund space  $Y$  and for a fixed  $\varepsilon$  with  $0 < \varepsilon < 1/32$  there exists a  $5\varepsilon^{-2}$ -Lipschitz  $C^1$  smooth function  $f = f_Y: Y \rightarrow [0, 4]$  whose support is contained in  $B_Y$  and such that  $f'(B_Y)$  contains the unit ball of the dual,  $B_{Y^*}$ . By putting  $f_\varepsilon(y) = 2\varepsilon f(y/\varepsilon)$  for all  $y \in Y$  we can obtain a  $10\varepsilon^{-2}$ -Lipschitz  $C^1$  smooth function  $f_\varepsilon$  whose support is in  $B_Y(0, \varepsilon)$  and such that  $f'_\varepsilon(Y)$  contains  $2B_{Y^*}$ . In particular, for every closed hyperplane  $H \subset X$  we will have a  $10\varepsilon^{-2}$ -Lipschitz  $C^1$  smooth function  $f_\varepsilon: H \rightarrow [0, 8\varepsilon]$  such that  $2B_{H^*} \subseteq f'_\varepsilon(B_H(0, \varepsilon))$ .

Now, since  $X^*$  is separable, Corollary II.4.3 of [9] gives us an equivalent LUR norm  $\|\cdot\|$  whose dual norm is also LUR and, in particular,  $\|\cdot\|$  is  $C^1$  smooth on  $X \setminus \{0\}$ . Take  $M > 20\varepsilon^{-2} + 1$ , and consider the ball  $B(0, M) = B_M$ . For every  $z \in S_X$  let  $H_z = \text{Ker } d\|\cdot\|(z)$  (so that  $Mz + H_z$  is the tangent hyperplane to  $S_M = \{x: \|x\| = M\}$  at the point  $Mz$ ).

For each  $z \in S_X$ , by composing the norm  $\|\cdot\|$  with a suitable real function we can obtain a  $2/\varepsilon$ -Lipschitz  $C^1$  smooth function  $g = g_z: H_z \rightarrow [0, 1]$

such that  $g(x) = 1$  whenever  $\|x\| \leq \varepsilon$ ,  $g(x) = 0$  for  $\|x\| \geq 2\varepsilon$ , and with the property that the real functions  $t \mapsto g(tx)$  are all non-increasing. On the other hand, since the norm  $\|\cdot\|$  is LUR and  $C^1$  smooth, the hemisphere  $\{x + tz \in H_z \oplus [z] : \|x + tz\| = M, t > 0\}$  can be regarded as the graph of a  $C^1$  smooth function  $h = h_z: H_z \cap B_M \rightarrow \mathbb{R}$ . Let  $G = G_z: H \cap B_M \rightarrow \mathbb{R}$  be defined by  $G(x) = Mg(x) + (1 - g(x))h(x)$ ; then  $G$  is a  $C^1$  smooth function such that  $G(x) = M$  whenever  $\|x\| \leq \varepsilon$ ,  $G(x) = h(x)$  for  $\|x\| \geq 2\varepsilon$ , and  $h(x) \leq F(x) \leq M$  for all  $x$ ; moreover, the function  $G$  has the property that the functions  $t \mapsto G(tx)$ ,  $t \in [0, \infty)$ , are all non-increasing (indeed, taking into account that the functions  $t \mapsto g(tx)$  and  $t \mapsto h(tx)$  are non-increasing for  $t \in [0, \infty)$ , and  $M - h(tx) \geq 0$ ,  $1 - g(tx) \geq 0$ , a straightforward calculation gives us that  $\frac{d}{dt}G(tx) \leq 0$  for all  $t \geq 0$ ).

Next, for each  $z \in S_X$ ,  $H = H_z$ , let us construct a "ball with a flat bump directed by  $z$  and  $H$ ", which we will denote  $U_z$ . Let us write  $X = H \oplus [z]$  and define

$$U_z = \{x + tz \in X : 0 \leq t \leq G_z(x), \|x\| \leq M\} \\ \cup \{x + tz \in X : t \leq 0, \|x + tz\| \leq M\}.$$

Looking at the definition of  $U_z$  and taking into account that the functions  $t \mapsto G(tx)$ ,  $t \in [0, \infty)$ , are all non-increasing, it is easy to check that every ray coming from the origin meets the boundary  $\partial u_z = \{x + tz \in X : t = G_z(x), \|x\| \leq M\} \cup \{x + tz \in X : t \leq 0, \|x + tz\| = M\}$  once and only once. Hence  $U_z$  is a starlike body, and it is  $C^1$  smooth because it is locally the graph of a  $C^1$  smooth function whose tangent hyperplanes do not contain any ray coming from the origin (this is a standard application of the implicit function theorem). Let  $q_{U_z}$  be the Minkowski functional of  $U_z$ .

Now we are going to put one of those crazy bumps from Theorem 1.3 on the flat part of our ball with a bump, in order to obtain what we will call a "ball with a weird bump directed by  $z$  and  $H$ ", and we will denote  $W_z$ . Take a  $10\varepsilon^{-2}$ -Lipschitz  $C^1$  smooth function  $f_\varepsilon = f_{H,\varepsilon}: H \rightarrow [0, 8\varepsilon]$  whose support is in  $B_H(0, \varepsilon)$  and such that  $2B_{H^*} \subseteq f'_\varepsilon(B_H(0, \varepsilon))$ , and define

$$W_z = \{x + tz \in X : 0 \leq t \leq M + f_\varepsilon(x), \|x\| \leq \varepsilon\} \\ \cup \{x + tz \in X : q_{U_z}(x + tz) \leq 1\}.$$

We claim that  $W_z$  is a  $C^1$  smooth starlike body which has the property that for every hyperplane  $H$  not containing any vector of the cone  $\{x + tz \in X : |t| > 2\|x\|\}$  there exists  $y \in \partial W_z \cap \{x + tz \in X : t > 0, \|x\| \leq \varepsilon\}$  such that  $y + H$  is tangent to  $\partial W_z$  at  $y$ .

We will first see that  $W_z$  is a  $C^1$  smooth starlike body. Let us take  $y \in \partial W_z$  and check that the ray  $\{\lambda y : \lambda \geq 0\}$  meets the boundary  $\partial W_z$  just

once, exactly at the point  $y$ . Write  $y = x_y + t_y z \in X = H \oplus [z]$ . If  $\|x_y\| \geq \varepsilon$  or  $t_y < 0$  this is clear because the sets  $U_z$  and  $W_z$  coincide outside the half-cylinder  $\{x + tz \in X : \|x\| \leq \varepsilon, t > 0\}$  (and we already know that  $U_z$  is a starlike body). If  $\|x_y\| \leq \varepsilon$  and  $t_y > 0$  then we have

$$M \leq \|y\| = \|x_y + t_y z\| \leq \|x_y\| + t_y \leq \varepsilon + t_y,$$

so that  $t_y \geq M - \varepsilon$  and therefore

$$\frac{t_y}{\|x_y\|} \geq \frac{t_y}{\varepsilon} \geq \frac{M - \varepsilon}{\varepsilon} \geq 20\varepsilon^{-3}.$$

Assume that there were another point  $y' = \lambda y$ ,  $\lambda \neq 1$ ,  $\lambda > 0$ , such that  $y' \in \partial W_z$ ; then we would have

$$\frac{t_{y'} - t_y}{\|x_{y'}\| - \|x_y\|} = \frac{(\lambda - 1) t_y}{(\lambda - 1) \|x_y\|} \geq 20\varepsilon^{-3} > 10\varepsilon^{-2},$$

but in fact

$$\begin{aligned} \frac{t_{y'} - t_y}{\|x_{y'}\| - \|x_y\|} &= \frac{M + f_\varepsilon(x_{y'}) - (M + f_\varepsilon(x_y))}{\|x_{y'}\| - \|x_y\|} \\ &= \frac{f_\varepsilon(x_{y'}) - f_\varepsilon(x_y)}{(\lambda - 1) \|x_y\|} \\ &= \frac{|f_\varepsilon(x_{y'}) - f_\varepsilon(x_y)|}{\|x_{y'} - x_y\|} \leq 10\varepsilon^{-2} \end{aligned}$$

because  $f_\varepsilon$  is  $10\varepsilon^{-2}$ -Lipschitz, a contradiction. Therefore  $W_z$  is a bounded starlike body, and it is clear that it is  $C^1$  smooth because it is locally the graph of a  $C^1$  smooth function whose tangent hyperplanes do not contain any ray emanating from the origin (this property is again guaranteed by the fact that  $f_\varepsilon$  is  $10\varepsilon^{-2}$ -Lipschitz).

Let us now see that  $W_z$  has the property that for every hyperplane  $F$  not containing any vector of the cone  $\{x + tz \in X = H \oplus [z] : |t| > 2 \|x\|\}$  there exists  $y \in \partial W_z \cap \{x + tz \in X : t > 0, \|x\| \leq \varepsilon\}$  such that  $y + F$  is tangent to  $\partial W_z$  at  $y$ . Taking into account the construction of  $W_z$  this is the same as saying that  $\{T \in X^* \mid T(x + tz) \neq 0 \text{ for all } x + tz \in X \text{ with } |t| > 2 \|x\|\}$  is contained in the set  $\{T \in X^* \mid \text{Ker } T \text{ is tangent to the graph of } t = f_\varepsilon(x)\}$ . Let us check this inclusion.

Take  $T \in X^*$  such that  $T(x + tz) \neq 0$  for all  $x + tz \in X$  with  $|t| > 2 \|x\|$ . Then  $T(x)/T(z) \leq 2$  for all  $x \in H$  with  $\|x\| \leq 1$  (indeed, either  $T(z) > 0$  or  $T(z) < 0$ ; suppose for instance  $T(z) > 0$ ; then, for  $\|x\| < 1 = -t/2$  we have  $T(x - 2z) \neq 0$  and  $T(-2z) < 0$ ; since the set  $\{x \in H : \|x\| < 1\}$  is connected

and  $T$  is continuous this implies that  $T(x) - 2T(z) = T(x - 2z) < 0$  and therefore  $T(x)/T(z) \leq 2$  for all  $x \in H$  with  $\|x\| < 1$ ). If we define  $S \in H^*$  by  $S(x) = T(x)/T(z)$  for all  $x \in H$  this means that  $\|S\|_{H^*} \leq 2$ . Now, since  $2B_{H^*} \subseteq f'_\varepsilon(B_H(0, \varepsilon))$ , there must be some  $x_0 \in B_H(0, \varepsilon)$  such that  $f'_\varepsilon(x_0) = -S$ . Then we have that  $T(x + tz) = T(x) + tT(z) = T(z)[S(x) + t] = T(z)[-f'_\varepsilon(x_0)(x) + t]$ , and since  $T(z) \neq 0$  this means that  $T(x + tz) = 0$  if and only if  $t = f'_\varepsilon(x_0)(x)$ , that is  $\text{Ker } T$  is tangent to the graph of  $t = f_\varepsilon(x)$  at the point  $x_0 + f'_\varepsilon(x_0)z$  in  $X = H \oplus [z]$ .

So far we have constructed, for every  $z \in S_X$  and  $H = \text{Ker } d \|\cdot\|(z)$ , a  $C^1$  smooth starlike body  $W_z$  which contains  $B_M$  and is contained in  $2B_M$ , and has the nice property that all the hyperplanes not containing any vector of  $\{x + tz \in X : |t| > 2 \|x\|\}$  are tangent to  $\partial W_z$  at some point of  $\partial W_z \cap \{x + tz \in X : t > 0, \|x\| \leq \varepsilon\}$ . Next we are going to make use of this fact in order to construct a bounded  $C^1$  smooth starlike body  $A$  with the property that every hyperplane of  $X$  is tangent to the body  $A$  at some point of  $\partial A$ .

Let  $\{z_\alpha\}_{\alpha \in I}$  be a  $10\varepsilon/M$ -net on the unit sphere  $S_X$  (so that  $\{Mz_\alpha\}_{\alpha \in I}$  is a  $10\varepsilon$ -net on the sphere  $S_M$ ), and for each  $\alpha$  let  $H_\alpha = \text{Ker } d \|\cdot\|(z_\alpha)$  and consider the ball with a weird bump directed by  $z_\alpha$  and  $H_\alpha$ ,  $W_\alpha = W_{z_\alpha}$ . Let  $\psi_\alpha$  be the Minkowski functional of  $W_\alpha$ . Now consider the union of all these bodies,

$$A = \bigcup_{\alpha \in I} W_\alpha.$$

Let us see that  $A$  is a bounded  $C^1$  smooth starlike body. Define  $\psi: X \rightarrow [0, \infty)$  by

$$\psi(x) = \inf_{\alpha \in I} \psi_\alpha(x).$$

It is obvious that  $\psi$  is positive homogeneous, and, as in the proof of the first part of this theorem, it is not difficult to check that for every  $z \in S_X$  there exist some  $r > 0$  and some  $\alpha \in I$  such that  $\psi(x) = \psi_\alpha(x)$  for all  $x \in S_X$  with  $\|x - z\| < r$ ; since every functional  $\psi_\alpha$  is  $C^1$  smooth away from the origin, this implies that  $\psi$  is  $C^1$  smooth in  $X \setminus \{0\}$ . Therefore  $\{x \in X \mid \psi(x) \leq 1\}$  is a  $C^1$  smooth starlike body. Taking into account the construction of  $A$  and the definition of  $\psi$ , it is easily checked, as in the preceding proof, that  $A = \{x \in X \mid \psi(x) \leq 1\}$ . Therefore  $A$  is a  $C^1$  smooth starlike body, and  $A$  is bounded (as every  $W_\alpha$  is contained in the ball  $B(0, 2M)$ ).

It only remains to prove that for every hyperplane  $H$  of  $X$  there is some  $y \in \partial A$  such that  $y + H$  is tangent to  $\partial A$  at  $y$ . From the construction of  $A$  it is clear that for each  $\alpha$  the bodies  $W_\alpha$  and  $A$  are the same inside the half-cylinder  $C_\alpha = \{x + tz_\alpha \in H_\alpha \oplus [z_\alpha] : \|x\| \leq \varepsilon, t > 0\}$ . Then, all the hyperplanes of  $X$  not containing any vector of  $\{x + tz_\alpha \in H_\alpha \oplus [z_\alpha] : |t| > 2 \|x\|\}$  are

tangent to  $\partial W_\alpha$ , and therefore tangent to  $\partial A$  too, at some point of  $\partial W_\alpha \cap C_\alpha = \partial A \cap C_\alpha$ . This means that the set

$$\bigcup_{\alpha \in I} \{T \in X^* \mid T(x + tz_\alpha) \neq 0 \text{ for all } x + tz_\alpha \in H_\alpha \oplus [z_\alpha] \text{ with } |t| > 2 \|x\|\}$$

is contained in

$$\{T \in X^* \mid y + \text{Ker } T \text{ is tangent to } \partial A \text{ at some point } y \in \partial A\}.$$

Therefore, in order to conclude the proof we only have to check that

$$X^* \setminus \{0\} = \bigcup_{\alpha \in I} \{T \in X^* \mid T(x + tz_\alpha) \neq 0 \text{ for all } x + tz_\alpha \in H_\alpha \oplus [z_\alpha] \text{ with } |t| > 2 \|x\|\}.$$

Pick any  $T \in X^*$ ,  $T \neq 0$ ; we may assume  $\|T\| = 1$ . Choose  $z \in X$ ,  $\|z\| = 1$ , such that  $T(z) > 1 - \varepsilon$ , and take  $z_\alpha$  such that  $\|z - z_\alpha\| \leq 10\varepsilon/M$  (this is possible because  $\{z_\alpha\}_{\alpha \in I}$  is a maximal collection of  $10\varepsilon/M$ -separated points of  $S_X$ ). We have that  $|T(z_\alpha) - T(z)| \leq \|z - z_\alpha\| \leq 10\varepsilon/M < 10\varepsilon$  and hence  $T(z_\alpha) \geq T(z) - 10\varepsilon > 1 - 11\varepsilon > 0$ . Then, for every  $x = tz_\alpha \in H_\alpha \oplus [z_\alpha]$  with  $t > 2 \|x\| > 0$  we will have

$$\begin{aligned} T(x + tz_\alpha) &= T(x) + tT(z_\alpha) > T(x) + t(1 - 11\varepsilon) \geq -\|x\| + t(1 - 11\varepsilon) \\ &> -\|x\| + 2\|x\|(1 - 11\varepsilon) = (1 - 22\varepsilon)\|x\| \geq (1 - \frac{22}{32})\|x\| > 0; \end{aligned}$$

and in a similar way one can check that  $T(x + tz_\alpha) < 0$  for all  $x + tz_\alpha \in H_\alpha \oplus [z_\alpha]$  with  $t < -2\|x\| < 0$ . Therefore  $T(x + tz_\alpha) \neq 0$  for all  $x + tz_\alpha \in H_\alpha \oplus [z_\alpha]$  with  $|t| > 2\|x\|$ . This concludes the proof of 1.2.

*Proofs of 1.4 and 1.5.* We will make use of the following result, due to S. A. Shkarin (see [14]).

**THEOREM 2.2 (Shkarin).** *There is a  $C^\infty$  diffeomorphism  $\varphi$  from  $\ell_2$  onto  $\ell_2 \setminus \{0\}$  such that all the derivatives  $\varphi^{(n)}$  are uniformly continuous on  $\ell_2$ , and  $\varphi(x) = x$  for  $\|x\| \geq 1$ .*

Since uniformly continuous functions are bounded on bounded sets, it is obvious that this deleting diffeomorphism  $\varphi$  is Lipschitz. Let  $M_1 > 0$  be its Lipschitz constant. For  $0 < \varepsilon < 1/2$ , let us define diffeomorphisms  $\varphi_\varepsilon: \ell_2 \rightarrow \ell_2 \setminus \{0\}$ ,  $\varphi_\varepsilon(x) = \varepsilon\varphi(x/\varepsilon)$ . Clearly,  $\varphi_\varepsilon(x) = x$  whenever  $\|x\| \geq \varepsilon$ , and  $\|\varphi'_\varepsilon\|_\infty = \|\varphi'\|_\infty$ , so that all these  $\varphi_\varepsilon$  have Lipschitz constant  $M_1$  (not depending on  $\varepsilon$ ).

We can identify  $\ell_2 = \ell_2 \oplus \mathbb{R} = \{(x, t): x \in \ell_2, t \in \mathbb{R}\}$ , with the norm  $\|(x, t)\| = (\|x\|^2 + t^2)^{1/2}$ . We have to construct a  $C^\infty$  smooth bounded starlike body

$A$  in this space with the property that the hyperplanes  $H_\alpha = \{(x, t): t = \alpha\}$  are not tangent to  $A$  at any point  $y = (x, \alpha) \in \partial A$ . Let us consider the function  $G(x) = (1 - \|x\|^2)^{1/2}$ , defined for  $\|x\| \leq 1$ , which is Lipschitz when restricted to  $\|x\| \leq 1/2$ , with Lipschitz constant, say,  $M_2$ . Next define

$$F_\varepsilon(x) = G(\varphi_\varepsilon(x)) = (1 - \|\varphi_\varepsilon(x)\|^2)^{1/2}$$

for  $\|x\| \leq 1$ . Clearly,  $F_\varepsilon$  is Lipschitz with constant less than or equal to  $M_1 M_2$  on the set  $\{x: \|x\| \leq \varepsilon\}$ , and  $F_\varepsilon(x) = (1 - \|x\|^2)^{1/2}$  for  $\|x\| \geq \varepsilon$ . Take  $M$  large enough so that  $M > M_1 M_2$  and  $(1 - (1/2M)^2)^{1/2} \geq 1/2$ . Fix  $\varepsilon$  with  $0 < \varepsilon < \frac{1}{2M}$ . Then we have  $1 \geq F_\varepsilon(x) \geq 1/2$  whenever  $\|x\| \leq \varepsilon$ , and  $F_\varepsilon$  is  $M$ -Lipschitz on this set. As in the proof of the smooth part of Theorem 1.2, it is easily checked that these conditions on  $F_\varepsilon$  imply that every ray emanating from the origin intersects the graph of  $t = F_\varepsilon(x)$  at exactly one point, and the same argument applies to the function  $t = -F_\varepsilon(x)$ . Then it is clear that the set

$$A = \{(x, t): t^2 + \|\varphi_\varepsilon(x)\|^2 \leq 1\}$$

is a bounded starlike body whose boundary is  $\partial A = \{(x, t): t^2 + \|\varphi_\varepsilon(x)\|^2 = 1\}$ . Moreover,  $\partial A$  is  $C^\infty$  smooth because it is locally the graph of  $C^\infty$  smooth functions whose tangent hyperplanes do not contain any ray coming from the origin.

Finally, let us check that the hyperplanes  $H_\alpha = \{(x, t): t = \alpha\}$  are not tangent to  $A$  at any point  $y = (x, \alpha) \in \partial A$ . Bearing in mind the construction of  $A$ , this comes down to showing that  $F'_\varepsilon(x) \neq 0$  whenever  $\|x\| \leq \varepsilon$ , which happens because the function  $t = F_\varepsilon(x)$  does not satisfy Rolle's theorem. Indeed,

$$F'_\varepsilon(x)(h) = -\frac{\langle \varphi_\varepsilon(x), \varphi'_\varepsilon(x)(h) \rangle}{(1 - \|\varphi_\varepsilon(x)\|^2)^{1/2}} \neq 0$$

for some  $h$ , because  $\varphi'_\varepsilon(x)$  is a linear isomorphism and  $\varphi_\varepsilon(x) \neq 0$ . This proves 1.4.

Corollary 1.5 is now easy to deduce: it suffices to compose the Minkowski functional  $q_A$  of this starlike body  $A$  with a smooth real bump function  $a$  such that  $a(t) = 1$  for small values of  $|t|$ , and apply the chain rule.

## FINAL REMARKS

Let us finish this paper with some remarks concerning the following question. What is the minimal size of the cone generated by the range of

the derivative of a bump function? For the time being only partial results are available.

On the one hand, if  $X = c_0$  the size of  $\mathcal{C}(b)$  can be really small. Indeed, as a consequence of P. Hájek's work [10] on smooth functions on  $c_0$  we know that if  $b$  is  $C^1$  smooth with a locally uniformly continuous derivative (note that there are bump functions with this property in  $c_0$ ), then  $b'(X)$  is contained in a countable union of compact sets in  $X^*$  (and in particular  $\mathcal{C}(b)$  has empty interior).

On the other hand, if  $X$  is nonreflexive and has a separable dual, there are bumps  $b$  on  $X$  so that  $\mathcal{C}(b)$  has empty interior. Indeed, as a straightforward consequence of Proposition 3.3 of [12] (see also Lemma 11 in [8]) we have that for our nonreflexive Banach space  $X$ , and for every equivalent Fréchet differentiable norm  $\|\cdot\|$  in  $X$ , the set  $NA_{\|\cdot\|} = \{T \in X^* : T \text{ attains its norm}\}$  must have empty interior in  $X^*$ . Besides, it is well known that every space with separable dual has an equivalent Fréchet smooth norm. Therefore, by combining these two results we get an equivalent  $C^1$  smooth norm  $\|\cdot\|$  on  $X$  such that  $NA_{\|\cdot\|}$  has empty interior. Now, by taking into account that for a differentiable norm the cone generated by the range of its derivative coincides with the set of norm attaining functionals, we can deduce that the cone  $\{\lambda d\|\cdot\|(x) : x \neq 0, \lambda \geq 0\}$  has empty interior in  $X^*$ . By composing this norm with a suitable real function we then obtain a  $C^1$  smooth bump function  $f$  whose support is precisely  $B_X$  and with the property that the cone generated by  $f'(X)$  has empty interior in  $X^*$ .

In the reflexive case, however, the problem is far from being settled. In fact, the cone  $\mathcal{C}(b)$  cannot be very small, since it is going to be a residual subset of the dual  $X^*$  (this is a straightforward consequence of Stegall's variational principle: for every Banach space  $X$  having the Radon-Nikodym Property (RNP) it is not difficult to see that  $\mathcal{C}(b)$  is a residual set in  $X^*$ ). Therefore, for infinite-dimensional reflexive Banach spaces  $X$  one can hardly expect a better answer to the above question than the following one: there are smooth bumps  $b$  on  $X$  such that the cone  $\mathcal{C}(b)$  has empty interior in  $X^*$ . In the case of the Hilbert space  $\ell_2$  the existence of such bumps has been shown very recently (see [4]), but in the general reflexive case the problem remains open.

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## REFERENCES

1. D. Azagra, Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces, *Studia Math.* **125** (1997), 179–186.
2. D. Azagra and T. Dobrowolski, Smooth negligibility of compact sets in infinite-dimensional Banach spaces, with applications, *Math. Ann.* **312** (1998), 445–463.
3. D. Azagra, J. Gómez, and J. A. Jaramillo, Rolle's theorem and negligibility of points in infinite-dimensional Banach spaces, *J. Math. Anal. Appl.* **213** (1997), 487–495.
4. D. Azagra and M. Jiménez-Sevilla, Rolle's theorem is either false or trivial in infinite-dimensional Banach spaces, preprint, 2000.
5. D. Azagra, "Smooth Negligibility and Subdifferential Calculus in Banach Spaces, with Applications," doctoral dissertation, Universidad Complutense de Madrid, 1997.
6. S. M. Bates, On smooth, nonlinear surjections of Banach spaces, *Israel J. Math.* **100** (1997), 209–220.
7. H. Cartan, "Calcul différentiel," Hermann, Paris, 1967.
8. G. Debs, G. Godefroy, and J. Saint Raymond, Topological properties of the set of norm-attaining linear functions, *Canad. J. Math.* **47** (1995), 318–329.
9. R. Deville, G. Godefroy, and V. Zizler, "Smoothness and Renormings in Banach Spaces," Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 64, Pitman, London, 1993.
10. P. Hájek, Smooth functions on  $c_0$ , *Israel J. Math.* **104** (1998), 17–27.
11. R. C. James, Weakly compact sets, *Trans. Amer. Math. Soc.* **113** (1964), 129–140.
12. M. Jiménez-Sevilla and J. P. Moreno, A note on norm attaining functions, *Proc. Amer. Math. Soc.* **126** (1998), 1989–1997.
13. E. B. Leach and J. H. M. Whitfield, Differentiable functions and rough norms on Banach spaces, *Proc. Amer. Math. Soc.* (1972), 120–126.
14. S. A. Shkarin, On Rolle's theorem in infinite-dimensional Banach spaces, transl. from *Mat. Z.* **51** (1992), 128–136.