UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

H-principles for Holomorphic Partial Differential Relations

H-principios para Relaciones en Derivadas Parciales Holomorfas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Guillermo Sánchez Arellano

Directores

Luis Giraldo Suárez
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ICMAT

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Memoria para optar al grado de doctor presentada por

Guillermo Sánchez Arellano

Directores:

Luis Giraldo Suárez y Francisco Presas Mata

To my parents, sister and friends.

🥰

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[1] Como no podía ser de otra manera, he terminado dejando todo para el último momento 😅.

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[2]Sin embargo, en el punto de ocupar salones Cuzco🐕, Edu y Natalia ganan por goleada. ¡Muchas gracias por acogerme de esa manera!

Por último, aunque ya ha aparecido antes, quiero mencionar a una de las mejores personas que he conocido en mi vida y cuya amistad es, sin duda, lo mejor que me llevo de estos 5 años. Las matemáticas son una disciplina hiperespecializada que casi nadie puede compartir con alguien de su ámbito cercano. Por eso no hay cardinal transfinito que cuente la suerte que he tenido de que estuvieras ahí todo este tiempo escuchando, ayudando, comprendiendo, aconsejando, riendo, enseñando... Esta tesis te debe mucho, muchísimo Edu.

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🧑🏽‍🎓

Contents

Resumen

En esta tesis presentamos las realificaciones de una Relación en Derivadas Parciales Holomorfas arbitraria, es decir, de un subconjunto de un fibrado de jets de secciones locales holomorfas. El resultado principal determina que si las realificaciones de una Relación en Derivadas Parciales Holomorfas satisface un h–principio relativo a dominio, entonces es posible deformar cualquier solución formal de la relación para obtener una solución holónoma en un entorno de un esqueleto Lagrangiano de la variedad Stein. Si esta variedad es una superficie de Riemann abierta o tiene tipo finito, ese esqueleto puede ser escogido de manera independiente a la solución formal de partida. Esto tiene como consecuencia la existencia de h–principios locales en torno a ese esqueleto prefijado. Estos resultados amplían los obtenidos por F. Forstnerič y M. Slapar para submersiones e inmersiones holomorfas y para estructuras de contacto complejas, obteniendo h–principios locales para estructuras de contacto par complejas, Engel holomorfas o localmente conformemente simplécticas-complejas.

Abstract

In this Thesis we introduce the notion of the realifications of an arbitrary *Holomorphic Partial Differential Relation*, i.e. a subset of a jet bundle of local holomorphic sections. Our main result states that if any realification of an open Holomorphic Partial Differential Relation over a Stein manifold satisfies a relative to domain h–principle, then it is possible to deform any formal solution into one that is holonomic in a neighbourhood of a Lagrangian skeleton of the Stein manifold. If the Stein manifold is an open Riemann surface or it has finite type, then that skeleton is independent of the formal solution. This yields the existence of local h–principles over that skeleton. These results broaden those obtained by F. Forstnerič and M. Slapar on holomorphic immersions, submersions and complex contact structures for instance to holomorphic local h–principles for complex even contact, holomorphic Engel or complex locally conformal symplectic structures.

Chapter 1

Introduction

1.1 Overview of the h-principle

A key feature of the Differential Geometry of the last 70 years has been to find the limits where a geometric problem begins to be directed by the rules of Differential Topology. Though, several results in the work of H. Whitney hinted in that direction ([Whi36], [Whi44]), the first spectacular application of this approach was in the work of J. Nash about isometric embeddings, in particular he was able to show that there were no obstructions (apart from the smooth ones, provided by the Whitney embedding theorems) to C^1–isometrically embed a Riemannian manifold onto standard $\mathbb{R}^N$ [Nas54]. This was kind of unexpected and it is in sharp contrast with the obstructions found by Riemann for the case of C^∞ embeddings.

The theory began to take a familiar shape with the work of S. Smale and his student M. Hirsch in which they manage to reduce the whole theory of immersions to a set of obstruction theoretic topological invariants [Hir59], [Sma58], [Sma59]. Moreover, the techniques developed to study this problem were later on captured in a general method to compute the homotopy type of several spaces of solutions of differential relations: what is known in modern language as the holonomic approximation Lemma [EM02].

At that point, it became clear that there was a general theory behind the set of particular problems that the school of S. Smale were solving [Phi67], [Phi68],[Phi69], [Phi70], [Phi74].

This was brilliantly captured by M. Gromov by introducing the *homotopy principle* that is general guiding light and can be expressed as

> A geometric problem is declared to satisfy an *h–principle* if every formal solution is homotopic (through formal solutions) to an actual solution to the problem.

This can be stated more precisely if we assume that the space of geometric objects solving the problem can be understood as the subset of sections of some jet bundle $E \to B$ that satisfy some conditions written in terms of their partial derivatives until some order, say r. In this case, these differential conditions define a subset $\mathcal{R}$ of the r–jet bundle of sections of E called *Partial Differential Relation* or *PDR*. In this setting, we say that the problem satisfy an h–principle if each formal solution, i.e. each section of $\mathcal{R} \to B$, is homotopic (through formal solutions) to the r–jet of a section of E that solves the problem.

As sections of a jet bundle, formal solutions may be understood as if they were solutions to the problem but with their partial derivatives being decoupled from their actual derivatives. If a formal solution is *holonomic* (i.e. if its derivatives are coupled), then it will correspond to the r–jet extension of a genuine solution of the original problem.

In some situations one may be interested in finding parametric families of solutions. One will say that the h–principle is *complete* or *full* whenever the inclusion of the space of holonomic solutions into the space of formal solutions is a (weak) homotopy equivalence. Therefore there is a full h–principle whenever the set of solutions of the geometric problem is homotopically equivalent to the set of formal solutions, i.e. solutions in which we do not care about making sure that the derivatives are coupled with the actual derivatives of the base section.

Parametric and full h–principles are indeed two of many possible different flavours of the h–principle. Each condition that one may impose to the solutions of the original problem will lead to a different kind of h–principle. For instance, one may be interested in solutions that coincide for some given solution in a subset of B, in this case the h–principle that we are going to be interested to find is called a *relative to domain* h–principle. Other possibility is looking for solutions just in a subset of the base space, this will lead to the pursue of what are called *local* h–principles.

Let $\mathcal{R}$ be a Partial Differential Relation. Solving $\mathcal{R}$, i.e. finding holonomic solutions of $\mathcal{R}$, may be really difficult. Indeed, solving a partial differential equation is a problem of this kind since any PDE determines a PDR. Nevertheless sometimes it is not necessary to know advanced calculus to prove the lack of existence of solutions of some PDR's as we can check in the following trivial examples

Example 1.1.1.

- Easy problem 1. Does it exist a smooth function $f : [0,1] \to [0,1]$ such that $f^2 + (f')^2 + 1 < 0$?
- Easy problem 2. Does it exist a smooth function $f : [0,1] \to [0,1]$ such that $f(0) = f(1)$, $f'(t) \neq 0$ and $f'(0) > 0 > f'(1)$ for every $t \in [0,1]$?
- Easy problem 3. Let $\varepsilon > 0$. Does it always exist a smooth function $f : [0,1] \to [0,1]$ such that $f(0) = 0, f(1) = 1$ and $|f'(t)| < \varepsilon$ for every $t \in [0,1]$?

Probably the easier way to solve the first easy problem is to notice that the PDR that it defines is the empty set. Indeed, for any two real numbers x (that can be the value of f at $t_0 \in [0,1]$) and y (that can be the value of $f'(t_0)$), we have that $x^2 + y^2 + 1 > 0$. The Partial Differential Relation corresponding to the second easy problem is not empty. Indeed its fiber is $\mathbb{R} \setminus \{0\}$, but since the derivative has to be continuous, the existence of a solution would contradict the Intermediate value Theorem. A solution to the third problem would contradict the Mean value Theorem for every $\varepsilon < 1/2$. $\bowtie$

A necessary condition for $\mathcal{R}$ to be solved is, obviously, the existence of a formal solution. Indeed we found that the first two easy problems cannot be solved even formally. On the other hand, the third one does have formal solutions so their existence is not a sufficient condition to solve $\mathcal{R}$ as it could be expected. Indeed finding formal solutions is a problem of Algebraic Topology and Homotopy Theory, so it constitutes a priori an important simplification of the problem, so looking for h–principles may seem to be too optimistic at a first glance.

Nevertheless, the surprising results that were mentioned above lead Gromov to propose finding as many instances as possible of relations satisfying that principle. Gromov's insight was really deep since the principle has been satisfied by a huge number of relations.

Indeed there are some situations in which the lack of h–principles has required the development of really sophisticated techniques to be disproved, so it is not always clear if there is or is not an h–principle in contrast with the easy examples above. This dichotomy has been an active field of research in important fields of Mathematics like Symplectic and Contact Topology, where it is not always clear if a problem is *flexible* or *rigid*, i.e. where there is or there is not h–principles.

The pursue of h–principles has originated several different and fruitful techniques to find them. The interested reader can find some examples among the following list.

- *Convex integration* [Gro73], [Gro86]
- *Removal of singularities* [GE71].
- *Holonomic approximation* [EM02], that is a version of other Gromov's technique known as *continuous sheaves* (or *covering homotopy*) [Gro86].
- Lohkamp's theory of negative Ricci curvature [Loh95].
- Asymptotically holomorphic theory [Don96].
- Wrinkling maps [EM09], [EM00] and [EM98].
- Existence and classification of overtwisted contact structures in all dimensions [BEM15].
- Loose Engel structures [CdPP20].

The strategy is always the same: isolate a particular, or not so particular, class of Differential Relations, and then prove that all of them fulfil the h-principle using some technique like the ones mentioned above. The aim of this work is to follow this strategy for Differential Relations in holomorphic jet spaces of holomorphic fiber bundles.

1.2 *H*-principles in the holomorphic category

Note that the holonomic solutions will be holomorphic in this case. It is well known that the identity principle, and therefore the absence of holomorphic partitions of unity, leads to a more rigid structure for complex manifolds than the one for smooth manifolds. Therefore one should not expect this to work at a first sight.

Nevertheless, one of the first instances of h–principles were born in the holomorphic category even before of the existence of the notion of h–principle. It is called Oka–principle and it deals with the Partial Differential Relation provided by the Cauchy-Riemann equations. The holonomic solutions of this relation will be precisely the holomorphic functions.

The Oka-principle for Stein manifolds was originally stated by Pierre Serre in 1951 as: *Analytic problems of Stein manifolds admit analytic solutions if there are no topological obstructions* (see page ix in [For17]). This come to say that a formal solution of the Cauchy-Riemann equations yields to the existence of a holonomic one for Stein manifolds. This principle is indeed stated in [Gro86] as: *Every continuous section of a holomorphic fiber bundle is homotopic to a holomorphic one.*

The complex manifolds for which this principle was originally stated are Stein manifolds so they seem to be a good starting point to find h-principles for Holomorphic Partial Differential Relations. These complex manifolds are characterized by the existence of (strongly) plurisubharmonic exhausting functions. This functions may be taken to be of Morse type and they do not have critical points of order greater than the complex dimension of the manifold. Therefore, the Morse complex of such a function is a half dimensional CW-complex with the homotopy type of the Stein manifold.

This Morse complex is called the Lagrangian skeleton of the Stein manifold. This name comes from the fact that the stable disks of the skeleton are isotropic submanifolds of a natural symplectic structure of the Stein manifold provided by the plurisubharmonic function. Moreover the intersection of the stable disks with the level sets of the plurisubharmonic function are also isotropic submanifolds for the contact distribution of complex tangencies of the Stein manifold. This facts show some of the strong connections between Stein manifolds and Symplectic and Contact Topology that are explored in [CE12].

One step forward in the way to obtain this h–principles was recently given by Forstnerič in [For20]. There it is proved that given a formal holomorphic contact form on a Stein manifold B, there exists a homotopy of formal holomorphic contact forms that starts on the original solution and terminates on a form that is a holonomic contact form inside a Stein domain $\Omega \subset B$ diffeotopic to B. Other similar results where proved by Forstnerič and Slappar in [FS07] for holomorphic immersions and submersions. Both results were obtained using convex integration and holomorphic approximation.

1.3 Contents of the thesis

In this thesis we broaden Forstnerič's and Slappar's results to general differential relations in holomorphic jet spaces of holomorphic vector bundles (Theorem 4.2.6) or even in jet spaces of general holomorphic fiber bundles (Remark 3.1.3). We obtain our results introducing a new approach through the realifications of Holomorphic Partial Differential Relations (Definition 4.1.2) that allows to obtain homotopies for a more general kind of relations, not just the ones studied using convex integration. More precisely, if that realifications satisfy an h–principle obtained by any technique, then we obtain automatically an h–principle for the original relation.

The key observation that motivate the definition of the realification of a Holomorphic Partial Differential Relation is the fact that every complex linear (or multilinear) form from a complex vector space V to $\mathbb{C}$ is completely determined by its restriction to a totally real subspace generating V. Indeed, a linear (or multilinear) map f from a real vector space W to $\mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R} \cong \mathbb{R}^2$ determines a complex linear (multilinear) form $\widetilde{f}$ from the complexification of W, $W \oplus iW$, to $\mathbb{C}$ such that $\widetilde{f}_{|W} = f$. the complex form $\widetilde{f}$ is given by

$$\widetilde{f}(iv) = if(v)$$

for every $v \in W$.

Since the holomorphic derivatives of a section of a holomorphic fiber bundle can be identified with complex linear maps, they must be determined by their restriction to totally real tangent subspaces of maximal dimension. Therefore, every Holomorphic Partial Differential Relation should be determined by their restriction to totally real submanifolds. This restrictions define what we have named their *realifications*.

Given a Holomorphic Partial Differential Relation $\mathcal{R}$ in the jet bundle of holomorphic sections of a holomorphic bundle $E \to B$, the realifications of $\mathcal{R}$ lie in the jet bundle of smooth sections of the restriction of E to a totally real submanifold of maximal dimension $M \subset B$. As we mentioned, we use that realifications as a bridge from the holomorphic setting to the smooth one, so that we can make use of well known techniques developed for the smooth category to find h–principles. In order to go back to the holomorphic category we now need to approximate the smooth homotopies of formal solutions over M by homotopies of holomorphic formal solutions defined over an open neighbourhood of M.

In order to do so we have adapted a Mergelyan approximation Theorem for the case of parametric sections of vector bundles (Theorem 3.1.1). We use that Theorem to obtain homotopies satisfying the desired properties in a Stein domain containing the totally real submanifold. This allows one to proceed inductively over the descending disks of a Morse type strongly plurisubharmonic function that forms a Lagrangian skeleton of B to prove Theorems 4.2.6 and 4.2.8.

At the end of this process we obtain a homotopy of formal solutions that finishes in a holonomic solution over a Stein neighbourhood of a Lagrangian skeleton that is diffeotopic to B. A priori, this skeleton depends on the initial formal data, but if the Stein manifold is an open Riemann surface or it has finite type (i.e. the strongly plurisubharmonic function defining the skeleton has a finite number of critical points), then that skeleton can be chosen to be independent of the formal solution. This yields the existence of local h–principles over that skeleton. This is the content of Theorems 4.2.1, 4.2.4 and 4.2.9.

The main conclusion of the previous Theorems is that, in order to prove local h–principles for Holomorphic Partial Differential Relations near a totally real submanifold (that may be stratified) of a complex manifold, it is only needed to study its realifications under the point of view of smooth manifolds. If you are able to stablish a relative to domain h–principle for them, then you will automatically obtain the local h–principle for the former relation. Let us emphasize that this approach works independently of the technique that is used to obtain the h–principles of the realifications.

Some examples of the previously mentioned Holomorphic Partial Differential Relations satisfying the local h–principles are that we have called *Thick Holomorphic Relations* (THRs, see Definition 4.1.5). THRs are defined to have ample realifications. It turns out that the ampleness condition is, in some sense, easier to be satisfied than in the smooth setting. Indeed, the complex analogues of symplectic and contact forms yield Thick Holomorphic Relations. We prove also a local h–principle in the setting of complex Engel geometry finding some suitable THRs. These are Theorems 4.4.8, 4.4.4, 4.4.7.

We have also come up recently with a new approach to prove local h–principles for HPDRs. They come from adapting the holonomic approximation Theorem 5.1.1(see [EM02] chapter 3) to totally real submanifolds and sections of holomorphic jet bundles. This is Lemma 5.1.11. With it we are able to find local (and some global) h–principles for locally Aut–invariant HPDRs without the requirement of studying their realifications. Nevertheless, there is still some improvements to be done in this direction so it is still work in progress.

1.4 Content guide

The original results of this document are based on the following preprint

[GSA23] Luis Giraldo and Guillermo Sánchez-Arellano. Local h-principles for Partial Holomorphic Relations, 2023.

Chapter 2. Preliminaries

The purpose of this Chapter is to introduce the basic notions of the different areas that are needed to prove the original results of this manuscript. We will settle down here the notation and definitions and provide some important results in the areas of fiber bundles, h–principles and Stein manifolds. The aim is to obtain a document that is as self-contained as possible.

During this thesis we are going to be interlinking both smooth and complex manifolds. Indeed the main consequence of the results provided in this work is that you only need to obtain h–principles for Partial Differential Relations (the realifications) to prove h–principles of Holomorphic Partial Differential Relations. This is the reason why we will present the smooth and holomorphic basic notions at the same time during this chapter in order to highlight their differences, similarities and connections.

Section 2.1. Smooth and holomorphic bundles

Our fundamental framework is going to be the theory of fiber bundles. Here we present the basic definitions and constructions that will appear later.

Besides all the Theorems in this thesis work for general holomorphic bundles, they are going to be presented for vector bundles. This is the reason why the first subsection of this part is about them. We will also introduce affine bundles, since it is the structure of jet bundles: the spaces where formal derivatives naturally live.

We will provide some basic facts about jet bundles and we will present the jet of local holomorphic sections both independently and as a Partial Differential Relation in the jet bundle of smooth sections. We will also provide here the definition of principal subspace, that will be useful for convex integration.

The last part of this Section will be the place where we define some particular Partial Differential and Holomorphic Relations. They are going to appear later in Section 4.4 since they are going to be the examples where we are going to apply the Theorems of Chapter 4. This will be the cases of

$$\mathcal{R}_{\text{Max-Rank}}, \mathcal{R}_{\text{LCSymp}}, \mathcal{R}_{\text{Cont}}, \mathcal{R}_{\text{ECont}} \text{ and } \mathcal{R}_{\text{Engel}}$$

that correspond with maps of maximal-rank differential (immersions and submersions), locally conformal symplectic forms, contact forms, even-contact forms and pairs of even-contact forms defining Engel structures respectively and their corresponding analogues in the holomorphic category

$$\mathcal{R}_{\mathcal{O}\text{Max-Rank}}, \mathcal{R}_{\mathcal{O}\,\text{LCSymp}}, \mathcal{R}_{\mathcal{O}\,\text{Cont}}, \mathcal{R}_{\mathcal{O}\,\text{ECont}} \text{ and } \mathcal{R}_{\mathcal{O}\,\text{Engel}}.$$

Section 2.2. *H*-principles

Here we introduce the notion of h–principle and all of their different kinds. Some of them are the same that appear in [EM02], but there are also new notions such like *weakly relativity to domain* 2.2.5, *holomorphic h–principle* 2.2.9 and *pseudo-holonomic solutions*. They are not necessary in the smooth setting, but they arise naturally for h–principles in the holomorphic category.

After that we provide a summary of convex integration. For them we will need to introduce the notions of open and ample Partial Differential Relations. Gromov's Theorem 2.2.15 states that all of the different kinds of h–principles are satisfied for this kind of relations.

One important class of open and ample differential relations are complements of thin singularities. We called them Thick Differential Relations and they intersect each principal subspace in the complement of a subspace of codimension ≥ 2. This is not the case for Partial Differential Relations like $\mathcal{R}_{\text{Max-Rank}}, \mathcal{R}_{\text{LCSymp}}$ and $\mathcal{R}_{\text{Cont}}$, their corresponding complements have indeed codimension 1. Their complex analogues intersect their corresponding principal subspaces also in the complement of 1-codimensional subspaces, but the naive fact that complex codimension 1 is indeed real codimension 2 gives that their realifications are indeed Thick Differential Relations, so Gromov's convex integration may be applied for them.

Section 2.3. Stein manifolds

This section will provide the three most common equivalent definitions of Stein manifolds that appear in the literature. We also present there the basic properties of this complex manifolds such like their Lagrangian skeletons and a Mergelyan approximation Theorem suitable to approximate functions near their stable manifolds.

In general we will need to perturb the skeleton to obtain homotopies of formal solutions ending in holonomic ones. To avoid that we introduce the notion of adapted skeleton. They are going to be useful to find local h–principles for Stein manifolds of finite type in the following Chapters.

Chapter 3. Holomorphic approximation

Our process to find local h–principles for Holomorphic Partial Differential Relations consists on going from a formal solution of a Holomorphic Partial Differential Relation $\mathcal{R}$, obtain a Partial Differential Relation associated to it (its realifications $\mathcal{R}_{\mathbb{R}}$), find a homotopies of formal solutions of $\mathcal{R}_{\mathbb{R}}$ and approximate them by homotpies of formal local solutions of $\mathcal{R}$. In summary, we follow the path from HPDRs to PDRs and back. This chapter is dedicated to the approximation results that will be needed to go back from PDRs to HPDRs.

Section 3.1. Parametric Mergelyan Theorem for sections

In this Section we prove a parametric Mergelyan type approximation Theorem for vector bundles 3.1.1 that also works for general holomorphic fiber bundles 3.1.3. This result will allow us to approximate homotopies of formal solutions by holomorphic ones near totally real submanifolds.

Section 3.2. Holomorphic approximation over stratified subsets

Here we use the Mergelyan type Theorem mentioned above to obtain approximations over totally real stratifications. This is going to be needed to prove the Theorems of the following chapter.

Chapter 4. Local *h*-principles for Holomorphic Partial Differential Relations

This Chapter constitutes the core of the thesis. Here we will state, prove and apply all of the original results of this work.

Section 4.1. Realifications and Thick Holomorphic Relations

Here we introduce the notion of the *realifications* of a Holomorphic Partial Differential Relation (see 4.1.2). They are going to be the key tool to go from HPDRs to PDRs. Thick Holomorphic Relations are going to be our main source of examples of Holomorphic Partial Differential Relations satisfying the hypothesis of our Theorems. They are the HPDRs such that their realifications are Thick Differential Relations. Since complex codimension 1 is indeed real codimension 2, THRs are going to appear more frequently in the theory of complex manifolds.

Section 4.2. Statements of the Theorems

After introducing all the notions that are needed, here we state the general cases of the original results of this work. They are a local h–principle for Holomorphic Partial Differential Relations over adapted skeletons of Stein manifolds of finite type and open Riemann surfaces whose realifications satisfy relative to domain h–principles. Moreover, for Stein manifolds of arbitrary type, we show that this condition over the realifications also imply that for every formal solutions there is a Stein subset homotopically equivalent to the Stein manifold where there is a homotopy of formal solutions connecting the former to a holonomic one.

If the original formal solutions are holomorphic, the results of Chapter 4 will allow to find homotopies of holomorphic formal solutions and holomorphic local h–principles for open Riemann surfaces and finite type Stein manifolds.

Section 4.3. Proofs of the Theorems

Here it is detailed the proofs of each one of the above mentioned Theorems.

Section 4.4. Applications

We conclude this Chapter finding some concrete applications of the Theorems. More precisely we provide local (holomorphic) h–principles for the Holomorphic Partial Differential Relations

$$\mathcal{R}_{\mathcal{O}\text{Max-Rank}}, \mathcal{R}_{\mathcal{O}\,\text{LCSymp}}, \mathcal{R}_{\mathcal{O}\,\text{Cont}}, \mathcal{R}_{\mathcal{O}\,\text{ECont}} \text{ and } \mathcal{R}_{\mathcal{O}\,\text{Engel}},$$

showing that the technique developed here works for proving already known results such as the ones for $\mathcal{R}_{\mathcal{O}\text{Max-Rank}}$ and $\mathcal{R}_{\mathcal{O}\,\text{Cont}}$, but also to prove some new ones like the ones for $\mathcal{R}_{\mathcal{O}\,\text{LCSymp}}, \mathcal{R}_{\mathcal{O}\,\text{ECont}}$ and $\mathcal{R}_{\mathcal{O}\,\text{Engel}}$.

Chapter 5. Conclusions and work in progress

In this Chapter we discuss the results of this thesis and present some of the possible lines of research that follow from this work. Some of them have already provided some partial results that are exposed in Section 5.1.

Section 5.1. *H*-principles for locally Aut-invariant HPDRs

Here we use the ideas behind the h–principles for Diff–invariant PDRs to produce similar results for HPDRs. We have proved a holomorphic holonomic approximation Theorem for totally real submanifolds and use it to finde local h–principles for an analogue of the Diff–invariance property in the holomorphic category that we have named by *locally* Aut*–invariance*.

Chapter 2

Preliminaries

This Chapter is devoted to expose the mathematical objects that will play a major role in this thesis. We will provide some basic definitions and theorems concerning smooth and holomorphic fiber bundles and the h–principles that their sections may satisfy. We will conclude this chapter with an overview of Stein manifolds since they will play a major role in all of the original results of this thesis.

2.1 Smooth and holomorphic bundles

We commence with the notions of smooth and holomorphic fiber bundles and after that we will introduce smooth and holomorphic vector and affine bundles.

We will continue our exposition with jet bundles of smooth and holomorphic sections. Along this thesis we are going to be interested in finding sections of bundles that satisfy some specific conditions that are going to be expressed in therms of their differentials. To understand them better we will *formalize* that conditions, i.e. we will *decouple* each section from their partial derivatives. Jet bundles constitute the natural spaces where this *formal derivatives* live.

That differential conditions will determine some subsets in their corresponding jet bundles called *Partial Differential/Holomorphic Relations*. We will conclude this section giving some interesting examples of relations. They are going to be born from immersions, submersions, symplectic forms, and 1-forms defining topologically stable distributions and all their corresponding holomorphic analogues.

As we announced, let us begin with the Definition of smooth and holomorphic fiber bundles

Definition 2.1.1. *Let F, E and B be three smooth (resp. complex) manifolds and let $\pi : E \to B$ be a smooth (resp. holomorphic) surjection. The tuple (E, B, π, F) is a* smooth fiber bundle *(resp.* holomorphic fiber bundle*) over B if for each $p \in B$ there exists a* local trivialization (U, ϕ) *around p, i.e. a neighbourhood U of p in B and a diffeomorphism (resp. biholomorphism) $\phi : \pi^{-1}(U) \longrightarrow U \times F$ that makes the following diagram commutative*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U & \xrightarrow{\mathrm{Id}} & U. \end{array}$$

In that situation, one calls F the fiber*, B the* base space*, E the* total space *and π the* projection *of the fiber bundle. We will denote the fiber over a subset $S \subset B$ by $E_S := \pi^{-1}(S)$. We denote the fiber bundle by the diagram*

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow{\scriptstyle\pi} \\ & & B \end{array}$$

or simply just by E or π when there is no room for confusion. Subbundles of E *are those submanifolds $S \hookrightarrow E$ that fiber over B with projection map $\pi_{|S}$.*

We want to emphasize that our notion of local trivialization is local just in the base but global in the fiber. We differentiate that notion from the one that we call *trivializing chart around a point $x \in E$*, that is a chart $\hat{\Phi}$ from an open neighbourhood W of x in E such that there exists another chart Φ making commutative the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{\Phi}} & \mathbb{R}^n \times \mathbb{R}^q \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \pi(W) & \xrightarrow{\Phi} & \mathbb{R}^n. \end{array}$$

Example 2.1.2. The easiest example of a fiber bundle is the cross product of two manifolds. $M_1 \times M_2$ is the total space of a fiber bundle with any of the canonical projections. The base space is one of the two manifolds and the fiber is the other one. This bundle is called the *trivial bundle* and, since general fiber bundles are locally trivial, they can be visualized as *"twisted"* cross products. It is well known that a fiber bundle over a contractible manifold is indeed a trivial bundle.

We introduce now the concept of sections of a fiber bundle. In the same manner a fiber bundle $F \hookrightarrow E \to B$ is a generalization of the Cartesian product of two manifolds $M_1 \times M_2$, one can see sections as generalizations of maps $B \to F$. Indeed both notions are equivalent for trivial bundles $M_1 \times M_2 \to M_1$ since sections are completely determined by their projection to the second factor. $\bowtie$

Almost all the original results of this thesis will require to approximate smooth sections by holomorphic ones. Chapter 3 is devoted to expose the approximation Theorems that have been developed to obtain the mentioned results. Here we just present the notation that will be used along this manuscript.

Definition 2.1.3. *Let $\pi : E \to B$ be a fiber bundle. A section over a subset $S \subset B$ is a continuous map $s : S \to E$ such that $\pi \circ s = \mathrm{Id}_S$. Let $U \subset B$ be an open subset, we define the following spaces of sections endowed with the compact-open topology*

- $\Gamma(U, E)$ *is the space of all sections over U that are smooth,*
- $\Gamma^r(U, E)$ *is the space of all sections over U of class $C^r, r \in \mathbb{Z}_+$,*
- $\Gamma_{\mathcal{O}}(U, E)$ *are the sections over U that are holomorphic if E is a holomorphic fiber bundle.*

We denote the space $\Gamma(B, E)$ by $\Gamma(E)$ and we proceed analogously with $\Gamma^r(E)$ and $\Gamma_{\mathcal{O}}(E)$. For non-open subsets $S \subset B$ we define the space $\Gamma(S, E)$ as the space of sections over S that extend to a section in $\Gamma(U, E)$ for some open neighbourhood $U = \mathcal{O}p_B(S)$. We define the spaces $\Gamma^r(S, E)$ and $\Gamma_{\mathcal{O}}(S, E)$ analogously.

Following Gromov (see [Gro86] p.85), here we denote by $\mathcal{O}p_B(A)$ (or simply by $\mathcal{O}p(A)$ when there is no room for confusion), an arbitrarily small but non-specified open neighbourhood of a subset A in B, that may be reduced if it is necessary for the argument. Most of the times $\mathcal{O}p_B\left(A\right)$ works as a replacement of the sentence *"an open neighbourhood of A in B"*.

Remark 2.1.4. One of the most important features of (smooth) fiber bundles is that they are Serre fibrations, i.e. they satisfy the homotopy lifting property with respect to all CW–pairs (X, A) (See [Hat02] Proposition 4.48). This means that for any pair (X, A), where X is a CW–complex and A is a subcomplex of X, and every fiber bundle $\pi : E \to B$, then every homotopy $h : X\times[0, 1] \to B$ has a lift $\widetilde{h}$ to E extending a given lift $g : X\times\{0\}\cup A\times[0, 1] \to E$. This is depicted in the following commutative diagram.

$$\begin{array}{ccc} X \times \{0\} \cup A \times [0,1] & \xrightarrow{g} & E \\ \big\downarrow & \overset{\widetilde{h}}{\nearrow} & \big\downarrow \pi \\ X \times [0,1] & \xrightarrow{h} & B. \end{array}$$

This yields that if $b_0 \in B$ and $x_0 \in \pi^{-1}(b_0) = F$, the projection $\pi : E \to B$ induces isomorphisms $\pi_* : \pi_k(E, F, x_0) \xrightarrow{\simeq} \pi_k(B, b_0), k \geq 1$ (see [Hat02] Theorem 4.41). Since in addition the projection map of a fiber bundle is surjective we have the following exact sequence of homotopy groups.

$$\ldots \to \pi_{k+1}(B, b_0) \to \pi_k(F, x_0) \to \pi_k(E, x_0) \to \pi_k(B, b_0) \to \ldots \to \pi_0(B, x_0) \to 0,$$

where the homomorphisms $\pi_k(F, x_0) \to \pi_k(E, x_0), k \in \mathbb{N}$ are induced by the inclusion and $\pi_k(E, x_0) \to \pi_k(B, b_0), k \in \mathbb{N}$ are induced by the projection. ⋈

Now we define the morphisms of the category of fiber bundles.

Definition 2.1.5. *A* morphism of fiber bundles *or a* bundle map *between two smooth (resp. holomorphic) fiber bundles $E_1 \xrightarrow{\pi_1} B_1$ and $E_2 \xrightarrow{\pi_2} B_2$ is a fiber preserving smooth (resp. holomorphic) map between the total spaces. That is a smooth (resp. holomorphic) map $\hat{f} : E_1 \to E_2$ such that there exist a smooth (resp. holomorphic) map $f : B_1 \to B_2$ making the following diagram commutative:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{f} & B_2. \end{array}$$

In this situation we will say that the fibered map $\hat{f}$ *covers the map* f *and we will denote the restricted map* $\hat{f}\big|_{E_p}$ *by* $\hat{f}_p$ *for every* $p \in B_1$. *An* isomorphism of fiber bundles *is a diffeomorphism (resp. biholomorphism) between the total spaces such that both itself and its inverse are bundle maps.*

Given a fiber bundle, it is possible to build other fiber bundles from it in the following ways.

Definition 2.1.6. *Let* $\pi : E \to B$ *be a smooth (resp. holomorphic) fiber bundle with fiber* F *and let* $f : M \to B$ *be a smooth (resp. holomorphic) map, where* M *is a smooth (resp. complex) manifold.*

- *The* pullback bundle $f^*E \xrightarrow{f^*\pi} M$ *is the fiber bundle over* M *whose fiber at a point* $q \in M$ *is* $E_{f(q)} \cong F$, *its total space is the fiber product of the maps* $f : M \to B$ *and* $\pi : E \to B$

  $$f^*E := E \times_B M = \{(x, q) \in E \times M \big| \pi(x) = f(q)\}$$

  and its projection map is just the projection to the second factor

  $$\begin{aligned} f^*\pi \colon f^*E &\longrightarrow M \\ (x, q) &\longmapsto q. \end{aligned}$$

  Note that the projection to the first factor is a bundle map that we denote by

  $$\begin{aligned} \hat{f} \colon f^*E &\longrightarrow E \\ (x, q) &\longmapsto x. \end{aligned}$$

  Given a section $s : B \to E$, *we can define its pullback*

  $$\begin{aligned} f^*s \colon M &\longrightarrow f^*E \\ p &\longmapsto (p, s \circ f(p)). \end{aligned}$$

  Given a submanifold $M \overset{i}{\hookrightarrow} B$, *we call the pullback bundle of the inclusion map* i^*E *the* restriction of E to M *and we denote it by* $E\big|_M \overset{\hat{i}}{\hookrightarrow} E$.

- *If $f : M \to B$ is the projection of a fiber bundle with fiber F', then the fiber product is a fiber bundle over B whose fiber at a point $p \in B$ is $E_p \times M_p \cong F \times F'$ and whose projection map is the following:*

$$\begin{aligned} E \times_B M &\longrightarrow B \\ (x,q) &\longmapsto \pi(x) = f(q). \end{aligned}$$

Any pair of intersecting local trivializations (U_α, ϕ_α) and (U_β, ϕ_β) of a fiber bundle induces a map

$$\begin{aligned} g_{\alpha\beta} \colon U_{\alpha\beta} := U_\alpha \cap U_\beta &\longrightarrow \mathcal{G}(F) \\ p &\longmapsto (\phi_\alpha \circ \phi_\beta^{-1})\big|_{\{p\}\times F}, \end{aligned}$$

where $\mathcal{G}(F)$ is the group $\mathrm{Diff}(F)$ of diffeomorphisms of F if E is a smooth bundle, or the group $\mathrm{Aut}(F)$ of biholomorphisms of F in the holomorphic case. These maps are called the *transition maps* and they satisfy the following *cocycle condition*:

$$g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma} \quad \text{in } U_\alpha \cap U_\beta \cap U_\gamma.$$

Remark 2.1.7. Note that it is possible to obtain a fiber bundle just from a covering $\{U_\alpha\}$ of the base and its corresponding transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{G}(F)$. Indeed, let $\widetilde{E}$ be the disjoint union

$$\widetilde{E} := \bigsqcup_\alpha U_\alpha \times F,$$

and consider the following equivalence relation in $\widetilde{E}$:

$$(p,u) \sim (q,v) \iff p = q \quad \text{and} \quad v = g_{\alpha\beta}(p)(u), \quad \text{for some } \alpha, \beta.$$

We define the total space E as the quotient space $\widetilde{E}\big/\sim$ and the projection map as the following:

$$\begin{aligned} \pi \colon E &\longrightarrow B \\ [(p,u)]_\sim &\longmapsto p. \end{aligned}$$

⋈

It is worth to mention that the bundle E obtained in Remark 2.1.7 is unique up to isomorphism. Indeed let $\pi' : E' \to B$ be a fiber bundle with the same transition maps as E. The isomorphism is given by the map

$$\begin{aligned} f\colon E' &\longrightarrow E \\ x &\longmapsto [\phi'_\alpha(x)]_\sim, \end{aligned}$$

where $\phi'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a local trivialization of E' around $\pi'(x)$ that can be used to define the transition maps $g_{\alpha\beta}$. The map f is well defined due to the cocycle condition.

This indicates that we can define a fiber bundle just from its local structure and their transition maps.

Definition 2.1.8. *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fiber bundle and let G be a subgroup of $\mathcal{G}(F)$. We will say that E has a G–*structure *if its transition maps take values in G. Two cocycles (fiber bundles) $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are said to be G–*equivalent *(or just* equivalent *when the structure group G can be deduced from the context) if there exist maps $\lambda_\alpha : U_\alpha \to G$ such that*

$$g_{\alpha\beta} = \lambda_\alpha^{-1} \cdot g'_{\alpha\beta} \cdot \lambda_\beta.$$

We will say that the structure of a fiber bundle can be reduced to H in G *(or just* reduced to H *when there is no room for error) if it is G–equivalent to a fiber bundle with an H–structure.*

Remark 2.1.9. Note that by Remark 2.1.7, two G–equivalent cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ induce two fiber bundles E and E' that are isomorphic. Indeed, it is straightforward to check that the map

$$\begin{aligned} \widetilde{\lambda}\colon \bigsqcup_\alpha U_\alpha \times F &\longrightarrow \bigsqcup_\alpha U_\alpha \times F \\ (p, v_\alpha) &\longmapsto (p, \lambda_\alpha(p)(v_\alpha)) \end{aligned}$$

induces a bundle isomorphism between the quotient spaces $\lambda : E \to E'$. $\bowtie$

In particular, a fiber bundle is isomorphic to a trivial bundle if its structure group can be reduced to the trivial subgroup $\{\mathrm{Id}\} < \mathcal{G}(F)$. One can also see a holomorphic fiber bundle with fiber F as a smooth fiber bundle between complex manifolds with a holomorphic projection and an $\mathrm{Aut}(F)$–structure. Other important examples of bundles with G–structures are vector and affine bundles that will be explored in the following Section.

2.1.1 Smooth and holomorphic vector and affine bundles

Definition 2.1.10. *A* vector bundle of rank n *is a smooth or holomorphic fiber bundle whose fiber is an* n*–dimensional vector space* $V = \mathbb{K}^n$ *over the field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, that carries a* $\mathrm{GL}(n, \mathbb{K})$*–structure. Vector subbundles are the subbundles whose fiber is a vector subspace of* V*. A bundle map between two vector bundles is a* morphism of vector bundles *or a* vector bundle map *if its restriction to any fiber is a linear map. Moreover, let* $G < GL(n, \mathbb{K})$ *be a Lie group such that the structure group of* E *can be reduced to* G *in* $GL(n, \mathbb{K})$*,*

- *If* $\mathbb{K} = \mathbb{R}$ *and* $G = \mathrm{O}(n)$ *then the transition maps preserve the standard euclidean metric. Therefore there is a well defined metric in each fiber and we call* E *an* Euclidean vector bundle*. By Gram-Schmidt, every real vector bundle admits a reduction to* $\mathrm{O}(n)$*.*
- *If* $\mathbb{K} = \mathbb{R}$ *and* $G = \mathrm{SO}(n)$ *then, since the transition maps preserve the determinant, there is a well defined orientation of each fiber and we call* E *an* orientable vector bundle*.*
- *If* $\mathbb{K} = \mathbb{R}$ *and* $G = \mathrm{Sp}(n)$ *then the transition maps preserve the standard symplectic form. Therefore there is a well defined symplectic linear structure in each fiber and we call* E *a* symplectic vector bundle*.*
- *If* $\mathbb{K} = \mathbb{C}$ *and* $G = \mathrm{U}(n)$ *the transition maps preserve the standard Hermitian metric of* $\mathbb{C}^n$*. Then there is a well defined Hermitian metric in each fiber and we call* E *an* Hermitian vector bundle*. In the same way as in the real case, every complex vector bundle is equivalent to a Hermitian vector bundle.*

Remark 2.1.11. Since every vector space has the origin as a distinguished element, then every vector bundle has a canonical section that assigns the origin of its fiber to each point in the base space. We call this section *the zero section* and it has the same regularity as the projection map of the vector bundle. If $V \hookrightarrow E \xrightarrow{\pi} B$ is a vector bundle and $Z : B \to E$ is the zero section of E, one can canonically identify E with $\mathrm{Vert}(\pi)\big|_{Z(B)}$, where $\mathrm{Vert}(\pi)$ denotes the vertical bundle $\ker d\pi$, that is the vector subbundle of TE whose fibers are the vector subspaces that are tangent to the fibers of π. $\bowtie$

Given a vector bundle one can construct pullback, restriction and fiber product bundles from them like in Definition 2.1.6, but one can also take advantage of their additional structure to construct other kind of new vector bundles from them.

Remark 2.1.12. Let $V \hookrightarrow E \xrightarrow{\pi} B$ and $V' \hookrightarrow E' \xrightarrow{\pi} B$ be two real or complex vector bundles. We define the vector bundle $E \oplus E'$ as the one whose fiber over a point $p \in B$ is the vector space $E_p \oplus E'_p \cong V \oplus V'$ and its transition maps are the maps $g_{\alpha\beta} \oplus g'_{\alpha\beta}$ that can be written in their matrix form as

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{pmatrix}.$$

Following similar steps one can define the following vector bundles:

- the *tensor product bundle* $E \otimes E'$,
- the *dual bundle* E^*,
- the *bundle of homomorphisms* $\mathrm{Hom}(E, E') \cong E^* \otimes E$,
- the *k-th exterior product bundle* $\bigwedge^k E$,
- the *k-th symmetric product bundle* $\mathrm{Sym}^k(E)$,
- ...

$\bowtie$

Example 2.1.13.

1. The (smooth) tangent bundle TM of an n–dimensional smooth manifold M is a real vector bundle of rank n. The charts $\phi_\alpha = (x_1, \ldots, x_n)$ of an smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ define the local trivializations $(\partial_{x_1}, \ldots, \partial_{x_n})$ and its transition maps are the Jacobians of the changes of coordinates. The differential $df = f_* : TM \to TM'$ of a smooth map $f : M \to M'$ between two smooth manifolds is a morphism of vector bundles. The sections of this bundle are the smooth vector fields, that is $\Gamma(TM) = \mathfrak{X}(M)$. Note that a smooth manifold is orientable if and only if TM is an orientable vector bundle.

2. The (smooth) cotangent bundle T^*M of a smooth manifold M is the dual vector bundle of TM. The local charts ϕ_α define the local trivializations $(dx_1, \dots, dx_n)$ and the pullback $f^* : T^*M' \to T^*M$ of a smooth map $f : M \to M'$ is a morphism of vector bundles. The sections of this bundle are the smooth 1–forms, that is $\Gamma(T^*M) = \Omega^1(M)$.

3. Using the operations of Remark 2.1.12 on TM and T^*M one can obtain vector bundles such as $\mathrm{Sym}^k(TM), \mathrm{Sym}^k(T^*M), \bigwedge^k TM$ and $\bigwedge^k T^*M$. Note that the sections of the last one are the smooth differential k–forms $\Gamma(\bigwedge^k T^*M) = \Omega^k(M)$. Let E be any vector bundle over M, the sections of the tensor bundle $\bigwedge^k T^*M \otimes E$ are the smooth k–forms with values in E, $\Omega^k(M, E)$.

4. The holomorphic tangent bundle $\mathcal{T}M$ of a complex manifold M is a holomorphic complex vector bundle of rank $\dim_{\mathbb{C}} M$, whose holomorphic sections are the complex vector fields $\Gamma_{\mathcal{O}}(\mathcal{T}M) =: \mathfrak{X}_{\mathcal{O}}(M)$. Similarly to the smooth case, the holomorphic cotangent bundle $\mathcal{T}^*M$ is the dual bundle of $\mathcal{T}M$. It is also possible to obtain the bundles $\mathrm{Sym}^k(\mathcal{T}M), \mathrm{Sym}^k(\mathcal{T}^*M), \bigwedge^k \mathcal{T}M$ and $\bigwedge^k \mathcal{T}^*M$. With the holomorphic sections of the last one being the holomorphic forms $\Gamma_{\mathcal{O}}(\bigwedge^k \mathcal{T}^*M) = \Omega^k_{\mathcal{O}}(M)$. Given a holomorphic vector bundle E, the holomorphic sections of $\bigwedge^k \mathcal{T}^*M \otimes E$ are the holomorphic k–forms with values in E, $\Gamma_{\mathcal{O}}(\bigwedge^k \mathcal{T}^*M \otimes E) = \Omega^k_{\mathcal{O}}(M, E)$.

$\bowtie$

Remark 2.1.12 shows, broadly speaking, that any operation or construction of vector spaces that is independent of the choice of a basis can also be done for vector bundles. In particular, given a real vector space V, one can produce a complex vector space

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C},$$

with $\dim V = \dim_{\mathbb{C}} V_{\mathbb{C}}$, where the scalar multiplication of a complex number $w \in \mathbb{C}$ times an element $v \otimes z \in V_{\mathbb{C}}$ is defined by the multiplication in the second factor

$$w \cdot (v \otimes z) = v \otimes wz.$$

This construction can also be done for vector bundles and it yields the following

Definition 2.1.14. *Let E be a real vector bundle of rank k over a smooth manifold M, let $L = M \times \mathbb{C}$ be the trivial complex line bundle. We define the* complexified vector bundle of *E, as the complex vector bundle $E_{\mathbb{C}}$ that is the tensor product $E \otimes_{\mathbb{R}} L$.*

Recall that a *linear complex structure* in a real vector space V is an automorphism $J : V \to V$ such that $J^2 = -\operatorname{Id}$. The notions of *complex vector space* and *real vector space with a linear complex structure* are equivalent. Indeed, multiplication by the complex number i induces a linear complex structure in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \cong_{\mathbb{R}} \mathbb{R}^{2n}$. Conversely, a real vector space V with a linear complex structure J becomes a complex vector space denoted by V_J, where multiplying by the complex number i in V_J corresponds with applying the automorphism J. In this way, the scalar product by complex numbers in V_J is defined by the map

$$\begin{aligned} \mathbb{C} \times V_J &\longrightarrow V_J \\ (a + ib, v) &\longmapsto av + bJv. \end{aligned}$$

It is possible to extend the linear complex structure J to $V_{\mathbb{C}}$ just by applying it in the first factor

$$\begin{aligned} \widetilde{J} \colon V_{\mathbb{C}} &\longrightarrow V_C \\ v \otimes z &\longmapsto J(v) \otimes z. \end{aligned}$$

Since $\widetilde{J}^2 = -\operatorname{Id}$, the eigenvalues of $\widetilde{J}$ are just $\{\pm i\}$. Denote the eigenspace of the eigenvalues i and $-i$ by $V_J^{1,0}$ and $V_J^{0,1}$ respectively. We have that

$$V_{\mathbb{C}} = V_J^{(1,0)} \oplus V_J^{(0,1)},$$

with projections:

$$\begin{aligned} \pi_{1,0} : V_{\mathbb{C}} &\longrightarrow V_J^{1,0} \\ v &\longmapsto \frac{1}{2}(v - iJv) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi_{0,1} : V_{\mathbb{C}} &\longrightarrow V_J^{0,1} \\ v &\longmapsto \frac{1}{2}(v + iJv). \end{aligned}$$

Note that both spaces $V_J^{1,0}$ and $V_J^{0,1}$ are isomorphic as real vector spaces by the conjugation $\overline{v \otimes z} := v \otimes \overline{z}$ and that the map

$$\begin{aligned} V_J &\longrightarrow V_J^{1,0} \\ v &\longmapsto Jv \otimes 1 + v \otimes i \end{aligned}$$

is an isomorphism of complex vector spaces.

Now we define the fibered version of a linear complex structure.

Definition 2.1.15. *Let E be a real vector bundle over a smooth manifold M. A* complex structure on *E is a vector bundle automorphism $J : E \to E$ that covers the identity map and such that $J_p : E_p \to E_p$ is a linear complex structure for each $p \in M$.*

Like in the linear case, a real vector bundle E with a complex structure J is equivalent to a complex vector bundle. In this situation, like in the linear case, the complexified vector bundle splits

$$E_{\mathbb{C}} = E_J^{1,0} \oplus E_J^{0,1},$$

where $E_J^{1,0}$ and $E_J^{0,1}$ are the complex vector subbundles whose fibers are the eigenspaces of J_p with eigenvalues i and $-i$ respectively. We will omit the sub-index J when the complex structure is clear from context.

Example 2.1.16.

1. A complex structure in the tangent bundle of a smooth manifold M receives the name of an *almost complex structure of M*. We say that a smooth manifold with an almost complex structure is an *almost complex manifold*. A smooth map between two almost complex manifolds is called *pseudo holomorphic* if its differential is a morphism of complex vector bundles.

   Complex manifolds have a canonical almost complex structure that comes from multiplication by the complex number i in the trivializations of the holomorphic tangent bundle given by a holomorphic atlas. An almost complex structure on M is *compatible with a holomorphic atlas of M* if it is the induced by this atlas.

   Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}$ be a holomorphic atlas on M and let J be an almost complex structure compatible with $\mathscr{A}$. Let $\phi_\alpha = (z_1, \ldots, z_n) \in \mathbb{C}^n$. This coordinates provide local trivializations of $\mathcal{T}M$

   $$\mathcal{T}M\big|_{U_\alpha} = \langle \partial_{z_j} \rangle_{j=1}^n.$$

Let us set $z_j = x_j + iy_j$, $j = 1, \dots, n$. In this coordinates it is satisfied that $J\partial_{x_j} = \partial_{y_j}$ and $J\partial_{y_j} = -\partial_{x_j}$. Moreover, they provide local trivializations of $T_J^{1,0}M$ and $T_J^{0,1}M$ as

$$T_J^{1,0}M_{|U_\alpha} = \langle \frac{1}{2}\left(\partial_{x_j} - i\partial_{y_j}\right)\rangle_{j=1}^n \quad \text{and} \quad T_J^{0,1}M_{|U_\alpha} = \langle \frac{1}{2}\left(\partial_{x_j} + i\partial_{y_j}\right)\rangle_{j=1}^n.$$

Therefore, the Wirtinger operators

$$\partial_{z_j} = \frac{1}{2}\left(\partial_{x_j} - i\partial_{y_j}\right) \quad \text{and} \quad \partial_{\bar{z}_j} = \frac{1}{2}\left(\partial_{x_j} + i\partial_{y_j}\right)$$

provide natural isomorphisms

$$(TM, J) \cong (T^{1,0}M, \cdot i) \cong \mathcal{T}M \quad \text{and} \quad (T^{0,1}M, \cdot i) \cong \overline{\mathcal{T}M}.$$

Note that the Cauchy-Riemann equations for a function $f : U_\alpha \to \mathbb{C}$ are equivalent to $\partial_{\bar{z}_j} f \equiv 0$, and that if f is holomorphic, then the complex derivative of f coincides with

$$\sum_{j=1}^n \partial_{z_j} f dz_j.$$

An almost complex structure J in a smooth manifold M is said to be *integrable* if there is a compatible holomorphic atlas of M that induces J. The integrability condition of an almost complex structure J is provided by its Nihenjuis tensor N_J

$$N_J(X, Y) := [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY]$$

and the Theorem of Newlander-Niremberg [NN57], that states that an almost complex complex structure J is integrable if and only if $N_J \equiv 0$.

2. Note that the sections of the complexified cotangent bundle $T^*_{\mathbb{C}}M$ of a smooth manifold M are the $\mathbb{C}$–valued 1–forms $\Omega^1(M, \mathbb{C})$ (see Example 2.1.13). An almost complex structure J on M provides a splitting $\Omega^1(M, \mathbb{C}) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, given by the projections

$$\begin{aligned}\pi_J^{1,0} : \Omega^1(M, \mathbb{C}) &\longrightarrow \Omega_J^{1,0}(M) \\ \alpha &\longmapsto \frac{1}{2}(\alpha - i\alpha \circ J)\end{aligned} \quad \text{and} \quad \begin{aligned}\pi_J^{0,1} : \Omega^1(M, \mathbb{C}) &\longrightarrow \Omega_J^{0,1}(M) \\ v &\longmapsto \frac{1}{2}(\alpha + i\alpha \circ J).\end{aligned}$$

where we denote by $\Omega^{1,0}_J(M)$ the $\mathbb{C}$–linear forms and by $\Omega^{0,1}_J(M)$ the $\mathbb{C}$–antilinear ones. More generally, one can decompose the $\mathbb{C}$–valued k–forms in the following way

$$\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}_J(M),$$

where the k–forms of $\Omega^{p,q}_J(M)$ are the wedge product of p forms of $\Omega^{1,0}_J(M)$ and q forms of $\Omega^{0,1}_J(M)$. Therefore, we can define the projections

$$\pi^{p,q}_J : \Omega^{p+q}(M,\mathbb{C}) \to \Omega^{p,q}_J(M),$$

and use them to define the Dolbeaut operators ∂_J and $\overline{\partial}_J$ by

$$\partial_J\Big|_{\Omega^{p,q}_J(M)} := \pi^{p+1,q}_J \circ d \quad \text{and} \quad \overline{\partial}_J\Big|_{\Omega^{p,q}_J(M)} := \pi^{p,q+1}_J \circ d,$$

where we denote by d the exterior derivative of forms.

The almost complex structure J is integrable if and only if $d = \partial_J + \overline{\partial}_J$. In this case, given a holomorphic chart $\phi_\alpha = (z_1,\dots,z_n)$, with $z_j = x_j + iy_j$ for every $j = 1,\dots,n$, the dual coordinates of $\{\partial_{z_j}\}$ are $\{dz_j = dx_j + idy_j\}$ and the dual coordinates of $\{\partial_{\bar{z}_j}\}$ are $\{d\bar{z}_j = dx_j - idy_j\}$. This chart provides local trivializations of $\mathcal{T}^*M$ as the span of $\langle dz_j\rangle$, and $\overline{\mathcal{T}^*M}$ as the span of $\langle d\bar{z}_j\rangle$.

3. All holomorphic vector bundles $V \hookrightarrow E \xrightarrow{\pi} B$ are indeed complex vector bundles. That is because the kernel of $d\pi$ is a complex vector subbundle of TE. Therefore the identification of E with $\ker d\pi\big|_{Z(B)}$ induces a complex structure in E.

   Hence in the following we will call *holomorphic vector bundles* to holomorphic complex vector bundles and *complex vector bundles* to smooth complex vector bundles.

$\bowtie$

The existence of a canonical section constitutes the main difference between vector bundles and affine bundles, that we describe below.

Recall that an affine space modelled by a vector space V is a set A and a map $\Phi : A\times V \to A$ satisfying

- transitivity, i.e. for each $p, q \in A$, there exists a vector $v \in V$ such that $\Phi(p, v) = q$,
- faithfulness, i.e. if $\Phi(p, v) = p$ for every $p \in A$, then $v = 0$,
- that $\Phi(p, 0) = p$ for every $p \in A$,
- and that $\Phi(p, u + v) = \Phi(\Phi(p, u), v)$ for every $p \in A$ and every $u, v \in V$.

Do note that, once a point $p \in A$ is fixed, the affine space A becomes a vector space. Indeed, the structure of vector space is inherited via the map $\Phi(p, \text{-}) : V \to A$. We can "fiber" the definition of affine spaces to obtain the notion of affine bundles.

Definition 2.1.17. *Let $\overrightarrow{\pi} : \overrightarrow{E} \to B$ be a vector bundle. An affine bundle modelled on $\overrightarrow{E}$ is a fiber bundle $\pi : E \to B$ and a map $\Phi : E \times_B \overrightarrow{E} \to E$ making commutative the following diagram*

$$\begin{array}{ccc} E \times_B \overrightarrow{E} & \xrightarrow{\Phi} & E \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ \overrightarrow{E} & \xrightarrow{\overrightarrow{\pi}} & B, \end{array}$$

such that for each $p \in B$, E_p is an affine space modelled by $\overrightarrow{E}_p$ by the map Φ_p.

Equivalently we can define an affine bundle as a fiber bundle E whose fiber is an affine space A that carries an $\mathrm{Aff}(A)$ *structure.*

It is worth to mention that, in the same way as choosing a point in an affine space provides it a vector space structure, the choice of a section of an affine bundle (i.e. the choice of a point in each affine fiber) gives it the structure of a vector bundle.

The most important example of affine bundles for this thesis are the jet bundles of sections, that will be studied in the following Section.

2.1.2 Jet bundles of smooth and holomorphic sections

The main goal of this work is to study the topology of some spaces of sections of fiber bundles. This spaces are made of those sections who satisfy certain conditions written in terms of their partial derivatives of several orders. To understand these spaces, we will compare them with the spaces of sections whose *formal derivatives* satisfy the prescribed conditions. More precisely, we will compare the original set with the one that it induces in its corresponding jet bundle. We devote this section to define smooth and holomorphic jet bundles and to provide the notions required for the rest of the work.

We commence by introducing the notion of the r–jet of a C^r map $f : \mathcal{O}p\,(p_0) \to \mathbb{R}^q$ at a point $p_0 \in \mathbb{R}^n$. That is a tuple

$$j^r(f)(p_0) = \left(p_0, f(p_0), f'(p_0), \ldots, f^{(r)}(p_0)\right),$$

where we denote by $f^{(s)}$ the set of all partial derivatives of order s written lexicographically. The *jet space of mappings from $\mathbb{R}^n$ to $\mathbb{R}^q$* is

$$J^r\left(\mathbb{R}^n, \mathbb{R}^q\right) := \{j^r(f)(p_0) | p_0 \in \mathbb{R}^n \text{ and } f \in C^r(\mathcal{O}p\,(p_0)\,, \mathbb{R}^q)\}$$

Note that $J^r\left(\mathbb{R}^n, \mathbb{R}^q\right) \equiv \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qN_1} \times \ldots \times \mathbb{R}^{qN_r}$, where $N_k = \frac{(n+k-1)!}{(n-1)!k!}$ represents the number of partial derivatives of order k of a function $\mathbb{R}^n \to \mathbb{R}$.

Roughly speaking, one can see $J^r(\mathbb{R}^n, \mathbb{R}^q)$ as a trivial bundle over $\mathbb{R}^n$ whose sections are mappings $f : \mathbb{R}^n \to \mathbb{R}^q$ together with some formal derivatives that may be decoupled from f.

Remark 2.1.18. Let us denote by $P_r(n, q)$ the space of $\mathbb{R}^q$–valued polynomials of n variables of degree $\leq r$. It is worth to mention that for every point $z = (p, x^{(0)}, \ldots, x^{(r)}) \in J^r(\mathbb{R}^n, \mathbb{R}^q)$ there exists a unique polynomial $P(X_1, \ldots, X_n) \in P_r(n, q)$ such that $j^r(P)(p) = z$. So there exists a canonical trivialization

$$\mathbb{R}^n \times P_r(n, q) \xrightarrow{j^r} J^r(\mathbb{R}^n, \mathbb{R}^q).$$

$\bowtie$

Note that this notion of formal derivative is not invariant under changes of coordinates. Nevertheless if the jets of two mappings coincide in some coordinates, then they coincide in all of them due to the chain rule. The jet of a section of a fiber bundle will be defined using this observation:

Definition 2.1.19. *Let $E \xrightarrow{\pi} B$ be a smooth fiber bundle and let p_0 be a point of B. Two sections $\sigma_1, \sigma_2 : \mathcal{O}p\,(p_0) \to E$ are r–*tangent *if*

$$j^r(\hat{\Phi} \circ \sigma_1 \circ \Phi^{-1})(\Phi(p_0)) = j^r(\hat{\Phi} \circ \sigma_2 \circ \Phi^{-1})(\Phi(p_0))$$

for some trivializing chart $\hat{\Phi}$ around $\sigma_1(p_0) = \sigma_2(p_0)$.

The r–tangengy condition defines an equivalence relation in the set of sections $\Gamma(p_0, E)$ over a point $p_0 \in B$ of a fiber bundle $\pi : E \to B$. The class of r–tangency of a section $\sigma : \mathcal{O}p_B\,(p) \to E$ is called the *r–jet of σ at p* and it is denoted by $j^r(\sigma)(p_0)$. The set of r–jets over every point in B, $E^{(r)}$, is called *the r–jet space of sections of E*.

Note that since the chain rule for order r involves the partial derivatives of all lower orders, the chain of inclusions

$$E := E^{(0)} \subset E^{(1)} \subset \ldots \subset E^{(r)} \subset \ldots$$

is not invariant under changes of coordinates. Nevertheless, since the r–tangency of two sections implies their $(r-1)$–tangency, we can define the maps

$$\begin{aligned} \pi^r_s \colon E^{(r)} &\longrightarrow E^{(s)} \\ j^r(\sigma)(p) &\longmapsto j^s(\sigma)(p), \end{aligned}$$

for every $0 \leq s \leq r$, where $E^{(0)} := E$ and $j^0(\sigma)(p) := \sigma(p)$.

The set $E^{(r)}$ carries a natural structure of smooth manifold given by the extensions of the trivializing charts

$$\hat{\Phi}^r : (p^r_0)^{-1}(W) \to J^r(\mathbb{R}^n, \mathbb{R}^q),$$

that send each r–tangency class $j^r(\sigma)(p_0)$ to $j^r(\hat{\Phi} \circ \sigma \circ \Phi^{-1})(\Phi(p_0))$. This smooth structure makes the maps $\pi^r_s : E^{(r)} \to E^{(s)}$ and $\pi^r := \pi \circ \pi^r_0 : E^{(r)} \to B$ become smooth fiber bundles for every $0 \leq s \leq r$ (see page 9 in [EM02]). Indeed, since the derivative is a linear operator, if E is a vector bundle, then so they are the bundles $E^{(r)} \to B$.

Definition 2.1.20. *Let $\pi : E \to B$ be a smooth fiber bundle and let $r \in \mathbb{N}$. The fiber bundle*

$$\begin{aligned} \pi^r := \pi \circ \pi^r_0 \colon E^{(r)} &\longrightarrow B \\ j^r(\sigma)(p) &\longmapsto p \end{aligned}$$

is the r–jet extension of π *or the* r–jet bundle of smooth sections of π. *Now let* $\sigma : B \to E$ *be a section. The following section of* $E^{(r)}$

$$\begin{aligned} j^r(\sigma)\colon B &\longrightarrow E^{(r)} \\ p &\longmapsto j^r(\sigma)(p) \end{aligned}$$

is called the r–jet extension of σ *or simply the* r–jet of σ.

The base section *or the* 0–jet part of a section $\sigma : B \to E^{(r)}$ *is the section* $\sigma^{(0)} := \pi^r_0 \circ \sigma : B \to E$. σ *is said to be* holonomic *if* $\sigma = j^r(\sigma^{(0)})$. *We denote the space of holonomic sections of* $E^{(r)}$ *by* $\mathrm{H}\, E^{(r)}$.

The space $E^{(1)}$ can be seen as the set of linear maps $L_x : T_{\pi(x)}B \to T_xE$ such that $d\pi \circ L_x = \mathrm{Id}$ for all $x \in E$. Thus, the fiber bundle $\pi^1_0 : E^{(1)} \to E$ is indeed an affine bundle modelled by

$$\mathrm{Hom}(\pi^*(TB), \mathrm{Vert}(E)) = \pi^*(T^*B) \otimes_E \mathrm{Vert}(E).$$

Definition 2.1.21. Principal subspaces *are those affine subspaces of each* $E^{(1)}_x = (\pi^1_0)^{-1}(x)$, $x \in E$ *made of all the morphisms that, given a hyperplane* $H \subset T_{\pi(x)}B$, *extend the same given linear map* $\eta : H \to T_xE$.

For each linear map $L_x \in E^{(1)}_x$ *there is a principal subspace containing* L_x *for each hyperplane* $H \subset T_{\pi(x)}B$. *This principal subspace can be identified with all linear maps of* $\mathrm{Hom}(T_{\pi(x)}B, \mathrm{Vert}_x(E))$ *whose kernel contains the hyperplane* H.

In this setting, a section $\sigma : B \to E^{(1)}$ will be holonomic if and only if the linear maps $L_{\sigma^{(0)}(p)} : T_pB \to T_{\sigma^{(0)}(p)}E$ coincide with the derivative of $\sigma^{(0)}$ at p for every $p \in B$.

If we identify each linear map L_x in $E^{(1)}$ with its graph P_x, one can also see the space $E^{(1)}$ as the set of *non-vertical* n–planes tangent to E, i.e. n–planes $P_x \subset T_xE$ such that $P_x \cap \mathrm{Vert}_x(E) = \{0\}$. Holonomic sections are those sections $\sigma : B \to E^{(1)}$ such that their image $\sigma(B)$ is the subbundle tangent to $\sigma^{(0)}(B)$ in TE. Under this viewpoint, sections of $\pi^1_0 : E^{(1)} \to E$ correspond with horizontal distributions in E of rank n. These distributions are known as *Ehreshmann connections of* E and they can be used to define projection operators

$K : TE \to \mathrm{Vert}(E)$ since $TE = H(B) \oplus \mathrm{Vert}(E)$ for every section H of π_0^1. A connection in E therefore provides the structure of a vector bundle to $\pi_0^1 : E^{(1)} \to E$.

If $E = M_1 \times M_2 \xrightarrow{\mathrm{pr}_1} M_1$ is a trivial bundle we denote their jet bundles by $J^r(M_1, M_2), r \in \mathbb{N}$ and we call them the *jet bundles of maps from M_1 to M_2* (see Example 2.1.2). In this situation, for each point $p = (p_1, p_2) \in M_1 \times M_2$, the vertical bundle at p is $\mathrm{Vert}_p(E) = T_p(\{p_1\} \times M_2)$. Moreover the horizontal tangent space $T_p(M_1 \times \{p_2\})$ is a canonical choice of a non-vertical subspace of T_pE of dimension $\dim M_1$. This yields that $\pi_0^1 : J^1(M_1, M_2) \to M_1 \times M_2$ is a vector bundle whose fiber over p is

$$\mathrm{Hom}(T_{p_1}M_1, T_{p_2}M_2).$$

The r–jet extension of sections induces an injective map $j^r : \Gamma(E) \to \Gamma(E^{(r)})$ whose image is the set of holonomic sections $\mathrm{H}\, E^{(r)} \hookrightarrow \Gamma(E^{(r)})$. Note that the C^0–topology in $\Gamma(E^{(r)})$ induces, via j^r, the C^r–topology in $\Gamma(E)$.

The space $E^{(r+1)}$ is canonically included in $\left(E^{(r)}\right)^{(1)}$ since it corresponds to the 1–jet extension of local holonomic sections of $E^{(r)}$. Indeed, the operator $j^{r+1} : \Gamma(E) \to \Gamma(E^{(r+1)})$ is the composition of $j^r : \Gamma(E) \to \Gamma(E^{(r)})$ with $j^1 : \Gamma(E^{(r)}) \to \Gamma\left(\left(E^{(r)}\right)^{(1)}\right)$. This makes $E^{(r+1)}$ an affine subbundle of $\left(E^{(r)}\right)^{(1)}$ that is modelled by

$$(\pi^r)^* \,\mathrm{Sym}^{r+1}(T^*B) \otimes_{E^{(r)}} (\pi_0^r)^* \,\mathrm{Vert}(E).$$

Since by Remark 2.1.18 there is a reparametrization of $J^r(\mathbb{R}^n, \mathbb{R}^q)$ given by

$$\begin{aligned} \Psi \colon \mathbb{R}^n \times P_r(n,q) &\longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q) \\ (p, P) &\longmapsto j^r(P)(p). \end{aligned}$$

Then, for every point $z \in E^{(r)}$, there is a chart $\Phi : \mathcal{O}p\,(z) \to \mathbb{R}^n \times P_r(n,q)$ such that the horizontal spaces $\mathbb{R}^n \times \{P\} \subset \mathbb{R}^n \times P_r(n,q)$ corresponds with graphs of holonomic sections. We will call this charts *holonomic trivializations* (see Section 1.7 in [EM02] or page 2 in [Gro86]).

All the previous process can be replicated for local holomorphic sections of holomorphic fiber bundles to obtain the *holomorphic jet bundles*. Given a holomorphic fiber bundle $\pi : E \to B$, its *r–jet of holomorphic sections* will be denoted by $E_{\mathcal{O}}^{(r)}$ and the *holomorphic r–jet extension of a holomorphic section* $\sigma : B \to E$ will be denoted by $j_{\mathcal{O}}^r(\sigma)$. The mappings $\pi_s^r : E_{\mathcal{O}}^{(r)} \to E_{\mathcal{O}}^{(s)}$ and $\pi^r : E_{\mathcal{O}}^{(r)} \to B$ become holomorphic fiber bundles for every $0 \leq s \leq r$ under their natural structure of holomorphic manifolds given by their trivializing charts and the jet of holomorphic mappings $J_{\mathcal{O}}^r(\mathbb{C}^n, \mathbb{C}^q)$. In this setting the bundles $\pi_r^{r+1} : E_{\mathcal{O}}^{(r+1)} \to E_{\mathcal{O}}^{(r)}$ are holomorphic affine bundles modelled by

$$(\pi^r)^* \operatorname{Sym}^{r+1}(\mathcal{T}^* B) \otimes_{E_{\mathcal{O}}^{(r)}} (\pi_0^r)^* \operatorname{Vert}(E).$$

Finally, the jet bundles of trivial holomorphic fiber bundles $M_1 \times M_2 \xrightarrow{\mathrm{pr}_1} M_1$ are denoted by $J_{\mathcal{O}}^r(M_1, M_2)$, $r \in \mathbb{N}$ and $\pi_0^1 : J_{\mathcal{O}}^1(M_1, M_2) \to M$ is a vector bundle whose fiber over $(p_1, p_2) \in M_1 \times M_2$ is

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{T}_{p_1} M_1, \mathcal{T}_{p_2} M_2).$$

2.1.2.1 Connections in vector bundles

We have mentioned above that if $E \to B$ is a fiber bundle, a section H of $\pi_0^1 : E^{(1)} \to E$ defines a projection $K : TE = H(B) \oplus \operatorname{Vert}(E) \to \operatorname{Vert}(E)$ called a Ehreshmann connection. Let us briefly introduce now some basic facts about connections in vector bundles that will be useful later on.

Since in vector bundles one can identify any fiber E_p with $\operatorname{Vert}_p(E)$, if E is a vector bundle of rank k, then the morphism K can be used to define a differential operator

$$\nabla : \Omega^0(M, E) = \Gamma(E) \longrightarrow \Gamma(T^* B \otimes E) = \Omega^1(M, E),$$

satisfying the Leibnitz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma, \quad \forall \sigma \in \Gamma(E), f \in C^\infty(M).$$

The local expression of a connection in a trivialization U_α of E can be written as

$$\nabla_{\big|U_\alpha} = d + A_\alpha,$$

where A_α is a $k \times n$ matrix of 1–forms called the *1–form of* ∇ *in* U_α. Note that the 1-form of ∇ in U_α in the intersection with another trivialization U_β satisfies that

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta} d g_{\alpha\beta}^{-1},$$

so they do not define a global 1–form (unless the vector bundle is trivial).

A connection ∇ in a vector bundle generalizes the exterior differential of forms. Indeed, one can extend the operator $\nabla : \Omega^0(B,E) \to \Omega^1(B,E)$ to $d_\nabla : \Omega^p(B,E) \to \Omega^{p+1}(B,E)$ by

$$d_\nabla(\theta \otimes \sigma) := d\theta \otimes \sigma + (-1)^p \theta \wedge d_\nabla \sigma,$$

where $\theta \in \Omega^p(B,E), \sigma \in \Omega^0(B,E)$ and $d_{\nabla\big|\Omega^0(B,E)} := \nabla$. We call the operator d^2_∇ *the curvature of* ∇. If ∇ is *flat* (i.e. if $d^2_\nabla = 0$), and E is a trivial line bundle, d_∇ defines a cohomology for forms with values in E that is called *Novikov cohomology.*

All the previous discussion can also be done in the holomorphic setting. In this case, the operator $\nabla : \Omega^0_{\mathcal{O}}(M,E) \to \Omega^1_{\mathcal{O}}(M,E)$ will be a holomorphic connection satisfying the Leibnitz rule for holomorphic forms and holomorphic functions.

2.1.3 Partial Differential Relations

Definition 2.1.22. *Let $\pi : E \to B$ be a smooth fiber bundle. A* Partial Differential Relation (PDR) of order *r is a subset $\mathcal{R}$ of $E^{(r)}$.*

A formal solution of *$\mathcal{R}$ is a section of $E^{(r)}$ whose image lies into $\mathcal{R}$. We denote the set of formal solutions of $\mathcal{R}$ by* $\mathrm{F}\mathcal{R}$*. A* solution of *$\mathcal{R}$ is a section $f : B \to E$ such that $j^r(f) \in \mathrm{F}\mathcal{R}$. Thus solutions of $\mathcal{R}$ are naturally identified with formal solutions that are holonomic, we denote that set by* $\mathrm{H}\mathcal{R}$*.*

Similarly, if $\pi : E \to B$ is a holomorphic fiber bundle, a Holomorphic Partial Differential Relation (HPDR) of order *r is a subset $\mathcal{R}$ of $E^{(r)}_{\mathcal{O}}$. In this case we denote the set of formal solutions that are also holomorphic by $\mathcal{O}\mathrm{F}\mathcal{R} := \mathrm{F}\mathcal{R} \cap \Gamma_{\mathcal{O}}(E^{(r)})$. Since any holonomic solution of a HPDR is indeed holomorphic, we have the following chain of inclusions*

$$\mathrm{H}\mathcal{R} \hookrightarrow \mathcal{O}\mathrm{F}\mathcal{R} \hookrightarrow \mathrm{F}\mathcal{R}.$$

Example 2.1.23. Let $\pi : E \to B$ be a holomorphic fiber bundle. One can see the r–jet bundle of holomorphic sections $E^{(r)}_{\mathcal{O}}$ as a Partial Differential Relation $E^{(r)}_{\mathcal{O}} \subset E^{(r)}$. This Partial Differential Relation is defined by the Cauchy-Riemann equations and it is indeed a subbundle of positive codimension of $E^{(r)}$. The holonomic solutions of this PDR, $\mathrm{H}\, E^{(r)}_{\mathcal{O}}$ are identified with the holomorphic sections of E, $\Gamma_{\mathcal{O}}(E)$. Moreover, the jet bundle of smooth sections $E^{(r)}$, inherits a structure of holomorphic fiber bundle over B and therefore we have the following chain of inclusions and identifications:

$$\Gamma_{\mathcal{O}}(E) \equiv \mathrm{H}\, E^{(r)}_{\mathcal{O}} \hookrightarrow \Gamma_{\mathcal{O}}(E^{(r)}_{\mathcal{O}}) \equiv \mathcal{O}\,\mathrm{F}\, E^{(r)}_{\mathcal{O}} := \mathrm{F}\, E^{(r)}_{\mathcal{O}} \cap \Gamma_{\mathcal{O}}(E^{(r)}) \hookrightarrow \Gamma(E^{(r)}_{\mathcal{O}}) \equiv \mathrm{F}\, E^{(r)}_{\mathcal{O}} \hookrightarrow \Gamma(E^{(r)}).$$

In $E^{(1)}$, the Partial Differential Relation $E^{(1)}_{\mathcal{O}}$ corresponds with the linear maps $L_x : T_{\pi(x)}B \to T_xE$ that are complex linear or, equivalently, with the non-vertical complex planes tangent to E with complex dimension equal to $\dim_{\mathbb{C}} B$. In higher orders, the elements of $E^{(r+1)}_{\mathcal{O}} \subset \left(E^{(r)}\right)^{(1)}$ are the 1–jet extensions of local holonomic (and therefore holomorphic) sections of $E^{(r)}_{\mathcal{O}}$ that are complex linear. $\bowtie$

Suppose now that the fiber bundle E of the previous example is a holomorphic affine bundle. Since a holomorphic section of E (i.e. a holonomic solution of the PDR $E^{(1)}_{\mathcal{O}} \subset E^{(1)}$) provides E the structure of a holomorphic vector bundle, one could say that a formal solution of $E^{(r)}_{\mathcal{O}}$ (i.e. a smooth section and a formal derivative satisfying Cauchy-Riemann conditions) provides E the structure of a "*formal holomorphic vector bundle*" that carries more information than just the complex vector bundle structure provided by the base section of the formal solution. The inclusion of $\mathrm{H}\, E^{(r)}_{\mathcal{O}}$ into $\mathrm{F}\, E^{(r)}_{\mathcal{O}}$ provides an inclusion of the set of holomorphic vector bundles into the set of formal holomorphic vector bundles.

This naive observation can be replicated for many geometric structures in Differential/Holomorphic Geometry. They can be interpreted as smooth/holomorphic sections of a fiber bundle that satisfy some differential conditions. Those conditions define a PDR/HPDR and their formal solutions define a formal structure that is usually easier to understand than the former geometric structure itself. We devote the rest of this Section to introduce some examples of Partial Differential and Holomorphic Relations whose holonomic solutions correspond with interesting geometric structures.

2.1.3.1 Maps of maximal rank: Immersions and Submersions

One easy example is the relation defined by *maps of maximal rank*, i.e. *immersions* and *submersions.*

Definition 2.1.24. *Let M_1 and M_2 be smooth (resp. complex) manifolds of dimensions n and m respectively. Let $f : M_1 \to M_2$ be a smooth (resp. holomorphic) map whose differential has maximal rank at every point of M_1. We will say that f is a* smooth immersion *(resp.* holomorphic immersion*) if $n \leq m$ and a* smooth submersion *(resp.* holomorphic submersion*) if $n \geq m$.*

Recall that the bundle $J^1(M_1, M_2) \to M_1 \times M_2$ (resp. $J^1_{\mathcal{O}}(M_1, M_2) \to M_1 \times M_2$) is a vector bundle whose fiber over a point $(p_1, p_2) \in M_1 \times M_2$) is $\mathrm{Hom}(T_{p_1}M_1, T_{p_2}M_2)$ (resp. $\mathrm{Hom}_{\mathbb{C}}(\mathcal{T}_{p_1}M_1, \mathcal{T}_{p_2}M_2)$). The previous Definition naturally determines the following partial differential and Holomorphic Partial Differential Relations

$$\mathcal{R}_{\text{Max-rank}}(M_1, M_2) \subset J^1(M_1, M_2), \quad \text{and} \quad \mathcal{R}_{\mathcal{O}\text{Max-rank}}(M_1, M_2) \subset J^1_{\mathcal{O}}(M_1, M_2)$$

as the set of those linear (resp. complex linear) maps that have maximal rank.

When $\dim M_1 \leq \dim M_2$ we denote

$$\mathcal{R}_{\text{Imm}}(M_1, M_2) := \mathcal{R}_{\text{Max-rank}}(M_1, M_2) \quad \text{and} \quad \mathcal{R}_{\mathcal{O}\,\text{Imm}}(M_1, M_2) := \mathcal{R}_{\mathcal{O}\text{Max-rank}}(M_1, M_2)$$

and we call their formal solutions *formal immersions* and *formal holomorphic immersions* respectively. These relations correspond to the sets of injective homomorphisms. Equivalently, these relations are formed by those non-vertical subspaces $\Lambda_p \subset T_p(M_1 \times M_2)$ tangent to $p = (p_1, p_2) \in M_1 \times M_2$ of dimension $\dim M_1$ such that

$$\dim(\Lambda_p \cap T(M_1 \times \{p_2\}) = 0.$$

If $\dim M_1 \geq \dim M_2$, then we denote

$$\mathcal{R}_{\text{Sub}}(M_1, M_2) := \mathcal{R}_{\text{Max-rank}}(M_1, M_2) \quad \text{and} \quad \mathcal{R}_{\mathcal{O}\,\text{Sub}}(M_1, M_2) := \mathcal{R}_{\mathcal{O}\text{Max-rank}}(M_1, M_2)$$

and we call their formal solutions *formal submersions* and *formal holomorphic submersions* respectively. Their elements correspond to surjective homomorphisms, or equivalently to the non-vertical subspaces Λ_p satisfying that

$$\dim(\Lambda_p \cap T(M_1 \times \{p_2\})) = \dim M_1 - \dim M_2.$$

Taking coordinates in both M_1 and M_2, the fiber of the relations $\mathcal{R}_{\text{Max-rank}}$ and $\mathcal{R}_{\mathcal{O}\text{Max-rank}}$ over a point in $M_1 \times M_2$ correspond to those matrices with maximal rank of size $\dim M_1 \times \dim M_2$ with coeficients in $\mathbb{R}$ and $\mathbb{C}$ respectively.

2.1.3.2 Formal primitives of exterior derivative operators

The following examples are going to be expressed in the language of differential forms with values in a smooth (resp. holomorphic) line bundle (i.e. a vector bundle of rank 1). These differential forms will need to satisfy some conditions that may be written in terms of their exterior derivative. To express these conditions as a subset of a jet bundle we will need to use the *symbol* of the exterior derivative operator as it is described below.

We first consider the smooth case. Let M be a smooth manifold. The exterior derivative operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ can be decomposed as

$$\Omega^k(M) \xrightarrow{j^1} \Gamma\left(\left(\bigwedge^k T^*M\right)^{(1)}\right) \xrightarrow{\widetilde{D}} \Omega^{k+1}(M),$$

where $\widetilde{D}$ is induced by the symbol of the differential operator d

$$D : (\bigwedge^k T^*M)^{(1)} \to \bigwedge^{k+1} T^*M.$$

Note that since D is an affine fibration, any smooth section $\sigma : M \to \bigwedge^{k+1} T^*M$ can be lifted in a homotopically canonical way to a *formal primitive of* σ, i.e. a section $F_\sigma : M \to (\bigwedge^k T^*M)^{(1)}$ such that $D \circ F_\sigma = \sigma$. This can be done parametrically and relative to parameter, i.e. continuously depending in a parameter in a compact Haussdorff space and remaining fixed in those parameters in a closed subset of the parametric space where there is already a formal primitive chosen. Moreover, since the operator d is *pure differential* (i.e. it involves just derivatives of sections, being independent of their values), the base section of F_σ can be chosen arbitrarily and independently of σ.

Now let B be a complex manifold and let $L \to B$ be a holomorphic line bundle. The exterior derivative operator acting on the holomorphic forms $d : \Omega^k_{\mathcal{O}}(B) \to \Omega^{k+1}_{\mathcal{O}}(B)$ (where it coincides with the operator ∂) is decomposed as

$$\Omega^k_{\mathcal{O}}(B) \xrightarrow{j^1_{\mathcal{O}}} \Gamma_{\mathcal{O}}\left(\left(\bigwedge^k \mathcal{T}^*B\right)^{(1)}_{\mathcal{O}}\right) \xrightarrow{\widetilde{D}} \Omega^{k+1}_{\mathcal{O}}(B).$$

Just like before, for every section $\sigma : B \to \bigwedge^{k+1} \mathcal{T}^*B$, there is a homotopically canonical formal primitive $F_\sigma : B \to (\bigwedge^k \mathcal{T}^*B)^{(1)}_{\mathcal{O}}$. F_σ is smooth in general, but if σ is holomorphic, since D is a surjective morphism of holomorphic vector bundles it can be assumed to be holomorphic.

Now we want to repeat the previous process to obtain formal primitives of closed forms with values in a vector bundle. The exterior derivative is not well defined in this case, so we will need to use connections. We are going to expose here the case of flat connections of line bundles.

Let M be an even-dimensional smooth manifold and let $L \to M$ be a smooth trivial line bundle with a flat connection ∇ determined by the closed 1–form μ. One can define D_∇, the symbol of d_∇, as

$$D_\nabla a := Da + \mu \wedge \pi^1_0(a)$$

for every $a \in ((\bigwedge^k T^*M) \otimes L)^{(1)}$. Similarly to the case of d in 2.1.3.2, induces a map $\widetilde{D}_\nabla$ that decomposes the operator $d_\nabla : \Omega^k(M, L) \to \Omega^{k+1}(M, L)$ as

$$\Omega^k(M, L) \xrightarrow{j^1} \Gamma\left(\left(\bigwedge^k (T^*M) \otimes L\right)^{(1)}\right) \xrightarrow{\widetilde{D}_\nabla} \Omega^{k+1}(M, L),$$

Analogously to the exterior differential operator, the map

$$D_\nabla : \left(\bigwedge^k (T^*M) \otimes L\right)^{(1)} \to \bigwedge^{k+1} (T^*M) \otimes L$$

is an affine fibration. Therefore we can also find formal primitives F_σ of L–valued forms $\sigma \in \Omega^{k+1}(M, L)$ in a parametric and relative to parameter way, but since d_∇ is not pure differential there will no free choice of the base section of F_σ.

Like for the case of the exterior derivative operator, all of the previous work can be mimicked for holomorphic connections in holomorphic line bundles.

2.1.3.3 Locally conformal symplectic forms

Definition 2.1.25. *Let M be a smooth (resp. complex) manifold of even dimension* $\dim M = 2n$ *(resp.* $\dim_{\mathbb{C}} M = 2n$*) and let $L \to M$ be a smooth (resp. holomorphic) trivial line bundle with a smooth (resp. holomorphic) flat connection ∇. We say that a closed 2–form on M with values in $L, \omega \in \Omega^2(M, L)$ (resp. $\omega \in \Omega^2_{\mathcal{O}}(M, L)$), is a* locally conformal symplectic form *(resp.* complex locally conformal symplectic form*) if it is d_∇–closed (i.e. $d_\nabla \omega = 0$) and nondegenerate, i.e. if*

$$\omega^n \neq 0 \tag{Symp}$$

everywhere. In this case, (M, L, ∇, ω) is called a locally conformal symplectic manifold *(resp.* complex locally conformal symplectic manifold*). We denote by $\Omega^2_{\mathrm{LCSymp}}(M, L, \nabla)$ (resp. $\Omega^2_{\mathcal{O}\,\mathrm{LCSymp}}(M, L, \nabla)$) the set of smooth (resp. complex) locally conformal symplectic forms of M with values in L.*

If $\omega = d_\nabla \lambda$ for some smooth (resp. holomorphic) 1–form λ with values in L we say that (M, L, ∇, ω) is exact *and that λ is a* Liouville form *for ω. We denote by $\Omega^1_{\mathrm{Liouv}}(M, L, \nabla)$ (resp. $\Omega^1_{\mathcal{O}\,\mathrm{Liouv}}(M, L, \nabla)$) the set of smooth (resp. holomorphic) Liouville forms of M with values in L.*

If the 2–form ω is not d_∇–closed but it still satisfies condition (Symp), then we will say that ω is a locally conformal almost-symplectic form *(resp.* complex locally conformal almost-symplectic form*) and that (M, ω) is a* locally conformal almost-symplectic manifold *(resp.* complex locally conformal almost-symplectic manifold*). We denote by $\Omega^2_{\mathrm{LCASymp}}(M, L, \nabla)$ and by $\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(M, L, \nabla)$ the corresponding sets of smooth and holomorphic locally conformal almost-symplectic forms.*

Remark 2.1.26. Let us provide some observations

1. Note that when the line bundle of the previous definition is trivialized and ∇ is the canonical connection d, then we can remove the words "*locally conformal*" in all of the previous concepts since they coincide with the classical ones of symplectic topology.

2. Given a connection ∇ in a line bundle L, the local expression in a local trivialization U_α of the operator d_∇ can be written as

$$d_\nabla = d + \mu,$$

where μ is the *1–form of the connection in* U_α. The local 1-forms do not define a global 1–form in general due to the transition functions of L, but they obviously do if the line bundle is trivial. It is straightforward to check that ∇ is flat (i.e. $d_\nabla^2 = 0$) if and only if the 1–form defining ∇, μ, is closed.

3. Taking local primitives of the 1–form of a connection $\mu_{|_U} = df$ defined in an open set U, one can realize that the local 2-forms $e^f\omega$ are simplectic forms in U. Indeed

$$d(e^f\omega) = e^f(d\omega + \mu \wedge \omega) = 0.$$

Therefore, ω is locally expressed as a multiple of a symplectic form. This motivates the use of the words "*locally conformal*" in the previous definition.

4. As it can be seen in Proposition 2.1 in [CM19], the observations 2 and 3 provide equivalent definitions of locally conformal symplectic manifolds.

$\bowtie$

Example 2.1.27. Let M be a smooth manifold, there is a unique form $\lambda \in \Omega^1(T^*M)$ such that

$$\alpha^*\lambda = \alpha$$

for every $\alpha \in \Omega^1(M) \simeq \Gamma(T^*M)$. This form is called the *Liouville form* on T^*M. Let $p \in T^*M$, the Liouville form at p is defined by

$$\lambda(p) := (d\pi)^*p = p \circ d\pi,$$

where $\pi : T^*M \to M$ is the natural projection. Taking local coordinates $(q_1, \dots, q_n)$ in $\mathcal{O}p\,(\pi(p))$ any 1–form α can be written locally as $\sum_{j=1}^n p_j dq_j$, that gives the coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ for the cotangent bundle T^*M. In this coordinates, the Liouville form is

$$\lambda(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{j=1}^n p_j dq_j.$$

This yields that locally $d\lambda = \sum_{j=1}^{n} dp_j \wedge dq_j$ so it is easy to check that $d\lambda$ is nondegenerate. Therefore, every cotangent bundle is canonically an exact symplectic manifold. Note that all the previous process can be replicated in the holomorphic category, so the complex cotangent bundle $\mathcal{T}^*B$ is an exact complex symplectic manifold of every complex manifold B. $\bowtie$

The condition of being exact locally conformal symplectic defines the following PDR

$$\mathcal{R}^0_{\text{LCSymp}}(M, L, \nabla) := \left\{ a \in (T^*M \otimes L)^{(1)} : (D_\nabla a)^n \neq 0 \right\}.$$

Now let B be an even-dimensional complex manifold and let $L \to B$ be a holomorphic trivial line bundle with a holomorphic flat connection ∇ determined by μ. We can replicate the process to define the following HPDR

$$\mathcal{R}^0_{\mathcal{O}\,\text{LCSymp}}(B, L, \nabla) := \left\{ a \in (\mathcal{T}^*B \otimes L)^{(1)}_{\mathcal{O}} : (D_\nabla a)^n \neq 0 \right\}.$$

The holonomic solutions σ of the previous relations provide exact smooth or complex locally conformal symplectic forms

$$d_\nabla \sigma^{(0)}.$$

This yields the equivalences

$$\mathrm{H}\mathcal{R}^0_{\text{LCSymp}}(M, L) \equiv \Omega^1_{\text{Liouv}}(M, L) \quad \text{and} \quad \mathrm{H}\mathcal{R}^0_{\mathcal{O}\,\text{LCSymp}}(B, L) \equiv \Omega^1_{\mathcal{O}\,\text{Liouv}}(B, L).$$

As for D, the symbol of d_∇ provides the fibrations

$$\widetilde{D}_\nabla : \mathrm{F}\mathcal{R}^0_{\text{LCSymp}}(M, L, \nabla) \longrightarrow \Omega^2_{\text{LCASymp}}(M, L, \nabla)$$

and

$$\widetilde{D}_\nabla : \mathrm{F}\mathcal{R}^0_{\mathcal{O}\,\text{LCSymp}}(B, L, \nabla) \longrightarrow \Omega^2_{\mathcal{O}\,\text{LCASymp}}(B, L, \nabla),$$

whose fibers are contractible. Therefore these maps induce homotopy equivalences whose homotopy inverses consist on taking a smooth or holomorphic formal primitive. Then

$$\mathrm{F}\mathcal{R}^0_{\text{LCSymp}}(M, L, \nabla) \xrightarrow{\sim} \Omega^2_{\text{LCASymp}}(M, L, \nabla)$$

and

$$\mathrm{F}\mathcal{R}^0_{\mathcal{O}\,\mathrm{LCSymp}}(B,L,\nabla) \xrightarrow{\sim} \Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(B,L,\nabla).$$

We can broad the previous concepts to other cohomology clases of locally conformal symplectic forms. Let $\eta \in \Omega^2(M,L)$ be a closed 2-form with values in L. We define the Partial Differential Relation

$$\mathcal{R}^{\eta}_{\mathrm{LCSymp}}(M,L,\nabla) := \left\{ a \in (T^*M \otimes L)^{(1)} : (D_\nabla a + \eta(\pi^1(a)))^n \neq 0 \right\}.$$

In the holomorphic setting, take a closed holomorphic 2-form $\eta \in \Omega^2_{\mathcal{O}}(B,L)$ to define the Holomorphic Partial Differential Relation

$$\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B,L,\nabla) := \left\{ a \in (\mathcal{T}^*M \otimes L)^{(1)}_{\mathcal{O}} : (D_\nabla a + \eta(\pi^1(a)))^n \neq 0 \right\}.$$

Let $\Omega^2_{[\eta]\,\mathrm{LCSymp}}(M,L,\nabla)$ and $\Omega^2_{\mathcal{O}[\eta]\,\mathrm{LCSymp}}(B,L,\nabla)$ be the spaces of locally conformal symplectic forms in the smooth and holomorphic cohomology class of η respectively. The holonomic solutions σ of the relations $\mathcal{R}^{\eta}_{\mathrm{LCSymp}}(M,L,\nabla)$ and $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B,L,\nabla)$ provide symplectic forms

$$\eta + \widetilde{D}_\nabla(\sigma) = \eta + d_\nabla \sigma^{(0)}$$

in $\Omega^2_{[\eta]\,\mathrm{LCSymp}}(M,L,\nabla)$ and $\Omega^2_{\mathcal{O}[\eta]\,\mathrm{LCSymp}}(B,L,\nabla)$ respectively. Since the space of primitives of a 2-form is convex, this yields the homotopy equivalences

$$\mathrm{H}\mathcal{R}^{\eta}_{\mathrm{LCSymp}}(M,L,\nabla) \xrightarrow{\sim} \Omega^2_{[\eta]\,\mathrm{LCSymp}}(M,L,\nabla)$$

and

$$\mathrm{H}\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B,L,\nabla) \xrightarrow{\sim} \Omega^2_{\mathcal{O}[\eta]\,\mathrm{LCSymp}}(B,L,\nabla).$$

In a similar way we obtain the homotopy equivalences

$$\mathrm{F}\mathcal{R}^{\eta}_{\mathrm{LCSymp}}(M,L,\nabla) \xrightarrow{\sim} \Omega^2_{\mathrm{LCASymp}}(M,L,\nabla)$$

and

$$\mathrm{H}\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B,L,\nabla) \xrightarrow{\sim} \Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(B,L,\nabla).$$

As it can be seen in 1 in Remark 2.1.26, locally conformal symplectic geometry generalizes symplectic geometry when the line bundle L is trivialized and ∇ is the exterior derivative. Therefore we denote by

$$\Omega^2_{\mathrm{Symp}}(M) := \Omega^2_{\mathrm{LCSymp}}(M, M\times\mathbb{R}, d) \quad \text{and} \quad \Omega^2_{\mathcal{O}\,\mathrm{Symp}}(M) := \Omega^2_{\mathcal{O}\,\mathrm{LCSymp}}(M, M\times\mathbb{C}, d),$$

and we proceed analogously to define $\Omega^1_{\mathrm{Liouv}}(M)$, $\Omega^1_{\mathcal{O}\,\mathrm{Liouv}}(M)$, $\Omega^2_{\mathrm{ASymp}}(M)$, $\Omega^2_{\mathcal{O}\,\mathrm{ASymp}}(M)$, $\mathcal{R}^{\eta}_{\mathrm{Symp}}(M)$ and $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{Symp}}(M)$.

A fundamental result in Symplectic Geometry is the following

Theorem 2.1.28 (Darboux). *Let (M,ω) be a (smooth or complex) symplectic manifold. For every point in M there exists some local coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$ in which the local expresion of ω is*

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j.$$

□

Darboux Theorem implies that symplectic manifolds of the same dimension are locally indistinguishable from each other. In the context of smooth or holomorphic distributions (i.e. vector subbundles of vector bundles or, equivalently, sections of the Grassmannian bundle), open subsets of distributions that are locally invariant are called *topologically stable*. Cartan showed in [Car01] that there are just four topologically stable sets of distributions in the tangent bundle of smooth manifolds. As it is shown in [PSC14], the holomorphic case is completely analogous. The topologically stable distributions in smooth or complex manifolds are

- Smooth/complex line fields,
- Smooth/complex contact distributions,
- Smooth/complex even-contact distributions,
- Smooth/complex Engel plane fields.

Smooth and holomorphic line bundles correspond with sections of the smooth and holomorphic vector bundles respectively, we will study below the rest of the topologically stable distributions.

2.1.3.4 Contact forms

Definition 2.1.29. *Let M be a smooth (resp. complex) manifold of odd dimension* $\dim M = 2n+1$ *(resp* $\dim_{\mathbb{C}} M = 2n+1$ *and let $L \to M$ be a smooth (resp. holomorphic) line bundle. We say that a smooth (resp. holomorphic) 1–form α over M with values in L is a* contact form *(resp.* complex contact form*) if*

$$\alpha \wedge (d\alpha)^n \neq 0 \qquad \text{(Cont)}$$

everywhere. In this case, $\xi = \ker \alpha \subset TM$ (resp. $\xi = \ker \alpha \subset \mathcal{T}M$) defines a completely nonintegrable hyperplane distribution. The pair (M, ξ) is a contact manifold *(resp.* complex contact manifold*). These distributions ξ will be called* coorientable *when the line bundle $L \simeq TM/\xi$ (resp. $L \simeq \mathcal{T}M/\xi$) is trivial.* A coorientation of *ξ is a trivialization of L.*

We denote the space of smooth (resp. complex) contact forms in M with values in L by $\Omega^1_{\mathrm{Cont}}(M, L)$ (resp. by $\Omega^1_{\mathcal{O}\,\mathrm{Cont}}(M, L)$).

Condition (Cont) of being a smooth or complex contact form in M with values in L defines the following PDR in the smooth setting

$$\mathcal{R}_{\mathrm{Cont}}(M, L) := \left\{ a \in (T^*M \otimes L)^{(1)} : a^{(0)} \wedge (Da)^n \neq 0 \right\},$$

and the following HPDR in the holomorphic case

$$\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(M, L) := \left\{ a \in (\mathcal{T}^*M \otimes L)^{(1)}_{\mathcal{O}} : a^{(0)} \wedge (Da)^n \neq 0 \right\}.$$

The holonomic solutions σ of the previous relations provide smooth or complex contact forms $\sigma^{(0)}$ since

$$\sigma^{(0)} \wedge (d(\sigma^{(0)}))^n = \sigma^{(0)} \wedge (\widetilde{D}(\sigma))^n \neq 0.$$

This yields the equivalences

$$\mathrm{H}\mathcal{R}_{\mathrm{Cont}}(M, L) \equiv \Omega^1_{\mathrm{Cont}}(M, L) \quad \text{and} \quad \mathrm{H}\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(M, L) \equiv \Omega^1_{\mathcal{O}\,\mathrm{Cont}}(B, L).$$

2.1.3.5 Even-contact forms

Definition 2.1.30. *Let M be a smooth (resp. complex) manifold of even dimension* $\dim M = 2n+2$ *(resp.* $\dim_{\mathbb{C}} M = 2n+2$*) and let $L \to M$ be a smooth (resp. holomorphic) line bundle. We say that a smooth (resp. holomorphic) 1–form α over M with values in L is an* even-contact form *(resp.* complex even-contact form*) if*

$$\alpha \wedge (d\alpha)^n \neq 0 \qquad \text{(ECont)}$$

everywhere. In this case, $\mathcal{E} := \ker \alpha \subset TM$ (resp. $\mathcal{E} := \ker \alpha \subset \mathcal{T}M$) defines a hyperplane distribution such that $[\mathcal{E}, \mathcal{E}] = TM$ (resp. $[\mathcal{E}, \mathcal{E}] = \mathcal{T}M$). The pair $(M, \mathcal{E})$ is called an even-contact manifold *(resp. a* complex even-contact manifold*).*

We denote the space of smooth (resp. complex) even-contact forms in M with values in L by $\Omega^1_{\mathrm{ECont}}(M, L)$ (resp. by $\Omega^1_{\mathcal{O}\,\mathrm{ECont}}(M, L)$).

In the same way as in the contact case, condition (ECont) defines the PDR

$$\mathcal{R}_{\mathrm{ECont}}(M, L) := \left\{ a \in (T^*M \otimes L)^{(1)} : a^{(0)} \wedge (Da)^n \neq 0 \right\},$$

and the following HPDR in the holomorphic case

$$\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(M, L) := \left\{ a \in (\mathcal{T}^*M \otimes L)^{(1)}_{\mathcal{O}} : a^{(0)} \wedge (Da)^n \neq 0 \right\}.$$

The holonomic solutions σ of the previous relations provide smooth or complex even-contact forms $\sigma^{(0)}$ since

$$\sigma^{(0)} \wedge (d(\sigma^{(0)}))^n = \sigma^{(0)} \wedge (\widetilde{D}(\sigma))^n \neq 0.$$

This yields the equivalences

$$\mathrm{H}\mathcal{R}_{\mathrm{ECont}}(M, L) \equiv \Omega^1_{\mathrm{ECont}}(M, L) \quad \text{and} \quad \mathrm{H}\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(M, L) \equiv \Omega^1_{\mathcal{O}\,\mathrm{ECont}}(B, L).$$

2.1.3.6 Engel pairs

Definition 2.1.31. *Let M be a smooth (resp. complex) manifold of dimension 4 and let $L \to B$ be a smooth (resp. holomorphic) line bundle. Let α and β be smooth (resp. holo-*

morphic) even-contact forms with values in L. We say that (α, β) *is an* Engel pair *(resp. a* holomorphic Engel pair*) if*

(Eng.1) $$\alpha \wedge \beta \wedge d\alpha \neq 0,$$

(Eng.2) $$\alpha \wedge \beta \wedge d\beta = 0$$

everywhere. The plane distribution $\mathcal{D} := \ker \alpha \cap \ker \beta$ *is called an* Engel distribution *(resp. a* holomorphic Engel distribution*) and the pair* $(M, \mathcal{D})$ *is an* Engel manifold *(resp.* holomorphic Engel manifold*).*

For $a, b \in (T^*M \otimes L)^{(1)}$ (resp. $a, b \in (\mathcal{T}^*M \otimes L)^{(1)}_{\mathcal{O}}$), the formal version of conditions (Eng.1) and (Eng.2) can be written as follows

(FEng.1) $$a^{(0)} \wedge b^{(0)} \wedge Da \neq 0,$$

(FEng.2) $$a^{(0)} \wedge b^{(0)} \wedge Db = 0.$$

The set of pairs $(a, b) \in (T^*M \otimes L)^{(1)} \times_M (T^*M \otimes L)^{(1)} = ((T^*M \otimes L) \times_M (T^*M \otimes L))^{(1)}$ that satisfy conditions (FEng.1) and (FEng.2) forms a PDR that will be denoted by $\mathcal{R}_{\text{Engel}}(M, L)$. We define analogously the corresponding HPDR $\mathcal{R}_{\mathcal{O}\,\text{Engel}}(M, L)$.

2.2 *H*–principles

Let $\mathcal{R}$ be a Partial Differential Relation. Solving $\mathcal{R}$ consists on finding holonomic solutions of $\mathcal{R}$, sometimes under some prescribed requirements like, for example, coinciding with some fixed solution over a closed set. This problem a priori can be really difficult. Indeed, solving a partial differential equation is a problem of this kind since any PDE determines a PDR. Therefore, since the existence of formal solutions is a necessary condition for $\mathcal{R}$ to be solved, it is important to solve $\mathcal{R}$ first formally and try to find holonomic solutions once the existence of formal solutions has been stablished.

Finding formal solutions is purely an algebraic/homotopy-theoretical problem, so it constitutes an important simplification of the original differential problem of solving $\mathcal{R}$. That is the reason why one does not expect that finding formal solutions is sufficient to find holonomic ones. Nevertheless there are many geometrically interesting differential relations for whom solving the problem formally is enough to solve $\mathcal{R}$. This is captured in the following

Definition 2.2.1. *Let $\mathcal{R}$ be a PDR or a HPDR. We say that $\mathcal{R}$* satisfies the h–principle *if every formal solution $\sigma \in \mathrm{F}\mathcal{R}$ is homotopic in $\mathrm{F}\mathcal{R}$ to a holonomic one, i.e. if there exist a continuous family of formal solutions $\sigma_t \in \mathrm{F}\mathcal{R}, t \in [0,1]$ such that $\sigma_0 = \sigma$ and $\sigma_1 \in \mathrm{H}\mathcal{R}$.*

This is equivalent to saying that the inclusion $i : \mathrm{H}\mathcal{R} \hookrightarrow \mathrm{F}\mathcal{R}$ induces an epimorphism $i_* : \pi_0(\mathrm{H}\mathcal{R}) \to \pi_0(\mathrm{F}\mathcal{R})$. We say that $\mathcal{R}$ satisfies a full *or* a complete h–principle *if the inclusion $i : \mathrm{H}\mathcal{R} \hookrightarrow \mathrm{F}\mathcal{R}$ is a weak homotopy equivalence.*

In some situations we want to solve the relation $\mathcal{R}$ under some prescribed requirements. That requirements will lead us to some different kinds of h–principles that will be defined in the following Section. For example we can ask for finding solutions just near some subset of the base space, this yields to the notion of *local h–principles.* Some other times we ask our solutions to coincide with some fixed solution over a closed subset leading to *relative h–principles.* We can also define *fibered h–principles* for parametric families of relations $P \times \mathcal{R}$ that can be relative to some subset Q of the parametric space P. Our last example will follow from considering parametric families of solutions that will lead us to *parametric (and relative to parameter) h–principles.*

2.2.1 Kinds of h–principles

All the original results of this Thesis are local h–principles so let us commence this section by introducing them.

The next two definitions adapt the notions of formal and holonomic solutions and h–principles for the context of local sections.

Definition 2.2.2. *We say that a local section of $E^{(r)}$ over a subset $S \subseteq B$, $\sigma \in \Gamma(\mathcal{O}p\,(S), E^{(r)})$, is* holonomic *if there exists an open neighbourhood of S, U, where it coincides with the r–jet of a section of E over U, i.e. if there exists $s : U \to E$ such that $j^r(\sigma^{(0)})_{|U} = s$.*

Definition 2.2.3. *Fix a partial differential relation $\mathcal{R} \subset E^{(r)}$ and the fibration $\pi_{\mathcal{R}} : \mathcal{R} \to B$. We define the* set of formal local solutions*,* $\mathrm{F}\mathcal{R}(S)$*, as the set of local sections of the fibration* $\pi_{\mathcal{R}}$ *over* S*. We also define the* set of holonomic local solutions $\mathrm{H}\mathcal{R}(S)$*, as the set of formal local solutions that are holonomic in some neighbourhood of* S*.*

We say that a partial differential relation $\mathcal{R} \subset E^{(r)}$ satisfies a local h–principle over $S \subset B$ *if for every formal local solution* σ_0 *there exists a homotopy of formal local solutions* $\sigma_t, t \in [0,1]$ *that joins* σ_0 *with a holonomic local solution* σ_1*. We say that* $\mathcal{R}$ *satisfies a full local h–principle over* S *if the inclusion* $i : \mathrm{H}\mathcal{R}(S) \hookrightarrow \mathrm{F}\mathcal{R}(S)$ *is a weak homotopy equivalence.*

We proceed now with the notion of relative to domain h–principles for local sections.

Definition 2.2.4. *Let $S \subset B$ be a closed subset of B and likewise let C be a closed subset of S. Let $\sigma : \mathcal{O}p\,(S) \to E^{(r)}$ be in $\mathrm{F}\mathcal{R}(S, C) := \mathrm{F}\mathcal{R}(S) \cap \mathrm{H}\mathcal{R}(C)$ where $\mathcal{R} \subset E^{(r)}$ is a partial differential relation of the bundle $E \to B$. We will denote the space of formal local solutions over S that coincide with $\sigma_{|C}$ over C by $\mathrm{F}\mathcal{R}(S, C, \sigma)$. We also define $\mathrm{H}\mathcal{R}(S, C, \sigma)$ analogously.*

We say that the local h–principle over S is relative to C if for every formal solution $\sigma_0 \in \mathrm{F}\mathcal{R}(S, C)$ there exists a homotopy $\sigma_t \in \mathrm{F}\mathcal{R}(S, C, \sigma_0), t \in [0,1]$ such that $\sigma_1 \in \mathrm{H}\mathcal{R}(S, C, \sigma_0)$. We also say that the local h–principle is relative to closed domain if it is relative to every closed domain $C \subset S$.

Since an analytic section is determined by its restriction to an open set, finding h-principles that are relative to domains with nonempty interiors is impossible for most of the differential relations that lie into $E^{(r)}_{\mathcal{O}}$ for some holomorphic vector bundle $E \to B$. That is the reason why we give the following weaker notion of relativity to domain.

Definition 2.2.5. *Let $\mathrm{F}\mathcal{R}(S, C, \sigma, \varepsilon)$ be the space of formal local solutions over S that are in $\mathrm{H}\mathcal{R}(C)$ and are ε–close to $\sigma_{|C}$ over C in $C^0(C, (E^{(r)})_{|C})$. Let $\mathrm{H}\mathcal{R}(S, C, \sigma, \varepsilon)$ be the analogous space for holonomic local solutions.*

We say that the local h–principle over S is weakly relative to C *if for every $\varepsilon > 0$ and for every $\sigma_0 \in \mathrm{F}\mathcal{R}(S, C)$ there exists a homotopy $\sigma_t \in \mathrm{F}\mathcal{R}(S, C, \sigma_0, \varepsilon), t \in [0,1]$ such that $\sigma_1 \in \mathrm{H}\mathcal{R}(S, C, \sigma_0, \varepsilon)$ for every $\varepsilon > 0$ small enough. We say that the local h–principle is* weakly relative to domain *if it is weakly relative to every closed domain $C \subset S$.*

Now let us consider fibered differential relations $\mathcal{R} \subset P \times E^{(r)}$. They can be seen as a family of differential relations $\mathcal{R}_p = (\{p\} \times E^{(r)}) \cap \mathcal{R} \hookrightarrow E^{(r)}$ that are continuously dependent on some parameter p in a compact Hausdorff space P. We define now the set of formal and holonomic local solutions of a fibered differential relation $\mathcal{R}$ and we will use the same notation as in the non-fibered case. Let the set of *fibered formal local solutions* be

$$\mathrm{F}\mathcal{R}(S) := \{\sigma : P \times \mathcal{O}p\,(S) \to \mathcal{R} | \sigma_p = \sigma(p, \text{-}\,) \in \mathrm{F}\mathcal{R}_p(S), \forall p \in P\}.$$

The sets $\mathrm{F}\mathcal{R}(S, C), \mathrm{F}\mathcal{R}(S, C, \sigma)$ and $\mathrm{F}\mathcal{R}(S, C, \sigma, \varepsilon)$ are defined analogously, and also their corresponding sets of parametric holonomic local solutions for any closed subset $C \subset S$, any $\sigma \in \mathrm{H}\mathcal{R}(C)$ and any $\varepsilon > 0$.

All the previous kinds of local h–principles can also be defined for fibered differential relations. In this case we will also say that the local h–principles are *relative to* Q, a compact subset of P, if the homotopies can be chosen to be fixed at Q when the former parametric formal local solution is holonomic over $Q \times \mathcal{O}p\,(S)$. We also say that the fibered h–principle is relative to the fiber if it is relative to every compact subset $Q \subset P$.

Definition 2.2.6. *Let $\mathcal{R} \subset E^{(r)}$ be a partial differential relation of $E \to B$, let $S \subset B$ be a closed subset, let P be a compact Hausdorff space and let $Q \subset P$ be a compact subset. Then we will say that* $\mathcal{R}$ satisfies an h–principle that is parametric with parameter P and relative to Q *if the fibered relation $\mathcal{R}_P := P \times \mathcal{R}$ satisfies a (fibered) h–principle that is relative to the fiber Q. In addition we will say that*

- *the h–principle is* full *if it is parametric for every disk $P = \mathbb{D}^k$ and relative to ∂P.*
- *the parametric over P h–principle is* relative to parameter *if it is relative to Q for every compact subset $Q \subset P$.*
- *the h–principle is* parametric *if it is parametric for every compact Hausdorff space P.*
- *the h–principle is* parametric and relative to parameter *if it is parametric and for every compact Hausdorff space P the parametric over P h–principle is relative to parameter.*

We can define analogously the full, parametric and/or relative to parameter versions of the relative and weakly relative to domain local h–principles.

Remark 2.2.7. Note that the two definitions of full local h–principles are equivalent.

Indeed the h–principle is parametric for $\mathcal{D}^k$ and relative to $\partial\mathcal{D}^k$ for each $k \in \mathbb{N} \cup \{0\}$ if and only if the relative homotopy group $\pi_k(\mathrm{F}\mathcal{R}(S), \mathrm{H}\mathcal{R}(S)) = 0$. Then, the long exact sequence

$$\ldots \longrightarrow \pi_k(\mathrm{H}\mathcal{R}(S)) \xrightarrow{i_*} \pi_k(\mathrm{F}\mathcal{R}(S)) \longrightarrow \pi_k(\mathrm{F}\mathcal{R}(S), \mathrm{H}\mathcal{R}(S)) \longrightarrow \pi_{k-1}(\mathrm{H}\mathcal{R}(S)) \longrightarrow \ldots$$
$$\ldots \longrightarrow \pi_0(\mathrm{F}\mathcal{R}(S), \mathrm{H}\mathcal{R}(S))$$

yields that $\pi_k(\mathrm{F}\mathcal{R}(S), \mathrm{H}\mathcal{R}(S)) = 0$, for every $k \in \mathbb{N} \cup \{0\}$ if and only if we have that $\pi_k(\mathrm{H}\mathcal{R}(S)) \overset{i_*}{\cong} \pi_k(\mathrm{F}\mathcal{R}(S))$ for every $k \in \mathbb{N} \cup \{0\}$. $\bowtie$

Now we give the notion of C^0–density in the context of h–principles for local sections.

Definition 2.2.8. *We say that a local h–principle over a closed subset S is C^0–*dense *if for each formal local solution σ and every $U = \mathcal{O}p_E\left(\sigma^{(0)}(S)\right)$, where $\sigma^{(0)}$ stands for the 0-jet part of σ, (i.e. $\sigma^{(0)} := \pi_0^r \circ \sigma$) the homotopy σ_t, $t \in [0,1]$, joining $\sigma = \sigma_0$ with a holonomic local solution σ_1, can be chosen in such a way that $\sigma_t^{(0)}(S) \subset U$, for all $t \in [0,1]$.*

In a similar way we can define the C^0–dense versions of all the previously defined kinds of h–principles.

Since Holomorphic Partial Differential Relations can be seen as partial differential relations, all the previous definitions work for them. Nevertheless sometimes we may be interested in h–principles whose homotopies are formed by holomorphic formal local solutions. The following definitions are made to take that into account.

Definition 2.2.9. *Let $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ be a Holomorphic Partial Differential Relation of a holomorphic bundle $E \to B$ and let $S \subset B$ be a closed subset. Denote by $\mathcal{O}\,\mathrm{F}\mathcal{R}(S)$ the subset of $\mathrm{F}\mathcal{R}(S)$ conformed by those formal local solutions that are holomorphic. We say that the local h–principle is* holomorphic *if for every $\sigma_0 \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S)$ there exists a homotopy $\sigma_t \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S)$ such that $\sigma_1 \in \mathrm{H}\mathcal{R}(S)$. Analogously we can define the full, parametric, relative to parameter and C^0–dense* holomorphic *local h–principles.*

Let $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}, r \geq 1$ be a Holomorphic Partial Differential Relation. Note that by the identity principle, if $C \subset S$, then $\mathrm{H}\mathcal{R}(S) = \mathcal{O}\,\mathrm{F}\mathcal{R}(S) \cap \mathrm{H}\mathcal{R}(C)$. Therefore the definition of weakly relative to domain given above is not useful in this context, indeed it attains for homotopies that preserve holonomy over C. Nevertheless sometimes we may be interested in finding homotopies made of holomorphic formal local solutions over S that are *close* to a holonomic local solution over C. This yields to the following definition.

Definition 2.2.10. *Let $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ be a Holomorphic Partial Differential Relation of a holomorphic bundle $E \to B$, let $C \subset S \subset B$ be closed subsets and let $\sigma \in \mathrm{F}\mathcal{R}(C)$ be a formal local solution. We will say that σ is a* pseudo-holonomic local solution over C *if $j^r(\sigma^{(0)}) \in \mathrm{H}\mathcal{R}(C)$ and the linear interpolation between σ and $j^r(\sigma^{(0)})$ is formed by formal solutions of $\mathcal{R}_{|\mathcal{O}p(C)}$.*

Let us denote by $p\,\mathrm{H}\mathcal{R}(C)$ the set of pseudo-holonomic local solutions over C, and let

$$\mathcal{O}\,\mathrm{F}\mathcal{R}(S,C) := \mathcal{O}\,\mathrm{F}\mathcal{R}(S) \cap p\,\mathrm{H}\mathcal{R}(C).$$

Let $\sigma \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S,C)$ and let $\mathcal{O}\,\mathrm{F}\mathcal{R}(S,C,\sigma,\varepsilon)$ be the subset of sections of $\mathcal{O}\,\mathrm{F}\mathcal{R}(S,C)$ that are ε–close to $j^r(\sigma^{(0)})_{|C}$ over C in $C^0(C,(E^{(r)})_{|C})$.

We say that the holomorphic h–principle of $\mathcal{R}$ is close over C *if for each $\varepsilon > 0$ and for every $\sigma_0 \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S,C)$ there exists a homotopy $\sigma_t \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S,C,\sigma_0,\varepsilon), t \in [0,1]$ such that $\sigma_1 \in \mathrm{H}\mathcal{R}(S)$. We also say that the holomorphic h–principle is* close over domain *if it is close over every closed domain $C \subset S$.*

We define similarly the full, C^0–dense, parametric and/or relative to parameter close over domain holomorphic h–principles.

2.2.2 Convex integration

In order to find instances of Partial Differential Relations satisfying some kind of h–principles one always follows the same strategy. One isolates a particular, or not so particular, class of PDRs, and then prove that all of them fulfill the h–principle using some specific technique.

One of the most important techniques used to find h–principles is *convex integration*. It works for open and ample Partial Differential Relations.

Definition 2.2.11. *A Partial Differential Relation is called* open *if it is open as a subset of $E^{(r)}$.*

A Partial Differential Relation $\mathcal{R} \subset E^{(1)}$ is called ample in principal directions *if $\mathcal{R}$ intersects all principal subspaces of each fiber of $\pi_0^1 : E^{(1)} \to E^{(0)} = E$ along ample sets.*

Recall that a subset Ω of an affine space A is called *ample* if either it is empty or if the convex hull of each path-connected component of Ω is A. There are more general definitions of ampleness, even for $r > 1$, where we can use this technique (see section 2.4.3 in [Gro73]). Nevertheless we will not refer to them throughout this thesis for simplicity.

One important class of open and ample Partial Differential Relations are Thick Differential Relations

Definition 2.2.12. *A Partial Differential Relation $\Sigma \subset E^{(1)}$ is called a* thin differential relation *(or a* thin singularity*) if every nonempty intersection with a principal subspace is a stratified manifold of codimension ≥ 2. We say that $\mathcal{R} \subset E^{(1)}$ is a* Thick Differential Relation *(TDR) if it is the complement of a thin singularity.*

Example 2.2.13.

- The PDR $\mathcal{R}_{\mathrm{Imm}}(M_1, M_2)$ is thick whenever $\dim M_1 < \dim M_2$. In the other hand the PDR $\mathcal{R}_{\mathrm{Sub}}(M_1, M_2)$ is open but not ample (See page 163 in [EM02]).
- The PDR $\mathcal{R}_{\mathrm{ECont}}(M, L)$ is thick for every smooth manifold of even dimension M and every smooth line bundle L (See [McD87] or 20.6.1 in [EM02]).
- The PDR $\mathcal{R}_{\mathrm{Cont}}(M, L)$ is open but not ample for every odd-dimensional manifold M and every smooth line bundle L. Indeed, it is easy to see that every nonempty principal subspace is the complement of a hyperplane, so it has two separated connected components that are both ample.
- For the same reason, the PDR $\mathcal{R}^{\eta}_{\mathrm{LCSymp}}(M, L)$ is open but not ample for every even-dimensional manifold M and every smooth line bundle L.

$\bowtie$

Remark 2.2.14. The Partial Differential Relation $E^{(1)}_{\mathcal{O}} \subset E^{(1)}$ is neither open (since it is the closed submanifold with positive codimension determined by Cauchy-Riemann equations) nor ample (since its principal subspaces are all convex but not total). In fact, no HPDR $\mathcal{R} \subset E^{(1)}_{\mathcal{O}} \subset E^{(1)}$ can be open or ample. Nevertheless, one of the most remarkable conclusions of this thesis is that, thanks to the Theorems in Chapter 4, we will be able to use convex integration to find local h–principles for some of this HPDR. $\bowtie$

Let us provide next a brief and non-rigorous overview of how convex integration works. For further details we suggest [EM02], [Gro73] for the original proof and [Gro86] for a version in jets of order higher than 1.

Consider an open and ample PDR $\mathcal{R} \subset E^{(1)}$ and let $\sigma \in \mathrm{F}\mathcal{R}$ be a formal solution. Since the result is going to work relative to domain, it is enough to solve it for each one of the simplices of a triangulation relatively to their boundaries. The h–principle obtained is going to be C^0–close, therefore we can solve the problem just in a tubular neighbourhood of $\sigma^{(0)}$, so it is enough to solve the case with Euclidean fiber. In summary, we can assume that our Partial Differential Relation lies into $J^1([0,1]^n, \mathbb{R}^q) = [0,1]^n \times \mathbb{R}^q \times M_{q\times n}(\mathbb{R})$.

Now let us focus in the first coordinate direction. Consider the principal subspaces $P^{\sigma(t)}_{x_1}$ passing through $\sigma(t)$ that are formed by those linear maps that pointwise fix the tangent hyperplane spanned by $\partial_{x_2}, \dots, \partial_{x_n}$, they correspond to the matrices that fix the last $n-1$ columns. Since $\mathcal{R}$ is ample, $\partial_{x_1}\sigma^{(0)}$ lies in the convex hull of the connected component of σ in $\mathcal{R} \cap P^{\sigma}_{x_1}$, we can find a convex combination of maps $a_1, \dots, a_k : [0,1] \to \mathcal{R} \cap P^{\sigma}_{x_1}$ whose weighted sum corresponds precisely with

$$\lambda_1 a_1 + \ldots + \lambda_k a_k = \partial_{x_1}\sigma^{(0)}.$$

Now choose a path $\gamma : [0,1] \to \mathcal{R} \cap P^{\sigma}_{x_1}$ that spend almost λ_j–time in a_j for every $j = 1, \dots, n$. The integral of this path will be holonomic in the first variable and it starts and finishes in σ. Consider now the path γ^N that consists in repeating γ N-times in the same amount of time. The integral of γ^N will be C^0-close to $\sigma^{(0)}$ for high enough N. Now substitute the first column of the 1-jet part of σ by the path γ^N and the 0-jet part by its integral to obtain a formal solution σ_1 that is holonomic in the first variable and whose 0-jet part is C^0–close to $\sigma^{(0)}$. This process can be done parametrically and relative to parameter, moreover it can be made in such a way that if the parametric dependence is smooth, the first derivatives of the integral in the direction of the parameter are arbitrarily C^0–close to the ones of the original parametric formal solution.

After solving the first coordinate direction we can repeat the process to find a formal solution σ^2 that is holonomic in the second coordinate direction, but we may loose the holonomy in the first variable. Nevertheless, one can consider the first coordinate as a smooth parameter and use the parametric version of the previous construction to obtain C^0–closeness of the derivatives with respect to the first coordinate of both σ_1 and σ_2. Now, we can homotope the 1-st column of the 1-jet part of σ_2 to $\partial_{x_1}\sigma_2^{(0)}$ to obtain a section that is C^0–close to $\sigma^{(0)}$ and that is holonomic in the first two variables.

Repeating this process for the rest of variables we will obtain a holonomic solution that is C^0–close to $\sigma^{(0)}$. All of this can be made to coincide with σ at $\mathcal{O}p\,(\partial I^n)$ if σ was already holonomic there.

After filling up the details that are missing in the previous discussion one can prove the following

Theorem 2.2.15 (Gromov)**.** *All open and ample PDR's satisfy a full, parametric, relative to domain and parameter h–principle that is C^0–dense.* □

Corollary 2.2.16. *Every TDR satisfies a full, parametric, relative to domain and parameter h–principle that is C^0–dense. In particular the Partial Differential Relations $\mathcal{R}_{\mathrm{Imm}}(M_1, M_2)$ for $\dim M_1 < \dim M_2$ and $\mathcal{R}_{\mathrm{ECont}}(M, L)$ for even-dimensional manifolds satisfy these kinds of h–principles.* □

2.3 Stein manifolds

The h–principle for the Partial Differential Relation $E^{(r)}_{\mathcal{O}} \subset E^{(r)}$ may be called *Oka*-principle (See page 4 in [Gro86]). The Oka-principle was indeed known before the introduction of the notion of h–principle and it is usually stated when the base manifold is *Stein*.

Stein manifolds are known to satisfy several remarkable properties such as "having a lot of holomorphic functions" (so that they can be embedded in complex Euclidean spaces) and having the homotopy type of a half-dimensional CW–complex. In this thesis we will obtain local h–principles near CW–complexes of this type.

Definition 2.3.1. *Let K be a compact subset in a complex manifold Ω. Let $\mathcal{O}(\Omega)$ be the set of holomorphic functions in Ω and*

$$\widehat{K}_{\mathcal{O}(\Omega)} := \{z \in \Omega : |f(z)| \leq \max_{x \in K}\{|f(x)|\}, \forall f \in \mathcal{O}(\Omega)\}$$

be the $\mathcal{O}(\Omega)$–convex hull of K. We say that K is $\mathcal{O}(\Omega)$–convex *if $K = \widehat{K}_{\mathcal{O}(\Omega)}$.*

We say that Ω is holomorphically convex *if for every compact subset $K \subset \Omega$ its $\mathcal{O}(\Omega)$–convex hull $\widehat{K}_{\mathcal{O}(\Omega)}$ is also compact.*

Remark 2.3.2.

1. Do note that if in the previous definition $U \subset \Omega$ is an open subset containing K, then $\widehat{K}_{\mathcal{O}(U)} \subset \widehat{K}_{\mathcal{O}(\Omega)}$. This yields to that if X is a properly embedded complex submanifold of a holomorphically convex complex manifold Ω, then X is also holomorphically convex. Indeed, if $K \subset X$ is a compact subset, $\widehat{K}_{\mathcal{O}(X)}$ is compact because it is a closed subset inside the compact set $\widehat{K}_{\mathcal{O}(\Omega)} \cap X$.

2. It is easy to see that convex compact subsets of $\mathbb{C}^n$ are $\mathcal{O}(\mathbb{C}^n)$–convex therefore, by a similar reasoning, every complex Euclidean space $\mathbb{C}^n$ is holomorphically convex.

3. By 1 and 2 we have that proper submanifolds of $\mathbb{C}^n$ are holomorphically convex.

$\bowtie$

There are several equivalent definitions of Stein manifolds. The classical one is the following

Definition 2.3.3. *We say that a complex manifold B is a* Stein manifold *if the following conditions are satisfied*

1. *B is holomorphically convex.*

2. *For every $p \in B$ there exist global holomorphic functions $f_1, \ldots, f_n \in \mathcal{O}(B)$ which form a holomorphic coordinate system at p.*

3. *For every pair of different points $p, q \in B$, there exists a global holormophic function $f \in \mathcal{O}(B)$ such that $f(p) \neq f(q)$.*

The Grauert's Oka principle for Stein manifolds states the following:

Theorem 2.3.4 (Grauert [Gra57])**.** *Let G be a complex Lie group and $H \subset G$ be a closed complex analytic subgroup. Let $G/H \hookrightarrow E \to B$ be a holomorphic fiber bundle with structure group G over a Stein manifold B. Then, every continuous section E is homotopic to a holomorphic one.* □

A direct consequence of Theorem 2.3.4 is the following result, that corresponds with Corollary 5.29 in [CE12]. It will be useful for us in order to prove generalizations of Theorems for holomorphic functions to results for holomorphic sections of vector bundles in Section 3.1.

Corollary 2.3.5. *Every holomorphic vector bundle over a Stein manifold $E \to B$ is holomorphically isomorphic to a vector subbundle of the trivial bundle $B \times \mathbb{C}^N$.* □

Remark 2.3.6. It follows from the maximum principle and Condition 3 in Definition 2.3.3 that every Stein manifold is necessarily not compact. It follows from the definition that every open holomorphically convex subset of a Stein manifold is also Stein. Moreover, by Remark 2.3.2, $\mathbb{C}^n$ and all of its proper complex submanifolds are Stein manifolds. The following Theorem provides a converse of the previous statement ⋈

Theorem 2.3.7 (Bishop [Bis61] and Narashiman [Nar60])**.** *Every Stein manifold B of complex dimension n admits a proper holomorphic embedding into $\mathbb{C}^{2n+1}$.* □

Therefore Stein manifolds and proper submanifolds of complex Euclidean spaces are equivalent notions. There is another characterization of Stein manifolds in terms of *plurisubharmonic functions.*

Definition 2.3.8. *Let B be a complex manifold with integrable almost-complex structure J. A smooth function $\phi : B \to \mathbb{R}$ is* (strongly[1]) plurisubharmonic *if the 2–form*

$$\omega_\phi := -dd^{\mathbb{C}}\phi := -d(d\phi \circ J)$$

[1]with this parenthesis we indicate that whenever it is not specified, the plurisubharmonic function should be assumed to be strongly plurisubharmonic.

is a symplectic form compatible with J*, i.e.* $H_\phi := g_\phi - \omega_\phi$ *is an Hermitian metric with* $g_\phi := \omega_\phi(\,\text{-}\,, J\,\text{-}\,)$*. We say that* ϕ *is* weakly plurisubharmonic *if the symmetric 2-form* g_ϕ *is semidefinite positive.*

We will say that ϕ *is* exhausting *if it is proper and bounded from below.*

Note that, since the squared distance to a point in $\mathbb{C}^N$ is a strongly plurisubharmonic exhausting function of $\mathbb{C}^N$, then every proper submanifold of $\mathbb{C}^N$, and therefore every Stein manifold, admits a strongly plurisubharmonic exhausting function. The following theorem shows that the converse is also true

Theorem 2.3.9. *(Grauert [Gra58]) Every complex manifold that admits a strongly plurisubharmonic exhausting function is Stein.* □

Since the space of strongly plurisubharmonic functions is open in $C^\infty(B)$, a generic plurisubharmonic function is of *Morse type* (it only has nondegenerate critical points). Moreover the subspace of plurisubharmonic functions is convex and therefore contractible, so the plurisubharmonic function of a Stein manifold is homotopically canonical. We will assume in the following that the plurisubharmonic functions that we will consider are of Morse type unless other thing is specified. We will also assume for simplicity that the set of critical points of ϕ, $\mathrm{Crit}(\phi) = \{p_i\}_{i=1}^k, k \in \mathbb{N} \cup \{\infty\}$, satisfy that $\phi(p_i) < \phi(p_{i+1})$ for each $i < k$. We will say that the Stein manifold has *finite type* if it admits a plurisubharmonic exhausting function with a finite amount of critical points.

Stein manifolds and symplectic and contact topology are closely related. Indeed, every Stein manifold B with plurisubharmonic function ϕ is an exact symplectic manifold (B, ω_ϕ) whose Liouville form is $\lambda_\phi := -d^{\mathbb{C}}\phi$. The nondegeracy of ω_ϕ provides an isomorphism of vector bundles

$$\iota_{\text{-}}\omega_\phi : TB \to T^*B,$$

where ι denotes the contraction in the first factor. Let X_ϕ be the *Liouville vector field for* λ_ϕ, i.e. the vector field such that $\iota_{X_\phi}\omega_\phi = \lambda_\phi$. X_ϕ is the gradient of ϕ for the metric g_ϕ. Indeed, for every $Y \in \mathfrak{X}(B)$,

$$g_\phi(X_\phi, Y) = \omega_\phi(X_\phi, JY) = \lambda_\phi(JY) = -d^{\mathbb{C}}\phi(JY) = d\phi(Y).$$

Let X_ϕ^t be the flow of X_ϕ. We can assume that the flow of X_ϕ is defined for every $t \in \mathbb{R}$. Indeed, after composing $\phi : B \to [a, \infty)$ with a convex diffeomorphism $f : [a, \infty) \to [b, \infty)$ such that $f'(t) \overset{t\to\infty}{\longrightarrow} \infty$ the function $f \circ \phi$ is plurisubharmonic exhausting function and its Liouville vector field is complete (See Proposition 2.11 in [CE12]). In this case we will say that ϕ is *completely exhausting*.

By the Cartan magic formula, since $d\lambda_\phi = \omega_\phi$ then $\mathcal{L}_{X_\phi}\omega = \omega$ and $\mathcal{L}_{X_\phi}\lambda_\phi = \lambda_p hi$. Therefore

$$(X_\phi^t)^*\omega = e^t\omega \quad \text{and} \quad (X_\phi^t)^*\lambda = e^t\lambda.$$

Let $p_i \in \mathrm{Crit}(\phi) \subset B$ be a (nondegenerate) critical point of ϕ and let us denote by M_i its *stable disk*

$$M_i := \left\{ q \in B \,|\, \lim_{t\to\infty} X_\phi^t(q) = p \right\}.$$

Let $q \in M_i$ and let $v \in T_qM_i$, since $\lim_{t\to\infty} d_{X_\phi^t(q)}\phi(v) = 0 \in T_pB$, then

$$e^t\lambda(v) = (X_\phi^t)^*\lambda(v) = \lambda(d_{X_\phi^t(q)}\phi(v)) \overset{t\to\infty}{\longrightarrow} 0$$

and therefore $\lambda(v) = 0$. This yields to that $\lambda_{\phi}\big|_{TM_i} \equiv 0$ and in particular that $\omega_{\phi}\big|_{TM_i} \equiv 0$. Therefore the stable disks are *isotropic submanifolds of* (B, ω_ϕ). In particular they are also *totally real manifolds*, i.e. $TM_i \cap JTM_i = 0$, indeed let $v \in TM_i$ such that $Jv \in TM_i$, we have that

$$0 = \omega_\phi(v, Jv) = g_\phi(v, v).$$

Therefore, since g_ϕ is nondegenerate, we obtain that $v = 0$.

Definition 2.3.10. *Let B be a complex manifold and $M \subset B$ be a subset. We say that M is a* totally real stratification by affine strata *if M is presented as a countable union $M = \bigcup_{i=1}^k M_i, k \in \mathbb{N} \cup \{\infty\}$, such that each M_i is a totally real submanifold of B diffeomorphic to a k_i–dimensional open ball. We denote by $M_{\leq i}$ the totally real stratification by affine strata $M_{\leq i} := \bigcup_{j=1}^i M_j$.*

If B is a Stein manifold we will say that such an stratification is a Lagrangian skeleton *if it is the union of the stable disks of a strongly plurisubharmonic Morse completely exhausting function ϕ and we will denote it by M_ϕ.*

Remark 2.3.11. Since the maximal real dimension of isotropic submanifolds of a symplectic manifold (M, ω) is $\frac{1}{2}\dim M$ and the dimension of each stable disk M_i coincides with the index of its corresponding critical point, it follows that for every Stein manifold B of complex dimension n, they have the homotopy type of a CW–complex of dimension $\leq \dim_{\mathbb{C}} B$. $\bowtie$

Now consider a regular value c of ϕ and its level set

$$\Sigma = \Sigma_c := \{p \in B \,|\phi(p) = c\}.$$

Let us denote by ξ the field of complex tancencies

$$\xi = \xi_\Sigma := T\Sigma \cap JT\Sigma.$$

ξ coincides with the real 1-codimensional distribution $\ker \lambda_\phi \subset T\Sigma \subset TB$. Since $d\lambda_\phi = \omega_\phi$ is nondegenerate in TB we have that $(d\lambda_\phi)\big|_\xi$ has rank $\dim \Sigma - 1$ and that there exist a vector field $R \in \mathfrak{X}(\Sigma)$ such that

$$\lambda_\phi(R) \equiv 1 \quad \text{and} \quad \iota_R d\lambda_\phi \equiv 0.$$

This yields to that

$$\lambda_{\phi\big|\Sigma} \wedge (d\lambda_{\phi\big|\Sigma})^{n-1} \neq 0,$$

where $n = \dim_{\mathbb{C}} B$, i.e. $\lambda_{\phi\big|\Sigma}$ is a contact form for the cooriented contact structure Σ.

Do note that since $\lambda_{\phi\big|M_i} \equiv 0$, the stable disks intersect Σ in *isotropic submanifolds*, i.e. submanifolds that are tangent to the contact distribution. Moreover, it holds that, since X_ϕ is tangent to every stable disk and $d\phi(JX_\phi) = -\omega_\phi(X_\phi, X_\phi) = 0$,

$$M_i \pitchfork \Sigma_c \quad \text{and} \quad JTM_i \subset T\Sigma_c,$$

for every regular value c of ϕ and every stable disk M_i.

Now let us review the region enclosed Σ_c.

Let $g : (-\infty, c) \to \mathbb{R}$ be a convex diffeomorphism. The composition $g \circ \phi$ is a plurisubharmonic exhausting function of the interior of the sublevel set

$$K_c := \{p \in B \,|\phi(p) \leq c\}.$$

Indeed since

$$d^{\mathbb{C}}(g \circ \phi) = (g' \circ \phi) d^{\mathbb{C}} \phi,$$

we have that

$$\omega_{g\circ\phi} = -dd^{\mathbb{C}}(g \circ \phi) = -(g'' \circ \phi) d\phi \wedge d^{\mathbb{C}}\phi + (g' \circ \phi)\omega_{\phi},$$

that is clearly nondegenerate in every critical point of ϕ. Since for every regular value $c' \in (-\infty, c)$ we can split the tangent bundle of every point into $\mathbb{R}R \oplus \mathbb{R}JR \oplus \xi$ and since

$$\omega_{g\circ\phi}(R, JR) \neq 0, \omega_{g\circ\phi}(R, Y) = \omega_{g\circ\phi}(JR, Y) = 0, \text{ and } \omega_{g\circ\phi}(Y, JY) = (g' \circ \phi)\omega_{\phi}(Y, JY) \neq 0$$

for every $Y \in \xi$, we have that $\omega_{g\circ\phi}$ is nondegenerate also in the regular points.

Do note that, since the interior of $K_{c+\varepsilon}$ admits a plurisubharmonic exhaustion function for every $\varepsilon > 0$, then K_c admits a basis of Stein neighbourhoods. Moreover, Theorem 5.7 in [CE12] (see also Theorem 2.5.2 in [For17]) states that K_c is $\mathcal{O}(B)$–convex (and therefore $\mathcal{O}(\mathring{K}_{c+\varepsilon})$–convex for every $\varepsilon > 0$). These properties are captured in the following

Definition 2.3.12. *Let $K \subset X$ be a compact subset of a complex manifold X.*

- *We say that K is* Stein compact *if it admits a basis of Stein neighbourhoods.*
- *We say that K is a* holomorphically convex compact set *if it admits an open Stein neighbourhood $\Omega \subset X$ such that K is $\mathcal{O}(\Omega)$–convex.*[2]

Remark 2.3.13. By Proposition 2.5.5 in [For17], every holomorphically convex compact set is also Stein compact. The converse is not true in general. The following Theorem is useful to characterize the Stein compact sets that are holomorphically convex compact sets. ⋈

Theorem 2.3.14 (Theorem 5.18 in [CE12])**.** *A Stein compact set K in a Stein manifold B is $\mathcal{O}(B)$–convex if and only if every holomorphic function on $\mathcal{O}p\,(K)$ can be approximated uniformly on K by holomorphic functions in B.* □

[2]Here we follow the Definition of holomorphically convex compact sets stated in [For17]. It is a slightly stronger version of the one given in [CE12]. Indeed the Definition given in the last one coincides with being Stein compact for subsets in a Stein manifold.

We are indeed interested in Stein compacts by their suitability of approximating functions. This will be explored in the next Section.

The following Theorem shows that the result of gluing a sublevel set of a plurisubharmonic function ϕ with one of their stable disks is inside the sublevel set of some other plurisubharmonic function ψ with the same critical set that coincides with ϕ everywhere but in a neighbourhood of the attached disk. In particular, this yields to that the result of that gluing is Stein compact. Moreover, the Lagrangian skeleton of ψ is homotopic to the original one.

Theorem 2.3.15 (Cielieback-Eliashberg (Theorem 8.5 in [CE12]))**.** *Let B be a Stein manifold and $\phi : B \to \mathbb{R}$ be a completely exhausting plurisubharmonic function. Let c be a regular value of ϕ and let $K_c = \{q \in B \,|\phi(q) \leq c\}$ be its corresponding sublevel set. Let $p \in B$ be a critical point of ϕ such that $\phi(p) > c$ and let Δ_p be its stable disk. Then, for any $U = \mathcal{O}p\,(K_c \cup \Delta_p)$ there exists a completely exhausting plurisubharmonic function $\psi : B \to \mathbb{R}$ and $V = \mathcal{O}p\,(\Delta_p)$ contained in U such that*

1. *ψ coincides with ϕ in $\mathcal{O}p\,(K_c)$ and outside V.*
2. *The level sets of ψ and ϕ coincide near Δ_p.*
3. *$\psi_{|_V}$ has only p as critical point and its stable disk coincides with Δ_p.*
4. *Some sublevel set $K'_{c'} = \{q \in B \,|\psi(q) \leq c'\}$ contains $K_c \cup \Delta_p$.*

Moreover, there exists a homotopy of plurisubharmonic functions $\psi_t, t \in [0,1]$ satisfying properties 1–3 connecting $\psi_0 = \phi$ with $\psi_1 = \psi$. □

2.3.1 Properly attached sets and adapted skeletons

We commence this Section stating a Mergelyan type approximation Theorem for functions. We will provide a generalization for parametric sections of vector bundles in Section 3.1 that will play a fundamental role in the proof of all the rest of original results of this thesis.

Theorem 2.3.16 (Mergelyan (Theorem 20 in [FFW20]))**.** *Let $S = K \cup M$ be a Stein compact set in a complex manifold B, where K is also a Stein compact set and M is a totally real* *submanifold of class C^k. Then for any function f of class C^k on S and holomorphic on $\mathcal{O}p\,(K)$, there exist a sequence of functions f_j that are holomorphic in $\mathcal{O}p\,(S)$ such that*

$$\lim_{j\to\infty} \| f_j - f \|_{C^k(S)} = 0.$$

□

This Theorem motivates the following

Definition 2.3.17. *Let K be a compact subset of a complex manifold B and let $M \subset B$ be a totally real submanifold. We say that K* and *M* are properly attached *if $K \cup M$ is Stein compact.*

If instead $M = \bigcup_{i=1}^{k} M_i, k \in \mathbb{N} \cup \{\infty\}$, is a totally real stratification by affine strata, we say that K is properly attached to *M if $K \cup M_{\leq i}$ is properly attached to M_{i+1} for every $i < k$.*

Remark 2.3.18.

- Lemma 2 in [FFW20] states that if S is a Stein compact set and $S \setminus K$ is a totally real submanifold, then, if K is compact, it is also Stein compact. This yields to that properly attached sets satisfy the hypothesis of Theorem 2.3.16.
- Every holomorphically convex compact set K is Stein compact, but the converese is not true (see Proposition 2.5.5 in [For17] and Remark 5 in [FFW20]).

⋈

The following result is a direct consequence of Remark 8.6, Corollary 8.26 and Proposition 10.8 in [CE12] (recall that in that book their definition of *holomorphically convex compact set* coincides with our notion of Stein compact inside Stein manifolds).

Proposition 2.3.19. *Let B be a complex manifold and let K be a compact domain whose boundary ∂K is a contact manifold with the field of complex tangencies as its contact structure. Let $M \subset B \setminus \mathring{K}$ be a totally real submanifold, then*

- *If ∂M is an isotropic submanifold of ∂K, then if $S := K \cup M$ is compact, it is also Stein compact.*
- *The converse is also true if both ∂K and M are real analytic.* □

Theorem 2.3.15 yields to the following

Remark 2.3.20. Every stable disk of the Lagrangian skeleton M_ϕ of a plurisubharmonic exhausting function ϕ is properly attached to any of its regular sublevel sets K_c. ⋈

Suppose now that you are interested in approximate a function $f : B \to \mathbb{C}$, that is holomorphic in a regular sublevel set $K = K_c$ of a plurisubharmonic exhausting function ϕ, by a function that is holomorphic near the Lagrangian skeleton instead of just near one of their stable disks. This will be the situation that we will need to deal with to prove the original results of this thesis when f is the section of a vector bundle instead of a function.

By Remark 2.3.20 you can apply Theorem 2.3.16 to K and M_i such that $c < \phi(p_i)$ to obtain an approximation f_1 of f that is holomorphic over $U = \mathcal{O}p\,(K_c \cup M_i)$. By Theorem 2.3.15, there exist a strongly plurisubharmonic exhausting function $\phi' : B \to \mathbb{R}$ with $\mathrm{Crit}(\phi) = \mathrm{Crit}(\phi')$ that coincides with ϕ in $\mathcal{O}p\,(K)$ and outside of $\mathcal{O}p\,(M_i)$ and such that there exists a regular value c' for ϕ' such that the sublevel set $K' := \{x \in B : \phi'(x) \leq c'\}$ satisfies that $K \cup M_i \subset K' \subset U$.

Since every open subset of an open Riemann surface is Stein (see [For17]), if $\dim_{\mathbb{C}} B = 1$ then M_{i+1} is properly attached to the holomorphically convex set K'.

In higher dimensions this may not be true. Nevertheless, in arbitrary dimension, each descending disk of $M_{\phi'}$ is properly attached to K' and homotopic to M_ϕ.

Therefore, in order to apply Mergelyan type theorems successively over the “bones” of the Lagrangian skeleton in general we may need to perturb or substitute their stable disks in each iteration if our Stein manifold is not a Riemann surface.

Let us conclude the Preliminaries introducing *adapted skeletons*. This kind of skeleton shares many properties with Lagrangian ones and we will be able to perform the inductive process mentioned above to them with their disks remaining fixed whenever our Stein manifold is of finite type.

Definition 2.3.21. *Let B be a Stein manifold and let $\phi : B \to \mathbb{R}$ be a complete plurisubharmonic Morse exhausting function. Enumerate the critical points $p_1, p_2, \dots, p_i, p_{i+1}, \dots$ of ϕ to satisfy that $\phi(p_i) < \phi(p_{i+1})$ for every $p_i, p_{i+1} \in \mathrm{Crit}(\phi)$ and denote by k_i the Morse index of p_i. We say that a totally real stratification by affine strata $M = \bigcup_{i=1}^{k} M_i, k \in \mathbb{N} \cup \{\infty\}$ is a* skeleton adapted to ϕ *or an* adapted skeleton *if the following holds:*

1. *ϕ does not have critical points in $B \setminus M$.*
2. *Each disk M_i contains the critical point p_i and $\dim M_i = k_i$.*
3. *There exists an isotopy $h_t : B \to B, t \in [0, \infty)$ such that $h_0 = \mathrm{Id}, h_{t|M} = \mathrm{Id}$ for all $t \in [0, \infty)$, $M = \bigcap_{t \in [0,\infty)} h_t(B)$ and each regular open sublevel set $\{\phi \circ h_t^{-1} < c\}$ is Stein.*
4. *Each $M_{\leq i}$ is a Stein compact set.*

Theorem 8.32 in [CE12] states that an affine stratification by disks satisfying conditions 1, 2 and 3 in the previous Definition always exists. By checking the proof of that Theorem one can actually see that the last condition also holds[3]. Note that this last condition in Definition 2.3.21 gives that if $M = \bigcup_{i=1}^{k} M_i, k \in \mathbb{N} \cup \{\infty\}$, is an adapted skeleton, then for every $i \leq k$, $M_{\leq i}$ is properly attached to M.

[3] In the notation of [CE12], the sets bounded by the convex hypersurfaces $\Sigma_i^{(r)}$ form the basis of Stein neighbourhoods of each $M_{\leq i}$.

Chapter 3

Holomorphic approximation

One key step in the proofs of the original results of this thesis consists on approximating smooth germs of solutions over totally real submanifolds by holomorphic ones. This can be done for functions near totally real submanifolds thanks to the Mergelyan type Theorem 2.3.16. Moreover, if the function that is going to be approximated is already holomorphic in an open neighbourhood of a *Stein compact* subset K, the approximation can be done to be arbitrarily close to the original function over K, provided that K is *properly attached* to the totally real submanifold (see Definitions 2.3.12 and 2.3.17).

Theorem 2.3.16 can be generalized for sections of holomorphic fiber bundles (see Theorem 34 in [FFW20]). Here we are interested in a parametric and relative to parameter version for sections of vector bundles. This version is Theorem 3.1.1, that can also be generalized for holomorphic bundles (see Remark 3.1.3). This result is proven below following an approach that has been useful to prove parametric generalizations of other approximation Theorems that appear in the literature.

The proofs of the Theorems stated in Section 4.2 need to approximate formal local solutions over stratified totally real manifolds. We may use Theorem 3.1.1 successively over the "bones" of the skeleton to provide suitable versions of Theorem 3.1.1 for stratified sets like Corollaries 3.2.1, 3.2.3 and 3.2.4.

3.1 Parametric Mergelyan Theorem for sections

When $E = B \times \mathbb{C}$, the following Mergelyan approximation Theorem is indeed the same as Theorem 2.3.16 in the non-parametric case. Following similar arguments to the proof of Theorem 4.3 in [For21] we can prove Lemma 3.1.2, that is a parametric and relative to parameter version for functions. After that we will use the arguments at the end of the proof of Theorem 2.8.4 in [For17] to give the following parametric Mergelyan approximation Theorem for sections of a holomorphic vector bundle.

Theorem 3.1.1. *Let $E \to B$ be a holomorphic vector bundle over a complex manifold B. Let $K \subset B$ be a Stein compact subset and $M \subset B$ be an embedded totally real closed C^r–submanifold (possibly with boundary) such that $S = K \cup M$ is also Stein compact in B. Then for any C^r section $\sigma : S \to E$ that is holomorphic over $\mathcal{O}p\,(K)$ there exists a sequence of sections σ_k that are holomorphic over $\mathcal{O}p\,(S)$ such that*

$$\lim_{k\to\infty} \parallel \sigma_k - \sigma \parallel_{C^r(S)} = 0.$$

This Theorem also holds parametrically for compact Hausdorff parameter spaces and relative to parameter.

More precisely, given a family of C^r sections $\sigma_p : S \to E$ that are holomorphic over $\mathcal{O}p\,(K)$ depending continuously on a parameter p in a compact Hausdorff space P, there exists a sequence of continuous families of sections $\sigma_{p,k}$ that are holomorphic over $\mathcal{O}p\,(S)$ such that

$$\lim_{k\to\infty} \parallel \sigma_{p,k} - \sigma_p \parallel_{C^r(S)} = 0.$$

Moreover, if there is a compact subset $Q \subset P$ such that σ_q is already holomorphic over $\mathcal{O}p\,(S)$ for every $q \in Q$, then the sequence can be chosen to satisfy that $\sigma_{q,k} = \sigma_q$ for every $q \in Q, k \in \mathbb{N}$.

The following Lemma is the parametric case for functions. It will be useful to prove Theorem 3.1.1.

Lemma 3.1.2. *Let K and $S = K \cup M$ be Stein compacts in a complex manifold B, where $M = S \setminus K$ is an embedded totally real submanifold (possibly with boundary) of class C^r.*

Then for any family $f_p : S \to \mathbb{C}$ of C^r functions that are holomorphic over $\mathcal{O}p\,(K)$ and that depends continuously on a parameter p in a compact Hausdorff space P, there exists a sequence of continuously depending families of holomorphic functions $\tilde{f}_{p,k} : \mathcal{O}p\,(S) \to \mathbb{C}$ such that

$$\lim_{k\to\infty} \| \tilde{f}_{p,k} - f_p \|_{C^r(S)} = 0.$$

If in addition there is a compact subset $Q \subset P$ such that f_q is already holomorphic over $\mathcal{O}p\,(S)$ for every $q \in Q$, then the sequence of families can be chosen to satisfy that $\tilde{f}_{q,k} = f_q$ for every $q \in Q$.

Proof (of Lemma 3.1.2). Let ε be a positive number. Now select a finite set of points $\{p_1, \ldots, p_m\} \subset P$ and a covering by open sets $\{P_j\}_{j=1}^m$, such that $p_j \in P_j$ for every $j = 1, \ldots, m$ and that

$$\| f_p - f_{p_j} \|_{C^r(S)} < \frac{\varepsilon}{4}, \quad \text{for every } p \in P_j, j = 1, \ldots, m.$$

Now, use the nonparametric version of this theorem for the case of functions (Theorem 2.3.16) to obtain holomorphic functions $g_j : \mathcal{O}p\,(S) \to \mathbb{C}$ such that

$$\| g_j - f_{p_j} \|_{C^r(S)} < \frac{\varepsilon}{4}, \quad j = 1, \ldots, m.$$

Then take a continuous partition of unity χ_j subordinated to $\{P_j\}_{j=1}^m$ and define

$$\tilde{f}_p := \sum_{j=1}^{m} \chi_j(p) g_j, \quad \text{for every } p \in P.$$

It is easy to check that $\tilde{f}_p$ is holomorphic for every $p \in P$, moreover we have that

$$\| \tilde{f}_p - f_p \|_{C^r(S)} \leq \sum_{j=1}^{k} \chi_j(p) \| g_j - f_p \|_{C^r(S)} < \frac{\varepsilon}{2},$$

since $\| g_j - f_p \| \leq \| g_j - f_{p_j} \| + \| f_{p_j} - f_p \| < \frac{\varepsilon}{2}$ for $p \in P_j$ and $\chi_j(p) = 0$ for $p \notin P_j$. This completes the proof for the nonrelative case.

Now, to make $\tilde{f}_p$ coincide with f_p at $p \in Q$ consider $N \subset B$ a compact neighbourhood of S such that f_q are holomorphic at the interior of N for every $q \in Q$. The space of such functions is a Banach space, so Michael's extension Theorem (see Theorem 2.8.2 in [For17] for the precise statement, that is proven in [Mic56]) guarantees that there exists a continuous family $\xi_p, p \in P$, of functions that are holomorphic in the interior of N such that $\xi_q = f_q$ for every $q \in Q$. Now let P_0 be a neighbourhood of Q such that

$$\| \xi_p - f_p \|_{C^r(S)} < \frac{\varepsilon}{2} \quad \text{for all } p \in P_0.$$

Now let $\rho : P \to [0,1]$ be a continuous function supported on P_0 such that $\rho_{|Q} \equiv 1$ and define

$$\widetilde{f}'_p := \rho(p)\xi_p + (1-\rho(p))\widetilde{f}_p, \quad \text{for } p \in P.$$

$\{\widetilde{f}'_p\}$ is a family of holomorphic functions continuously dependent on the parameter $p \in P$, with $\| \widetilde{f}'_p - f_p \|_{C^r(S)} < \frac{\varepsilon}{2}$ and such that $\widetilde{f}'_q = f_q$ for every $q \in Q$. The sequence $\widetilde{f}'_{p,k}$ required can be obtained from here doing the same process for $\varepsilon = \frac{1}{k}$. □

Proof (of Theorem 3.1.1). Since S is a Stein compact and the result is local, it is enough to prove it in the case in which B is a Stein submanifold.

By Corollary 2.3.5 E is isomorphic to a holomorphic vector subbundle of $B \times \mathbb{C}^N$ for some $N \in \mathbb{N}$. Moreover, note that the orthogonal projection $\pi : B \times \mathbb{C}^N \to E$ is holomorphic.

This allows us to see sections from B to E as maps to $\mathbb{C}^N$. Therefore we can apply Lemma 3.1.2 to each component and then project the resulting map to E by π to obtain the desired section. □

Remark 3.1.3. Theorem 3.1.1 can be easily generalized for holomorphic fiber bundles over Stein manifolds. This can be done choosing a finite set of points $\{p_1, \dots, p_m\} \subset P$ and a covering by open sets $\{P_j\}_{j=1}^m$ such that $p_j \in P_j$ for every $j = 1, \dots, m$ such that

$$f_p(B) \subset N_j \quad \text{for every } p \in P_j, j = 1, \dots, m;$$

where we denoted by N_j a tubular neighbourhood of $f_{p_j}(B)$. Each N_j is therefore a vector bundle, so Theorem 3.1.1 can be applied successively in j with parameter space P_j being relative to $Q \cap \left(\bigcup_{i=1}^{j-1} (P_i \cap P_j) \right)$. ⋈

3.2 Holomorphic approximation over stratified sets

To prove the Theorems of Section 4.2 we will need to apply Theorem 3.1.1 repeatedly attaching totally real disks to the already attached ones to obtain approximations near a skeleton of the original Stein manifold. Let us provide now a list of Corollaries of Theorem 3.1.1 that will be useful in Section 4.3. We state them for vector bundles, but it follows from Remark 3.1.3 that they also work for holomorphic fiber bundles.

Corollary 3.2.1. *Let $E \to B$ be a holomorphic vector bundle over a Stein manifold B of finite type. Let $M = \bigcup_{i=0}^{k} M_i, k \in \mathbb{N}$, be an adapted skeleton of B and let $K \subset B$ be a holomorphically convex compact subset properly attached to M. For every family of C^r–sections $\sigma_p : \mathcal{O}p\,(K \cup M) \to E$ that are holomorphic over $\mathcal{O}p\,(K)$ and continuously dependent on a parameter p in a compact Hausdorff space P and for every $\varepsilon > 0$, there exists a continuous family of sections $\tilde{\sigma}_p$ that are holomorphic over $\mathcal{O}p\,(K \cup M)$ such that*

$$\| \tilde{\sigma}_p - \sigma_p \|_{C^0(K\cup M)} < \varepsilon, \quad \forall p \in P.$$

Moreover, if there is a compact subset $Q \subset P$ such that for every $q \in Q$, the section σ_q is holomorphic over $\mathcal{O}p\,(K \cup M)$, then $\tilde{\sigma}_p$ can be chosen to satisfy that $\tilde{\sigma}_q = \sigma_q$ for every $q \in Q$.

Proof. Apply Theorem 3.1.1 to σ_p over $K \cup M_1$ to obtain a section $\sigma_{p,1}$ that is holomorphic and C^r–close to σ_p over Ω_1, a Stein neighbourhood of $K \cup M_1$. Let $\rho_1 : B \to \mathbb{R}$ be a smooth cutoff function supported on Ω_1 and such that $\rho_1 \equiv 1$ over $\mathcal{O}p\,(K \cup M_1)$. Now apply Theorem 3.1.1 to $\tilde{\sigma}_{p,1} := \rho_1\sigma_{p,1} + (1-\rho_1)\sigma_p$ over $(K \cup M_1) \cup M_2$ to obtain a section $\sigma_{p,2}$ that is holomorphic and C^r–close to $\tilde{\sigma}_{p,1}$ (and therefore C^0–close to σ_p) over Ω_2, a Stein neighbourhood of $K \cup M_{\leq 2}$. Cutoff the section again to obtain $\tilde{\sigma}_{p,2} := \rho_2\tilde{\sigma}_{p,2} + (1-\rho_2)\tilde{\sigma}_{p,1}$, where $\rho_2 : B \to \mathbb{R}$ is supported on Ω_2 and is equal to 1 over $\mathcal{O}p\,(K \cup M_{\leq 2})$. Repeat the previous process until getting the last section $\tilde{\sigma}_p := \tilde{\sigma}_{p,k}$. □

Remark 3.2.2. Although the C^r–closeness over $K \cup M$ cannot be achieved in general due to the Cauchy Riemann conditions, the approximation $\tilde{\sigma}_p$ obtained in the previous proof is C^r–close to σ_p over K and over each $M_{i+1} \setminus \mathcal{O}p\,(M_{\leq i})$. ⋈

The next Corollary is just a version of Corollary 8.39 in [CE12] for parametric sections of vector bundles and noticing that there are places where we can obtain C^r–closeness instead of just C^0–closeness. Its proof is exactly the same but using our version of the Mergelyan Theorem instead of the one that works just for functions.

Corollary 3.2.3. *Let $E \to B$ be a holomorphic vector bundle over a Stein manifold B with integrable almost-complex structure J. Let $\phi : B \to \mathbb{R}$ be a strongly plurisubharmonic Morse exhausting function. Consider the sublevel set of ϕ for some regular value $c \in \mathbb{R}$, $K = \{x \in B : \phi(x) \leq c\}$.*

Then, for every family of C^r–sections $\sigma_p : B \to E$ that are holomorphic over $\mathcal{O}p\,(K)$ and continuously dependent on a parameter p in a compact Hausdorff space P, and for every positive function $\varepsilon : B \to \mathbb{R}$, there exists an isotopy $h_t : B \hookrightarrow B, t \in [0,1]$ such that $h_0 = \mathrm{Id}$ and such that $K \subset h_t(B)$ and $(h_t(B), J_{|h_t(B)})$ is Stein for every $t \in [0,1]$ satisfying that there exists a continuous family of sections $\tilde{\sigma}_p$ that are holomorphic over $h_1(B)$ such that $\tilde{\sigma}_p$ is arbitrarily C^r–close to σ_p over K for every $p \in P$ and such that

$$|\tilde{\sigma}_p(x) - \sigma_p(x)| < \varepsilon(x), \quad \forall (p,x) \in P \times h_1(B).$$

Moreover, if there is a compact subset $Q \subset P$ such that for every $q \in Q$, the section σ_q is already holomorphic, then $\tilde{\sigma}_p$ can be chosen to satisfy that $\tilde{\sigma}_q = \sigma_q$ for every $q \in Q$.

Proof. For simplicity of notation let us assume that there is only one critical point for each critical value. Let $p_1 \in B$ be the only critical point satisfying that there is no critical value in the interval $(c, \phi(p_1))$. Take an increasing sequence of regular values of ϕ, $c = c_0 < c_1 < c_2 < \ldots$ such that $\phi(p_1) < c_1$ and such that there is only one critical value on each interval (c_{j-1}, c_j), whose corresponding critical point is p_j $j = 1, \ldots, \infty$. Let $V_j := \{x \in B : \phi(x) \leq c_j\}, j = 1, \ldots, \infty$ and let M_1 be the stable disk of p_1.

Since M_1 is totally real and properly attached to K we can use Theorem 3.1.1 to obtain an open neighbourhood $U_1 = \mathcal{O}p\,(K \cup M_1) \subset V_1$ where there is defined a holomorphic section σ_1 satisfying that

$$\| \sigma_1 - \sigma \|_{C^r(\overline{U}_1)} < \frac{1}{2} \min_{x \in \overline{U}_1} \{\varepsilon(x)\}.$$

Extend σ_1 smoothly to the whole manifold satisfying that $|\sigma_1(x) - \sigma(x)| < \frac{1}{2}\varepsilon(x)$ for every $x \in B$ (maybe using a cutoff function and shrinking U_1 if necessary). Now use Theorem 2.3.15 to obtain a strongly plurisubharmonic Morse exhausting function ϕ_1 such that

- $\phi_1 = \phi$ outside U_1 and inside $\widetilde{V}_0 = \mathcal{O}p\,(K)$,
- the only critical point of $\phi_1\big|_{U_1\setminus\widetilde{V}_0}$ is p_1,
- there exists a regular value $c_1' > \phi_1(p_1)$ such that $\widetilde{V}_0 \cup M_1$ is contained in the subset $V_1' := \{x \in B : \phi_1(x) \leq c_1'\} \subset U_1$.

Now let M_2 be the stable disk of p_2 for ϕ_1. We can use Theorem 3.1.1 again to obtain an open neighbourhood $U_2 = \mathcal{O}p\,(V_1' \cup M_2) \subset V_2$ and (after extending smoothly to B) a section $\sigma_2 : B \to X$ that is holomorphic in U_2, such that

$$\| \sigma_2 - \sigma_1 \|_{C^r(\overline{U}_2)} < \frac{1}{4} \min_{x\in\overline{U}_2}\{\varepsilon(x)\}$$

and such that

$$|\sigma_2(x) - \sigma_1(x)| < \frac{1}{4}\varepsilon(x)$$

for every $x \in B$. Then use again Theorem 2.3.15 to obtain a strongly plurisubharmonic exhausting function ϕ_2 such that

- $\phi_2 = \phi_1$ outside U_2 and inside $\widetilde{V}_1 = \mathcal{O}p\,(V_1')$,
- the only critical point of $\phi_{2|U_2\setminus\widetilde{V}_1}$ is p_2,
- there exists a regular value $c_2' > \phi_2(p_2)$ such that $V_2' := \{x \in B : \phi_2(x) < c_2'\}$ satisfies that $\widetilde{V}_1 \cup M_2 \subset V_2' \subset U_2$.

Now just repeat that process inductively to obtain a sequence of holomorphic sections σ_i, a sequence of strongly plurisubharmonic Morse exhausting functions ϕ_i, a sequence of disks M_i that are the stable manifolds of the points p_i for ϕ_{i-1}, a sequence c_i' of regular values of ϕ_i such that $c_i' < \phi_i(p_i)$ and sequences of subsets $\widetilde{V}_{i-1}, V_i'$ and $U_i, i = 1, \ldots$ such that

1. $\widetilde{V}_{i-1} \cup M_i \subset V_i' := \{x \in B : \phi_i \leq c_i'\} \subset U_i$,

2. $U_{i+1} = \mathcal{O}p\,(V_i' \cup M_{i+1}) \subset V_{i+1}$,

3. $\phi_i = \phi_{i-1}$ outside U_i and inside $\widetilde{V}_{i-1} = \mathcal{O}p\left(V_{i-1}'\right)$,

4. the only critical point of $\phi_i\big|_{U_i \setminus \widetilde{V}_{i-1}}$ is p_i,

5. σ_i is holomorphic in U_i,

6. $\| \sigma_i - \sigma_{i-1} \|_{C^r(\overline{U}_i)} < \left(\frac{1}{2}\right)^i \min_{x \in \overline{U}_i}\{\varepsilon(x)\}$,

7. $|\sigma_i(x) - \sigma_{i-1}(x)| < \left(\frac{1}{2}\right)^i \varepsilon(x)$, for every $x \in B$,

for each $i = 1, \ldots$, where $\phi_0 = \phi$ and $\sigma_0 = \sigma$.

If B is of finite type, the previous process finishes after $k \in \mathbb{N}$ steps. At that moment we will have obtained a section σ_k that is holomorphic over U_k. Now, by Theorems 5.7 and 5.18 in [CE12]and the same argument that the one given in the proof of Theorem 3.1.1, we can uniformly approximate σ_k by a global holomorphic section $\tilde{\sigma}$ that satisfies the desired properties. In this case $h_t = \mathrm{Id}$ for all $t \in [0, 1]$.

If B is not of finite type, $\tilde{\sigma}$ is obtained as the limit of the sequence $\{\sigma_j\}_{j \in \mathbb{N}}$ that is defined in $\tilde{B} := \bigcup_{i \in \mathbb{N}} \mathring{V}_i'$. Note that all the previous steps can be done fixed at Q if the sections σ_q are already holomorphic for every $q \in Q$.

The construction of the desired isotopy is the same that the one in the proof of Theorem 8.32(b) in [CE12]. It is constructed as follows. Take regular values $c_i'' > c_i$ of ϕ without critical values in $[c_i, c_i'']$ and set the sublevel set $V_i'' := \{x \in B : \phi(x) \leq c_i''\}$ and the number $d_i := \sum_{j=1}^{i} \frac{1}{2^j}$ for each $i \in \mathbb{N}$.

Now construct for each $i \in \mathbb{N}$ a diffeotopy $h_t^i : B \to B, t \in [d_{i-1}, d_i]$, where $d_0 = 0$ such that

- h_t^i maps level sets of ϕ_k to level sets,
- $h_{d_{i-1}}^i = \mathrm{Id}$ and $h_{d_i}^i(V_i) = V_i'$,
- $h_t^i = \mathrm{Id}$ on V_{i-1} and outside V_i'' for all $t \in [d_{i-1}, d_i]$

Now define the diffeotopies $h_t = h_t^i \circ h_{d_i}^{i-1} \circ \ldots \circ h_{\frac{1}{2}}^1$ for $t \in [d_{i-1}, d_i]$ for each $i \in \mathbb{N}$. Note that since h_t stabilices on compact sets there exists $h_1 := \lim_{t\to 1} h_t$ that is not surjective, therefore h_t can be defined as an isotopy for $t \in [0,1]$. □

The fact that every open Riemann surface is Stein yields to the following special situation

Corollary 3.2.4. *Let $E \to \Sigma$ be a holomorphic vector bundle over an open Riemann surface. Let M_ϕ be a Lagrangian skeleton of Σ for the strongly plurisubharmonic Morse exhaustion function ϕ and let $K \subset \Sigma$ be a compact subset. For every family of C^r–sections $\sigma_p : \mathcal{O}p\,(K \cup M_\phi) \to E$ that are holomorphic over $\mathcal{O}p\,(K)$ and that are continuously dependent on a parameter p in a compact Hausdorff space P, and for every positive function $\varepsilon : K \cup M_\varphi \to \mathbb{R}_+$, there exists a continuous family of sections $\widetilde{\sigma}_p$ that are holomorphic over $\mathcal{O}p\,(K \cup M_\phi)$ such that*

$$|\widetilde{\sigma}_p(x) - \sigma_p(x)| < \varepsilon(x), \quad \forall (p,x) \in P \times \mathcal{O}p\,(K \cup M_\phi)\,.$$

Moreover, if there is a compact subset $Q \subset P$ such that for every $q \in Q$, the section σ_q is holonomic over $\mathcal{O}p\,(K \cup M_\phi)$, then $\widetilde{\sigma}_p$ can be chosen to satisfy that $\widetilde{\sigma}_q = \sigma_q$ for every $q \in Q$.

Proof. The construction of $\widetilde{\sigma}$ follows the same steps as the ones in the proof of Corollary 3.2.3. The only difference is that in the case of Riemann surfaces we can choose all the functions ϕ_i to be equal to ϕ. □

Remark 3.2.5. Since every holomorphic fiber bundle with fiber biholomorphic to $\mathbb{C}^n$ over a Stein manifold has a holomorphic section (Theorem 2.5.4 in [Gro89]), every holomorphic affine bundle is (non canonically) a holomorphic vector bundle. This means that all the results in this section apply also to affine bundles, in particular to holomorphic jet bundles. Other way of observing that is just by Remark 3.1.3. ⋈

Chapter 4

Local h–principles for HPDRs

4.1 Realifications and Thick Holomorphic Relations

Fix a holomorphic fiber bundle $E \to B$, we will see its jet spaces of holomorphic sections $E^{(r)}_{\mathcal{O}}$ as subbundles of $E^{(r)}$ (see Example 2.1.23). In this context, a *Holomorphic Partial Differential Relation* is a Partial Differential Relation that lies into $E^{(r)}_{\mathcal{O}}$. $E^{(r)}_{\mathcal{O}}$ is a closed submanifold of positive codimension inside $E^{(r)}$. Indeed this subbundle is determined by the Cauchy-Riemann equations. This yields to the following trivial but important

Remark 4.1.1. There is no open differential relation $\mathcal{R} \subset E^{(r)}$ that lies into $E^{(r)}_{\mathcal{O}}$, so whenever we say that a Holomorphic Partial Differential Relation is open it always will be as a subset of $E^{(r)}_{\mathcal{O}}$. ⋈

As in the smooth case, given a point $x \in E$, we can identify the fiber $(\pi^1_0|_{E^{(1)}_{\mathcal{O}}})^{-1}(x)$ with the complex linear morphisims $\mathrm{Hom}_{\mathbb{C}}(T_pB, \mathrm{Vert}_x)$, where $p = \pi(x)$. *Principal complex subspaces* are those affine subspaces conformed by all the morphisms that, given a complex hyperplane $H_p \subset T_pB$, extend the same given complex linear map $\eta : H_p \to \mathrm{Vert}_x$.

Given a totally real submanifold of maximal dimension $M \subset B$ we can define the restriction map ρ_M that sends each section $s : \mathcal{O}p\,(p) \to E$ to its restriction

$$s_{|\mathcal{O}p(p)\cap M} : \mathcal{O}p\,(p) \cap M \longrightarrow E_{|M}.$$

ρ_M induces the isomorphisms of bundles

$$\rho_M^r : (E_{\mathcal{O}}^{(r)})_{\big|M} \longrightarrow (E_{\big|M})^{(r)}, \quad r \in \mathbb{N} \cup \{0\},$$

that sends each r–tangency class of a holomorphic section s over $p \in M$ to the r–tangency class of $\rho_M(s)$ at p. Take an r–tangency class in $(E_{\big|M})^{(r)}$ and an analytic representative ζ of it. The inverse map $(\rho_M^r)^{-1}$ sends the taken class to the r–tangency class in $(E_{\mathcal{O}}^{(r)})_{\big|M}$ represented by the complex analytic extension of ζ.

Definition 4.1.2. *Let $E \to B$ be a holomorphic fiber bundle and $\mathcal{R} \subset E_{\mathcal{O}}^{(r)}$ be a Holomorphic Partial Differential Relation. For any given smooth totally real submanifold $M \subset B$ of maximal dimension, we define* the induced realified relation

$$\mathcal{R}_{\mathbb{R}}(M) := \rho_M^r(\mathcal{R}) \subset (E_{\big|M})^{(r)}.$$

Remark 4.1.3. Note that the map $\rho_M^1 : (E_{\mathcal{O}}^{(1)})_{\big|M} \to (E_{\big|M})^{(1)}$ is actually an isomorphism of complex affine bundles. Indeed, let $A \in (E_{\mathcal{O}}^{(1)})_{\big|M}$. The fiber at the point $x := \pi_0^1(A)$ of the affine bundle $\pi_0^1 : E^{(1)} \to E$ can be identified with the vector space over $\mathbb{C}$ formed by all the complex linear maps from T_pB to Vert_x, where $A \equiv 0$. Analogously, the fiber over $\rho_M^1(A)$ of $(E_{\big|M})^{(1)} \to E_{\big|M}$ can be identified with the complex vector space of real linear maps from T_pM to Vert_x where $\rho_M^1(A) = 0$. In this setting, the map ρ_M^1 sends each linear map to its restriction to T_pM and its inverse is just the $\mathbb{C}$–linear extension from T_pM to $T_pB \equiv T_pM \oplus iT_pM$. This shows in particular that ρ_M^1 sends principal complex subspaces of $(E_{\mathcal{O}}^{(1)})_{\big|M}$ to principal subspaces of $(E_{\big|M})^{(1)}$. Actually ρ_M^r is an isomorphism of complex affine bundles between $(E_{\mathcal{O}}^{(r)})_{\big|M} \to (E_{\mathcal{O}}^{(r-1)})_{\big|M}$ and $(E_{\big|M})^{(r)} \to (E_{\big|M})^{(r-1)}$ for any $r \in \mathbb{N}$, but we will not need it in this manuscript. $\bowtie$

The previous remark is key to understand how we can use the techniques provided by the smooth theory in the holomorphic setting. With those techniques we can obtain homotopies of formal solutions of the realifications. Then we use the isomorphism described above to understand them as homotopies of formal solutions of the restricted Holomorphic Partial Differential Relation. In order to do so, we need to guarantee that the realifications satisfy an h–principle. We now provide a family of examples where this condition holds along the remaining of this subsection.

Definition 4.1.4. *Let $E \to B$ be a holomorphic fiber bundle and $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ be a HPDR. We will say that $\mathcal{R}$* has open realifications *if for any totally real submanifold $M \subset B$ of maximal dimension, we have that its realification is an open differential relation in $(E_{|_M})^{(r)}$. Similarly, if $r = 1$ we say that $\mathcal{R}$* has ample realifications *if its realifications are ample in principal directions.*

Note that any open holomorphic relation, has always open realifications. One important family of holomorphic relations with ample realifications are thick holomorphic relations.

Definition 4.1.5. *Let $E \to B$ be a holomorphic fiber bundle and $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ be a Holomorphic Partial Differential Relation in it. We say that $\mathcal{R}$ is a* thin holomorphic relation *(or a* thin holomorphic singularity*) if it intersects every principal complex subspace of $E^{(1)}_{\mathcal{O}}$, in a complex analytic set of complex codimension greater or equal than 1. We will say that $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ is a* thick holomorphic relation *(THR) if it is the complement of a thin holomorphic relation.*

Some remarks are in order: the definition of complex analytic set (or complex subvariety) follows the classical one in complex analysis (p.86 in [GR09]). We say that a subset S of a complex manifold X is complex analytic if there exists a covering by holomorphic charts, $\phi_j : B_j \to U_j$, where $B_j \subset X, U_j = B(0, \varepsilon) \subset \mathbb{C}^n$ for $\varepsilon > 0$ small enough, such that for any B_j there exists a finite set of holomorphic functions $f_1, \dots, f_m$ defined over U_j, satisfying that $U_j \cap S = Z(f_1) \cap Z(f_2) \cap \ldots \cap Z(f_m)$.

$U_j \cap S$ is the union of a finite number of complex subvarieties. The codimension of a complex analytic set on the chart B_j is the minimum of the codimensions of that subvarieties. The codimension of a complex analytic set is the minimum of the codimensions of the complex analytic sets $Z(f_j)$ over each ball of the covering.

Proposition 4.1.6. *Let $\pi : E \to B$ be a holomorphic vector bundle and $\mathcal{R} \subset E^{(1)}_{\mathcal{O}}$ be a Holomorphic Partial Differential Relation. If $\mathcal{R}$ is a THR then it has ample realifications.*

Proof. Let $M \subset B$ be a totally real submanifold of maximal dimension. We have to prove that $\mathcal{R}_{\mathbb{R}}(M)$ is ample in $(E_{|M})^{(1)}$.

By Remark 4.1.3 the restriction map ρ^1_M sends principal complex subspaces of $(E^{(1)}_{\mathcal{O}})_{|M}$ to principal subspaces of $(E_{|M})^{(1)}$. Therefore the complement of $\mathcal{R}_{\mathbb{R}}(M), \Sigma$, will intersect each principal subspace in a stratified subset of codimension greater or equal than 2. This is the same of saying that Σ is a thin singularity in the sense of 2.2.12. This yields that $\mathcal{R}_{\mathbb{R}}(M)$ is a TDR so it is ample in principal directions. □

Broadly speaking, Proposition 4.1.6 works just because complex hyperspaces do not disconnect the ambient space, so their complements are ample. Note that this is not true in the real analytic category. Indeed, the complement of a real analytic set of codimension 1 can be formed by two convex connected components, so it will not be ample. Since there are many geometric objects that can be understood as nonzero sections of vector bundles, this observation will lead us to some differences between the real and complex cases. They will be explored in Section 4.4.

Remark 4.1.7. Note that if $\mathcal{R}$ is a thick holomorphic relation, then every kind of h–principle applies for its realifications. Indeed, since it is open, the relation has open realifications. Therefore, since the realifications are open and ample we can apply Gromov's convex integration 2.2.15 to them. We can take advantage of this fact in order to prove h–principles near totally real submanifolds for the relation $\mathcal{R}$ (or even more general kind of sets). More precisely, we will prove some h–principles for germs of holomorphic sections over adapted skeletons of Stein manifolds of finite type (see Definition 2.3.21 and Theorem 4.2.1). ⋈

4.2 Statements of the Theorems

4.2.1 Local h–principles over skeletons of finite type.

The following Theorem is the main result of this Thesis and it is proven in Section 4.3. The rest of the principal results are either a consequence of it or are proved in a very similar way.

Theorem 4.2.1. *Let B be a Stein manifold of finite type and let $E \to B$ be a holomorphic vector bundle with a Holomorphic Partial Differential Relation $\mathcal{R}$ that is open in $E^{(r)}_{\mathcal{O}}$. If $\mathcal{R}_{\mathbb{R}}(M)$ satisfies a relative to domain h–principle for every totally real submanifold of maximal dimension $M \subset B$, then $\mathcal{R}$ satisfies the local h–principle over any adapted skeleton. The local h–principle is weakly relative to Stein compact domains properly attached to the adapted skeleton.*

Moreover, if the h–principles that satisfy the realifications $\mathcal{R}_{\mathbb{R}}(M)$ are all parametric, relative to parameter, full and/or C^0–dense, then so it is the local h–principle for sections lifting to $\mathcal{R}$.

Remark 4.2.2. All of the Theorems in this Section are stated for holomorphic vector bundles but, like Theorem 3.1.1 and all of the approximation results in Chapter 3, it follows from Remark 3.1.3 that all of them work for general holomorphic fiber bundles. $\bowtie$

As a direct consequence of Theorem 4.2.1 we have the following result for open Holomorphic Partial Differential Relations with ample realifications. In particular, by Proposition 4.1.6, it works for any THR.

Theorem 4.2.3. *Let B be a Stein manifold of finite type and $E \to B$ be a holomorphic vector bundle. Then every open holomorphic relation $\mathcal{R} \subset E^{(1)}_{\mathcal{O}}$ with ample realifications satisfies a full, C^0–dense, parametric and relative to parameter local h–principle over any adapted skeleton that is weakly relative to any Stein compact properly attached to the adapted skeleton.*

Proof (of Theorem 4.2.3). Since $\mathcal{R}_{\mathbb{R}}(M)$ is open and ample for every totally real submanifold M of maximal dimension, Gromov's convex integration gives that $\mathcal{R}_{\mathbb{R}}(M)$ satisfies all kinds of h–principles. Therefore we are under the strongest set of hypothesis of Theorem 4.2.1. $\square$

The homotopies of formal local solutions obtained in Theorem 4.2.1 are made to preserve the holonomy over properly attached Stein compact sets, but they are not formed by holomorphic local sections. Nevertheless we can do the opposite and obtain the following

Theorem 4.2.4. *Under the hypothesis of Theorem 4.2.1, the relation $\mathcal{R}$ satisfies the local holomorphic h–principle over any adapted skeleton. This holomorphic h–principle is close over Stein compact domains properly attached to the adapted skeleton.*

Moreover if the h–principles satisfied by the realifications are all parametric, relative to parameter, full and/or C^0–dense, then so it is the local holomorphic h–principle over the adapted skeleton.

Remark 4.2.5. Likewise Theorem 4.2.1, all of its consequences stated above have analogue results for local holomorphic h–principles, i.e. preserving holomorphy instead of holonomy over properly attached sets. ⋈

4.2.2 Theorems for Stein manifolds of arbitrary type

All the previous results in this Section assume that the base manifold B is of finite type. If this is not the case, we are not able to fix the adapted skeleton and therefore we cannot obtain a local h–principle over it for complex dimensions greater than 1. Nevertheless, if the conditions over the realifications are satisfied, then given a formal solution over a Stein manifold of arbitrary type B, we still can find a homotopy between that formal solution and a formal solution that is holonomic over a Stein manifold diffeotopic to B. That homotopy can still be made to be parametric, relative to parameter and close to the 0-jet part of the original formal solution if the h–principles of the realifications satisfy their corresponding conditions. More precisely, we prove the following Theorems.

Theorem 4.2.6. *Let B be a Stein manifold with integrable almost complex structure J and let $E \to B$ be a holomorphic vector bundle with a Holomorphic Partial Differential Relation $\mathcal{R}$ that is open in $E^{(r)}_{\mathcal{O}}$ and assume that $\mathcal{R}_{\mathbb{R}}(M)$ satisfies a relative to domain h–principle for every totally real submanifold of maximal dimension $M \subset B$. Then, for every formal solution $\sigma_0 : B \to \mathcal{R}$ there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of formal solutions $\sigma_t : B \to \mathcal{R}$ such that*

1. $h_0 = \mathrm{Id}_B$

2. $(h_t(B), J_{\big|_{h_t(B)}})$ *is Stein (or equivalently,* $(B, h_t^* J)$ *is Stein) for every* $t \in [0,1]$*.*

3. σ_1 *is holonomic in* $\mathcal{O}p\,(h_1(B))$*.*

Moreover, if the h–principles that satisfy the realifications $\mathcal{R}_{\mathbb{R}}(M)$ *are all* C^0*–dense, then the homotopy of formal solutions can be chosen to satisfy that* $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$*. If the h–principles over the realifications are parametric (and relative to parameter) then the previous theorem holds for continuous families of formal solutions (relatively to closed sets of the parameter space) with* h_t *being independent of the parameter.*

Let $E \to B$ be a holomorphic fiber bundle over a Stein manifold B with integrable almost complex structure J and let $\mathcal{R} \subset E_{\mathcal{O}}^{(r)}$ be a Holomorphic Partial Differential Relation. Let us consider the space

$$\mathscr{D}(B,J) \equiv \{h : B \xrightarrow{\cong} h(B) \subset B \,\text{diffeomorphism} \,\big|\, (B, h^* J) \,\text{is Stein}\}$$

and let $\mathscr{D}_0(B,J)$ be the connected component of the identity of $\mathscr{D}(B,J)$. Let us consider also the spaces of pairs

$$\mathscr{F}\mathcal{R} \equiv \{(\sigma, h) \,\big|\, h \in \mathscr{D}_0(B,J) \,\text{and}\, \sigma \in \mathrm{F}h^*\mathcal{R}\}$$

and

$$\mathscr{H}\mathcal{R} \equiv \{(\sigma, h) \,\big|\, h \in \mathscr{D}_0(B,J) \,\text{and}\, \sigma \in \mathrm{H}h^*\mathcal{R}\}.$$

Finally, let us call $\mathscr{F}\mathcal{R}_h$ and $\mathscr{H}\mathcal{R}_h$ the preimage of the diffeomorphism $h \in \mathscr{D}_0(B,J)$ under the natural projection over the second component of $\mathscr{F}\mathcal{R}$ and $\mathscr{H}\mathcal{R}$ respectively.

Using that notation, the previous Theorem can be stated in the following way

Corollary 4.2.7. *Let* B *be a Stein manifold with integrable almost complex structure* J *and let* $E \to B$ *be a holomorphic vector bundle with a Holomorphic Partial Differential Relation* $\mathcal{R}$ *that is open in* $E_{\mathcal{O}}^{(r)}$*. Assume that* $\mathcal{R}_{\mathbb{R}}(M)$ *satisfies a relative to domain h–principle for every totally real submanifold of maximal dimension* $M \subset B$*. Then, for every* $\sigma \in \mathrm{F}\mathcal{R}$ *there exists a path* $(\sigma_t, h_t) \in \mathscr{F}\mathcal{R}, t \in [0,1]$ *such that* $(\sigma_0, h_0) = (\sigma, \mathrm{Id})$ *and* $(\sigma_1, h_1) \in \mathscr{H}\mathcal{R}$*. Moreover, if the h–principles that satisfy the realifications are* C^0*–dense, then the homotopy can be chosen to satisfy that* $(\sigma_t \circ h_t^{-1})^{(0)}$ *is* C^0*–close to* σ *for every* $t \in [0,1]$*.*

If the h–principles over the realifications are parametric (and relative to parameter), then for every compact Hausdorff space P (every closed subset $Q \subset P$), and for every $s : P \to \mathscr{F}\mathcal{R}_{\mathrm{Id}}$ (such that $s(Q) \subset \mathscr{H}\mathcal{R}_{\mathrm{Id}}$) there exists a path $h_t \in \mathscr{D}_0(B, J)$ and a path of maps $s_t : P \to \mathscr{F}\mathcal{R}_{h_t}$ such that ($s_t\big|_Q \equiv h_t^* s\big|_Q$ for every $t \in [0,1]$) $s_0 = s$ and $s_1(P) \subset \mathscr{H}\mathcal{R}_{h_1}$.

The following is a version of Theorem 4.2.6 for holomorphic formal solutions.

Theorem 4.2.8. *Let B, J, X and $\mathcal{R}$ be like in Theorem 4.2.6. Then, for every holomorphic formal solution $\sigma_0 : B \to \mathcal{R}$ there exists a smooth path of diffeomorphisms $h_t : B \to h_t(B) \subset B$, $t \in [0,1]$ and a homotopy of holomorphic formal solutions $\sigma_t : h_1(B) \to \mathcal{R}\big|_{h_1(B)}$ such that*

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J\big|_{h_t(B)})$ *is Stein (or equivalently, $(B, h_t^* J)$ is Stein) for every $t \in [0,1]$,*
3. σ_1 *is holonomic in $h_1(B)$.*

Moreover, if the h–principles that satisfy the realifications $\mathcal{R}_{\mathbb{R}}(M)$ are all C^0–dense, then the homotopy of formal solutions can be chosen to satisfy that $\sigma_t^{(0)}$ is arbitrarily C^0–close to $\sigma_0^{(0)}$. If the h–principles over the realifications are parametric (and relative to parameter), then the previous theorem holds for continuous families of holomorphic formal solutions (relatively to closed sets of the parameter space) with h_t being independent of the parameter.

4.2.3 The case of Riemann surfaces

The fact that every open subset of an open Riemann surface is Stein makes special the case of complex dimension 1. Like in Corollary 3.2.4 we will not have to ask the open Riemann surface to have finite type to obtain local h–principles over any Lagrangian skeleton. Moreover, it is easily seen from the definitions (see 2.3.12 and 2.3.17) that this fact also implies that any compact subset of an open Riemann surface is Stein compact and properly attached to any other compact subset. Since in addition the totally real submanifolds of maximal dimension of a Riemann surface are precisely the smooth curves on it we will obtain the following

Theorem 4.2.9. *Let Σ be an open Riemann surface and let $E \to \Sigma$ be a holomorphic vector bundle with a Holomorphic Partial Differential Relation $\mathcal{R}$ that is open in $E^{(r)}_{\mathcal{O}}$. If $\mathcal{R}_{\mathbb{R}}(\Gamma)$ satisfies a relative to domain h–principle for every smooth curve $\Gamma \subset \Sigma$, then $\mathcal{R}$ satisfies the local h–principle over any Lagrangian skeleton. The h–principle is weakly relative to any compact domain.*

Moreover, if the h–principles that satisfy the realifications $\mathcal{R}_{\mathbb{R}}(\Gamma)$ are all parametric, relative to parameter, full and/or C^0–dense, then so it is the local h–principle over sections lifting to $\mathcal{R}$.

Remark 4.2.10. We can also obtain the corresponding result analogous to Theorem 4.2.3. Moreover, such as in Remark 4.2.5, we can also get their corresponding analogue results for holomorphic h–principles preserving holomorphy instead of holonomy along compact subsets.

$\bowtie$

4.3 Proofs of the Theorems

The proof of Theorems 4.2.1 and 4.2.6 follow the same type of argument. The firs step is done by Lemma 4.3.1. This Lemma uses the h–principles over the realifications together with Remark 4.1.3 and Theorem 3.1.1 to construct homotopies of parametric formal local solutions over totally real manifolds with trivial complex normal bundles that begin in a prescribed parametric local solution and that end in a holonomic one relatively to the fiber. This allows to extend the holonomy from one stratum of the skeleton to the next one inductively. This inductive process will conclude the proof of Theorem 4.2.6. To finish the proof of Theorem 4.2.1 it will only remain to choose the appropriate parametric spaces.

On the other hand, Theorems 4.2.4 and 4.2.8 follow directly from the previous ones just by using Theorem 3.1.1, Corollaries 3.2.1 and 3.2.3 and Remark 3.2.2.

4.3.1 Extending holonomy and pseudo-holonomy through totally real submanifolds

The following Lemma allows us to homotope formal local solutions that are holonomic in some Stein compact set K to local solutions that are holonomic in K and in a totally real manifold M properly attached to K. It works parametrically and relative to parameter and it constitutes the key step of the proofs of Theorems 4.2.1 and 4.2.6.

Lemma 4.3.1. *Let $M \subset B$ be a totally real submanifold of class C^r of a complex manifold B with trivial complex normal bundle and let K be a compact set properly attached to M. Let $S := K \cup M$, let $Q \subset P$ be a compact subset in a compact Hausdorff space and let $E \to B$ be a holomorphic vector bundle with a fibered open holomorphic relation $\mathcal{R} \subset P \times E^{(r)}_{\mathcal{O}}$ such that all its realifications satisfy a relative to domain and fiber h–principle. Let $\sigma \in \mathrm{F}\mathcal{R}$ be a formal local solution that is already holonomic in $P \times \mathcal{O}p\,(K) \cup Q \times \mathcal{O}p\,(S)$. Then, for every $\varepsilon > 0$ there exists Ω a Stein neighbourhood of S and a homotopy $\sigma_t : P \times \Omega \to \mathcal{R}$, $t \in [0,1]$ of formal local solutions such that*

1. *$\sigma_t \in \mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon)$, for every $t \in [0,1]$,*
2. *$\sigma_0 = \sigma$ in $P \times \Omega$,*
3. *$(\sigma_t)\big|_{Q\times\Omega} = \sigma\big|_{Q\times\Omega}$, for every $t \in [0,1]$,*
4. *σ_1 is holonomic in $P \times \Omega$.*

Moreover, if the h–principles satisfied by the realifications are C^0–dense, then the homotopy σ_t can be chosen to satisfy that $\sigma_t^{(0)}$ is arbitrarily C^0–close to $\sigma^{(0)}$ on $P \times S$ for every $t \in [0,1]$.

Proof. Let $k = \dim(M)$ and $n = \dim_{\mathbb{C}}(B)$. Since the complex normal bundle of M is trivial, the tangent bundle of B restricted to M is

$$TB\big|_M = TM \oplus iTM \oplus (\mathbb{C}^{n-k} \times M).$$

Choosing a Riemannian metric we can construct $W \subset \mathcal{O}p\,(M)$, a totally real submanifold of dimension n that contains M, as the image of $TM \oplus \Lambda$ under the exponential map of the tangent bundle, where $\Lambda \cong (\mathbb{R}^{n-k} \times M)$ is a trivial real subbundle of the complex normal bundle of M.

By Remark 4.1.3, we can identify the restriction $\sigma_{|P\times W}$ with a formal solution of the realification $s_0 \in \mathrm{F}\mathcal{R}_{\mathbb{R}}(W)$. Let $U \subset B$ be an open Stein neighbourhood of K such that $\sigma_{|P\times\overline{U}}$ is holonomic and $C := \overline{U} \cap W$. Since $\mathcal{R}_{\mathbb{R}}(W)$ satisfies a relative to domain and fiber h–principle, we can find a homotopy $s_t, t \in [0,1]$ of formal solutions of $\mathcal{R}_{\mathbb{R}}(W)$ joining s_0 with $s_1 \in \mathrm{H}\mathcal{R}_{\mathbb{R}}(W)$ such that $s_{t|Q\times W\cup P\times C} = s_{0|Q\times W\cup P\times C}$.

Again by Remark 4.1.3, this homotopy translates immediately to σ'_t, a homotopy of sections of $\mathcal{R}_{|P\times W}$ satisfying that $\sigma'_0 = \sigma_{|P\times W}$ and that $\sigma'_{t|Q\times W\cup P\times C} = \sigma_{|Q\times W\cup P\times C}$ for every $t \in [0,1]$. Since σ'_t coincides with σ at $P \times C$, σ'_t extends naturally to $P \times \overline{U}$.

Now we use Theorem 3.1.1 to C^r-approximate the family of sections $(\sigma'_t)_p := \sigma'_t(p, \text{-})$, $(t,p) \in [0,1] \times P$ by a continuous family of holomorphic sections $(\widetilde{\sigma}_t)_p, (t,p) \in [0,1] \times P$. Note that $(\widetilde{\sigma}_t)_p$ approximates $(\sigma'_t)_p$ over S only, since W may be not properly attached to K.

$\widetilde{\sigma}_t$ can be seen as a homotopy of sections of $\mathcal{R}_{|P\times\mathcal{O}p(S)}$ that are ε–close to σ over $P \times K$ and that coincides with σ in $Q \times S$. Note that since $\widetilde{\sigma}_1$ C^r-approximates σ'_1, σ'_1 extends s_1, s_1 is holonomic and $\mathcal{R}$ is open, if the C^r–approximation is sufficiently close, then the r–jet of $\widetilde{\sigma}_1^{(0)}$ belongs to $\mathrm{H}\mathcal{R}(S, K, \sigma, \varepsilon)$.

Taking maybe an even closer C^r–approximation, we can also assume that there exists Ω, a Stein neighbourhood of S, where the paths of sections

$$h_1(t) := (1-t)\sigma + t\widetilde{\sigma}_0, \quad h_2(t) := \widetilde{\sigma}_t \quad \text{and} \quad h_3(t) := (1-t)\widetilde{\sigma}_1 + tj^r(\widetilde{\sigma}_1^{(0)}), \quad t \in [0,1]$$

are contained in $\mathrm{F}\mathcal{R}_{|P\times\Omega}$. We call $h' : [0,1] \to \mathrm{F}\mathcal{R}_{|P\times\Omega}$ the homotopy of formal solutions of $\mathcal{R}_{|P\times\Omega}$ that results after joining the paths h_1, h_2 and h_3.

The problem with h' is that although $h'(t)_p := h'(t)(p, \text{-})$ is arbitrarily close to σ_p over K for every $p \in P$, they do not necessarily have to be holonomic in $\mathcal{O}p\,(K)$, so $h'(t)$ may not be in $\mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon)$. To fix this, consider the homotopy of local sections

$$h''(t) := (1-t)\sigma + tj^r(\widetilde{\sigma}_1^{(0)}).$$

Note that, taking close enough the approximation given by Theorem 3.1.1, we can assume that $h''(t) \in \mathrm{H}\mathcal{R}(K)$ for every $t \in [0,1]$. Indeed $\sigma'_t = \sigma$ along U for every $t \in [0,1]$, therefore $\widetilde{\sigma}_t$ approximates σ over $U' \subset \Omega$ an open neighbourhood of K. Now we can interpolate h' and h'' in $U' \setminus \mathcal{O}p\,(K)$ to obtain the desired homotopy. That is, consider a smooth cutoff function ρ supported in $\overline{U'}$ and such that $\rho_{|_{\mathcal{O}p(K)}} \equiv 1$, the desired homotopy is

$$\sigma_t := h'(t(1-\rho)) + h''(t\rho).$$

If the h–principles of the realifications are C^0–dense, then we can make s_t to be C^0–close to s_0. Taking such an s_t and replicating the argument is enough to prove the last part of the statement. □

The following Lemma is an analogue of Lemma 4.3.1 for holomorphic local sections to prove Theorems 4.2.4 and 4.2.8. With it we obtain holonomic sections preserving holomorphy and pseudo-holonomy on certain sets.

Lemma 4.3.2. *Let $E, \mathcal{R}, B, M, K, S, Q$ and P be like in Lemma 4.3.1. Let $\sigma \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S)$ be a holomorphic formal local solution that is already pseudo-holonomic in both $P \times \mathcal{O}p\,(K)$ and $Q \times \mathcal{O}p\,(S)$. Then, for every $\varepsilon > 0$ there exists a Stein neighbourhood of S, Ω and a homotopy $\sigma_t : P \times \Omega \to \mathcal{R}$, $t \in [0,1]$ of holomorphic formal local solutions such that*

1. *$\sigma_t \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon)$ for every $t \in [0,1]$,*
2. *$\sigma_0 = \sigma$ in $P \times \Omega$,*
3. *$(\sigma_t)_{|_{Q\times\Omega}} = \sigma_{|_{Q\times\Omega}}$ for every $t \in [0,1]$,*
4. *σ_1 is holonomic in $P \times \Omega$.*

Moreover, if the h–principles satisfied by the realifications are C^0–dense, then the homotopy σ_t can be chosen to satisfy that $\sigma_t^{(0)}$ is arbitrarily C^0-close to $\sigma^{(0)}$ on $P \times S$ for every $t \in [0,1]$.

Proof. The proof consists in just using Lemma 4.3.1 to obtain a homotopy $\sigma'_t, t \in [0,1]$ that we approximate relative to $t \in \{0,1\}$ by Theorem 3.1.1 to obtain the desired homotopy σ_t. □

4.3.2 Extending holonomy and pseudo-holonomy through adapted skeletons

The following Lemma is the result of using Lemma 4.3.1 inductively over each stratum of an adapted skeleton properly attached to some Stein compact set. Therefore we obtain a version of Lemma 4.3.1 where we can substitute the totally real submanifold by an adapted skeleton of the base Stein manifold.

Lemma 4.3.3. *Let $Q \subset P$ be a compact subset in a compact Hausdorff space and let B be a Stein manifold of finite type. Let $E \to B$ be a holomorphic vector bundle equipped with a fibered open holomorphic relation $\mathcal{R} \subset P \times E^{(r)}_{\mathcal{O}}$ and assume that all its realifications satisfy a relative to domain and fiber h–principle. Let $M \subset B$ be an adapted skeleton and let $K \subset B$ be a compact subset properly attached to M and let $S := K \cup M$. Then, for every formal local solution over S, $\sigma \in \mathrm{F}\mathcal{R}(S)$, that is already holonomic over $P \times K \cup Q \times S$ and for every $\varepsilon > 0$, there exists a homotopy $\sigma_t : P \times \mathcal{O}p\,(S) \to \mathcal{R}, t \in [0,1]$ of formal local solutions satisfying that*

1. *$\sigma_t \in \mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon)$,for every $t \in [0,1]$,*
2. *$\sigma_0 = \sigma$ in $P \times \mathcal{O}p\,(S)$,*
3. *$(\sigma_t)\big|_{Q\times\mathcal{O}p(S)} = \sigma\big|_{Q\times\mathcal{O}p(S)}$, for every $t \in [0,1]$,*
4. *σ_1 is holonomic in $P \times \mathcal{O}p\,(S)$.*

Moreover, if the h–principles satisfied by the realifications are C^0–dense, then the homotopy σ_t can be chosen to satisfy that $\sigma^{(0)}_t$ is arbitrarily C^0–close to $\sigma^{(0)}$ on $P \times S$ for every $t \in [0,1]$.

Proof. The proof goes inductively on each stratum of the skeleton. Let $K_i := K \cup M_{\leq i}$ and $S_i := K_i \cup M_{i+1}$ for every $i = 0, \dots, k$ where k is the number of critical points of a Morse plurisubharmonic function ϕ defining M and $M_0 := \emptyset$. After the inductive process we will have obtained finite sequences σ_i and h_i, $i = 1, \dots, k$ such that

1. $\sigma_i \in \mathrm{F}\mathcal{R}(S, K_{i-1}, \sigma_{i-1}, \varepsilon/(k+1)) \cap \mathrm{H}\mathcal{R}(K_i)$.
2. $h_i : [0,1] \to \mathrm{F}\mathcal{R}(S, K_{i-1}, \sigma_{i-1}, \varepsilon/(k+1))$ is a homotopy such that $h_i(0) = \sigma_{i-1}$ and $h_i(1) = \sigma_i$,

where $\sigma_0 = \sigma$. Putting all the homotopies h_i together we obtain the homotopy

$$\begin{aligned} \widetilde{h} \colon [0,1] &\longrightarrow \mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon) \\ t &\longmapsto \widetilde{\sigma}_t \end{aligned}$$

that joins $\widetilde{h}(0) = \sigma$ with $\widetilde{h}(1) = \sigma_k \in \mathrm{H}\mathcal{R}(S, K, \sigma, \varepsilon)$.

Now we detail the inductive process. Taking $\sigma_0 = \sigma_{-1} = \sigma$ and $h_0 \equiv \sigma$ we obtain the base case. For the inductive step take $0 < j < k$ and assume that we have obtained h_i and σ_i for each $i \leq j$.

Since M_{j+1} is a totally real disc properly attached to K_j it has trivial complex normal bundle, so we can use Lemma 4.3.1 on K_j, S_j and σ_j to obtain a homotopy

$$h : [0,1] \to \mathrm{F}\mathcal{R}(S_j, K_j, \sigma_j, \varepsilon/(k+1))$$

defined on Ω, a Stein neighbourhood of S_j, such that $h(1)$ is a holonomic solution of $\mathcal{R}$ in $P \times \Omega$. Let $\rho : B \to [0,1]$ be a smooth cutoff function supported on Ω and such that $\rho\big|_{\mathcal{O}p(S_j)} \equiv 1$. This allows us to define the sections

$$\begin{aligned} h_{j+1}(t) \colon P \times \mathcal{O}p\,(S) &\longrightarrow \mathcal{R} \\ (p,x) &\longmapsto h(t \cdot \rho(x))(p,x) \end{aligned}$$

for $t \in [0,1]$ obtaining the homotopy

$$h_{j+1} : [0,1] \to \mathrm{F}\mathcal{R}(S, K_j, \sigma_j, \varepsilon/(k+1))$$

and the section $\sigma_{j+1} := h_{j+1}(1) \in \mathrm{H}\mathcal{R}(K_{j+1})$.

In the case that the h–principles satisfied by the realifications are C^0–dense, we can take the homotopies h_i to satisfy that $h_i(t)^{(0)}$, and hence $h(t)^{(0)}$, are C^0–close to $\sigma^{(0)}$, for every $t \in [0,1]$. □

The following Lemma is analogous to Lemma 4.3.3 for holomorphic local solutions. Thanks to it we can obtain holonomic sections preserving holomorphy and pseudo-holonomy.

Lemma 4.3.4. *Let $E, \mathcal{R}, B, M, K, S, Q$ and P be like in Lemma 4.3.3. Then, for every holomorphic formal local solution over S, $\sigma \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S)$ that is already pseudo-holonomic over $P \times K \cup Q \times S$ and every $\varepsilon > 0$, there exists a homotopy $\sigma_t : P \times \mathcal{O}p\,(S) \to \mathcal{R}, t \in [0,1]$ of holomorphic formal local solutions satisfying that*

1. *$\sigma_t \in \mathcal{O}\,\mathrm{F}\mathcal{R}(S, K, \sigma, \varepsilon)$, for every $t \in [0,1]$,*
2. *$\sigma_0 = \sigma$ in $P \times \mathcal{O}p\,(S)$,*
3. *$(\sigma_t)\big|_{Q\times\mathcal{O}p(S)} = \sigma\big|_{Q\times\mathcal{O}p(S)}$, for every $t \in [0,1]$,*
4. *σ_1 is holonomic in $P \times \mathcal{O}p\,(S)$.*

Moreover, if the h–principles satisfied by the realifications are C^0–dense, then the homotopy σ_t can be chosen to satisfy that $\sigma_t^{(0)}$ is arbitrarily C^0–close to $\sigma^{(0)}$ on $P \times S$ for every $t \in [0,1]$.

Proof. The proof consist in just using Lemma 4.3.3 to obtain a homotopy $\sigma'_t, t \in [0,1]$ that we approximate relative to $t \in \{0,1\}$ by Corollary 3.2.1 and Remark 3.2.2 to obtain the desired homotopy σ_t. □

4.3.3 Proofs of the Theorems for finite type Stein manifolds.

Proof (of Theorem 4.2.1).

Let K be a compact set that is properly attached to M and let $S := K \cup M$ and let

- (P, Q) be a pair of an arbitrary compact Hausdorff space P and a compact subset Q, if the h–principles of the realifications are parametric and relative to parameter.
- $(P, Q) = (P, \emptyset)$ where P is an arbitrary compact Hausdorff space, if the h–principles of the realifications are just parametric.
- $(P, Q) = (\mathbb{D}^k, \partial\mathbb{D}^k)$, if the h–principles of the realifications are full.
- $(P, Q) = (P, \emptyset)$ where P is just a point, if the h–principles of the realifications do not satisfy any of the previous conditions.

Now let $\sigma_0 \in \mathrm{F}\mathcal{R}_P(S, K)$ be a formal local solution of $\mathcal{R}_P$ over S. Now just use Lemma 4.3.3 to obtain a homotopy $\sigma_t \in \mathrm{F}\mathcal{R}_P(S, K, \sigma_0, \varepsilon)$ such that $\sigma_1 \in \mathrm{H}\mathcal{R}_P(S)$. Note that if the h–principles of the realifications are C^0–dense then we can choose σ_t to satisfy that $\sigma_t^{(0)}$ is arbitrarily C^0–close to $\sigma_0^{(0)}$. □

Proof (of Theorem 4.2.4).

The proof is almost the same to the one of Theorem 4.2.1 but using Lemma 4.3.4 instead of 4.3.3. We just also need to join the homotopy σ_t obtained with the linear interpolation betweeen σ_1 and $j^r(\sigma_1^{(0)})$ to make the final local solution to be holonomic instead of just pseudo-holonomic. □

4.3.4 Proofs of the Theorems for Stein manifolds of arbitrary type.

Suppose that, for each $i = 0, \ldots, k$, the section σ_i in the proof of Lemma 4.3.3 is holonomic in the open Stein neighbourhood $\Omega_i = \mathcal{O}p\,(M_{\leq i})$. There is no reason why we can assume that $\Omega_i \subset \Omega_{i+j}$ holds. Actually, if we try to use adapted skeletons to prove analogous results for Stein manifolds of infinite type, it may happen that $\bigcap_{j=0}^{\infty} \Omega_{i+j} = M_{\leq i}$ that has empty interior. That is the reason why, although there are adapted skeletons for any Stein manifold of arbitrary type, we do not use them to prove results analogous to the two previous Lemmas for the infinite type case.

Therefore, to prove Theorem 4.2.6 we will need to obtain a sequence of local sections σ_i such that they are holonomic in Stein domains Ω_i satisfying that $\Omega_i \subseteq \Omega_{i+1}$ for each $i = 0, \ldots$. The price to pay for it will be to change the skeleton in each inductive step.

The proof of Theorem 4.2.6 follows similar steps to the ones given in the proof of Theorem 8.45 in [CE12] but using the h–principles of the realifications and the appropiate C^r–approximation. The construction of the isotopy h_t is the same that the one that is used in the proof of Theorem 8.32 (b) of [CE12] to obtain the isotopy that is called $h^{(1)}$ in the notation of the book and it is described in the Proof of Corollary 3.2.3. One can also check the proof of Theorem 1.2 in [For20]. Let us detail the process.

Proof (of Theorem 4.2.6).

If the base manifold B is of finite type then Lemma 4.3.3 proves the result. Let us assume that B is of infinite type.

Take a pair (P, Q) such as in the proof of Theorem 4.2.1, with abuse of notation, we will call by $\mathcal{R}$ the fibered relation $P \times \mathcal{R} \subset P \times E^{(r)}_{\mathcal{O}}$. Let us also take a Morse strongly plurisubharmonic function ϕ and assume that it satisfies the same conditions that the one in the proof of Corollary 3.2.3.

Let $\{p_i\}_{i=0}^{\infty} = \mathrm{Crit}(\phi)$ and let $\{c_i\}_{i=0}^{\infty}$ be a sequence of regular values such that

$$c_0 < \phi(p_0) < c_1 < \ldots < c_i < \phi(p_i) < c_{i+1} < \ldots$$

and let $\{V_i\}_{i=1}^{\infty}$ be like in the proof of Corollary 3.2.3. Use Lemma 4.3.1 on $M_0 = \{p_0\}$ on σ_0 relative to Q to obtain (after gluing with σ_0 by a cutoff function) a homotopy of formal solutions $\sigma_t^0, t \in [0, \frac{1}{2}]$ such that $\sigma^0_{\frac{1}{2}} \in \mathrm{H}\mathcal{R}(M_0)$. Find a regular value $c \in (\phi(p_0), c_1)$ such that the section $\sigma^0_{\frac{1}{2}}$ is holonomic in $K := \{x \in B : \phi(x) < c\}$.

Now let us start the inductive process. Since the stable manifold of the critical point p_1 for ϕ, M_1, is properly attached to K, we can use Lemma 4.3.1 again to find an open neighbourhood $U_1 = \mathcal{O}p\,(K \cup M_1) \subset V_1$ where there is defined a homotopy of formal solutions of $\mathcal{R}_{|U_1}$, $\sigma^1_t \in \mathrm{F}\mathcal{R}(K \cup M_1, K, \sigma^0_{\frac{1}{2}}, \frac{1}{2}\varepsilon), t \in [\frac{1}{2}, \frac{3}{4}]$ relative to Q and such that $\sigma^1_{\frac{1}{2}} = \sigma^0_{\frac{1}{2}}$ and $\sigma^1_{\frac{3}{4}}$ is holonomic in U_1. Extend the homotopy σ^1 to a formal solution of $\mathcal{R}$ in the whole manifold by σ_0 via a cutoff function (shrinking U_1 if necessary).

Now we reproduce the same inductive process that the one described in the proof of Corollary 3.2.3 but using Lemma 4.3.1 instead of Theorem 3.1.1 like above.

Let $d_0 = 0$ and $d_k := \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i$ for $k \in \mathbb{N}$. After the inductive process we will have obtained sequences $\phi_i, M_i, c_i', \widetilde{V}_{i-1}, V_i'$ and U_i like the ones in the proof of Corollary 3.2.3 satisfying conditions (1) to (4) and a sequence of homotopies of formal solutions of $\mathcal{R}$ $\sigma_t^i, t \in [d_i, d_{i+1}]$ such that

1. $\sigma_{d_i}^i = \sigma_{d_i}^{i-1}$,
2. $\sigma_t^i \in \mathrm{F}\mathcal{R}(B, V_{i-1}', \sigma_{d_i}^{i-1}, \left(\frac{1}{2}\right)^i \varepsilon)$, for every $t \in [d_i, d_{i+1}]$,
3. $\sigma_{d_{i+1}}^i$ is holonomic in U_i

for each $i \in \mathbb{N}$. Let $\widetilde{B} := \bigcup_{i\in\mathbb{N}} \mathring{V}_i'$. Note that the sequence $\sigma_{d_{k+1}}^k \xrightarrow{k} \sigma_1$, where σ_1 is a holonomic solution of $\mathcal{R}_{|\widetilde{B}}$. Therefore, the homotopy defined by $\sigma_t = \sigma_t^k$, if $t \in [d_k, d_{k+1}]$ is the one we are looking for.

The construction of h_t is exactly the same as the one in the proof of Corollary 3.2.3. □

Proof (of Theorem 4.2.8).

Like in the previous proofs, use the non-holomorphic version of this Theorem (Theorem 4.2.6) to obtain a homotopy of formal solutions σ_t and an isotopy h_t'. Now use Corollary 3.2.3 for $\sigma_{t|h_1(B)}$ on $h_1'(B)$ and relatively to $\{0,1\} \times P \cup [0,1] \times P$ to obtain the isotopy h_t'' and the homotopy of formal solutions σ_t that are holomorphic in $h_1''(h_1'(B))$. The desired isotopy is the result of joining h_t' with h_t''. □

4.3.5 Proofs of the Theorems for open Riemann surfaces

Proof (of Theorem 4.2.9 and Remark 4.2.10).

Take a plurisubharmonic Morse exhausting function ϕ and let M_ϕ its Lagrangian skeleton. Note that we can replicate the proofs of Theorems 4.2.6 and 4.2.8 but assuming that all the functions ϕ_i coincide with ϕ. This yields a version of Lemmas 4.3.3 and 4.3.4 for open Riemann surfaces of arbitrary type. Now replicate the proofs of Theorems 4.2.6 and 4.2.4 using these Lemmas to obtain the desired results. $\square$

4.4 Applications

In this Section we will show that the Holomorphic Partial Differential Relations defined in 2.1.3 are suitable for the Theorems proven in this Chapter. Therefore we will obtain some local h–principles (or homotopies of formal parametric local solutions joining a given a family of formal solutions with a holonomic one). The strategy has been first to express the geometric objects in terms of sections of a bundle. Then, check if the corresponding Holomorphic Partial Differential Relation satisfies the conditions of the Theorems.

We will start by reviewing the case of maps of fiberwise maximal rank, a Holomorphic Partial Differential Relation in which it is known that a version of Theorem 4.2.6 is satisfied (see Theorem 1.4 in [FS07] and Theorem 8.43 and Remark 8.44 in [CE12]). Here we will rewrite this problem in our terms and give a local h–principle for the Holomorphic Partial Differential Relation $\mathcal{R}_{\mathcal{O}\text{Max-rank}}(B, V) \subset J^1_{\mathcal{O}}(B, V)$ where B is a Stein manifold and V a complex manifold.

After that we will study the locally conformal symplectic forms and the forms (or pairs of forms) whose kernels consist in topologically stable complex distributions, i.e. holomorphic contact and even contact forms and Engel pairs. The case of complex contact structures is at the same situation as the one of maps of maximal rank (see Theorems 1.2 and 6.1 in [For20]) and we will proceed in the same way. The rest of the cases are completely new to the best of our knowledge.

4.4.1 Maps of fiberwise maximal rank: holomorphic immersions and submersions

The PDR $\mathcal{R}_{\text{Max-Rank}}(M_1, M_2)$ is ample (it is indeed a TDR) if and only if $\dim M_1 < \dim M_2$. The h–principle of this relation (Corollary 2.2.16) is known as the Smale-Hirsh h–principle for immersions. Despite of that, the holomorphic analogue of this relation $\mathbb{R}_{\mathcal{O}\text{Max-Rank}}(M_1, M_2)$ is always a THR (see [FS07]).

Therefore, if B is a stein manifold and V a complex manifold, then the Holomorphic Partial Differential Relation $\mathcal{R}_{\mathcal{O}\text{Max-Rank}}(B, V)$ satisfies all the hypotheses of the Theorems proven in this chapter. Hence we have

Theorem 4.4.1. *Let B be a Stein manifold and let V be a complex manifold. Then, for every $\sigma_0 \in P \times \mathrm{F}\mathcal{R}_{\mathcal{O}\textit{Max-Rank}}(B, V)$ being a continuous family of formal solutions that is holonomic over $Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0, 1]$ and a homotopy of formal solutions $\sigma_t : P \times h_1(B) \to P \times \mathcal{R}_{\textit{Max-Rank}}(B, V)$ such that*

1. *$h_0 = \mathrm{Id}_B$,*
2. *$(h_t(B), J_{|h_t(B)})$ is Stein for every $t \in [0, 1]$,*
3. *$\sigma_t^{(0)}$ is arbitrarily close to $\sigma_0^{(0)}$,*
4. *$\sigma_t(q, \text{-}) = \sigma_0(q, \text{-})_{|h_1(B)}$ for every $q \in Q$,*
5. *σ_1 is a family of holomorphic immersions if $\dim B \leq \dim V$ or a family of holomorphic submersions if $\dim B \geq \dim V$ in $h_1(B)$.*

Moreover, if $\sigma_0(p, \text{-})$ is holomorphic for every $p \in P$ then, for every $t \in [0, 1]$, σ_t can be chosen to satisfy that $\sigma_t(p, \text{-})$ is holomorphic for every $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then the Holomorphic Partial Differential Relation $\mathcal{R}_{\textit{Max-Rank}}(B, V)$ satisfies a C^0–dense, parametric, and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton. □

4.4.2 Holomorphic contact forms

In the smooth category Gromov's h–principle [Gro69] stablishes an h–principle of contact forms for any open manifold (see Section 10.3 in [EM02]). In closed manifolds it is not true, it indeed fails in $\mathbb{S}^3$, but there is a special class, called *overtwisted contact manifolds* that satisfy the h–principle (see Theorem 1.6.1 in [Eli89] for dimension 3 and [BEM15] for dimension ≥ 5). The Partial Differential Relation $\mathcal{R}_{\mathrm{Cont}}(M)$ is open but not ample, so they are needed other techniques different from convex integration to prove these h–principles.

In the complex cathegory, using Mori theory it is possible to classify all projective contact manifolds whose canonical bundle is not nef and that are not Fano with $b_2 = 1$ [KPSW00]. In the affine case, F. Forstnerič has recently proven that this Holomorphic Partial Differential Relation satisfies a version of Theorem 4.2.6 (see Theorem 6.1 in [For20]).

Let B be a Stein manifold of odd complex dimension $\dim B = 2n + 1$ and $L \to B$ be a holomorphic line bundle. Consider the HPDR $\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B, L)$. Note that if α satisfies condition (Cont) in Definition 2.1.29, i.e. $j^1\alpha \in \mathrm{H}\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B, L)$, then $\alpha \wedge (d\alpha)^n$ provides a holomorphic trivialization of $K_B \otimes_{\mathbb{C}} L^{n+1}$, where $K_B := \bigwedge^{2n+1} T^*B$ is the canonical bundle of B.

Remark 4.4.2. It is worth mentioning that when the base space is a Stein manifold, the Oka principle gives that the existence of a formal solution of $\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}$ implies that $K_B \otimes_{\mathbb{C}} L^{n+1}$ is holomorphically trivial. ⋈

This Holomorphic Partial Differential Relation is clearly an open THR. Indeed, it intersects each fiber of

$$\pi_0^1 : (T^*B \otimes L)^{(1)}_{\mathcal{O}} \to T^*B \otimes L$$

in the complement of a complex-analytic set of codimension 1. That complex-analytic set contains some principal complex subspaces and intersects the rest of them in complex codimension 1 (see Lemma 2.1 in [For20] for more details). Therefore $\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B, L)$ satisfies all the hypotheses of our Theorems and therefore we have the following

Theorem 4.4.3. *Let B a Stein manifold of complex dimension $2n + 1$ and let $L \to B$ be a holomorphic line bundle. Then, for every continuous parametric family of formal solu-*

tions $\sigma_0 \in P \times \mathrm{F}\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B,L)$ that is holonomic over $Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of formal solutions $\sigma_t : P \times h_1(B) \to P \times \mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B)$ such that

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0,1]$,
3. $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$,
4. $\sigma_t(q, \text{-}) = \sigma_0(q, \text{-})_{|h_1(B)}$ *for every* $q \in Q$,
5. σ_1 *is a family of holomorphic contact forms in* $h_1(B)$.

Moreover, if $\sigma_0(p, \text{-})$ is holomorphic for every $p \in P$ then, for every $t \in [0,1]$, σ_t can be chosen to satisfy that $\sigma_t(p, \text{-})$ is holomorphic for every $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then $\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(B)$ satisfies a C^0–dense, parametric, and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton. □

4.4.3 Holomorphic even-contact forms

Besides the Partial Differential Relation $\mathcal{R}_{\mathrm{Cont}}(M)$ is not ample, this is not the case for smooth even-contact forms. The ampleness of $\mathcal{R}_{\mathrm{ECont}}(M)$ yields a full parametric, and relative to parameter and domain h–principle for even-contact smooth forms (see [McD87]). Therefore, it is not surprising that the complex even-contact relation also satisfies the conditions of the Theorems that we have proven. Nevertheless let us also review that case as it will be useful for the study of holomorphic Engel distributions.

Let B be a Stein manifold of even dimension $\dim_{\mathbb{C}} B = 2n+2$ and $L \to B$ be a holomorphic line bundle. Consider the HPDR $\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B,L)$. A holomorphic form α satisfying condition (ECont) in Definition 2.1.30 provides a nowhere vanishing section $\alpha \wedge (d\alpha)^n$ of the complex bundle of $\bigwedge^{2n+1} T^*B \otimes L^{n+1}$. Since the rank of that bundle coincides with the dimension of the base space B, such a section will always exist if B is a Stein manifold.

It is easily seen that $\mathcal{R}_{\mathbb{C}\text{- ECont}}(B)$ is also a THR, therefore we obtain the following

Theorem 4.4.4. *Let B a Stein manifold of complex dimension $2n+2$ and let $L \to B$ be a holomorphic line bundle. Then, for every continuous parametric family of formal solutions $\sigma_0 \in P \times \mathrm{F}\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B)$ that is holonomic over $Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of formal solutions $\sigma_t : P \times h_1(B) \to P \times \mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B)$ such that*

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0,1]$,
3. $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$,
4. $\sigma_t(q, \text{-}) = \sigma_0(q, \text{-})_{|h_1(B)}$ *for every* $q \in Q$,
5. σ_1 *is a family of complex–even contact forms in* $h_1(B)$.

Moreover, if $\sigma_0(p, \text{-})$ is holomorphic for every $p \in P$ then, for every $t \in [0,1]$, σ_t can be chosen to satisfy that $\sigma_t(p, \text{-})$ is holomorphic for every $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then $\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B)$ satisfies a C^0–dense, parametric, and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton. □

4.4.4 Holomorphic Engel structures

The case of Engel structures is similar to the one in contact topology. Engel structures in open manifolds are flexible, but once again, since the corresponding PDR is not ample, it has been necessary to come up with different techniques to prove h–principles for this kind of distributions. There is a parametric existence h–principle [CPdPP17] and like in the contact case, there is a *"overtwisted"* class of Engel structures.

The study of holomorphic Engel structures for projective varieties started less than ten years ago in [PSC14], where it was provided some examples of Engel structures. There is also some examples of exotic holomorphic Engel structures in $\mathbb{C}^4$ [CP18].

The results obtained in this Section can be used as a source of examples of holomorphic Engel structures in Stein manifolds of dimension 4. Indeed each Stein manifold satisfying the required topological properties to have a formal Engel pair will yields to a homotopically equivalent Stein manifold with a genuine holomorphic Engel structure.

Let B be a Stein manifold of dimension $\dim_{\mathbb{C}} B = 4$ and $L \to B$ be a holomorphic line bundle. Consider the HPDR $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}}(B, L)$.

Remark 4.4.5. Note that $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}}(B, L)$ is a fibration over $\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B, L)$. Indeed, let us call $E = T^*B \otimes L$ and consider the bundle $E^{(1)}_{\mathcal{O}} \times_B E^{(1)}_{\mathcal{O}} \xrightarrow{p_2} E^{(1)}_{\mathcal{O}}$, where p_2 is the projection to the second factor of the fibered product over B. Let X be the restriction of the previous bundle over $\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B, L) \subset E^{(1)}_{\mathcal{O}}$. Now consider the bundle maps

$$\begin{aligned} \Phi\colon X &\longrightarrow (K_B \otimes L^3) \times_B \mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B, L) \\ (a, b) &\longmapsto (a^{(0)} \wedge b^{(0)} \wedge Db, b) \end{aligned}$$

and

$$\begin{aligned} \Psi\colon \ker \Phi &\longrightarrow (K_B \otimes L^3) \times_B \mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(B, L) \\ (a, b) &\longmapsto (a^{(0)} \wedge b^{(0)} \wedge Da, b). \end{aligned}$$

Then $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}}(B.L) = \ker \Phi \setminus \ker \Psi$ is the complement of an affine subbundle inside another affine subbundle of X, and therefore a fiber bundle over $\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}$. ⋈

To find an h–principle for holomorphic Engel structures take a (parametric) formal holomorphic Engel pair (a, b) on B and assume for a moment that b is already holonomic, i.e. $b = j^1\beta$, where β is a complex even-contact form in B with values in a line bundle $L \to B$.

Now consider the map

$$\begin{aligned}\Phi_\beta \colon T^*B\otimes L &\longrightarrow K_B\otimes L^3\\ \alpha_x &\longmapsto \alpha_x\wedge\beta_x\wedge(d\beta)_x\end{aligned}$$

for each $x\in B$ and each $\alpha_x\in\pi^{-1}(x)$. Since $\beta\wedge(d\beta)\neq 0$ everywhere, $V_\beta := \ker\Phi_\beta$ is a vector subbundle of $T^*B\otimes L$ of rank 3 where condition (FEng.2) in Definition 2.1.31 is always satisfied.

Note that the Engel condition (FEng.1) in Definition 2.1.31 defines a THR in $(V_\beta)^{(1)}_{\mathcal{O}}$

$$\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}_\beta} := \{a\in(V_\beta)^{(1)}_{\mathcal{O}} \mid a^{(0)}\wedge\beta\wedge Da\neq 0\}.$$

Therefore we obtain the following

Lemma 4.4.6. *Let B be a Stein manifold of dimension 4 and let $L\to B$ be a holomorphic line bundle. Let β be a parametric family of complex even-contact forms on B with values in L continuously dependent on p in a compact CW-complex P. Then, for every continuous P–parametric family of formal solutions $\sigma_0\in\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}_\beta}$ such that $\sigma_0(q,\text{-})$ is holonomic for every q in a compact subcomplex Q of P, there exists a smooth family of diffeomorphisms $h_t : B\to h_t(B)\subset B, t\in[0,1]$ and a homotopy of formal solutions $\sigma_t : P\times h_1(B)\to\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}_\beta}$ such that*

1. $h_0=\mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t\in[0,1]$,
3. $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$,
4. $\sigma_t(q,\text{-})=\sigma_0(q,\text{-})_{|h_1(B)}$ *for every* $q\in Q$,
5. $\sigma_1(p,\text{-})$ *is holonomic for every* $p\in P$.

Moreover, if $\sigma_0(p,\text{-})$ is holomorphic for every $p\in P$, then, for every $t\in[0,1]$, σ_t can be chosen to satisfy that $\sigma_t(p,\text{-})$ is holomorphic for every $p\in P$.

If in addition B is of finite type, then $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}_\beta}$ satisfies a C^0–dense, parametric, and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton. □

Due to the previous Lemma, to obtain holomorphic Engel h–principles it will be enough to find a homotopy of formal complex even contact forms a_t such that (a_t, b_t) is a path of formal holomorphic Engel pairs, where $(a_0, b_0) = (a, b)$ and b_t is the homotopy obtained in Theorem 4.4.4. After that we just join that homotopy to the one given by Lemma 4.4.6.

To find such a homotopy we only need to use the homotopy lifting property of the fiber bundle $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}} \to \mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}$ (see Remark 4.4.5) with respect to $B \times P$ relatively to $B \times Q$ (see Proposition 4.48 in [Hat02]). In the case of the holomorphic h–principle we also approximate that homotopy by a holomorphic one using Theorem 3.1.1. Thus we obtain the following

Theorem 4.4.7. *Let B be a Stein manifold of dimension 4 and let $L \to B$ be a holomorphic line bundle. Then, for every continuous P–parametric family of formal holomorphic Engel pairs $(a, b)_p, p \in P$ such that $(a, b)_q$ is a pair of holonomic sections for every $q \in Q \subseteq P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0, 1]$ and a homotopy of formal holomorphic Engel pairs $(a, b)_t$ such that*

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0, 1]$,
3. $(a, b)_0 = (a, b)$,
4. $b_t^{(0)}$ *is arbitrarily close to* $b^{(0)}$,
5. $(a, b)_{t,q} = (a, b)_q$ *for every* $q \in Q$ *and every* $t \in [0, 1]$,
6. $(a, b)_1$ *is a P–parametric family of pairs of holonomic sections.*

Moreover, if $(a, b)_p$ is a pair of holomorphic sections for every $p \in P$ then, for every $t \in [0, 1]$, $(a, b)_t$ can be chosen to satisfy that $(a, b)_{t,p}$ is a pair of holomorphic sections for every $p \in P$.

If in addition B is of finite type, then $\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}}(B, L)$ satisfies a parametric and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton. Moreover, the h–principle is C^0–dense and weakly relative to (close to) domains properly attached to the skeleton. □

4.4.5 Complex locally conformal symplectic structures

Like in the contact case, since the relation $\mathcal{R}_{\mathrm{Liouv}}(M, L, \nabla)$ is both open and Diff_M–*invariant* one may use *holonomic approximation* to prove an h–principle for every open even-dimensional smooth manifold M (See [EM02]). But, since this PDR is not ample, one cannot use convex integration to find full h–principles for closed manifolds M.

There are only a few known examples of compact hyperkähler manifolds (i.e. compact complex locally conformal symplectic compact Kähler manifolds in which the connection is d). They are Hilbert schemes of $K3$–manifolds and generalized Kummer manifolds [Bea83], and also another singular example in real dimension 20 constructed by G. O'Grady [O'G99]. The "twisted case" is not in a very different situation since every compact complex locally conformal symplectic compact Kähler manifold is a cyclic cover of a hyperkähler manifold [Ist16]. Here we discuss the situation in which the manifold is not compact but Stein, providing a source of many examples of complex locally conformal symplectic manifolds.

Let B be a Stein manifold of even dimension $\dim_{\mathbb{C}} B = 2n$ and $L \to B$ be a trivial holomorphic line bundle with a flat connection ∇. Similarly to the complex contact case, if ω satisfies condition (Symp) in Definition 2.1.25, then ω^n is a trivialization of K_B. Let $a \in H^2(B, L)$ and let η be a closed holomorphic 2–form representing a. The relation $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B, L, \nabla)$ is a THR, so therefore we have the following

Theorem 4.4.8. *Let B be a Stein manifold of complex dimension $2n$ and let $L \to B$ be a trivial holomorphic line bundle with a flat connection ∇. Let $a \in H^2(B, L)$ be a fixed cohomology class and let η be a holomorphic closed 2–form in that class. Then, for every continuous parametric family of formal solutions $\sigma_0 \in P \times \mathrm{F}\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B, L, \nabla)$ that is holonomic over $Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0, 1]$ and a* *homotopy of formal solutions* $\sigma_t : P \times h_1(B) \to P \times \mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B, L, \nabla)$ *such that*

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0,1]$,
3. $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$,
4. $\sigma_t(q, \text{-}) = \sigma_0(q, \text{-})_{|h_1(B)}$ *for every* $q \in Q$,
5. σ_1 *is holonomic.*

Moreover, if $\sigma_0(p, \text{-})$ *is holomorphic for every* $p \in P$ *then, for every* $t \in [0,1]$, σ_t *can be chosen to satisfy that* $\sigma_t(p, \text{-})$ *is holomorphic for every* $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then the Holomorphic Partial Differential Relation $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B, L, \nabla)$ *satisfies a* C^0*–dense, parametric and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton.* □

The previous Theorem will be useful to prove some h–principles over germs of locally conformal symplectic structures. Here locally conformal almost-symplectic forms play the role of formal solutions.

Definition 4.4.9. *Let* $a \in H^2(B, L)$ *be a fixed cohomology class and let* $S \subset B$ *be a closed subset of* B. *We denote by* $\Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(S, L, \nabla)$ *the set of germs of locally conformal symplectic forms over* S *that lie in* a, *i.e. the set of equivalence classes of local locally conformal symplectic forms* $\omega \in \Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(\mathcal{O}p\,(S), L_{|\mathcal{O}p(S)}, \nabla_{|\mathcal{O}p(S)})$ *where two local locally conformal symplectic forms are equivalent if and only if they coincide in some* $\mathcal{O}p\,(S)$. *We denote the set of germs of locally conformal almost-symplectic forms over* S *by* $\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(S, L, \nabla)$.

Theorem 4.4.10. *Let* B *be a Stein manifold of complex dimension* $2n$ *and let* $L \to B$ *be a trivial holomorphic line bundle with a flat connection* ∇. *Let* $a \in H^2(B, L)$ *be fixed cohomology class and let* η *be a holomorphic closed 2–form in that class. Then, for any continuous family of complex locally conformal almost-symplectic forms* $\omega_{p,0}, p \in P$ *such that*

$\omega_{q,0}$ represents a for every $q \in Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of P–parametric locally conformal almost-symplectic forms $\omega_{p,t}$ such that

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0,1]$,
3. $\omega_{q,t} = \omega_{q,0|h_1(B)}$ *for every* $q \in Q$,
4. $\omega_{p,1}$ *represents a for every* $p \in P$.

In particular, $\omega_{p,1}$ is a locally conformal symplectic form for every $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then the inclusion

$$\Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(M, L, \nabla) \overset{i}{\hookrightarrow} \Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(M, L, \nabla)$$

is a weak homotopy equivalence for every adapted skeleton $M \subset B$.

Proof. Let $\sigma_{p,0}$ be a continuous family of holomorphic formal primitives of the 2–forms $\omega_{p,0} - \eta$ such that $\sigma_{q,0}$ is holonomic for every $q \in Q$. Apply Theorem 4.4.8 to obtain the desired h_t and a homotopy of holomorphic formal solutions of $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B, L, \nabla)$, $\sigma_{p,t}$. The homotopy of P–parametric locally conformal almost-symplectic forms $\omega_{p,t} := \widetilde{D}\sigma_{p,t} + \eta$ satisfies the desired properties.

Now assume that B is a Stein manifold of finite type and let M be an adapted skeleton of B. To prove the weak homotopy equivalence in the statement, it suffices to consider $\mathbb{S}^k$–families of locally conformal almost-symplectic forms for the surjectivity of

$$i_* : \pi_k(\Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(M, L, \nabla)) \to \pi_k(\Omega^2_{a,\mathcal{O}\,\mathrm{LCASymp}}(M, L, \nabla)),$$

and $\mathbb{S}^k \times [0,1]$–families relative to $\mathbb{S}^k \times \{0,1\}$ for the injectivity. □

Let us conclude this Chapter with the corresponding theorems for complex symplectic structures that are granted by 1 in Remark 2.1.26.

Theorem 4.4.11. *Let B be a Stein manifold of complex dimension $2n$. Let $a \in H^2(B)$ be a fixed cohomology class and let η be a holomorphic closed 2–form in that class. Then, for every continuous parametric family of formal solutions $\sigma_0 \in P \times \mathrm{F}\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{Symp}}(B)$ that is holonomic over $Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of formal solutions $\sigma_t : P \times h_1(B) \to P \times \mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{LCSymp}}(B)$ such that*

1. $h_0 = \mathrm{Id}_B$,
2. $(h_t(B), J_{|h_t(B)})$ *is Stein for every* $t \in [0,1]$,
3. $\sigma_t^{(0)}$ *is arbitrarily close to* $\sigma_0^{(0)}$,
4. $\sigma_t(q, \text{-}) = \sigma_0(q, \text{-})_{|h_1(B)}$ *for every* $q \in Q$,
5. σ_1 *is holonomic.*

Moreover, if $\sigma_0(p, \text{-})$ is holomorphic for every $p \in P$ then, for every $t \in [0,1]$, σ_t can be chosen to satisfy that $\sigma_t(p, \text{-})$ is holomorphic for every $p \in P$.

If in addition B is a Riemann surface or if it is of finite type, then $\mathcal{R}^{\eta}_{\mathcal{O}\,\mathrm{Symp}}(B)$ satisfies a C^0–dense, parametric and relative to parameter local h–principle (holomorphic h–principle) over any adapted skeleton that is weakly relative to (close to) domains properly attached to the skeleton. □

Finally, setting the corresponding spaces of germs $\Omega^2_{a,\mathcal{O}\,\mathrm{Symp}}(S) := \Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(S, M \times \mathbb{C}, d)$ and $\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}} := \Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(S, M \times \mathbb{C}, d)$ we obtain the following

Theorem 4.4.12. *Let B be a Stein manifold of complex dimension $2n$. Let $a \in H^2(B)$ be fixed cohomology class and let η be a holomorphic closed 2–form in that class. Then, for any continuous family of complex almost-symplectic forms $\omega_{p,0}, p \in P$ such that $\omega_{q,0}$ represents a for every $q \in Q \subset P$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B, t \in [0,1]$ and a homotopy of P–parametric almost-symplectic forms $\omega_{p,t}$ such that*

1. $h_0 = \mathrm{Id}_B$,

2. $(h_t(B), J_{\big|_{h_t(B)}})$ *is Stein for every* $t \in [0,1]$,

3. $\omega_{q,t} = \omega_{q,0\big|_{h_1(B)}}$ *for every* $q \in Q$,

4. $\omega_{p,1}$ *represents* a *for every* $p \in P$.

In particular, $\omega_{p,1}$ *is a symplectic form for every* $p \in P$.

If in addition B *is a Riemann surface or if it is of finite type, then the inclusion*

$$\Omega^2_{a,\mathcal{O}\,\mathrm{Symp}}(M) \overset{i}{\hookrightarrow} \Omega^2_{\mathcal{O}\,\mathrm{ASymp}}(M)$$

is a weak homotopy equivalence for every adapted skeleton $M \subset B$. □

Chapter 5

Conclusions and work in progress

The Theorems of Section 4.2 are quite general. We have shown a way to adapt the technique of convex integration (among others) to find local h–principles for them. This is somehow remarkable because Holomorphic Partial Differential Relations are far from being ample and open seen as Partial Differential Relations. The naive fact that complex codimension 1 corresponds with real codimension 2 has shown that there are many geometrically interesting examples of Thick HPDRs whose analogues in the smooth category are not even ample. Indeed, any HPDR defined as the complement of a zero section in some holomorphic vector bundle is a good candidate to be Thick.

The downside of this (that is also the motivation of our future work), is that the h–principles found are of local nature. In order to try making them global, Forstnerič state in [For20] a question for the relation $\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}$ (that may be generalized for more general HPDRs) such that, if its answer is affirmative, the local h–principles obtained here would turn to be global. This question seems too hard to answer in general, nevertheless we have came up with a different one that, far from being general, works at least for bounded open domains of Euclidean complex spaces and maybe for more examples that we are studying.

In this Chapter we will discuss this topic, and also other two that originate from the work of this thesis.

Convex integration is one of the two main techniques that are developed in [EM02], that is one of the fundamental books to read in order to start learning about h–principles. The other one corresponds to *holonomic approximation*.

Holonomic approximation is useful in the smooth category to find local h–principles in subsets with positive codimension for PDRs that can be defined coordinate freely. If a Stein manifold is subcritical (it has no critical points with index equal to its complex dimension) then the skeletons are going to have positive codimension inside totally real manifolds of maximal dimension, so one has a chance to use holonomic approximation there. But even the totally real submanifolds of maximal dimension have positive codimension in complex manifolds so it is tempting to try to adapt this technique to find local h–principles for Holomorphic Partial Differential Relations that can be defined independently of holomorphic coordinates. We have obtained some partial results in this direction that are worth mentioning in this work.

Using holonomic approximation one can obtain not only local, but global h–principles for open manifolds. Since Stein manifolds are necessarily noncompact it also worth to explore the chance to adapt the smooth case to Stein manifolds. We have found that it can be done for example for convex bounded domains in Euclidean complex spaces.

Finally, other possible threads to follow come from trying to generalize other techniques from the smooth category like wrinkled maps (see [EM09], [EM00] and [EM98]) or study the applications of the previous work to the Theory of directed embeddings and immersions. Indeed we have already mentioned that, since complex codimension 1 corresponds with real codimension 2, there are several geometrically interesting examples of Thick Holomorphic Relations whose analogues in the category of smooth manifolds are not even ample. This phenomena have interesting consequences for directed immersions and embeddings. For instance we have complex symplectic submanifold of complex codimension 2 and complex-Lagrangian and complex-Legendrian submanifolds in complex-symplectic and complex-contact manifolds respectively.

5.1 H–principles for locally Aut-invariant HPDRs

5.1.1 Holonomic approximation and G–invariant relations

Approximate sections of $E^{(r)} \to B$ by holonomic sections is obviously impossible in general. Indeed, let $\varepsilon > 0$ and consider the section

$$\begin{aligned} F\colon [a,b] &\longrightarrow J^1([a,b],\mathbb{R}) \\ x &\longmapsto (x, f(x), 0), \end{aligned}$$

where $f : [a,b] \to \mathbb{R}$ is a strictly increasing function. To approximate F by $j^1(g)$ one would need to find a function $g : [a,b] \to \mathbb{R}$ close to f such that $|g'(x)| < \varepsilon$, for every $x \in [a,b]$. Like in the Easy problem 3 in Example 1.1.1, this is impossible by the Mean Value Theorem since there will always exist $x_0 \in [a,b]$ such that

$$g'(x_0) \sim \frac{f(b) - f(a)}{\text{Length}([a,b])} > \varepsilon$$

for $\varepsilon > 0$ small enough.

There is a clever trick to overcome this issue. The idea is as simple as trying to make Length$([a,b])$ longer than just $b-a$. This is indeed what civil engineers do when they project a road over a mountain: they do zigzags instead of going straightly up. The problem of that is that you will need extra room for the interval $[a,b]$ to wiggle in order to make it longer, i.e. you will need to have (or find) some positive codimension.

This is also the idea behind the Smale-Hirsch classification of immersions [Sma59] that is called the microextension trick in [EM02]. This consists in embedding your base manifold in an auxiliary bigger space to allow the domain to *wiggle*. This idea is also behind wrinkling maps [EM09], [EM00] and [EM98], where the extra room is found in the target manifold allowing multivaluedness of sections.

Let us provide now a brief overview on how to use this extra dimension to find holonomic approximation of a section of a jet bundle σ. Let A be a triangulated subset of the base space with positive codimension. Since the approximation will work relative to domain one can assume that A is the unit cube in the first k coordinates $I^k \subset \mathbb{R}^n$, since it is an approximation result, the target space can be assumed also to be Euclidean, so we can assume that $\sigma \in \Gamma(J^r(\mathbb{R}^n, \mathbb{R}^q))$. At a single point $p \in I^k$, the section σ admits a holonomic representative $\widetilde{\sigma}_p$ that is as close as desired to σ in $\mathcal{O}p\,(p)$. So for each $u \in I^{k-1}$, one can take a family of holonomic sections $\widetilde{\sigma}_t$ parametrized by t in $\{u\} \times I \cong I$, such that $\widetilde{\sigma}_t$ is close to σ in a ball of radius δ centred at t.

Choosing a number N such that $\frac{1}{N} \ll \delta$ and $N+1$ equidistant points $0 = t_0, \ldots, t_N = 1$, one can find homotopies $\widetilde{\sigma}_{t_i}^{\tau}, \tau \in [0, \frac{1}{N}]$ of sections close to σ such that coincides with $\widetilde{\sigma}_{t_i}$ close to the boundary of the δ–ball centred at t_i and that $\widetilde{\sigma}_{t_i}^{\tau} = \widetilde{\sigma}_{t+\tau}$ in $\mathcal{O}p\,(\{u\} \times I)$.

Now one can use the extra codimension to isotope intervals $\{u\} \times I$ fixing the points t_i and taking their middle points away at a distance close to δ. One can now glue the sections $\widetilde{\sigma}_{t_i}$ in the deformations of the intervals $\{u\} \times [t_i, t_i + \frac{1}{2N}]$ and the sections $\widetilde{\sigma}_{t_i}^{\frac{1}{N}}$ in the deformations of the intervals $\{u\} \times [t_i + \frac{1}{2N}, t_{i+1}]$. The result is, for each $u \in I^{k-1}$, a section that is holonomic and close to σ in a deformation of the interval $\{u\} \times I$. One can see that this as an I–parametric family of holonomic sections over I^{k-1} and, since the previous process can also be done parametrically and relative to parameter, repeat it for $A = I^{k-1} \subset \mathbb{R}^{n-1}$. After k iterations one would obtain a holonomic section approximating σ over a deformation of A that would be δ close in the C^0–sense.

Following the previous steps (see Chapter 3 in [EM02] for more details) one obtains the following

Theorem 5.1.1 (Holonomic approximation [EM02])**.** *Let $A \subset B$ be a polyhedron of positive codimension in a smooth manifold B and let $\sigma_p : \mathcal{O}p\,(A) \to E^{(r)}$ be a family of sections continuously dependent on p in a compact Haussdorff space P that are holonomic for every p in a neighbourhood of a closed subset $Q \subset P$. Then, for every arbitrarily small $\delta, \varepsilon > 0$ there exist a continuous family of δ–small (in the C^0–sense) diffeotopies $h_p^t : B \to B, t \in [0,1]$ and a continuous family of holonomic sections $\widetilde{\sigma}_p : \mathcal{O}p\,\left(h_p^1(A)\right) \to E^{(r)}$ such that*

- $h_p^t = \mathrm{Id}$ *and* $\widetilde{\sigma}_p = \sigma_p$ *for all* $p \in \mathcal{O}p\,(Q)$,
- $\widetilde{\sigma}_p$ *is ε–close to* $\sigma_{p}\big|_{\mathcal{O}p(h)_p^1(A)}$ *in the C^0–sense.*

If in addition the sections σ_p were already holonomic in an open neighbourhood of a subpolyhedron $B \subset A$, then h_p^t may be choosen to be the identity and $\widetilde{\sigma}_p$ to be equal to $\sigma \cdot p$ over $\mathcal{O}p\,(B)$.

We have shown in Section 2.2.2 that open and ample Partial Differential Relations satisfy a full, parametric, C^0–dense and relative to domain and parameter h–principle due to the technique of *convex integration*. Holonomic approximation is a different technique to find h–principles for open $\operatorname{Diff} B$*–invariant PDRs* in open manifolds.

Given a fiber bundle $\pi : E \to B$, let us denote by $\operatorname{Diff}_B E$ the group of fiber preserving diffeomorphisms of E. More precisely, let $h_E : E \to E$ be a diffeomorphism of E, then $h_E \in \operatorname{Diff}_B E$ if and only if there exists a diffeomorphism $h_B : B \to B \in \operatorname{Diff} B$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{h_E} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{h_B} & B. \end{array}$$

If π is a holomorphic fiber bundle, then we define the group of fiber preserving biholomorphisms (automorphisms) by $\operatorname{Aut}_B E$ analogously.

Definition 5.1.2. *Let $\pi_* : \operatorname{Diff}_B E \to \operatorname{Diff} B$ be the projection $h_E \mapsto h_B$. We say that the projection is* natural *if there exists a homomorphism $j : \operatorname{Diff} B \to \operatorname{Diff}_B E$ such that $\pi_* \circ j = \operatorname{Id}$.*

We say that the projection is holomorphically natural *if there exist a homomophism $j_{\mathcal{O}}$: $\operatorname{Aut} B \to \operatorname{Aut}_B E$ such that $\pi_* \circ j_{\mathcal{O}} = \operatorname{Id}$.*

Note that if a fiber bundle is natural, then its jet bundles are so, indeed the homomorphism

$$\begin{aligned} j^r : \operatorname{Diff} B &\longrightarrow \operatorname{Diff}_B E^{(r)} \\ h &\longmapsto h_* := j^r(h), \end{aligned}$$

with $j^r(h)$ defined by sending $s \in E^{(r)}$ to the r-jet of the section $j(h) \circ \widetilde{s} \circ h^{-1}$ at $h(p)$, where $p \in B$ is the base point of s and $\widetilde{s} : \mathcal{O}p_B\,(p) \to E$ is a local section which represents the r–tangency class s, i.e. $j^r(\widetilde{s})(p) = s$. We will also denote $(h^{-1})_*$ by h^*.

Similarly, if a holomorphic fiber bundle is holomorphically natural, then its holomorphic jet bundles are so by the homomorphisms $j^r_{\mathcal{O}}: \operatorname{Aut} B \to \operatorname{Aut}_B E^{(r)}_{\mathcal{O}}$ defined analogously.

Example 5.1.3. Some examples of natural bundles are the tangent bundle, the cotangent bundle and all their exterior power bundles of a smooth manifold B.

- For $E = TB$, the homomorphism corresponding to a diffeomorphism $f : B \to B$ is given by its differential $df : TB \to TB$.
- If $E = \bigwedge^k(T^*B)$ the homomorphism is now given by the pushforward of the biholomorphism $f : B \to B$:

$$\begin{aligned} d^k f \colon \textstyle\bigwedge^k(T^*B) &\longrightarrow \textstyle\bigwedge^k(T^*B) \\ \omega_p &\longmapsto d^k f(\omega_p) := (f_*\omega)_{f(p)}, \end{aligned}$$

  for every $p \in B, \omega_p \in \bigwedge^k(T^*_pB)$ and $\omega_{f(p)} \in \bigwedge^k(T^*_{f(p)}B)$, where

$$(f_*\omega)_{f(p)}(v_1, \dots, v_k) := \omega_p(df^{-1}(v_1), \dots, df^{-1}(v_k))$$

  for every $v_1, \dots, v_k \in T_{f(p)}B$.

Similarly, the complex tangent bundle, the complex cotangent bundle and all their exterior power bundles of a complex manifold are holomorphically natural. ⋈

Definition 5.1.4. *Given a natural fiber bundle $E \to B$ and G a subgroup of* Diff B, *we say that a Partial Differential Relation $\mathcal{R} \subset E^{(r)}$ is G–*invariant *if $h_*(\mathcal{R}) = \mathcal{R}$ for every $h \in G$, that is if $\mathcal{R}$ is invariant under the action of G in $E^{(r)}$ induced by the natural homomorphism.*

If G is a subgroup of Aut B *we only need to ask the fiber bundle $E \to B$ to be holomorphically natural.*

$\mathcal{R} \subset E^{(r)}$ is Diff B–invariant if it can be defined B–coordinate freely, in particular the subset of holonomic sections of $E^{(r)} \to B$ is Diff B–invariant. Similarly, a HPDR is Aut B invariant if it can be independently of holomorphic coordinates. Note that the PDR $E^{(r)}_{\mathcal{O}} \subset E^{(r)}$ is Aut B invariant but not Diff B invariant.

Example 5.1.5. All the Holomorphic Partial Differential Relations defined in Section 2.1.3 are indeed Aut B–invariant but not Diff B–invariant. Their analogues in the smooth category are Diff B–invariant. ⋈

Now let us assume that one has a parametric formal solution σ_p of a Partial Differential Relation $\mathcal{R}$ over a smooth manifold B and let $A \subset B$ be a polyhedron of positive codimension.

One can approximate σ using Theorem 5.1.1 by a holonomic section over the deformations of A, $h_p^1(A)$. If the approximation is close enough and $\mathcal{R}$ is open one can find a parametric family of homotopies of formal solutions $\widetilde{\sigma}_{p,t}$ starting at $\widetilde{\sigma}_{p,0} = \sigma_p$ and ending in the holonomic approximation $\widetilde{\sigma}_{p,1} \in \mathrm{H}\mathcal{R}(h_p^1(A))$.

Now, if the relation is Diff B invariant, the paths $(h_p^t)^*\sigma$ and $(h_p^1)^*\widetilde{\sigma}_{p,t}$ are formed by formal local solutions over $\mathcal{O}p\,(A)$ and together will form a homotopy of parametric formal solutions of $\mathcal{R}$ over A joining σ_p with a parametric family of holonomic solutions.

This proves the local h–principle for open and Diff B–invariant PDRs in natural fiber bundles (See Theorem 7.2.1 in [EM02]).

Theorem 5.1.6. *Let $E \to B$ be a natural fiber bundle. Then any open* Diff B*–invariant Partial Differential Relation $\mathcal{R} \subset E^{(r)}$ satisfies a full, parametric, C^0–dense and relative to domain and parameter local h–principle near any polyhedron $A \subset B$ of positive codimension.* □

If moreover the base manifold is open one may obtain global h–principles in exchange of C^0–density. In this situation, the base manifold will admit an exhausting Morse type function without maximum points, so their stable disks will form a skeleton of positive codimension A. Moreover there exists isotopies $g_{k,t} : B \to B, k \in \mathbb{N}, t \in [0,1]$ sending a tubular neighbourhood of A of radius k to another of radius $\varepsilon\left(1 - \frac{1}{2^{k+1}}\right)$ such that $g_{k+1,t}$ coincides with $g_{k,t}$ in the tubular neighbourhood of radius k.

Every formal local solution σ of a Diff B–invariant Partial Differential Relation $\mathcal{R}$ over A provides a global formal solution $\widetilde{\sigma}$ defined by

$$\widetilde{\sigma}(p) := \lim_{k\to\infty} (g_k^t)^*\sigma(p)$$

for every $p \in B$. This yields to the global h–principle for open and Diff B–invariant PDRs in natural fiber bundles over open manifolds (see Theorem 7.2.2 in [EM02]).

Theorem 5.1.7. *Let $E \to B$ be a natural fiber bundle over an open manifold B. Then any open* Diff B*–invariant Partial Differential Relation $\mathcal{R} \subset E^{(r)}$ satisfies a full, parametric and relative to domain and parameter h–principle.* □

5.1.2 Adapted holonomic approximation

We have tried to replicate the previous strategy in the holomorphic Category. Our strategy has been to adapt Theorem 5.1.1 for sections in jet bundles of holomorphic sections. After that we will need to find a suitable property for HPDRs in the holomorphic category analogous to Diff B–invariance.

Anonther Mergelyan type approximation result can be found in Corollary 9 in [FFW20]. It states the following

Proposition 5.1.8. *Let K be the sublevel set of a strongly plurisubharmonic function in a complex manifold B and let M be a totally real submanifold properly attached to K. Let Y be another complex manifold. Then the set of holomorphic maps $f : \mathcal{O}p\,(K \cup M) \to Y$ is dense in the set of maps that are continuous over $K \cup M$ that are holomorphic in $\mathring{K}$. Moreover, every map of class C^r over $K \cup M$ and holomorphic in $\mathring{K}$ can be C^r–approximated by maps that are holomorphic in $\mathcal{O}p\,(K \cup M)$.* □

Proposition 5.1.8 allows us to approximate the diffeotopies h^1 obtained by the nonparametric case of Theorem 5.1.1. This proves the following adaptation of the holonomic approximation Theorem.

Lemma 5.1.9. *Let $M \subset B$ be a totally real submanifold of class C^r of a complex manifold B and let K be a sublevel set of a plurisubharmonic function properly attached to M. Let $S := K \cup M$ and let $\sigma : \mathcal{O}p\,(S) \to E^{(r)}$ be a section that is holonomic in $\mathcal{O}p\,(K)$. Then, for every arbitrarily small $\delta, \varepsilon > 0$ there exist a δ–small (in the C^0–sense) diffeotopy $h_t : B \to B$, $t \in [0,1]$ and a holonomic section $\widetilde{\sigma}_p : \mathcal{O}p\,\left(h^1_p(S)\right) \to E^{(r)}$ such that*

1. *h_t is close to the identity and $\widetilde{\sigma}$ is close to σ over $\mathcal{O}p\,(K)$,*

2. *$\widetilde{\sigma}$ is ε–close to $\sigma_{|\mathcal{O}p(h)_t(M)}$ in the C^0–sense,*

3. *$h_{1|\mathcal{O}p(S)}$ is holomorphic, and therefore $h_1(M)$ is a totally real submanifold of B.*

□

Remark 5.1.10. When the parametric space P in Theorem 5.1.1 is compact it seems plausible that one can make the diffeotopies $h_{p,t}$ independent of p. If that is true then it will also a parametric version of Lemma 5.1.9. This would also happen if, like for Theorem 2.3.16, there exists a parametric version of Proposition 5.1.8. In this situation, one will moreover be able to make condition 3 to be true for every $t \in [0,1]$, i.e. to find a diffeotopy h_t be made of holomorphic maps over $\mathcal{O}p\,(S)$. ⋈

We can improve Lemma 5.1.9 to work for jet bundles of local holomorphic sections.

Lemma 5.1.11. *Let $M \subset B$ be a totally real submanifold of class C^r of a complex manifold B and let K be a sublevel set of a plurisubharmonic function properly attached to M. Let $S := K \cup M$ and let $\sigma : \mathcal{O}p\,(S) \to E^{(r)}$ be a section that is holonomic in $\mathcal{O}p\,(K)$. Then, for every arbitrarily small $\delta, \varepsilon > 0$ there exist a δ–small (in the C^0–sense) diffeotopy $h_t : B \to B$, $t \in [0,1]$ and a holonomic section $\widetilde{\sigma} : \mathcal{O}p\,(h_1(S)) \to E^{(r)}$ such that*

1. *$\widetilde{\sigma}$ is defined over $\mathcal{O}p\,(K)$, where it is C^r–close to σ and h_t is close to the identity,*

2. *$\widetilde{\sigma}$ is ε–close to $\sigma_{|\mathcal{O}p(h)_t(M)}$ in the C^0–sense,*

3. *$h_{1|\mathcal{O}p(S)}$ is holomorphic, and therefore $h_1(M)$ is a totally real submanifold of B.*

Proof. We must see $E^{(r)}_{\mathcal{O}}$ as a Partial Differential Relation in $E^{(r)}$. Now use Lemma 5.1.9 to find h_t and a holonomic approximation of σ in $\Gamma(\mathcal{O}p\,(S)\,, E^{(r)})$, $\hat{\sigma}$. Since $\hat{\sigma}$ is C^0–close to σ we can assume that their images lie in a tubular neighbourhood N of $E^{(r)}_{\mathcal{O}}$ in $E^{(r)}$. Let π be the projection $\pi : N \to E^{(r)}_{\mathcal{O}}$ and use Theorem 3.1.1 to approximate $\pi \circ \hat{\sigma}$ by a holomorphic section $\sigma' : \mathcal{O}p\,(h_1(S)) \to E^{(r)}_{\mathcal{O}}$.

Let $\widetilde{\sigma} := j^r_{\mathcal{O}}(\sigma'^{(0)})$. Since $\sigma'^{(0)} \sim_{C^r} (\pi \circ \hat{\sigma})^{(0)} \sim_{C^r} \hat{\sigma}^{(0)}$ and $j^r(\hat{\sigma}^{(0)}) = \hat{\sigma} \sim_{C^0} \sigma$ in $\mathcal{O}p\,(h_1(S))$, then $\widetilde{\sigma} \sim_{C^0} \sigma$. □

5.1.3 Locally Aut-invariant HPDRs

One may now try to mimic the proof of Theorem 5.1.6 to obtain local h–principles for open Aut B–invariant HPDRs. Nevertheless, the maps h_1 found by Lemma 5.1.11 are not necessarily biholomorphisms outside an open neighbourhood of S. We will then need the HPDR to satisfy a stronger condition.

Definition 5.1.12. *Let $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ be two holomorphic fiber bundles and let $\mathrm{Bihol}_{\mathrm{FP}}(E, E')$ be the space of fiber preserving biholomorphisms from E to E'. That is the space of biholomorphisms $h_{E,E'} : E \to E'$ such that there exists a biholomorphism $h_{B,B'}$ making commutative the following diagram*

$$\begin{array}{ccc} E & \xrightarrow{h_{E,E'}} & E' \\ \pi\downarrow & & \downarrow\pi' \\ B & \xrightarrow{h_{B,B'}} & B'. \end{array}$$

Let $\pi_* : \mathrm{Bihol}_{\mathrm{FP}}(E, E') \to \mathrm{Bihol}(B, B')$ be the projection sending $h_{E,E'} \mapsto h_{B,B'}$. We say that E and E' form a holomorphically natural pair *if there exist a homomorphism $j_{\mathcal{O}} : \mathrm{Bihol}(B, B') \to \mathrm{Bihol}_{FP}(E, E')$ such that $\pi_* \circ j = \mathrm{Id}$.*

We will say that a holomorphic fiber bundle $E \to B$ is locally holomorphically natural *if for each $p_1, p_2 \in B$ there exist $U_i = \mathcal{O}p_B(p_i), i = 1, 2$ such that the bundles $E_{|U_1}$ and $E_{|U_2}$ form a holomorphically natural pair.*

Note that the holomorphic jet bundles of a holomorphically natural pair form another holomorphically natural pair. Indeed, the natural homomorphisms are of the form

$$\begin{aligned} j^r_{\mathcal{O}} \colon \mathrm{Bihol}(B, B') &\longrightarrow \mathrm{Bihol}_{\mathrm{FP}}(E^{(r)}_{\mathcal{O}}, E'^{(r)}_{\mathcal{O}}) \\ h &\longmapsto h_* := j^r_{\mathcal{O}}(h), \end{aligned}$$

where $j^r_{\mathcal{O}}(h)$ sends each $s \in E^{(r)}_{\mathcal{O}}$ to the r–jet of the section $j_{\mathcal{O}}(h) \circ \widetilde{s} \circ h^{-1}$ at $h(p)$, where $p \in B$ is the base point of s and $\widetilde{s}$ is a local section representing s.

Therefore, the holomorphic jet bundle of a locally holomorphically natural bundle is also holomorphically natural. The holomorphically natural fiber bundles in Example 5.1.3 are also locally holomorphically natural.

Definition 5.1.13. *Given a locally holomorphically natural holomorphic fiber bundle $E \to B$ we say that a Holomorphic Partial Differential Relation $\mathcal{R} \subset E^{(r)}_{\mathcal{O}}$ is* locally $\mathrm{Aut}\, B$–invariant *if for every pair of points $p_1, p_2 \in B$ it holds that*

$$h_*(\mathcal{R}_{|\mathcal{O}p(p_1)}) = \mathcal{R}_{|\mathcal{O}p(p_2)},$$

for every $h \in \mathrm{Bihol}(\mathcal{O}p\,(p_1)\,, \mathcal{O}p\,(p_2))$.

The HPDRs that can be defined independently of holomorphic coordinates are not only $\mathrm{Aut}\, B$–invariant but also locally $\mathrm{Aut}\, B$–invariant. Therefore we have the following

Example 5.1.14. All the HPDRs defined in Section 2.1.3 are locally $\mathrm{Aut}\, B$–invariant. ⋈

5.1.4 Local *h*-principles for locally Aut-invariant HPDRs

Theorem 5.1.15. *Let $E \to B$ be a locally holomorphically natural fiber bundle and let $\mathcal{R}$ be an open and locally $\mathrm{Aut}\, B$ invariant HPDR. Then $\mathcal{R}$ satisfy a local h–principle over any totally real submanifold $M \subset B$ that is C^0–dense and weakly relative to sublevel sets of plurisubharmonic functions properly attached to M.*

Under the same hypothesis, $\mathcal{R}$ also satisfy a local holomorphic h–principle over totally real manifolds that is C^0–dense and close to sublevel sets of plurisubharmonic functions properly attached to M.

Proof. Let $M \subset B$ be a totally real submanifold properly attached to a sublevel set of a plurisubharmonic function K. Let $\sigma \in \mathrm{F}\mathcal{R}$ be a formal solution. Use Lemma 5.1.11 on M, K and σ to obtain a δ–small diffeotopy $h_t : B \to B$ such that $h_{1|\mathcal{O}p(K\cup M)}$ is holomorphic and a holonomic local solution $\widetilde{\sigma} \in \mathrm{H}\mathcal{R}(h_1(K \cup M))$ that is ε'–close to $\sigma_{|\mathcal{O}p(K\cup M)}$. Then the section $h_1^*(\widetilde{\sigma})$ is in $\mathrm{H}\mathcal{R}(K \cup M)$. Taking δ and ε' small enough the linear interpolation between σ and $h_1^*\widetilde{\sigma}$ will lie in $\mathrm{F}\mathcal{R}(K \cup M, K, \varepsilon, \sigma)$.

The local holomorphic h–principle follows since the linear interpolation between holomorphic sections is formed by holomorphic sections. □

We can now repeat the same arguments that the ones given in the proof of Theorem proof of Theorem 4.2.6 given in Section 4.3.4 using Theorem 5.1.15 instead of Lemma 4.3.1 to obtain the following

Theorem 5.1.16. *Let B be a Stein manifold with integrable almost complex structure J and let $E \to B$ be a locally holomorphically natural vector bundle with a Holomorphic Partial Differential Relation $\mathcal{R}$ that is open and locally* Aut B*–invariant in $E^{(r)}_{\mathcal{O}}$. Then for every formal solution $\sigma_0 : B \to \mathcal{R}$, there exists a smooth family of diffeomorphisms $h_t : B \to h_t(B) \subset B$, $t \in [0,1]$ and a homotopy of formal solutions $\sigma_t : B \to \mathcal{R}$ such that*

1. $h_0 = \mathrm{Id}_B$.
2. $(h_t(B), J_{|_{h_t(B)}})$ *is Stein (or equivalently, $(B, h_t^* J)$ is Stein) for every $t \in [0,1]$.*
3. σ_1 *is holonomic in $\mathcal{O}p\,(h_1(B))$.*

Moreover, the homotopy $\sigma_t^{(0)}$ may be choosen to be close to $\sigma^{(0)}$.

Remark 5.1.17. Similarly to Remark 5.1.10. If the adapted holonomic approximation Theorem works parametrically and relative to parameter, then the previous local h–principles given in Theorem 5.1.15 would be full, parametric and relative to parameter.

In the same way, Theorem 5.1.16 would hold for continuous families of formal solutions relatively to parameter. ⋈

Apart from trying to find a parametric version of Proposition 5.1.8, it will be interesting for this work in progress to find examples of open and locally Aut–invariant HPDRs that are not under the hypothesis of the Theorems stated in Section 4.2 or at least that are not Thick.

Holonomic approximation does not need actually the PDRs to be open to work. There are more general hypothesis under which all the steps can be done. They are called *parametric local integrability* (i.e. each parametric family of solutions over a point admits a family of holonomic representatives solving the PDR) and *microflexibility* (i.e. roughly speaking, that one can interpolate between the formal solution and the integral local representatives in some extra codimension) see Chapter 13 in [EM02]. It may be interesting to generalize this properties for the holomorphic category and try to find geometrically interesting examples of HPDRs satisfying this conditions that are not open.

5.1.5 Global h–principles for locally Aut-invariant HPDRs

We have seen that holonomic approximation can be used in the smooth category not only to find local h–principles but also global in open manifolds.

To do that one need to find diffeomorphisms sending the smooth manifold to a neighbourhood of its Morse skeleton. This is hard to find in the holomorphic category. Indeed the Uniformization Theorem shows that it is impossible even for $\mathbb{C}$, since its skeleton is just a point but $\mathbb{C}$ is not biholomorphic to any of its subdomains, so one can not expect to find these kind of biholomorphic maps even for the simplest Stein manifold $\mathbb{C}$.

Nevertheless, for convex bounded domains in Euclidean complex vector spaces $\Omega \subset \mathbb{C}^n$ inside a disk of radius R centered in a point $p_0 \in \Omega$, $\mathbb{D}^n(p_0, R)$ the homotecy centered at p_0

$$\begin{aligned} \lambda_{p_0, \frac{\delta}{R}} : \mathbb{D}^n(p_0, R) \longrightarrow & \mathbb{D}^n(p_0, \delta) \\ z \longmapsto & p_0 + \frac{\delta}{R}(z - p_0) \end{aligned}$$

is biholomorphic for every $\delta > 0$.

Therefore we have the following

Proposition 5.1.18. *Let Ω be a convex bounded domain of an Euclidean complex space $\mathbb{C}^n$ and let $\mathcal{R} \subset J^r(\Omega, \mathbb{C}^q)$ be an open and locally $\operatorname{Aut}\Omega$–invariant HPDR that satisfies a local h–principle over a point $p_0 \in \Omega$. Then $\mathcal{R}$ satisfy a (global) h–principle.*

Moreover, if the local h–principle is full and/or parametric and relative to parameter so it is the global one.

Proof. Let $\sigma_p \in \mathrm{F}\mathcal{R}$ be a family of formal solutions continuously dependent on p in a compact Hausdorff space P such that $\sigma_q\, \mathrm{H}\mathcal{R}$ for every q in a closed subset Q

- (P, Q) be a pair of an arbitrary compact Hausdorff space P and a compact subset Q, if the local h–principles over p_0 are parametric and relative to parameter.
- $(P, Q) = (P, \emptyset)$ where P is an arbitrary compact Hausdorff space, if the local h–principles over p_0 are just parametric.
- $(P, Q) = (\mathbb{D}^k, \partial\mathbb{D}^k)$, if the local h–principles over p_0 are full.
- $(P, Q) = (P, \emptyset)$ where P is just a point, if the local h–principles over p_0 do not satisfy any of the previous conditions.

Let $R > 0$ such that $\Omega \subset \mathbb{D}(p_0, R)$. Since $\mathcal{R}$ satisfies a local h–principle over p_0, there exists a homotopy of formal local solutions $\sigma_{p,t}, t \in [0, 1]$ such that $\sigma_{p,1} \in \mathrm{H}\mathcal{R}(p_O)$ for every $p \in P$ and $\sigma_{q,t} = \sigma_q$ for every $q \in Q$. Let $\delta > 0$ be a number such that the formal local solutions $\sigma_{p,t}$ are defined over $\mathbb{D}(p_0, \delta)$ and $\sigma_{p,1} \in \mathrm{H}\mathcal{R}_{|\mathbb{D}(p_0,\delta)}$.

The homotopy of sections $\widetilde{\sigma}_{p,t}$ that results of combining $\lambda^*_{p_0,1+t(\frac{\delta}{R}-1)}\sigma_p$ and $\lambda^*_{p_0,\frac{\delta}{R}}\sigma_{p,t}$ is a homotopy of formal solutions of $\mathcal{R}$ satisfying that $\widetilde{\sigma}_{p,0} = \sigma_p$ for every $p \in P$, $\widetilde{\sigma}_{q,t} = \sigma_q$ for every $q \in Q$ and every $t \in [0, 1]$ and $\widetilde{\sigma}_{p,1} \in \mathrm{H}\mathcal{R}$ for every $p \in P$. □

Corollary 5.1.19. *Every open locally* $\mathrm{Aut}\,\Omega$*–invariant HPDR over a bounded convex subset of an Euclidean complex space satisfies a global h–principle*

Remark 5.1.20. Like before, if the parametric and relative to parameter version of Proposition 5.1.8 works, the previous corollary would say that the global h–principle is full, parametric and relative to parameter. ⋈

It will be interesting for improving this work trying to prove a parametric and relative to parameter version of Proposition 5.1.8. Moreover it will be interesting to find as many instances as possible of Stein manifolds to whom the previous arguments could be generalized.

This seems to be possible if the Stein manifold carries a holomorphic vector field whose complete flow compress the manifold to the Lagrangian skeleton, so we are trying to find examples of them.

One family of candidates would be smooth manifolds whose cotangent bundles are complex manifolds such that the complement of their zero section are biholomorphic to the symplectization of a Boothby Wang manifold. Other family of potential examples are conincal symplectic resolutions.

Symbols and Notations

$\Gamma(E)$	Space of smooth sections of E.	Page 17.
$\Gamma^r(E)$	Space of smooth sections of E.	Page 17.
$\Gamma_{\mathcal{O}}(E)$	Space of smooth sections of E.	Page 17.
$\mathcal{O}p_B\,(A)$	Replacement of the sentence *"an open neighbourhood of A in B"*.	Page 17.
$\mathrm{Vert}(E)$	Vertical bundle of E.	Page 22.
$\bigwedge^k E$	k-th exterior product bundle of E.	Page 23.
$\mathrm{Sym}^k(E)$	k–th symmetric product of E.	Page 23.
TM	Tangent bundle of M.	Page 23.
T^*M	Cotangent bundle of M.	Page 24.
$\Omega^k(M)$	Smooth k–forms on M.	Page 24.
$\Omega^k(M,E)$	Smooth k–forms of M with values in E.	Page 24.
$\mathcal{T}M$	Holomorphic tangent bundle of M.	Page 24.
$\Omega^k_{\mathcal{O}}(M)$	Holomorphic k–forms on M.	Page 24.
$\Omega^k_{\mathcal{O}}(M,E)$	Holomorphic k–forms on M with values in E.	Page 24.
$E_{\mathbb{C}}$	Complexification of the vector bundle E.	Page 25.

$\Omega^{p,q}_J(M)$	(p,q)–forms on the almost complex manifold (M,J).	Page 28.
π^r_s	Affine bundle projection from the r–th to the s–th jet bundles, $r \geq s$.	Page 31.
π^r	Projection $\pi \circ \pi^r_0$ of a fiber bundle π.	Page 32.
$j^r(\sigma)$	r–jet extension of the section σ.	Page 32.
$\sigma^{(0)}$	Base section of σ, $\pi^r_0(\sigma)$.	Page 32.
$J^r(M_1, M_2)$	r–jet bundle of maps from M_1 to M_2.	Page 33.
$E^{(r)}_{\mathcal{O}}$	r–th jet of holomorphic sections of the bundle E.	Page 34.
$j^r_{\mathcal{O}}(\sigma)$	Holomorphic r–jet extension of σ.	Page 34.
$J^r_{\mathcal{O}}(M_1, M_2)$	r–jet bundle of holomorphic maps from M_1 to M_2.	Page 34.
d_∇	Exterior differential operator of forms with values in a vector bundle with a connection ∇.	Page 35.
PDR	Acronym for Partial Differential Relation.	Page 35.
F$\mathcal{R}$	Acronym for the space of formal solutions of $\mathcal{R}$.	Page 35.
H$\mathcal{R}$	Acronym for the space of holonomic solutions of $\mathcal{R}$.	Page 35.
HPDR	Acronim for Holomorphic Partial Differential Relation.	Page 35.
$\mathcal{O}$F$\mathcal{R}$	Acronym for the space of holomorphic formal solutions of $\mathcal{R}$.	Page 35.
$\mathcal{R}_{\text{Max-rank}}$	PDR of smooth maps of maximal rank.	Page 37.
$\mathcal{R}_{\mathcal{O}\text{Max-rank}}$	HPDR of holomorphic maps of maximal rank.	Page 37.
$\mathcal{R}_{\text{Imm}}$	PDR of smooth immersions.	Page 37.
$\mathcal{R}_{\mathcal{O}\,\text{Imm}}$	HPDR of holomorphic immersions.	Page 37.
$\mathcal{R}_{\text{Sub}}$	PDR of smooth submersions.	Page 38.

$\mathcal{R}_{\mathcal{O}\,\mathrm{Sub}}$	HPDR of holomorphic submersions.	Page 38.
D	Symbol of the exterior derivative operator of forms.	Page 38.
F_σ	Formal primitive of the form σ.	Page 38.
D_∇	Symbol of the operator d_∇.	Page 39.
$\Omega^2_{\mathrm{LCSymp}}(M, L, \nabla)$	L–valued locally conformal symplectic forms on M.	Page 40.
$\Omega^2_{\mathcal{O}\,\mathrm{LCSymp}}(M, L, \nabla)$	L–valued holomorphic locally conformal symplectic forms on M.	Page 40.
$\Omega^1_{\mathrm{Liouv}}(M, L, \nabla)$	L–valued Liouville forms on M.	Page 40.
$\Omega^1_{\mathcal{O}\,\mathrm{Liouv}}(M, L, \nabla)$	L–valued holomorphic Liouville forms on M.	Page 40.
$\Omega^2_{\mathrm{LCASymp}}(M, L, \nabla)$	L–valued locally conformal almost-symplectic forms on M.	Page 40.
$\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(M, L, \nabla)$	L–valued holomorphic locally conformal almost-symplectic forms on M.	Page 40.
$\mathcal{R}^\eta_{\mathrm{LCSymp}}(M, L, \nabla)$	PDR for L–valued symplectic forms in $[\eta]$.	Page 43.
$\mathcal{R}^\eta_{\mathcal{O}\,\mathrm{LCSymp}}(M, L, \nabla)$	HPDR for L–valued holomorphic symplectic forms in $[\eta]$.	Page 43.
$\Omega^2_{\mathrm{Symp}}(M)$	Symplectic forms on M.	Page 44.
$\Omega^2_{\mathcal{O}\,\mathrm{Symp}}(M)$	Complex symplectic forms on M.	Page 44.
$\Omega^1_{\mathrm{Liouv}}(M)$	Liouville forms on M.	Page 44.
$\Omega^1_{\mathcal{O}\,\mathrm{Liouv}}(M)$.	Holomorphic Liouville forms on M.	Page 44.
$\Omega^2_{\mathrm{ASymp}}(M)$	Almost-symplectic forms on M.	Page 44.
$\Omega^2_{\mathrm{ASymp}}(M)$	Holomorphic almost-symplectic forms on M.	Page 44.
$\mathcal{R}^\eta_{\mathrm{Symp}}(M)$	PDR for symplectic forms in $[\eta]$.	Page 44.
$\mathcal{R}^\eta_{\mathcal{O}\,\mathrm{LCSymp}}(M)$.	HPDR for complex symplectic forms in $[\eta]$.	Page 44.

$\mathcal{R}_{\mathrm{Cont}}(M, L)$	PDR for L–valued contact forms.	Page 45.
$\mathcal{R}_{\mathcal{O}\,\mathrm{Cont}}(M, L)$	HPDR for L–valued holomorphic contact forms.	Page 45.
$\mathcal{R}_{\mathrm{ECont}}(M, L)$	PDR for L–valued even-contact forms.	Page 46.
$\mathcal{R}_{\mathcal{O}\,\mathrm{ECont}}(M, L)$	HPDR for L–valued holomorphic even-contact forms.	Page 46.
$\mathcal{R}_{\mathrm{Engel}}(M, L)$	PDR for L–valued Engel pairs.	Page 47.
$\mathcal{R}_{\mathcal{O}\,\mathrm{Engel}}(M, L)$	HPDR for L–valued holomorphic Engel pairs.	Page 47.
$\mathrm{F}\mathcal{R}(S)$	Set of formal local solutions of $\mathcal{R}$ over the subset S.	Page 49.
$\mathrm{H}\mathcal{R}(S)$	Set of holonomic local solutions of $\mathcal{R}$ over the subset S.	Page 49.
$\mathrm{F}\mathcal{R}(S, C)$	Formal local solutions of $\mathcal{R}$ over S holonomic in $\mathcal{O}p\,(C)$.	Page 49.
$\mathrm{F}\mathcal{R}(S, C, \sigma)$	Formal local solutions of $\mathcal{R}$ over S coinciding with $\sigma \in \mathrm{H}\mathcal{R}(C)$ in $\mathcal{O}p\,(C)$.	Page 49.
$\mathrm{H}\mathcal{R}(S, C, \sigma)$	Holonomic solutions of $\mathcal{R}$ over S coinciding with σ in $\mathcal{O}p\,(C)$.	Page 49.
$\mathrm{F}\mathcal{R}(S, C, \sigma, \varepsilon)$	Formal local solutions of $\mathcal{R}$ over S ε–close to $\sigma \in \mathrm{H}\mathcal{R}(C)$ in $\mathcal{O}p\,(C)$.	Page 49.
$\mathrm{H}\mathcal{R}(S, C, \sigma, \varepsilon)$	Holonomic local solutions of $\mathcal{R}$ over S ε–close to $\sigma \in \mathrm{H}\mathcal{R}(C)$ in $\mathcal{O}p\,(C)$.	Page 49.
$\mathcal{R}_P$	Fibered PDR $P \times \mathcal{R}$.	Page 50.
$\mathcal{O}\,\mathrm{F}\mathcal{R}(S)$	Formal local solutions of $\mathcal{R}$ over S holomorphic in $\mathcal{O}p\,(S)$.	Page 51.
$p\,\mathrm{H}\mathcal{R}(C)$	Pseudo-holonomic local solutions of $\mathcal{R}$ over C.	Page 52.
$\mathcal{O}\,\mathrm{F}\mathcal{R}(S, C)$	Formal local solutions of $\mathcal{R}$ over S pseudo-holonomic in C.	Page 52.

$\mathcal{O}\,\mathrm{F}\mathcal{R}(S, C, \sigma, \varepsilon)$	Holomorphic formal local solutions of $\mathcal{R}$ over S that are pseudo-holonomic in C and ε–close to σ in $\mathcal{O}p\,(C)$.	Page 52.
TDR	Acronym for Thick Differential Relation.	Page 53.
$\widehat{K}_{\mathcal{O}(\Omega)}$	$\mathcal{O}(\Omega)$–convex hull of K.	Page 56.
ω_ϕ	Symplectic form associated to the plurisubharmonic function ϕ.	Page 58.
g_ϕ	Riemannian form associated to the plursisubharmonic exhausting function ϕ.	Page 58.
λ_ϕ	Liouville form of ω_ϕ.	Page 58.
M_ϕ	Lagrangian skeleton of the plurisubharmonic completely exhausting function ϕ.	Page 59.
Σ_c	Level set of a function.	Page 60.
K_c	Sublevel set of a function.	Page 61.
$\mathcal{R}_{\mathbb{R}}(M)$	Realification of $\mathcal{R}$ over M.	Page 78.
THR	Acronym for Thick Holomorphic Relation.	Page 79.
$\Omega^2_{a,\mathcal{O}\,\mathrm{LCSymp}}(S, L, \nabla)$	Germs over S of locally conformal symplectic forms in the cohomology class a.	Page 104.
$\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(S, L, \nabla)$	Germs over S of locally conformal almost-symplectic forms.	Page 104.
$\Omega^2_{a,\mathcal{O}\,\mathrm{Symp}}(S)$	Germs over S of symplectic forms in the cohomology class a.	Page 106.
$\Omega^2_{\mathcal{O}\,\mathrm{LCASymp}}(S)$	Germs over S of almost-symplectic forms.	Page 106.
$\mathrm{Diff}_B\, E$	Fiber preserving diffeomorphisms of E.	Page 113.
$\mathrm{Aut}_B\, E$	Fiber preserving automorphisms of E.	Page 113.
$\mathrm{Bihol}_{\mathrm{FP}}(E, E')$	Fiber preserving biholomorphisms from E to E'.	Page 118.

Bibliography

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782, 1983.

[Bis61] Errett Bishop. Mappings of partially analytic spaces. *Amer. J. Math.*, 83:209–242, 1961.

[BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy. Existence and classification of overtwisted contact structures in all dimensions. *Acta Math.*, 215(2):281–361, 2015.

[Car01] E. Cartan. Sur quelques quadratures dont l'élément différentiel contient des fonctions arbitraires. *Bull. Soc. Math. France*, 29:118–130, 1901.

[CdPP20] Roger Casals, Álvaro del Pino, and Francisco Presas. Loose Engel structures. *Compos. Math.*, 156(2):412–434, 2020.

[CPdPP17] Roger Casals, José Luis Pérez, Álvaro del Pino, and Francisco Presas. Existence h-principle for Engel structures. *Invent. Math.*, 210(2):417–451, 2017.

[CM19] Baptiste Chantraine and Emmy Murphy. Conformal symplectic geometry of cotangent bundles. *J. Symplectic Geom.*, 17(3):639–661, 2019.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[CP18] R. Coelho and N. Pia. Exotic holomorphic Engel structures on $\mathbb{C}^4$. *J. Geom. Anal.*, 28(3):2550–2557, 2018.

[Dom62] Peter Dombrowski. On the geometry of the tangent bundle. *J. Reine Angew. Math.*, 210:73–88, 1962.

[Don96] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

[Eli89] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[EM98] Y. Eliashberg and N. M. Mishachev. Wrinkling of smooth mappings. III. Foliations of codimension greater than one. *Topol. Methods Nonlinear Anal.*, 11(2):321–350, 1998.

[EM00] Y. M. Eliashberg and N. M. Mishachev. Wrinkling of smooth mappings. II. Wrinkling of embeddings and K. Igusa's theorem. *Topology*, 39(4):711–732, 2000.

[EM09] Y. M. Eliashberg and N. M. Mishachev. Wrinkled embeddings. In *Foliations, geometry, and topology*, volume 498 of *Contemp. Math.*, pages 207–232. Amer. Math. Soc., Providence, RI, 2009.

[FFW20] John Erik Fornæss, Franc Forstnerič, and Erlend F. Wold. Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. In *Advancements in complex analysis—from theory to practice*, pages 133–192. Springer, Cham, [2020] ©2020.

[For21] F. Forstnerič. Immersions of open Riemann surfaces into the Riemann sphere. *Izv. Ross. Akad. Nauk Ser. Mat.*, 85(3):239–260, 2021.

[For17] Franc Forstnerič. *Stein manifolds and holomorphic mappings*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in*

Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, second edition, 2017. The homotopy principle in complex analysis.

[For20] Franc Forstnerič. h-principle for complex contact structures on Stein manifolds. *J. Symplectic Geom.*, 18(3):733–767, 2020.

[FS07] Franc Forstnerič and Marko Slapar. Stein structures and holomorphic mappings. *Math. Z.*, 256(3):615–646, 2007.

[Gei08] Hansjörg Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[GSA23] Luis Giraldo and Guillermo Sánchez-Arellano. Local h-principles for partial holomorphic relations, 2023.

[Gra57] Hans Grauert. Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. *Math. Ann.*, 133:450–472, 1957.

[Gra58] Hans Grauert. On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)*, 68:460–472, 1958.

[Gro89] M. Gromov. Oka's principle for holomorphic sections of elliptic bundles. *J. Amer. Math. Soc.*, 2(4):851–897, 1989.

[Gro69] M. L. Gromov. Stable mappings of foliations into manifolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:707–734, 1969.

[Gro73] M. L. Gromov. Convex integration of differential relations. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:329–343, 1973.

[GE71] M. L. Gromov and Ja. M. Èliašberg. Elimination of singularities of smooth mappings. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:600–626, 1971.

[Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[GR09] Robert C. Gunning and Hugo Rossi. *Analytic functions of several complex variables.* AMS Chelsea Publishing, Providence, RI, 2009. Reprint of the 1965 original.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Hir59] Morris W. Hirsch. Immersions of manifolds. *Trans. Amer. Math. Soc.*, 93:242–276, 1959.

[Ist16] N. Istrati. Twisted holomorphic symplectic forms. *Bull. Lond. Math. Soc.*, 48(5):745–756, 2016.

[KPSW00] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142(1):1–15, 2000.

[Loh95] Joachim Lohkamp. Curvature h-principles. *Ann. of Math. (2)*, 142(3):457–498, 1995.

[McD87] Dusa McDuff. Applications of convex integration to symplectic and contact geometry. *Ann. Inst. Fourier (Grenoble)*, 37(1):107–133, 1987.

[Mic56] Ernest Michael. Continuous selections. I. *Ann. of Math. (2)*, 63:361–382, 1956.

[Nar60] Raghavan Narasimhan. Imbedding of holomorphically complete complex spaces. *Amer. J. Math.*, 82:917–934, 1960.

[Nas54] John Nash. C^1 isometric imbeddings. *Ann. of Math. (2)*, 60:383–396, 1954.

[NN57] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math. (2)*, 65:391–404, 1957.

[O'G99] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.

[Phi67] Anthony Phillips. Submersions of open manifolds. *Topology*, 6:171–206, 1967.

[Phi68] Anthony Phillips. Foliations on open manifolds. I. *Comment. Math. Helv.*, 43:204–211, 1968.

[Phi69] Anthony Phillips. Foliations on open manifolds. II. *Comment. Math. Helv.*, 44:367–370, 1969.

[Phi70] Anthony Phillips. Smooth maps transverse to a foliation. *Bull. Amer. Math. Soc.*, 76:792–797, 1970.

[Phi74] Anthony Phillips. Smooth maps of constant rank. *Bull. Amer. Math. Soc.*, 80:513–517, 1974.

[PV20] Álvaro del Pino and Thomas Vogel. The Engel-Lutz twist and overtwisted Engel structures. *Geom. Topol.*, 24(5):2471–2546, 2020.

[PSC14] Francisco Presas and Luis E. Solá Conde. Holomorphic Engel structures. *Rev. Mat. Complut.*, 27(1):327–344, 2014.

[Sma58] Stephen Smale. A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.*, 90:281–290, 1958.

[Sma59] Stephen Smale. The classification of immersions of spheres in Euclidean spaces. *Ann. of Math. (2)*, 69:327–344, 1959.

[Whi36] Hassler Whitney. Differentiable manifolds. *Ann. of Math. (2)*, 37(3):645–680, 1936.

[Whi44] Hassler Whitney. The self-intersections of a smooth n-manifold in $2n$-space. *Ann. of Math. (2)*, 45:220–246, 1944.