A note on the existence of global solutions for reaction-diffusion equations with almost-monotonic nonlinearities

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1 Introduction

Let us consider the following problem

$$\left\{\begin{array}{rcll} u_t - \Delta u & = & f(x,u) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \Gamma \\ u(0) & = & u_0 & \text{in} \quad \Omega \end{array}\right. \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with f of the form

$$f(x,u) = g(x) + m(x)u + f_0(x,u), \quad x \in \Omega,\ u \in \mathbb{R}, \tag{1.2}$$

where

$$g \in L^{q_0}(\Omega),\ \text{with}\ N/2 < q_0 < \infty, \quad m \in L^{r_0}(\Omega)\ \text{with}\ N/2 < r_0 \leq \infty \tag{1.3}$$

and f_0 is a Carathéodory function, locally Lipschitz in the second variable in a $L^{q_0}(\Omega)$ form with respect to $x \in \Omega$, i.e, such that for $|u| \leq R$ and $|v| \leq R$ we have

$$|f_0(x,u) - f_0(x,v)| \leq L_0(x,R)|u-v|$$

with $0 \leq L_0(\cdot,R) \in L^{q_0}(\Omega)$ for each $R > 0$, and with $f_0(\cdot,0) = \partial_u f_0(\cdot,0) = 0$.

Our goal here is to prove global existence and uniqueness of solutions for initial data $u_0 \in L^q(\Omega)$ for any $1 \leq q < \infty$ under the only additional assumption that $f_0(x,u)$ is *almost monotonic*, i.e, to satisfy the following condition

$$\partial_u f_0(x,u) \leq L(x), \quad \text{for all} \quad x \in \Omega,\ u \in \mathbb{R} \tag{1.4}$$

for some $L \in L^{\sigma_0}(\Omega)$, $\sigma_0 > N/2$. Note that this implies

$$uf(x,u) \leq C(x)u^2 + D(x)|u|, \quad x \in \Omega,\ u \in \mathbb{R}, \tag{1.5}$$

for $C = m + L \in L^{\sigma}(\Omega)$, $\sigma = \min\{r_0, \sigma_0\} > N/2$ and $0 \leq D = |g| \in L^{q_0}(\Omega)$.

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2 Preliminary results

Notice that, the Nemytskii operator associated to $F(x,u) = g(x) + f_0(x,u)$ maps bounded sets of $L^\infty(\Omega)$ into bounded sets of $L^{q_0}(\Omega)$ and so we have the existence of local solutions of problem (1.1) for smooth initial data. Namely, we have (see [6])

Theorem 2.1 *Assume that f is as in (1.2)–(1.3). If $\alpha < 1$ is such that $2\alpha - \frac{N}{q_0} > 0$, then for any initial data $u_0 \in H^{2\alpha,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)$ there exists a unique local solution of (1.1) with initial data u_0, with $u(\cdot;u_0) \in C([0,\tau); H^{2\alpha,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)) \cap C((0,\tau); H^{2,q_0}(\Omega))$ and $u_t \in C((0,\tau); H^{2\gamma,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega))$ for any $\gamma < 1$, for some τ depending on u_0.*

The solution satisfies the Variation of Constant Formula,

$$u(t;u_0) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(u(s;u_0)))\,\mathrm{d}s$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Note that since $q_0 > N/2$, $H^{2\alpha,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega) \subset C_0^\eta(\overline{\Omega})$ for any $0 \le \eta < 2 - N/q_0$. In particular the solution satisfies $u \in C([0,\infty) \times \overline{\Omega})$. Also, if g and $L_0(\cdot,R)$, for each R, are a bounded functions, then the above holds for any $q_0 > N/2$.

It is known that condition (1.5) ensures the global existence of the local solutions in Theorem 2.1 (see [2] and [10]). Namely, we have

Theorem 2.2 *Assume that f is as in (1.2)–(1.3) and satisfies (1.5), that is*

$$uf(x,u) \le C(x)u^2 + D(x)|u|, \quad x \in \Omega,\ u \in \mathbb{R},$$

for some $C \in L^\sigma(\Omega)$ and $0 \le D \in L^\rho(\Omega)$ with $\sigma, \rho > N/2$.

Then for the solutions in Theorem 2.1 one has $\tau = \infty$.

3 Existence in $L^q(\Omega)$, $1 \le q < \infty$

The goal of the section is to prove main result in this work. Namely, the existence of a unique solution of problem (1.1) starting at $u_0 \in L^q(\Omega)$ for any $1 \le q < \infty$.

Theorem 3.1 *Let $1 \le q < \infty$. Suppose that f is as in (1.2)–(1.3) and f_0 satisfies (1.4) with $L \in L^{\sigma_0}(\Omega)$, $\sigma_0 > N/2$. Then, for any $u_0 \in L^q(\Omega)$, there exists a solution of (1.1) defined for all $t \ge 0$, u, such that*

$$u \in C([0,\infty); L^q(\Omega)) \cap C((0,\infty); H^{2,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)), \quad u(0) = u_0$$

and satisfies

$$u(t) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(\cdot,u(s)))\,\mathrm{d}s \tag{3.1}$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Moreover, for each $T > 0$ there exists $c(T)$ such that

$$|u(t,x)| \le c(T)\left(1 + t^{-\frac{N}{2q}}\|u_0\|_{L^q(\Omega)}\right), \qquad 0 < t \le T \quad \textit{for all } x \in \overline{\Omega}. \tag{3.2}$$

Proof. We proceed in several steps. In the first step, fixed $1 \le q < \infty$, we construct a Cauchy sequence of approximating solutions. Then, we obtain a uniform $L^\infty(\Omega)$ bound for the approximating sequence. In a third step, we show that the limit of the approximating solutions is a solution of the limit problem (notice that such limit exists since the approximating solutions forms a Cauchy sequence). Finally, we show how to obtain more regularity of the solution constructed in the previous steps.

Without loss of generality we can assume that L in (1.4) is non-negative.

Step 1. Approximate the initial data. Let $\alpha < 1$ such that $2\alpha - \frac{N}{q_0} > 0$. Then, by Theorem 2.1, the problem (1.1) is well-posed in $H^{2\alpha,q_0}(\Omega) \cap H^{1,q_0}_0(\Omega)$. Also, since f satisfies (1.5), the solutions are globally defined for $t > 0$, see Theorem 2.2 and (1.5).

Hence, for any $1 \le q < \infty$ and $u_0 \in L^q(\Omega)$, we can take smooth enough initial data $u_0^n \in H^{2\alpha,q_0}(\Omega) \cap H^{1,q_0}_0(\Omega)$ such that $u_0^n \to u_0$ in $L^q(\Omega)$ as $n \to \infty$ and consider the solutions of (1.1) starting at u_0^n. We define $u_n(t) = u(t; u_0^n)$.

Let $v_{n,k}(t) = u_n(t) - u_k(t)$. Subtracting equations for u_n and u_k, we have

$$\left\{\begin{array}{rcll} \partial_t v_{n,k}(t) - \Delta v_{n,k}(t) &=& m(x)v_{n,k}(t) + f_0(x, u_n(t)) - f_0(x, u_k(t)) & \text{in} \quad \Omega \\ v_{n,k} &=& 0 & \text{on} \quad \Gamma \\ v_{n,k}(0) &=& u_n^0 - u_k^0 & \text{in} \quad \Omega. \end{array}\right.$$

Observe that for fixed n, k, we have for almost all $x \in \Omega, t \in [0, T]$ and some $0 < \theta(t, x) < 1$

$$\begin{array}{rcl} f_0(x, u_n(t, x)) - f_0(x, u_k(t, x)) &=& \partial_u f_0(x, \theta(t, x)u_n(t, x) + (1 - \theta(t, x))u_k(t, x))v_{n,k}(t, x) \\ &=& C_{n,k}(t, x)v_{n,k}(t, x) \end{array}$$

for some function $C_{n,k}(t, x)$. Notice that $C_{n,k} \in L^\infty((0, T); L^{q_0}(\Omega))$ since u_n and u_k are smooth, f_0 is locally Lipschitz in the second variable in an $L^{q_0}(\Omega)$ manner. Also, from (1.4) we have $C_{n,k}(x, t) \le L(x)$ for all $t \ge 0$ and $x \in \Omega$.

Now, consider the linear problem

$$\left\{\begin{array}{rcll} z_t - \Delta z &=& (m(x) + C_{n,k}(t, x))z & \text{in} \quad \Omega \\ z &=& 0 & \text{on} \quad \Gamma \\ z(0) &=& z_0 & \text{in} \quad \Omega \end{array}\right. \tag{3.3}$$

with z_0 smooth and denote by $z(t, 0; z_0)$ the solution whose existence follows from [8] or [4]). Such solutions satisfy by comparison $z(t, 0; -|z_0|) \le z(t, 0; z_0) \le z(t, 0; |z_0|)$, i.e,

$$|z(t, 0; z_0)| \le z(t, 0; |z_0|)$$

and the latter is a nonnegative solution of (3.3).

But for nonnegative initial data, $z_0 \ge 0$, since $C_{n,k}(x, t) \le L(x)$ for all $t \ge 0$ and $x \in \Omega$, we can compare $z(t, 0; z_0) \ge 0$ with the solutions of

$$\left\{\begin{array}{rcll} w_t - \Delta w &=& (m(x) + L(x))w & \text{in} \quad \Omega \\ w &=& 0 & \text{on} \quad \Gamma \\ w(0) &=& z_0 & \text{in} \quad \Omega \end{array}\right.$$

to obtain $0 \le z(t,0;z_0) \le w(t;z_0)$.

Hence, we obtain that for any smooth initial data z_0 in (3.3) we have

$$|z(t,0;z_0)| \le w(t;|z_0|) \quad \text{for } t \ge 0.$$

In particular,

$$\|z(t,0;z_0)\|_{L^q(\Omega)} \le \|w(t;|z_0|)\|_{L^q(\Omega)} \le c\mathrm{e}^{-\lambda t}\|z_0\|_{L^q(\Omega)}$$

where λ is the first eigenvalue of $-\Delta - (m(x) + L(x))I$ on Ω with Dirichlet boundary conditions.

Now, $v_{n,k}$ is a solution of (3.3) and so

$$\|v_{n,k}(t)\|_{L^q(\Omega)} \le c\mathrm{e}^{-\lambda t}\|v_{n,k}(0)\|_{L^q(\Omega)}$$

for all $t \ge 0$. In particular, given $T > 0$, we have that for any $0 \le t \le T$,

$$\|v_{n,k}(t)\|_{L^q(\Omega)} \le c(T)\|v_{n,k}(0)\|_{L^q(\Omega)} \to 0 \quad \text{as} \quad n,k \to \infty$$

and so, u_n is a Cauchy sequence in $C([0,T];L^q(\Omega))$.

Hence, there exists $u \in C([0,\infty);L^q(\Omega))$ such that for any $T > 0$,

$$\sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^q(\Omega)} \le c(T)\|u_n^0 - u_0\|_{L^q(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \tag{3.4}$$

i.e, for any $T > 0$,

$$u_n \to u \quad \text{in } C([0,T];L^q(\Omega)).$$

In particular, passing to a subsequence if needed, $u_n(t,x) \to u(t,x)$ as $n \to \infty$ a.e. for $(t,x) \in [0,T] \times \Omega$.

Also it is easy to see that u does not depend on the sequence of initial data, but only on $u_0 \in L^q(\Omega)$.

Step 2. L^∞-bound for the approximating sequence. Let us show now that the sequence $u_n(t)$ is uniformly bounded in $L^\infty(\Omega)$ with respect to n, for $0 < \varepsilon \le t \le T$.

For this, since f satisfies (1.5), we will use the auxiliary problem

$$\left\{\begin{array}{rcll} U_t - \Delta U &=& C(x)U + D(x) & \text{in} \quad \Omega \\ U &=& 0 & \text{on} \quad \Gamma \\ U(0) & & \text{given in } L^q(\Omega) & \end{array}\right. \tag{3.5}$$

with $C = m + L \in L^\sigma(\Omega)$, $\sigma = \min\{r_0, \sigma_0\} > N/2$ and $0 \le D = |g| \in L^{q_0}(\Omega)$, $q_0 > N/2$.

Denote by $U^n(t,x)$ the solution of (3.5) with initial data $|u_0^n|$ and by $U(t,x)$ the solution of (3.5) with initial data $|u_0|$.

Now, using the variation of constants formula in (3.5) we have

$$U^n(t) = \Phi(t) + U_h^n(t), \quad U(t) = \Phi(t) + U_h(t)$$

where $U_h^n(t), U_h(t)$ are the solutions of the homogeneous problem

$$\left\{\begin{array}{rcll} V_t - \Delta V &=& C(x)V & \text{in} \quad \Omega \\ V &=& 0 & \text{on} \quad \Gamma \\ V(0) & & \text{given in } L^q(\Omega) & \end{array}\right.$$

resulting from taking $D \equiv 0$ in (3.5) and initial data $|u_0^n|$ and $|u_0|$ respectively, and $\Phi(t)$ is the unique solution of problem (3.5) with $U(0) = 0$ (which does not depend on u_0^n or u_0), that is,

$$\left\{ \begin{array}{rcll} W_t - \Delta W & = & C(x)W + D(x) & \text{in} \quad \Omega \\ W & = & 0 & \text{on} \quad \Gamma \\ W(0) & = & 0 & \text{in} \quad \Omega. \end{array} \right.$$

In other words $U_h^n(t) = S_C(t)|u_0^n|$, $U_h(t) = S_C(t)|u_0|$ and $\Phi(t) = \int_0^t S_C(t-s)D\,\mathrm{d}s$ where S_C denotes the semigroup generated by $\Delta + C(x)I$ with Dirichlet boundary conditions. Hence standard estimates implies that, for any $T > 0$,

$$\|U^n(t)\|_{L^\infty(\Omega)} \le c(T)(1 + t^{-\frac{N}{2q}} \|u_0^n\|_{L^q(\Omega)}), \qquad 0 < t \le T,$$

$$\|U^n(t)\|_{L^q(\Omega)} \le c(T)(1 + \|u_0^n\|_{L^q(\Omega)}), \qquad 0 \le t \le T$$

and

$$\begin{aligned} \|U^n(t) - U(t)\|_{L^\infty(\Omega)} &= \|U_h^n(t) - U_h(t)\|_{L^\infty(\Omega)} = \|S_C(t)(|u_0^n| - |u_0|)\|_{L^\infty(\Omega)} \\ &\le c(T)t^{-\frac{N}{2q}} \||u_0^n| - |u_0|\|_{L^q(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for $0 < t \le T$, and for $0 \le t \le T$

$$\|U^n(t) - U(t)\|_{L^q(\Omega)} \le c(T)\||u_0^n| - |u_0|\|_{L^q(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, for any $0 < \varepsilon < T < \infty$,

$$U^n \to U \quad \text{in } L^\infty([\varepsilon, T] \times \Omega) \cap C([0, T]; L^q(\Omega)).$$

Observe now that, since f satisfies (1.5), $U^n(t, x)$ is a supersolution of problem (1.1) and $-U^n(t, x)$ is a subsolution. Thus,

$$|u_n(t,x)| \le U^n(t,x) \le c(T)(1 + t^{-\frac{N}{2q}} \|u_0^n\|_{L^q(\Omega)}), \qquad 0 < t \le T, \quad \text{a.e. in } \Omega, \tag{3.6}$$

and so

$$\|u_n(t)\|_{L^\infty(\Omega)} \le c(\varepsilon, T, \|u_0^n\|_{L^q(\Omega)}), \quad \varepsilon \le t \le T.$$

Now, since $u_0^n \to u_0$ in $L^q(\Omega)$ as $n \to \infty$ and the convergences $U^n(t, x) \to U(t, x)$ and $u_n(t, x) \to v(t, x)$ obtained above (see (3.4)) we get

$$|u(t,x)| \le U(t,x) \le c(T)(1 + t^{-\frac{N}{2q}} \|u_0\|_{L^q(\Omega)}), \qquad 0 < t \le T, \quad \text{for a.e. } x \in \Omega. \tag{3.7}$$

Now observe that the bounds above, the regularity of u_n in Theorem 2.1 and (3.4) imply that for any $0 < \varepsilon < T < \infty$ and $1 \le s < \infty$,

$$\sup_{t \in [\varepsilon, T]} \|u_n(t) - u(t)\|_{L^s(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \tag{3.8}$$

i.e, for any $T > 0$ and $1 \le s < \infty$,

$$u_n \to u \quad \text{in } C([\varepsilon, T]; L^s(\Omega)).$$

In particular $u \in C((0,\infty); L^s(\Omega))$ for any $1 \le s < \infty$.

Step 3. The limit is a solution of (1.1). First, assume $0 < \varepsilon < t < T$. Then for any $\phi \in H^{2,q_0'}(\Omega) \cap H_0^{1,q_0'}(\Omega)$, where q_0' is the conjugate of q_0, i.e, $\frac{1}{q_0} + \frac{1}{q_0'} = 1$ (as usual, for $q_0 = 1$ we take $q_0' = \infty$), we have from (1.1) and the regularity if u_n in Theorem 2.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_n\phi + \int_\Omega u_n(-\Delta\phi) = \int_\Omega f(\cdot,u_n)\phi = \int_\Omega g\phi + \int_\Omega m(x)u_n\phi + \int_\Omega f_0(x,u_n)\phi.$$

Now, using the uniform bounds in (3.6), (3.7) and the convergence in (3.8), and the fact that f_0 is locally Lipchitz in its second variable in an $L^{q_0}(\Omega)$ manner, we have that for $1 \le s \le q_0$,

$$f_0(\cdot,u_n) \to f_0(\cdot,u) \quad \text{in} \quad C([\varepsilon,T]; L^s(\Omega)).$$

Hence, letting $n \to \infty$,we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\phi + \int_\Omega u(-\Delta\phi) = \int_\Omega f(\cdot,u)\phi = \int_\Omega g\phi + \int_\Omega m(x)u\phi + \int_\Omega f_0(x,u)\phi.$$

Notice that from [3], this implies

$$u(t) = S_m(t-\varepsilon)u(\varepsilon) + \int_\varepsilon^t S_m(t-s)h(s)\,\mathrm{d}s \tag{3.9}$$

where $S_m(t)$ denotes the strongly continuous analytic semigroup generated by $\Delta + m(x)I$ with homogeneous Dirichlet boundary conditions, and $h(\cdot) = g + f_0(\cdot,u(\cdot)) \in L^\infty([\varepsilon,T]; L^{q_0}(\Omega))$.

The smoothing effect of the semigroup gives that

$$\int_\varepsilon^t S_m(t-s)h(s)\,\mathrm{d}s \in C([\varepsilon,T]; H^{2\gamma,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)), \text{ for any } \gamma < 1,$$

while the continuity of the linear semigroup $S_m(t)$ at 0 and $u(\varepsilon) \to u_0$ in $L^q(\Omega)$ as $\varepsilon \to 0$, give, taking $\varepsilon \to 0$ in (3.9),

$$\int_0^t S_m(t)h(s)\,\mathrm{d}s = \lim_{\varepsilon\to 0}\int_\varepsilon^t S_m(t-s)h(s)\,\mathrm{d}s = u(t) - S_m(t)u_0.$$

Thus,

$$u(t) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(s,u(s)))\,\mathrm{d}s.$$

Step 4. Further regularity. From the smoothing effect of the semigroup $S_m(t)$ and the regularity observed above, we have that for any $\varepsilon > 0$, $u(\varepsilon) \in H^{2\alpha,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)$ for some $\alpha < 1$ such that $2\alpha - \frac{N}{q_0} > 0$.

Therefore, for $t \ge \varepsilon$, $u(t)$ coincides with the unique solution in Theorems 2.1 and 2.2. In particular $u(t)$ is continuous in Ω and we can take $x \in \overline{\Omega}$ in (3.7). ■

Corollary 3.2 *For $1 \leq q < \infty$ and $T > 0$, we have that the solution u in Theorem 3.1 satisfies, for $q \leq p \leq \infty$,*

$$\|u(t)\|_{L^p(\Omega)} \leq c(T) \left(1 + t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q(\Omega)}\right) \quad 0 < t \leq T.$$

Proof. Following the argument in Step 2 in the proof of Theorem 3.1, we can bound the approximating sequence u_n using the bound provided by the linear problem (3.5) to get

$$\|u_n(t)\|_{L^q(\Omega)} \leq \|U^n(t)\|_{L^q(\Omega)} \leq c(T)(1 + \|u_0\|_{L^q(\Omega)}), \quad 0 \leq t \leq T,$$

for all $n \geq 0$. Therefore, since $u \in C([0,T]; L^q(\Omega))$ is the limit of u_n in $C([0,T]; L^q(\Omega))$ as $n \to \infty$, we have

$$\|u(t)\|_{L^q(\Omega)} \leq c(T)(1 + \|u_0\|_{L^q(\Omega)}), \quad 0 \leq t \leq T.$$

From (3.7) we also have that

$$\|u(t)\|_{L^\infty(\Omega)} \leq c(T) \left(1 + t^{-\frac{N}{2q}}\|u_0\|_{L^q(\Omega)}\right), \quad 0 < t \leq T.$$

Thus, by interpolation

$$\|u(t)\|_{L^p(\Omega)} \leq \|u(t)\|_{L^q(\Omega)}^{\frac{q}{p}} \|u(t)\|_{L^\infty(\Omega)}^{1-\frac{q}{p}} \leq c(T) \left(1 + t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q(\Omega)}\right), \quad 0 < t \leq T.$$

■

Let show now that the solutions of (1.1) in $L^q(\Omega)$ are unique for $1 \leq q < \infty$.

Theorem 3.3 *Given $u_0 \in L^q(\Omega)$, $1 \leq q < \infty$, there exists a unique function*

$$v \in C([0,\infty); L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,\infty); L^\infty(\Omega)), \quad v(0) = u_0$$

that satisfies

$$v(t) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(\cdot, v(s)))\, ds, \qquad t \geq 0 \tag{3.10}$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Therefore the function $u(\cdot)$ constructed in Theorem 3.1 is the unique function satisfying this.

Proof. Notice that the function u constructed in Theorem 3.1 satisfies the assumptions above. So, let v also satisfy the statement of the theorem. Then from (3.10) we have that, for any $\varepsilon > 0$,

$$v(t) = S_m(t-\varepsilon)v(\varepsilon) + \int_\varepsilon^t S_m(t-s)(g + f_0(\cdot, v(s)))\, \mathrm{d}s.$$

From the assumptions on v we have that $h(s) = g + f_0(\cdot, v(s))$ satisfies, for any $T > 0$, that $h \in L^\infty([\varepsilon, T]; L^{q_0}(\Omega))$. Then, the smoothing effect of the semigroup gives that

$$v \in C([\varepsilon, T]; H^{2\gamma, q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)), \text{ for any } \gamma < 1.$$

Hence, for $t \geq \varepsilon$, v is a solution as in Theorem 2.1.

Hence, arguing as in (3.8) we have

$$\sup_{\varepsilon \leq t \leq T} \|u(t) - v(t)\|_{L^q(\Omega)} \leq c(T)\|u(\varepsilon) - v(\varepsilon)\|_{L^q(\Omega)}$$

with $c(T)$ not depending on ε.

The continuity of u and v at 0 in $L^q(\Omega)$, and the fact that $u(0) = v(0)$ imply $u = v$.
■

4 Final remarks and examples

(i) Note that Theorems 3.1 and 3.3 allow to define a strongly continuos nonlinear semigroup in $L^q(\Omega)$ as

$$S(t)u_0 = u(t; u_0), \qquad t \geq 0$$

where $u(t; u_0)$ is the solution in Theorem 3.1.

The asymptotic behavior of this semigroup is the same as the semigroup obtained for more regular initial data from Theorems 2.1 and 2.2. In fact, from (3.2) we get that for any $0 < \varepsilon < T < \infty$ and for any bounded set of initial data $B \subset L^q(\Omega)$ we get that

$$\{S(t)B, \ \varepsilon \leq t \leq T\} \quad \text{is bounded in } L^\infty(\Omega).$$

This implies, in turn that

$$\{g + f_0(\cdot, u(t; u_0)), \ \varepsilon \leq t \leq T, \ u_0 \in B\} \quad \text{is bounded in } L^{q_0}(\Omega)$$

and again the smoothing effect of the semigroup implies that

$$\{S(t)B, \ \varepsilon \leq t \leq T\} \quad \text{is bounded in } H^{2\gamma, q_0}(\Omega) \cap H_0^{1,q_0}(\Omega))$$

for any $\gamma < 1$.

(ii) Notice that the proofs in [2], [5] and [9] are based on energy estimates of the approximating solutions while the proof presented above is based on the maximum principle, in the form of the comparison principle. In particular, for the case of posing the problem in $L^1(\Omega)$, this avoid the use of Kato's inequality providing a unified argument. The equivalence between Kato's inequality and positive semigroups has been established in [1].

(iii) The standard theory for semilinear reaction-diffusion equations requires f to satisfy some growth restriction in order to obtain a well-posed problem in $L^q(\Omega)$. Namely, the equation (1.1) is locally well posed provided f satisfies

$$|f(x,t) - f(x,s)| \leq C(1 + |s|^{p-1} + |s|^{p-1})|t - s|, \quad t, s \in \mathbb{R} \tag{4.11}$$

for all $x \in \Omega$, with

$$p \le p_c = 1 + \frac{2q}{N} \quad \left(\text{i.e, for any} \quad q \ge q_C = \frac{N(p-1)}{2}\right).$$

Notice that although the uniqueness in $L^q(\Omega)$, for $q > q_{\rm C}$, when f satisfies the growth restriction (4.11), follows from with subcritical nonlinearities, the proof of Theorem 3.3 does not use any growth restriction on the nonlinear term (other than the fact of being almost-monotonic).

(iv) Theorems 3.1 and 3.3 extend to problems in unbounded domains in a natural way (see [2]). Also, the same techniques can be applied to obtain solutions in $\mathbb{R}^N$ in any $L^q_U(\Omega)$, locally uniform L^q space, see [5] for a proof based on energy estimates. In the case of initial data in $L^1_U(\Omega)$, L in (1.4) was required to be bounded. By the techniques presented in here, no additional restriction is required on L in order to obtain a solution.

(v) In [7], positive solutions of equation $u_t - \Delta u = -|u|^p$ with measures as initial data is considered. In particular, for positive L^1 densities, the solution is unique. We have shown that this uniqueness also holds for general L^1 initial data (with no assumption on their sign).

(vi) An example of nonlinearity for which all the previous results apply are the following:

$$f_0(x,u) = \sum_{j=1}^{k} n_j(x) h_j(u) + f_1(x,u)$$

with $h_j \in C^1(\mathbb{R})$, $h_j(0) = h_j'(0) = 0$, $j = 1, \ldots, k$, and $f_1(x,s)$ is a Hölder continuous with respect to x uniformly for s in bounded sets of $\mathbb{R}$, $\partial_s f_1(x,s)$ is bounded in x for s in bounded sets of $\mathbb{R}$ and $f_1(x,0) = \partial_s f_1(x,0) = 0$, $x \in \Omega$. This includes in particular the following cases, taking $f_1 \equiv 0$:

- *Logistic equation*

  $$f_0(x,u) = -n(x)|u|^{\rho-1}u$$

  with $n(x)$ a nonnegative $L^r(\Omega)$ function, not identically zero, and $\rho > 1$. In this case, $L_0(x,R) = \rho R^{\rho-1} n(x)$ and we can always take $L \equiv 0$ in (1.4).

- Monotone polynomial nonlinearity

  $$f_0(x,u) = \sum_{j=2}^{k} n_j(x) u^j$$

  with k odd and $n_j \in L^{d_j}(\Omega)$, $d_j > N/2$, $1 \le j \le k$ and $n_k(x) \le a_0 < 0$ for all $x \in \Omega$. In this case, we can take

  $$L_0(x,R) = \sum_{j=2}^{k} (j-1) R^{j-1} |n_j(x)|$$

and

$$L(x) = k \left[\max_{u \in \mathbb{R}} \sum_{j=2}^{k} n_j(x) u^{j-1} \right]^+ ,$$

where $[g(x)]^+ = \max\{g(x), 0\}$. Notice that $L, L_0 \in L^{q_0}(\Omega)$ with $q_0 = \min\{d_1, \ldots, d_k\}$.

- Polynomial nonlinearity with fractional powers

$$f_0(x, u) = \sum_{j=1}^{k} n_j(x) |u|^{\rho_j - 1} u$$

with $1 < \rho_j < \rho_k$ and $n_j \in L^{d_j}(\Omega)$, $d_j > N/2$, $1 \leq j \leq k$, and $n_k(x) \leq a_0 < 0$, $x \in \Omega$. We can take L and L_0 analogous to the previous example.

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