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Topology and its Applications 113 (2001) 51–59

TOPOLOGY AND ITS APPLICATIONS

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Some duality properties of non-saddle sets ☆

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Received 15 January 1999; received in revised form 29 November 1999

Abstract

We show in this paper that the class of compacta that can be isolated non-saddle sets of flows in ANRs is precisely the class of compacta with polyhedral shape. We also prove—reinforcing the essential role played by shape theory in this setting—that the Conley index of a regular isolated non-saddle set is determined, in certain cases, by its shape. We finally introduce and study the notion of dual of a non-saddle set. Examples of compacta related by duality are attractor–repeller pairs. We use the complement theorems in shape theory to prove that the shape of the dual set is determined by the shape of the original non-saddle set. © 2001 Elsevier Science B.V. All rights reserved.

AMS classification: 57N25; 54C56; 37B25; 37B30

Keywords: Dynamical system; Isolated set; Non-saddle set; Shape

In some recent papers [4,10,17] it has been proved that, under suitable conditions, the attractors of dynamical systems have the shape of finite polyhedra. Moreover, Günther and Segal also proved [10] that all finite-dimensional compacta with polyhedral shape can be attractors of continuous flows in manifolds. One of the aims of this paper is the search for similar results for isolated non-saddle sets.

We first prove that any isolated non-saddle set of a flow has polyhedral shape. Moreover, a non-saddle set will be an attractor or a repeller whenever its shape is trivial. We next prove—reinforcing the essential role played by shape theory in this setting—that, under certain conditions, the Conley index of a regular isolated non-saddle set is uniquely

☆ The authors are supported by DGESIC.

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PII: S0166-8641(00)00017-1

determined by its shape. The main purpose of the paper is to introduce and study the notion of the dual of a regular isolated non-saddle set. The pair formed by a set and its dual generalizes the notion of an attractor–repeller pair. We prove that the shape of a compactum determines that of its dual for certain flows defined on manifolds. The proof of this result makes use of some classical complement theorems in shape theory and seems to be mainly applicable in the case of attractor–repeller pairs.

The reader is referred to the books by Borsuk [5], Cordier and Porter [7], Dydak and Segal [9] and Mardešić and Segal [15] for information on shape theory, the book by Bhatia and Szego [3] for basic properties of dynamical systems, and the paper by Bhatia [2] for basic properties of non-saddle sets.

Let S be a compact invariant set of a flow $\varphi : M \times \mathbb{R} \to M$ in a locally compact Hausdorff topological space M. We say that S is an isolated invariant set if there exists a compact neighborhood N of S in X such that S is the maximum invariant subset of N. In this case, N is called an isolating neighborhood for S in M.

We see first that, in general, an isolated invariant set does not need to have the shape of a finite polyhedron.

Example 1. Consider a dynamical system defined in the unit square $[0,1] \times [0,1]$ as follows: The points $(0, \frac{1}{2})$ and $(1/n, \frac{1}{2})$, $n \in \mathbb{N}$, are stationary points. All the points in $[0,1] \times \{0,1\}$ are also stationary. The orbits of the rest of the points are vertical straight lines joining two stationary points.

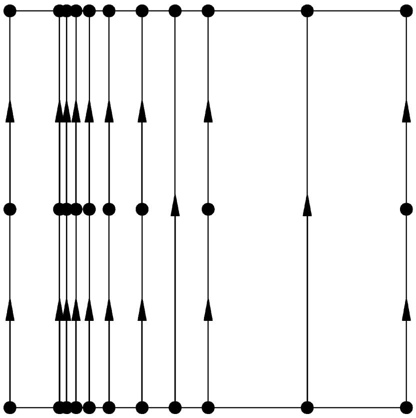

Fig. 1.

The set $(0, \frac{1}{2}) \cup (\bigcup_{n=0}^{\infty} (1/n, \frac{1}{2}))$ is an isolated invariant set which does not have the shape of a finite polyhedron. Similar systems can be defined in $\mathbb{R}^2$, the sphere S^2 or in a torus.

Example 2. This result is not true even if the isolated invariant set is connected. To see this we consider a dynamical system defined in the cylinder $D \times [0,1]$, where D stands for the unit disk: The points in the Hawaiian earring $H = \bigcup_{n=1}^{\infty} S_{((1/2n,0,1/2),1/2n)}$ are stationary points. All the points in $D \times \{0, 1\}$ are also stationary. The orbits of the rest of the points are vertical straight lines joining two stationary points.

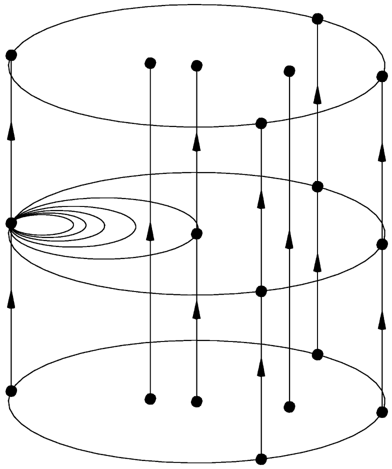

Fig. 2.

Then the set H is an isolated invariant set which does not have the shape of a finite polyhedron. Similar systems can be defined in $\mathbb{R}^3$, the sphere S^3 or in a solid torus.

The above examples are particular cases of the following more general result, whose statement and proof have been provided by the referee. The authors gratefully acknowledge this contribution.

Theorem 3. *Any finite-dimensional compactum X can be embedded in $\mathbb{R}^n$, for suitable n, in such a way that there is a flow in $\mathbb{R}^n$ having X as an isolated invariant set.*

Proof. First, embed the compact set X as a subset of the diagonal of some $\mathbb{R}^{2n}$. Let φ be a translation flow on $\mathbb{R}^{2n}$, given by $\varphi((x, y), t) = (x, y + at)$ for some non-zero a. Then φ has no fixed points and all of the flow lines hit the diagonal in at most one point. By a theorem of Beck [1], φ can be modified to a new flow φ' in such a way that all the orbits of φ not containing a point of X are preserved in φ' while the orbits containing a point of X are decomposed in two orbits together with that point of X. Then X is an isolated invariant set for the flow φ'. □

Thus, the shape of isolated invariant sets may be quite general. In order to get some control over their shape we need to impose an additional condition on the invariant set, namely, that of being non-saddle.

Let $\varphi : M \times \mathbb{R} \to M$ be a flow. A set $X \subset M$ is said to be a saddle set if there is a neighborhood U of X such that every neighborhood V of X contains a point $x \in V$ with $\gamma^+(x) \not\subset U$ and $\gamma^-(x) \not\subset U$. We say that X is non-saddle if it is not a saddle set, i.e., if for every neighborhood U of X there exists a neighborhood V of X such that for every $x \in V$, $\gamma^+(x) \subset U$ or $\gamma^-(x) \subset U$.

Attractors and repellers are examples of non-saddle sets.

In the rest of the paper we will assume, without further mention, that all non-saddle sets are invariant.

Theorem 4. *Let K be an isolated non-saddle set of the flow $\varphi : M \times \mathbb{R} \to M$, where M is a locally compact ANR. Then K has the shape of a polyhedron.*

Proof. Consider an isolating neighborhood U for K in M. Since K is non-saddle, there exists another neighborhood $V \subset U$ of K, with the property that for every $x \in V$ at least one of the semi-orbits $\gamma^+(x)$ or $\gamma^-(x)$ is contained in U. We define

$$N = \{x \in U \mid \gamma^+(x) \subset U \text{ or } \gamma^-(x) \subset U\}.$$

Since $V \subset N \subset U$, N is an isolating neighborhood for K (the compactness is a consequence of the compactness of U). Moreover N can be decomposed as $N = N^+ \cup N^-$, where

$$N^+ = \{x \in N \mid \gamma^+(x) \subset N\} \quad \text{and} \quad N^- = \{x \in N \mid \gamma^-(x) \subset N\}.$$

Observe that $N^+ \cap N^- = K$.

By [16, Theorem 5.2], there exists a map $f : N \to \mathbb{R}$ such that $f(x) = 0$ if $x \in K$ and $f(xt) < f(x)$ if $x[0,t] \subset N \setminus K$ and $t > 0$. Now, $\omega^+(x) \subset K$ for every $x \in N^+$, hence $f(x) > 0$ if $x \in N^+ \setminus K$. Similarly, $f(x) < 0$ if $x \in N^- \setminus K$. If W is an arbitrary open neighborhood of K with its closure contained in the interior of N, then there is a $t_0 > 0$ such that $f^{-1}([-t_0, t_0]) \subset W$. Otherwise, there would exist a sequence of points $(x_n) \subset N \setminus W$ and a null sequence (t_n) of positive numbers such that $|f(x_n)| < t_n$. From this we could deduce the existence of a point $x_0 \in N \setminus K$ with $f(x_0) = 0$, in contradiction with the previous remark. This implies, in particular, that $f^{-1}((-t_0, t_0))$ is open in M and that for any null sequence $(t_n)_{n \geqslant 0}$ starting in t_0, the sets $f^{-1}([-t_n, t_n])$ form a neighborhood basis of K in M.

Consider any such null sequence $(t_n)_{n \geqslant 0}$. We first construct a retraction

$$r : f^{-1}([-t_0, t_0]) \to f^{-1}([-t_1, t_1])$$

in the following way: If $x \in f^{-1}([-t_1, t_1])$ we define $r(x) = x$. On the other hand, if $x \in f^{-1}([-t_0, t_0]) \setminus f^{-1}([-t_1, t_1])$ and $f(x) > t_1$ then $x \in N^+$ and there exists a unique $t_x > 0$ such that $f(xt_x) = t_1$. We then define $r(x) = xt_x$. In a similar way, if $f(x) < -t_1$ we define $r(x) = xt_x$ where $t_x < 0$ is the unique negative number satisfying that $f(xt_x) = -t_1$. A strong deformation retraction from $f^{-1}([-t_0, t_0])$ to $f^{-1}([-t_1, t_1])$ is given by the homotopy $\theta : f^{-1}([-t_0, t_0]) \times [0,1] \to f^{-1}([-t_1, t_1])$ defined as $\theta(x, s) = x(t_x s)$ if $f(x) \notin [-t_1, t_1]$ and $\theta(x, s) = x$ if $f(x) \in [-t_1, t_1]$. In an analogous way we may construct a strong deformation retraction from $f^{-1}([-t_1, t_1])$ to $f^{-1}([-t_2, t_2])$ and, in general from $f^{-1}([-t_n, t_n])$ to $f^{-1}([-t_{n+1}, t_{n+1}])$ for every $n \in \mathbb{N}$.

Since the sets $f^{-1}([-t_n, t_n])$ form a neighborhood basis of K in M we can define a strong shape deformation retraction from $f^{-1}([-t_1, t_1])$ to K. Therefore K has the shape of $f^{-1}([-t_1,t_1])$. But, since $f^{-1}((-t_0,t_0))$ is an open set of M, then it is an ANR, and since $f^{-1}([-t_1,t_1])$ is a retract of it then it is also an ANR. Therefore $f^{-1}([-t_1,t_1])$ has the homotopy type [19], and hence the shape, of a finite polyhedron. □

Theorem 5. *A finite-dimensional compactum can be an isolated non-saddle set of a continuous flow on a manifold if and only if it has the shape of a finite polyhedron.*

The direct implication is a particular case of Theorem 1. The converse implication is a direct consequence of the following theorem of Günther and Segal [10, Corollary 4].

Theorem 6 (Günther and Segal). *A finite-dimensional compactum can be an attractor of a continuous flow on a manifold if and only if it has the shape of a finite polyhedron.*

In the following theorem we prove that, in the case of flows in manifolds, the simplest case of trivial shape is restricted to the special case of attractors and repellers.

Theorem 7. *Let K be an isolated non-saddle set of the flow $\varphi : M\times\mathbb{R}\to M$, where M is an n-manifold with $n>1$. If K has trivial shape then K is an attractor or a repeller.*

Proof. Consider an isolating neighborhood N for K in M such that $N=N^+\cup N^-$, as in the proof of the previous theorem. Let U be a connected open neighborhood of K, $U\subset N$. Then by a result in [8] $U\setminus K$ is still connected. Since $N^+\cap(U\setminus K)$ and $N^-\cap(U\setminus K)$ are disjoint closed subsets of $U\setminus K$ we deduce that either $N^+\cap(U\setminus K)=\emptyset$ or $N^-\cap(U\setminus K)=\emptyset$. In the first case $N=N^-$ and K is a repeller while in the second case $N=N^+$ and K is an attractor. □

In the rest of the paper we will need that our invariant non-saddle set satisfies one more property: regularity.

We say that an isolated invariant set S is regular if there is an isolating neighborhood N such that if $x\in N$, $t\geqslant 0$ and $xt\in N$ then $x[0,t]\subset N$. This is equivalent to saying that orbits which leave N never return. We say then that N is a regular isolating neighborhood for S.

In the next theorem we prove that, in some cases, the Conley index of a regular isolated non-saddle set is uniquely determined by its shape. For the definition and properties of the Conley index see [6].

Theorem 8. *Let K and K' be isolated non-saddle sets $(K, K'\neq\emptyset)$ of flows φ and φ' defined on locally compact ANRs M and M', respectively. Suppose that K and K' admit index pairs (N,L) and (N',L') such that N and N' are connected regular isolating neighborhoods of K and K' and L and L'* (the exit sets) *are contractible or, more generally, have trivial shape. Suppose, in addition, that N and N' are ANRs. Then $Sh(K)=Sh(K')$ if and only if they have the same* (unpointed) *Conley index.*

Proof. We first see that $N \setminus (N^+ \cup N^-)$ is closed. Suppose that we have a sequence of points $(x_n) \subset N \setminus (N^+ \cup N^-)$ converging to a point $x_0 \notin N \setminus (N^+ \cup N^-)$. We may suppose that $x_0 \in N^+$ (the argument for $x_0 \in N^-$ being similar). Then, for every $\varepsilon > 0$ there is a $t_0 > 0$ such that $d(x_0 t_0, K) < \varepsilon$. When ε is small enough this implies that $d(x_n t_0, K) < \varepsilon$ and $x_n t_0 \in N$ for almost every n. Since $x_n, x_n t_0 \in N$, the regularity of N implies that $x_n[0, t_0] \subset N$. Then the point $y_n = x_n t_0$ satisfies $d(y_n, K) < \varepsilon$, $\gamma^-(y_n) \not\subset N$ and $\gamma^+(y_n) \not\subset N$. Since such a y_n exists for every ε this contradicts the fact that K in non-saddle. Hence $N \setminus (N^+ \cup N^-)$ is closed and, since N is connected, $N = N^+ \cup N^-$ ($N^+ \cup N^-$ can not be empty since it contains K). We now consider, as in the proof of Theorem 1, a map $f : N \to \mathbb{R}$ such that $f(x) = 0$ if $x \in K$ and $f(xt) < f(x)$ if $x[0, t] \subset N \setminus K$ and $t > 0$. If we consider a $t_0 > 0$ such that $N = f^{-1}([-t_0, t_0])$ we can construct, as in Theorem 1, a strong shape deformation retraction from N to K. Therefore $Sh(K) = Sh(N)$. Since L has trivial shape, by a result of Mardešić [14] $Sh(N) = Sh(N/L)$. Hence $Sh(K) = Sh(N/L)$. On the other hand, by a result of Hyman [11], L is an ANR-divisor and hence N/L is an ANR. A similar argument applies to K' and (N', L').

Then $Sh(K) = Sh(K')$ if and only if $Sh(N/L) = Sh(N'/L')$ if and only if N/L and N'/L' have the same homotopy type if and only if K and K' have the same unpointed Conley index. □

In the next theorem we introduce the notion of dual of a regular isolated non-saddle set. The pair (K, K') formed by a set and its dual generalizes the notion of an attractor–repeller pair.

Theorem 9. *Let K be a regular isolated non-saddle set of the flow $\varphi : M \times \mathbb{R} \to M$, where M is a compact metric space. Let K' be the set of all points $x \in M$ such that $\omega^+(x) \not\subset K$ and $\omega^-(x) \not\subset K$. Then K' is also a regular isolated non-saddle set. We call K' the dual of K.*

Moreover, K' is nonempty whenever $K \neq M$.

Proof. The proof of the case $K = M$ is immediate, hence we may suppose that $K \neq M$.

To see that K' is nonempty if $K \neq M$ consider a regular isolating neighborhood N for K and any point $x \in N \setminus K$. If both limit sets of x were contained in K the regularity of N would imply that all the trajectory $\gamma(x)$ would be contained in N, and this would contradict the fact that N is an isolating neighborhood. Therefore at least one of the limit sets of x (say $\omega^+(x)$) is not contained in K. It is easy to see that this implies that $\omega^+(x) \cap K = \emptyset$. But being invariant it will be clearly contained in K'. Hence K' is nonempty.

Since all the points of a trajectory share the same positive and negative limit sets, K' is clearly invariant.

To prove that K' is compact consider an isolating neighborhood N for K in M such that $N = N^+ \cup N^-$, as in the proof of Theorem 1. Take $x \in M \setminus K'$. Then one of its limit sets is contained in K and hence there exist a $t_0 \in \mathbb{R}$ such that xt_0 is contained in the interior of N. There exists a neighborhood U of x in M such that $Ut_0 \subset N$. Since all the points in N have at least one of its limits points in K, the same will happen to the points in U. Therefore $U \subset M \setminus K'$ and $M \setminus K'$ is open and K' compact.

To see that K' is isolated consider any compact neighborhood N' of K' not meeting K (one such neighborhood could be the complementary of the interior of N). Then any point $x \in N' \setminus K'$ has at least one of its limit sets in K, which implies that K' is the largest invariant subset of N', and hence that N' is an isolating neighborhood for K'.

To see that the compactum K' is also regular consider the isolating neighborhood N for K introduced in the proof of Theorem 1 and the map $f : N \to \mathbb{R}$ such that $f(x) = 0$ if $x \in K$ and $f(xt) < f(x)$ if $x[0,t] \subset N \setminus K$ and $t > 0$. Consider $t_0 > 0$ such that $f^{-1}([-t_0, t_0]) \subset Int(N)$ and take $N' = M \setminus f^{-1}((-t_0, t_0))$. Then N' is a regular isolating neighborhood for K', since any orbit leaving N' will never return.

We see finally that K' is a non-saddle set. Suppose that it is a saddle set. Then there is a neighborhood U' of K' such that every neighborhood U'' of K' contains a point x with $\gamma^-(x) \not\subset U'$ and $\gamma^+(x) \not\subset U'$. There exists, then, a sequence $(x_n) \subset U'$ converging to a point $x_0 \in K'$ such that $\gamma^-(x_n) \notin U'$ and $\gamma^+(x_n) \not\subset U'$, for every $n \subset \mathbb{N}$. Thus the semi-trajectories $\gamma^-(x_n)$ and $\gamma^+(x_n)$ must meet the boundary $\partial U'$ in points $y_n = x_n(-t_n)$ and $z_n = x_n s_n$ (with $t_n, s_n > 0$) that we can assume to converge to $y_0 \in \partial U'$ and $z_0 \in \partial U'$, respectively. We can moreover suppose that the sequences of numbers (t_n) and (s_n) are also convergent or divergent. If $t_n \to t \in \mathbb{R}$, we would have $y_n t_n \to y_0 t_0$ and this would imply that $y_0 t_0 = x_0$ and this is a contradiction. Therefore, we may suppose that $t_n \to \infty$ and also that $s_n \to \infty$. This implies that $w^+(y_0) \subset K'$ and that $w^-(z_0) \subset K'$, which in turn implies, by the regularity of K', that $w^-(y_0) \subset K$ and that $w^+(z_0) \subset K$. Consider again the isolating neighborhood N for K introduced before. We can suppose that $N \cap K' = \emptyset$. Since $w^-(y_0) \subset K$ and $w^+(z_0) \subset K$ there exist $t_0, s_0 > 0$ such that $y_0(-t_0) \in Int(N)$ and $z_0 s_0 \in N$. Then there exists n_0 such that $y_n(-t_0) \in N$ and $z_n s_0 \in N$ for every $n \geqslant n_0$. But this implies that $x_n(-t_n - t_0) \in N$ and $x_n(s_n + s_0) \in N$ for every $n \geqslant n_0$. Therefore $\omega^-(x_n) \subset K$ and $\omega^+(x_n) \subset K$, and this is a contradiction with the regularity of K. □

In our last theorem we will apply some classical complement theorems in Shape Theory to prove that, under appropriate conditions, the shape of a regular isolated non-saddle set completely determines that of its dual. Our theorem seems to be mainly applicable to the case of attractor–repeller pairs.

We first present the statement of the complement theorems. For information the reader is referred to [12,13,18].

Theorem 10 (Ivanšić, Sher and Venema). *Let X_1 and X_2 be r-shape connected continua in E^n of fundamental dimension at most k and satisfying ILC (the inessential loops condition), where $n \geqslant \max\{2k+2-r, k+3, 5\}$. Then $Sh(X_1) = Sh(X_2)$ implies $E^n \setminus X_1 \simeq E^n \setminus X_2$.*

Theorem 11 (Ivanšić and Sher). *Let X_1 and X_2 be continua in the interior of the piecewise linear n-manifold M such that for $j = 1$ or 2, X_j has fundamental dimension at most k, X_j satisfies ILC, and pro-$\pi_i(X_j)$ is stable for $0 \leqslant i \leqslant r-1$ and satisfies the Mittag–Leffler* *condition for* $i = r < n - 3$*, where* $n \geqslant \max\{2k + 2 - r, k + 3, 5\}$*. Suppose the inclusion of* X_1 *into* M *is shape* r*-connected and that* X_1 *and* X_2 *have the same shape relative to* M*. Then* $M \setminus X_1 \simeq M \setminus X_2$*.*

We are now in a position to state our theorem. Note that Theorem 9 also holds if we replace E^n by S^n.

Theorem 12. *Let* K_1 *and* K_2 *be regular isolated non-saddle sets of flows* $\varphi_1, \varphi_2 : M \times \mathbb{R} \to M$*, where* $M = S^n$ *or, more generally, a piecewise linear compact* n*-manifold. Suppose that* K_1 *and* K_2 *satisfy the hypothesis of the Ivanšić–Sher–Venema Theorem or the Ivanšić–Sher Theorem. Then, if* $Sh(K_1) = Sh(K_2)$*, the dual sets* K_1' *and* K_2' *also have the same shape. In particular, if* K_1 *and* K_2 *are attractors with the same shape, then their dual repellers have the same shape too.*

Proof. $M_1 = S^n \setminus K_1$ is an invariant neighborhood of K_1'. If we restrict the flow to M_1 we get a flow $\varphi_1^* : M_1 \times \mathbb{R} \to M_1$ such that $M_1 = W^s \cup W^u$ and $W^s \cap W^u = K_1'$, where W^s and W^u denote, respectively, the stable and unstable manifolds of φ_1^*. Since, by regularity, K_1' is an attractor in W^s and a repeller in W^u, there is a Lyapunov function $f : M_1 \to \mathbb{R}$ (constructed gluing together the Lyapunov functions in W^s and W^u which agree on K_1') with $f^{-1}(0) = K_1'$ and $f(xt) < f(x)$ for every $x \notin K_1'$ and every $t > 0$ (see [3]). Then, by the general properties of the Lyapunov functions, there is a $c > 0$ such that $f^{-1}([-c, c])$ is a compact strong deformation retract of M_1 and by an argument similar to that in Theorem 1 we have that $f^{-1}([-c, c])$ has the same shape as K_1'. We have a similar situation in $M_2 = S^n \setminus K_2$ and K_2' and, since by the previous theorem M_1 and M_2 are homeomorphic, we conclude that $Sh(K_1') = Sh(K_2')$. □

Remark 13. As we showed in the proof of Theorem 4, if K is an isolated non-saddle set of a flow in a manifold M, then K has a base of compact neighborhoods $U_1 \supset U_2 \supset \cdots \supset U_n \supset U_{n+1} \supset \cdots$, such that each U_{j+1} is a strong deformation retraction of U_j, and such that each ∂U_j is a strong deformation retraction of $U_j \setminus \mathring{U}_k$, for every $1 \leqslant j \leqslant k$.

If moreover K satisfies the ILC, then the above base can be chosen with the additional property that every loop γ in ∂U_i null-homotopic in U_i is null-homotopic in ∂U_i. Conversely, this additional property on the above base guarantees the ILC.

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