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**UNEMPLOYMENT, OPTIMAL  
WAITING AND QUEUES**

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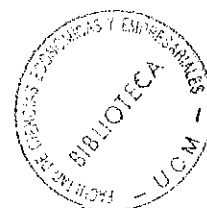
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## **Abstract**

In this article we portray the state of technically diverse economies as the outcome of probabilistic equilibria. We provide the steady state distributions of current technologies, unemployment, wages and job durations thus allowing the comparison of situations with identical mean values but a differently distributed probability mass. A queueing closed network is used to establish the connections between advertised, filled and dormant business options, and their implications for the unemployment of differently qualified workers. The queueing connecting rates are obtained endogenously as the outcome of the optimizing strategies of firms.

*Keywords:* Unemployment, probabilistic matching, queues, optimal technologies.

## 1. INTRODUCTION

Technological change has been alternatively blamed or praised for the effect it has on unemployment. It seems to be generally agreed that technical progress brings an increase in the number of vacancies while bearing in it the demise of antiquated methods of production. However in any economy the old and the new ways of producing the same type of goods often coexist, a fact that cannot only be accounted for by the clinging of old established firms to methods they have always known while the more modern entrepreneurs adopt the latest technology, since it often happens that newly created firms choose aged but still profitable technologies. This seems to contradict the usually assumed fast adaptability to innovations on the part of new entrepreneurs. The effect of this simultaneous heterogeneity of production methods makes the interaction between growth and employment quite complex since it is not necessarily the case that the best educated workers always get the plum jobs in a world where a fast moving technology opens and closes labour sectors, but more a matter of firms and workers competing among themselves for the adequately qualified partner. The problem of matching workers to jobs depends on how closely the distribution of labour qualifications among the unemployed fits in with the labour requirements of firms. It is the object of this article to provide a theoretical framework within which this situation can be fully accounted for.

The article connects with two main lines of research, unemployment and endogenous growth. Given how large both areas are it seems best to start by citing only a few articles setting the general framework from which our analysis springs. The part of the labour literature that is closest to our work generally takes as given the reservation wages and intensity of job search among the unemployed and studies how workers are matched to advertised vacancies using a technology that changes over time but is generally the same for all coexisting firms. A matching function establishing the rate of successful job contacts together with a condition equating the flows in and out of unemployment, provide a relation between the levels of unemployment and advertised vacancies. This together with some equilibrium conditions relating the costs and benefits of creating new vacancies and the supply and demand of human capital, jointly determine the full equilibrium of the system. These analyses are generally deterministic in nature and, although random Poisson shocks occurring at a known rate might affect the

creation or destruction of jobs, the predicted equilibrium levels of unemployment, wages and vacancies are described in terms of their expected equilibrium values given the nature of the shocks. Two articles in this line are those of Aghion and Howitt (1994) and Blanchard and Diamond (1994). The first one analyzes how technology shocks simultaneously create and destroy business options thus affecting the reallocation of labour between nascent and dying firms. Blanchard and Diamond examine the effect on labour market variables of a preference on the part of firms for workers with the shortest unemployment and compares it with the effect of no ranking when deciding whom to hire.

It is usually assumed in the literature either that all workers are alike in qualification or at most that there is a small finite number of qualifications although this is sometimes converted into a continuous distribution (see Acemoglu (1997)). When formalizing the matching of workers to the technically diverse advertising firms we assume that workers vary continuously in qualification and this gives them different chances of becoming employed. Some authors have considered differentiated access of workers to jobs on the basis of the level of qualification involved on one or both sides of the hiring. For instance Ours and Ridder (1995) consider qualified workers who apply simultaneously for two types of jobs differing in the salary offered by each and in the time it takes to find them. This results in workers applying for jobs requiring less qualification than they themselves possess. We characterize the interval of firms in which each unemployed worker searches in terms of the duration of his previous unemployment and of his qualification level. In consequence each firm ends up having a waiting list of differently qualified job applicants from which the candidate most suitable for the job is chosen. Kahn (1987) considers that it may be in the interest of firms to offer salary incentives to obtain a queue of job applicants since that reduces the cost of educating employed labour, but he does not specify the grounds on which job applicants are selected. In this article we consider salary incentives as a way to attract better qualified job applicants and also to accelerate the matching process of firms to workers. We then obtain an ordering of job applicants according to their qualification and determine an explicit expression for the expected best labour qualification each firm hires.

We have used probability theory to model the behaviour of the system because the levels of unemployment and vacancies in any economy generally vary in a manner that limits the predictive power of a deterministic approach. Our modelling of the matching relationship permits the determination of the equilibrium probability distributions of unemployment and vacancies thus obtaining an understanding

of the behaviour of these variables which goes beyond explanations that only take into account their mean values and cannot in consequence compare situations where the probability mass around them is very differently distributed. Labour markets with very similar initial conditions often differ in actual unemployment and vacancies rates. This can be justified by the fact that random exogenous shocks periodically alter labour market conditions causing actual market rates to differ even though their mean values and underlying probability distribution may be the same. In our context the common mean values can be considered as the deterministic solution. We adopt a probabilistic matching function and derive a stochastic Beveridge curve accounting for the indeterminate sign of the correlation between unemployment and advertised vacancies as reported in Bean and Pissarides (1993). The deterministic versions of the matching function and Beveridge curve usually employed in the literature can be understood as particular cases of their stochastic equivalent functions when the random variables of unemployment and vacancies are substituted for their corresponding mean values.

The second main area of research which we lightly touch upon is the area of endogenous growth. We too account endogenously for the technical efficiency of firms and connect it to unemployment. Aghion and Howitt (1992) consider a model where technological shocks together with the value of the employment rate in the research sector determines the growth rate of the economy. A system of patents ensures the appropriability of rents for a time but innovations are copied later on by other firms. Thus different technologies are used simultaneously, a feature shared with this article. Eicher (1996) focuses on the interconnections between the education decisions of workers and the economic growth resulting from their greater ability when employed. Finally, Andolfatto (1996) and Laing, Palivos and Wong (1997) link endogenous growth to unemployment by adding a matching function to the model. In our article the growth rate is the outcome of the endogenous decisions of firms regarding technology and the choice of appropriately qualified workers. We account for the technical diversity among firms at any given time in terms of the optimal waiting it takes to convert technologies that were once profitable, which we call dormant vacancies, into profitable ones which for a time find a niche in the market. The niches occupied by existing advertising firms at any given time might be quite different in technical character and labour qualification and also vary in the time they last in business. The flows resulting from the choices made by all firms, and their matching to unemployed workers, can be modelled as arrival and departure stochastic processes. Queueing theory deals exactly with this problem. We model the economy as a closed Markovian

queueing network where the pools of waiting firms and unemployed workers are the service areas in the system. We will have three pools representing respectively the nodes of advertised, filled and dormant firms. The rate linking the pool of advertised vacancies to the pool of filled jobs, and hence employment, is given by the matching function, the rate linking the pool of filled jobs to the pool of dormant options is the inverse of job durations and the rate linking the pool of dormant options to the pool of advertised jobs is the rate at which new firms appear in the economy. All these rates are determined endogenously. We then have a stochastic dynamical system whose solution converges to a unique random equilibrium associated to a probability distribution which we calculate.

The article is organized as follows. In section two we set up the queueing network describing the system and obtain the stationary probability distributions of unemployment and vacancies. In section three we account for the optimal behaviour of firms which, together with the characteristics of labour supply, determine the wage and labour qualification distributions in the economy and the optimal waiting times associated to the rates linking the pools of the system. In the last section we consider some extensions of the model. All proofs are given in the appendix.

## 2. QUEUEING NETWORKS

We consider a closed network of firms and related employment. Each worker provides a unit of labour and each operating firm is matched with one worker. Firms can be in three alternative positions: advertised, filled with a job applicant and dormant. The time a firm spends at each situation is assumed to be exponentially distributed. A dormant firm is an unprofitable technology that can become a profitable one by investing in the right technology and employing the appropriate worker. To get the right technology a firm waits an average time  $\lambda^{-1}$  after which it advertises a job opening and gets an appropriate candidate after waiting an average time  $\mu^{-1}$ . If the firm cannot hire a worker within a reasonable time, equal on average to  $\gamma^{-1}$ , the advertised vacancy becomes a dormant one. If the firm is able to hire the right worker it starts producing and stays in business for a mean time  $S$  after which it again becomes dormant. In the next section we determine endogenously the rates linking these positions.

We assume given the total number  $V$  of firms, that is

$$V = v_a(t) + v_f(t) + v_d(t), \quad (2.1)$$

where each successive term in the sum represents respectively the number of advertised, filled and dormant firms at time  $t$ . The size  $L$  of the labour force is also assumed given and

$$L = U(t) + v_f(t) \quad (2.2)$$

where  $U(t)$  represents the total number of unemployed workers at time  $t$  and  $v_f(t)$  represents the total number of employed persons since one worker matches with one filled firm. The two related networks of firms and workers are given in figure 1 below.

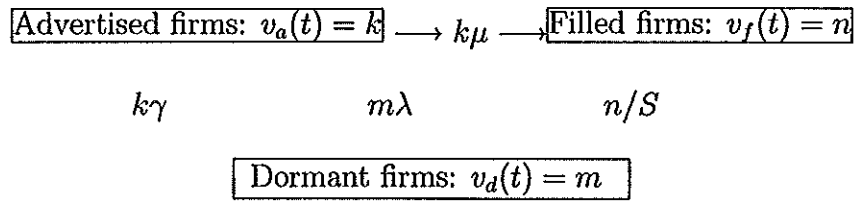


Figure 1.a

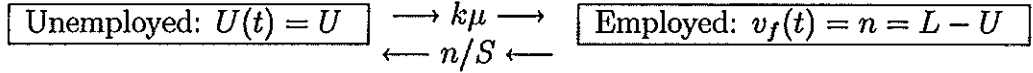


Figure 1.b

Given (2.1) and (2.2), the analysis of the stochastic process  $X(t) = (v_a(t), v_f(t))$  completely characterizes the behaviour of vacancies and unemployment. Since the stochastic process  $X(t)$  is Markovian and irreducible, with finite state space  $E = \{(k, n) \in \mathbb{IN} \times \mathbb{IN} : k + n \leq V\}$ ,  $X(t)$  is ergodic and there exists a unique stationary probability distribution  $\{p_{kn}\}$ ,  $(k, n) \in E$ ,

$$p_{kn} = \lim_{t \rightarrow \infty} \Pr[X(t) = (k, n)] \quad (2.3)$$

with marginal stationary probabilities  $p_{k\bullet}$  and  $p_{\bullet n}$  (see Asmussen (1987)). We denote by  $U, v_a, v_f$  and  $v_d$  the stationary random variables of unemployment and



advertised, filled and dormant vacancies respectively. We determine the probability distributions of these variables using the Chapman-Kolmogorov equations that characterize an equilibrium by establishing an equality between the probabilities of getting in and out of any given state  $(k, n) \in E$ . These equations have a slightly different form depending on whether  $L > V$  or  $L \leq V$ . We first consider the case when  $L > V$ , thereby assuming the existence of positive unemployment. The Chapman-Kolmogorov equations can be easily obtained with the help of diagram 1 (see appendix). They are

$$\begin{aligned} \left( (\gamma + \mu)k + \lambda(V - k - n) + \frac{n}{S} \right) p_{kn} &= \frac{n+1}{S} p_{k,n+1} + \\ &+ \lambda(V - k - n + 1) p_{k-1,n} + \mu(k+1) p_{k+1,n-1} + \gamma(k+1) p_{k+1,n} \end{aligned} \quad (2.4)$$

for every  $(k, n) \in E$ . Note that  $p_{kn} = 0$  if  $k, n < 0$  or  $k, n > V$ . This linear system has a unique non negative solution satisfying the condition that all probabilities sum to unity (see Asmussen (1987)). This solution fully describes the stochastic behaviour of unemployment and vacancies in every state. The matching of unemployed labour to advertised vacancies occurs at the random rate  $\mu v_a(t)$  and hence the matching function in steady state is given by

$$m(U, v_a) = \mu v_a \quad (2.5)$$

where  $v_a$  is a random variable with probability distribution  $\{p_{k\bullet}\}_{0 \leq k \leq V}$ . We can consider the deterministic matching functions used in the literature as particular cases of (2.5) when we substitute the random variable  $v_a$  by its mean value  $\bar{v}_a$ . In that case average matching would be  $m(\bar{U}, \bar{v}_a) = \mu \bar{v}_a$ , where  $\bar{U}$  is the average level of unemployment. Note that although equation (2.5), does not seem to depend on the level of unemployment, such dependence will be later established through the endogenous relation between  $\mu$  and  $\bar{U}$ . We can obtain moments of any order for all stationary variables using the moment generating function

$$P(z, y) = \sum_{k=0}^{k=V} \sum_{n=0}^{n=V-k} p_{kn} z^k y^n, \quad (2.6)$$

where  $z, y \in [0, 1]$ . Using this function, equations (2.4) become

$$\lambda V(1-z)P(z, y) + \left( (\mu + \gamma - \lambda)z + \lambda z^2 - \mu y - \gamma \right) P'_z(z, y) +$$

$$+ \left( \frac{y-1}{S} + \lambda y(z-1) \right) P'_y(z, y) = 0. \quad (2.7)$$

If we solve this partial differential equation we get  $P(z, y)$  and thus obtain the probability distribution  $\{p_{kn}\}$ . However just to get the moments we only need to differentiate (2.7) with respect to  $z$  and  $y$  at  $z = y = 1$ . The first and second order moments are given in the following proposition.

**Proposition 1.** *The mean numbers of advertised and filled vacancies are,*

$$\bar{v}_a = \lambda V / (A + \lambda \mu S) \quad (2.8)$$

$$\bar{v}_f = \mu S \bar{v}_a = \lambda \mu S V / (A + \lambda \mu S), \quad (2.9)$$

where  $A = \lambda + \mu + \gamma$ . Also the variances, covariance and correlation coefficient of the two variables are

$$Var[v_a] \equiv \sigma_a^2 = \frac{A(\lambda \mu S + \mu + \gamma) + \lambda \mu + \frac{\mu + \gamma}{S}}{\lambda \mu (1 + AS) + A(A + S^{-1})} \bar{v}_a \quad (2.10)$$

$$Var[v_f] \equiv \sigma_f^2 = \frac{\mu S A (A + S^{-1})}{\lambda \mu (1 + AS) + A(A + S^{-1})} \bar{v}_a \quad (2.11)$$

$$Cov[v_a, v_f] = -\frac{\lambda \mu (1 + AS)}{\lambda \mu (1 + AS) + A(A + S^{-1})} \bar{v}_a \quad (2.12)$$

$$\rho_{v_a v_f}^2 = \frac{\lambda^2 \mu S}{A(\lambda \mu S + \mu + \gamma)} \quad (2.13)$$

**Proof.** See appendix. ■

Given (2.1) and (2.2), the above mean values allow us to write the mean numbers of dormant vacancies and unemployment as,

$$\bar{v}_d = (\mu + \gamma) V / (A + \lambda \mu S) \quad (2.14)$$

$$\bar{U} = L - \mu S \bar{v}_a = L - \lambda \mu S V / (A + \lambda \mu S). \quad (2.15)$$

This relation between  $\bar{U}$  and  $\bar{v}_a$ , coincides with the version of the Beveridge curve given in Aghion and Howitt (1994) although we need not impose any condition on the relative values of the parameters to ensure non negative unemployment.

We also have a general steady state relationship between random unemployment and the random number of vacancies, namely  $V = v_a + v_f + v_d$ . This is not a random Beveridge curve since it does not just give a relationship between random advertised vacancies and random unemployment. It also includes the random term  $v_d$ . The joint probability distribution  $\{p_{kn}\}$  is the probability distribution of a random Beveridge curve. The following linear approximation might be useful to get a feeling of the relationship between the variables involved. This simplified Beveridge curve is given by,

$$U = \bar{U} - \frac{Cov[v_a, v_f]}{\sigma_a} (v_a - \bar{v}_a) \quad (2.16)$$

This relationship provides a positive correlation between unemployment and advertised vacancies, since the covariance term (2.12) is always negative. However there is a monotonically increasing relationship between the mean values of those variables as can be seen in (2.15). That is, in probability we have the opposite of what is the case in mean in most models in the literature including this one. The negative sign of  $Cov[v_a, v_f]$  above is forced by the fact that in closed networks, when the total number  $V$  of vacancies is small in relation to the size  $L$  of the labour force, an increase in the number of advertised firms must be accompanied by a larger counteracting effect given by the destruction of filled vacancies, whatever the values of the rates involved in the network. The sign of the covariance term is opposite to that reported by Andolfatto (1996) for the US. But then it is likely that for a large economy  $L \leq V$ , a case we will analyze later and which will in some cases confirm the findings of Andolfatto (1996). Observe also that  $\rho_{v_a v_f}^2 < 1$  and hence the relationship between unemployment and advertised vacancies is never linear. Moreover when the destruction rate  $\gamma$  of advertised vacancies is large,  $\rho_{v_a v_f}^2$  tends to zero giving linear independence in the limit. When  $\gamma$  is small and the matching parameter  $\mu$  is also small the relationship between advertised vacancies and unemployment is almost linear. As we shall see later a low value of  $\mu$  is the consequence of slow rate of job applications, which in turn can be the consequence of a mismatch between the distribution of technical efficiency in firms and that of qualifications among the unemployed.

When  $L \leq V$  the corresponding Chapman-Kolmogorov equations have, when  $n = L$ , an extra term on the left hand side, given by

$$-\mu k p_{kL}, \quad (2.17)$$

expressing the probability of zero unemployment, or whatever level may be considered the minimum feasible level of unemployment in the economy. In this case

the stationary matching function is

$$m(U, v_a) = \mu v_a, \text{ if } U > 0 \quad (2.18)$$

and zero otherwise. In this case the following proposition holds.

**Proposition 2.** *The mean numbers of advertised and filled vacancies and the covariance between the two variables are respectively,*

$$\bar{v}_a = \frac{\lambda V}{A + \lambda \mu S} + \frac{\lambda V^2 p_{V0} + \mu (1 + \lambda S) \bar{v}_{a,L}}{A + \lambda \mu S} \quad (2.19)$$

$$\bar{v}_f = \frac{\lambda \mu S V}{A + \lambda \mu S} + \frac{\mu S (\lambda V^2 p_{V0} - (\lambda + \gamma) \bar{v}_{a,L})}{A + \lambda \mu S} \quad (2.20)$$

$$\begin{aligned} \text{Cov}[v_a, v_f] = & \frac{S \lambda \mu (V - \lambda V - 1)}{A + \lambda \mu S} \bar{v}_a + \frac{S \lambda \mu (\lambda S + \gamma S + \mu (1 + \lambda S) (S - 1)) \bar{v}_{a,L}}{A + \lambda \mu S} \bar{v}_a + \\ & + \frac{S \mu}{(AS + 1)(A \lambda \mu S)} \left( \lambda V^2 (1 - \lambda (1 + AS)) p_{V0} - (\gamma + \lambda) \bar{v}_{a,L}^2 \right) + \\ & + \frac{A \mu S (L - \lambda S (L + V - 1)) \bar{v}_{a,L}}{(AS + 1)(A \lambda \mu S)}, \end{aligned} \quad (2.21)$$

where

$$\begin{aligned} \bar{v}_{a,L} &= \sum_{k=1}^{k=V-L} k p_{kL}, \\ \bar{v}_{a,L}^2 &= \sum_{k=1}^{k=V-L} k(k-1) p_{kL} \end{aligned}$$

and  $p_{V0}$  is the probability that all vacancies are advertised and hence the numbers of dormant and filled vacancies in the economy are both zero.

**Proof.** See appendix. ■

When  $L \leq V$ , all moments depend on  $\bar{v}_{a,L}$ ,  $\bar{v}_{a,L}^2$  and  $p_{V0}$ , values which could be easily but messily found by solving the first order differential equations associated to the boundary states of the corresponding Chapman-Kolmogorov equations (see diagram 1 in the appendix). The Beveridge relationship among the mean values of advertised vacancies and unemployment is

$$\bar{U} = L - S \mu (\bar{v}_a - \bar{v}_{a,L}). \quad (2.22)$$

When  $V$  and  $L$  are both large enough it is likely that  $p_{V0}$  is close to zero, in which case the first term in (2.21) dominates over the other terms and hence

when  $\lambda < 1$  this results in a negative correlation between advertised vacancies and unemployment. This is so because when  $\lambda$  is small and  $V$  large there is an accumulation of dormant vacancies which eventually come out as advertised firms and this, for any  $\mu$ , results in lower unemployment since the number of newly created jobs is greater than the possible counteracting job destruction. The opposite happens when  $\lambda$  is large enough. The indetermination in the sign of the correlation theoretically accounts for the evidence in regard to growth and unemployment reported in Bean and Pissarides (1993) if we identify greater growth with large values of  $\lambda$ . Note that every case is compatible with a downward sloping relationship between the mean values of the deterministic Beveridge curve (2.22).

We have used a matching function  $\mu v_a$  encompassing parameters chosen by firms but we could have used instead a function  $\delta U$ , where  $\delta$  is the rate at which the unemployed find jobs. Since in equilibrium the probability of having one extra person employed coincides with the probability of having one extra filled vacancy, it follows that  $\mu \bar{v}_a = \delta \bar{U}$ . This last equation is used by Laing et al. (1995) to characterize the equilibrium of the system and we could use it to find  $\delta$  once the values of the other terms are known. Our choice of the matching function is in accordance with the character of the optimization problem accounting endogenously for the rates in the queueing network used in this section and generally taken as parameters in the literature. We analyze this next.

### 3. OPTIMAL ENTREPRENEURIAL DECISIONS

We consider a market in steady state composed of firms, workers and the unemployed who are the only job seekers. Individuals differ in qualification and firms vary in the technology they use and the corresponding labour qualification they demand. All jobs require a unit of labour of the appropriate qualification. All individuals have a unit of labour at their disposal which, if employed they supply inelastically, at the agreed wage, having no regard for leisure. We assume that there is only one consumption good per period which is taken to be the numeraire commodity. All income elements are expressed in terms of this commodity. In this article we do not analyze in detail the search process of the unemployed but only assume that employers know both the unemployment benefits and the distribution of unemployment duration of qualified labour. All income and cost terms are expressed in terms of their values at time  $t = 0$  when all decisions are taken.

We first describe informally the decisions a typical firm optimally takes and

relate them to the rates assumed in the queueing network of the previous section. We take a dormant option with an inherited unprofitable technology that is considering how long it will be optimal to wait before it can get the most profitable one given the options offered by the state of technical knowledge and its own initial position. The longer it waits the more advanced the technology will be but the higher will be the cost involved. We assume the distribution of dormant technologies to be known and hence we can obtain the average optimal time  $\bar{T}_1^*$  it takes a dormant technology to become a profitable one in the economy. The rate  $\lambda$  in the previous section is then equal to  $1/\bar{T}_1^*$ . The firm then has to decide how long to wait for the optimal candidate having a minimum qualification associated to the chosen technology. The longer it waits the better the chance of getting a better candidate but the greater is the cost involved since having already chosen the technology there is a loss in the relative technical efficiency of the firm as time passes. Hence there is an optimal time that each particular firm will wait before it hires anyone. Given the distribution of dormant technologies we can determine the optimal market mean time  $\bar{T}_2^*$  it takes an advertised firm to become a filled one. Then  $\mu = 1/\bar{T}_2^*$  in the network above. Once the right job candidate has been found the firm starts production and stays in business for an optimally determined time. Given the distribution of dormant technologies, the average time firms stay in business in the market is  $\bar{S}^*$  and hence the market rate at which firms get out of business is  $1/\bar{S}^*$ . Just in this section and for simplicity we assume to be zero the rate  $\gamma$  at which advertised vacancies become dormant because no job candidate arrives. We will also determine the optimal labour qualification hired by each firm and the salary paid. We first discuss the characteristics of labour supply.

### 3.1. Labour Supply

We assume the unemployed labour force has different labour qualifications represented by a random variable  $\Theta_u$ , with support in  $[0, 1]$ . Let  $F(\theta_u)$  be the steady state distribution function of  $\Theta_u$ . We assume that the unemployed are continuously improving their skills and in consequence their qualification levels are changing over time, although their relative distribution remains unchanged. All individuals maximize the expected discounted utility of consumption over time financed by wage or unemployment subsidies. An unemployed worker with qualification  $\theta_u$  can send applications to any job requiring a qualification level less than or equal to his own, provided the wage offered for a job of length  $S$ , is not

less than his reservation wage. Although his acceptance or rejection of a job offer would generally depend on the job length, we assume for simplicity that all workers think the offered job will last forever. If at time  $t = 0$  an individual with qualification  $\theta_u$  has spent a time  $z$ ,  $z \in (0, \infty)$  unemployed getting a subsidy  $\nu(z, \theta_u)$ , that decreases with  $z$ , then after an interval  $t$  he would work in a firm offering a salary  $\omega$ , growing at rate  $\rho(t)$ , if

$$\int_0^\infty (\omega\rho(t) - \nu(z+t, \theta_u)) dt \geq 0$$

where  $\rho(t)$  is assumed to be a decreasing function so as to guarantee a positive reservation wage. This can be accounted for by the fact that a worker loses part of his qualification by accepting a job demanding less qualification than the one he possesses. We take  $z$  to be a random variable with probability distribution  $B(z, \theta_u)$ . Hence, the expected reservation wage at  $t = 0$  of  $\theta_u$  is,

$$\omega_R(\theta_u) = \frac{\int_0^\infty \int_0^\infty \nu(z+t, \theta_u) dt dB(z, \theta_u)}{\int_0^\infty \rho(t) dt}. \quad (3.1)$$

Throughout the article we take the values of all variables as referred to  $t = 0$ . We assume that unemployment benefits, and hence reservation wages, are increasing in  $\theta_u$ . If a firm requires a qualification level  $\theta_v$ , where  $v$  indexes the firm, and offers to pay  $\omega(\theta_v)$ , an unemployed  $\theta_u$ ,  $\theta_u \geq \theta_v$  with  $\omega_R(\theta_u) \leq \omega(\theta_v)$ , will be willing to work for it and also for any other firm requiring a qualification level in the interval  $[\theta_u^m, \theta_u]$ , where  $\theta_u^m$  is the smallest  $\theta_v$  satisfying  $\omega_R(\theta_u) \leq \omega(\theta_v)$ . The distribution  $B(z, \theta_u)$  is the outcome of all factors that together determine the probability that a given worker might find employment. Hence elements such as the intensity of search on the part of job applicants and the ranking established by firms to select workers affect the shape of  $B(z, \theta_u)$ . Blanchard and Diamond (1994) consider the effect of selecting the worker with the shortest unemployment spell when choosing whom to hire, and find that this rule increases mean unemployment duration. In our context things are complicated by the fact that job searchers have a reservation wage that depends on the shape of  $B(z, \theta_u)$  and this can counteract the effect of such ranking, as the following example shows.

**Example.** Suppose  $B(z, \theta_u)$  is an Erlang distribution, with  $N$  steps, mean  $\alpha$  and density function

$$dB(z, \theta_u) = \frac{(N\alpha)^N z^{N-1} e^{-N\alpha z}}{(N-1)!},$$

where in this example we assume for simplicity that  $\rho(t) = \exp(-t)$  and  $\nu(z + t, \theta_u) = \nu_0(\theta_u) \exp(-\eta(z + t))$ ,  $\eta > 0$ . Then,

$$\omega_R(\theta_u) = \frac{\nu_0(\theta_u)(N\alpha)^N}{\eta(\eta + N\alpha)^N}.$$

When  $N$  is greater than one, the probability that a worker gets out of unemployment is greater the shorter the time spent in it, which is the probabilistic equivalent of the ranking established by Blanchard and Diamond (1994). However the greater is  $N$  the lower is the reservation wage thus inducing workers to search in a wider interval of job qualifications. This might counteract the effect of ranking on mean unemployment duration. In the case of an Erlang distribution this is so, since the mean unemployment duration is a constant  $\alpha^{-1}$ . When  $N = 1$  we have an exponential distribution in which case the worker will have a higher reservation wage than that corresponding to any other  $N$  and will thus only look for a job in a smaller interval of firms than the one he would have looked into if he had faced a distribution  $B(z, \theta_u)$  with larger  $N$ . The opposite ranking of selecting the worker who has spent the longest spell unemployed in preference to younger workers can be modelled with a hyperexponential distribution although we would still have constant mean unemployment duration.

### 3.2. Technology and Labour Demand

We now describe how a typical dormant firm, which we identify by  $\theta_v^0$ , plans all decisions at time  $t = 0$  when facing the job seeking behaviour we have just described. We concentrate first on how it selects the optimal technology with which to come out as an advertised position. We assume that a menu of profitable technologies is available to each dormant option at any time. The particular element in the menu a dormant option  $\theta_v^0$  gets, depends on the time interval  $T_1$  it waits after  $t = 0$ . By waiting longer all dormant options obtain menus with more elements than were available before. Among the elements of any menu the firm selects the one that promises the largest profit and identifies that choice with the minimum level of labour qualification  $\theta_v$  required to operate it. If the value of labour qualification hired is greater than the required minimum, it will increase the value of output. We assume that the cost of waiting is increasing with time and the cost of getting a particular menu is greater the lower the dormant status of the firm.

If an advertised vacancy with qualification  $\theta_v$  offers a wage  $\omega$ , the expected time interval between two consecutive job applications to the same vacancy is assumed



to be a random variable with known mean  $\bar{x} \equiv \bar{x}(\theta_v, \omega, \bar{U}, \bar{v}_a)$ . It seems reasonable to assume that  $\bar{x}$  depends on  $\omega$ , since firms often use wages as incentives for better job applicants who also come at a faster rate than they would if salaries were lower. The dependence of  $\bar{x}$  on  $\bar{U}$  and  $\bar{v}_a$  is justified since these elements represent how tight the market is and hence how easy it is for a firm to find a suitable worker. The potential applicants to this vacancy are all the unemployed with qualification  $\theta_u \in [\theta_v, \theta_v^M(\omega)]$ , where the upper bound of the interval is given by the maximum qualification of all unemployed whose reservation wage is  $\omega_R(\theta_v) \leq \omega$ . Notice that, although  $\theta_v$  is the relative labour qualification of the advertising firm at time  $T_1$ , the mean  $\bar{x}$  does not depend on  $T_1$ , since the distribution of relative qualifications is assumed constant over time. In the next section we give a particular functional form to  $\bar{x}$ , which will help to understand how this term encompasses the supply and demand sides of hired labour.

All firms expecting a positive profit must think that they will get a job applicant having the required minimum qualification. When this happens the firm has to decide whether to hire him or to wait a bit longer in the hope of getting a more highly qualified worker. We assume that although given the technology the firm obtains a greater output per unit of labour the more qualified is that unit, it is costly for the firm to wait since the technology already adopted will become progressively dated. Hence the firm will wait a time interval  $T_2$  after the arrival of the first job applicant only if the benefit of so doing exceeds the cost. When a firm advertises a job and offers a salary  $\omega$ , it faces a pool of potential applicants with a qualification distribution  $F(\theta_u; \theta_v) = \Pr(\Theta_u \leq \theta_u \mid \theta_v \leq \Theta_u \leq \theta_v^M)$ ,  $\theta_v^M \equiv \theta_v^M(\omega)$ . If the arrival times of applicants to a vacancy are independent and exponentially distributed random variables with mean  $\bar{x}$ , we get the following proposition.

**Proposition 1.** *If at time  $T_1$ , the flow of job applicants to a vacancy requiring a minimum qualification  $\theta_v, 0 \leq \theta_v \leq 1$ , and offering a salary  $\omega, \omega \geq \omega_R(\theta_v)$ , is exponentially distributed at rate  $\bar{x}^{-1}$  then the best labour qualification a firm expects to get after waiting an interval  $T_2$  beyond the arrival time of the first suitably qualified applicant is,*

$$\bar{\theta}_u \equiv \bar{\theta}_u(\theta_v, \omega, T_2) = \theta_v^M - \int_{\theta_v}^{\theta_v^M} F(\theta_u; \theta_v) \exp\left(-(1 - F(\theta_u; \theta_v)) \bar{x}^{-1} T_2\right) d\theta_u \quad (3.2)$$

and  $\bar{\theta}_u(\theta_v, \omega, T_2)$  is increasing and concave in  $T_2$ .

**Proof.** See appendix. ■

Note that proposition 1 provides an explicit formula accounting for the expected qualification to be hired from the queue of job applicants to an advertising vacancy  $\theta_v$ . Other articles, such as that of Kahn (1987), do not provide any way to order this queue and simply assume that having more candidates is always more convenient for the firm.

### 3.3. Optimal Entrepreneurial Choice

We now describe the profit maximization problem determining the decisions of each firm regarding  $(\theta_v, T_1, T_2, \omega, S)$ . Firms are identified with their latent qualification level  $\theta_v^0$ . Since we have assumed all vacancies potentially profitable we are assuming a job candidate always arrives at any given firm, and we now say this happens at time  $\bar{x}$ . Let  $T_0$  be the time at which production starts, that is  $T_0 = T_1 + \bar{x} + T_2$ . The revenue of the firm at time  $t = T_0 + \tau$ , net of labour costs, is

$$R(T_0 + \tau) = Y(\theta_v^0, \theta_v, T_1, \bar{\theta}_u(T_2, \bar{x}(\theta_v, \omega), \theta_v^M)) - \omega \sigma^*(\bar{x}(\theta_v, \omega) + T_2) \sigma(\tau) \quad (3.3)$$

where  $Y$  is the output obtained with the technology  $\theta_v$  and the optimal labour qualification  $\bar{\theta}_u$  and the second term on the right hand side represents labour costs. A way to understand  $Y$  is to think of it as the product of three elements,  $Y^0(\theta_v^0)$ ,  $\Psi(\theta_v, T_1, \theta_v^0)$  and  $\Phi(\bar{\theta}_u, \bar{x} + T_2)$ . The first one reflects the output that would have been produced with the old unprofitable technology. The second one represents the extra output that can be produced because the technology has been updated, and the labour qualification used is the minimum required level  $\theta_u = \theta_v$ . The last term reflects the added output because of the mean extra level of labour qualification. Of course these three terms do not have to appear in this multiplicative form and hence we represent  $Y$  in (3.3) as a general function. Note that in choosing  $T_1$  and  $\theta_v$  the firm selects endogenously the technology and the labour qualification to operate it. Both variables are chosen independently although the value of  $\theta_v$  is a relative qualification referred to a time  $T_1$ . We could think of  $T_1$  as the mean time before an innovation occurs and consider it exogenous and a sign of a new firm appearing, as it is often done in the literature, or determine endogenously the time elapsing before the right number of shocks occur before the firm decides to appear in the market. However this would be a simplification of the approach adopted in this article since we do not need to specify the rate at which shocks occur. We can now establish a link with the endogenous growth literature. If we were to give a functional to the second term

$\Psi(\theta_v, T_1, \theta_v^0)$  in the explanation of the output function, aggregate over all firms by integrating over the existing distribution of technologies and take the derivative of the resulting term with respect to time, we would get the endogenous growth rate of the economy.

The wage costs can be understood as follows. At  $T_1$  the firm would pay  $\omega$  for the candidate. But although a job was advertised at  $T_1$ , the hired worker did not turn up till  $T_1 + \bar{x} + T_2$ , by which time the qualification demanded at  $T_1$  had a lower relative value so that it would have cost less to employ that particular worker if the job had been advertised later. The element  $\sigma^*$  reflects the corresponding difference in labour costs. Note that  $\sigma^*$  is increasing in time and hence decreasing in wage since by offering a salary increment the firm faces a lower  $\bar{x}$  and hence has to wait less for job candidates. Finally there is the element  $\sigma(\tau)$  reflecting the increase over time in the cost of human capital. The product of these three elements is the total labour cost. All functions are assumed to be twice differentiable.

The total profit a firm gets if it starts production at  $T_0$  is

$$\Pi(\theta_v^0, \theta_v, T_1, T_2, \omega, S) = \int_0^S R(T_0 + \tau) d\tau - C(\theta_v^0, \theta_v, T_1) \quad (3.4)$$

where  $C(\theta_v^0, \theta_v, T_1)$  is the cost of incorporating the technology. Hence the firm is assumed to solve

$$\text{Max}_{\theta_v, T_1, T_2, \omega, S} \Pi(\theta_v^0, \theta_v, T_1, T_2, \omega, S) \quad (3.5)$$

where  $\theta_v \in [0, 1]$ ,  $T_2 \geq 0$ ,  $T_1 \geq 0$ ,  $S \geq 0$  and  $\omega \geq \omega_R(\theta_v)$ . The following conditions ensure the existence of a solution to this maximization problem. These conditions are

**A1)** All dormant firms are potentially profitable, that is  $\forall \theta_v^0, \exists T_1, \delta > 0$  and  $\varepsilon > 0$  such that

$$R(T_1 + \bar{x} + T_2 + 0) - C(\theta_v^0, \theta_v, T_1) > 0,$$

$\forall T_2 \in [0, \varepsilon)$  and  $\omega \in [\omega_R(\theta_v), \omega_R(\theta_v) + \delta)$  and for some  $\theta_v \in [0, 1]$ .

**A2)** For all  $T_1, T_2, \theta_v$  and  $\omega$  satisfying (A1),  $\lim_{S \rightarrow \infty} R(T_0 + S) < 0$ .

**A3)** For all  $T_1, \theta_v$  and  $\omega$  satisfying (A1),  $\lim_{T_2 \rightarrow \infty} R(T_0 + 0) < 0$ .

**A4)** For all  $T_1, T_2$  and  $\theta_v$  satisfying (A1),  $\lim_{\omega \rightarrow \infty} R(T_0 + 0) < 0$ .

**A5)** For all  $T_2, \theta_v, S$  and  $\omega$  satisfying (A1),  $\lim_{T_1 \rightarrow \infty} \Pi(\theta_v^0, \theta_v, T_1, \bar{\theta}_u, \omega, S) < 0$ .

These conditions guarantee the existence of a solution to (3.5), given by  $(\theta_v^*(\theta_v^0, \bar{U}, \bar{v}_a), T_1^*(\theta_v^0, \bar{U}, \bar{v}_a), T_2^*(\theta_v^0, \bar{U}, \bar{v}_a), \omega^*(\theta_v^0, \bar{U}, \bar{v}_a), S^*(\theta_v^0, \bar{U}, \bar{v}_a))$ . Notice that these optimal values depend on the shape of the unemployment qualification distribution  $F(\theta_u)$  and changing its shape would alter the values of all endogenous variables and among them that of the unemployment figure in the economy. However the assumption of an invariant  $F$  basically implies that, as in the result of Ours and Ridder (1995), the continuous improvement in educational levels does not affect the unemployment level.

There will generally be a multiplicity of solutions (as in Acemoglu (1997)) to this problem due to the substitutability among the variables chosen by the firm. Of course it is always possible to ensure uniqueness by imposing the appropriate conditions on  $Y, \sigma^*, \sigma$  and  $C$  ensuring the strict concavity of  $\Pi$ , but the conditions are messy and we omit them. If an interior optimum exists, it must satisfy the first order conditions,

$$SY_{\theta_v} = \omega \sigma_{\theta_v}^* \int_0^S \sigma(\tau) d\tau + C_{\theta_v} \quad (3.6)$$

$$SY_{T_1} = C_{T_1} \quad (3.7)$$

$$SY_{T_2} = \omega \sigma_{T_2}^* \int_0^S \sigma(\tau) d\tau \quad (3.8)$$

$$SY_{\omega} - \omega \sigma_{\omega}^* \int_0^S \sigma(\tau) d\tau = \sigma^* \int_0^S \sigma(\tau) d\tau \quad (3.9)$$

$$R(T_0 + S) = 0 \quad (3.10)$$

which basically state the equality between the marginal revenue and the marginal cost associated to each variable. It may be worth explaining in some detail the conditions on the optimal salary and job length. Condition (3.9) states that by offering a salary increment, the firm is able to employ a better qualified worker who brings an extra revenue that, together with the time advantage obtained because all candidates arrive sooner, is equal to the marginal increase in wages during the total length of the job. The fifth condition stating that the job ends when the revenue net of labour costs goes to zero also provides the typical condition on the optimal salary  $\omega$  establishing it as a proportion of output,  $\omega = (\sigma^* \sigma(S))^{-1} Y$ . Note also that this value accounts for the outside opportunities of the hired candidate (as in Andolfatto (1996)), since they are reflected in the term  $\bar{x}$  the employer takes as given and also in the upper bound  $\theta_v^M$  of the interval determining the potential job applicants to the vacancy. Note also that  $\sigma_{\theta_v}^*$  in (3.6) can have any

sign. If  $dF(\theta_u)$  is decreasing in  $\theta_u$ , we would have a positive sign indicating that it is costly to attempt to hire a worker with a higher qualification. The opposite would be true if the density function is increasing.

If we now assume that the distribution  $G^0(\theta_v^0)$  of dormant relative technologies at  $t = 0$  is known, and has its support in the unit interval, the mean market values of  $(\theta_v^*, T_1^*, T_2^*, \omega^*, S^*)$  are,

$$\bar{\theta}_v^*(\bar{U}, \bar{v}_a) = \int_0^1 \theta_v^*(\theta_v^0, \bar{U}, \bar{v}_a) dG^0(\theta_v^0) \quad (3.11)$$

$$\bar{T}_1^*(\bar{U}, \bar{v}_a) = \int_0^1 T_1^*(\theta_v^0, \bar{U}, \bar{v}_a) dG^0(\theta_v^0) \quad (3.12)$$

$$\bar{T}_2^*(\bar{U}, \bar{v}_a) = \int_0^1 T_2^*(\theta_v^0, \bar{U}, \bar{v}_a) dG^0(\theta_v^0) + \bar{x}(\bar{\theta}_v^*, \bar{\omega}^*, \bar{U}, \bar{v}_a) \quad (3.13)$$

$$\bar{\omega}^*(\bar{U}, \bar{v}_a) = \int_0^1 \omega^*(\theta_v^0, \bar{U}, \bar{v}_a) dG^0(\theta_v^0) \quad (3.14)$$

$$\bar{S}^*(\bar{U}, \bar{v}_a) = \int_0^1 S^*(\theta_v^0, \bar{U}, \bar{v}_a) dG^0(\theta_v^0). \quad (3.15)$$

For ease of reference to the rates in the queueing network we represent in (3.13) the mean time an advertised vacancy stays open instead of the mean value of  $T_2^*$ . We can now go back to the previous section where we obtained the mean values of unemployment ((2.15) or (2.22)) and advertised vacancies ((2.8) or (2.19)) and, together with (3.12), (3.13) and (3.15), get the rates  $\lambda = 1/\bar{T}_1^*$ ,  $\mu = 1/\bar{T}_2^*$ , the rate at which a vacancy becomes dormant again  $1/\bar{S}^*$  and the equilibrium distributions of unemployment and vacancies. All rates depend on  $L$ ,  $V$  and the forms of all distributions and functions taken as given in the model. From (3.11) and (3.14) we also get the optimal expected labour qualification and wages in steady state. This closes the model. Note that this equilibrium solution is compatible with the existence of any number of firms with distinct initial positions but the same optimal  $\theta_v$  and  $T_1$ , which might pay different salaries to get differently qualified workers since there is a degree of substitutability between improved technology and better qualified labour. This might result in different job durations thus offering an alternative explanation to the analysis by Stein (1997) accounting for the fact that firms differ in their ability to stand strong technological shocks. Note also that once the full equilibrium solution is found, we could take (3.14) and substitute  $\bar{U}$  and  $\bar{v}_a$  for their actual current values and, given the steady state distribution of unemployment and advertised vacancies we have obtained using

the Chapman-Kolmogorov equations, we would get the corresponding stationary distributions of wages, labour qualifications and job durations in the economy, as functions of the levels of unemployment and vacancies at any point in time.

## 4. EXTENSIONS

### 4.1. Open Networks

In section two we used a closed queueing network, with a given number of total vacancies  $V$ , to represent the behaviour of the labour market. This might be thought unduly restrictive since in any economy the number of entrepreneurial options seems to be changing all the time. Our queueing network could be changed into an open system by considering an infinite number of dormant vacancies and a constant rate of advertised ones. This would ensure the ergodicity of the process as long as the destruction rate of advertised vacancies because no job candidate appears,  $\gamma$ , is non zero. If  $\gamma = 0$  the process  $X(t)$  would still be ergodic if  $L > \lambda S$ . The main results would remain qualitatively the same. The matching function and Beveridge curve would be identical to the functional forms given in section two for the case  $L \leq V$ , although the covariance between advertised vacancies and unemployment would always be negative when  $\gamma = 0$ .

### 4.2. Destruction of Vacancies

The possibility of destroying a vacancy because no job applicant has turned up has not been taken into account in the optimization problem described in section 3.3 since a worker is always assumed to have arrived at  $t = T_1 + \bar{x}$ . This could be generalized by substituting  $\bar{x}$  for the corresponding random variable  $\chi$ . In this case the firm's profit should take into account the probability that a job applicant arrives before a time  $T$  when the expected profit of the firm goes to zero, that is the firm would maximize,

$$\begin{aligned} \Pi(\theta_v^0, \theta_v, T_1, T_2, \omega, S) = & \int_0^T \left( \int_0^S R(T_0 + \tau) d\tau - C(\theta_v^0, \theta_v, T_1) \right) h(x \mid \chi < T) dx - \\ & - C(\theta_v^0, \theta_v, T_1) \Pr(\chi > T) \end{aligned}$$

where now  $T_0$  is defined as the random variable  $T_0 = T_1 + \chi + T_2$  and  $h(x \mid \chi < T)$  is the conditional density of interarrival times between job applicants given that someone arrives before the expected benefit goes to zero. In that case the

destruction rate  $\gamma$  would be given by  $\gamma = \Pr(\chi > T)/T$  and correspondingly the matching rate would be  $\mu = \Pr(\chi \leq T)/(\bar{x}_T + T_2)$ , where  $\bar{x}_T$  is the conditional expectation of someone arriving before  $T$ .

The destruction rate  $\gamma$  could be given an alternative interpretation. It could be understood as the rate at which workers change jobs. If an employed worker decides to move to an advertised job, his old job will become dormant and there will be no change in unemployment. The queueing network we used in section two would need no modification to account for this although the functional forms of  $\mu$  and  $\gamma$  would vary to take into account on the job search.

### 4.3. An explicit formula for $\bar{x}$

The function  $\bar{x}$  represents what entrepreneurs think about the searching behaviour of workers. For any given  $\theta_v$ , the interval of possible job applicants to that vacancy can be identified by the qualification interval  $[\theta_v, \theta_v^M]$ . The competition posed by firms close in qualification to  $\theta_v$  also partly determine how many job applicants arrive to the vacancy  $\theta_v$ . This possible competition can be identified by the qualification interval  $[\theta_v^m, \theta_v^M]$ . We have assumed that the wages offered by all firms in the economy are known since that is necessary to obtain  $\theta_v^m$  and we have assumed that each one of them is found optimally given  $\bar{x}$ . This problem can be got round by assuming instead that wages are a fixed proportion  $\beta$  of the output to be produced if the firm would hire a worker with the exact minimum qualification  $\theta_v$ , that is,  $\omega = \beta Y(\theta_v^0, \theta_v, T_1)$  and then find the optimal value of  $\beta$ .

Although in the article we have not modelled searching behaviour explicitly it seems reasonable to assume that  $\bar{x}$  is increasing in the mean number of those vacancies competing with  $\theta_v$  and decreasing in the mean number of job applicants. A plausible formula could be the ratio between the mean numbers of competing advertised vacancies and potential job candidates, namely

$$\bar{x}(\beta, \theta_v, \bar{U}, \bar{v}_a) = \frac{\bar{v}_a \int_{\theta_v^m}^{\theta_v^M} dG(\theta_v)}{\bar{U} \int_{\theta_v}^{\theta_v^M} dF(\theta_v)}$$

where  $G(\theta_v)$  is the qualification distribution of advertised firms. If it were to be the case that  $F$  and  $G$  are both uniform distributions with support in  $[0, 1]$ , then  $\bar{x} = \bar{v}_a(\theta_v^M - \theta_v^m)/\bar{U}(\theta_v^M - \theta_v)$ .

## CONCLUSIONS

Queueing theory can encompass any idea about how the economic relationships among agents determine the times they spend at each pool, the transition rates among them and the preferences about whom to select from each one. This allows us to use a probabilistic matching function providing a truly stochastic Beveridge curve which permits a more thorough understanding of the labour market than it is possible with analyses based on deterministic relationships. We have used the idea of optimal waiting to determine how firms decide when to appear in the market, how long to stay in business and what kind of labour to employ when the steady state encompasses a variety of efficient technologies and labour qualifications. In this article we have endogeneized the waiting times associated to all choices the firm makes and have incorporated this information within a general queueing model thus obtaining, with the aid of the Chapman-Kolmogorov equations, the steady state distributions of unemployment and advertised vacancies which in turn provide the corresponding market distributions of optimal labour qualifications, wages, technology and job durations.

Some of the possible extensions of this article have been pointed out in the last section. But there remain other natural extensions connecting labour market analyses with the endogenous growth literature. The most immediate one would relate the choice of educational levels on the part of the unemployed with the efficiency levels of firms. This could be taken into account within a queueing network by incorporating educational pools and then assuming that entrepreneurial preferences about whom to hire first order the queues of the unemployed according to educational levels. A more complicated analysis would take into account the possibility that the transition rates between pools could depend on the times spent at each pool by the agents involved, an analysis that would require the use of non Markovian systems. This would be mathematically tractable although the equilibrium solution would be analytically quite complicated and a numerical analysis might be in order.

## APPENDIX

**Proof of proposition 2.1.** The Chapman-Kolmogorov equations in (2.4) can be easily obtained with the following transition diagram. At a given state  $(k, n, m)$ ,  $1 < k, n, m < V$ ,  $m = V - k - n$ , the possible transitions in and out of this state



are

$$\begin{array}{llll}
(k, n+1, m-1) \longrightarrow (n+1)/S \longrightarrow & \longrightarrow n/S \longrightarrow (k, n-1, m+1) \\
(k-1, n, m+1) \longrightarrow (m+1)\lambda \longrightarrow & \longrightarrow k\gamma \longrightarrow (k-1, n, m+1) \\
(k+1, n-1, m) \longrightarrow (k+1)\mu \longrightarrow & \longrightarrow m\lambda \longrightarrow (k+1, n, m-1) \\
(k+1, n, m-1) \longrightarrow (k+1)\gamma \longrightarrow & \longrightarrow k\mu \longrightarrow (k-1, n+1, m)
\end{array}$$

Diagram 1

The transition rates are zero if the state from which the system comes (or where it arrives to) does not belong to  $E$ . We identify  $(k, n, V-k-n)$  with  $(k, n)$ . Taking derivatives in (2.7) with respect to  $z$  at the point  $z = y = 1$  and with respect to  $y$  at the same point we get

$$\mu SP'_z(1, 1) = P'_y(1, 1)$$

$$\lambda VP(1, 1) - (\lambda z + \mu + \gamma)P'_z(1, 1) - \lambda P'_y(1, 1) = 0.$$

Note that  $P(1, 1) = 1$ ,  $P'_z(1, 1) = \bar{v}_a$  and  $P'_y(1, 1) = \bar{v}_f$ . Then, the first order moments follow. Differentiating twice in  $zz$ ,  $yy$  and  $zy$  at the point  $z = y = 1$  we get

$$\lambda P'_z(1, 1) + (\lambda + \mu + \gamma)P''_{zz}(1, 1) + \lambda P''_{zy}(1, 1) = \lambda VP'_z(1, 1)$$

$$S^{-1}P''_{yy}(1, 1) = \mu P''_{zy}(1, 1)$$

$$(\lambda + \mu + \gamma)P''_{zy}(1, 1) + \lambda P'_y(1, 1) + (\lambda + S^{-1})P''_{yy}(1, 1) = \mu P''_{zz}(1, 1) + \lambda VP'_y(1, 1).$$

Given that,  $P''_{zz}(1, 1) = E[v_a^2] - \bar{v}_a$ ,  $P''_{yy}(1, 1) = E[v_f^2] - \bar{v}_f$  and  $P''_{zy}(1, 1) = E[v_a v_f]$ , the second order moments and correlation coefficient follow easily. ■

**Proof of proposition 2.2.** The Chapman-Kolmogorov equations can be obtained as in above proposition but taking into account the term (2.17). In this case the moment generating function is

$$P(z, y) = \sum_{k=0}^{k=V-L} \sum_{n=0}^{n=L} p_{kn} z^k y^n + \sum_{k=V-L+1}^{k=V} \sum_{n=0}^{n=V-k} p_{kn} z^k y^n.$$

In this case the Chapman-Kolmogorov equations given in (2.4) and including the new term (2.17), become

$$\begin{aligned}
& \lambda V P(z, y) + ((y - 1) S^{-1} + \lambda y (z - 1)) P'_y(z, y) + \\
& + ((\gamma + \lambda z) (z - 1) + \mu (z - y)) P'_z(z, y) = \\
& \lambda V z^V (z - 1) p_{V0} + \mu y^L (z - y) \sum_{k=1}^{k=V-L} k p_{k,L} z^{k-1}
\end{aligned}$$

Proceeding as we did in proposition 2.1, the results follow. ■

**Proof of proposition 3.1.** Let  $\Theta_0$  be the qualification level of the first applicant. Denote by  $\Theta_i$  the qualification of the  $i$ th applicant arriving after the first one did. Let  $\Theta_n^M$  be the qualification level hired by the firm if it gets  $n$  applicants after the first one. Hence,  $\Theta_n^M = \max\{\Theta_0, \Theta_1, \dots, \Theta_n\}$ . Since  $\Theta_0, \Theta_1, \dots, \Theta_n$  are independent and identically distributed random variables, the mean labour qualification hired by an advertising vacancy at  $T_1$  when, demanding a minimum labour qualification  $\theta_v$ , offers a salary  $\omega$  and gets  $n$  applicants, is

$$\bar{\theta}_n(\theta_v, \omega) = \theta_v^M - \int_{\theta_v}^{\theta_v^M} (F(\theta_u; \theta_v))^{n+1} d\theta_u.$$

Since job applicants arrive following a Poisson distribution at rate  $\bar{x}^{-1}$ , the qualification a firm expects to get from the best applicant if it waits  $T_2$  after the first arrival is

$$\bar{\theta}_u(\theta_v, T_2, \omega) = \theta_v^M - \int_{\theta_v}^{\theta_v^M} F(\theta_u; \theta_v) \exp\{-(1 - F(\theta_u; \theta_v))\bar{x}^{-1}T_2\} d\theta_u.$$

Taking derivatives with respect to  $T_2$  this function is shown to be increasing and concave. ■

## REFERENCES

- Acemoglu, D. (1997). "Training and Innovation in an Imperfect Labour Market", *Review of Economic Studies* 64, 445-464.  
Aghion, P. and Howitt, P. (1992). "A Model of Growth through Creative Destruction", *Econometrica* 60, 323-351.

- Aghion, P. and Howitt, P. (1994). "Growth and Unemployment", *Review of Economic Studies* 61, 477-494.
- Andolfatto, D. (1996). "Business Cycles and Labor-Market Search", *The American Economic Review* 86, 112-132.
- Asmussen, S. (1987). *Applied Probability and Queues*, Wiley.
- Bean, C. and Pissarides, C. (1993). "Unemployment, Consumption and Growth", *European Economic Review* 37, 837-859.
- Blanchard, O.J. and Diamond, P. (1994). "Ranking, Unemployment Duration and Wages", *Review of Economic Studies* 61, 417-434.
- Eicher, T.S. (1996). "Interaction between Endogenous Human Capital and Technological Change", *Review of Economic Studies* 63, 127-144.
- Kahn, L.M. (1987). "Unemployment Insurance, Job Queues, and Systematic Job Search: An Equilibrium Approach", *Southern Economic Journal* 54, 397-411.
- Laing, D., Palivos, T. and Wang, P. (1995). "Learning, Matching and Growth", *Review of Economic Studies* 62, 115-129.
- Ours, J.C. van and Ridder, G. (1995). "Job Matching and Job Competition: Are Lower Educated Workers at the Back of Job Queues?", *European Economic Review* 39, 1717-1731.
- Stein, J.C. (1997). "Waves of Creative Destruction: Firm-Specific Learning-by-Doing and the Dynamics of Innovation", *Review of Economic Studies* 64, 265-288.