

ON THE SET OF BOUNDED LINEAR OPERATORS TRANS-
FORMING A CERTAIN SEQUENCE OF A HILBERT SPACE
INTO AN ABSOLUTELY SUMMABLE ONE

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Let \mathcal{H} be a real, separable Hilbert space, \mathcal{B} the set of bounded linear operators on \mathcal{H} , and $S = \{a_n \mid n \in N\}$ a fixed sequence in \mathcal{H} ; we shall denote by

$$C_S = \left\{ A \in \mathcal{B} \mid \sum_{n=1}^{\infty} \|Aa_n\| < \infty \right\}. \text{ Obviously } C_S \neq \phi, \forall S \in \mathcal{H}.$$

It is straightforward to check that C_S is a left ideal. We will prove that it never is a proper bilateral ideal, using the following result of Calkin [1]:

Every proper bilateral ideal in the ring of bounded operators of a Hilbert space contains the ideal \mathfrak{S}_0 of finite rank operators and is contained in the ideal of the completely continuous ones, \mathfrak{S} .

Lemma 1. *It is a necessary and sufficient condition for $\mathfrak{S}_0 \subset C_S$ that S be a summable sequence.*

Proof. Let $S = \{a_n \mid n \in N\}$. S is weakly summable if and only if

$\forall x \in \mathcal{H}, \sum_{n=1}^{\infty} |(a_n, x)| < \infty$, ((\cdot, \cdot) denotes scalar product). In the Hilbert space \mathcal{H} a sequence is summable iff it is weakly summable. By the assumption $\mathfrak{S}_0 \subset C_S$ we have all the uni-dimensional projectors, $P_{w(x)}$, $x \in \mathcal{H} - \{0\}$, included in C_S and hence

$$\forall x \in \mathcal{H} - \{0\}, P_{w(x)} \in C_S \Leftrightarrow \sum_{n=1}^{\infty} \|P_{w(x)}(a_n)\| < \infty,$$

$$\forall x \in \mathcal{H} - \{0\} \Leftrightarrow \sum_{n=1}^{\infty} |(a_n, x)| < \infty, \quad \forall x \in \mathcal{H}.$$

Thus we also see that whenever S is summable, C_S contains the uni-dimensional rank operators and from this it is easy to prove that it contains the whole set of finite rank operators.

Recalling that the Hilbert – Schmidt operators on \mathcal{H} coincide with the absolutely p -summing ones, $1 \leq p < \infty$, ([4]), one has that there exists no sequence S for which $C_S = \mathfrak{S}_0$. In fact, if $C_S \supset \mathfrak{S}_0$ then $C_S \supset \mathfrak{S}_2$, ideal of the Hilbert – Schmidt operators, since every $A \in \mathfrak{S}_2$ transforms a summable sequence into an absolutely summable one, that is $\sum_{n=1}^{\infty} \|Aa_n\| < \infty$, which is to say $A \in C_S$. Nevertheless one can find sequences S such that C_S consists only of finite rank operators, but not of all of them though.

So far we have proved that if a sequence $S \subset \mathcal{H}$ is not summable, C_S cannot be a proper bilateral ideal. Let us now see that for S summable C_S is not proper bilateral either.

Theorem 1. *Let $S = \{a_n \mid n \in N\}$ be summable. C_S then contains a non completely continuous operator.*

For the proof of this theorem we prove first the following

Lemma 2. *If S is summable, for every $\epsilon > 0$ there exists a natural number μ such that*

$$|(a_{p_1} + \dots + a_{p_s}, x)| < \epsilon,$$

with $\mu \leq p_1 < \dots < p_s$ and $x \in \mathcal{H}$ only restricted by $\|x\| = 1$.

Proof. Suppose that there is an $\epsilon > 0$ such that $\forall m, \exists x_m, \mu_m > m$ verifying $|(a_{\mu_m} + \dots + a_{\mu_m + s_m}, x_m)| \geq \epsilon$. We take m_1, m_2, \dots such that $m_{i+1} > \mu_{m_i} + s_{m_i}$, and construct the subsequence of S

$$a_{\mu_{m_1}}, \dots, a_{\mu_{m_1} + s_{m_1}}, a_{\mu_{m_2}}, \dots, a_{\mu_{m_2} + s_{m_2}}, \dots$$

for which the corresponding series does not verify the Cauchy condition and therefore is non-summable. This clearly contradicts $\{a_n \mid n \in N\}$ summable. ■

Proof of the theorem. If $\{a_n \mid n \in N\}$ spans a finite dimensional subspace of \mathcal{H} , then $\{a_n \mid n \in N\}$ summable is equivalent to $\{a_n \mid n \in N\}$ absolutely summable, and $C_S = \mathcal{B}$.

Let us now suppose that the closed linear hull $[a_1, \dots, a_n, \dots]$ is not finite dimensional, and consider in \mathcal{H} a complete orthonormal system $\{e_n \mid n \in N\}$; let $a_n = \sum_{j=1}^{\infty} a_{nj} e_j$. The series

$$\sum_n |(a_n, e_j)| = \sum_n |a_{nj}|, \quad (j = 1, 2, \dots)$$

converge uniformly following the lemma just proved.

If $\eta > 0$ denotes any fixed real number, for $\frac{\eta}{2}$ there exists $\nu(\frac{\eta}{2})$ such that $\sum_{n=\nu}^{\infty} |a_{nj}| < \frac{\eta}{2}$ ($\forall j \in N$). Since for each n , $a_{nj} \rightarrow 0$ ($j \rightarrow \infty$), for large enough j we can obtain

$$\sum_{n=1}^{\nu-1} |a_{nj}| < \frac{\eta}{2}$$

and so

$$\sum_{n=1}^{\infty} |a_{nj}| < \eta,$$

thus

$$\sum_{n=1}^{\infty} |a_{nj}| \rightarrow 0 \quad (j \rightarrow \infty).$$

Let us denote $\theta_j = \sum_n |a_{nj}|$, $\theta_j \rightarrow 0$ ($j \rightarrow \infty$), and choose θ_{l_j} so that $\sum_j \theta_{l_j} < \infty$, that is

$$\sum_{n, l_j} |a_{nl_j}| < \infty.$$

Construct the coordinate subspace $[e_{l_1}, e_{l_2}, \dots, e_{l_j}, \dots]$ corresponding to the sequence $\{l_j | j \in N\}$, and denote by P the orthogonal projection on it. We have,

$$Pa_n = a_{nl_1} e_{l_1} + \dots + a_{nl_j} e_{l_j} + \dots \quad (j = 1, 2, \dots)$$

$$\|Pa_n\| \leq |a_{nl_1}| + \dots + |a_{nl_j}| + \dots \quad (j = 1, 2, \dots)$$

thus

$$\sum_n \|Pa_n\| \leq \sum_{n, l_j} |a_{nl_j}| < \infty.$$

So we see that P transforms S into $S' = \{Pa_n | n \in N\}$ which is absolutely summable, and therefore $P \in C_S$, being P a non completely continuous operator. ■

The existence of a non completely continuous operator in C_S implies the existence of infinitely many of them.

This theorem shows that when we impose C_S to contain the whole set of finite rank operators, we obtain a very strong condition on S , namely that it must be summable, and because of this fact C_S contains "too many" operators, and it cannot be included in the ideal set of the completely continuous ones \mathfrak{S} , proving that C_S is never a proper bilateral ideal.

However C_S can be the whole set \mathcal{B} . Indeed, it is so iff $S = \{a_n | n \in N\}$ is absolutely summable. The other extreme case, $C_S = \{0\}$, 0 the null operator, is also possible and we consider it now. We quote first the following result [3]:

"Let J designate a left ideal in \mathcal{B} . If J does not include projectors, then $J = \{0\}$ ".

Then it is easy to prove that whenever $C_S \neq \{0\}$, it includes an uni-dimensional projector; hence a *necessary and sufficient condition* for $C_S \neq \{0\}$ is the existence of a ray $r \subset \mathcal{H}$ such that $\sum_n \|P_r a_n\| < \infty$, or equivalently:

$$C_S = \{0\} \Leftrightarrow \sum |(a_n, x)| = \infty, \quad \forall x \in \mathcal{H} - \{0\}.$$

A geometrical consequence of this is that for $C_S = \{0\}$ the nucleus of S , $N(S) = \bigcap_1^\infty [a_n, \dots]$ must be the whole \mathcal{H} hence in particular $[S] = \mathcal{H}$.

$N(S) \neq \mathcal{H}$ would imply the existence of a natural number p such that $E = [a_p, a_{p+1}, \dots] \neq \mathcal{H}$, consequently $P_{x \in E}$ would belong to C_S , and there would exist uni-dimensional projectors in C_S .

We give an instance of a sequence S for which $C_S = \{0\}$. Let $\{r_n \mid n \in N\}$ be a complete system of rays in \mathcal{H} , $[r_1, \dots, r_n, \dots] = \mathcal{H}$. If we choose

$$a_1^{(n)}, \dots, a_m^{(n)}, \dots$$

in r_n , $r_n = w(a_m^{(n)})$ ($n, m \in N$) such that

$$\|a_m^{(n)}\| \geq k_n > 0 \quad (n \in N),$$

then $S = \{a_m^{(n)} \mid n, m \in N\}$ verifies $C_S = \{0\}$.

We can even impose conditions on this sequence S so that it becomes a L -system, (that is, the image of an orthonormal basis by a bounded linear operator) with $C_S = \{0\}$. This would be the case if

$$\sum_{n,m=1}^\infty \|a_m^{(n)}\|^2 < \infty \quad \text{and} \quad \sum_{m=1}^\infty \|a_m^{(n)}\| = \infty \quad (n \in N).$$

If we project S on any ray r of \mathcal{H} , we have $\sum_{m=1}^\infty \|P_r a_m^{(p)}\| = \infty$ for a certain $r_p \in \{r_n \mid n \in N\}$ and therefore

$$\sum_{n,m} |(a_m^{(n)}, x)| = \infty, \quad \forall x \in \mathcal{H} - \{0\},$$

hence $C_S = \{0\}$.

We observe the fact that whenever we have $C_S = \{0\}$ for a given S , we also have $C_{S'} = \{0\}$ for any $S' \supset S$, and this suggests to search the "minimal systems", S , for which $C_S = \{0\}$. We point out that any heterogonal in direction system, S , – even if it is complete – gives $C_S \neq \{0\}$, since it verifies $N(S) = \{0\} \neq \mathcal{H}$.

However the join S of two, or more, heterogonal systems can give $C_S = \{0\}$. To see this, choose an orthonormal complete system of \mathcal{H} , $\{e_n \mid n \in N\}$. Let $S' = \{b_n \mid n \in N\}$ be a L -system such that $C_{S'} = \{0\}$ and such that $\{e_n + b_n \mid n \in N\}$ is heterogonal. Then, for

$$S = \{e_n, e_n + b_n \mid n \in N\}$$

we have $C_S = \{0\}$.

Indeed, had we

$$\sum_n |(e_n, x_0)| + \sum_n |(e_n, x_0) + (b_n, x_0)| < \infty$$

for a certain $x_0 \in \mathcal{H} - \{0\}$, then we would have $\sum_n |(b_n, x_0)| < \infty$ contradicting $C_{S'} = \{0\}$.

We summarize the above results in the following:

Proposition. C_S can never be a bilateral ideal unless either

(i) it is the whole \mathcal{B} . For this we need a very strong summability condition on S , namely S must be absolutely summable or

(ii) C_S is the zero ideal. For this we need a very strong condition of non-summability on S , i.e. for every $x \in \mathcal{H} - \{0\}$, $\{(a_n, x) \mid n \in N\}$ must not be absolutely summable.

Finally we show that C_S can not contain the ideal \mathfrak{S} of the completely continuous operators, except when C_S equals \mathcal{B} , that is, when S is absolutely summable.

Theorem 2. Let $S = \{a_n \mid n \in N\}$ be such that $\sum_n \|a_n\| = \infty$. Then there exists a completely continuous operator C such that

$$\sum_{n=1}^{\infty} \|Ca_n\| = \infty.$$

To prove this we give the following

Lemma 2. Let $\sum_{n=1}^{\infty} p_n$ be a numerical divergent series of positive terms. Then there exists a sequence $\{q_n \mid n \in N\}$ verifying

$$q_1 \geq q_2 \geq \dots \geq q_n \geq \dots > 0, \quad q_n \rightarrow 0 \quad (n \rightarrow \infty)$$

such that

$$\sum_{n=1}^{\infty} p_n q_n = \infty.$$

Proof of the theorem. Let us refer \mathcal{H} to an orthonormal basis $\{e_n \mid n \in N\}$. Let $a_n = \sum_{j=1}^{\infty} a_{nj} e_j$. By the assumption made above we have

$$\sum_{n=1}^{\infty} \sqrt{a_{n1}^2 + \dots + a_{nj}^2 + \dots} = \infty.$$

We are trying to construct a completely continuous diagonal operator C ,

$$C = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ 0 & & & & \ddots \end{pmatrix}, \quad \lambda_j \rightarrow 0 \quad (j \rightarrow \infty),$$

such that

$$\sum_{n=1}^{\infty} \|Ca_n\| = \sum_{n=1}^{\infty} \sqrt{a_{n1}^2 \lambda_1^2 + \dots + a_{nj}^2 \lambda_j^2 + \dots} = \infty.$$

One can find natural numbers ν_n ($n \in N$) with the condition that

$$\sum_{n=1}^{\infty} \sqrt{a_{n1}^2 + \dots + a_{nv_n}^2} = \infty$$

and we can additionally suppose that $v_n < v_{n+1}$ ($n \in N$).

Lemma 2 guarantees the existence of a sequence $\mu_1 \geq \mu_2 \geq \dots$
 $\dots \geq \mu_n \geq \dots > 0$, $\mu_n \rightarrow 0$ ($n \rightarrow \infty$) such that

$$\sum_{n=1}^{\infty} \mu_n \sqrt{a_{n1}^2 + \dots + a_{nv_n}^2} = \infty.$$

Let us take

$$\begin{aligned} \lambda_1 = \dots = \lambda_{v_1} = \mu_1, \lambda_{v_1+1} = \dots = \lambda_{v_2} = \mu_2, \dots \\ \dots, \lambda_{v_{n-1}+1} = \dots = \lambda_{v_n} = \mu_n, \dots \end{aligned}$$

As

$$\sqrt{a_{n1}^2 \lambda_1^2 + \dots + a_{nv_n}^2 \lambda_{v_n}^2} \geq \mu_n \sqrt{a_{n1}^2 + \dots + a_{nv_n}^2},$$

we have

$$\sum_{n=1}^{\infty} \sqrt{\lambda_1^2 a_{n1}^2 + \dots + \lambda_{v_n}^2 a_{nv_n}^2} = \infty,$$

and consequently

$$\sum_{n=1}^{\infty} \sqrt{\lambda_1^2 a_{n1}^2 + \dots + \lambda_{v_n}^2 a_{nv_n}^2 + \dots} = \infty. \blacksquare$$

Thus we can establish that if a sequence S is such that $\forall C \in \mathfrak{S}$,
 $\sum_{n=1}^{\infty} \|Ca_n\| < \infty$, then necessarily $\sum_{n=1}^{\infty} \|a_n\| < \infty$, and $C_S = \mathfrak{B}$.

This theorem, in a certain sense, can be considered as dual of the one due to Gohberg and Markus [2] which asserts *that for every bounded operator $A \in \mathfrak{B}$, $A \neq 0$, there exists an orthonormal base $\{e_n \mid n \in N\}$ such that*

$$\sum_{n=1}^{\infty} \|Ae_n\| = \infty.$$

We have obtained that for every non absolutely summable sequence – hence in particular for all the orthonormal bases – there exists not only a bounded operator, but a completely continuous one, C , such that $\sum_n \|Ca_n\| = \infty$.

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