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ON RATIONAL CUSPIDAL CURVES, OPEN SURFACES AND LOCAL SINGULARITIES

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1. Introduction

Let C be an irreducible projective plane curve in the complex projective space $\mathbb{P}^2$. The classification of such curves, up to the action of the automorphism group $PGL(3,\mathbb{C})$ on $\mathbb{P}^2$, is a very difficult open problem with many interesting connections. The main goal is to determine, for a given d, whether there exists a projective plane curve of degree d having a fixed number of singularities of given topological type. In this note we are mainly interested in the case when C is a rational curve.

The problem remains very difficult even if we aim much less, e.g. the determination of the maximal number of cusps among all the rational cuspidal plane curves (a problem proposed by F. Sakai in [14]), — this number is expected to be small. In [53] K. Tono recently proved that it is less than 9; the maximal number of cusps known by the authors is 4; and, in fact, it is expected to be 4. The referee pointed out to us that in mid-90-s S. Orevkov found a bound bigger than 8 but he never published his result.

This remarkable problem of classification is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces, and in the classical algebraic geometry.

For instance, the open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve. On the other hand, regarding these surfaces, Flenner and Zaidenberg in [8] formulated the *rigidity conjecture*. This says that every $\mathbb{Q}$-acyclic affine surfaces Y with logarithmic Kodaira dimension $\bar{\kappa}(Y)=2$ must be rigid. This conjecture for $Y=\mathbb{P}^2\setminus C$ would imply the projective rigidity of the curve C in the sense that every equisingular deformation of C in $\mathbb{P}^2$ would be projectively equivalent to C. (Notice that if C has at least three cusps then $\bar{\kappa}(\mathbb{P}^2\setminus C)=2$ by [59]; and, in fact, all curves with $\bar{\kappa}(\mathbb{P}^2\setminus C)<2$ are classified, see below.) Many known examples support the rigidity conjecture, see [7, 8, 9, 10]. Zaidenberg in [14] also conjectured that the set of shapes of the Eisenbud-Neumann splice diagrams is finite for all $\mathbb{Q}$-acyclic affine surfaces Y with $\bar{\kappa}(Y)=2$ (this is stronger than the existence of a uniform bound for the number of cusps for rational cuspidal curves).

Another related, very old, famous open problem has its roots in early algebraic geometry, and wears the name of Coolidge and Nagata, see [3, 30]. It predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line.

Key words and phrases. Cuspidal rational plane curves, logarithmic Kodaira dimension, Nagata-Coolidge problem, Flenner-Zaidenberg rigidity conjecture, surface singularities, $\mathbb{Q}$-homology spheres, Seiberg-Witten invariant, graded roots, Heegaard Floer homology, Ozsváth-Szabó invariants.

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The aim of this article is to present some of these conjectures and related problems, and to complete them with some results and new conjectures from the recent work of the authors.

In section 3 we present the Nagata-Coolidge problem. The main theme of section 4 is Orevkov's conjecture [41], which formulates an inequality involving the degree d and numerical invariants of local singularities. In a different formulation, this is equivalent with the positivity of the virtual dimension of the space of curves with fixed degree and certain local type of singularities which can be geometrically realized. (This was used, as a 'first test', by the second author to check that some singularities might be realized or not.) The equivalence of the two inequalities is proved via some properties of the numerical, local and global deformation invariants; this material is presented in sections 2 and 3.

Section 5 starts with some classification results: the classification of projective plane curves with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ by the work of Kashiwara, Lin, Miyanishi, Sugie, Tsunoda, Tono, Wakabayashi, Yoshihara, Zaidenberg (among others). Also, we recall the classification (of the authors) of rational unicuspidal curves whose cusp has one Puiseux pair. The end of this section deals with the *rigidity conjecture* of Flenner and Zaidenberg.

In section 6 we present the author's 'compatibility property' (a sequence of inequalities), conjecturally satisfied by the degree and local invariants of the singularities of a rational cuspidal curve. It turns out that in the unicuspidal case, the inequalities are true if and only if they are (in fact) equalities. One of the reformulations (valid in the unicuspidal case) of this conjectural series of identities is the *semigroup distribution property*, which is a very precise compatibility property connecting the semigroup of the local singularity and the degree of the curve. We also present one of its equivalent statements suggested to us by Campillo. Section 7 explains the relation of the semigroup distribution property (for the unicuspidal case) with the 'Seiberg-Witten invariant conjecture' (of the forth author and Nicolaescu [36]), which basically leads us to this compatibility property. The last section present another connection with the Seiberg-Witten theory (based on the articles [32, 33, 34]), but now exploiting the relation with the Heegaard-Floer homology (introduced and studied by Ozsváth and Szabó [43]). Here, a crucial intermediate object is the 'graded root' (introduced by the forth author [32]), which provides a completely different interpretation of the semigroup distribution property.

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2. Local data

One of our main goals is to characterise the local embedded topological types of local singular germs $(C, p_i) \subset (\mathbb{P}^2, p_i)$ which can be realized by a projective plane curve C of degree d. (Here, the points $\{p_i\}$ are not fixed.) In this section we collect some results about the (deformation of) local invariants (C, p_i).

2.1. Let (C, p) be the germ at p of a reduced curve $C \subset \mathbb{P}^2$ and let $f \in \mathcal{O}_{\mathbb{P}^2,p}$ be a function defining the singularity (C, p) in some local coordinates x and y.

2.1.1. Let $\phi : \mathcal{C}_{(C,p)} \to \mathcal{S}_{(C,p)}$ be a *semi-universal deformation* of (C,p). The base space $\mathcal{S}_{(C,p)}$ is smooth and its tangent space is isomorphic to the vector space $\mathcal{O}_{C,p}/(f_x, f_y)$. In particular its dimension is equal to the *Tjurina number* $\tau(C,p) := \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^2,p}/I^{ea}(C,p))$, where $I^{ea}(C,p)$ is the ideal (f, f_x, f_y) (see [24].)

2.1.2. Let $\mathcal{S}^{es}_{(C,p)} \subset \mathcal{S}_{(C,p)}$ be the smooth subgerm of a *semi-universal equisingular deformation* of (C,p), see [62], Theorem 7.4. Let $T_\varepsilon := Spec(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ be the base space of *first order infinitesimal* deformations. The *equisingularity ideal* of (C,p) is the ideal

$$I^{es}(C,p) = \{g \in \mathcal{O}_{\mathbb{P}^2,p} \quad | \quad f + \varepsilon g \text{ is equisingular over } T_\varepsilon\}.$$

It contains $I^{ea}(C,p)$. Moreover the tangent space of $\mathcal{S}^{es}_{(C,p)} \subset \mathcal{S}_{(C,p)}$ is isomorphic to the vector space $I^{es}(C,p)/I^{ea}(C,p)$. In particular the codimension of $\mathcal{S}^{es}_{(C,p)}$ in $\mathcal{S}_{(C,p)}$ is the topological invariant

$$\tau^{es}(C,p) := \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^2,p}/I^{es}(C,p)). \tag{1}$$

The equisingular stratum $\mathcal{S}^{es}_{(C,p)}$ coincides with the μ-*constant stratum* of $\mathcal{S}_{(C,p)}$, where $\mu = \mu(C,p)$ is the Milnor number of (C,p).

2.1.3. Let $n : \mathcal{O}_{C,p} \hookrightarrow \tilde{\mathcal{O}}_{C,p} := \prod_{i=1}^{r} \mathbb{C}\{t\}$ be the normalisation map, where r is the number of local irreducible components of (C,p). Let $\delta(C,p)$ be the dimension of the cokernel $\tilde{\mathcal{O}}_{C,p}/n(\mathcal{O}_{C,p})$. Let $\mathrm{cond}(\mathcal{O}_{C,p})$ be the conductor ideal of $\mathcal{O}_{C,p}$, that is the annihilator of the $\mathcal{O}_{C,p}$-module $\tilde{\mathcal{O}}_{C,p}/n(\mathcal{O}_{C,p})$.

The δ-*constant stratum* $\mathcal{S}^{\delta}_{(C,p)} \subset \mathcal{S}_{(C,p)}$ is the locus of points where δ is constant, see [4, 48]. In general, $\mathcal{S}^{\delta}_{(C,p)}$ is not longer smooth at C, but it is locally irreducible. Even more, the tangent cone of $\mathcal{S}^{\delta}_{(C,p)}$ at C is always a linear space which is identified with the vector space $\mathrm{cond}(\mathcal{O}_{C,p})/I^{ea}(C,p)$, [4], Theorem 4.15. In particular, the codimension of $\mathcal{S}^{\delta}_{(C,p)}$ in $\mathcal{S}_{(C,p)}$ is equal to

$$\mathrm{codim}\,(\mathcal{S}^{\delta}_{(C,p)} \subset \mathcal{S}_{(C,p)}) = \delta(C,p). \tag{2}$$

2.1.4. S. Diaz and J.Harris [4] showed the inclusion of ideals

$$I^{ea}(C,p) \subset I^{es}(C,p) \subset \mathrm{cond}(\mathcal{O}_{C,p}) \subset \mathcal{O}_{C,p}.$$

Since the equisingular stratum $\mathcal{S}^{es}_{(C,p)}$ is contained in the δ-constant stratum $\mathcal{S}^{\delta}_{(C,p)}$ then

$$\bar{M}(C,p) := \mathrm{codim}\,(\mathcal{S}^{es}_{(C,p)} \subset \mathcal{S}^{\delta}_{(C,p)}) = \tau^{es}(C,p) - \delta(C,p) \geq 0. \tag{3}$$

In fact, $\bar{M}(C,p)$ is a topological invariant of the singularity (see next paragraph). S. Orevkov and M. Zaidenberg used $\bar{M}(C,p)$ in a different situation, cf. [41, 42].

2.2. The minimal embedded resolution of (C,p) is obtained via a sequence of blowing-ups at infinitely near points to p. The topological numerical invariants $\delta(C,p), \mu(C,p), \tau^{es}(C,p)$ and $\bar{M}(C,p)$ can be written in terms of the multiplicity sequence $[m^p_1, \dots, m^p_{k_p}]$ of (C,p) associated with this minimal resolution. This is the set of multiplicities of the strict transform of C at all the infinitely near points along all the branches. Let r_p denote the number of branches of C at p. (We do not omit the 1's; for this notation we follow [27] and [8].)

Using Milnor's formula $\mu(C,p) = 2\delta(C,p) - r_p + 1$ (see e.g. Theorem 6.5.9 in [61]) one gets

$$\mu(C,p) + r_p - 1 = 2\delta(C,p) = \sum_{i=1}^{k_p} m^p_i(m^p_i - 1). \tag{4}$$

A blow-up in the minimal good resolution is called *inner* (or *subdivisional*) if its center is at the intersection point of two exceptional curves of the resolution process (such a center is called a *satellite point* in [60]). If the center is situated on exactly one exceptional divisor, then the blow-up is called *outer* (or *sprouting*). Notice that the first blow-up is neither inner nor outer. A center is called a *free infinitely near point* if it is either outer or it is p, the center of the very first blow up. It is convenient to count this first infinitely near point by two. Let ω_p, resp. ρ_p, denote the number of inner, respectively of outer, blow-ups. Then $k_p - 1 = \omega_p + \rho_p$ and the number of free infinitely near points is $L_p := 2 + \rho_p = k_p - \omega_p + 1$.

C.T.C. Wall in Theorem 8.1 of [60] proved the following formula for $\tau^{es}(C,p)$ (see also Proposition 11.5.8 in [61])

$$\tau^{es}(C,p) = \sum_{i=1}^{k_p} \frac{(m_i^p - 1)(m_i^p + 2)}{2} + \omega_p - 1 = \sum_{i=1}^{k_p} \frac{m_i^p(m_i^p + 1)}{2} - L_p. \tag{5}$$

Different proofs of this formula have been also given by J.F. Mattei [28], E. Shustin [46] or Theo de Jong [20].

The parametric codimension is equal to

$$\bar{M}(C,p) = \sum_{i=1}^{k_p} (m_i^p - 1) + \omega_p - 1 = -L_p + \sum_{i=1}^{k_p} m_i^p. \tag{7}$$

For other equivalent formulae of $\bar{M}(C,p)$ see [42].

3. Global data

3.1. Let $\mathbb{P}^N$, where $N = d(d+3)/2$, be the Hilbert scheme parametrising algebraic projective plane curves of degree d. The locus $V_{d,g}$ in $\mathbb{P}^N$ of reduced and irreducible curves of degree d and genus g is irreducible (see e.g. [16]). The locus of reduced and irreducible curves of degree d and genus g having only nodes as singularities is smooth of dimension $3d - 1 + g$ and is dense in $V_{d,g}$.

Recall that our goals is to characterise the local embedded topological types of local singular germs $(C, p_i) \subset (\mathbb{P}^2, p_i)$ which can be realized by a projective curve C of degree d. Assume that a projective reduced plane curve C of degree d exists with fixed topological types $S_1, \ldots, S_\nu$. Let $V(S_1, \ldots, S_\nu)$ denote the locally closed subscheme of reduced curves on $\mathbb{P}^2$ having exactly ν-singularities with topological types $S_1, \ldots, S_\nu$.

Greuel and Lossen in Theorem 3.6 of [13] proved that the dimension of $V(S_1, \ldots, S_\nu)$ at C satisfies

$$\dim(V(S_1, \ldots, S_\nu), C) \geq C^2 + 1 - p_a(C) - \tau^{es}(C) = \frac{d(d+3)}{2} - \tau^{es}(C), \tag{8}$$

where $\tau^{es}(C) := \sum_{i=1}^{\nu} \tau^{es}(C, p_i)$. The right hand side

$$\operatorname{expdim}(V(S_1, \ldots, S_\nu), C) := \frac{d(d+3)}{2} - \tau^{es}(C)$$

is called *expected dimension* of $V(S_1, \ldots, S_\nu)$ at C. The study of the locally closed subscheme $V(S_1, \ldots, S_\nu)$ of $\mathbb{P}^N$ and its properties have been studied intensively in the literature (see for instance works by Artal Bartolo, Diaz, Greuel, Harris, Karras, Lossen, Shustin, Tannenbaum, Wahl, Zariski among many others).

3.2. **The action of $PGL(3,\mathbb{C})$.** Consider a reduced curve C of degree d in $\mathbb{P}^2$ and the action of the group $PGL(3) := PGL(3,\mathbb{C})$ on the space $\mathbb{P}^N$, which parametrises plane curves of degree d. The orbit of a curve C is a quasi-projective variety of dimension $\dim PGL(3) - \dim \operatorname{Stab}_{PGL(3)}(C)$. For a general curve, the dimension of the orbit is 8, that is, its stabiliser is 0-dimensional.

Since the topological types $S_1, \dots, S_\nu$ of singularities of C and the degree d of C remain constant under the $PGL(3)$-action, we consider the *virtual dimension* of $V(S_1, \dots, S_\nu)$ at C defined by

$$\operatorname{virtdim}(V(S_1,\dots,S_\nu),C) := \operatorname{expdim}(V(S_1,\dots,S_\nu),C) - (8 - \dim \operatorname{Stab}_{PGL(3)}(C)). \quad (9)$$

Curves with small orbits have been studied and classified by P. Allufi and C. Faber in [1]. According to this classification, C is always a configuration of rational curves. Moreover, C consists of irreducible components of the form below, with arbitrary multiplicities. We reproduce here their list together with the dimension of the stabiliser $\operatorname{Stab}_{PGL(3)}(C)$.

(1) C consists of a single line; $\dim \operatorname{Stab}_{PGL(3)}(C) = 6$.
(2) C consists of 2 (distinct) lines; $\dim \operatorname{Stab}_{PGL(3)}(C) = 4$.
(3) C consists of 3 or more concurrent lines; $\dim \operatorname{Stab}_{PGL(3)}(C) = 3$.
(4) C is a triangle (consisting of 3 lines in general position); $\dim \operatorname{Stab}_{PGL(3)}(C) = 2$.
(5) C consists of 3 or more concurrent lines, together with 1 other (non-concurrent) line; $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.
(6) C consists of a single conic; $\dim \operatorname{Stab}_{PGL(3)}(C) = 3$.
(7) C consists of a conic and a tangent line; $\dim \operatorname{Stab}_{PGL(3)}(C) = 2$.
(8) C consists of a conic and 2 (distinct) tangent lines; $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.
(9) C consists of a conic and a transversal line and may contain either one of the tangent lines at the 2 points of intersection or both of them; $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.
(10) C consists of 2 or more bitangent conics (conics in the pencil $y^2 + \lambda xz$) and may contain the line y through the two points of intersection as well as the lines x and/or z, tangent lines to the conics at the points of intersection; again, $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.
(11) C consists of 1 or more (irreducible) curves from the pencil $y^b + \lambda z^a x^{b-a}$, with $b \geq 3$, and may contain the lines x and/or y and/or z; $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.
(12) C contains 2 or more conics from a pencil through a conic and a double tangent line; it may also contain that tangent line. In this case, $\dim \operatorname{Stab}_{PGL(3)}(C) = 1$.

3.3. **The Coolidge-Nagata conjecture.** Let C_1 and C_2 be two curves in $\mathbb{P}^2$. The pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are *birationally equivalent* if there exist a birational map $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the proper image of C_1 by f coincides with C_2. Traditionally, the map σ is called a *Cremona transformation.*

Let $\pi : V \to \mathbb{P}^2$ be an embedded resolution of singularities of C, let $\bar{C}$ be the strict transform of C by π and let K_V be the canonical divisor of V. One can show that $h^0(m(K_V + \bar{C}))$ and the Kodaira dimension $\kappa(K_V + \bar{C}, V)$ are birational invariants of C as plane curves (see [19], [22]). Thus, one defines $\kappa[C] := \kappa(K_V + \bar{C}, V)$.

Let D be the reduced total preimage of C by the embedded resolution π. One can also show that $h^0(m(K_V + D))$ and the Kodaira dimension $\kappa(K_V + D, V)$ are birational invariants of the open surface $Y := \mathbb{P}^2 \setminus C$ (see [19]). Thus one defines the logarithmic Kodaira dimension $\bar{\kappa}(Y) := \kappa(K_V + D, V)$ of Y.

One of the open problems regarding projective plane curves is the following famous *Coolidge-Nagata problem/conjecture*, cf. [3] and [30]: *every rational cuspidal curve can be transformed by a Cremona transformation into a straight line.*

In [3], J.I. Coolidge proved:

3.4. **Theorem.** *A rational curve can be transformed into a straight line by a Cremona transformation if and only if all the conditions for special adjoints of any index are incompatible.*

A *special adjoint of index* m is an effective divisor in the complete linear system $|mK_V + \bar{C}|$. In fact one can check that $\kappa[C] = -\infty$ if and only if C has no special adjoints. One has the following criterium due to N.M. Kumar and M.P. Murthy (cf. Corollary 2.4 in [22]), cf. also with Iitaka (Proposition 12 in [19]):

3.5. **Theorem.** *Let C be an irreducible rational plane curve. The following conditions are equivalent:*

a) *the curve C can be transformed into a straight line by a Cremona transformation,*
b) $\kappa[C] = -\infty$,
c) $|2K_V + \bar{C}| = \emptyset$,
d) $|2(K_V + \bar{C})| = \emptyset$.

In [22] N.M. Kumar and M.P. Murthy also showed that a sufficient condition is $\bar{C}^2 \geq -3$.

The Nagata-Coolidge problem has been solved for cuspidal rational plane curves with logarithmic Kodaira dimension of the complement $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ (using the classification listed in 5.2) and for all known curves with $\bar{\kappa} = 2$ (see below, cf. also with 5.9).

4. On rational curves

From now on we are interested in rational projective plane curves. Let C be a reduced rational projective plane curve of degree d in the complex projective plane with singular points $\{p_j\}_{j=1}^{\nu}$. Let S_j be the topological type of the singularity (C, p_j).

The main theme of this section is Orevkov's conjecture [41]. Since this conjecture is not true for a small number of curves (with positive dimensional stabiliser $Stab_{PGL(3)}(C)$, all of them with $\bar{\kappa}(Y) < 2$), we correct the original version by adding a contribution provided by this stabiliser; and we also present some equivalent reformulations.

4.1. **Virtual dimension conjecture.** *The virtual dimension* $\operatorname{virtdim}(V(S_1, \ldots, S_\nu), C)$ *of reduced rational projective plane curve is non-negative:*

$$\operatorname{expdim}(V(S_1, \ldots, S_\nu), C) - 8 + \dim \operatorname{Stab}_{PGL(3)}(C)) \geq 0. \tag{10}$$

This version was also conjectured independently by I. Luengo.

Since C is rational, if the multiplicity sequence of the singularity (C, p) is $[m_1^p, \ldots, m_k^p]$, then (the genus-formula reads as):

$$(d-1)(d-2) = \sum_{p \in \operatorname{Sing}(C)} \sum_{i=1}^{k} m_i^p (m_i^p - 1) = \sum_{p \in \operatorname{Sing}(C)} \left(\sum_{i=1}^{k} (m_i^p)^2 - \sum_{i=1}^{k} m_i^p \right). \tag{11}$$

Eliminating d^2 from (9) and (5), one gets

$$\text{virtual dim} = 3d - 9 - \sum_{p\in \mathrm{Sing}(C)} \sum_{i=1}^{k} m_i^p + \sum_{P\in \mathrm{Sing}(C)} L_p + \dim \mathrm{Stab}_{PGL(3)}(C).$$

Substituting (7) in this equality, one gets

$$\text{virtual dim} = 3d - 9 - \sum_{p\in \mathrm{Sing}(C)} \bar{M}(C,p) + \dim \mathrm{Stab}_{PGL(3)}(C). \quad (12)$$

In particular, (10) is equivalent to

$$3d - 9 - \sum_{p\in \mathrm{Sing}(C)} \bar{M}(C,p) + \dim \mathrm{Stab}_{PGL(3)}(C) \geq 0. \quad (13)$$

4.2. **Orevkov's conjecture.** S. Orevkov in [41] conjectured the following inequality.

For a rational cuspidal curve

$$\sum_{p\in \mathrm{Sing}(C)} \bar{M}(C,p) \leq 3d - 9. \quad (14)$$

Orevkov's conjecture is stated for any rational curve (without restrictions). In such a case, the sum is over all irreducible branches at each singular point. Note that (14) is not true for the curve C defined by $x^2y + z^3 = 0$ since $\bar{M}(C,p) = 1$ at its singular point. Nevertheless, $\dim \mathrm{Stab}_{PGL(3)}(C) = 1$ and (10) and (13) hold. We believe that the correct statement of the conjecture is (10), or equivalently (13). Nevertheless, one can prove (cf. Lemma 5.3) that for irreducible, cuspidal, rational plane curve with with $\bar{\kappa}(Y) = 2$ the statements of (10) and (14) are equivalent.

4.3. **The $\bar{C}^2$–Conjecture.** Let $\pi : V \to \mathbb{P}^2$ be the minimal good embedded resolution of $C \subset \mathbb{P}^2$, and let $\bar{C}$ be the strict transform of C and $D = \pi^{-1}(C)$ be the reduced preimage of C as above. One of the integers which plays a special role in the classification problem is the self-intersection of $\bar{C}$ in V. It equals

$$\bar{C}^2 = d^2 - \sum_{p\in \mathrm{Sing}(C)} \sum_{i=1}^{k} (m_i^p)^2. \quad (15)$$

From (10) and (15) one gets

$$3d = \bar{C}^2 + 2 + \sum_{p\in \mathrm{Sing}(C)} \sum_{i=1}^{k} m_i^p.$$

Thus, via (12):

$$\text{virtual dim} = \bar{C}^2 - 7 + \sum_{p\in \mathrm{Sing}(C)} L_p + \dim \mathrm{Stab}_{PGL(3)}(C). \quad (16)$$

Therefore, inequality (10) is equivalent to:

$$\bar{C}^2 - 7 + \sum_{p\in \mathrm{Sing}(C)} L_p + \dim \mathrm{Stab}_{PGL(3)}(C) \geq 0. \quad (17)$$

5. Cuspidal rational curves and the rigidity conjecture.

Let C be an irreducible curve of degree d in the complex projective plane. One of the main invariants of such curves is the logarithmic Kodaira dimension $\bar{\kappa} = \bar{\kappa}(Y)$, where $Y := \mathbb{P}^2 \setminus C$. The following result of Wakabayashi [59] is crucial in the classification procedure.

5.1. **Theorem.** [59] *Let C be an irreducible curve of degree d in $\mathbb{P}^2$.*

(1) *If $g(C) \geq 1$ and $d \geq 4$ then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$.*
(2) *If $g(C) = 0$ and C has at least 3 cuspidal points then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$.*
(3) *If $g(C) = 0$ and C has at least 2 singular points and one of them is locally reducible then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$.*
(4) *If $g(C) = 0$ and C has 2 cuspidal points then $\bar{\kappa}(\mathbb{P}^2 \setminus C) \geq 0$.*

5.2. Recall that the open surface $Y := \mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve. From now on we assume that C is a rational cuspidal plane curve of degree d, in particular C is irreducible. Let $(C, p_i)_{i=1}^{\nu}$ be the collection of local plane curve singularities, all of them locally irreducible.

(a) If $\bar{\kappa} = -\infty$ then $\nu = 1$ by [59]. Moreover, all these curves are classified by M. Miyanishi and T. Sugie [29] (see also H. Kashiwara [21]). The family contains as an important subfamily the Abhyankar-Moh-Suzuki curves (see [11]).

(b) The case $\bar{\kappa} = 0$ cannot occur by a result of Sh. Tsunoda [54], see also Orevkov's article [41].

(c) If $\bar{\kappa} = 1$ then by the above result of Wakabayashi [59] one has $\nu \leq 2$. In the case $\nu = 1$, K. Tono provides the possible equations of the curves [52]. (Notice that Tsunoda's classification in [55] is incomplete.). On the other hand, by another result of Tono [51], the case $\nu = 2$ corresponds exactly to the Lin-Zaidenberg bicuspidal rational plane curves.

5.3. **Lemma.** *If C is an irreducible, cuspidal, rational plane curve with with $\bar{\kappa}(Y) = 2$ then* $\dim Stab_{PGL(3)}(C) = 0$.

Proof. According with the above classification of Aluffi and Faber, C is a rational cuspidal plane curve with $\dim \mathrm{Stab}_{PGL(3)}(C) > 0$ if and only if C is a generic member of the pencil $y^d + \lambda z^a x^{d-a}$ with $d \geq 3$ and $(d, a) = 1$. If $a = d - 1$ then C is of Abhyankar-Moh-Suzuki type (with $\bar{\kappa} = -\infty$). Otherwise $a \neq d-1$ and C is of Lin-Zaidenberg type (with $\bar{\kappa} = 1$). □

5.4. From a different point of view, one can classify triples (d, a, b) such that there exists a unicuspidal rational plane curve C of degree d whose singularity has only one Puiseux pair of type (a, b), where $1 < a < b$. Let $\{\varphi_j\}_{j\geq 0}$ denote the Fibonacci numbers $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_{j+2} = \varphi_{j+1} + \varphi_j$.

5.5. **Theorem.** [12] *The Puiseux pair (a, b) can be realized by a unicuspidal rational curve of degree d if and only if (d, a, b) appears in the following list.*

(a) $(a, b) = (d - 1, d)$*;*
(b) $(a, b) = (d/2, 2d - 1)$*, where d is even;*
(c) $(a, b) = (\varphi_{j-2}^2, \varphi_j^2)$ *and* $d = \varphi_{j-1}^2 + 1 = \varphi_{j-2}\varphi_j$*, where j is odd and ≥ 5;*
(d) $(a, b) = (\varphi_{j-2}, \varphi_{j+2})$ *and* $d = \varphi_j$*, where j is odd and ≥ 5;*
(e) $(a, b) = (\varphi_4, \varphi_8 + 1) = (3, 22)$ *and* $d = \varphi_6 = 8$*;*
(f) $(a, b) = (2\varphi_4, 2\varphi_8 + 1) = (6, 43)$ *and* $d = 2\varphi_6 = 16$.

In the first four cases $\bar{\kappa} = -\infty$ and they can be realized by some particular curves which appear in Kashiwara's classification [21]. The last two sporadic cases have $\bar{\kappa} = 2$ and were found by Orevkov and Artal-Bartolo, cf. [41].

5.6. Another aproach is to classify rational cuspidal curves C such that the highest multiplicity of the singular points m is close to the degree d. Flenner and Zaidenberg classified the curves with $m = d - 2$ in [9] and $m = d - 3$ in [10]. The case $m = d - 4$ is partially solved by Fenske [7]. Note that m can not be too small because in [27] it is proved that $d < 3m$ solving a conjecture of Yoshihara [63]. Let $\alpha = \frac{(3+\sqrt{5})}{2}$, Orevkov [41] gives two families of curves with $\alpha m < d$ and conjectured that those families gives the only curves verifyng $\alpha m < d$.

5.7. **The rigidity conjecture of Flenner and Zaidenberg.** Let Y be an $\mathbb{Q}$-acyclic affine surface, and fix one of its 'minimal logarithmic compactifications' (V, D). This means that V is a smooth projective surface with a normal crossing divisor D, such that $Y = V \setminus D$, and (V, D) is minimal with these properties.

The sheaf of the logarithmic tangent vectors $\Theta_V\langle D\rangle$ controls the deformation theory of the pair (V, D), cf. [8]. E.g., $H^0(V, \Theta_V\langle D\rangle)$ is the set of infinitesimal automorphisms, $H^1(V, \Theta_V\langle D\rangle)$ is the space of infinitesimal deformations, and $H^2(V, \Theta_V\langle D\rangle)$ is the space of obstructions. Iitaka showed in [17] that if $\bar{\kappa}(Y) = 2$ then the automorphism group of the surface Y is finite (this also provides a different proof of 5.3). Therefore $h^0(\Theta_V\langle D\rangle) = 0$. In [8, 9, 66] Flenner and Zaidenberg conjectured the following

Rigidity conjecture: *Every $\mathbb{Q}$-acyclic affine surfaces Y with logarithmic Kodaira dimension $\bar{\kappa}(Y) = 2$ is rigid and has unobstructed deformations. That is,*

$$h^1(\Theta_V\langle D\rangle) = 0 \qquad \text{and} \qquad h^2(\Theta_V\langle D\rangle) = 0. \tag{18}$$

In particular, the Euler characteristic $\chi(\Theta_V\langle D\rangle) = h^2(\Theta_V\langle D\rangle) - h^1(\Theta_V\langle D\rangle)$ must vanish.

In [8], [9] and [10] the conjecture was verified for most of the known examples. In [8] unobstructedness was proved for all $\mathbb{Q}$-acyclic surfaces of non log-general type. In [65] it is proved that a rigid rational cuspidal curve has at most 9 cusps.

This can be applied in our situation as follows. Consider a projective curve C, and write $Y := \mathbb{P}^2 \setminus C$. The $\mathbb{Q}$-acyclicity of Y is equivalent to the fact that C is rational and cuspidal. For V one can take the minimal embedded resolution of the pair $(\mathbb{P}^2, C)$.

The conjecture for $Y = \mathbb{P}^2 \setminus C$ implies the projective rigidity of the curve C. This means that every equisingular deformation of C in $\mathbb{P}^2$ would be projectively equivalent to C. Thus $V(S_1, \ldots, S_\nu)$ has expected dimension 8 (see Section 4).

In Corollary 2.5 of [9], Flenner and Zaidenberg show that for any cuspidal rational plane curve

$$\chi(\Theta_V\langle D\rangle) = K_V(K_V + D) = -3(d-3) + \sum_{p \in \mathrm{Sing}(C)} \bar{M}(C, p). \tag{19}$$

By (12) and Lemma 5.3 then

$$\text{virtual dim} = -\chi(\Theta_V\langle D\rangle). \tag{20}$$

The vanishing of $\chi(\Theta_V\langle D\rangle)$ implies any of the equivalent equalities (10), (13) or (17). On the other hand, if (10), (13) or (17) hold, then

$$\chi(\Theta_V\langle D\rangle) \leq 0. \tag{21}$$

5.8. **Proposition.** [Tono] *For cuspidal rational plane curves with* $\bar{\kappa} = 2$ *the following inequality holds*

$$\chi(\Theta_V\langle D\rangle) \geq 0. \tag{22}$$

Proof. (22) follows from the article [53] of K. Tono in the following way. F. Sakai in [45] introduce the invariant $\gamma_2 := h^0(2K_V + D)$. Lemma 4.1 in [53] states that if the pair (V, D) satisfies the following three conditions (for details see [loc. cit.])

- (A1) $\bar{\kappa}(V \setminus D) = 2$,
- (A2) (V, D) is almost minimal, and
- (A3) D contains neither a rod consisting of (-2)-curves nor a fork consisting of (-2)-curves,

then

$$\gamma_2 = K_V(K_V + D) + \frac{D(D + K_V)}{2} + \chi(\mathcal{O}_V).$$

(The main point here is that by a vanishing theorem $h^1(2K_V + D) = 0$, by an easy argument $h^2(2K_V + D) = 0$ too, hence $\gamma_2 = \chi(2K_V + D)$ can be computed by Riemann-Roch.)

One can check that in our case the minimal embedded resolution satisfies these conditions. Moreover, $\chi(\mathcal{O}_V) = 1$ and (since D is a rational tree, the adjunction formula implies) $K_V D + D^2 = -2$. Thus $\gamma_2 = K_V(K_V + D)$. Therefore, via (19), one has:

$$\chi(\Theta_V\langle D\rangle) = h^0(2K_V + D) \geq 0.$$

□

5.9. **Corollary.** *Let C be an irreducible, cuspidal, rational plane curve with $\bar{\kappa}(\mathbb{P}^2 - C) = 2$. The following conditions are equivalent:*

- (i) $\chi(\Theta_V\langle D\rangle) = 0$,
- (ii) *virtdim*$(V(S_1, \ldots, S_\nu), C) \geq 0$, *i.e.* (10) *holds, where S_j is the topological type of the corresponding uni-branch singularity (C, p_j).*
- (iii) $\chi(\Theta_V\langle D\rangle) \leq 0$.

In such a case, the curve C can be transformed by a Cremona transformation of $\mathbb{P}^2$ into a straight line (i.e., the Coolidge-Nagata problem has a positive answer).

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ follows from (20) and (21). $(iii) \Rightarrow (i)$ follows from 5.8 and (21). Finally, the characterisation 3.5 shows that C can be transform into a straight line by a Cremona transformation. Indeed, $h^0(2K_V + D) = \chi(\Theta_V\langle D\rangle) = 0$, but $\mathcal{O}_V(2K_V + \bar{C})$ is a subsheaf of $\mathcal{O}_V(2K_V + D)$, hence $h^0(2K_V + \bar{C}) = 0$ as well. □

6. The semigroup distribution property

6.1. The characterisation problem of the realization of prescribed topological types of singularities has a long and rich history providing many interesting compatibility properties connecting local invariants of the germs $\{(C, p_i)\}_i$ with some global invariants of C — like its degree, or the log-Kodaira dimension of $\mathbb{P}^2 \setminus C$, etc. (For a — non-complete — list of some of these restrictions, see e.g. [11, 12].)

In [11] we proposed a new compatibility property — valid for rational cuspidal curves C. Its formulation is surprisingly very elementary. Consider a collection $(C, p_i)_{i=1}^{\nu}$ of locally irreducible plane curve singularities (i.e. cusps), let $\Delta_i(t)$ be the characteristic polynomial of the monodromy action associated with (C, p_i), and $\Delta(t) := \prod_i \Delta_i(t)$. Its degree is 2δ, where δ is the sum of the delta-invariants $\delta(C, p_i)$ of the singular points. Then $\Delta(t)$ can be written as $1+(t-1)\delta+(t-1)^2Q(t)$ for some polynomial $Q(t)$. Let c_l be the coefficient of $t^{(d-3-l)d}$ in $Q(t)$ for any $l=0,\dots,d-3$.

6.2. **Conjecture A.** [11] *Let $(C,p_i)_{i=1}^{\nu}$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta=(d-1)(d-2)$ for some integer d. If $(C,p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve then*

$$c_l \leq (l+1)(l+2)/2 \ \ \textit{for all} \ \ l=0,\dots,d-3.$$

In fact, the integers $n_l := c_l-(l+1)(l+2)/2$ are symmetric: $n_l=n_{d-3-l}$; and $n_0=n_{d-3}=0$. We also mention that examples with strict inequality occur, cf. [11] (in all these examples known by the authors $\nu>1$).

The main result of [11] is :

6.3. **Theorem.** [11] If $\bar{\kappa}(\mathbb{P}^2\setminus C)$ *is* ≤ 1*, then the above conjecture A is true (in fact with* $n_l=0$*).*

There is an additional surprising phenomenon in the above conjecture. Namely, in the *unicuspidal* case one can show the following.

6.4. **Proposition.** [11] *If* $\nu=1$ *then* $c_l\geq(l+1)(l+2)/2$ *for* $0\leq l\leq d-3$.

Therefore, conjecture 6.2 in this case can be reformulated as follows:

6.5. **Conjecture B1.** *With the notations of 6.2, if* $\nu=1$*, then* $n_l=0$ *for all* $l=0,\dots,d-3$*, that is*

$$c_l=(l+1)(l+2)/2 \ \ \textit{for all} \ \ l=0,\dots,d-3.$$

In fact, if $\nu=1$, we can do more. Recall that the characteristic polynomial Δ of $(C,p)\subset(\mathbb{P}^2,p)$ is a complete (embedded) topological invariant of this germ, similarly as the semigroup $\Gamma_{(C,p)}\subset\mathbb{N}$. In the next discussion we will replace Δ by $\Gamma_{(C,p)}$. Recall that the semigroup $\Gamma_{(C,p)}\subset\mathbb{N}$ consists of all possible intersection multiplicities $I_p(C,h)$ at the point p for all $h\in\mathcal{O}_{(\mathbb{C}^2,p)}$.

Hence, one can reformulate conjecture B1 in terms of the semigroup of the germ (C,p) and the degree d. It turns out that the of vanishing of the coefficients n_l (as in B1.) is replaced by a very precise and mysterious distribution of the elements of the semigroup with respect to the intervals $I_l:=(\,(l-1)d,ld\,]$:

6.6. **Conjecture B2.** *Assume that* $\nu=1$*. Then for any* $l>0$*, the interval* I_l *contains exactly* $\min\{l+1,d\}$ *elements from the semigroup* $\Gamma_{(C,p)}$.

In other words, for every rational unicuspidal plane curve C of degree d, the above conjecture is equivalent to the identity

$$D(t)\equiv 0, \qquad (DP)$$

where:

$$D(t):=\sum_{k\in\Gamma_{(C,p)}} t^{\lceil k/d\rceil}-\Big(1+2t+\cdots+(d-1)t^{d-2}+d(t^{d-1}+t^d+t^{d+1}+\cdots)\Big).$$

For the equivalences of conjectures B1 and B2, see 7.5. Here we only mention a key relation between the coefficients c_l and the semigroup $\Gamma_{(C,p)}$.

First, consider the identity (cf. [15]) $\Delta(t) = (1-t)\cdot L(t)$, where $L(t) = \sum_{k\in\Gamma_{(C,p)}} t^k$ is the Poincaré series of $\Gamma_{(C,p)}$. Write $\Delta(t) = 1 - P(t)(1-t)$ for some polynomial $P(t)$, then $L(t) + P(t) = 1/(1-t) = \sum_{k\geq 0} t^k$. In particular, $P(t) = \sum_{k\in\mathbb{N}\setminus\Gamma_{(C,p)}} t^k$. Then

$$Q(t) = \frac{P(t)-\delta}{t-1} = \sum_{k\notin\Gamma_{(C,p)}} \frac{t^k-1}{t-1} = \sum_{k\notin\Gamma_{(C,p)}} (1+t+\cdots+t^{k-1}).$$

Hence $c_l = \#\{k\notin\Gamma_{(C,p)} : k > (d-3-l)d\}$. Since $k\in\Gamma_{(C,0)}$ if and only if $\mu-1-k\notin\Gamma_{(C,p)}$ for any $0\leq k\leq \mu-1$, one gets $c_l = \#\{k\in\Gamma_{(C,p)}\,;\, k\leq ld\}$.

6.7. The following equivalent formulation was suggested by A. Campillo.

6.8. **Theorem.** *Let C be a unicuspidal rational plane curve of degree d. The curve C satisfies the semigroup compatibility property (DP) (i.e. conjectures B1 and/or B2) if and only if the elements of the semigroup $\Gamma_{(C,p)}$ in $[0, ld]$ are realized by projective (possibly non-reduced) curves of degree l for $l\leq d-3$.*

Proof. The proof of the 'if' part is easy. For the 'only if' part fix a projective coordinate system $[X:Y:Z]$ such that the affine chart $Z\neq 0$ contains the singular point p. Let V be the vector space of polynomials of degree l in variables $(X/Z, Y/Z)$. Its dimension is $N := (l+1)(l+2)/2$, which, in fact, equals the number of elements of the semigroup in the interval $[0, ld]$. Denote these elements by $0 = s_1, ..., s_N$, ordered in an increasing way.

Consider the decreasing filtration of vector spaces $V_1\supset V_2\supset\cdots\supset V_N$, defined by

$$V_i := \{f\in V : I_p(C,f)\geq s_i\}.$$

First, we verify that $\dim(V_i/V_{i+1})$ is at most 1. Indeed, assume that $I_p(C,f_i) = I_p(C,f_2) = I$. Let $n : (\mathbb{C},0)\to(C,p)$ be the normalisation of (C,p), and write $f_i\circ n(t) = a_i t^I + \cdots$ with $a_i\neq 0$, for $i = 1$ and 2. Then $I_p(C, a_2f_1 - a_1f_2) > I$. Since there is no semigroup element between s_i and s_{i+1}, the inequality $\dim(V_i/V_{i+1})\leq 1$ follows.

Next, notice that to prove the theorem it is enough to show that each dimension $\dim(V_i/V_{i+1})$ is exactly 1.

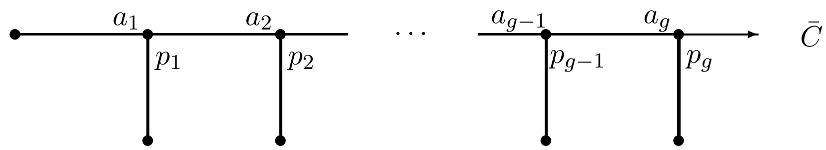
But, if $\dim(V_i/V_{i+1}) = 0$ for some i then $\dim(V_N)$ is at least 2. Since for any $f\in V_N$ one has $I_p(C,f)\geq s_N$, $\dim(V_N)\geq 2$ would imply (by similar argument as above) the existence of an $f\in V_N$ with $I_p(C,f) > s_N$. Since $I_p(C,f)$ is an element of the semigroup and the last element of the semigroup in the interval $[0,ld]$ is s_N, we get that $I_p(C,f) > ld$, which contradicts the irreducibility of C by Bézout Theorem. □

6.9. **A counterexample to an 'extended' version.** In [11] we formulated the following conjecture, as an extension of the conjecture B2. to an 'if and only if' statement.

6.10. **'Conjecture' C.** *The local topological type $(C,p)\subset(\mathbb{P}^2,p)$ can be realized by a degree d unicuspidal rational curve if and only if the property (DP) is valid.*

In the sequel we present a counterexample to the 'if' part (i.e. to the 'extension').

If the germ $(C,0)$ has g Newton pairs $\{(p_k,q_k)\}_{k=1}^g$ with $\gcd(p_k,q_k) = 1$, $p_k\geq 2$ and $q_k\geq 1$ (and by convention, $q_1 > p_1$), define the integers $\{a_k\}_{k=1}^g$ by $a_1 := q_1$ and $a_{k+1} := q_{k+1} + p_{k+1}p_ka_k$ for $k\geq 1$. Then its Eisenbud-Neumann splice diagram decorated by the numerical data $\{(p_k,a_k)\}_{k=1}^g$ has the following shape [5]:



Consider now the local singularity whose Eisenbud-Neumann splice diagram is decorated by two pairs $(p_1, a_1) = (2, 7)$ and $(p_2, a_2) = (4, 73)$. A local equation for such singularity can be $(x^2 - y^7)^4 + x^{33}y = 0$. Its multiplicity sequence is $[8_3, 4_6, 1_4]$. A minimal set of generators of its semigroup $\Gamma_{(C,p)}$ is given by $\langle 8, 28, 73\rangle$. Its Milnor number is $16 \cdot 15$, hence a possible unicuspidal plane curve C of degree 17 might exist with such local singularity. Moreover the distribution property (DP) of the semigroup is also satisfied. Nevertheless, such a curve C does not exist. To prove this, one can either use Cremona transformations to transform C into another curve for which one sees that it does not exist, or one uses Varchenko's semi-continuity criterium for the spectrum of the singularity [57, 58]. Here we will follow the second argument.

The spectrum of the irreducible singularity $(C, 0)$ can be computed from the Newton pairs of the singularity. The forth author provided such a formula in [31]. It is convenient to consider the spectrum $Sp(C, 0) = \sum_r n_r(r)$ as an element of $\mathbb{Z}[\mathbb{Q} \cap (0, 2)]$. We write $Sp_{(0,1)}(C, 0)$ for the collection of spectral elements situated in the interval $(0, 1)$.

6.10.1. **Theorem.** *If the irreducible germ $(C, 0)$ has g Newton pairs $\{(p_k, q_k)\}_{k=1}^g$ then*

$$Sp_{(0,1)}(C, 0) = \sum_{k=1}^{g} S_k \quad \textit{where} \quad S_k = \sum \left(\frac{i/a_k + j/p_k + t}{p_{k+1}p_{k+2}\cdots p_g} \right),$$

where the second sum is over $0 < i < a_k$, $0 < j < p_k$, $i/a_k + j/p_k < 1$ and $0 \leq t \leq p_{k+1}p_{k+2}\cdots p_g - 1$ (if $k = g$ then $S_g = \sum(l/a_g + k/p_g)$ where the sum is over $0 < l < a_g$, $0 < k < p_g$, $l/a_g + k/p_g < 1$).

If the local singular type $\{(C, p)\}$ can be realized by a degree d plane curve C, then (C, p) is in the deformation of the 'universal' plane germ $(U, 0) := (x^d + y^d, 0)$. In particular, the collection of all spectral numbers $Sp(C, p)$ of the local plane curve singularity (C, p) satisfies the semi-continuity property compared with the spectral numbers of $(U, 0)$ for any interval $(\alpha, \alpha+1)$. Since the spectral numbers of $(U, 0)$ are of type l/d, the semi-continuity property for intervals $(-1 + l/d, l/d)$ $(l = 2, 3, \ldots, d - 1)$ reads as follows:

$$\#\{\alpha \in Sp(C, p) \ : \ \alpha < l/d\} \leq (l - 2)(l - 1)/2. \tag{23}$$

In our case, for $d = 17$ and $l = 12$, using Theorem 6.10.1 we get

$$\#\{\alpha \in Sp(C, p) \ : \ \alpha < 12/17\} - (12 - 2)(12 - 1)/2 = 1,$$

which contradicts (23). Thus the rational unicuspidal plane curve C of degree 17 with such singularity cannot exist.

Thus, in the realization problem, the above case $(p_1, a_1; p_2, a_2; d)$ cannot be eliminated by the semigroup distribution property (DP), but it can be eliminated by the semi-continuity of the spectrum. However it is not true that the semi-continuity implies (DP). For a more precise discussion see [11].

7. The semigroup compatibility property and surface singularities

7.1. **Superisolated singularities.** The theory of normal surface singularities (in fact, of isolated hypersurface surface singularities) 'contains' in a canonical way the theory of complex projective plane curves via the family of *superisolated* singularities. These singularities were introduced by the second author in [25], see also [2] for a survey on them. A hypersurface singularity $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, $f=f_d+l^{d+1}$ (where f_d is homogeneous of degree d and l is linear) is superisolated if the projective plane curve $C:=\{f_d=0\}\subset\mathbb{P}^2$ is reduced, and none of its singularities $\{p_i\}_{i=1}^{\nu}$ is situated on $\{l=0\}$. The equisingular type of f depends only on f_d, i.e. only on the projective curve $C\subset\mathbb{P}^2$. In particular, all the invariants (of the equisingular type) of f can be determined from the invariants of the pair $(\mathbb{P}^2,C)$.

In the next discussion we follow [11, 26]. There is a standard procedure which provides the plumbing graph of the link M of f from the embedded resolution graphs of (C,p_i)'s and the integer d. The point is that the link M is a rational homology sphere if and only if C is rational and cuspidal. In this section, we will assume that these conditions are satisfied. Let $\mu_i=\mu(C,p_i)$ and Δ_i be the Milnor number and the characteristic polynomial of the local plane curve singularities (C,p_i). Set $2\delta:=\sum_i\mu_i$, $\Delta:=\prod_i\Delta_i$, and $\bar{\Delta}(t):=t^{-\delta}\Delta(t)$.

Let (V,D) be the minimal embedded resolution of the pair $(\mathbb{P}^2,C)$ as above. The minimal plumbing graph of M (or, equivalently, the minimal good resolution graph of the surface singularity $\{f=0\}$) can be obtained from the dual graph of D by decreasing the decoration (self-intersection) of $\bar{C}$ by $d(d+1)$. In the language of topologists, if C is unicuspidal ($\nu=1$), then $M=S^3_{-d}(K)$ (i.e. M is obtained via surgery of the 3-sphere S^3 along K with surgery coefficient $-d$), where $K\subset S^3$ is the local knot of (C,p). One can also verify that $H_1(M,\mathbb{Z})=\mathbb{Z}_d$.

Another topological invariant of f is the following one. Let $Z\to(\{f=0\},0)$ be the minimal good resolution, K_Z be the canonical divisor of Z and $\#$ the number of irreducible components of the exceptional divisor (which equals the number of irreducible components of D). Then $K_Z^2+\#$ is a well-defined invariant of f, which, in fact, can be computed from the link M (or, from its graph) as well. In our case, surprisingly, in this invariant of the link M all the information about the local types (C,p_i) are lost: $K_Z^2+\#=1-d(d-2)^2$, it depends only on d.

The same is true for the Euler characteristic $\chi(F)$, or for the signature $\sigma(F)$ of the Milnor fiber F of f, or about the geometric genus p_g of f. In fact, it is well-known that for any hypersurface singularity, any of p_g, $\sigma(F)$ and $\chi(F)$ determines the remaining two modulo $K_Z^2+\#$. E.g., one has the relation:

$$8p_g+\sigma(F)+K_Z^2+\#=0. \tag{24}$$

In our case, for the superisolated singularity f, one has $p_g=d(d-1)(d-2)/6$, hence the smoothing invariants $\chi(F)$ and $\sigma(F)$ depend only on the degree d.

7.2. For a normal surface singularity with rational homology sphere link (and with some additional analytic restriction, e.g. complete intersection or Gorenstein property) there is a subtle connection between the Seiberg-Witten invariants of its link M and some analytic/smoothing invariants. The hope is that the geometric genus (or, equivalently, $\chi(F)$ or $\sigma(F)$, see (24) and the discussion nearby), can be recovered from the link. The starting point is an earlier conjecture of Neumann and Wahl [39]:

7.2.1. *For any isolated complete intersection whose link M is an integral homology sphere we have the equality $\sigma(F)=8\lambda(M)$, where $\lambda(M)$ is the Casson invariant of the link.*

Notice that the link of a hypersurface superisolated singularity is never an integral homology sphere. The generalised conjecture, applied to rational homology spheres (Conjecture SWC below) was proposed by the forth author in a joint work with L. Nicolaescu in [36] involving the Seiberg-Witten invariant of the link. It was verified for rather large number of non-trivial special families (rational and elliptic singularities, suspension hypersurface singularities $f(x,y)+z^n$ with f irreducible, singularities with good $\mathbb{C}^*$ action) [36, 37, 38, 32, 35]. But the last three authors of the present article have shown in [26] that the conjecture fails in general. The counterexamples were provided exactly by superisolated singularities and/or their universal abelian covers, see also Stevens paper [47] where he computes explicit equations for the universal abelian covers. Nevertheless, in the next paragraph we will recall this conjecture (in its original form), since this have guided us to the semigroup compatibility property, and we believe that it hides a deep mathematical substance (even if at this moment it is not clear for what family we should expect its validity).

Let $\mathbf{sw}_M(can)$ be the Seiberg-Witten invariant of the link M associated with the canonical *spin*^{c} structure (this is induced by the complex structure of $\{f=0\}\setminus\{0\}$, and it can be identified combinatorially from the graph of M; in this article we will not discuss the invariants associated with the other *spin*^{c} structures).

7.3. **'Conjecture' SWC.** [36] *For a $\mathbb{Q}$-Gorenstein surface singularity whose link M is a rational homology sphere one has*

$$\mathbf{sw}_M(can) - (K_Z^2+\#)/8 = p_g.$$

In particular, if the singularity is Gorenstein and admits a smoothing, then $-\mathbf{sw}_M(can) = \sigma(F)/8$ (cf. (24)).

If M is an integral homology sphere then $\mathbf{sw}_M(can) = -\lambda(M)$. If M is a rational homology sphere then by a result of Nicolaescu [40], $\mathbf{sw}_M(can) = \mathcal{T}_M - \lambda(M)/|H_1(M,\mathbb{Z})|$, where $\lambda(M)$ is the Casson-Walker invariant of M (normalised as in [23]), and $\mathcal{T}_M$ denotes the sign refined Reidemeister-Turaev torsion (associated with the canonical *spin*^{c} structure) [56].

7.4. In our present situation, when M is the link of a superisolated singularity f, one shows, cf. [26] (using the notations of 7.1), that

$$\mathcal{T}_M = \frac{1}{d}\sum_{\xi^d=1\neq\xi}\frac{\Delta(\xi)}{(\xi-1)^2} \qquad \text{and} \qquad \lambda(M) = -\frac{\bar\Delta(t)''(1)}{2}+\frac{(d-1)(d-2)}{24}. \tag{25}$$

Therefore, since p_g and $K_Z^2+\#$ depend only on d, the SWC imposes serious restriction on the local invariant Δ. This condition, for some cases when the number of singular points of C is ≥ 2, is not satisfied (hence SWC fails, cf. [26]); nevertheless, as we will see, the SWC identity in the unicuspidal case is equivalent with Conjecture B2 of section 6 about the distribution property of the semigroup. In order to explain this, let us *assume that C is unicuspidal*, and consider (motivated by (25))

$$R(t) := \frac{1}{d}\sum_{\xi^d=1}\frac{\Delta(\xi t)}{(1-\xi t)^2} - \frac{1-t^{d^2}}{(1-t^d)^3}.$$

Similarly,

$$N(t) := \sum_{l=0}^{d-3} \Big(c_l - \frac{(l+1)(l+2)}{2}\Big) t^{d-3-l}; \quad \text{and} \quad D(t) := \sum_{k\in\Gamma_{(C,p)}} t^{\lceil k/d\rceil} - \frac{1-t^d}{(1-t)^2}.$$

Notice that this $D(t)$ agrees with the one defined in (DP), section 6. In [11] the following facts are verified:

$$R(t) = D(t^d)/(1-t^d) = N(t^d). \tag{26}$$

$$N(t) \text{ (hence } R(t) \text{ too) has non-negative coefficients.} \tag{27}$$

$$R(1) = \mathbf{sw}_M(can) - \frac{K^2 + \#}{8} - p_g. \tag{28}$$

Therefore, in this case, we have the equivalence of the 'Seiberg-Witten invariant conjecture' with the 'semigroup distribution property':

7.5. **Theorem.** *Assume that C is unicuspidal and rational (that is, $\nu = 1$). Then the following facts are equivalent:*
(a) $R(1) = 0$, i.e. Conjecture SWC (7.3) is true (for the above germ f);
(b) $R(t) \equiv 0$;
(c) $N(t) \equiv 0$, i.e. Conjecture B1 (6.5) is true;
(d) $D(t) \equiv 0$, i.e. Conjecture B2 (6.6) is true.

8. The semigroup distribution property and Heegaard Floer homology

8.1. The presentation of this section is based on some recent results of the forth author in [32, 33, 34]. In the sequel we assume that C is *unicupidal*, and we keep the notations of the previous section.

There is another way to compute the Seiberg-Witten invariant of the link M via its Heegaard-Floer homology. For any oriented rational homology 3-sphere M the Heegaard Floer homology $HF^+(M)$ was introduced by Ozsváth and Szabó in [43] (cf. also with their long list of articles). $HF^+(M)$ is a $\mathbb{Z}[U]$-module with compatible $\mathbb{Q}$-grading. Moreover, $HF^+(M)$ has a natural direct sum decomposition (compatible with the $\mathbb{Q}$-grading) corresponding to the *spin*^{c}-structures of M: In this article we write $HF^+(M, can)$ for the Heegaard-Floer homology associated with the canonical *spin*^{c} structure.

For some (negative definite) plumbed rational homology 3-spheres M, one can compute the Heegaard Floer homology of $HF^+(M, can)$ of M (equivalently, of $-M$) in a purely combinatorial way from the plumbing graph G. This is true for all the 3-manifolds discussed in this section. This is done via some intermediate objects, the *graded root* associated with G (in fact, one has a graded root corresponding to each *spin*^{c}-structure of M, but here we will discuss only the 'canonical' one). The theory of graded roots, from the point of view of singularity theory, is rather interesting by itself, and we plan to exploit further this connection in the future.

Next, we provide a short presentation of abstract graded roots (cf. [32]).

8.2. **Definition of the 'abstract graded root'** (R,χ)**.** Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u,v]$ the edge with end-points u and v. We say that R is a graded root with grading $\chi:\mathcal{V}\to\mathbb{Z}$ if

(a) $\chi(u)-\chi(v)=\pm 1$ for any $[u,v]\in\mathcal{E}$;

(b) $\chi(u)>\min\{\chi(v),\chi(w)\}$ for any $[u,v],\ [u,w]\in\mathcal{E}$;

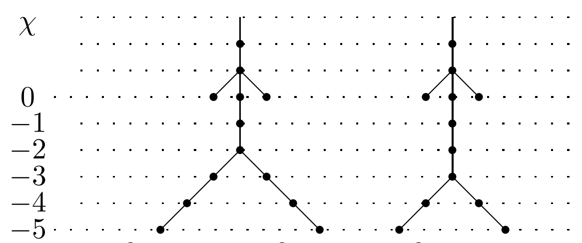
(c) χ is bounded below, $\chi^{-1}(n)$ is finite for any $n\in\mathbb{Z}$, and $\#\chi^{-1}(n)=1$ if $n\gg 0$.

8.3. **Examples.** (1) For any integer $n\in\mathbb{Z}$, let R_n be the tree with $\mathcal{V}=\{v^k\}_{k\geq n}$ and $\mathcal{E}=\{[v^k,v^{k+1}]\}_{k\geq n}$. The grading is $\chi(v^k)=k$.

(2) Let I be a finite index set. For each $i\in I$ fix an integer $n_i\in\mathbb{Z}$; and for each pair $i,j\in I$ fix $n_{ij}=n_{ji}\in\mathbb{Z}$ with the next properties: (i) $n_{ii}=n_i$; (ii) $n_{ij}\geq\max\{n_i,n_j\}$; and (iii) $n_{jk}\leq\max\{n_{ij},n_{ik}\}$ for any $i,j,k\in I$. For any $i\in I$ consider R_{n_i} with vertices $\{v_i^k\}$ and edges $\{[v_i^k,v_i^{k+1}]\}$, $(k\geq n_i)$. In the disjoint union $\coprod_i R_{n_i}$, for any pair (i,j), identify v_i^k and v_j^k, resp. $[v_i^k,v_i^{k+1}]$ and $[v_j^k,v_j^{k+1}]$, whenever $k\geq n_{ij}$, and take the induced χ.

(3) Any map $\tau:\{0,1,\dots,r\}\to\mathbb{Z}$ produces a starting data for construction (2). Indeed, set $I=\{0,\dots,r\}$, $n_i:=\tau(i)$ $(i\in I)$, and $n_{ij}:=\max\{n_k\,:\,i\leq k\leq j\}$ for $i\leq j$. Then the root constructed in (2) using this data will be denoted by (R_τ,χ_τ).

8.3.1. Here are two (typical) graded roots (cf. with 8.11):



$S^3_{-5}(T_{2,13})$, $S^3_{-5}(T_{4,5})$ $S^3_{-5}(T_{3,7})$
$\Sigma(5,5,6)$

8.4. **The canonical graded root** (R,χ) **of** M**.** [32] Next, we define for any (negative definite, plumbed) rational homology sphere M a graded root.

We fix a plumbing graph G and denote by L the corresponding lattice: the free $\mathbb{Z}$-module of rank $\#$ with fixed basis $\{A_j\}_j$, and bilinear form $(A_i,A_j)_{i,j}$. (In our case, a possible choice is the dual resolution graph and the corresponding intersection form associated with the minimal good resolution $Z\to(\{f=0\},0)$.) Set $L'=\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})\subset L\otimes\mathbb{Q}$. Let $K_Z\in L'$ be the canonical cycle defined by $K_Z(A_j)+A_j^2+2=0$ for any j. Then define $\chi:L\to\mathbb{Z}$ by (the Riemann-Roch formula) $\chi(x):=-(K_Z(x)+x^2)/2$.

The definition of the graded root captures the position of the lattice points in the different ellipsoids $\chi^{-1}(n)$. For any $n\in\mathbb{Z}$, one constructs a finite 1-dimensional simplicial complex $\bar{L}_{\leq n}$ as follows. Its 0-skeleton is $L_{\leq n}:=\{x\in L:\,\chi(x)\leq n\}$. For each x and j, with both x and $x+A_j\in L_{\leq n}$, we consider a unique 1-simplex with endpoints at x and $x+A_j$ (e.g., the segment $[x,x+A_j]$ in $L\otimes\mathbb{R}$). We denote the set of connected components of $\bar{L}_{\leq n}$ by $\pi_0(\bar{L}_{\leq n})$. For any $v\in\pi_0(\bar{L}_{\leq n})$, let C_v be the corresponding connected component of $\bar{L}_{\leq n}$.

Next, we define (R,χ) as follows. The vertices $\mathcal{V}(R)$ are $\cup_{n\in\mathbb{Z}}\pi_0(\bar{L}_{\leq n})$. The grading $\mathcal{V}(R)\to\mathbb{Z}$, still denoted by χ, is $\chi|\pi_0(\bar{L}_{\leq n})=n$. If $v_n\in\pi_0(\bar{L}_{\leq n})$, and $v_{n+1}\in\pi_0(\bar{L}_{\leq n+1})$, and $C_{v_n}\subset C_{v_{n+1}}$, then $[v_n,v_{n+1}]$ is an edge of R. All the edges are obtained in this way.

8.5. **Example.** [33] Recall that the link of the superisolated singularity f (where C is rational and unicuspidal of degree d) is the surgery manifold $S^3_{-d}(K)$, where $K \subset S^3$ is the local knot of (C,p). The graded root of M can be represented by a function τ as in 8.3(3) associated with the Alexander polynomial Δ of $K \subset S^3$. Similarly as in section 6, set $\mu = 2\delta$ for the degree of Δ (which equals $(d-1)(d-2)$), and write $\Delta(t)$ as $1+\delta(t-1)+(t-1)^2Q(t)$ for some polynomial $Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i$. Set $c_l := \alpha_{(d-3-l)d}$ (cf. with 6.1). Then define $\tau : \{0,1,\dots,2d-4\} \to \mathbb{Z}$ by

$$\tau(2l) = \tfrac{l(l-1)}{2}d - l(\delta-1), \qquad \tau(2l+1) = \tau(2l+2) + c_{d-3-l}.$$

Then $(R,\chi) = (R_\tau, \chi_\tau)$.

8.6. **Example.** Let $\Sigma(d,d,d+1)$ be the Seifert 3-manifold $(d,d,d+1)$; equivalently, the link of the Brieskorn singularity $x^d + y^d + z^{d+1} = 0$. Its graded root also can be represented by the 'τ-construction' (for the more general situation of Seifert manifolds, see [32]).

For any $0 \le l \le d-3$ define $c^u_l := (l+1)(l+2)/2$, and $2\delta := (d-1)(d-2)$. Then define $\tau^u : \{0,1,\dots,2d-4\} \to \mathbb{Z}$ by

$$\tau^u(2l) = \tfrac{l(l-1)}{2}d - l(\delta-1), \qquad \tau^u(2l+1) = \tau^u(2l+2) + c^u_{d-3-l}.$$

Then $(R,\chi) = (R_{\tau^u}, \chi_{\tau^u})$.

Notice the shocking similarities of 8.5 and 8.6: the graded roots associated with $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ coincide exactly when $c_l = c^u_l$ for all l.

To any graded root, one can associate a natural graded $\mathbb{Z}[U]$-module.

8.7. **Definition. The $\mathbb{Z}[U]$-module associated with a graded root.** Consider the $\mathbb{Z}[U]$-module $\mathbb{Z}[U,U^{-1}]$, and (following [44]) denote by $\mathcal{T}^+_0$ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. It is a $\mathbb{Z}[U]$-module with grading $\deg(U^{-h}) = 2h$.

Now, fix a graded root (R,χ). Let $\mathbb{H}(R,\chi)$ be the set of functions $\phi : \mathcal{V} \to \mathcal{T}^+_0$ with the property that whenever $[v,w] \in \mathcal{E}$ with $\chi(v) < \chi(w)$ one has $U \cdot \phi(v) = \phi(w)$. Then $\mathbb{H}(R,\chi)$ is a $\mathbb{Z}[U]$-module via $(U\phi)(v) = U \cdot \phi(v)$. Moreover, $\mathbb{H}(R,\chi)$ has a grading: $\phi \in \mathbb{H}(R,\chi)$ is homogeneous of degree $h \in \mathbb{Z}$ if for each $v \in \mathcal{V}$ with $\phi(v) \neq 0$, $\phi(v) \in \mathcal{T}^+_0$ is homogeneous of degree $h - 2\chi(v)$.

In the sequel, the following notation is useful: If P is a $\mathbb{Q}$-graded $\mathbb{Z}[U]$-module with h-homogeneous elements P_h, then for any $r \in \mathbb{Q}$ we denote by $P[r]$ the same module graded in such a way that $P[r]_{h+r} = P_h$.

8.8. **Theorem.** [32, 44] *Assume that M is either $S^3_{-d}(K)$ or $\Sigma(d,d,d+1)$. Then*

$$HF^+(-M, can) = \mathbb{H}(R,\chi)[-(K_Z^2 + \#)/4].$$

In other words, for these 3-manifolds, the Heegaard-Floer homology can be recovered from the graded root via $\mathbb{H}(R,\chi)$ modulo a shift in grading by $-(K_Z^2+\#)/4$. (The shift in the above two examples are different; in the case of $S^3_{-d}(K)$ one has $K_Z^2 + \# = 1 - d(d-2)^2$, while for $\Sigma(d,d,d+1)$ one has $K_Z^2 + \# = -d(d-1)(d-3)$.)

Now, Conjecture B1 (6.5) and the above discussion/examples read as follows:

8.9. **Theorem.** *Assume that $\nu = 1$. Then the following facts are equivalent:*

(a) Conjecture B1 (6.5) is true,

(b) The canonical graded roots of $S^3_{-d}(K)$ and $\Sigma(d,d,d+1)$ are the same.

(c) The canonical Heegaard-Floer homologies of $-S^3_{-d}(K)$ *and* $-\Sigma(d,d,d+1)$ *are the same modulo a shift in the grading, namely:*

$$HF^+(-S^3_{-d}(K), can)[1-d(d-2)^2] = HF^+(-\Sigma(d,d,d+1), can)[-d(d-1)(d-3)].$$

Proof. The equivalence $(a) \Leftrightarrow (b)$ is clear from the above discussion, while $(b) \Leftrightarrow (c)$ can be deduced by a direct computation, or from an easy formula which provides $\mathbb{H}(R_\tau, \chi_\tau)$ from τ, cf. [32]. (Nevertheless, see another argument below.) □

8.10. **Remark.** Regarding the Seiberg-Witten invariant of $M = S^3_{-d}(K)$, one has

$$\mathbf{sw}_M(can) - \frac{K_Z^2 + \#}{8} = \sum_{l \geq 0} \tau(2l+1) - \tau(2l+2) = \sum_{l \geq 0} c_l, \qquad (29)$$

and there is a similar formula for $M = \Sigma(d,d,d+1)$ with the obvious replacements.

Therefore, to the equivalences (8.9) one can add:

$$(d) \quad \mathbf{sw}_M(can) - \frac{K_Z^2 + \#}{8} \,|_{M=S^3_{-d}(K)} = \mathbf{sw}_M(can) - \frac{K_Z^2 + \#}{8} \,|_{M=\Sigma(d,d,d+1)}.$$

Since the Conjecture SWC (7.3) is true for the Brieskorn singularity $f_{BR} := x^d + y^d + z^{d+1}$ (cf. [37]), and the geometric genus of the superisolated singularity f equals the geometric genus of f_{BR} (both equal $d(d-1)(d-2)/6$), this last identity (d) is also equivalent with the validity of the SWC for f — a fact already proved in 7.5.

(Notice also that the expression $\mathbf{sw}_M(can) - (K_Z^2 + \#)/8$ can be deduced from $\mathbb{H}$, a fact which implies $(c) \Rightarrow (d)$, while $(d) \Rightarrow (a)$ follows from (29).)

8.11. **Example.** Assume that $d = 5$ and C is unicuspidal whose singular point has only one Puiseux pair (a,b) with $a < b$. Then by the genus formula the possible values of (a,b) are $(4,5)$, $(3,7)$ and $(2,13)$. It turns out that the first and the third cases can be realized, while the second not. The corresponding graded roots (together with the root of $\Sigma(5,5,6)$) are drawn in the above figure (8.3.1).

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