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ORIGINAL PAPER

# Marginality and the position value

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**Abstract**
We present a new characterization of the position value, one of the most prominent allocation rules for communication situations (graph-games or games with restricted communication). This characterization includes the PL-marginality property, an extension for communications situations of the classic marginality for TU-games, as well as component efficiency and balanced link contributions for necessary players.

**Keywords** Game theory · TU-game · Communication situations · Position value · Marginality · PL-marginality

**Mathematics Subject Classification** 91A06 · 91A12

## 1 Introduction

The study of cooperative games (TU-games) in which a graph imposes some restrictions on the cooperation of the players, the so-called communication situations, graph-games or games with restricted communication, was introduced by (Myerson 1977, 1980). He modified the original game to the graph-restricted game using the graph. Then, he proposed the Shapley value (Shapley 1953) as a point solution for these situations. First, he characterized the value in terms of component efficiency and fairness (Myerson 1977). After that, he used the component efficiency and balanced contributions to obtain another characterization (Myerson 1980).

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In his Ph.D. thesis (in Dutch), Meessen (1988) introduced—also for communication situations—the link game and the position value. The link game is a TU-game in which the players are the links of the graph and the value of each subgraph (coalition of links) coincides with the worth of the grand coalition in the Myerson (sub) graph-restricted game. The position value is calculated, for each player, as half of the sum of the Shapley value in the link game of the links he is involved in.

These concepts were later popularized by Borm et al. (1992). Slikker (2005) gave the first characterization of the position value for general communication situations. van den Brink et al. (2011) gave another characterization (as a corollary from a more general result regarding Harsanyi power solution). This is generalized for union stable systems in Algaba et al. (2015). The position value has received a lot of attention from game theorists. Algaba et al. (2000) defined the position value for union stable systems, Casajus (2007) proved that the position value is, in some sense, the Myerson value, Gómez et al. (2004) introduced a unified approach for the Myerson and the position values, Kongo (2010) established that the difference between the position value and the Myerson value is attributable to the existence of coalition structures, Ghintran et al. (2012) extended it to the probabilistic communication situations, Ghintran (2010) defined a weighted position value, Xianghui et al. (2017) defined the position value for communication situations with fuzzy coalitions, Fernández et al. (2018) introduced the cg-position value for games on fuzzy communication structures, Zhang et al. (2019) and Shan et al. (2020) characterized the position value for hypergraph communications situations, and Borkotokey et al. (2020) characterized it (and also the Myerson value) for the subclass of probabilistic network games in multilinear form.

Marginalism is a principle in economics according to which the value is determined by the additional utility that an extra unit of a good provides. The propagation of this concept in economics is commonly referred to as the Marginalist Revolution, establishing the difference between classical and modern economics. Marginalism plays a major role in the Shapley value, since it is calculated as a linear combination of the player's marginal contributions to the different coalitions. Young (1985) brilliantly introduced the marginality property.[1] This property states that if the marginal contributions of a player in two different games coincide, the value of that player should be equal in both games. After this work, many other contributions are devoted to analyzing the relation between the Shapley value and the marginality property. See, for example, de Clippel and Serrano (2008), Skibski et al. (2013), and Huettner and Casajus (2019).

All in all, the Myerson value and the position value can be thought of as Shapley values of certain particular types of games. Therefore, it makes sense to ask whether the property of marginality holds for them. The answer is negative in both cases. Then, the issue is to explore whether certain variations of the classic property of marginality can serve to characterize these values. These variations arise precisely from ideas related to centrality.

[1] In his characterization of the Shapley value, Young included the property of strong monotonicity, although, in fact, he used the property of marginality that receives this name after Chun (1989). Pintér (2015) gave another proof of Young's axiomatization.

Gómez et al. (2003) and González-Arangüena et al. (2017) additively decomposed the Myerson value into the WG-Myerson value that measures the productivity of the player via the characteristic function (for symmetric games, the communication centrality), and the BG-Myerson value that evaluates the ability of the player to intermediate among others (betweenness centrality). Manuel et al. (2020) introduced different types of marginal contributions for communication situations as well as the corresponding marginality properties. When the cooperation in a game is restricted by means of a graph, in addition to the classic marginal contribution of a player to a coalition, it is possible to consider that a player can contribute to a coalition by lending it his links to increase the connectedness, but without joining himself (L-marginal contribution), and also the possibility of joining with his relationships (PL-marginal contributions). Marginality (respectively L-marginality and PL-marginality) states that if player's marginal contributions (respectively, L-marginal and PL-marginal contributions) are the same in two communication situations that differ in the characteristic function, ceteris paribus, the outcome will also be the same in both of them.

In this way, the authors prove that the Myerson value, the WG-Myerson value, and the BG-Myerson value can be characterized by using, respectively, PL-marginality, marginality, and L-marginality, as well as other properties. Some of these properties can be considered as part of a recent literature (see Yokote and Kongo (2017), and Navarro (2019)) that consists of weakening well-known axioms. Other variants of the marginality axioms can be found in (Casajus 2011a, b) and Casajus and Yokote (2017).

In this paper, we explore the extent to which the position value can be characterized from (one of the previous) marginality properties. Then, the aim of this article is twofold. On the one hand, we wish to relate and characterize the position value with the marginality properties. On the other hand, to look for new parallel behaviors between the Myerson value and the position value. In this way, we prove that the position value can be characterized in terms of component efficiency, balanced link contributions for necessary players, and PL-marginality.

This characterization supports the use of the rule in applications in which the underlying game is changing, whereas the network is fixed. We can find examples in cost allocation problems in a network or when the allocation should be updated as the value changes. For example, the distribution of the maintenance costs of a road network between towns fluctuates over time, but those towns that have the same PL-marginal contributions over time should always bear the same costs.

The remainder of the paper is organized as follows. A section of preliminaries is placed after this introduction. In Section 3, we include the new characterization of the position value, and in Section 4, we prove the independence of the used axioms. Section 5 is devoted to present an additive decomposition of the position value in a parallel manner to that existing for the Myerson value. The paper ends with some final remarks, an appendix with the proof of the main result and the references.

## 2 Preliminaries

A TU-game or a *cooperative n-person game* is a pair $(N, v)$, where $N = \{1, \dots, n\}$ denotes the set of players and $v$ is a real function defined on $2^N = \{S | S \subseteq N\}$, the family of all subsets *(coalitions)* of $N$. For each coalition $S$, $v(S)$ is the outcome that his $s = |S|$ members can obtain if they decide to cooperate. It is assumed that $v(\emptyset) = 0$.

The set of all TU-games with $N$ fixed is a vector space, $G^N$, whose dimension is $2^n - 1$. Given $\emptyset \neq S \subseteq N$, the game $(N, u_S)$ with characteristic function given by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T, \\ 0, & \text{otherwise,} \end{cases}$$

is known as the *unanimity game* of $S$. The family $\{(N, u_S)_{\emptyset \neq S \subseteq N}\}$ is a basis of $G^N$. As a consequence, given $(N, v) \in G^N$,

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S.$$

The coordinates $\{\Delta_v(S)\}_{\emptyset \neq S \subseteq N}$ are known as *Harsanyi dividends* (Harsanyi 1959). We can obtain the dividend of each coalition, $\emptyset \neq S \subseteq N$, using the following expression:

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

Consequently, the value of each coalition, $\emptyset \neq S \subseteq N$, can be obtained from the dividends of its subcoalitions as

$$v(S) = \sum_{\emptyset \neq T \subseteq S} \Delta_v(T).$$

$(N, v)$ is a zero-normalized game if $v(\{i\}) = 0$ for all $i \in N$. Given a game $(N, v)$, we will denote $(N, \hat{v})$ the zero-normalized game with characteristic function given by $\hat{v}(S) = v(S) - \sum_{i \in S} v(\{i\})$. $G_0^N$ is the subspace of $G^N$ consisting of all the zero-normalized games.

The game $(N, v)$ is symmetric if $v(S) = v(T)$ whenever $s = t$.

Given $(N, v)$, $i \in N$ is: *a)* a null player if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, *b)* a necessary player (van den Brink and Gilles 1996) if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.

An allocation rule for TU-games is a function $\psi$ defined on $G^N$ that assigns to every $(N, v) \in G^N$ a vector $\psi(N, v) \in \mathbb{R}^n$ whose $i$th component represents the payoff to the player $i$ in the game $(N, v)$.

The most prominent allocation rule for TU-games was introduced by Shapley (1953). The Shapley value assigns to every player the following linear combination of his marginal contributions to different coalitions:

$$Sh_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!}(v(S \cup \{i\}) - v(S)), \quad i \in N,$$

and it can be alternatively obtained from the Harsanyi dividends as:

$$Sh_i(N,v) = \sum_{S \subseteq N : i \in S} \frac{\Delta_v(S)}{s}, \quad \text{for all } i \in N.$$

*A graph* is a pair $(N,\gamma)$, where $N = \{1,2,\dots,n\}$ is the set of nodes and $\gamma$ a subset of $\gamma_N = \{\{i,j\} \mid i,j \in N, i \neq j\}$. Each *link* $\{i,j\} \in \gamma$ represents the possibility of a direct relationship between $i$ and $j$. We will denote by $\Gamma^N$ the family of all graphs with node set $N$.

Two nodes $i$ and $j$ are *directly connected* in a graph $(N,\gamma)$, if $\{i,j\} \in \gamma$; and they are *connected* if there exists a sequence of nodes (called intermediaries) $i_1, i_2, \dots i_k$ with $i_1 = i$, $i_k = j$ such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1,\dots,k-1$. A set $S \subseteq N$, $S \neq \emptyset$, is *connected (internally connected)* in $\gamma$ if every pair of nodes in $S$ can be connected using intermediaries in $S$. We will assume that singletons are connected sets. A *connected component*, $C$, in $(N,\gamma)$ is a maximally connected set, i.e., $C$ is connected in $(N,\gamma)$ and $C' \supset C$ is not connected. A graph $(N,\gamma)$ induces a partition $N/\gamma$ of the set $N$ into connected components. The restriction of the graph $(N,\gamma)$ to the set $S \subseteq N$ is the graph $(S,\gamma_{|_S})$ with $\gamma_{|_S} = \{\{i,j\} \in \gamma \mid i,j \in S\}$. $S/\gamma$ will be the set of the connected components of $S$ in $(S,\gamma_{|_S})$. For each $\gamma' \subseteq \gamma$, $(N,\gamma')$ is a *subgraph* of $(N,\gamma)$. Given a graph $(N,\gamma)$ and $l \in \gamma$, $(N,\gamma \setminus \{l\})$ is the subgraph obtained when removing the link $l$, and $(N,\gamma_i)$ is the subgraph of the links incident on $i$, i.e., $\gamma_i = \{l \in \gamma \mid i \in l\}$.

A *communication situation* is a triple $(N,v,\gamma)$, where $(N, v)$ is a TU-game and $(N,\gamma)$ an undirected graph. $\mathcal{CS}^N$ will denote the set of all communication situations with players-nodes set $N$, and $\mathcal{CS}_0^N$ the subset of those communication situations in which the game is zero-normalized.

A map $\psi : \mathcal{CS}^N \to \mathbb{R}^n$ is an *allocation rule* for communication situations. $\psi_i(N,v,\gamma)$ represents the outcome according to $\psi$ for player $i$ in game $(N, v)$ given the restrictions in the communication imposed by the graph $(N,\gamma)$.

Meessen (1988) and Borm et al. (1992) introduced a very relevant allocation rule for communication situations, the position value. Given $(N,v,\gamma) \in \mathcal{CS}_0^N$, they defined a new TU-game, the link game, $(\gamma, r_\gamma^v)$. In this game, the players are the links of the graph, and the characteristic function is given by $r_\gamma^v(\eta) = \sum_{C \in N/\eta} v(C)$, for each $\eta \subseteq \gamma$. As $r_\gamma^v(\emptyset) = \sum_{i=1}^n v(\{i\})$, it suffices that the game be zero-normalized for the requirement that $r_\gamma^v(\emptyset) = 0$ to be satisfied.

The position value of player $i$ in $(N,v,\gamma) \in \mathcal{CS}_0^N$, denoted $\pi_i(N,v,\gamma)$, is defined as

$$\pi_i(N,v,\gamma) = \frac{1}{2} \sum_{l \in \gamma_i} Sh_l(N, r_\gamma^v).$$

Then, it assigns to each player in a communication situation half of the sum of the Shapley values (in the link game) of the links he is involved in.

An allocation rule $\psi$ defined on $\mathcal{CS}^N$ (or in $\mathcal{CS}_0^N$ if sufficient):

i) satisfies *component efficiency* (Myerson 1977) if for all $(N, v, \gamma) \in \mathcal{CS}^N$ and all $C \in N/\gamma$, $\sum_{i \in C} \psi_i(N, v, \gamma) = v(C)$;
ii) verifies *balanced link contributions* (Slikker 2005) if given $(N, v, \gamma) \in \mathcal{CS}_0^N$ and $i, j \in N$, $\sum_{l \in \gamma_j}[\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma \setminus \{l\})] = \sum_{l \in \gamma_i}[\psi_j(N, v, \gamma) - \psi_j(N, v, \gamma \setminus \{l\})]$.
In this paper, we will use a weak version of this property in which the previous condition is only required for necessary players.[2]
iii) satisfies *link anonymity* (Borm et al. 1992), if for all $(N, v, \gamma) \in \mathcal{CS}_0^N$ link anonymous (i.e., a communication situation in which the link game is symmetric), there exists $\lambda \in \mathbb{R}$, such that for all $i \in N$, it holds that $\psi_i(N, v, \gamma) = \lambda|\gamma_i|$.
iv) satisfies *the superfluous link property* (Borm et al. 1992), if for all $(N, v, \gamma) \in \mathcal{CS}_0^N$ and every $l \in \gamma$ superfluous link in $(N, v, \gamma)$ (i.e., a null player in the link game), then $\psi(N, v, \gamma) = \psi(N, v, \gamma \setminus \{l\})$.
v) verifies *PL-marginality* (Manuel et al. 2020) if given $(N, v, \gamma), (N, w, \gamma) \in \mathcal{CS}^N$ and $i \in N$, such that $r_\gamma^v(\eta \cup \delta) - r_\gamma^v(\eta) = r_\gamma^w(\eta \cup \delta) - r_\gamma^w(\eta)$, for all $\eta \subseteq \gamma \setminus \gamma_i, \delta \subseteq \gamma_i$, then $\psi_i(N, v, \gamma) = \psi_i(N, w, \gamma)$.

Borm et al. (1992) characterized the position value only on the domain consisting of communication situations with a fixed player set and a cycle-free graph. Then, the position value is the unique allocation rule in such a domain that satisfies component efficiency, additivity, the superfluous link property, and the link anonymity.

Slikker (2005) expressed the position value in terms of the link game dividends as, $\pi_i(N, v, \gamma) = \sum_{\eta \subseteq \gamma} \frac{1}{2} \Delta_{r^v}(\eta) \frac{|\eta_i|}{|\eta|}$, and he also characterized this allocation rule using component efficiency and balanced link contributions.

Casajus (2007) characterized the position value in terms of the Myerson value (Myerson 1977, 1980) of the link-agent form, a modification of the original TU-game different from the graph-restricted game or the link game.

## 3 A new characterization of the position value

This section is devoted to characterizing the position value in terms of components efficiency, balanced link contributions for necessary players, and PL-marginality.

It is well known that the position value satisfies component efficiency and balanced link contributions for all players (Slikker 2005), and thus for necessary players.

To prove that it also satisfies PL-marginality, we will use the following lemma whose proof is straightforward. This lemma states that if the PL-marginal contributions of player $i$ and his subset of links $\delta$ to $\eta \subseteq \gamma \setminus \gamma_i$ (Manuel et al. 2020) coincide for games $(N, v)$ and $(N, w)$, ceteris paribus, then the change in the PL-marginal contributions corresponding to $\delta$ and $\delta^*$ of this player in both games also coincide. In

[2] In Navarro (2019), we can find interesting results relating fairness to necessary players and the equal treatment property.

other words, if the change in the value of the game does not affect the PL-marginal contributions, neither will it affect their variations.

**Lemma 3.1** *Suppose* $(N, v, \gamma), (N, w, \gamma) \in \mathcal{CS}_0^N$ *and* $i \in N$. *The following two statements are equivalent*:

*i)* $r_\gamma^v(\eta \cup \delta) - r_\gamma^v(\eta) = r_\gamma^w(\eta \cup \delta) - r_\gamma^w(\eta)$, *for all* $\eta \subseteq \gamma \setminus \gamma_i, \delta \subseteq \gamma_i$.
*ii)* $r_\gamma^v(\eta \cup \delta) - r_\gamma^v(\eta \cup \delta^*) = r_\gamma^w(\eta \cup \delta) - r_\gamma^w(\eta \cup \delta^*)$, *for all* $\eta \subseteq \gamma \setminus \gamma_i, \delta, \delta^* \subseteq \gamma_i$.

**Proposition 3.1** *The position value satisfies PL-marginality.*[3]

***Proof*** Consider $(N, v, \gamma)$, $(N, w, \gamma) \in \mathcal{CS}_0^N$ and $i \in N$, such that $r_\gamma^v(\eta \cup \delta) - r_\gamma^v(\eta) = r_\gamma^w(\eta \cup \delta) - r_\gamma^w(\eta)$, for all $\eta \subseteq \gamma \setminus \gamma_i, \delta \subseteq \gamma_i$. By the *Lemma* 3.1, this is equivalent to

$$r_\gamma^v(\eta \cup \delta) - r_\gamma^v(\eta \cup \delta^*) = r_\gamma^w(\eta \cup \delta) - r_\gamma^w(\eta \cup \delta^*) \text{ for all } \eta \subseteq \gamma \setminus \gamma_i, \delta, \delta^* \subseteq \gamma_i. \quad (1)$$

For each $l \in \gamma_i$ and each $\nu \subseteq \gamma \setminus \{l\}$, let us denote $\nu_i = \nu \cap \gamma_i$.

Taking in (1), $\eta = \nu \setminus \nu_i$, $\delta = \nu_i \cup \{l\}$ and $\delta^* = \nu_i \subseteq \delta$, we have

$$r_\gamma^v(\nu \cup \{l\}) - r_\gamma^v(\nu) = r_\gamma^w(\nu \cup \{l\}) - r_\gamma^w(\nu).$$

As a direct consequence,

$$\pi_i(N, v, \gamma) = \pi_i(N, w, \gamma),$$

which completes the proof. □

In the following proposition, we provide a characterization of the position value relaxing the balanced link contributions property in the one of Slikker (2005) but including the PL-marginality property. As mentioned in the introduction, this characterization supports the use of the rule in applications in which the game changes, ceteris paribus; e.g., adaptation of cost allocations in a network over time.

**Proposition 3.2** *The position value is the unique allocation rule on* $\mathcal{CS}_0^N$ *that satisfies efficiency in connected components, balanced link contributions for necessary players, and PL-marginality.*

The proof of Proposition 3.2 can be found in the Appendix.

[3] PL-monotonicity property (Manuel et al. 2020) implies PL-marginality, but both properties are not equivalent. The position value does not satisfy PL-monotonicity. A counterexample can be found in Ortega (2021).

## 4 Independence of axioms

This section is devoted to proving that the axioms used in the previous characterization are independent.

(i) Efficiency in connected components and PL-marginality do not imply balanced link contributions for necessary players.

The Myerson value satisfies efficiency in connected components and PL-marginality, but it does not satisfy balanced link contributions for necessary players.

(ii) Efficiency in connected components and balanced link contributions for necessary players do not imply PL-marginality .

Given a communication situation $(N, v, \gamma) \in \mathcal{CS}_0^N$, $i \in N$, and $C_i^{N,\gamma}$ the connected component in $(N, \gamma)$ to which $i$ belongs, let $(C_i^{N,\gamma})_n$ be the set of all necessary players in the game $(N, v)$ that are in $C_i^{N,\gamma}$ and $(C_i^{N,\gamma})_n^c$ its complementary in $C_i^{N,\gamma}$. Let us consider the allocation rule, $\psi$, that assigns to $i \in N$[4]

$$\psi_i(N, v, \gamma) = \begin{cases} \pi_i(N, v, \gamma), & \text{if } (C_i^{N,\gamma})_n \neq \emptyset \\ & \text{and } i \in (C_i^{N,\gamma})_n, \\ \frac{v(C_i^{N,\gamma}) - \sum_{j \in (C_i^{N,\gamma})_n} \pi_j(N,v,\gamma)}{|(C_i^{N,\gamma})_n^c|}, & \text{if } (C_i^{N,\gamma})_n^c \neq \emptyset \\ & \text{and } i \in (C_i^{N,\gamma})_n^c. \end{cases}$$

$\psi$ satisfies efficiency in connected components and balanced link contributions for necessary players. However, it does not satisfy PL-marginality, because it does not coincide with the position value.

(iii) Balanced link contributions for necessary players and PL-marginality do not imply efficiency in connected components. Consider the rule $\psi = 0.5\pi$.

## 5 A decomposition of the position value

Walking on the steps of Neyman (1989), Gómez et al. (2003), Béal et al. (2016) and González-Arangüena et al. (2017), it is possible to decompose additively the position value into two values: the within groups position value (WG-position value) and the between groups position value (BG-position value).

The WG-position value of a player will represent the part of his position value obtained by his productivity in the characteristic function. For $(N, v, \gamma) \in \mathcal{CS}_0^N$ and $i \in N$, it is defined as

$$\pi_i^W(N, v, \gamma) = \pi_i(N, v_i, \gamma),$$

[4] We abuse the notation using $\sum_{j\in\emptyset} \pi_j(N, v, \gamma) = 0$.

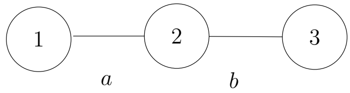


**Fig. 1** Graph $(N, \gamma)$

with $v_i(S) = v(S) - v(S \setminus \{i\})$, for $S \subseteq N$.

The BG-position value of a player will be the part of his position value received by his intermediation in the connection of others. For $(N, v, \gamma) \in \mathcal{CS}_0^N$ and $i \in N$, it is defined as

$$\pi_i^B(N, v, \gamma) = \pi_i(N, v_{-i}, \gamma),$$

with $v_{-i}(S) = v(S \setminus \{i\})$, for $S \subseteq N$.

To illustrate these definitions, let us consider the following example.

***Example 5.1*** Let $(N, v, \gamma)$ be the communication situation with $N = \{1, 2, 3\}$, $v$ the characteristic function of the messages game given by

$$v(S) = \begin{cases} \binom{s}{2}, & \text{if } s \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

in which the value of a coalition is the number of messages that can be sent between his members. The graph is $\gamma = \{\{1, 2\}, \{2, 3\}\}$. A possible representation of this graph is given in Fig. 1.

Then, we have that

$$Sh(N, v) = (1, 1, 1).$$

The characteristic function of the link game is

$$r_\gamma^v = u_{\{a\}} + u_{\{b\}} + u_{\{a,b\}},$$

and the position value is

$$\pi(N, v, \gamma) = \left(\frac{3}{4}, \frac{3}{2}, \frac{3}{4}\right).$$

Moreover

$$r_\gamma^{v_i} = \begin{cases} u_{\{a\}} + u_{\{a,b\}}, & \text{if } i = 1, \\ u_{\{a\}} + u_{\{b\}}, & \text{if } i = 2, \\ u_{\{b\}} + u_{\{a,b\}}, & \text{si } i = 3, \end{cases}$$

and thus, $\pi^W(N,v,\gamma)=(\frac{3}{4},1,\frac{3}{4})$. As clearly $\pi^W+\pi^B=\pi$, we have $\pi^B(N,v,\gamma)=(0,\frac{1}{2},0)$.

We can observe that player 2, which is fully connected with the other two, has WP-position value (productivity via characteristic function) equal to his Shapley value. Players 1 and 3 (which are not directly connected) spend $\frac{1}{4}$ of their Shapley value rewarding player 2. He allows them to send messages to each other. As they are terminal nodes, they do not receive any intermediation fees. Then, the three units of the total value of the component (efficiency) can be decomposed in the productivity of the characteristic function preserved by the network ($\frac{3}{4}+\frac{3}{4}+1$) which we will call WP-efficiency and the intermediation cost (BP-efficiency).

The WG-position value can be characterized in terms of WP-efficiency[5] in connected components, balanced link contributions for necessary players, and marginality.

The BG-position value can be characterized using BP-efficiency[6] in connected components, balanced link contributions for null players, and L-marginality.

The proofs of these characterizations are rather mechanical, similar to the proofs in Manuel et al. (2020), and Proposition 3.2 of this paper.

## 6 Final remarks and conclusions

The following Table 1 contains a summary of the properties considered above (as well as other classics in the literature of the Myerson and position values) and their relationship with the different values. In this way, we can see which properties are (or are not) shared by different values. The symbol C is assigned to those properties that are used in a characterization of the corresponding value. We have used the results in this paper as well as the corresponding results in Manuel et al. (2020). Although some of them are not included in these papers, their proofs or counterexamples are straightforward, so it is left to the reader.

As a conclusion, a marginalist analysis of the position value allows to find additional parallelisms with the Myerson value. PL-marginality is a common property of the Myerson and the position value, and can be used to characterize them. Similarly, the marginality property is satisfied by the WG-position value and the WG-Myerson value, and both the BG-Myerson value and the BG-position value satisfy L-marginality, these properties being useful in the respective characterizations.

[5] A allocation rule $\psi : \mathcal{CS}_0^N \to \mathbb{R}^n$ satisfies WP-efficiency in connected components if given $(N,v,\gamma)\in\mathcal{CS}_0^N$ and $C\in N/\gamma$, $\sum_{i\in C}\psi_i(N,v,\gamma)=\sum_{S\subseteq C}\Delta_v(S)\beta_\gamma(S)$. As it can be seen, this efficiency is a linear combination of the dividends of the game in which the coefficients $\beta_\gamma(S)$ are rather technical. They depend upon the relative degrees of nodes (of $S$) in the minimal connection graphs of the coalition $S$. Comparing these coefficients with the corresponding $\alpha_\gamma(S)$ in Manuel et al. (2020), we can see $\alpha_\gamma(S)$ depend on the proportion that nodes in $S$ represent in the (set of nodes of them) minimal connection graphs of the coalition $S$. See Gómez et al. (2004) for details.

[6] A allocation rule $\psi : \mathcal{CS}_0^N \to \mathbb{R}^n$ satisfies BP-efficiency in connected components if given $(N,v,\gamma)\in\mathcal{CS}_0^N$ and $C\in N/\gamma$, $\sum_{i\in C}\psi_i(N,v,\gamma)=\sum_{S\subseteq C}\Delta_v(S)(1-\beta_\gamma(S))$.

**Table 1** Summary of properties

| | M | WG-M | BG-M | P | WG-P | BG-P |
|---|---|---|---|---|---|---|
| Efficiency in CC | √/C | × | × | √/C | × | × |
| W-efficiency in CC | × | √/C | × | × | × | × |
| B-efficiency in CC | × | × | √/C | × | × | × |
| WP-efficiency in CC | × | × | × | × | √/C | × |
| BP-efficiency in CC | × | × | × | × | × | √/C |
| Balanced contributions | √/C | × | × | × | × | × |
| Balanced contributions for necessary | √/C | √/C | √ | × | × | √ |
| Balanced contributions for null | √ | √ | √/C | × | √ | × |
| Balanced link contributions | × | × | × | √/C | × | × |
| Balanced link contributions for necessary | × | × | √ | √/C | √/C | √ |
| Balanced link contributions for null | × | √ | × | √ | √ | √/C |
| Marginality | × | √/C | × | × | √/C | × |
| L-marginality | × | × | √/C | × | × | √/C |
| PL-marginality | √/C | × | × | √/C | × | × |
| Equal treatment | × | × | × | × | × | × |
| Equal treatment of necessary players | √/C | √/C | √ | × | × | √ |
| Equal treatment of null players | × | √ | × | × | √ | × |
| Fairness | √/C | × | × | × | × | × |
| Fairness for necessary | √ | √ | √ | × | × | √ |
| Fairness for null | √ | √ | √ | × | √ | × |

All these characterizations support the use of the respective rules when we need to adapt the distribution between individuals in a fixed network in economic situations in which the value changes over time.

## Appendix

### Proof of Proposition 3.2

It has already been proved that the position value satisfies PL-marginality. On the other hand, it is widely known that it verifies efficiency in connected components and balanced link contributions (Slikker 2005), and in particular, for necessary players.

Reciprocally, suppose $\psi$ is an allocation rule in $\mathcal{CS}_0^N$ satisfying these three properties. We will prove that $\psi_i(N, v, \gamma) = \pi_i(N, v, \gamma)$ for all $(N, v, \gamma) \in \mathcal{CS}_0^N$ and for all $i \in N$.

Consider the graph-restricted game $(N, v^\gamma)$. For all $i \in N$, $\delta \subseteq \gamma_i, \eta \subseteq \gamma \setminus \gamma_i$

$$r_\gamma^{v^\gamma}(\eta \cup \delta) - r_\gamma^{v^\gamma}(\eta) = \sum_{C \in N/(\eta\cup\delta)} v^\gamma(C) - \sum_{C \in N/\eta} v^\gamma(C).$$

As a connected component, $C$, of a subgraph of $\gamma$ is also connected in $\gamma$, we have $v^\gamma(C) = v(C)$, and therefore

$$\sum_{C \in N/(\eta\cup\delta)} v^\gamma(C) - \sum_{C \in N/\eta} v^\gamma(C) = \sum_{C \in N/(\eta\cup\delta)} v(C) - \sum_{C \in N/\eta} v(C)$$
$$= r_\gamma^{v}(\eta \cup \delta) - r_\gamma^{v}(\eta).$$

As $\psi$ and $\pi$ satisfy PL-marginality

$$\begin{aligned}\psi_i(N, v^\gamma, \gamma) &= \psi_i(N, v, \gamma),\\ \pi_i(N, v^\gamma, \gamma) &= \pi_i(N, v, \gamma),\end{aligned} \tag{2}$$

and therefore, it is sufficient to prove the result for $(N, v^\gamma, \gamma)$.

The proof will use induction on $d(N, v^\gamma)$, the cardinality of $\delta(N, v^\gamma) = \{S \mid \Delta_{v^\gamma}(S) \neq 0\}$.

Suppose $(N, v, \gamma)$ is such that $d(N, v^\gamma) = 0$. Then, each $i \in N$ is a necessary player in $v^\gamma$. We will prove that, in this case, $\psi_i(N, v^\gamma, \gamma) = \pi_i(N, v^\gamma, \gamma)$ by induction on $|\gamma|$.

If $|\gamma| = 0$, then $i$ is isolated in $(N, \gamma)$ and using component efficiency, $\psi_i(N, v^\gamma, \gamma) = v^\gamma(\{i\}) = 0 = \pi_i(N, v^\gamma, \gamma)$, which proves the result in this case.

By the induction hypothesis, let us assume that the result is proved for $\gamma$ with $|\gamma| \leq r$, $r \geq 0$ and consider $\gamma$, such that $|\gamma| = r + 1$.

Let $C_i^{N,\gamma}$ be the connected component in $(N, \gamma)$ to which $i$ belongs. If $C_i^{N,\gamma} = \{i\}$, using again efficiency in connected components, and similarly to the previous case, $\psi_i(N, v^\gamma, \gamma) = \pi_i(N, v^\gamma, \gamma) = 0$. Otherwise, for all $j \in C_i^{N,\gamma}$, using balanced link contributions for necessary players, we have

$$|\gamma_i|\psi_j(N, v^\gamma, \gamma) - \sum_{l \in \gamma_i} \psi_j(N, v^\gamma, \gamma \setminus \{l\}) = |\gamma_j|\psi_i(N, v^\gamma, \gamma) - \sum_{l' \in \gamma_j} \psi_i(N, v^\gamma, \gamma \setminus \{l'\}),$$

and, as $|\gamma \setminus \{l\}| < r + 1$ and $|\gamma \setminus \{l'\}| < r + 1$, using the induction hypothesis (on $|\gamma|$ ) and that $v^\gamma$ is the null game

$$\psi_i(N, v^\gamma, \gamma \setminus \{l'\}) = \pi_i(N, v^\gamma, \gamma \setminus \{l'\}) = 0, l' \in \gamma_j,$$
$$\psi_j(N, v^\gamma, \gamma \setminus \{l\}) = \pi_j(N, v^\gamma, \gamma \setminus \{l\}) = 0, l \in \gamma_i.$$

Therefore

$$|\gamma_j|\psi_i(N, v^\gamma, \gamma) = |\gamma_i|\psi_j(N, v^\gamma, \gamma),$$

or

$$\psi_j(N, v^\gamma, \gamma) = \frac{|\gamma_j|}{|\gamma_i|}\psi_i(N, v^\gamma, \gamma) = c|\gamma_j| \text{ for all } j \in C_i^{N,\gamma}$$

(let us observe that $|\gamma_i|$ and $|\gamma_j|$ are strictly positive).

Using efficiency in connected components

$$0=\sum_{j\in C_i^{N,\gamma}}\psi_j(N,v^\gamma,\gamma)=\sum_{j\in C_i^{N,\gamma}}c|\gamma_j|,$$

which implies $c=0$ and $\psi_i(N,v^\gamma,\gamma)=0=\pi_i(N,v^\gamma,\gamma)$, which completes the induction on $|\gamma|$ and the result for $d(N,v^\gamma)=0$.

Suppose now that the result is proved for communication situations $(N,v,\gamma)$, such that $d(N,v^\gamma)\le k,\ k\ge 0$, and consider $(N,v,\gamma)$ with $\delta(N,v^\gamma)=\{S_1,...,S_{k+1}\}$, and thus, $d(N,v^\gamma)=k+1$. Then, $v^\gamma=\sum_{m=1}^{k+1}\Delta_{v^\gamma}(S_m)u_{S_m}$. If $i\notin\cap_{m=1}^{k+1}S_m$, i.e., if $i$ is not a necessary player in $(N,v^\gamma)$, let us define $(v^\gamma)_i=\sum_{S_m:i\in S_m}\Delta_{v^\gamma}(S_m)u_{S_m}$. For $\delta\subseteq\gamma_i,\eta\subseteq\gamma\setminus\gamma_i$, let $C_i^{N,\eta\cup\delta}$ and $C_i^{N,\eta}$ be the connected components in $(N,\eta\cup\delta)$ and $(N,\eta)$, respectively, to which $i$ belongs. Then

$$\begin{aligned}r_\gamma^{(v^\gamma)_i}(\eta\cup\delta)-r_\gamma^{(v^\gamma)_i}(\eta)&=(v^\gamma)_i(C_i^{N,\eta\cup\delta})-0\\&=v^\gamma(C_i^{N,\eta\cup\delta})-v^\gamma(C_i^{N,\eta\cup\delta}\setminus\{i\})=r_\gamma^{v^\gamma}(\eta\cup\delta)-r_\gamma^{v^\gamma}(\eta).\end{aligned}$$

Taking into account that both $\psi$ and $\pi$ satisfy PL-marginality

$$\begin{aligned}\psi_i(N,(v^\gamma)_i,\gamma)&=\psi_i(N,v^\gamma,\gamma),\\\pi_i(N,(v^\gamma)_i,\gamma)&=\pi_i(N,v^\gamma,\gamma).\end{aligned}\tag{3}$$

As $d(N,(v^\gamma)_i)\le k$, using the induction hypothesis

$$\psi_i(N,(v^\gamma)_i,\gamma)=\pi_i(N,(v^\gamma)_i,\gamma).\tag{4}$$

Thus, by (2), (3), and(4)

$$\begin{aligned}\psi_i(N,v,\gamma)&=\psi_i(N,v^\gamma,\gamma)=\psi_i(N,(v^\gamma)_i,\gamma)\\&=\pi_i(N,(v^\gamma)_i,\gamma)=\pi_i(N,v^\gamma,\gamma)=\pi_i(N,v,\gamma),\end{aligned}$$

and the result is proved for $i\notin\cap_{m=1}^{k+1}S_m$.

To finish, suppose now that $i\in\cap_{m=1}^{k+1}S_m$. Then, $i$ is a necessary player in $(N,v^\gamma)$. Let $C_i^{N,\gamma}$ be the connected component in $(N,\gamma)$ to which $i$ belongs.

If $C_i^{N.\gamma}=\{i\}$, using component efficiency, both rules coincide. On the other hand, if $i$ is the unique necessary player of $(N,v^\gamma)$ in his component, using the fact that both rules are the same for non-necessary players—as it is shown before—and the component efficiency, both rules coincide.

Let, then, $j\neq i,\ j\in C_i^{N.\gamma}$ and $j$ be a necessary player in $(N,v^\gamma)$.

We will prove that $\pi_i(N,v,\gamma)=\psi_i(N,v,\gamma)$ by the induction hypothesis on $|\gamma|$. If $|\gamma|=1$, then $C_i^{N.\gamma}=\{i,j\}$ and as $\psi$ satisfies balanced link contributions for necessary players

$$\psi_i(N,v^\gamma,\gamma)-\psi_i(N,v^\gamma,\gamma\setminus\{\{i,j\}\})=\psi_j(N,v^\gamma,\gamma)-\psi_j(N,v^\gamma,\gamma\setminus\{\{i,j\}\}).$$

In $\gamma \setminus \{\{i,j\}\}$, $i$ and $j$ are isolated, and using efficiency in connected components, $\psi_i(N, v^\gamma, \gamma \setminus \{\{i,j\}\}) = \psi_j(N, v^\gamma, \gamma \setminus \{\{i,j\}\}) = 0$.

Then, $\psi$ assigns the same amount to $i$ and $j$, $\frac{v^\gamma(C_i^{N,\gamma})}{2}$, quantity that coincides with the allocation of $\pi$ to both players, because $\pi$ also satisfies balanced link contributions for necessary players and efficiency in connected components.

Suppose now that the result is true if $|\gamma| \leq r$, $r \geq 1$, and consider $|\gamma| = r + 1$. As $\psi$ satisfies balanced link contributions for necessary players

$$|\gamma_i|\psi_j(N, v^\gamma, \gamma) - \sum_{l \in \gamma_i} \psi_j(N, v^\gamma, \gamma \setminus \{l\}) = |\gamma_j|\psi_i(N, v^\gamma, \gamma) - \sum_{l' \in \gamma_j} \psi_i(N, v^\gamma, \gamma \setminus \{l'\}),$$

and, as $|\gamma \setminus \{l\}| = |\gamma \setminus \{l'\}| \leq r$, using the induction hypothesis

$$|\gamma_i|\psi_j(N, v^\gamma, \gamma) - \sum_{l \in \gamma_i} \pi_j(N, v^\gamma, \gamma \setminus \{l\}) = |\gamma_j|\psi_i(N, v^\gamma, \gamma) - \sum_{l' \in \gamma_j} \pi_i(N, v^\gamma, \gamma \setminus \{l'\}).$$

On the other hand, $\pi$ satisfies balanced link contributions for necessary players

$$|\gamma_i|\pi_j(N, v^\gamma, \gamma) - \sum_{l \in \gamma_i} \pi_j(N, v^\gamma, \gamma \setminus \{l\}) = |\gamma_j|\pi_i(N, v^\gamma, \gamma) - \sum_{l' \in \gamma_j} \pi_i(N, v^\gamma, \gamma \setminus \{l'\}),$$

and then

$$|\gamma_i|\psi_j(N, v^\gamma, \gamma) - |\gamma_j|\psi_i(N, v^\gamma, \gamma) = |\gamma_i|\pi_j(N, v^\gamma, \gamma) - |\gamma_j|\pi_i(N, v^\gamma, \gamma),$$

or equivalently

$$|\gamma_i|[\psi_j(N, v^\gamma, \gamma) - \pi_j(N, v^\gamma, \gamma)] = |\gamma_j|[\psi_i(N, v^\gamma, \gamma) - \pi_i(N, v^\gamma, \gamma)].$$

Therefore

$$\psi_j(N, v^\gamma, \gamma) - \pi_j(N, v^\gamma, \gamma) = |\gamma_j|\frac{1}{|\gamma_i|}[\psi_i(N, v^\gamma, \gamma) - \pi_i(N, v^\gamma, \gamma)] = |\gamma_j|c,$$

for all $j \in C_i^{N,\gamma}$, $j$ necessary player in $(N, v^\gamma)$ (let us note that $|\gamma_i| \neq 0$ and $|\gamma_j| \neq 0$).

Using efficiency in connected components and that both rules coincide for non-necessary players

$$\sum_{j \in (C_i^{N,\gamma})_n} \psi_j(N, v^\gamma, \gamma) = \sum_{j \in (C_i^{N,\gamma})_n} \pi_j(N, v^\gamma, \gamma),$$

where $(C_i^{N,\gamma})_n$ is the set of all necessary players in $C_i^{N,\gamma}$. Thus

$$0 = \sum_{j \in (C_i^{N,\gamma})_n} [\psi_j(N, v^\gamma, \gamma) - \pi_j(N, v^\gamma, \gamma)] = \sum_{j \in (C_i^{N,\gamma})_n} |\gamma_j|c.$$

Then, $c = 0$ and $\psi_i(N, v^\gamma, \gamma) = \pi_i(N, v^\gamma, \gamma)$, which completes the induction for $r$ and $k$, and thus, the result is proved. □

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